Differential Calculus

Made Easy

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PREFACE

This book on Differential Calculus is designed for students of schools and colleges. The aim of the book is to present the subject in such a way that even an average student can feel no difficulty in understanding it.

There are, of course, a number of books available on the subject, which are too concise to be intelligible for the students. There is hardly any single book which covers the entire syllabus of Differential Calculus prescribed in various state boards.

Every effort has been made to present the subject matter in the most lucid style. The following are some of the salient features of this book :

(*i*) The book serves the purpose of a text as well as a help book.

(*ii*) A large number of typical and important solved examples have been provided to enable the student to have a clear grasp of the subject and to equip him for attempting problems in the board examination without any difficulty.

(*iii*) Care has been taken to grade the problems in such a way that students move from basic intricate problems with ease.

(*iv*) The 'Remarks and Notes' have been added quite often in the book so that they should help in understanding the ideas better.

(*v*) At the end of each chapter, a 'Revision Exercise' has been incorporated. The purpose of this exercise is to give an opportunity to students to revise the entire chapter in the minimum possible time.

It is hoped with all the above characteristics the book will be found really useful by teachers and students alike.

A serious effort has been made to keep the book free from errors, but even then some errors might have crept in.

I am grateful to M/s. Laxmi Publications (P) Ltd., New Delhi, for the keen interest and active co-operation.

Suggestions from the teachers as well as the students for further improvement will be warmly welcomed.

—**Author**

SYMBOLS

Greek Alphabets

A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
D	δ	Delta	M	μ	Mu	Y	υ	Upsilon
E	ε	Epsilon	N	ν	Nu	Φ	φ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega
	∃	there exists		∀	for all			

Metric Weights and Measures

LENGTH

10 millimetres	= 1 centimetre
10 centimetres	= 1 decimetre
10 decimetres	= 1 metre
10 metres	= 1 decametre
10 dekametres	= 1 hectometre
10 hectometres	= 1 kilometre

CAPACITY

10 millilitres	= 1 centilitre
10 centilitres	= 1 decilitre
10 decilitres	= 1 litre
10 litre	= 1 decalitre
10 dekalitres	= 1 hectolitre
10 hectolitres	= 1 kilolitre

VOLUME

1000 cubic centimetres	= 1 centigram
1000 cubic decimetres	= 1 cubic metre

AREA

100 square metres	= 1 are
100 ares	= 1 hectare
100 hectares	= 1 square kilometre

WEIGHT

10 milligrams	= 1 centigram
10 centigrams	= 1 decigram
10 decigrams	= 1 gram
10 grams	= 1 decagram
10 decagrams	= 1 hectogram
10 hectograms	= 1 kilogram
100 kilograms	= 1 quintal
10 quintals	= 1 metric ton (tonne)

ABBREVIATIONS

kilometre	km	tonne	t
metre	m	quintal	q
centimetre	cm	kilogram	kg
millimetre	mm	gram	g
kilolitre	kl	are	a
litre	l	hectare	ha
millilitre	ml	centiare	ca

1

Functions

1.1 INTRODUCTION

Differential calculus deals with the problem of calculating rates of change. The 'function' concept lays the foundation of the study of the most important branch calculus of mathematics. The word 'function' is derived from a Latin word meaning 'operation'. In this chapter, we recall and introduce some frequently used common real valued functions alongwith their graphs and we shall study the properties of some of the most basic functions.

1.2 DEFINITION

Constant. A quantity which have the same value throughout a mathematical operation is called a constant. *e.g.,* $2, -7, \sqrt{3}, \pi$ etc.

Variable. A quantity which can assume different values in a particular problem is called a variable.

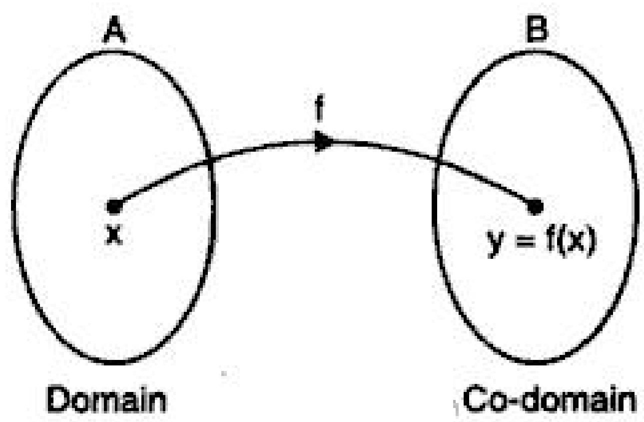
e.g., If x represents any number between 3 and 7, then x is a variable.

1.3 FUNCTION

Let A and B be two non-empty sets of real numbers. If there exists a rule 'f' which associates to every element $x \in$ A, a unique element $y \in$ B, then such a rule f is called a function (or mapping) from the set A to the set B.

If f is a function from A to B, then we write :

$$f : \text{A} \rightarrow \text{B}.$$



The set A is called the domain of function f, and the set B is called the codomain of f.

If x is an element of set A, then the element in B that is associated to x by f is denoted by $f(x)$ and is known as the image of x under f or the value of f at x, and we write $f(x) = y$.

If $f(x) = y$, then we also say that x is a pre-image of y.

The variables x and y are respectively called the independent variable and the dependent variable of the function. This is so, because each y-value depend on the corresponding x-value.

A function of x is generally denoted by the symbol $f(x)$ and read as "f of x".

Caution. $f(x) \neq f \times x$.

1.3.1 Range. The range of a function $f : A \to B$ is the set of all those element of B which are having their pre-images in set A.

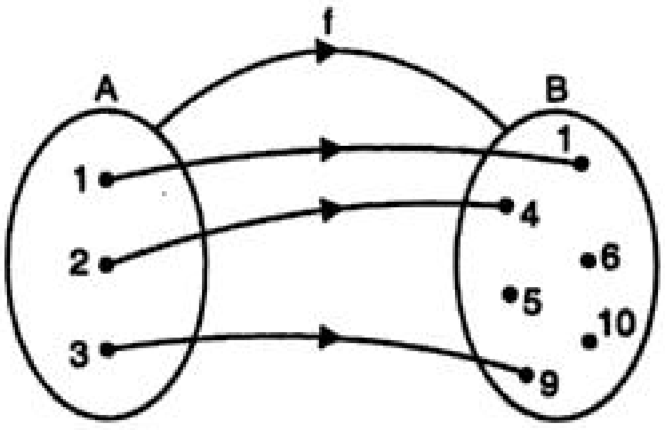
Or the range of a function is the set of images of elements of its domain.

i.e., Range of $f = \{ f(x) : x \in A\}$.

Range $(f) \subseteq$ co-domain of f.

Illustration. Let $A = \{1, 2, 3\}, B = \{1, 4, 5, 9, 10\}$

Let $f : A \to B$ be the mapping which assigns to each element in A, its square.



Thus, we have
$$f(1) = 1^2 = 1$$
$$f(2) = 2^2 = 4$$
$$f(3) = 3^2 = 9.$$

Since to each element (1 or 2 or 3) of A, there is exactly one element of B, so f is a function. In this case every element of B is not image of some element of A.

$\therefore$ We have,

Domain $= \{1, 2, 3\}$

Co-domain $= \{1, 4, 5, 9, 10\}$

Range $= \{1, 4, 9\}$.

1.3.2 Real Valued Functions (Real Functions). A function $f : A \to B$ is said to be a real function if and only if both A and B are the subsets of the real number system R.

e.g., The function $g : R \to R$ defined by $g(x) = x^2 + 1$. $\forall x \in R$, is a real function.

A real function is generally described only by a formula and the domain of the function is not explicitly stated.

In such cases, the domain of the function is the set of all those real numbers x for which the function $f(x)$ is meaningful.

i.e., $f(x) \in R$, as the domain of f.

e.g., we have $f(x) = \sqrt{x-5}$

Here, $f(x)$ is defined if $x - 5 \geq 0$ *i.e.,* $x \geq 5$

$\therefore$ Domain $(f) = \{x : x - 5 \geq 0\} = \{x : x - 5 \geq 0, x \in R\}$.

If $f(a)$ is any of the forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$.

Then, we say that $f(x)$ is not defined at $x = a$.

1.3.3 Equal Functions. Two functions are said to be equal (coincide) if their domains, of definition coincide and their values for all identical values of the arguments are equal.

In other words :

$f, g : A \to B$ are equal if $f(x) = g(x), x \in A$.

Illustration. The function $f(x) = 2$ and $g(x) = 1 + \sin^2 x + \cos^2 x$ coincide.

Illustration. Let $f(x) = \dfrac{x^2 - 25}{x - 5}, x \in R - \{5\}$ and $g(x) = x + 5, x \in R$.

The functions f and g are not equal because f is not defined at 5 whereas g is defined at 5 and has value 10 there at.

Here, we note that $f(x) = g(x)$ for $x \in R - \{5\}$.

1.4 TYPE OF FUNCTIONS

1. One-one function (or Injective mapping). A function $f : A \to B$ is said to be a one-one function, if the images of distinct elements of A are also distinct elements of B.

i.e., $$x \neq y \Rightarrow f(x) \neq f(y)$$

Equivalently, $$f(x) = f(y) \Rightarrow x = y \quad \forall\, x, y \in A.$$

e.g., $f : A \to B$ be the function defined by

$f(x) = 2x + 5$ is one-one function.

2. Many one function. A function $f : A \to B$ is said to be many one function if two or more elements of set A have the same image in B.

e.g., $f : R \to R$ defined by

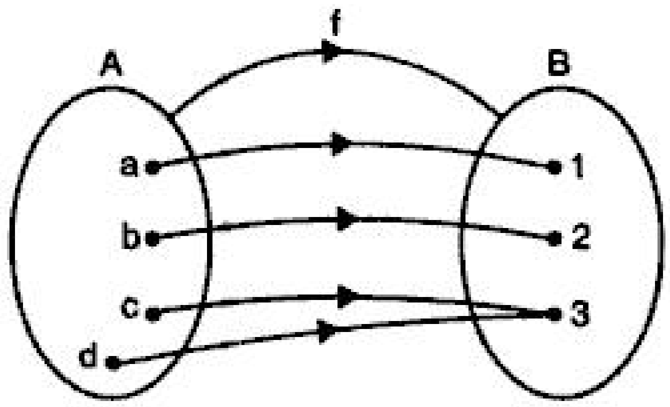
$f(x) = x^2$ is many one function.

$\because \quad f(-2) = f(2) = 4.$

3. Onto function. (or Surjective mapping). A function $f : A \to B$ is said to be an onto function if every element $y \in B$, has at least one pre-image $x \in A$.

In other words, for $y \in B$, there exists at least one $x \in A$ such that :

$$f(x) = y$$



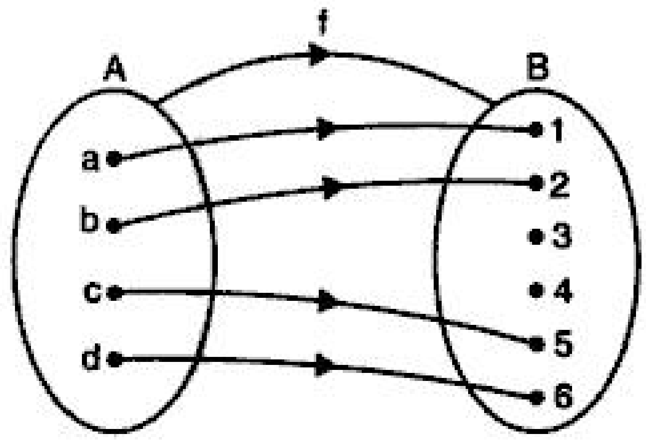
e.g., $f : R \to R$ defined by :

$f(x) = 4x + 5$ is an onto function.

Note. If the function $f : A \to B$ is onto, then range of f *i.e.,* the image of A is whole B.

4. Into function. A function $f : A \to B$ is said to be an into function if for at least one $y \in B$, which has no pre-image in the set A.

The adjoining diagram illustrates an into function, because $3, 4 \in B$ has no pre-image in A.



e.g., The function

$f : R \to R$ defined by

$$f(x) = x^2$$

is an into function, because there is no real number whose image is a negative real number.

Note. A function which is not onto is called an into function.

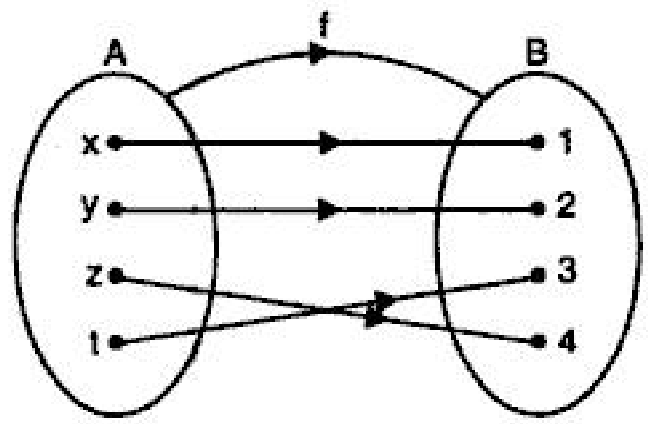
5. One-one onto function. (or Bijection function). A function $f : A \to B$ is said to be one-one onto function if it is both one-one and onto.

e.g., The function

$f : A \to B$ defined by

$$f(x) = 5x + 3, x \in A$$

is an one-one onto function.



Note. Let A and B are finite sets, and $f : A \to B$ is a function :

Then,

(*i*) If f is one-one, then $n(A) \leq n(B)$

(*ii*) If f is onto, then $n(A) \geq n(B)$.

(*iii*) If f is both one-one and onto, then $n(A) = n(B)$.

6. Even function. A function f is even if $f(-x) = f(x)$ for all values of x.

e.g., $f(x) = \cos x$ is an even function.

because, $f(x) = \cos x$

and $f(-x) = \cos(-x) = \cos x = f(x)$.

7. Odd function. A function f is odd if $f(-x) = -f(x)$ for all values of x.

e.g., $f(x) = \sin x$ is an odd function.

because, $f(x) = \sin x$

and $f(-x) = \sin(-x) = -\sin x = -f(x)$.

$[\because \; f(-x) = -x + 1 \neq f(x) \text{ and } f(-x) \neq -f(x)]$

Note. Every function need not be even or odd. *e.g.*, The function $f(x) = x + 1$ is neither even nor odd.

8. Periodic function. A function $f(x) = y$ is said to be a periodic function if there exists a real number $a > 0$ such that :

$$f(x + a) = f(x)$$

Then, a is called period of the function.

e.g., $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π, while $\tan x$ and $\cot x$ are periodic with period π.

9. Identity function. The function $f : A \to B$ defined by $f(x) = x$ *i.e.*, each element of the set A is associated onto itself, then the function f is called an identity function.

10. Inverse function. If a function $f : A \to B$ is one-one and onto function.

$\therefore$ for each $y \in B$, there exists unique $x \in A$ such that :

$$f(x) = y$$

Then, we can define an inverse function, which is denoted by $f^{-1} : B \to A$ and $f^{-1}(y) = x$ if and only if $f(x) = y$.

Note. (*i*) Every function does not have inverse. A function has inverse if and only if it is one-one and onto.

Also, $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

(*ii*) f^{-1} if it exists is unique.

(*iii*) The inverse of the identity function is the identity function itself.

11. Composite function. Let $f : A \to B$ and $g : B \to C$ be two functions.

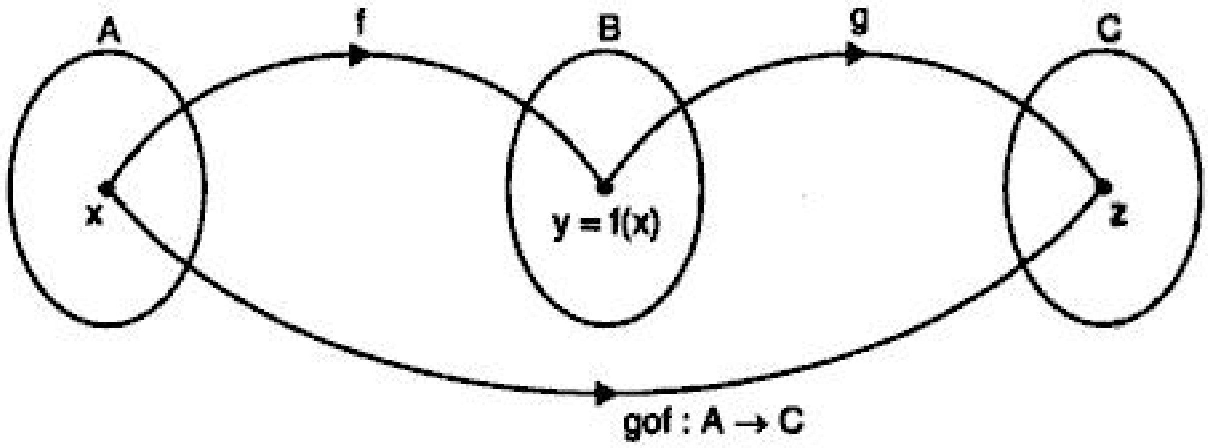
$\therefore$ for each $x \in A$, there exists a unique element $f(x) \in B$.

Since, $g : B \to C$ is a function, so $g(f(x))$ is a unique element of C.

Thus, to each $x \in A$, there exists exactly one element $g(f(x))$ in C. This correspondence between the elements of A and C is called the composite function of f and g is denoted by gof.

$\therefore \qquad (gof)(x) = g(f(x)), x \in A.$

The composite function can be represented by the following diagrams.



Note. (*i*) gof is composite of f and g whereas fog is composite of g and f.

(*ii*) In general $fog \neq gof$.

(*iii*) The existence of fog and gof is independent of each other. *i.e.,* if fog exists then gof may or may not exists and vice-versa.

Illustration. Let $f : R \to R$ and $g : R \to R$ defined by

$$f(x) = 4x + 5 \text{ and } g(x) = 5x - 2.$$

$\therefore \qquad (gof)\,x = g(f(x)) = g(4x + 5) = 5(4x + 5) - 2 = 20x + 25 - 2 = 20x + 23$

and $\qquad (fog)\,x = f(g(x)) = f(5x - 2) = 4(5x - 2) + 5 = 20x - 8 + 5 = 20x - 3.$

Now, let us find $(fog)(5)$;

$\therefore \qquad (fog)(5) = f(g(5)) = f(5(5) - 2) = f(23) = 4(23) + 5 = 92 + 5 = 97.$

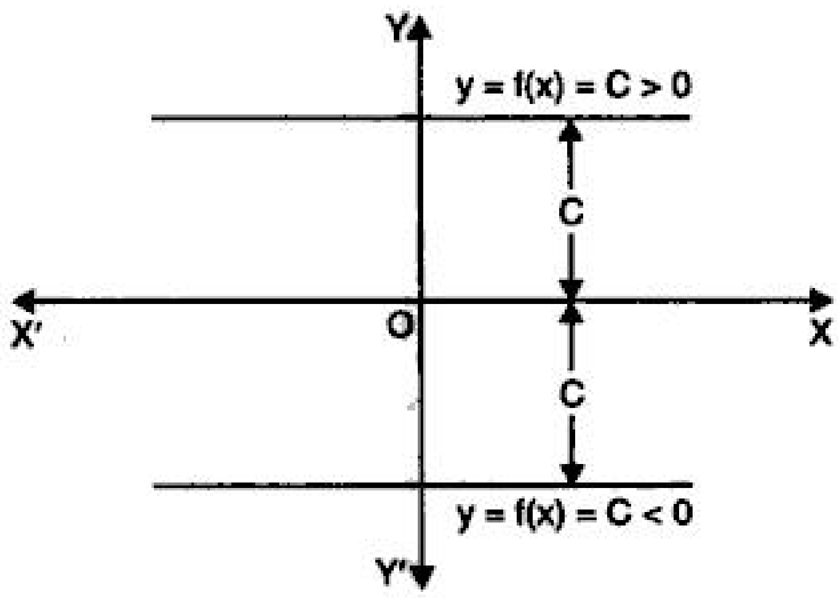
Also, $\qquad (fog)(5) = 20(5) - 3 = 100 - 3 = 97. \qquad [\because \ (fog)(x) = (20x - 3)]$

1.5 GRAPHS OF SOME STANDARD FUNCTIONS

1. Constant function. A function f defined by $y = f(x) = C$ for every $x \in R$, where C is a real number, is called a constant function from R to R.

The graph of this function will be a straight line parallel to x-axis.

If $C > 0$, then the graph will be a straight line parallel to x-axis and at a distance C units above it and if $C < 0$, then the graph will be a straight line parallel to x-axis and at a distance C units below it.



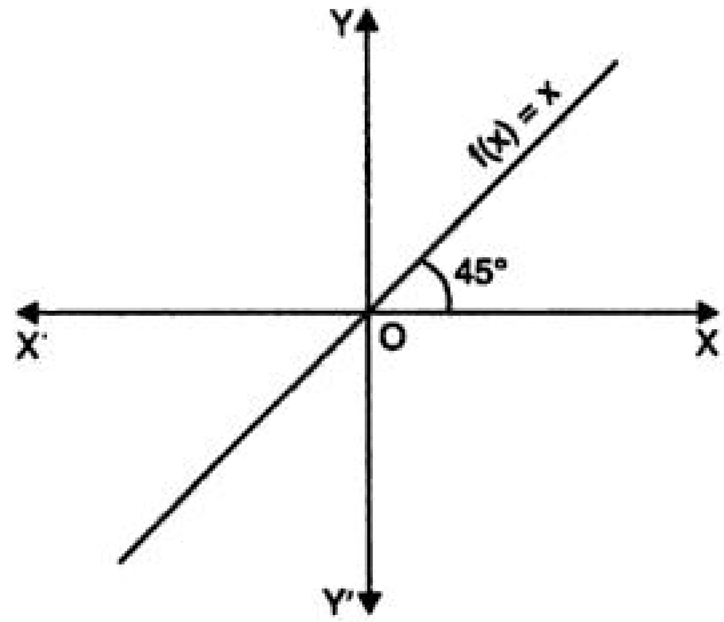
It coincides with x-axis if $C = 0$.

Domain of $f = D(f) = (-\infty, \infty)$, Range of $f = R(f) = \{C\}$.

There are as many constant functions as there are real numbers.

2. Identity function. The function f defined by $f(x) = x$ for all $x \in$ R is called identity function.

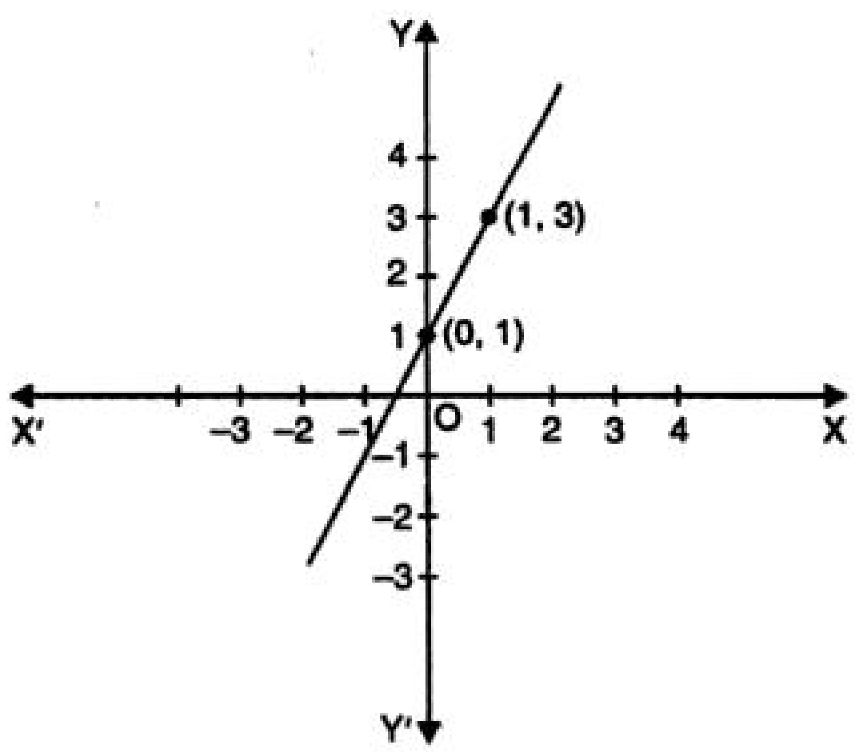
The graph of identity function is a straight line passing through the origin (0, 0) and having slope 1 *i.e,* the line is inclined at an angle of 45° with the x-axis.



Domain of $f = D(f) = (-\infty, \infty)$

Range of $f = R(f) = (-\infty, \infty)$.

3. Linear function. A function f defined by $f(x) = ax + b$, where $a, b \in$ R and $a \neq 0$, is called a linear function from R to R. This is a particular type of polynomial function. For every real value of x, $f(x)$ has a unique value.



The graph of this function will be a straight line and to draw its graph, we need only two points on it.

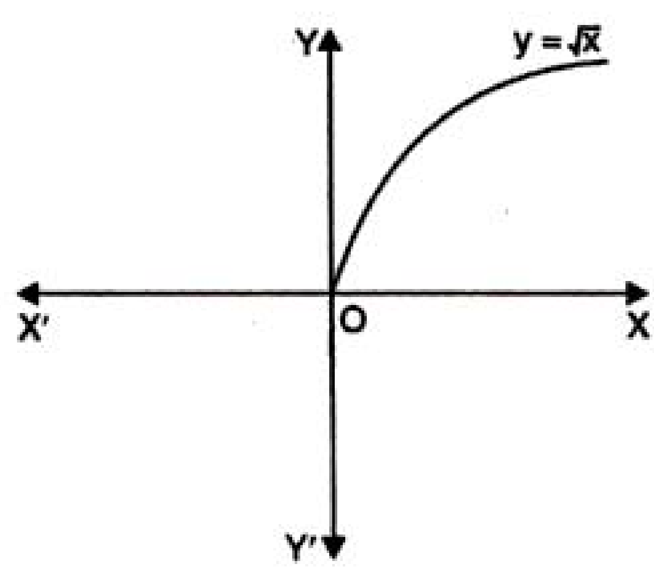
e.g., Let $y = f(x) = 2x + 1$ is a linear function.

x	0	1
y	1	3

Remark. The linear function $f(x) = ax + b$ will represent the identity function when $a = 1$ and $b = 0$.

4. Square root function. The function defined by $f(x) = \sqrt{x}$ $\forall\, x \geq 0$ is called a square root function.

Its domain is the set of all non-negative real numbers and range is also the set of non-negative real numbers.

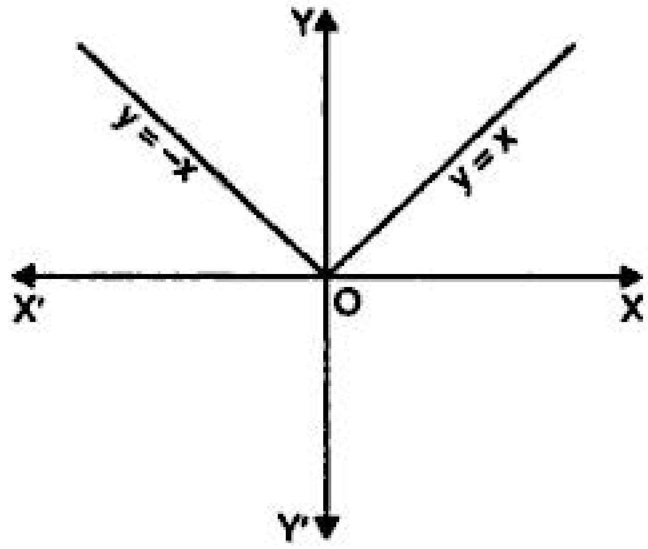


5. Rational function. The function $y = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials is called a rational function, provided $g(x)$ is not a zero polynomial.

6. Modulus function. (Absolute value function). The function $y = f(x) = |x|$ defined by

$$f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$$

is called the modulus function or absolute value function.



For every value of x, the value of $y = |x|$ is unique.

Domain of $f = D(f) = (-\infty, \infty)$ *i.e.,* the set of real numbers.

Range of $f = R(f) = [0, \infty)$ *i.e.,* the set of all non-negative real numbers.

The graph of the modulus function is as shown in the figure.

For $x \geq 0$, the graph coincides with the graph of the identity function *i.e.,* the line $y = x$.

and For $x < 0$, it is coincident to the line $y = -x$.

Since, $f(-x) = |-x| = |x| = f(x)$, it is an even function of x and its graph is symmetrical with respect to y-axis.

Properties of modulus function.

(*i*) $|x| \geq 0 \quad \forall x \in R$ *i.e.,* $|x|$ is always non-negative *i.e.,* positive or zero.

(*ii*) $\sqrt{x^2} = |x| \quad \forall x \in R$

(*iii*) For real numbers x and y, we have

$$|x + y| \leq |x| + |y|$$
$$|x - y| \geq |x| - |y|$$
$$|x + y| = |x| + |y| \text{ if } x, y \geq 0.$$

(*iv*) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, provided $y \neq 0$.

$$|xy| = |x|.|y|.$$

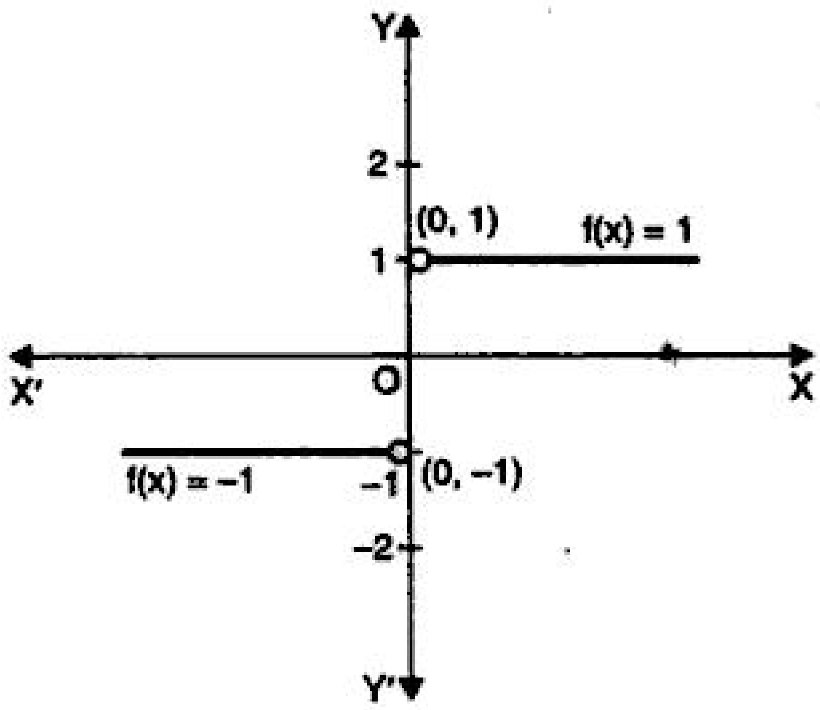
(*v*)
$$|x| \leq c \Rightarrow -c \leq x \leq c$$
$$|x| \geq c \Rightarrow x \geq c \text{ or } x \leq -c$$

where : c is a constant.

7. Signum function. The function

$$y = f(x) = \begin{cases} \dfrac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$$

is called the signum function.



Now, $x \neq 0 \Rightarrow$ either $x > 0$ so that $|x| = x$

or $x < 0$ so that $|x| = -x$.

$$\therefore \qquad f(x) = \begin{cases} 1 \ , \ \text{if } x > 0 \\ 0 \ , \ \text{if } x = 0 \\ -1 \ , \ \text{if } x < 0 \end{cases}.$$

Domain of $f = D(f) = (-\infty, \infty)$ *i.e.,* the set of all real numbers.

Range of $f = R(f) = [-1, 0, 1]$.

The graph of the signum function is as shown in the figure, the point corresponding to $x = 0$ is excluded.

8. The greatest integer function or the integral part function OR the floor function. The function $y = f(x) = [x]$, where $[x]$ is the greatest integer $\leq x$, is called the greatest integer function.

Thus, $[x] = x$ if x is an integer

and $[x]$ = the integer immediately to the left of x if x is not an integer.

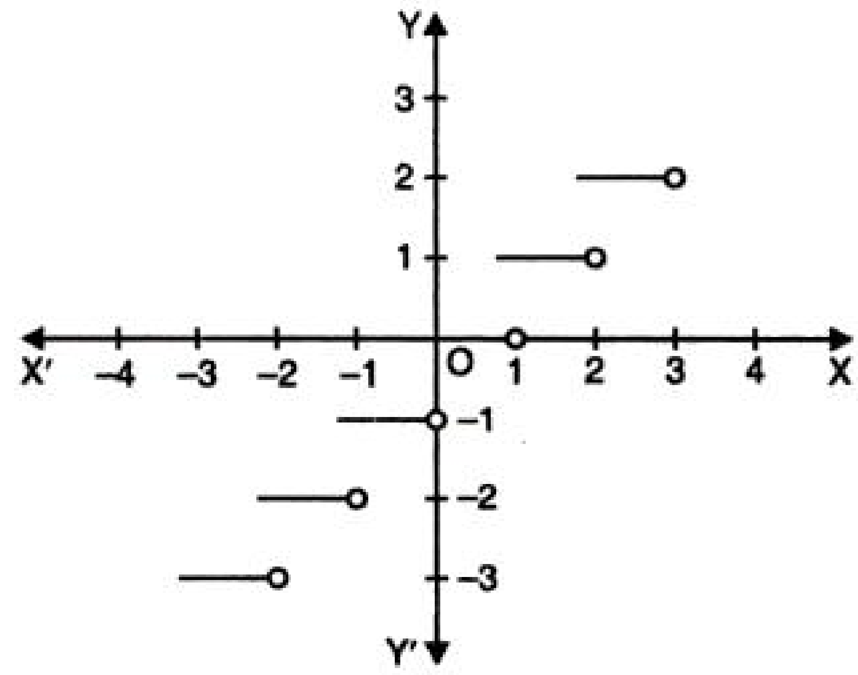
e.g.,

$$[2.9] = 2$$
$$[3] = 3$$
$$[-6.45] = -7 \text{ etc.}$$

Domain of the greatest integer function is the set of all real numbers and the range is the set of all integers, as $[x]$ is always an integer.

If x is a real number and n is integer such that

$$n \leq x < n + 1, \text{ then } [x] = n.$$



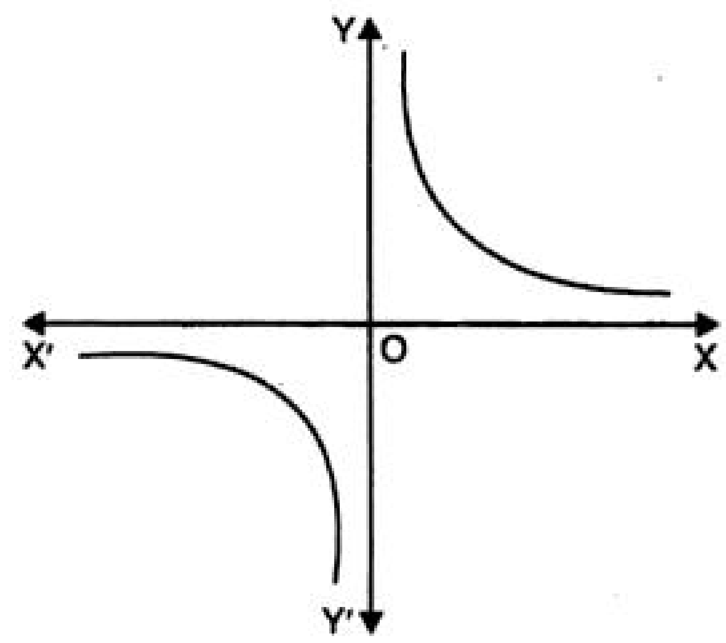
The graph of the greatest integer function is as shown in the figure.

9. Reciprocal function. The function $y = f(x) = \frac{1}{x}$, $x \neq 0$ is called the reciprocal function. For every non-zero real value of x, the value of $y = \frac{1}{x}$ is unique.

Domain of the reciprocal function is R – {0}.

Range of the reciprocal function is R – {0}.

The graph of $y = f(x) = \frac{1}{x}$, $x \neq 0$, is as shown in the figure.



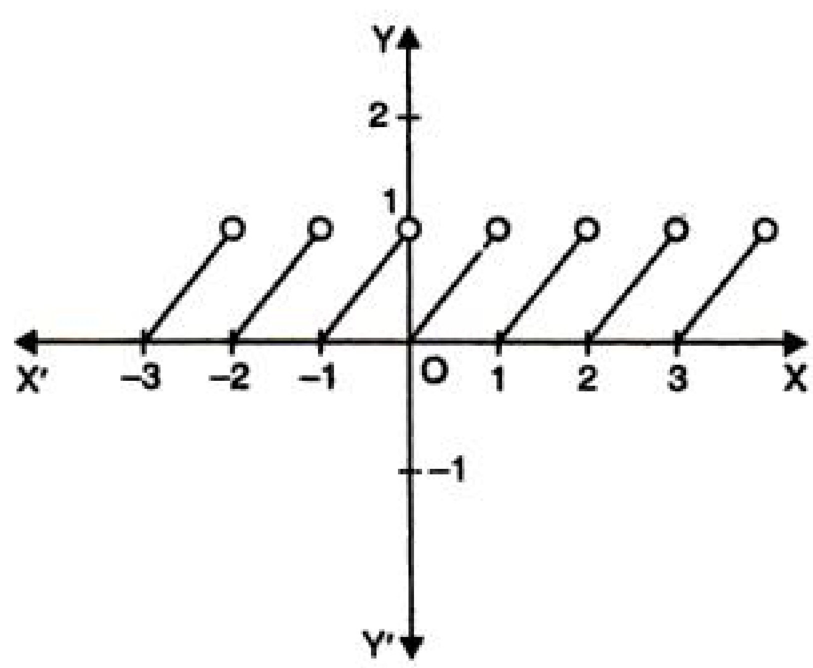
10. Fractional part function. The function f defined by $f(x) = x - [x]$ is called the fractional part function, where $[x]$ greatest integer $\leq x$.

Domain of f = D(f) = set of all reals.

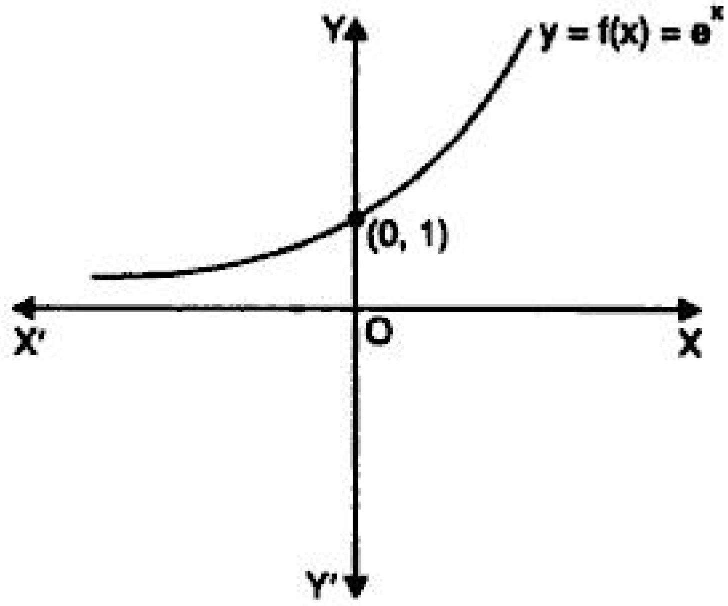
Range of f = R(f) = [0, 1).

The graph of this function is as shown in the figure.

All these lines are parallel with slope 1 *i.e.*, they make an angle of 45° with positive x-axis.



11. Exponential function. We know that for every real number x, the series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ has the sum e^x.



This infinite series is known as the exponential series.

The function $f(x) = e^x$ for all $x \in R$ is called the exponential function. e^x is always positive.

We observe that the domain of an exponential function is R *i.e.*, the set of all reals and the range is the set $(0, \infty)$ as it attains only positive values.

The graph of the exponential function is as shown in the figure.

$y = e^x$ increases as x increases and can be made as large as we please by taking x sufficiently large.

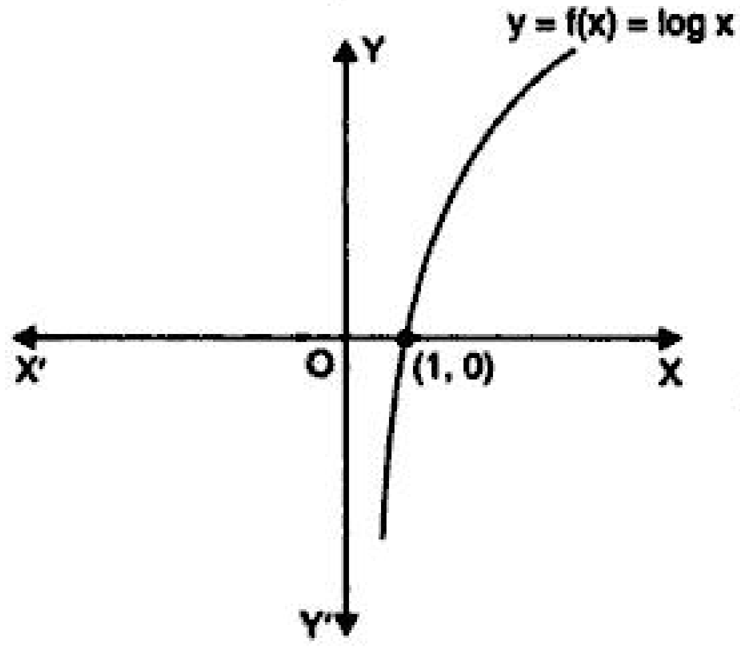
12. Logarithmic function. The function $y = f(x) = \log x, x \in (0, \infty)$ is called logarithmic function. We know that for $0 < x < \infty$.

$$y = \log x \text{ if and only if } x = e^y.$$

Domain of f = D(f) = $(0, \infty)$

Range of f = R(f) = R *i.e.*, the set of all reals.

The graph of f is completely to the right of y-axis.



Properties of logarithmic function.

(*i*) $\log e = 1$

(*ii*) $\log 1 = 0$

(*iii*) $\log (xy) = \log x + \log y$

(*iv*) $\log \left(\frac{x}{y}\right) = \log x - \log y$

(*v*) $e^y = x$ if and only if $y = \log x$

(*vi*) $e^{\log x} = e^y = x$

(*vii*) $\log x^y = y \log x$

(*viii*) $\log_y x = \frac{\log x}{\log y}$.

13. Polynomial function. A function $f : R \to R$ is said to be a polynomial function if for each x in R, $f(x)$ is a polynomial in x.

In other words, A function $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, where $a_0, a_1, a_2, ..., a_n$ and x are real numbers, $a_0 \neq 0$ and n is a non-negative integer, is called a polynomial function of degree n over R.

e.g., $f(x) = x^3 - 3x^2 + 2, g(x) = x^4 + \sqrt{3}\, x$ etc.

are examples of polynomial functions.

Domain of f = D(f) = set of all reals.

Range of f = R(f) = depends upon the polynomial representing the function.

(*i*) If $y = f(x) = ax + b, a \neq 0$, then it is a linear polynomial and its graph is a straight line with slope a and intercept on y-axis is b.

(*ii*) If $y = f(x) = ax^2 + bx + c, a \neq 0$, then it is a quadratic polynomial and its graph is upward parabola if $a > 0$ and downward parabola if $a < 0$.

14. Smallest integer function or **(Ceiling function).** For any real number x, we denote $\lceil x \rceil$, the smallest integer greater than or equal to x.

e.g., $\lceil 5.3 \rceil = 6, \lceil -6.2 \rceil = -7, \lceil 4 \rceil = 4$ etc.

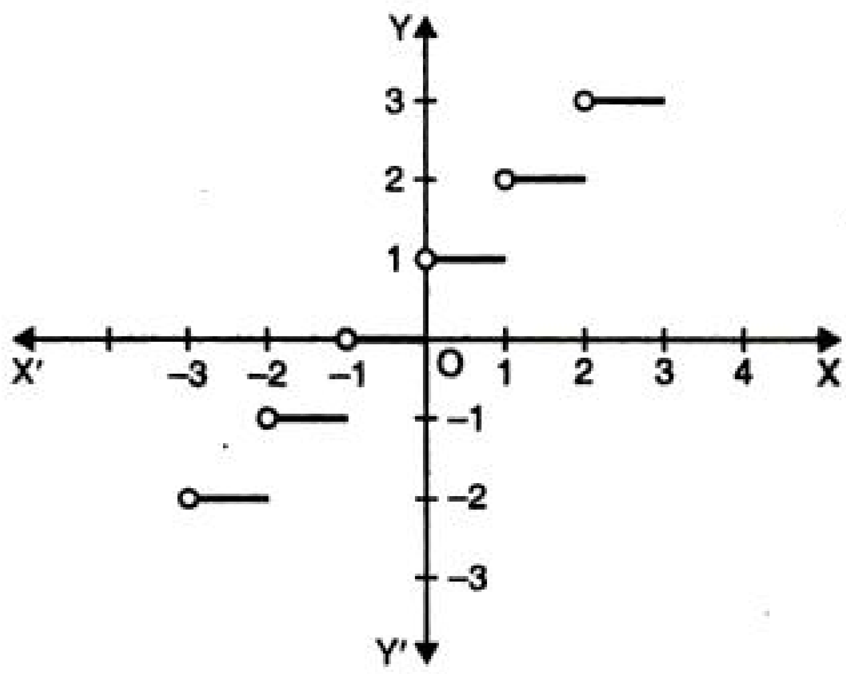
$\therefore$ The function $f : R \to R$ defined by

$$f(x) = \lceil x \rceil, x \in R$$

is called the smallest integer function or the ceiling function.

Domain of f = D(f) = set of all reals

Range of f = R(f) = set of all integers.



The graph of smallest integer function is as shown in the figure.

15. Trigonometric functions. The function f which associates to each x (in radians) the number $\sin x$ or $\cos x$ or $\tan x$ or $\sec x$ or $\operatorname{cosec} x$ or $\cot x$ is called trigonometric function.

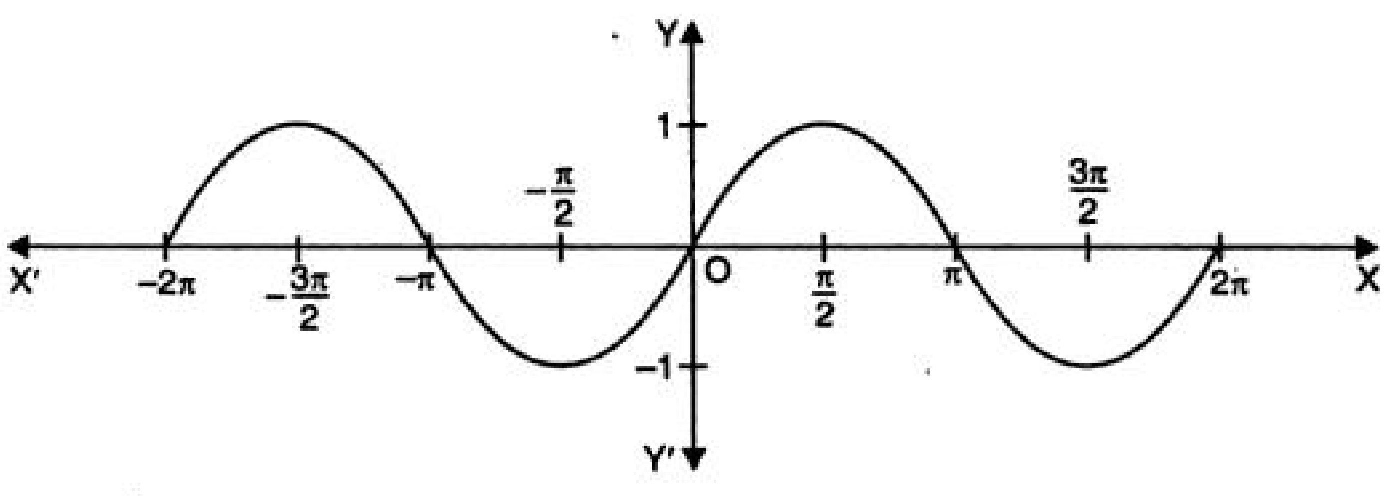
Thus, a function defined in terms of trigonometric relations is called a trigonometric function.

(*i*) **Sine function.** The function f which associates to each x (in radians) to $\sin x$ is called the sine function. *i.e.*, $y = f(x) = \sin x$.

Domain of f = D(f) is the set of all reals.

Range of f = R(f) is $[-1, 1]$

Since, $\sin x$ is a periodic function with period 2π, it is sufficient to sketch its graph only for $0 \leq x \leq 2\pi$. We can then extend it easily by repeating it over the intervals of length 2π.



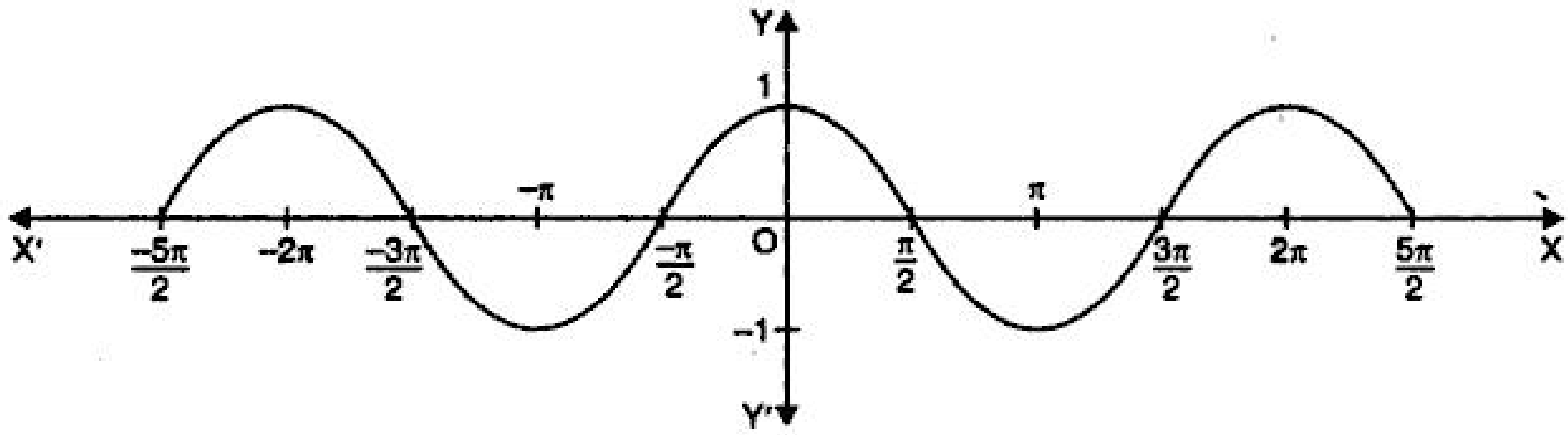
The graph of sine function is as shown in the figure.

(*ii*) **Cosine function.** The function f which associates to each x (in radians) to $\cos x$ is called the cosine function.

i.e., $$y = f(x) = \cos x.$$

Domain of f = D(f) is the set of all reals.

Range of f = R(f) = $[-1, 1]$.



Since, cos x is a periodic function with period 2π, we shall draw its graph for $0 \leq x \leq 2\pi$ and extend it by repeating it over the intervals of length 2π.

Graph of the cosine function is as shown in the figure.

(*iii*) **Tangent function.** The function f which associates to each x (in radians) to $\tan x$ is called the tangent function.

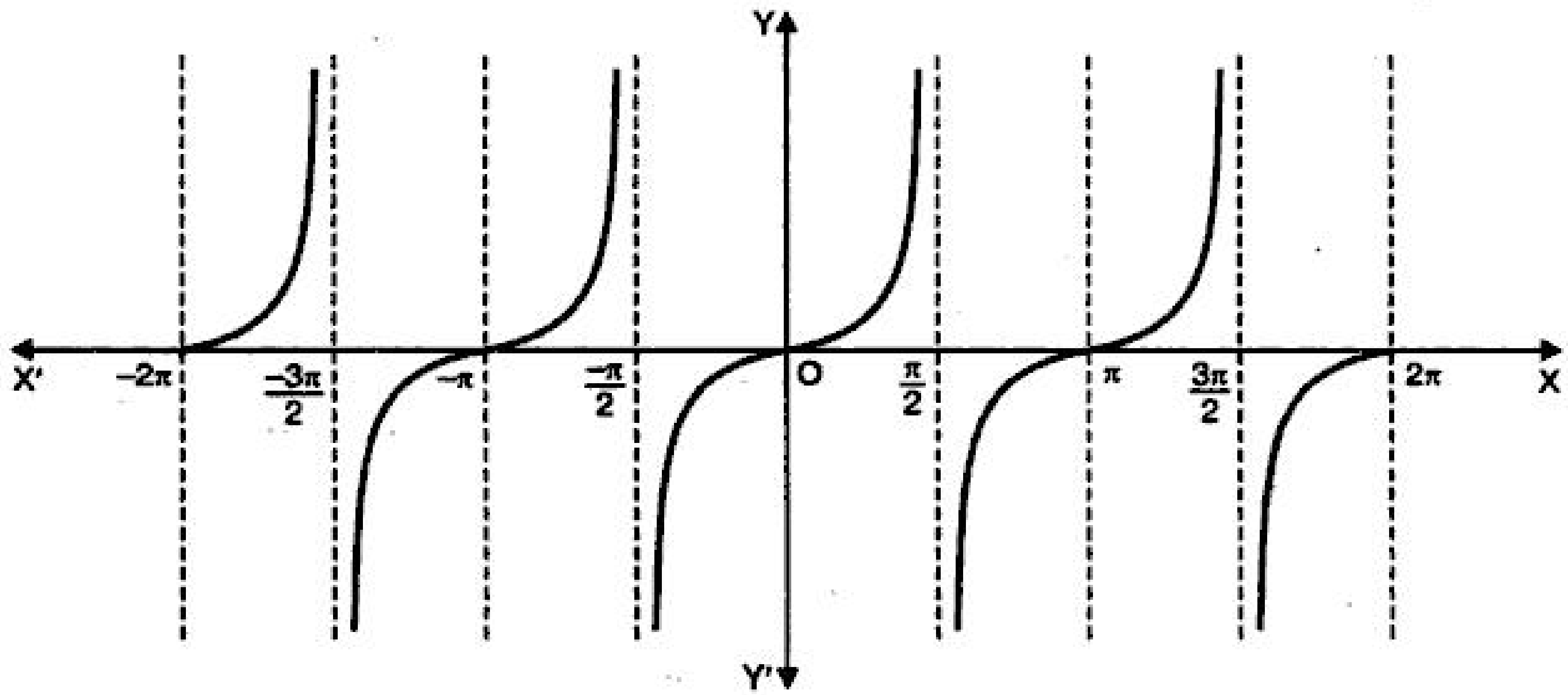
i.e., $$y = f(x) = \tan x.$$

Domain of $f = D(f) = R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Range of f $= R(f) =$ the set of all reals.

Since, $\tan x$ is a periodic function with period π. As x increases from 0 to $\frac{\pi}{2}$, $\tan x$ keeps on increasing from 0 to ∞.

As x crosses the value $\frac{\pi}{2}$, $\tan x$ becomes negative and is arbitrarily large in magnitude. It increases to 0 as x increases from $\frac{\pi}{2}$ to π.



Graph of $f(x) = \tan x$ is as shown in the figure.

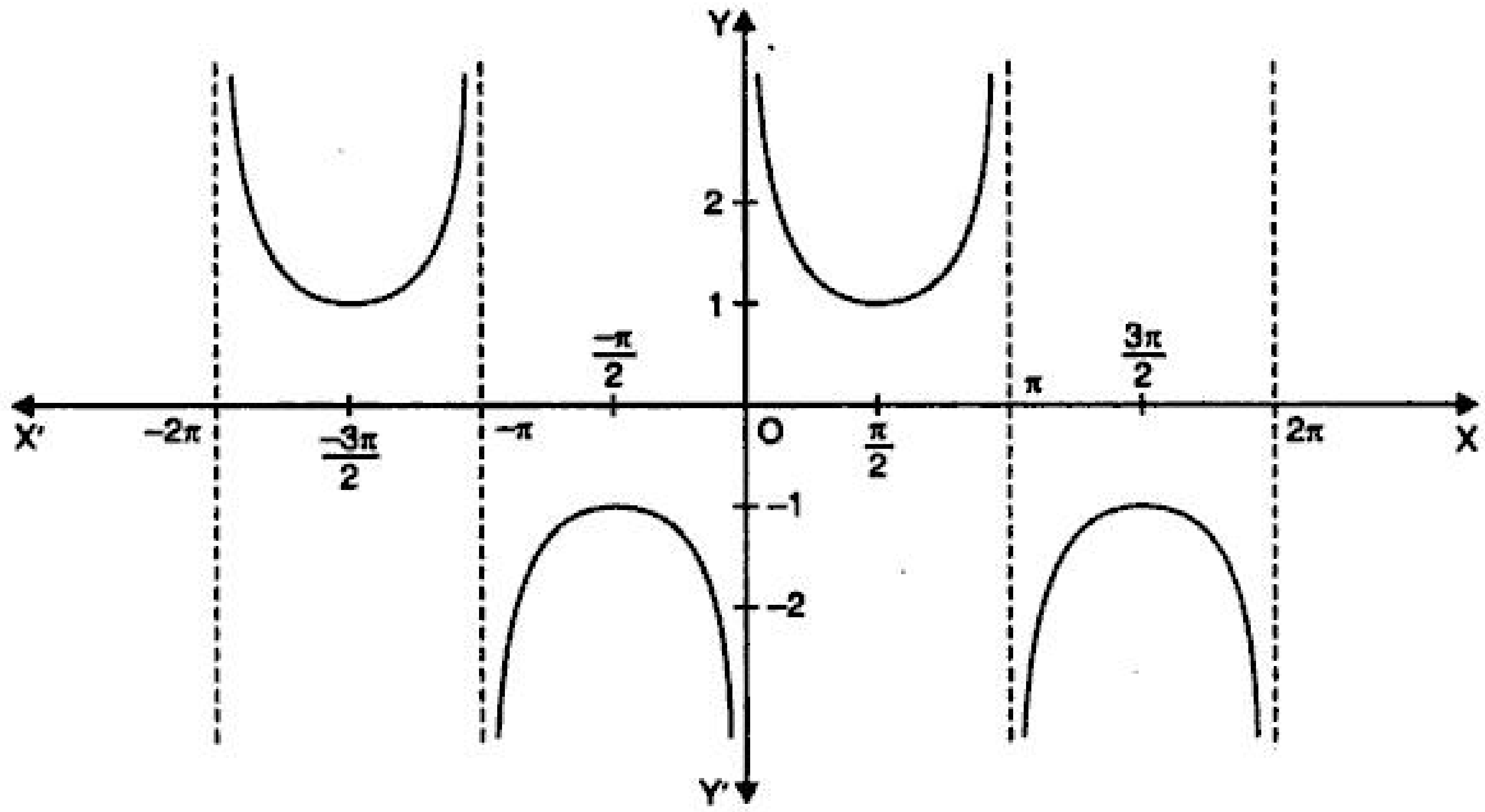
(*iv*) **Cosecant function.** The function f which associates to each x (in radians) to cosec x is called the cosecant function.

i.e., $$y = f(x) = \text{cosec } x$$

Since, cosec x is a periodic function with period 2π.

Domain of $f = D(f) = R - \{n\pi : n \in Z\}$.

Range of $f \;= R(f) = R - (-1, 1)$.



Graph of $f(x)$ = cosec x is as shown in the figure.

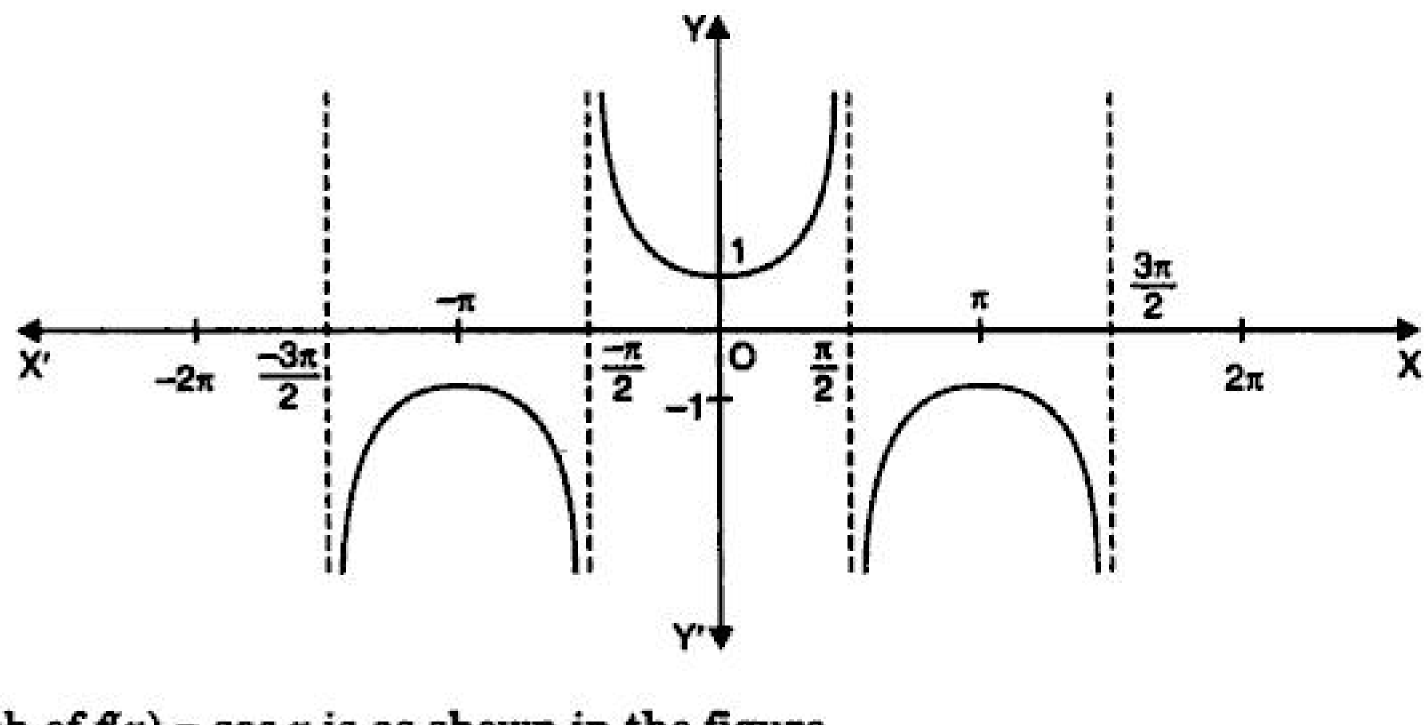
(*v*) **Secant function.** The function f which associates to each x (in radians) to sec x is called the secant function.

i.e., $$y = f(x) = \sec x.$$

Domain of $f = D(f) = R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Range of $f \;= R(f) = R - (-1, 1)$

Since, sec x is a periodic function with period 2π.



Graph of $f(x)$ = sec x is as shown in the figure.

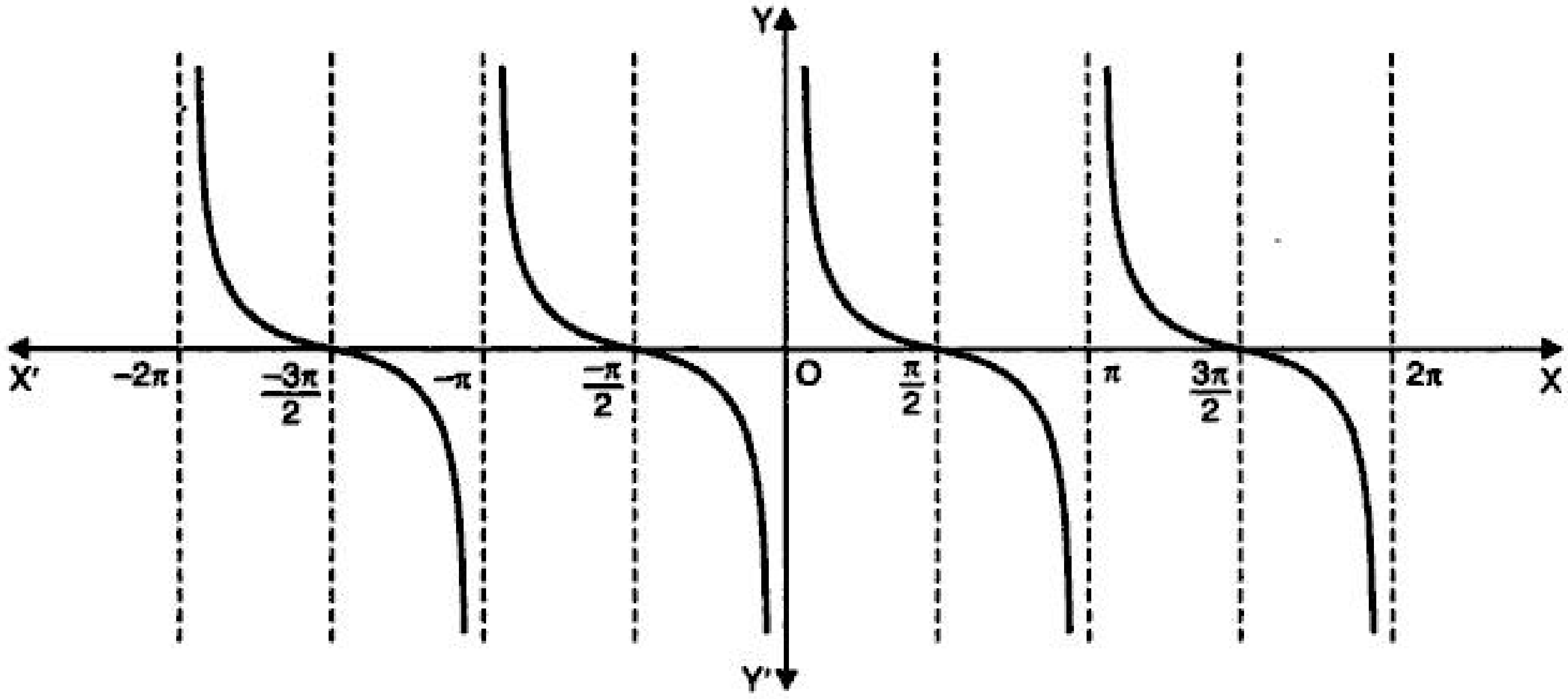
(*vi*) **Cotangent function.** The function f which associates to each x (in radians) to cot x is called the cotangent function.

i.e., $$y = f(x) = \cot x.$$

Domain of f = D(f) = R – $\{n\pi : n \in Z\}$.

Range of f = R(f) = set of all reals.

Since, $\cot x$ is a periodic function with period π. The function $\cot x$ takes all positive and negative values except integral multiples of π, where it is not defined.



The graph of $f(x) = \cot x$ is as shown in figure.

16. Inverse trigonometric functions. The sine function from R to R is neither one-one nor onto, whereas it becomes both one-one and onto from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1, 1]$. The inverse function of $\sin x.\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is called the inverse sine function and is denoted as $\sin^{-1} x$.

The graph of $\sin x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called its principal branch.

$\therefore$ $\sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the function defined as

$\sin^{-1} x = y$ where : $\sin y = x$.

Similarly,

$\cos^{-1} x : [-1, 1] \to [0, \pi]$ is the function defined as $\cos^{-1} x = y$, where $\cos y = x$.

$\tan^{-1} x : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the function defined as $\tan^{-1} x = y$, where $\tan y = x$

$\cot^{-1} x : R \to (0, \pi)$ is the function defined as $\cot^{-1} x = y$, where $\cot y = x$

$\sec^{-1} x : R - [-1, 1] \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is the function defined as $\sec^{-1} x = y$, where $\sec y = x$.

$\text{cosec}^{-1} x : R - (-1, 1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is a function defined as $\text{cosec}^{-1} x = y$, where $\text{cosec } y = x$.

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Note. (*i*) If $R(g) \cap D(f) = \phi$,

Then, $D = \phi$ and *fog* is not defined.

(*ii*) If $R(g) \subset D(f)$,

Then, $D = D(g)$.

$\therefore$ The composite of *g* and *f*, denoted by *gof* is a function defined by :

$$(gof)(x) = g(f(x)) \text{ for all } x \in D$$

where $$D = \{x : x \in D(f) \text{ and } f(x) \in D(g)\} \neq \phi.$$

(*iii*) If $R(f) \cap D(g) = \phi$,

Then, $D = \phi$ and *gof* is not defined.

Hence, if $R(f) \cap D(g) = \phi$, then *gof* does not exists. In other words, *gof* exists if

$$R(f) \cap D(g) \neq \phi.$$

Similarly, *fog* exists if $R(g) \cap D(f) \neq \phi$.

1.8 INVERSE OF A REAL FUNCTION

Let $f : X \to Y$ be a one-one and onto function.

$\therefore$ For each $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$.

$\therefore$ We get a function, denoted by f^{-1} and defined as :

$$f^{-1} : Y \to X$$

such that $f^{-1}(y) = x$ iff $f(x) = y$

The function f^{-1} is called the inverse function of *f*.

Clearly,

$$f^{-1}(f(x)) = f^{-1}(y) = x$$
$$f(f^{-1}(y)) = f(x) = y.$$

Remark. (*i*) Every function does not have inverse. A function has inverse if and only if it is one-one and onto.

(*ii*) The inverse of the identity function is the identity function itself.

(*iii*) Note that the function f^{-1} and $\frac{1}{f}$ for any *f*, need not be same.

SOME SOLVED EXAMPLES

Example 1. *If* $f(x) = \frac{x}{x^2+1}$, *find* $f(1)$, $f(3)$, $f\left(\frac{2}{x}\right)$ *and* $f(b)$.

Solution. We have, $f(x) = \frac{x}{x^2+1}$

$$\therefore \quad f(1) = \frac{1}{1^2+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$f(3) = \frac{3}{(3)^2+1} = \frac{3}{9+1} = \frac{3}{10}$$

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$\therefore$ $\quad$ $D(f) = D(g) \cap D(h) - \{x : h(x) = 0\}$

$= [-1, 1] - \{0\} = [-1, 0) \cup (0, 1]$.

(*v*) We have, $\quad f(x) = \sqrt{x-4}$

$D(f) = \{x : x - 4 \geq 0\}$

$= \{x : x \geq 4\} = [4, \infty)$.

Example 9. *Find the domain of the following functions :*

(*i*) $f(x) = \dfrac{1}{\sqrt{x+|x|}}$ $\qquad$ (*ii*) $f(x) = \sin^{-1} 2x$

(*iii*) $f(x) = \cos^{-1}(3x-1)$ $\qquad$ (*iv*) $f(x) = \tan^{-1}(2x+1)$

(*v*) $f(x) = \dfrac{1}{\sqrt{x+[x]}}$ $\qquad$ (*vi*) $f(x) = |x-2|$.

Solution. (*i*) We have, $f(x) = \dfrac{1}{\sqrt{x+|x|}}$

The function $f(x)$ is defined only when $x + |x|$ is positive.

We know that,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$\therefore$ $x + |x|$ is positive only when $|x| = x$ *i.e.*, when $x > 0$.

$\therefore$ Domain of $f = D(f)$ = the set of all positive reals *i.e.*, $(0, \infty)$.

(*ii*) We have, $\quad f(x) = \sin^{-1} 2x$.

Since, $\sin^{-1} x$ is defined only for $x \in [-1, 1]$.

$\therefore$ $\quad f(x) = \sin^{-1} 2x$ is defined only if :

$-1 \leq 2x \leq 1$

$\Rightarrow \quad -\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$

$\therefore$ Domain of f $\quad = D(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

(*iii*) We have, $\quad f(x) = \cos^{-1}(3x-1)$.

Since, the domain of $\cos^{-1} x$ is $[-1, 1]$.

$\therefore$ $\quad f(x) = \cos^{-1}(3x-1)$ is defined only if

$-1 \leq 3x - 1 \leq 1$

$\Rightarrow \quad 0 \leq 3x \leq 2$

$\Rightarrow \quad 0 \leq x \leq \dfrac{2}{3}$

$\therefore$ Domain of f $= D(f) = \left[0, \frac{2}{3}\right]$.

(*iv*) We have, $f(x) = \tan^{-1}(2x + 1)$

Since, the domain of $\tan^{-1} x$ is the set of all reals *i.e.*, $(-\infty, \infty)$

$\therefore$ $f(x) = \tan^{-1}(2x + 1)$ is defined only if :

$$-\infty < 2x + 1 < \infty$$

$\Rightarrow$ $-\infty < x < \infty$

$\therefore$ Domain of $f = D(f) = (-\infty, \infty)$. *i.e.*, the set of all reals.

(*v*) We have, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$

Since $[x] = \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $x + [x] = \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{cases}$

$\therefore$ $f(x)$ is defined for all values of x for which $x + [x] > 0$

$\therefore$ Domain of f $= D(f) = (0, \infty)$.

(*vi*) We have, $f(x) = |x - 2|$.

Since, $f(x)$ is defined for all values of x and has real, unique and finite values.

$\therefore$ Domain of $f = D(f) =$ the set of all reals. *i.e.*, $(-\infty, \infty)$.

Example 10. *Find the domain of the following functions :*

(*i*) $f(x) = \sqrt{x^2 - 5x + 6}$ (*ii*) $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$

(*iii*) $f(x) = \cos^{-1}[x]$ (*iv*) $f(x) = \dfrac{1}{1 - \cos x}$

(*v*) $f(x) = \sin^{-1}[x]$ (*vi*) $f(x) = \sqrt{x^2 - 1} + \sqrt{x^2 + 1}$

(*vii*) $f(x) = \log\left(\log \dfrac{x}{2}\right)$.

Solution. (*i*) We have, $f(x) = \sqrt{x^2 - 5x + 6}$

$\Rightarrow$ $f(x) = \sqrt{(x-2)(x-3)}$

The function $f(x)$ is defined for only those values of x for which the product

$$(x - 2)(x - 3) \geq 0.$$

$\therefore$ **Case I.** When $x - 2 \geq 0$ and $x - 3 \geq 0$

$\Rightarrow$ $x \geq 2$ and $x \geq 3$

This is true only when $x \geq 3$.

Case II. When $x - 2 \leq 0$ and $x - 3 \leq 0$

$\Rightarrow$ $x \leq 2$ and $x \leq 3$

This is true only when $x \leq 2$.

$\therefore$ Domain of f $= D(f) = (-\infty, 2] \cup [3, \infty)$.

(*ii*) We have, $f(x) = \dfrac{1}{\sqrt{1-\cos x}}$

The function $f(x)$ is defined only for those values of x for which $\cos x \neq 1$.

i.e., $f(x)$ is not defined when x is even multiple of π.

$\therefore$ Domain of f $= D(f) = R - \{x = 2n\pi : n \in Z\}$.

(*iii*) We have, $f(x) = \cos^{-1}[x]$

Since, the domain of $\cos^{-1} x$ is $[-1, 1]$.

$\therefore$ $f(x) = \cos^{-1}[x]$ is defined if $-1 \le [x] \le 1$

But we know that :

$$[x] = \begin{cases} -1 & \text{if } -2 < x \le -1 \\ 0 & \text{if } -1 < x \le 0 \\ 1 & \text{if } 0 < x \le 1 \end{cases}$$

$\therefore$ We have, domain of $f = D(f) = (-2, 1]$.

(*iv*) We have, $f(x) = \dfrac{1}{1-\cos x}$

Please try yourself, as part (*ii*) of the same example.

(*v*) We have, $f(x) = \sin^{-1}[x]$

Since, the domain of $\sin^{-1} x$ is $[-1, 1]$.

$\therefore$ $f(x) = \sin^{-1}[x]$ is defined if $-1 \le [x] \le 1$

But, we know that

$$[x] = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$$

$\therefore$ We have,

Domain of f $= D(f) = [-1, 2)$.

(*vi*) We have, $f(x) = \sqrt{x^2-1} + \sqrt{x^2+1}$

Since, the given function $f(x)$ is the sum of two functions.

$\therefore$ Let $g(x) = \sqrt{x^2-1}$ and $h(x) = \sqrt{x^2+1}$

Now, $\sqrt{x^2-1}$ is defined for all those values of x for which $x^2 - 1 \ge 0$

$\Rightarrow$ $(x-1)(x+1) \ge 0$

Case I. When $x - 1 \ge 0$ and $x + 1 \ge 0$

$\Rightarrow$ $x \ge 1$ and $x \ge -1$

This is true, when $x \ge 1$.

Case II. When $x - 1 \le 0$ and $x + 1 \le 0$

$\Rightarrow$ $x \le 1$ and $x \le -1$

This is true when $x \le -1$

Solution. (*i*) Let $y = f(x) = 3 \sin x - 4 \cos x$ ...(1)

Domain. Since, sin x and cos x are defined for all real values of x.

$\therefore$ $3 \sin x - 4 \cos x$ is also defined for all real values of x.

$\Rightarrow$ $D(f) = R$.

Range. From equation (1), we have

$$y = 3 \sin x - 4 \cos x \quad ...(2)$$

Put $3 = r \cos \theta$ and $4 = r \sin \theta$, $r > 0$. Square and add,

$$(3)^2 + (4)^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$\Rightarrow$ $25 = r^2$ $[\because \sin^2 A + \cos^2 A = 1]$

$\Rightarrow$ $r = 5$

$\therefore$ From equation (2), we have

$$y = r \cos \theta \sin x - r \sin \theta \cos x$$

$\Rightarrow$ $y = 5 (\cos \theta \sin x - \sin \theta \cos x)$

$[\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$

$\Rightarrow$ $y = 5 \sin (x - \theta)$.

But, $-1 \leq \sin (x - \theta) \leq 1$

$\Rightarrow$ $-5 \leq 5 \sin (x - \theta) \leq 5$

$\Rightarrow$ $-5 \leq y \leq 5 \Rightarrow -5 \leq f(x) \leq 5$

$\therefore$ $R(f) = [-5, 5]$.

(*ii*) Let $y = f(x) = \dfrac{1}{\sqrt{x+2}}$ ...(1)

Domain. Here, $f(x)$ is defined only for those real values of x for which $x + 2 > 0$. *i.e.*, $x > -2$.

$\therefore$ $D(f) = [-2, \infty)$.

Range. From equation (1), we have

$$y = \frac{1}{\sqrt{x+2}}$$

Since, $\sqrt{x+2}$ is the positive square root of $x + 2$ for all x in domain of $f(x)$.

$\Rightarrow$ $\dfrac{1}{\sqrt{x+2}} > 0$ *i.e.*, $y > 0$.

$\therefore$ $R(f) = (0, \infty)$.

Alternatively,

From equation (1), we have

$$y = \frac{1}{\sqrt{x+2}} \Rightarrow y^2 = \frac{1}{x+2}$$

$\Rightarrow$ $x + 2 = \dfrac{1}{y^2} \Rightarrow x = \dfrac{1}{y^2} - 2$

But $x > -2$, $\Rightarrow \dfrac{1}{y^2} - 2 > -2$

$\Rightarrow \quad \frac{1}{y^2} > 0$ or $y^2 > 0 \Rightarrow$ either $y > 0$ or $y < 0$

$\Rightarrow \quad y > 0 \quad \therefore \quad R(f) = (0, \infty)$.

(*iii*) Let $\quad y = f(x) = 1 + x - [x - 2] \quad$ ...(1)

Domain. Since, $f(x)$ is the difference of two functions.

$\therefore$ Let $\quad g(x) = 1 + x \quad$ and $\quad h(x) = [x - 2]$

$\therefore \quad D(g) = R \quad$ and $\quad D(h) = R$

$\therefore \quad D(f) = D(g) \cap D(h) = R \cap R = R$.

Range. We know that

$$[a] \le a < [a] + 1$$

$$\Rightarrow \quad 0 \le a - [a] < 1$$

Now, put $a = x - 2$, we have

$$0 \le x - 2 - [x - 2] < 1$$

$$\Rightarrow \quad 3 \le 1 + x - [x - 2] < 4 \qquad \text{[Adding 3 throughout]}$$

$$\Rightarrow \quad 3 \le f(x) < 4$$

$$\therefore \quad R(f) = [3, 4).$$

(*iv*) Let $\quad y = f(x) = \frac{x^2 - 9}{x - 3} \quad$ ...(1)

Domain. Clearly, $f(x)$ is defined for all those real values of x for which $x - 3 \neq 0$, *i.e.*, $x \neq 3$.

$$\therefore \quad D(f) = R - \{3\}.$$

Range. From equation (1), we have

$$y = \frac{x^2 - 9}{x - 3}$$

$$\Rightarrow \quad y = \frac{(x + 3)(x - 3)}{x - 3} \Rightarrow y = x + 3$$

$\therefore$ When $x = 3$, then, $y = 3 + 3 = 6$.

$$\therefore \quad R(f) = R - \{6\}.$$

(*v*) Let $\quad y = f(x) = \frac{1}{2 - \cos 3x} \quad$ ...(1)

Domain. The function $f(x)$ is not defined for those values of x for which $2 - \cos 3x = 0$.

$$\therefore \quad 2 - \cos 3x \neq 0$$

which is true for all real values of x.

$$\because \quad -1 \le \cos 3x \le 1$$

$$\therefore \quad D(f) = R.$$

Range. From equation (1), we have

$$y = \frac{1}{2 - \cos 3x} \Rightarrow 2 - \cos 3x = \frac{1}{y}$$

$\Rightarrow \quad \cos 3x = 2 - \frac{1}{y}$ $\quad [\because \; -1 \le \cos 3x \le 1]$

$\Rightarrow \quad -1 \le 2 - \frac{1}{y} \le 1$

$\Rightarrow \quad -3 \le -\frac{1}{y} \le -1$ $\quad$ [Adding – 2 throughout]

$\Rightarrow \quad 3 \ge \frac{1}{y} \ge 1 \;\Rightarrow\; \frac{1}{3} \le y \le 1$

$\Rightarrow \quad y \in \left[\frac{1}{3}, 1\right]$

$\therefore \quad R(f) = \left[\frac{1}{3}, 1\right].$

Example 14. *Find the domain and range of the following functions :*

(*i*) $\frac{1}{\sqrt{4 + 3\sin x}}$ $\qquad$ (*ii*) $\sec x$

(*iii*) $1 + 3\cos 2x$ $\qquad$ (*iv*) $\frac{1}{(2x - 3)(x + 1)}$

(*v*) $\tan^{-1} x + \cot^{-1} x$ $\qquad$ (*vi*) $x^2 - [x^2]$.

Solution. (*i*) Let $\quad y = f(x) = \frac{1}{\sqrt{4 + 3\sin x}}$ $\quad$ ...(1)

Domain. Since $\sin x$ is defined for all real values of x.

$\therefore \quad -1 \le \sin x \le 1 \quad$ for all $x \in R$

$\Rightarrow \quad -3 \le 3\sin x \le 3 \quad$ for all $x \in R$

$\Rightarrow \quad 1 \le 4 + 3\sin x \le 7 \quad$ for all $x \in R$ $\quad$ [Adding 4 throughout]

$\therefore$ The function $f(x)$ is defined for all those real values of x for which $(4 + 3\sin x) \ne 0$.

$\therefore \quad f(x) = \frac{1}{\sqrt{4 + 3\sin x}}$ is defined for all $x \in R$.

$\therefore \quad D(f) = R$.

Range. From equation (1), we have

$$f(x) = \frac{1}{\sqrt{4 + 3\sin x}}$$

Since, $\quad -1 \le \sin x \le 1 \quad \forall x \in R$

$\Rightarrow \quad -3 \le 3\sin x \le 3 \quad \forall x \in R$

$\Rightarrow \quad 1 \le 4 + 3\sin x \le 7 \quad \forall x \in R$ $\quad$ [Adding 4 throughout]

$\Rightarrow \quad 1 \le \sqrt{4 + 3\sin x} \le \sqrt{7} \quad \forall x \in R$

$\Rightarrow \quad \frac{1}{\sqrt{7}} \le \frac{1}{\sqrt{4 + 3\sin x}} \le 1 \quad \forall x \in R$

Range. From equation (1), we have

$$y = 1 - |x - 7|$$

$\Rightarrow$ $|x - 7| = 1 - y$

$\Rightarrow$ $1 - y \geq 0 \Rightarrow y \leq 1$

$\therefore$ $R(f) = (-\infty, 1]$.

(*iv*) Let $y = f(x) = \left[\log\left(\sin^{-1}\sqrt{x^2 + 3x + 2}\right)\right]$ ...(1)

where [ ] → greatest integer function

Domain. Since, $f(x)$ is defined for all those real values of x, for which

$$0 < (x^2 + 3x + 2) \leq 1$$

$\therefore$ **Case I.** When, $x^2 + 3x + 2 > 0$

$\Rightarrow$ $(x + 1)(x + 2) > 0$

$\Rightarrow$ Either $(x + 1) > 0$ and $(x + 2) > 0$

or $(x + 1) < 0$ and $(x + 2) < 0$

$\therefore$ Either $x + 1 > 0$ and $x + 2 > 0$

$\Rightarrow$ $x > -1$ and $x > -2$

$\Rightarrow$ $x > -1$.

or $x + 1 < 0$ and $x + 2 < 0$

$\Rightarrow$ $x < -1$ and $x < -2$

$\Rightarrow$ $x < -2$.

$\Rightarrow$ $x \in (-\infty, -2) \cup (-1, \infty)$.

Case II. When $x^2 + 3x + 2 \leq 1$

$\Rightarrow$ $x^2 + 3x + 1 \leq 0$

$\Rightarrow$ $x \in \left[\frac{-3 - \sqrt{5}}{2}, \frac{-3 + \sqrt{5}}{2}\right]$

$$\left[\begin{array}{l} \because\ x^2 + 3x + 1 \leq 0 \\ \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \Rightarrow x = \dfrac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)} \\ \Rightarrow x = \dfrac{-3 \pm \sqrt{5}}{2} \end{array}\right]$$

$\therefore$ From case (I) and case (II), we have

$$x \in \left[\frac{-3 - \sqrt{5}}{2}, \frac{-3 + \sqrt{5}}{2}\right] \text{ and } x \in (-\infty, -2) \cup (-1, \infty)$$

$\Rightarrow$ $x \in \left[\frac{-3 - \sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3 + \sqrt{5}}{2}\right]$

$\therefore$ $D(f) = \left[\frac{-3 - \sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3 + \sqrt{5}}{2}\right]$.

(*ii*) We have, $$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}.$$

Domain. Since, the given function $f(x)$ is defined for all real values of x

$\therefore$ $D(f) = \mathbb{R}$. *i.e.*, the set of all reals.

Range. When $x < 0$; $\therefore$ $f(x) = x^2$

$\Rightarrow$ $x^2 > 0$

$\Rightarrow$ $f(x) \in (0, \infty)$.

When $0 \le x \le 1$; $f(x) = x$

$\Rightarrow$ $0 \le x \le 1$

$\Rightarrow$ $f(x) \in [0, 1]$

When $x > 1$; $f(x) = \dfrac{1}{x}$

$\Rightarrow$ $\dfrac{1}{x} < 1$ and $\dfrac{1}{x} > 0$

$\Rightarrow$ $f(x) \in (0, 1)$.

$\therefore$ $R(f) = (0, \infty) \cup [0, 1] \cup (0, 1) = [0, \infty)$.

Example 17. *Find the domain and range of the following functions :*

(*i*) $\sin^{-1}(|x-2| - 2)$ (*ii*) $\sin^2(x^3) + \cos^2(x^3)$

(*iii*) $\cos^{-1}[x]$ (*iv*) $[\cos x]$.

Solution. (*i*) Let $y = f(x) = \sin^{-1}(|x-2| - 2)$ ...(1)

Domain. As we know that, the domain of $\sin^{-1} x$ is $[-1, 1]$.

$\therefore$ $f(x)$ is defined only if,

$$-1 \le |x-2| - 2 \le 1 \Rightarrow 1 \le |x-2| \le 3 \text{ [Adding 2 throughout]}$$

$\Rightarrow$ $-3 \le x - 1 \le -1$ or $1 \le x - 1 \le 3$ $\left[\because a \le |x| \le b \text{ if and only if, } -b \le x \le -a \text{ or } a \le x \le b\right]$

$\Rightarrow$ $-2 \le x \le 0$ or $2 \le x \le 4$

$\therefore$ $x \in [-2, 0] \cup [2, 4]$

$\therefore$ $D(f) = [-2, 0] \cup [2, 4]$.

Range. The value of inverse sine function lies between $\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$, both inclusive

$\therefore$ We have, $-1 \le |x-1| - 2 \le 1$

$\Rightarrow$ $\sin^{-1}(-1) \le \sin^{-1}(|x-1| - 2) \le \sin^{-1}(1)$.

$\Rightarrow$ $\sin^{-1}(-1) \le f(x) \le \sin^{-1} 1$

$\therefore$ $R(f) = \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.

(*ii*) Let $$y = f(x) = \sin^2 (x^3) + \cos^2 (x^3) \qquad ...(1)$$

Domain. Since, we know that, $\sin^2 (x^3) + \cos^2 (x^3) = 1$ for all real values of x.

$$\therefore \qquad D(f) = R.$$

Range. For every real value of x, the given function $f(x)$ has a constant value *i.e.*, 1.

$$\therefore \qquad R(f) = \{1\}.$$

(*iii*) Let $$y = f(x) = \cos^{-1} [x]. \qquad ...(1)$$

Domain. We know that, the function $\cos^{-1} x$ is defined only when $-1 \le x \le 1$.

$$\therefore \qquad D(f) = \{x : -1 \le [x] \le 1\}$$
$$= \{x : [x] = -1 \text{ or } 0 \text{ or } 1\}$$

$\therefore$ when $-1 \le x < 0 \Rightarrow [x] = -1$

when $0 \le x < 1 \Rightarrow [x] = 0$

when $1 \le x < 2 \Rightarrow [x] = 1$

$$\therefore \qquad D(f) = [-1, 2).$$

Range. The value of inverse cosine function lies between 0 and π, both inclusive.

Also, $\cos^{-1} (-1) = \pi$

$$\cos^{-1} (0) = \frac{\pi}{2} \quad \text{and} \quad \cos^{-1} (1) = 0$$

$$\therefore \qquad R(f) = \left\{0, \frac{\pi}{2}, \pi\right\}.$$

(*iv*) Let $$y = f(x) = [\cos x] \qquad ...(1)$$

Domain. As $\cos x$ is defined for all real values of x.

$\therefore$ $f(x) = [\cos x]$ is also defined for all real values of x.

$$\therefore \qquad D(f) = R.$$

Range. We know that, range of $\cos x$ is $[-1, 1]$

i.e., $$-1 \le \cos x \le 1$$

$\therefore$ when $-1 \le \cos x < 0 \Rightarrow [\cos x] = -1$

and when $0 \le \cos x < 1 \Rightarrow [\cos x] = 0$ and $\cos x = 1$

$\Rightarrow [\cos x] = 1$

$$\therefore \qquad R(f) = \{-1, 0, 1\}.$$

Example 18. *If* $y = f(x) = \dfrac{2x-3}{5x-2}$, *show that* $f(y) = x$.

Solution. It is given that, $y = f(x) = \dfrac{2x-3}{5x-2}$ ...(1)

$$\Rightarrow \qquad y(5x-2) = 2x-3$$
$$\Rightarrow \qquad 5xy - 2y = 2x - 3$$
$$\Rightarrow \qquad 5xy - 2x = 2y - 3$$
$$\Rightarrow \qquad x(5y-2) = 2y-3$$
$$\Rightarrow \qquad x = \frac{2y-3}{5y-2} = f(y).$$

Example 19. *(i) If $f : R \to R$ and $g : R \to R$ be functions defined by $f(x) = x^2 + 1$ and $g(x) = \sin x$, then find fog and gof.*

(ii) If $f(x) = x^2 + x - 1$ and $g(x) = (4x - 7)$, Then find :

$$(f+g), (f-g), \left(\frac{f}{g}\right) \text{ and } (f \cdot g).$$

Solution. (*i*) We have,

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = \sin x$$

Since, $x^2 \geq 0 \quad \forall x \in R$

$\Rightarrow \quad x^2 + 1 \geq 1 \quad \forall x \in R$

$\Rightarrow \quad f(x) \geq 1 \quad \forall x \in R$

$\therefore \quad R(f) = [1, \infty).$

Also, the range of function $g(x) = \sin x$ is

$$R(f) = [-1, 1] \quad \forall x \in R.$$

Clearly, $R(f) = [1, \infty) \subseteq D(g)$

and $R(g) = [-1, 1] \subseteq D(f)$.

$\therefore \quad (gof)(x) = g(f(x)) = g(x^2 + 1) = \sin(x^2 + 1)$

and $(fog)\, x = f(g(x)) = f(\sin x) = \sin^2 x + 1$

(*ii*) We have, $f(x) = x^2 + x - 1 \quad \text{and} \quad g(x) = 4x - 7$

$\therefore \quad (f+g)(x) = f(x) + g(x) = (x^2 + x - 1) + (4x - 7)$

$$= x^2 + 5x - 8$$

$$(f-g)(x) = f(x) - g(x) = (x^2 + x - 1) - (4x - 7)$$

$$= x^2 - 3x + 6$$

$$\left(\frac{f}{g}\right) x = \frac{f(x)}{g(x)} = \frac{x^2 + x - 1}{4x - 7}, \text{ where} : x \neq \frac{7}{4}$$

$$(f \cdot g)\, x = f(x) \cdot g(x) = (x^2 + x - 1)(4x - 7)$$

$$= 4x^3 + 4x^2 - 4x - 7x^2 - 7x + 7$$

$$= 4x^3 - 3x^2 - 11x + 7.$$

Example 20. *(i) If $f(x) = \cos x$ and $g(x) = e^x$, find $(f+g)$, $(f-g)$, $(f \cdot g)$ and $\left(\frac{f}{g}\right)$.*

(ii) If $f(x) = e^x$ and $g(x) = \log_e x$, find fog and gof.

Solution. (*i*) We have,

$$f(x) = \cos x \quad \text{and} \quad g(x) = e^x$$

$\therefore \quad (f+g)(x) = f(x) + g(x) = \cos x + e^x$

$$(f-g)(x) = f(x) - g(x) = \cos x - e^x$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = \cos x \cdot e^x = e^x \cos x$$

$$\left(\frac{f}{g}\right) x = \frac{f(x)}{g(x)} = \frac{\cos x}{e^x} = e^{-x} \cos x$$

and $f(x_2) = \sqrt{4 - x_2^2}$ for $0 \le x_1, x_2 \le 2$.

$f(x_1) = f(x_2)$

$\Rightarrow \sqrt{4 - x_1^2} = \sqrt{4 - x_2^2}$

$\Rightarrow 4 - x_1^2 = 4 - x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2 \Rightarrow x_1 = x_2$ [$\because$ x_1 and x_2 are non-negative]

$\therefore$ $f(x)$ is one-one.

f(x) is onto :

Let $y \in [0, 2]$

$f(x) = y$

$\Rightarrow \sqrt{4 - x^2} = y \Rightarrow 4 - x^2 = y^2$

$\Rightarrow x^2 = 4 - y^2$

$\Rightarrow x = \sqrt{4 - y^2}$

$\Rightarrow 0 \le x \le 2.$ [$\because$ $0 \le \sqrt{4 - y^2} \le 2$]

$\therefore$ $f(x)$ is onto.

$\therefore$ $f(x)$ is one-one and onto, so $f(x)$ is invertible and hence f^{-1} exists.

Since, f^{-1} exists.

$\therefore$ $y = f(x)$

$\Rightarrow x = f^{-1}(y)$

$\therefore$ We have, $x = \sqrt{4 - y^2}$

$\Rightarrow f^{-1}(y) = \sqrt{4 - y^2}$

$\Rightarrow f^{-1}(x) = \sqrt{4 - x^2}$

$\therefore$ $f^{-1}(x) = \sqrt{4 - x^2}$; $0 \le x \le 2.$

(*ii*) We have, $f(x) = \dfrac{2x - 1}{x + 3}$

f(x) is one-one :

Let $f(x_1) = \dfrac{2x_1 - 1}{x_1 + 3}$ and $f(x_2) = \dfrac{2x_2 - 1}{x_2 + 3}$

$f(x_1) = f(x_2)$

$\Rightarrow \dfrac{2x_1 - 1}{x_1 + 3} = \dfrac{2x_2 - 1}{x_2 + 3}$

$\Rightarrow \quad (2x_1 - 1)(x_2 + 3) = (2x_2 - 1)(x_1 + 3)$ [cross-multiplication]

$\Rightarrow \quad 2x_1 x_2 + 6x_1 - x_2 - 3 = 2x_1 x_2 + 6x_2 - x_1 - 3$

$\Rightarrow \quad 6x_1 + x_1 = 6x_2 + x_2$

$\Rightarrow \quad 7x_1 = 7x_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ $f(x)$ is one-one.

f(x) is onto :

Let $f(x) = y$

$$\Rightarrow \quad \frac{2x-1}{x+3} = y$$

$\Rightarrow \quad (2x - 1) = y(x + 3)$

$\Rightarrow \quad 2x - xy = 3y + 1$

$\Rightarrow \quad x(2 - y) = 3y + 1$

$$\Rightarrow \quad x = \frac{1+3y}{2-y}$$

such that : $f(x) = y$

$$\Rightarrow \quad f\left(\frac{1+3y}{2-y}\right) = \frac{2\left(\frac{1+3y}{2-y}\right)-1}{\left(\frac{1+3y}{2-y}\right)+3} = \frac{(2+6y)-(2-y)}{(1+3y)+3(2-y)}$$

$$= \frac{7y}{1+3y+6-3y} = \frac{7y}{7} = y$$

$\therefore$ f is onto.

$\therefore$ $f(x)$ is one-one and onto, so $f(x)$ is invertible and hence f^{-1} exists.

Since f^{-1} exists.

$\therefore$ Let $y = f(x) \Rightarrow x = f^{-1}(y)$

Also, we have $x = \dfrac{1+3y}{2-y}$

$$\therefore \quad f^{-1}(y) = \frac{1+3y}{2-y}$$

$$\therefore \quad f^{-1}(x) = \frac{1+3x}{2-x}$$

Now, the domain of f is the set of all real numbers except -3.

$\therefore$ The range of $f^{-1}(x)$ is the set of all real numbers except -3.

Solution. The given function is

$$y = x^2 + 3$$

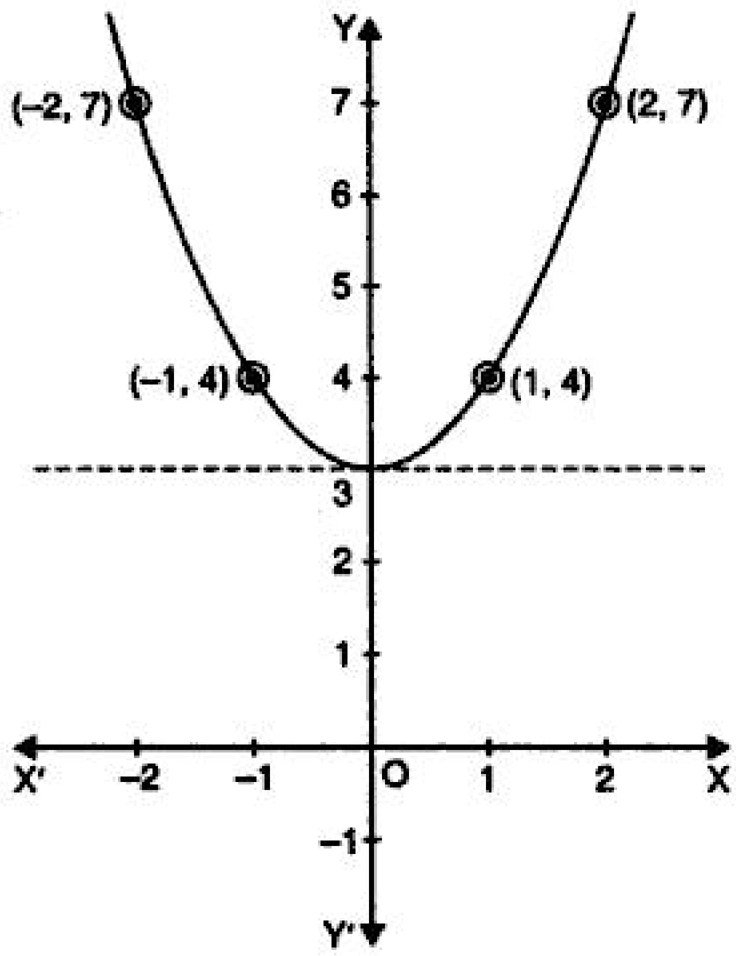
$\Rightarrow$ $$y - 3 = x^2$$

$\Rightarrow$ $$x = \pm \sqrt{y-3}$$

Clearly, the given function represents a parabola passes through (0, 3). *i.e.*, curve does not lie below $y = 3$.

Some points on this curve are :

x	± 1	± 2
y	4	7



The graph of the function is as shown in the figure.

Example 34. *Draw the graph of* $y = x^2 + 2x + 3$.

Solution. The given function is

$$y = x^2 + 2x + 3$$

$$= x^2 + 2x + 1 - 1 + 3 \quad \text{[Add and subtract 1]}$$

$$= (x + 1)^2 + 2$$

Clearly, the graph of the given function will be a parabola.

As $y = (x + 1)^2 + 2$, when $x = -1$, then the value of $(x+1)^2$ is least *i.e.*, 0.

$\Rightarrow$ $$y = (-1 + 1)^2 + 2$$

$\Rightarrow$ $$= 0 + 2 = 2$$

$\therefore$ Vertex is $(-1, 2)$

$y = 2 - 2x$ for $x \geq \frac{1}{2}$ represents a straight line passes through the points $\left(\frac{1}{2}, 1\right)$ and $(1, 0)$.

$y = 2x$ for $x \leq \frac{1}{2}$ represents a straight line passes through the points $\left(\frac{1}{2}, 1\right)$ and $(0, 0)$.

$\therefore$ The graph of the given function is as shown in the figure.

EXERCISE FOR PRACTICE

1. Let f and g be real functions defined by $f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{1-x}$. Then, find each of the following functions :

   (i) $f + g$ (ii) $f - g$ (iii) $f . g$ (iv) $\frac{f}{g}$.

2. Let f and g be real functions defined by $f(x) = \frac{1}{x+4}$ and $g(x) = (x + 4)^3$. Then, find each of the following functions :

   (i) $f + g$ (ii) $f - g$ (iii) $f . g$ (iv) $\frac{f}{g}$.

3. Find the domain of the following functions :

   (i) $| x - 2 |$ (ii) e^{x+2} (iii) $\sin^{-1}(3x - 1)$

   (iv) $e^{x+\sin x}$ (v) $\sqrt{x^2 - 1} + \frac{1}{\sqrt{x}}$ (iv) $x - [x]$.

4. Find the domain and range of $\frac{1}{2 - \cos 3x}$.

5. Find the domain and range of the following functions :

   (i) $\frac{x^2}{2 + x^2}$ (ii) $\frac{x-1}{x+1}$ (iii) $[2 \cos x]$

   (iv) $\frac{|x|}{x}$ (v) $\frac{x-2}{|x-2|}$ (vi) $\frac{1}{\sqrt{x+2}}$

   (vii) $3 \sin x + 4 \cos x + 1$ (viii) $\frac{|x-3|}{x-3}$.

6. If the map $f : R \to R$ is given by $f(x) = \log(1 + x)$ and the map $g : R \to R$ is given by $g(x) = e^x$. Find $(gof)(x)$ and $(fog)(x)$.

7. If $f(x) = \sqrt{1-x}$ and $g(x) = \log_e x$ are two real functions, then describe functions fog and gof.

8. If $f(x) = [x]$ and $g(x) = | x |$.

   Find (i) $fog(x)$ (ii) $gof(x)$ (iii) $gog(x)$.

9. If $f(x) = \frac{x-1}{x+1}$, $(x \neq 1, -1)$, show that fof^{-1} is an identity function.

10. If $f(x) = \frac{1}{1-x}$, show that $(fof)\left(\frac{1}{2}\right) = -1$.

11. Show that $f(x) = \tan^2 x + | x |$ is an even function.

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Thus, if we go on decreasing this value of x and take it nearer to 1, then the value of $f(x)$ will come nearer to 2.

$$\therefore \qquad \text{As } x \to 1 \Rightarrow f(x) \to 2$$

We can write it as ; $\underset{x \to 1}{\text{Lim}} \left(\frac{x^2 - 1}{x - 1} \right) = 2.$

2.4 LIMIT OF A FUNCTION

Le $f(x)$ be a function defined for all x in the nbd of 'a' except possibly at 'a'. Then, l is said to be the limiting value of $f(x)$ as x tends to a. If the numerical difference between $f(x)$ and l can be made as small as we please by taking x sufficiently close to 'a' but not equal to 'a'.

We write this as :

$$\lim_{x \to a} f(x) = l.$$

Def. Let f be a function defined in a nbd of a except possibly at a. Then, a real number l is said to be a limit of f as x tends to a if for any $\varepsilon > 0$, however small, there exists $\delta > 0$ (depending upon ε) such that :

$$|f(x) - l| < \varepsilon, \text{ whenever } 0 < |x - a| < \delta.$$

i.e., $$l - \varepsilon < f(x) < l + \varepsilon, \text{ whenever } x \in (a - \delta, a) \cup (a, a + \delta)$$

We write, $\lim_{x \to d} f(x) = l.$

Remark. The limit of f at a, if exists, will continue to exist and be the same if we change the value of f at a only.

2.4.1 Left hand limit of a function. Let $f(x)$ be a function of x.

If $f(x)$ approaches to l as $x \to a^-$, then l is called the left hand limit of the function and we write it as :

$$\lim_{x \to a^-} f(x) = l.$$

2.4.2 Right hand limit of a function. Let $f(x)$ be a function of x if $f(x)$ approaches to l as $x \to a^+$, then l is called the right hand limit of the function and we write it as :

$$\lim_{x \to a^+} f(x) = l.$$

Remark. (*i*) $\lim_{x \to a} f(x)$ exists if and only if

$\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are equal *i.e.,*

$\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$

(*ii*) If $x \to a^-$ and $x = a - h$, then we have $h = (a - x)$ and as $x \to a^-$, the difference $a - x$ ($= h$) is positive and is close to zero.

$\therefore \quad x \to a^- \Rightarrow h \to 0^+$, where $x = a - h$.

(*iii*) If $x \to a^+$ and $x = a + h$, then we have $h = x - a$ and as $x \to a^+$, the difference $x - a (= h)$ is positive and is close to zero.

$\therefore \quad x \to a^+ \quad \Rightarrow \quad h \to 0^+$ where $x = a + h$.

(*iv*) Left hand and right limits are required to be used when the function under consideration is involving modulus function, bracket function and (or) is defined by more than one rule.

2.5 THEOREM

The limit of a function, if it exists, is unique i.e., if $\lim\limits_{x \to a} f(x) = l$ *and* $\lim\limits_{x \to a} f(x) = l'$

Then, $l = l'$.

Proof. Let us suppose that $\lim\limits_{x \to a} f(x)$ exists.

Let if possible, $f(x)$ tends to two different limits l and l' as $x \to a$.

Let $\quad \varepsilon = \dfrac{1}{2} \mid l - l' \mid > 0$

Now, $\quad \lim\limits_{x \to a} f(x) = l$

$\Rightarrow$ For given $\varepsilon > 0$, there exists a positive real number δ_1 (depending on ε) such that :

$$\mid f(x) - l \mid < \varepsilon \text{ whenever } 0 < \mid x - a \mid < \delta_1 \qquad \text{...(1)}$$

Also, $\quad \lim\limits_{x \to a} f(x) = l'$

$\Rightarrow$ For given $\varepsilon > 0$, there exists a positive real number δ_2 (depending on ε) such that :

$$\mid f(x) - l' \mid < \varepsilon \text{ whenever } 0 < |x - a| < \delta_2 \qquad \text{...(2)}$$

Let $\quad \delta = \min. \{\delta_1, \delta_2\}$

Then, $\quad |l - l'| = \mid l - f(x) + f(x) - l' \mid$

$[\because \mid a + b \mid \le \mid a \mid + \mid b \mid]$

$\le \mid l - f(x) \mid + \mid f(x) - l' \mid$

$= \mid f(x) - l \mid + \mid f(x) - l' \mid \qquad [\because \mid -a \mid = \mid a \mid]$

[By using (1) and (2)]

$\le \varepsilon + \varepsilon$ whenever $0 < |x - a| < \delta \qquad \left[\because \varepsilon = \dfrac{1}{2}|l - l'|\right]$

$= 2\varepsilon$

$= \mid l - l' \mid$

which is a contradiction.

Hence, $\quad l = l'$.

2.6 THEOREM : SANDWITCH THEOREM OR SQUEEZE PRINCIPLE

Statement. *If* $f(x) \le h(x) \le g(x)$ *for some deleted neighbourhood of a and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = l, \text{ then } \lim_{x \to a} h(x) = l.$$

In other words, if a function can be sandwitched (or squeezed) between two other functions, each of which approaches the same limit l as $x \to a$*, then the sandwitched (or squeezed) function also approaches the same limit l as* $x \to a$.

Proof. Let $\lim_{x \to a} f(x) = l$

$\Rightarrow$ For given $\varepsilon > 0$, there exists a positive real number δ_1 (depending on ε) such that :

$$|f(x) - l| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta_1 \qquad ...(1)$$

i.e., $$l - \varepsilon < f(x) < l + \varepsilon \text{ for } 0 < |x - a| < \delta_1$$

Also, $\lim_{x \to a} g(x) = l,$

$\Rightarrow$ For given $\varepsilon > 0$, there exists a positive real number δ_2 (depending on ε) such that :

$$|g(x) - l| < \varepsilon \text{ for } 0 < |x - a| < \delta_2 \qquad ...(2)$$

i.e., $$l - \varepsilon < g(x) < l + \varepsilon \text{ for } 0 < |x - a| < \delta_2$$

Let $\delta = \text{min.} (\delta_1, \delta_2)$.

Then, $|f(x) - l| < \varepsilon$ and $|g(x) - l| < \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow$ $l - \varepsilon < f(x) < l + \varepsilon$ and $l - \varepsilon < g(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

Also, $f(x) \le h(x) \le g(x)$ for $0 < |x - a| < \delta$

$\therefore$ $l - \varepsilon < f(x) \le h(x) \le g(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow$ $l - \varepsilon < h(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow$ $|h(x) - l| < \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow$ $\lim_{x \to a} h(x) = l.$

2.7 FUNDAMENTAL THEOREMS ON LIMITS

Here we list some of the fundamental results involving limits of functions. The proofs of these are beyond the scope of this book.

(*i*) If $f(x) = k$, *a* constant function, then $\lim_{x \to a} f(x) = k$

(*ii*) $\lim_{x \to a} [k\, f(x)] = k \lim_{x \to a} f(x)$ where : k is a constant.

(*iii*) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(*iv*) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(*v*) $\lim_{x \to a} [f(x) \,.\, g(x)] = \lim_{x \to a} f(x) \,.\, \lim_{x \to a} g(x)$

(*vi*) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \ne 0$

(*vii*) If $f(x) \le g(x)$ for all x, then : $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$

(*viii*) $\lim_{x \to a} \left(\dfrac{1}{f}\right)(x) = \lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{\lim_{x \to a} f(x)}$, provided $\lim_{x \to a} f(x) \ne 0$

(*ix*) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

And, $\lim_{x \to a} f(x)$ is the value to which $f(x)$ approaches, when x approaches to a.

Now, we have the following possibilities :

I. The limit exists but the value does not exist *i.e.,* $\lim_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) does not exist.

e.g., Let $$f(x) = \left(\frac{x^2 - 4}{x - 2}\right)$$

Then, $f(x)$ is not defined at $x = 2$. *i.e.,* $f(2)$ does not exist, because it attains the form $\frac{0}{0}$.

But, $$\lim_{x \to 2} f(x) = \lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right)$$

$$= \lim_{x \to 2}\left(\frac{(x-2)(x+2)}{(x-2)}\right) = \lim_{x \to 2} (x + 2) = 4.$$

Thus, the limit exists but the value does not exist.

II. The value exists but the limit does not exist. *i.e.,* the value $f(a)$ exists, but $\lim_{x \to a} f(x)$ does not exist.

e.g., Let $f(x) = [x]$ Then, $f(1) = [1] = 1$

Now, Right Hand Limit :

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} [1 + h] = \lim_{h \to 0} 1 = 1$$

Left Hand Limit :

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} [1 - h] = \lim_{h \to 0} 0 = 0$$

Clearly, $$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$$

So, $\lim_{x \to 1} f(x)$ does not exist.

Thus, the value exists but the limit does not exist.

III. The limit and the value both exist and are unequal *i.e.,* $\lim_{x \to a} f(x)$ and $f(a)$ both exist but are unequal. *e.g.,*

Let $$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & , x \neq 2 \\ 3 & , x = 2 \end{cases}$$

Right Hand Limit :

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2^+}\left(\frac{(x-2)(x+2)}{x-2}\right)$$

$$= \lim_{x \to 2^+} (x + 2) = \lim_{h \to 0} (2 + h + 2) = 4$$

Left Hand Limit :

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2^-} (x + 2)$$

$$= \lim_{h \to 0} (2 - h + 2) = 4$$

$$\therefore \qquad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 4.$$

So, $\lim_{x \to 2} f(x)$ exists and is equal to 4.

Also, the value $f(2)$ exists and is equal to 3.

Thus, $\lim_{x \to 2} f(x)$ and $f(2)$ both exist but are unequal.

IV. The limit and the value both exist and are equal *i.e.*, $\lim_{x \to a} f(x)$ and $f(a)$ both exist and are equal.

e.g., Let $f(x) = x^2$. Then, $f(1) = (1)^2 = 1$

Now, Right Hand Limit :

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = \lim_{h \to 0} (1 + h)^2$$

$$= \lim_{h \to 0} (1 + h^2 + 2h) = 1$$

Left Hand Limit :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = \lim_{h \to 0} (1 - h)^2$$

$$= \lim_{h \to 0} (1 + h^2 - 2h) = 1$$

$$\therefore \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$$

So, $\lim_{x \to 1} f(x) = f(1) = 1.$

Thus, the limit and the value both exist and are equal.

2.10 EVALUATION OF LIMITS

To evaluate $\lim_{x \to a} f(x)$, where : a is a real number and $f(x)$ defined by a single rule.

1. Find $f(a)$, if it is not indeterminate *i.e.*, $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ or $(\infty - \infty)$ or $(0 \times \infty)$, then $\lim_{x \to a} f(x) = f(a)$

i.e., the limit is obtained simply by putting $x = a$.

2. If $f(a)$ is indeterminate, then proceed as under :

Method I. Method of Substitution :

(*i*) Put $x = a + h$; where : h is very small.

So that, $h \to 0$ as $x \to a$.

(*ii*) Simplify the numerator and denominator of the function and remove the common factor and then put $h = 0$.

Method II. Method of Factorisation :

(*i*) Factorise the numerator and denominator.

(*ii*) Cancel the common factor $(x - a)$ remembering that, $x \to a \Rightarrow x \neq a \Rightarrow x - a \neq 0$ and then put $x = a$.

Remark. While evaluating $\lim_{x \to a} \frac{f(x)}{g(x)}$, if $f(x)$ and $g(x)$ appear to be factorised easily, then the method of factorisation should be preferred over the method of substitution.

Method III. Method of Rationalisation :

In the functions which involves square roots, rationalisation of either numerator or denominator should be used to simplify and then take the value at a.

Method IV. By using some standard result or standard limit.

Method V. Use of Binomial Expansion :

If degree of the expression is 4 or more than 4, then binomial expansion is used.

i.e.,
$$(a + h)^n = \left[a\left(1 + \frac{h}{a}\right)\right]^n = a^n \left(1 + \frac{h}{a}\right)^n$$
$$= a^n \left[1 + \frac{nh}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \frac{n(n-1)(n-2)}{3!}\frac{h^3}{a^3} + \ldots\ldots +\right].$$

3. When $f(x)$ is defined by more than one rule. (*e.g.*, the absolute value function, the greatest integer function etc.) in the neighbourhood of a.

Then, find both left hand limit *i.e.*, $\lim_{x \to a^-} f(x)$ and right hand limit *i.e.*, $\lim_{x \to a^+} f(x)$.

If both exist and are equal, then their common value gives the value of $\lim_{x \to a} f(x)$.

Calculation of Left hand limit/Right hand limit

Left Hand Limit :

(*i*) Write $\lim_{x \to a^-} f(x)$.

(*ii*) Put $x = a - h$, where : $h > 0$ and $h \to 0$. We get $\lim_{h \to 0} f(a - h)$

(*iii*) The value of $\lim_{h \to 0} f(a - h)$ is Left Hand Limit of $f(x)$ as x approaches to a.

Right Hand Limit :

(*i*) Write $\lim_{x \to a^+} f(x)$

(*ii*) Put $x = a + h$, where : $h > 0$ and $h \to 0$. We get $\lim_{h \to 0} f(a + h)$

(*iii*) The value of $\lim_{h \to 0} f(a + h)$ is Right hand limit of $f(x)$ as x approaches to a.

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$$= \frac{2}{[\sqrt{2}+\sqrt{2}]\,[\sqrt{3+2\sqrt{2}}+\sqrt{2}+1]}$$

$$= \frac{2}{2\sqrt{2}\,[\sqrt{3+2\sqrt{2}}+\sqrt{2}+1]} = \frac{1}{\sqrt{2}\,[\sqrt{3+2\sqrt{2}}+\sqrt{2}+1]}$$

$$\left[\because \sqrt{3+2\sqrt{2}} = \sqrt{(\sqrt{2}+1)^2} = (\sqrt{2}+1)\right]$$

$$= \frac{1}{\sqrt{2}\,[\sqrt{(\sqrt{2}+1)^2}+\sqrt{2}+1]} = \frac{1}{\sqrt{2}\,[\sqrt{2}+1+\sqrt{2}+1]} = \frac{1}{2\sqrt{2}(\sqrt{2}+1)}.$$

(*iv*) When $x = \sqrt{3}$, the given expression assumes the indeterminate form $\left(\frac{0}{0}\right)$.

$$\therefore \lim_{x\to\sqrt{3}} \frac{x^4-9}{x^2+4\sqrt{3}x-15} = \lim_{x\to\sqrt{3}} \frac{(x^2)^2-(3)^2}{x^2+4\sqrt{3}x-15}$$

$$= \lim_{x\to\sqrt{3}} \frac{(x^2-3)(x^2+3)}{(x+5\sqrt{3})(x-\sqrt{3})} = \lim_{x\to\sqrt{3}} \frac{(x^2-(\sqrt{3})^2)(x^2+3)}{(x+5\sqrt{3})(x-\sqrt{3})}$$

$$= \lim_{x\to\sqrt{3}} \frac{(x-\sqrt{3})(x+\sqrt{3})(x^2+3)}{(x+5\sqrt{3})(x-\sqrt{3})} = \lim_{x\to\sqrt{3}} \frac{(x+\sqrt{3})(x^2+3)}{(x+5\sqrt{3})}$$

$$= \frac{(\sqrt{3}+\sqrt{3})[(\sqrt{3})^2+3]}{\sqrt{3}+5\sqrt{3}} = \frac{(2\sqrt{3})(6)}{6\sqrt{3}} = \frac{12\sqrt{3}}{6\sqrt{3}} = 2.$$

Example 8. *Evaluate the following limits :*

(*i*) $\lim_{x\to 0} \frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$ (*ii*) $\lim_{x\to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$

(*iii*) $\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$ (*iv*) $\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

(*v*) $\lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$.

Solution. (*i*) We have, $\lim_{x\to 0} \frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$ (on rationalisation)

$$= \lim_{x\to 0} \frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2} \times \frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{(\sqrt{1-x^2}+\sqrt{1+x^2})}$$

$$= \lim_{x\to 0} \frac{(\sqrt{1-x^2})^2-(\sqrt{1+x^2})^2}{2x^2(\sqrt{1-x^2}+\sqrt{1+x^2})} = \lim_{x\to 0} \frac{(1-x^2)-(1+x^2)}{2x^2(\sqrt{1-x^2}+\sqrt{1+x^2})}$$

$$= \lim_{x\to 0} \frac{-2x^2}{2x^2(\sqrt{1-x^2}+\sqrt{1+x^2})} = \lim_{x\to 0} \frac{-1}{\sqrt{1-x^2}+\sqrt{1+x^2}}$$

$$= \frac{-1}{\sqrt{1-0}+\sqrt{1+0}} = \frac{-1}{2}.$$

(*ii*) When $x = 3$, the given expression assumes the indeterminate form $\left(\frac{0}{0}\right)$.

$$\therefore \quad \lim_{x\to 3} \frac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27} = \lim_{x\to 3} \frac{(x-3)(x^2-4x+3)}{(x-3)(x^3-2x^2-6x+9)}$$

$$= \lim_{x\to 3} \frac{x^2-4x+3}{x^3-2x^2-6x+9}$$

$$= \lim_{x\to 3} \frac{(x-1)(x-3)}{(x-3)(x^2+x-3)} = \lim_{x\to 3} \frac{x-1}{x^2+x-3} = \frac{3-1}{(3)^2+3-3} = \frac{2}{9}.$$

(*iii*) When $x = 2$, the given expression assumes the indeterminate form $(\infty - \infty)$.

$$\therefore \quad \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x^2-3x+2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{4x-6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x(x-1)-(4x-6)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-x-4x+6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \frac{(x-2)(x-3)}{x(x-1)(x-2)} = \lim_{x\to 2} \frac{(x-3)}{x(x-1)}$$

$$= \frac{2-3}{2(2-1)} = \frac{-1}{2}.$$

(*iv*) When $x = 4$, the given expression assumes the indeterminate form $\left(\frac{0}{0}\right)$.

$$\therefore \quad \lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$$ [Rationalising the numerator and denominator]

$$= \lim_{x\to 4}\left[3-\sqrt{5+x} \times \frac{3+\sqrt{5+x}}{3+\sqrt{5+x}}\right] \times \left[\frac{1+\sqrt{5-x}}{1+\sqrt{5-x}} \times \frac{1}{1-\sqrt{5-x}}\right]$$

$$= \lim_{x\to 4}\left[\frac{(3)^2-(\sqrt{5+x})^2}{3+\sqrt{5+x}}\right]\left[\frac{1+\sqrt{5-x}}{(1)^2-\left(\sqrt{5-x}\right)^2}\right]$$

$$= \lim_{x\to 4}\left[\frac{9-(5+x)}{3+\sqrt{5+x}} \cdot \frac{1+\sqrt{5-x}}{1-(5-x)}\right]$$

$$= \lim_{x\to 4}\left[\frac{(4-x)}{3+\sqrt{5+x}} \cdot \frac{1+\sqrt{5-x}}{(x-4)}\right]$$

$$= \lim_{x\to 4}\left[\frac{-(x-4)(1+\sqrt{5-x})}{(3+\sqrt{5+x})(x-4)}\right]$$

$$= \lim_{x \to 4} \frac{-(1+\sqrt{5-x})}{(3+\sqrt{5+x})}$$

$$= \frac{-(1+\sqrt{5-4})}{(3+\sqrt{5+4})} = \frac{-(2)}{3+3} = \frac{-2}{6} = \frac{-1}{3}.$$

(*v*) When $x = 2$, the given expression assumes the indeterminate form $\left(\frac{0}{0}\right)$.

$$\therefore \quad \lim_{x \to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} \qquad \text{(on rationalisation)}$$

$$= \lim_{x \to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} \times \frac{\sqrt{3x-2}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}$$

$$= \lim_{x \to 2} \frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2})^2-(\sqrt{x+2})^2}$$

$$= \lim_{x \to 2} \frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2)-(x+2)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2x-4}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)}$$

$$= \lim_{x \to 2} \frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2}$$

$$= \frac{(2+2)(\sqrt{3(2)-2}+\sqrt{2+2})}{2}$$

$$= \frac{(4)(\sqrt{4}+\sqrt{4})}{2} = \frac{(4)(4)}{2} = 8.$$

Example 9. *Evaluate the following limits :*

(*i*) $\lim_{x \to 2} \left[\frac{1}{x-2} + \frac{6x}{8-x^3}\right]$ (*ii*) $\lim_{x \to 0} \frac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$

(*iii*) $\lim_{x \to 2} \left(\frac{4}{x^2-4} + \frac{1}{2-x}\right)$ (*iv*) $\lim_{x \to 1} \left(\frac{1}{x-1} + \frac{1}{1-x}\right)$.

Solution. (*i*) When $x = 2$, the given expression assumes the indeterminate form $(\infty - \infty)$.

$$\therefore \quad \lim_{x \to 2} \left(\frac{1}{x-2} + \frac{6x}{8-x^3}\right) = \lim_{x \to 2} \left(\frac{1}{x-2} - \frac{6x}{x^3-8}\right)$$

$$= \lim_{x \to 2} \left(\frac{1}{x-2} - \frac{6x}{x^3-2^3}\right) \qquad [\because (a^3-b^3) = (a-b)(a^2+ab+b^2)]$$

$$= \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{6x}{(x-2)(x^2+2x+4)}\right]$$

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(*v*) Please try yourself. $\left[\textbf{Ans. } \frac{1}{2\sqrt{2}}.\right]$

Example 13. *Evaluate the following limits by using substitution method :*

(*i*) $\lim_{x \to 1} (x-1)^2 + 5$

(*ii*) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2}$

(*iii*) $\lim_{x \to -3} \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}$

(*iv*) $\lim_{x \to 1} \frac{x - 1}{2x^2 - 7x + 5}$

(*v*) $\lim_{x \to 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$

(*vi*) $\lim_{x \to 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$

(*vii*) $\lim_{x \to 3} \frac{x - 3}{\sqrt{x-2} - \sqrt{4-x}}$

(*viii*) $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8}$.

Solution. (*i*) We have, $\lim_{x \to 1} (x-1)^2 + 5$

Let $x = 1 + h$

$\therefore \quad h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1} (x-1)^2 + 5 = \lim_{h \to 0} [((1+h) - 1)^2 + 5] = \lim_{h \to 0} (h^2 + 5) = 5.$$

(*ii*) We have, $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2}$

Let $x = 2 + h$, $\therefore \quad h \to 0$ as $x \to 2$

$$\therefore \quad \lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2} = \lim_{h \to 0} \frac{(2+h)^2 - 3(2+h) + 2}{(2+h)^2 - (2+h) - 2}$$

$$= \lim_{h \to 0} \frac{4 + h^2 + 4h - 6 - 3h + 2}{4 + h^2 + 4h - 2 - h - 2}$$

$$= \lim_{h \to 0} \frac{h^2 + h}{h^2 + 3h} = \lim_{h \to 0} \frac{h(h+1)}{h(h+3)}$$

$$= \lim_{h \to 0} \left(\frac{h+1}{h+3}\right) = \left(\frac{0+1}{0+3}\right) = \frac{1}{3}.$$

(*iii*) We have, $\lim_{x \to -3} \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}$

Let $x = -3 + h$, $\therefore \quad h \to 0$ as $x \to -3$

$$\therefore \quad \lim_{x \to -3} \left(\frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}\right) = \lim_{h \to 0} \frac{(-3+h)^3 + 4(-3+h)^2 + 4(-3+h) + 3}{(-3+h)^2 + 2(-3+h) - 3}$$

$$= \lim_{h \to 0} \frac{(h^3 - 9h^2 + 27h - 27) + 4(h^2 - 6h + 9) - 12 + 4h + 3}{h^2 - 6h + 9 - 6 + 2h - 3}$$

$$= \lim_{h \to 0} \frac{h^3 - 5h^2 + 7h}{h^2 - 4h}$$

$$= \lim_{h \to 0} \frac{h(h^2 - 5h + 7)}{h(h-4)} = \lim_{h \to 0} \frac{(h^2 - 5h + 7)}{(h-4)} = \frac{0 - 0 + 7}{0 - 4} = \frac{-7}{4}.$$

(*iv*) We have, $\lim_{x \to 1} \frac{x-1}{2x^2 - 7x + 5}$

Let $x = 1 + h$, $\therefore$ $h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1} \frac{x-1}{2x^2 - 7x + 5} = \lim_{h \to 0} \frac{(1+h)-1}{2(1+h)^2 - 7(1+h) + 5}$$

$$= \lim_{h \to 0} \frac{h}{2(1 + h^2 + 2h) - 7 - 7h + 5} = \lim_{h \to 0} \frac{h}{2h^2 - 3h}$$

$$= \lim_{h \to 0} \frac{h}{.h(2h-3)} = \lim_{h \to 0} \frac{1}{(2h-3)} = \frac{1}{(0-3)} = \frac{-1}{3}.$$

(*v*) We have, $\lim_{x \to 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$

Let $x = 1 + h$, $\therefore$ $h \to 0$ as $x \to 1$

$$\therefore \lim_{x \to 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1} = \lim_{h \to 0} \frac{(1+h)^3 + 3(1+h)^2 - 6(1+h) + 2}{(1+h)^3 + 3(1+h)^2 - 3(1+h) - 1}$$

$$= \lim_{h \to 0} \frac{(1 + 3h + 3h^2 + h^3) + 3(1 + h^2 + 2h) - 6 - 6h + 2}{(1 + 3h + 3h^2 + h^3) + 3(1 + h^2 + 2h) - 3 - 3h - 1}$$

$$= \lim_{h \to 0} \frac{h^3 + 6h^2 + 3h}{h^3 + 6h^2 + 6h} = \lim_{h \to 0} \frac{h(h^2 + 6h + 3)}{h(h^2 + 6h + 6)}$$

$$= \lim_{h \to 0} \left(\frac{h^2 + 6h + 3}{h^2 + 6h + 6} \right) = \frac{0 + 6(0) + 3}{0 + 6(0) + 6} = \frac{3}{6} = \frac{1}{2}.$$

(*vi*) We have, $\lim_{x \to 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$

Let $x = 3 + h$, $\therefore$ $h \to 0$ as $x \to 3$

$$\therefore \quad \lim_{x \to 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3} = \lim_{h \to 0} \frac{(3+h)^2 - (3+h) - 6}{(3+h)^3 - 3(3+h)^2 + (3+h) - 3}$$

$$= \lim_{h \to 0} \frac{h^2 + 9 + 6h - 3 - h - 6}{h^3 + 27 + 9h^2 + 27h - 3(9 + h^2 + 6h) + 3 + h - 3}$$

$$= \lim_{h \to 0} \frac{h^2 + 5h}{h^3 + 6h^2 + 10h} = \lim_{h \to 0} \frac{h(h+5)}{h(h^2 + 6h + 10)}$$

$$= \lim_{h \to 0} \frac{h + 5}{h^2 + 6h + 10} = \frac{0 + 5}{0 + 6(0) + 10} = \frac{5}{10} = \frac{1}{2}.$$

(*vii*) We have, $\lim_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

Let $x = 3 + h$, $\therefore$ $h \to 0$ as $x \to 3$.

$$\therefore \lim_{x \to 3}\left(\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\right) = \lim_{h \to 0}\left(\frac{3+h-3}{\sqrt{3+h-2}-\sqrt{4-(3+h)}}\right)$$

$$= \lim_{h \to 0} \frac{h}{\sqrt{1+h}-\sqrt{1-h}}$$

$$= \lim_{h \to 0}\left[\frac{h}{(\sqrt{1+h}-\sqrt{1-h})} \times \frac{(\sqrt{1+h}+\sqrt{1-h})}{(\sqrt{1+h}+\sqrt{1-h})}\right] \quad \text{(on rationalisation)}$$

$$= \lim_{h \to 0} \frac{h(\sqrt{1+h}+\sqrt{1-h})}{(\sqrt{1+h})^2-(\sqrt{1-h})^2}$$

$$= \lim_{h \to 0} \frac{h(\sqrt{1+h}+\sqrt{1-h})}{(1+h)-(1-h)} = \lim_{h \to 0} \frac{h(\sqrt{1+h}+\sqrt{1-h})}{2h}$$

$$= \lim_{h \to 0}\left(\frac{\sqrt{1+h}+\sqrt{1-h}}{2}\right) = \left(\frac{\sqrt{1+0}+\sqrt{1-0}}{2}\right)$$

$$= \frac{1+1}{2} = \frac{2}{2} = 1.$$

(*viii*) We have, $\lim_{x \to 2} \frac{x^5-32}{x^3-8} = \lim_{x \to 2} \frac{x^5-(2)^5}{x^3-(2)^3}$

Let $x = 2 + h$, $\therefore$ $h \to 0$ as $x \to 2$

$$\therefore \quad \lim_{x \to 2} \frac{x^5-2^5}{x^3-2^3} = \lim_{h \to 0} \frac{(2+h)^5-2^5}{(2+h)^3-2^3}$$

$$= \lim_{h \to 0} \frac{[2^5 + {}^5C_1(2)^4 h + {}^5C_2(2)^3 h^2 + \ldots\ldots + h^5] - 2^5}{(2^3 + {}^3C_1 . (2)^2 h + {}^3C_2 . 2h^2 + h^3) - 2^3}$$

[By binomial expansion]

$$= \lim_{h \to 0} \frac{5(16h) + 10(8h^2) + \ldots\ldots + h^5}{3(4h) + 3(2h^2) + h^3} = \lim_{h \to 0} \frac{h(80 + 80h + \ldots\ldots + h^4)}{h(12 + 6h + h^2)}$$

$$= \lim_{h \to 0} \frac{80 + 80h + \ldots\ldots + h^4}{12 + 6h + h^2} = \frac{80+0}{12+0} = \frac{80}{12} = \frac{20}{3}.$$

Example 14. *Evaluate the following limits :*

(*i*) $\lim_{x \to a} \frac{x^3-a^3}{x^2-a^2}$ (*ii*) $\lim_{x \to 2} \frac{x^7-128}{x-2}$

(*iii*) $\lim_{x \to 1} \frac{x^3-1}{x-1}$ (*iv*) $\lim_{x \to 2} \frac{x^{10}-1024}{x-2}$

(*v*) $\lim_{x \to 2} \frac{x^{10}-1024}{x^4-16}$ (*vi*) $\lim_{x \to 2} \frac{x^{10}-1024}{x^5-32}$.

$$= \lim_{h\to 0} \log(1-h)$$

$$= \lim_{h\to 0}\left[-h-\frac{h^2}{2!}-\frac{h^3}{3!}\ldots\ldots\right] \quad \left[\because \log(1-x) = -x-\frac{x^2}{2!}-\frac{x^3}{3!}\ldots\ldots\right]$$

$$= -\lim_{h\to 0}\left[h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\ldots\right] = 0.$$

R.H.L. $\lim_{x\to 0^+} \log(1+x) = \lim_{h\to 0} \log(1+0+h)$

$$= \lim_{h\to 0} \log(1+h)$$

$$= \lim_{h\to 0}\left[h-\frac{h^2}{2!}+\frac{h^3}{3!}-\ldots\ldots\right] = 0.$$

$$\left[\because \log(1+x) = \left(x-\frac{x^2}{2!}+\frac{x^3}{3!}\ldots\ldots\right)\right]$$

$\therefore \quad \lim_{x\to 0^-} \log(1+x) = \lim_{x\to 0^+} \log(1+x) = 0$

Hence, $\lim_{x\to 0} \log(1+x) = 0.$

(*iii*) We have, $\lim_{x\to 3} \frac{x}{[x]}$

L.H.L. $\lim_{x\to 3^-} \frac{x}{[x]} = \lim_{h\to 0^+} \frac{3-h}{[3-h]}$ $\quad [\because\ 2 < 3-h < 3]$

$$= \lim_{h\to 0^+} \frac{3-h}{2} = \frac{3-0}{2} = \frac{3}{2}.$$

R.H.L. $\lim_{x\to 3^+} \frac{x}{[x]} = \lim_{h\to 0^+} \frac{3+h}{[3+h]}$ $\quad [\because\ 3 < 3+h < 4]$

$$= \lim_{h\to 0^+} \frac{3+h}{3} = \frac{3+0}{3} = 1.$$

$\therefore \quad \lim_{x\to 3^-} \frac{x}{[x]} \neq \lim_{x\to 3^+} \frac{x}{[x]}$

Hence, $\lim_{x\to 3} \frac{x}{[x]}$ does not exist.

(*iv*) We have, $\lim_{x\to 2} [x]$

L.H.L. $\lim_{x\to 2^-} [x] = \lim_{h\to 0^+} [2-h]$ $\quad [\because\ 2-1 < 2+h < 2]$

$$= \lim_{h\to 0^+} [2-1] = (2-1) = 1.$$

R.H.L. $\lim_{x\to 2^+} [x] = \lim_{h\to 0^+} [2+h]$ $\quad [\because\ 2 < 2+h < 2+1]$

$$= \lim_{h\to 0^+} 2 = 2.$$

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$$= \frac{7}{0-2} = -\frac{7}{2}.$$

R.H.L. $\lim_{x \to 0^+} \frac{7|x|}{x^2 + 2x} = \lim_{h \to 0} \frac{7|0+h|}{(0+h)^2 + 2(0+h)}$

$$= \lim_{h \to 0} \frac{7|h|}{h^2 + 2h} = \lim_{h \to 0} \frac{7h}{h(h+2)}$$

$$= \lim_{h \to 0} \frac{7}{h+2} = \frac{7}{0+2} = \frac{7}{2}.$$

$\therefore \quad \lim_{x \to 0^-} \frac{7|x|}{x^2 + 2x} \neq \lim_{x \to 0^+} \frac{7|x|}{x^2 + 2x}$

Hence, $\lim_{x \to 0} \frac{7|x|}{x^2 + 2x}$ does not exist.

(*vi*) We have, $\lim_{x \to 0^+} \frac{1}{3x} = \lim_{h \to 0} \frac{1}{3(0+h)} = \lim_{h \to 0} \frac{1}{3h} = \frac{1}{0} = \infty.$

Example 31. *Evaluate the following limits :*

(*i*) $\lim_{x \to 0} \frac{1}{3 + 2^{1/x}}$ (*ii*) $\lim_{x \to 3} (x - [x]).$

Solution. (*i*) We have, $\lim_{x \to 0} \frac{1}{3 + 2^{1/x}}$

L.H.L. $\lim_{x \to 0^-} \frac{1}{3 + 2^{1/x}} = \lim_{h \to 0} \frac{1}{3 + 2^{1/0-h}}$

$$= \lim_{h \to 0} \frac{1}{3 + 2^{-1/h}} \qquad \left[\begin{array}{l} \because \text{ As } h \to 0, \ \frac{1}{h} \to \infty \\ \Rightarrow 2^{1/h} \to \infty, \ 2^{-1/h} \to 0 \end{array}\right]$$

$$= \frac{1}{3+0} = \frac{1}{3}.$$

R.H.L. $\lim_{x \to 0^+} \frac{1}{3 + 2^{1/x}} = \lim_{h \to 0} \frac{1}{3 + 2^{1/0+h}}$

$$= \lim_{h \to 0} \frac{1}{3 + 2^{1/h}} = \frac{1}{3 + \infty} = \frac{1}{\infty} = 0.$$

$\therefore \quad \lim_{x \to 0^-} \frac{1}{3 + 2^{1/x}} \neq \lim_{x \to 0^+} \frac{1}{3 + 2^{1/x}}$

Hence, $\lim_{x \to 0} \frac{1}{3 + 2^{1/x}}$ does not exist.

(*ii*) We have, $\lim_{x \to 3} (x - [x])$

L.H.L. $\lim_{x \to 3^-} (x - [x]) = \lim_{h \to 0} (3 - h - [3 - h])$

$$= \lim_{h \to 0} (3 - h - 2) = \lim_{h \to 0} (1 - h) = 1.$$

R.H.L. $\lim_{x \to 3^+} (x - [x]) = \lim_{h \to 0} (3 + h - [3 + h])$

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} (h) = 0.$$

$\therefore \qquad \lim_{x \to 3^-} (x - [x]) \neq \lim_{x \to 3^+} (x - [x])$

Hence, $\lim_{x \to 3}$ does not exist.

Example 32. *For what values of k does $\lim_{x \to 0} f(x)$ exists, where f(x) is defined as :*

$$f(x) = \begin{Bmatrix} (x-k)^2 - 3 & , \text{if } x < 0 \\ x + 2k & , \text{if } x \geq 0 \end{Bmatrix}.$$

Solution. We have,

$$f(x) = \begin{Bmatrix} (x-k)^2 - 3 & , \text{if } x < 0 \\ x + 2k & , \text{if } x \geq 0 \end{Bmatrix}$$

L.H.L. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x - k)^2 - 3$

$$= \lim_{h \to 0} ((0 - h - k)^2 - 3)$$

$$= (0 - k)^2 - 3 = k^2 - 3.$$

R.H.L. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 2k)$

$$= \lim_{h \to 0} (0 + h + 2k) = (0 + 2k) = 2k.$$

Since, $\lim_{x \to 0} f(x)$ exists.

$\therefore \qquad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\Rightarrow \qquad k^2 - 3 = 2k$

$\Rightarrow \qquad k^2 - 2k - 3 = 0$

$\Rightarrow \qquad (k - 3)(k + 1) = 0$

$\Rightarrow \qquad k = 3, -1.$

Example 33. *If* $f(x) = \begin{Bmatrix} x + \frac{1}{2} & , \text{for } x > \frac{1}{2} \\ 0 & , \text{for } x = \frac{1}{2} \\ 2x & , \text{for } x < \frac{1}{2} \end{Bmatrix}$

Find $\lim_{x \to \frac{1}{2}} f(x)$, *if exists.*

Solution. We have,

$$f(x)=\begin{cases} x+\frac{1}{2}, & \text{for } x>\frac{1}{2} \\ 0, & \text{for } x=\frac{1}{2} \\ 2x, & \text{for } x<\frac{1}{2} \end{cases}$$

L.H.L. $\lim_{x\to\frac{1}{2}^-} f(x)=\lim_{x\to\frac{1}{2}^-}(2x)$

$$=\lim_{h\to 0} 2\left(\frac{1}{2}-h\right)=\lim_{h\to 0}(1-2h)=(1-0)=1.$$

R.H.L. $\lim_{x\to\frac{1}{2}^+} f(x)=\lim_{x\to\frac{1}{2}^+}\left(x+\frac{1}{2}\right)$

$$=\lim_{h\to 0}\left(\frac{1}{2}+h+\frac{1}{2}\right)=\lim_{h\to 0}(1+h)=1+0=1.$$

$\therefore \quad \lim_{x\to\frac{1}{2}^-} f(x)=\lim_{x\to\frac{1}{2}^+} f(x)=1$

Hence, $\lim_{x\to\frac{1}{2}} f(x)$ exists and is equal to 1.

Example 34. *Let* $f(x)=\begin{cases} ax+b & \text{for } x>1 \\ 3bx-2a+1 & \text{for } x<1 \end{cases}$

Find a relation between a and b, so that, $\lim_{x\to 1} f(x)$ *exists.*

Solution. We have,

$$f(x)=\begin{cases} ax+b & \text{for } x>1 \\ 3bx-2a+1 & \text{for } x<1 \end{cases}$$

L.H.L. $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3bx-2a+1)$

$$=\lim_{h\to 0}(3b(1-h)-2a+1)$$

$$=\lim_{h\to 0}(3b-3bh-2a+1)$$

$$=(3b-2a+1).$$

R.H.L. $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(ax+b)$

$$=\lim_{h\to 0}(a(1+h)+b)=\lim_{h\to 0}(a+ah+b)=a+b.$$

Since, $\lim_{x\to 1} f(x)$ exists.

(*ii*) Let $\tan^{-1} x = \theta \Rightarrow x = \tan\theta$

$\therefore\ x \to 0 \Rightarrow \tan\theta \to 0 \Rightarrow \theta \to 0$

$$\therefore \quad \lim_{x\to 0}\left(\frac{\tan^{-1} x}{x}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1$$

Also,
$$\lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \lim_{\theta\to 0}\left(\frac{\cos\theta}{\frac{\sin\theta}{\theta}}\right) = \frac{\left(\lim\limits_{\theta\to 0}\cos\theta\right)}{\left(\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\right)} = \frac{1}{1} = 1.$$

3.5 LIMITS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

For the evaluation of exponential and logarithmic limits, the students are advised to learn the following expansions :

(*i*) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

(*ii*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots$

(*iii*) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\ldots$

(*iv*) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$

(*v*) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\ldots$

(*vi*) $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots\ldots$

(*vii*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$

(*viii*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$

(*ix*) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

(*x*) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$

(*xi*) $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ldots\ldots$

(*xii*) $\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots\ldots$

3.6 THEOREM : PROVE THAT : $\lim_{x \to 0^+} f(-x) = \lim_{x \to 0^-} f(x)$

Proof. Let us consider that ; $y = -x$ in the function $f(x)$

Then, as $x \to 0^+ \Rightarrow y \to 0^-$

$$\therefore \quad \lim_{x \to 0^+} f(-x) = \lim_{y \to 0^-} f(y)$$

$$= \lim_{x \to 0^-} f(x).$$ [By changing the variable from y to x throughout]

3.7 THEOREM : PROVE THAT : (*i*) $\lim_{x \to 0} e^x = 1$ (*ii*) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof. (*i*) We have,

$$\lim_{x \to 0} e^x = \lim_{x \to 0}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right) \left[\because\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right]$$

$$= 1 + 0 = 1.$$

(*ii*) We have,

$$\lim_{x \to 0}\left(\frac{e^x - 1}{x}\right) = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots - 1\right)}{x}$$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)}{x} = \lim_{x \to 0} \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\right)}{x}$$

$$= \lim_{x \to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\right) = 1.$$

3.8 THEOREM : PROVE THAT : $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a,\ a > 0$

Proof. Since, $f(x) = e^{\log f(x)}$

$$\Rightarrow \quad a^x = e^{\log a^x} \qquad [\because\ m \log n = \log n^m]$$

$$= e^{x \log a}$$

We have,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{[e^{x \log a} - 1]}{x} \qquad \left[\because\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right]$$

$$= \lim_{x \to 0} \frac{\left[1 + (x \log a) + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \ldots\ldots - 1\right]}{x}$$

$$= \lim_{x \to 0} \frac{\left[1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots - 1\right]}{x}$$

$$= \lim_{x \to 0} \frac{\left[x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots\right]}{x}$$

$$= \lim_{x \to 0} \left[\log a + \frac{x}{2!}(\log a)^2 + \frac{x^2}{3!}(\log a)^3 + \ldots\ldots\right]$$

$$= \log a$$

Thus, $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a.$

3.9 THEOREM : PROVE THAT : (*i*) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$ (*ii*) $\lim_{x \to 0} (1 + x)^{1/x} = e$

Proof. (*i*) As $n \to \infty$, n is positive and very large so that $\left(0 < \frac{1}{n} < 1\right)$.

$\therefore$ By Binomial theorem, we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots\ldots$$

$$= 1 + 1 + \frac{(n-1)}{2!} \cdot \frac{1}{n} + \frac{(n-1)(n-2)}{3!} \cdot \frac{1}{n^2} + \ldots\ldots$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$

Now, when $n \to \infty$, $\frac{1}{n}, \frac{2}{n}, \ldots\ldots$ all approaches to zero.

$\therefore$ We have,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$$

3.12 LIMITS AT INFINITY AND INFINITE LIMITS

Meaning of $x \to \infty$:

If x is a variable. Then, the symbol $x \to \infty$ will be used to mean that x takes very large values and we write this as $x \to +\infty$ or simply $x \to \infty$.

Meaning of $x \to -\infty$:

If x is a variable. Then, the symbol $x \to -\infty$ will be used to mean that x takes negative values having a very large magnitude and we write this as $x \to -\infty$.

3.12.1 Infinite Limit of a Function. Let $f(x)$ be a function of x. If the value of $f(x)$ can be made greater than any pre-assigned number by taking x close to a. Then, we say that the function $f(x)$ becomes positively infinite as x approaches to a.

i.e., $$\lim_{x \to a} f(x) = +\infty$$

e.g., Let $$f(x) = \frac{1}{|x|}, \quad x \neq 0$$

Then, $$\lim_{x \to 0} f(x) = +\infty.$$

Similarly, if the value of $f(x)$ can be made less than any pre-assigned number by taking x close to a. Then, we say that the function $f(x)$ becomes negatively infinite as x approaches to a.

i.e., $$\lim_{x \to a} f(x) = -\infty$$

e.g., Let $$f(x) = \frac{-1}{|x|}, \quad x \neq 0$$

Then, $$\lim_{x \to 0} f(x) = -\infty.$$

3.12.2 Working Rule for Finding $\lim_{x \to \infty} f(x)$. Replace x by $\frac{1}{y}$ in the given function and take the limit as $y \to 0$.

In case of a rational function, divide the numerator and the denominator by the highest power of x, replace x by $\frac{1}{y}$ and take the limit as $y \to 0$.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following limits :*

(i) $\lim_{x \to 0} \frac{\sin 2x}{\tan 3x}$ *(ii)* $\lim_{x \to 0} \frac{\sin 2x}{x}$

(iii) $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$ *(iv)* $\lim_{x \to 0} \frac{\sin x^2}{x}$.

Solution. (*i*) We have,

$$\lim_{x \to 0} \frac{\sin 2x}{\tan 3x} = \lim_{x \to 0} \left[\frac{\frac{\sin 2x}{2x} . 2x}{\frac{\tan 3x}{3x} . 3x} \right] = \frac{2}{3} \frac{\lim_{x \to 0} \left(\frac{\sin 2x}{2x} \right)}{\lim_{x \to 0} \left(\frac{\tan 3x}{3x} \right)}$$

$$= \frac{2}{3}\left(\frac{1}{1}\right) = \frac{2}{3}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \ \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1 \right]$$

(*ii*) We have,

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \left(\frac{\sin 2x}{2x} . 2 \right) = 2 . \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \right)$$

$$= 2 . (1) = 2. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

(*iii*) We have,

$$\lim_{x \to 0} \frac{\sin 5x}{\tan 3x} = \lim_{x \to 0} \left[\frac{\frac{\sin 5x}{5x} . 5x}{\frac{\tan 3x}{3x} . 3x} \right] = \frac{5}{3} \left[\frac{\lim_{x \to 0} \frac{\sin 5x}{5x}}{\lim_{x \to 0} \frac{\tan 3x}{3x}} \right]$$

$$= \frac{5}{3}\left(\frac{1}{1}\right) = \frac{5}{3}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \ \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1 \right]$$

(*iv*) We have,

$$\lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left[\frac{\sin x^2}{x^2} . x \right] = \lim_{x \to 0} \left(\frac{\sin x^2}{x^2} \right) . \lim_{x \to 0} x$$

$$= 1(0) = 0. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Example 2. *Evaluate the following limits :*

(*i*) $\lim_{x \to 0} \frac{\sin 5x}{2x}$ (*ii*) $\lim_{x \to 0} \frac{\cos^2 x}{1 - \sin x}$

(*iii*) $\lim_{x \to 0} \frac{x \tan x}{1 - \cos 2x}$ (*iv*) $\lim_{x \to 0} \frac{\tan \frac{x}{2}}{3x}$

(*v*) $\lim_{\theta \to 0} \frac{1 - \cos m\theta}{1 - \cos n\theta}$.

Solution. (*i*) We have,

$$\lim_{x\to 0}\frac{\sin 5x}{2x}=\lim_{x\to 0}\left[\frac{\sin 5x}{(2x).5}.5\right]=\lim_{x\to 0}\left(\frac{\sin 5x}{5x}.\frac{5}{2}\right)$$

$$=\frac{5}{2}\left(\lim_{x\to 0}\frac{\sin 5x}{5x}\right)=\frac{5}{2}\times 1 \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$=\frac{5}{2}.$$

(*ii*) We have,

$$\lim_{x\to 0}\frac{\cos^2 x}{1-\sin x}=\lim_{x\to 0}\left(\frac{1-\sin^2 x}{1-\sin x}\right) \qquad [\because \cos^2 A+\sin^2 A=1]$$

$$=\lim_{x\to 0}\frac{(1-\sin x)(1+\sin x)}{(1-\sin x)}$$

$$=\lim_{x\to 0}(1+\sin x)=\lim_{x\to 0}1+\lim_{x\to 0}\sin x$$

$$=1+0=1.$$

(*iii*) We have,

$$\lim_{x\to 0}\left(\frac{x\tan x}{1-\cos 2x}\right)=\lim_{x\to 0}\left[\frac{x^2.\dfrac{\tan x}{x}}{2\sin^2 x}\right] \qquad [\because 1-\cos 2A=2\sin^2 A]$$

$$=\frac{1}{2}\lim_{x\to 0}\left[\frac{x^2\dfrac{\tan x}{x}}{\dfrac{\sin^2 x}{x^2}.x^2}\right]=\frac{1}{2}\,\frac{\lim_{x\to 0}\left(\dfrac{\tan x}{x}\right)}{\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^2}$$

$$=\frac{1}{2}\left(\frac{1}{1}\right) \qquad \left[\begin{array}{l}\because \lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1\\ \lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1\end{array}\right]$$

$$=\frac{1}{2}.$$

(*iv*) We have,

$$\lim_{x\to 0}\left[\frac{\tan\dfrac{x}{2}}{3x}\right]=\lim_{x\to 0}\left[\frac{\tan\dfrac{x}{2}}{\dfrac{3x}{2}}.\frac{1}{2}\right]$$

$$=\frac{1}{6}\lim_{x\to 0}\left(\frac{\tan\dfrac{x}{2}}{\dfrac{x}{2}}\right)=\frac{1}{6}(1) \qquad \left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

$$=\frac{1}{6}.$$

(*ii*) We have,

$$\lim_{x \to 0} \frac{3 \sin x - \sin 3x}{x^3} = \lim_{x \to 0} \left[\frac{3 \sin x - (3 \sin x - 4 \sin^3 x)}{x^3}\right]$$

$[\because \quad \sin 3A = 3 \sin A - 4 \sin^3 A]$

$$= \lim_{x \to 0} \frac{4 \sin^3 x}{x^3} = 4 \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^3 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= 4 (1)^3 = 4.$$

(*iii*) We have,

$$\lim_{x \to 0} \left(\frac{\tan x - \sin x}{x^3}\right) = \lim_{x \to 0} \left[\frac{\left(\frac{\sin x}{\cos x} - \frac{\sin x}{1}\right)}{x^3}\right]$$

$$= \lim_{x \to 0} \left(\frac{\sin x - \sin x \cos x}{x^3 \cos x}\right) = \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x}$$

$$= \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2} \cdot \frac{1}{\cos x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{2 \sin^2 \frac{x}{2}}{x^2} \cdot \frac{1}{\cos x}\right] \qquad \left[\begin{array}{l}\because \ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow \ 1 - \cos A = 2 \sin^2 \frac{A}{2}\end{array}\right]$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot 2 \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{4} \cdot 4} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 \cdot \frac{1}{2} \cdot (1)^2 \cdot \frac{1}{1} = \frac{1}{2}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

(*iv*) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x} = \lim_{x \to 0} \left[\frac{\frac{\sin 2x}{x} + \frac{\sin 3x}{x}}{2 + \frac{\sin 3x}{x}}\right]$$

[Dividing numerator and denominator by x]

$$= \lim_{x \to 0} \left[\frac{\frac{\sin 2x}{2x} \cdot 2 + \frac{\sin 3x}{3x} \cdot 3}{2 + \frac{\sin 3x}{3x} \cdot 3} \right]$$

$$= \frac{2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \right) + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right)}{2 + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right)}$$

$$= \frac{2(1) + 3(1)}{2 + 3(1)} = \frac{5}{5} = 1. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

(*v*) We have,

$$\lim_{x \to 0} \frac{\sin x^\circ}{x} = \lim_{x \to 0} \left[\frac{\sin \frac{\pi x}{180^\circ}}{x} \right] \qquad \left[\begin{array}{l} \because \ 180 \text{ degree} = \pi \text{ radian} \\ \because \ x \text{ degree} = \frac{\pi x}{180} \text{ radian} \end{array} \right]$$

$$= \lim_{x \to 0} \left[\frac{\sin \frac{\pi x}{180^\circ}}{\frac{\pi x}{180^\circ}} \cdot \frac{\pi}{180} \right] = \frac{\pi}{180} \left(\lim_{x \to 0} \frac{\sin \frac{\pi x}{180^\circ}}{\frac{\pi x}{180^\circ}} \right)$$

$$= \frac{\pi}{180} \times 1 = \frac{\pi}{180^\circ}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Example 6. ***Evaluate the following limits :***

(*i*) $\lim_{\theta \to \pi/2} \frac{\cot \theta}{\left(\frac{\pi}{2} - \theta \right)}$ (*ii*) $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$

(*iii*) $\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2}$ (*iv*) $\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2}$

(*v*) $\lim_{x \to \pi/4} \left(\frac{\sec^2 x - 2}{\tan x - 1} \right)$.

Solution. (*i*) We have, $\lim_{\theta \to \pi/2} \frac{\cot \theta}{\left(\frac{\pi}{2} - \theta \right)}$

Let $\theta = \frac{\pi}{2} + h \Rightarrow h \to 0 \text{ as } \theta \to \frac{\pi}{2}$

$$\therefore \quad \lim_{\theta \to \pi/2} \frac{\cot \theta}{\left(\frac{\pi}{2} - \theta\right)} = \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right)}{\left[\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right]}$$

$$= \lim_{h \to 0} \left(\frac{-\tan h}{-h}\right) \qquad \left[\because \cot\left(\frac{\pi}{2} + A\right) = -\tan A\right]$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \qquad \left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

$$= 1.$$

(*ii*) We have, $\lim_{x \to \pi/2} \left(\frac{2x - \pi}{\cos x}\right)$

Let $x = \frac{\pi}{2} + h \Rightarrow h \to 0 \text{ as } x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \pi/2} \left(\frac{2x - \pi}{\cos x}\right) = \lim_{h \to 0} \left[\frac{2\left(\frac{\pi}{2} + h\right) - \pi}{\cos\left(\frac{\pi}{2} + h\right)}\right]$$

$$= \lim_{h \to 0} \frac{\pi + 2h - \pi}{-\sin h} \qquad [\because \cos(90° + A) = -\sin A]$$

$$= 2 \lim_{h \to 0} \frac{1}{-\left(\frac{\sin h}{h}\right)} = -2\left(\frac{1}{\lim_{h \to 0} \frac{\sin h}{h}}\right)$$

$$= -2\left(\frac{1}{1}\right) = -2. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

(*iii*) We have, $\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2}$

$$= \lim_{x \to 0} \frac{2\left[\sin\left(\frac{A + B}{2}\right)x\right] . \sin\left[\left(\frac{B - A}{2}\right)x\right]}{x^2}$$

$$\left[\because \cos C - \cos D = 2 \sin \frac{C + D}{2} \sin \frac{D - C}{2}\right]$$

$$= \frac{3(1)-2}{3-0(1)} = \frac{1}{3}. \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\ \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

(*iii*) We have, $\lim_{x\to\pi/4}\frac{1-\tan x}{(x-\pi/4)}$

Let $x=\frac{\pi}{4}+h \Rightarrow h\to 0$ as $x\to\frac{\pi}{4}$

$$\therefore \lim_{x\to\pi/4}\left(\frac{1-\tan x}{x-\pi/4}\right)=\lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{\left(\frac{\pi}{4}+h-\frac{\pi}{4}\right)}$$

$$=\lim_{h\to 0}\left(\frac{1-\frac{1+\tan h}{1-\tan h}}{h}\right) \qquad \left[\because \tan\left(\frac{\pi}{4}+A\right)=\frac{1+\tan A}{1-\tan A}\right]$$

$$=\lim_{h\to 0}\left[\frac{1-\tan h-1-\tan h}{h(1-\tan h)}\right]$$

$$=\lim_{h\to 0}\frac{-2\tan h}{h(1-\tan h)}=-2\lim_{h\to 0}\left[\frac{\tan h}{h}\cdot\frac{1}{(1-\tan h)}\right]$$

$$=-2\lim_{h\to 0}\left(\frac{\tan h}{h}\right).\lim_{h\to 0}\frac{1}{(1-\tan h)}$$

$$=-2(1).\left(\frac{1}{1-0}\right)=-2. \qquad \left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

(*iv*) We have,

$$\lim_{x\to 0}\frac{\sqrt{5}-\sqrt{4+\cos x}}{3\sin^2 x}=\lim_{h\to 0}\left[\frac{\sqrt{5}-\sqrt{4+\cos x}}{3\sin^2 x}\times\frac{\sqrt{5}+\sqrt{4+\cos x}}{\sqrt{5}+\sqrt{4+\cos x}}\right] \qquad \text{[Rationalization]}$$

$$=\lim_{h\to 0}\left[\frac{5-(4+\cos x)}{3\sin^2 x\left(\sqrt{5}+\sqrt{4+\cos x}\right)}\right]$$

$$=\lim_{h\to 0}\left[\frac{1-\cos x}{3\left(1-\cos^2 x\right)\left(\sqrt{5}+\sqrt{4+\cos x}\right)}\right] \qquad [\because \sin^2 A+\cos^2 A=1]$$

$$=\lim_{h\to 0}\left[\frac{(1-\cos x)}{3(1-\cos x)(1+\cos x)\left(\sqrt{5}+\sqrt{4+\cos x}\right)}\right]$$

(*iii*) We have,

$$\lim_{x \to 1} \frac{1-\frac{1}{x}}{\sin \pi (x-1)} = \lim_{x \to 1}\left[\frac{x-1}{x \sin \pi (x-1)}\right]$$

$$= \lim_{x \to 1}\left[\frac{\pi (x-1)}{\pi x \sin \pi (x-1)}\right]$$

Let $x = 1 + h \Rightarrow h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1}\left[\frac{\pi (x-1)}{\pi x \sin \pi (x-1)}\right] = \lim_{h \to 0}\left[\frac{\pi (1+h-1)}{\pi(1+h) \sin \pi (1+h-1)}\right]$$

$$= \lim_{h \to 0}\left(\frac{\pi h}{\pi(1+h) \sin \pi h}\right) = \left(\lim_{h \to 0} \frac{1}{\pi(1+h)}\right) \cdot \lim_{h \to 0}\left(\frac{\pi h}{\sin \pi h}\right)$$

$$= \frac{1}{\pi} \lim_{h \to 0}\left(\frac{1}{1+h}\right) \cdot \frac{1}{\lim_{h \to 0}\left(\frac{\sin \pi h}{\pi h}\right)}$$

$$= \frac{1}{\pi}\left(\frac{1}{1+0}\right)\left(\frac{1}{1}\right) \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{1}{\pi}.$$

(*iv*) We have, $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

Let $x = 1 + h \Rightarrow h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{h \to 0} (1-(1+h)) \tan\left(\frac{\pi (1+h)}{2}\right)$$

$$= \lim_{h \to 0} (-h) \tan\left(\frac{\pi}{2} + \frac{\pi h}{2}\right)$$

$$= \lim_{h \to 0} (-h)\left(-\cot \frac{\pi h}{2}\right) \qquad \left[\because \tan\left(\frac{\pi}{2} + A\right) = -\cot A\right]$$

$$= \lim_{h \to 0} h \cot\left(\frac{\pi h}{2}\right) = \lim_{h \to 0}\left[\frac{\left(\frac{\pi h}{2}\right)}{\tan\left(\frac{\pi h}{2}\right)} \cdot \frac{1}{\left(\frac{\pi}{2}\right)}\right]$$

$$= \frac{2}{\pi} \cdot \frac{1}{\lim_{h \to 0}\left[\frac{\tan\left(\frac{\pi h}{2}\right)}{\left(\frac{\pi h}{2}\right)}\right]}$$

$$= \frac{2}{\pi} \cdot \left(\frac{1}{1}\right) = \frac{2}{\pi}. \qquad \left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

Example 10. *Evaluate the following limits :*

(i) $\lim\limits_{x \to 0} \dfrac{1 - \cos x \sqrt{\cos 2x}}{x^2}$ (ii) $\lim\limits_{x \to \pi} \dfrac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$

(iii) $\lim\limits_{x \to \pi/6} \dfrac{\cot^3 x - 3}{\text{cosec } x - 2}$ (iv) $\lim\limits_{y \to 0} \dfrac{(x + y) \sec (x + y) - x \sec x}{y}$

(v) $\lim\limits_{x \to 0} \dfrac{\cot 2x - \text{cosec } 2x}{x}$.

Solution. (i) We have,

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} = \lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} \times \frac{1 + \cos x \sqrt{\cos 2x}}{1 + \cos x \sqrt{\cos 2x}} \quad \text{[Rationalisation]}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x \cos 2x}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x + \cos^2 x - \cos^2 x \cos 2x}{x^2 (1 + \cos x \sqrt{\cos 2x})} \quad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \lim_{x \to 0} \frac{\sin^2 x + \cos^2 x (1 - \cos 2x)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x + \cos^2 x (2 \sin^2 x)}{x^2 (1 + \cos x \sqrt{\cos 2x})} \quad [\because 1 - \cos 2A = 2 \sin^2 A]$$

$$= \lim_{x \to 0} \frac{\sin^2 x (1 + 2 \cos^2 x)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{\lim\limits_{x \to 0} (1 + 2 \cos^2 x)}{\lim\limits_{x \to 0} (1 + \cos x \sqrt{\cos 2x})}$$

$$= \frac{(1)^2 [1 + 2(1)]}{[1 + 1\sqrt{1}]} \quad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{3}{2}.$$

(ii) We have,

$$\lim_{x \to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{x \to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} \times \frac{\sqrt{5 + \cos x} + 2}{\sqrt{5 + \cos x} + 2} \quad \text{[Rationalisation]}$$

$$= \lim_{x \to \pi} \frac{5 + \cos x - 4}{(\pi - x)^2 (\sqrt{5 + \cos x} + 2)}$$

$$= \lim_{x \to \pi} \frac{(1 + \cos x)}{(\pi - x)^2 (\sqrt{5 + \cos x} + 2)}$$

Let $x = \pi + h \quad \Rightarrow \quad h \to 0$ as $x \to \pi$

$$\therefore \quad \lim_{x \to \pi} \frac{(1 + \cos x)}{(\pi - x)^2 (\sqrt{5 + \cos x} + 2)} = \lim_{h \to 0} \frac{[1 + \cos(\pi + h)]}{[\pi - (\pi + h)]^2 [\sqrt{5 + \cos(\pi + h)} + 2]}$$

$[\because \cos(\pi + A) = -\cos A]$

$$= \lim_{h \to 0} \frac{(1 - \cos h)}{h^2 [\sqrt{5 - \cos h} + 2]}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h^2 [\sqrt{5 - \cos h} + 2]} \qquad \left[\begin{array}{l} \because \ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow 1 - \cos A = 2 \sin^2 \dfrac{A}{2} \end{array}\right]$$

$$= 2 \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\frac{4h^2}{4} . [\sqrt{5 - \cos h} + 2]}$$

$$= \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 . \frac{1}{\lim\limits_{h \to 0} [\sqrt{5 - \cos h} + 2]} \qquad \left[\because \ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{1}{2} (1)^2 . \frac{1}{[\sqrt{5 - 1} + 2]} = \frac{1}{2} . \frac{1}{(\sqrt{4} + 2)} = \frac{1}{2(4)} = \frac{1}{8} .$$

(*iii*) We have,

$$\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\text{cosec } x - 2} = \lim_{x \to \pi/6} \frac{(\text{cosec}^2 x - 1) - 3}{\text{cosec } x - 2} \qquad [\because \ \text{cosec}^2 A - \cot^2 A = 1]$$

$$= \lim_{x \to \pi/6} \left(\frac{\text{cosec}^2 x - 4}{\text{cosec } x - 2} \right)$$

$$= \lim_{x \to \pi/6} \frac{(\text{cosec } x - 2)(\text{cosec } x + 2)}{(\text{cosec } x - 2)} = \lim_{x \to \pi/6} (\text{cosec } x + 2)$$

$$= \text{cosec} \left(\frac{\pi}{6} \right) + 2 = 2 + 2 = 4.$$

(*iv*) We have, $\lim\limits_{y \to 0} \dfrac{(x + y) \sec(x + y) - x \sec x}{y}$

$$= \lim_{y \to 0} \frac{x \sec(x + y) + y \sec(x + y) - x \sec x}{y}$$

$$= \lim_{y \to 0} \frac{x[\sec(x + y) - \sec x] + y \sec(x + y)}{y}$$

$$= \lim_{h \to 0} \frac{\left[2\cos\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right][\sqrt{a+h} + \sqrt{a}]}{h}$$

$$= 2\lim_{h \to 0} \cos\left(a + \frac{h}{2}\right) . \lim_{h \to 0} \frac{\sin\frac{h}{2}}{2\left(\frac{h}{2}\right)} . \lim_{h \to 0} \left(\sqrt{a+h} + \sqrt{a}\right)$$

$$= 2\cos a \;.\; \frac{1}{2} . (1) . (\sqrt{a} + \sqrt{a}) \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \cos a \times (2\sqrt{a}) = 2\sqrt{a}\,\cos a.$$

(*ii*) We have, $\lim_{x \to \pi/2} \frac{1 - \sin^3 x}{\cos^2 x}$

Let $x = \left(\frac{\pi}{2} + h\right) \Rightarrow h \to 0 \text{ as } x \to \frac{\pi}{2}$

$$\therefore \qquad \lim_{x \to \pi/2} \frac{1 - \sin^3 x}{\cos^2 x} = \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} + h\right)}{\cos^2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h \to 0} \frac{1 - \cos^3 h}{(-\sin h)^2} = \lim_{h \to 0} \frac{1 - \cos^3 h}{\sin^2 h}$$

$$= \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos h + \cos^2 h)}{(1 - \cos^2 h)}$$

$$\left[\begin{array}{c}\because (a^3 - b^3) = (a - b)(a^2 + ab + b^2) \\ \sin^2 A + \cos^2 A = 1\end{array}\right]$$

$$= \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos h + \cos^2 h)}{(1 - \cos h)(1 + \cos h)}$$

$$= \lim_{h \to 0} \frac{(1 + \cos h + \cos^2 h)}{(1 + \cos h)} = \frac{\lim_{h \to 0} [1 + \cos h + \cos^2 h]}{\lim_{h \to 0} [1 + \cos h]}$$

$$= \frac{1 + 1 + (1)^2}{1 + 1} = \frac{3}{2}.$$

or $$\lim_{x \to 0} x \sin \frac{1}{x}{}^{*} = 0 \times (\text{A finite quantity between } -1 \text{ and } 1)$$

$$= 0. \qquad \left[\because \sin \frac{1}{x} \text{ lies between } -1 \text{ and } 1\right]$$

(*ii*) We have,

$$\lim_{x \to 0} \frac{\cos x - \cot x}{x} = \lim_{x \to 0} \frac{\cos x - \dfrac{\cos x}{\sin x}}{x}$$

$$= \lim_{x \to 0} \frac{\cos x \sin x - \cos x}{x \sin x} = \lim_{x \to 0} \frac{\cos x (\sin x - 1)}{x \sin x}$$

$$= -\lim_{x \to 0} \frac{\cos x (1 - \sin x)}{x \sin x}$$

$$= -\lim_{x \to 0} \frac{\cos x (1 - \sin x)}{x \sin x} \times \frac{(1 + \sin x)}{(1 + \sin x)} \qquad \text{[Rationalisation]}$$

$$= -\lim_{x \to 0} \frac{\cos x (1 - \sin^2 x)}{x \sin x (1 + \sin x)}$$

$$= -\lim_{x \to 0} \frac{\cos x \, . \cos^2 x}{x^2 \dfrac{\sin x}{x} (1 + \sin x)} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= -\lim_{x \to 0} \frac{\cos^3 x}{x^2 \left(\dfrac{\sin x}{x}\right)(1 + \sin x)}$$

$$= -\lim_{x \to 0} \frac{\cos^3 x}{x^2 (1 + \sin x)} \cdot \frac{1}{\lim_{x \to 0} \left(\dfrac{\sin x}{x}\right)}$$

$$= \left(\frac{-1}{0}\right)\left(\frac{1}{1}\right) = -\infty . \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

(*iii*) We have,

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \to 0} \left(\frac{e^{\sin x} - 1}{\sin x}\right) \cdot \left(\frac{\sin x}{x}\right)$$

$$^{*} \lim_{x \to 0} x \sin \frac{1}{x} = \lim_{x \to 0} \frac{\sin \dfrac{1}{x}}{\dfrac{1}{x}} = 1 \text{ is wrong.}$$

Because, As $x \to 0, \dfrac{1}{x} \to \infty$.

[redacted]

$$= \lim_{\sin x \to 0}\left[\frac{e^{\sin x}-1}{\sin x}\right].\lim_{x\to 0}\left[\frac{\sin x}{x}\right] \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\ \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

$$= 1 \times 1 = 1. \qquad [\text{As } x \to 0,\ \sin x \to 0]$$

(iv) We have, $\lim_{x\to\infty} 2^{x-1}.\tan\left(\frac{a}{2^x}\right)$

Let $\frac{a}{2^x} = t \Rightarrow t \to 0$ as $x \to \infty$

$$\therefore \quad 2^{x-1} = 2^x.2^{-1} = \frac{2^x}{2} = \left(\frac{2^x}{a}.\frac{a}{2}\right) = \left(\frac{a}{2t}\right)$$

$$\therefore \quad \lim_{x\to\infty} 2^{x-1}\tan\left(\frac{a}{2^x}\right) = \lim_{t\to 0}\frac{a}{2t}.\tan t$$

$$= \frac{a}{2}\lim_{t\to 0}\left(\frac{\tan t}{t}\right)$$

$$= \frac{a}{2}(1) = \frac{a}{2}. \qquad \left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

(v) We have, $\lim_{x\to\pi/2}\left(\frac{e^{\sin x}-1}{\sin x}\right)$

Let $x = \frac{\pi}{2}+h \Rightarrow h \to 0$ as $x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x\to\pi/2}\left(\frac{e^{\sin x}-1}{\sin x}\right) = \lim_{h\to 0}\left[\frac{e^{\sin\left(\frac{\pi}{2}+h\right)}-1}{\sin\left(\frac{\pi}{2}+h\right)}\right]$$

$$= \lim_{h\to 0}\frac{e^{\cos h}-1}{\cos h} \qquad \left[\because \lim_{\theta\to 0}\cos\theta = 1\right]$$

$$= \frac{e-1}{1} = (e-1).$$

(vi) We have, $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$$= \lim_{h\to 0}\frac{(a^2+2ah+h^2)\sin(a+h)-a^2\sin a}{h}$$

$$= \lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]+(2ah+h^2)\sin(a+h)}{h}$$

$$= \lim_{h \to 0} \frac{a^2 \left[2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right] + h(2a+h)\sin(a+h)}{h}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \lim_{h \to 0} \left[\frac{2a^2 \cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2} + h(2a+h)\sin(a+h)}{h}\right]$$

$$= a^2 \lim_{h \to 0} \cos\left(a+\frac{h}{2}\right) . \lim_{h \to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) + \lim_{h \to 0} (2a+h)\sin(a+h)$$

$$= a^2 . \cos a . 1 + (2a)(\sin a) = a^2 \cos a + 2a \sin a.$$

Example 13. *Evaluate the following limits :*

(i) $\lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x}$ (ii) $\lim_{x \to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

(iii) $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$ (iv) $\lim_{x \to 0} \frac{x^2 + 1 - \cos x}{x \tan x}$

(v) $\lim_{x \to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$.

Solution. (i) We have, $\lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x}$

Let $x = \frac{\pi}{2} + h \Rightarrow h \to 0$ as $x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{h \to 0} \frac{\cos^2\left(\frac{\pi}{2}+h\right)}{1 - \sin\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h \to 0} \frac{(-\sin h)^2}{1 - \cos h} = \lim_{h \to 0} \frac{\sin^2 h}{(1 - \cos h)} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \lim_{h \to 0} \frac{(1 - \cos^2 h)}{(1 - \cos h)}$$

$$= \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos h)}{(1 - \cos h)} = \lim_{h \to 0} (1 + \cos h)$$

$$= (1 + 1) = 2.$$

(*ii*) We have,

$$\lim_{x \to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} = \lim_{x \to \pi/6} \frac{2\sin^2 x + 2\sin x - \sin x - 1}{2\sin^2 x - 2\sin x - \sin x + 1}$$

[Note this step]

$$= \lim_{x \to \pi/6} \frac{2\sin x(\sin x + 1) - 1(\sin x + 1)}{2\sin x(\sin x - 1) - 1(\sin x - 1)}$$

$$= \lim_{x \to \pi/6} \frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)} = \lim_{x \to \pi/6} \frac{(\sin x + 1)}{(\sin x - 1)}$$

$$= \frac{\lim_{x \to \pi/6} (\sin x + 1)}{\lim_{x \to \pi/6} (\sin x - 1)} = \left(\frac{\sin \frac{\pi}{6} + 1}{\sin \frac{\pi}{6} - 1}\right) = \left(\frac{\frac{1}{2} + 1}{\frac{1}{2} - 1}\right) = \frac{\frac{3}{2}}{\frac{-1}{2}} = -3.$$

(*iii*) We have,

$$\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$$

[Note this step]

$$= \lim_{x \to 0} \frac{\sin x - \sin 3x - \sin 3x + \sin 5x}{x}$$

$$= \lim_{x \to 0} \frac{(\sin x - \sin 3x) + (\sin 5x - \sin 3x)}{x}$$

$$= \lim_{x \to 0} \frac{\left[2\cos\left(\frac{x + 3x}{2}\right)\sin\left(\frac{x - 3x}{2}\right)\right] + \left[2\cos\left(\frac{5x + 3x}{2}\right)\sin\left(\frac{5x - 3x}{2}\right)\right]}{x}$$

$$\left[\because \ \sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}\right]$$

$$= \lim_{x \to 0} \frac{2\cos 2x \sin(-x) + 2\cos 4x \sin x}{x}$$

$$= \lim_{x \to 0} \frac{-2\cos 2x \sin x + 2\cos 4x \sin x}{x} \qquad [\because \ \sin(-\theta) = -\sin\theta]$$

$$= \lim_{x \to 0} \frac{2\sin x(\cos 4x - \cos 2x)}{x}$$

$$= 2\lim_{x \to 0}\left(\frac{\sin x}{x}\right).\lim_{x \to 0}(\cos 4x - \cos 2x)$$

$$= 2.(1).(1 - 1) \qquad \left[\because \ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= 2(0) = 0.$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{\tan^2 h} \qquad \left[\begin{array}{l} \because\ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow\ 1 - \cos A = 2 \sin^2 \dfrac{A}{2} \end{array}\right]$$

$$= \lim_{h \to 0} \left(2 \sin^2 \frac{h}{2} \cdot \frac{1}{\frac{\sin^2 h}{\cos^2 h}} \right) = \lim_{h \to 0} \left(2 \sin^2 \frac{h}{2} \cdot \frac{\cos^2 h}{\sin^2 h} \right)$$

$$= 2 \lim_{h \to 0} \left(\frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4}} \right) \cdot \left(\frac{\frac{h^2}{4}}{\sin^2 h} \right) . (\cos^2 h)$$

$$= 2 \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \lim_{h \to 0} \frac{1}{4} \left(\frac{h}{\sin h} \right)^2 \lim_{h \to 0} \cos^2 h$$

$$= 2 (1)^2 . \frac{1}{4} (1)^2 . (1)^2 \qquad \left[\because\ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{1}{2}.$$

(*vi*) We have,

$$\lim_{x \to \pi/2} \frac{\cos 3x + 3 \cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim_{x \to \pi/2} \frac{4 \cos^3 x - 3 \cos x + 3 \cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim_{x \to \pi/2} \frac{4 \cos^3 x}{\left(\frac{\pi}{2} - x\right)^3}$$

$$[\because\ \cos 3A = 4 \cos^3 A - 3 \cos A]$$

Let $x = \frac{\pi}{2} + h \Rightarrow h \to 0$ as $x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \pi/2} \frac{4 \cos^3 x}{\left(\frac{\pi}{2} - x\right)^3} = \lim_{h \to 0} \frac{4 \cos^3 \left(\frac{\pi}{2} + h\right)}{\left[\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right]^3}$$

$$= 4 \lim_{h \to 0} \frac{(-\sin h)^3}{(-h)^3} = 4 \lim_{h \to 0} \frac{-\sin^3 h}{-h^3}$$

$$= 4 \lim_{h \to 0} \left(\frac{\sin h}{h} \right)^3 \qquad \left[\because\ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= 4 (1)^3 = 4.$$

(*vii*) We have,

$$\lim_{x \to 0}\left(\frac{\cos mx - \cos nx}{x^2}\right) = \lim_{x \to 0}\frac{2\sin\left(\frac{mx+nx}{2}\right)\sin\left(\frac{nx-mx}{2}\right)}{x^2}$$

$$\left[\because \ \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$$= 2\lim_{x \to 0}\frac{\sin\left(\frac{m+n}{2}\right)x}{x}\cdot\frac{\sin\left(\frac{n-m}{2}\right)x}{x}$$

$$= 2\lim_{x \to 0}\left(\frac{\sin\left(\frac{m+n}{2}\right)x}{\left(\frac{m+n}{2}\right)x}\right)\left(\frac{m+n}{2}\right)\cdot\lim_{x \to 0}\left(\frac{\sin\left(\frac{n-m}{2}\right)x}{\left(\frac{n-m}{2}\right)x}\right)\left(\frac{n-m}{2}\right)$$

$$= 2\times(1)\times\left(\frac{m+n}{2}\right)\times 1\times\left(\frac{n-m}{2}\right)$$

$$= \frac{n^2-m^2}{2}.$$

Example 15. *Evaluate the following limits :*

(*i*) $\lim_{x \to 1}\frac{1+\cos \pi x}{(1-x)^2}$ (*ii*) $\lim_{x \to \pi/2}\frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2}$

(*iii*) $\lim_{x \to 0}\frac{1+\sin x-\cos x}{x}$ (*iv*) $\lim_{x \to \pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$.

Solution. (*i*) We have, $\lim_{x \to 1}\frac{1+\cos \pi x}{(1-x)^2}$

Let $x = 1 + h \Rightarrow h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1}\frac{1+\cos \pi x}{(1-x)^2} = \lim_{h \to 0}\frac{1+\cos \pi(1+h)}{[1-(1+h)]^2}$$

$$= \lim_{h \to 0}\frac{1+\cos(\pi+\pi h)}{(-h)^2} = \lim_{h \to 0}\frac{1-\cos \pi h}{h^2}$$

$$= \lim_{h \to 0}\frac{2\sin^2\frac{\pi h}{2}}{h^2} = 2\lim_{h \to 0}\frac{\sin^2\frac{\pi h}{2}}{\frac{\pi^2 h^2}{4}}\cdot\left(\frac{\pi^2}{4}\right)$$

$$\left[\because \ 1-\cos 2A = 2\sin^2 A \Rightarrow 1-\cos A = 2\sin^2\frac{A}{2}\right]$$

$$= 2.\left(\frac{\pi^2}{4}\right).\lim_{h\to 0}\left(\frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}}\right)^2 \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= \frac{\pi^2}{2}(1)^2 = \frac{\pi^2}{2}.$$

(*ii*) **We have,**

$$\lim_{x\to\pi/2}\frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{x\to\pi/2}\frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2}\times\frac{(\sqrt{2-\sin x}+1)}{(\sqrt{2-\sin x}+1)} \qquad \text{[Rationalisation]}$$

$$= \lim_{x\to\pi/2}\frac{(2-\sin x)-1}{\left(\frac{\pi}{2}-x\right)^2(\sqrt{2-\sin x}+1)}$$

$$= \lim_{x\to\pi/2}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2(\sqrt{2-\sin x}+1)}$$

Let $x = \frac{\pi}{2}+h \Rightarrow h\to 0$ as $x\to\frac{\pi}{2}$

$$\therefore \lim_{x\to\pi/2}\frac{(1-\sin x)}{\left(\frac{\pi}{2}-x\right)^2(\sqrt{2-\sin x}+1)} = \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2}-h\right)^2\left(\sqrt{2-\sin\left(\frac{\pi}{2}+h\right)}+1\right)}$$

$$= \lim_{h\to 0}\frac{1-\cos h}{h^2(\sqrt{2-\cos h}+1)}$$

$$\left[\because 1-\cos 2A = 2\sin^2 A \Rightarrow 1-\cos A = 2\sin^2\frac{A}{2}\right]$$

$$= \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h^2(\sqrt{2-\cos h}+1)} = 2\lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{4\left(\frac{h^2}{4}\right).(\sqrt{2-\cos h}+1)}$$

$$= \frac{2}{4}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2.\frac{1}{\lim_{h\to 0}(\sqrt{2-\cos h}+1)}$$

$$= \frac{1}{2}.(1)^2.\left(\frac{1}{\sqrt{2-1}+1}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{1+1}\right) = \frac{1}{4}.$$

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$$= \frac{1}{2} \lim_{h \to 0} \frac{\left(1 - \cos\frac{h}{2}\right)}{\left(1 - \cos^2\frac{h}{2}\right)(1 + \cos h)}$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{\left(1 - \cos\frac{h}{2}\right)}{\left(1 - \cos\frac{h}{2}\right)\left(1 + \cos\frac{h}{2}\right)(1 + \cos h)}$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{1}{\left(1 + \cos\frac{h}{2}\right)(1 + \cos h)} = \frac{1}{2}\left(\frac{1}{(1+1)(1+1)}\right) = \frac{1}{8}.$$

Example 17. *Evaluate the following limits :*

(i) $\lim_{x \to \pi/2} \frac{\cot x}{\left(x - \frac{\pi}{2}\right)}$ (ii) $\lim_{x \to \pi} \frac{1 + \sec^3 x}{\tan^2 x}$

(iii) $\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$.

Solution. (i) We have, $\lim_{x \to \pi/2} \frac{\cot x}{\left(x - \frac{\pi}{2}\right)}$

Let $x = \left(\frac{\pi}{2} + h\right) \Rightarrow h \to 0$ as $x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \pi/2} \frac{\cot x}{\left(x - \frac{\pi}{2}\right)} = \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right)}{\left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)} = \lim_{h \to 0} \frac{-\tan h}{h}$$

$$= -\lim_{h \to 0}\left(\frac{\tan h}{h}\right) = -1. \qquad \left[\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

(ii) We have,

$$\lim_{x \to \pi} \frac{1 + \sec^3 x}{\tan^2 x} = \lim_{x \to \pi} \frac{(1 + \sec x)(1 - \sec x + \sec^2 x)}{(\sec^2 x - 1)} \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$[a^3 + b^3 = (a + b)(a^2 - ab + b^2)]$

$$= \lim_{x \to \pi} \frac{(1 + \sec x)(1 - \sec x + \sec^2 x)}{(\sec x - 1)(\sec x + 1)}$$

$$= \lim_{x \to \pi} \frac{1 - \sec x + \sec^2 x}{\sec x - 1} \qquad [\because \ \sec \pi = -1]$$

$$= \frac{1-(-1)+(-1)^2}{-1-1} = \frac{1+1+1}{-2} = \frac{-3}{2}.$$

(*iii*) We have,

$$\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = \lim_{x \to \pi/4} \frac{\tan x (\tan^2 x - 1)}{\cos\left(x + \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)(\tan x - 1)}{\cos\left(x + \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)\left(\frac{\sin x}{\cos x} - 1\right)}{\cos\left(x + \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)(\sin x - \cos x)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)(\cos x - \sin x)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x\right)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

[Multiply and divided by $\sqrt{2}$]

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)\left(\cos \frac{\pi}{4} \cos x - \sin \frac{\pi}{4} \sin x\right)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$\left[\because \ \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

[$\because$ cos (A + B) = cos A cos B – sin A sin B]

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1) \cos\left(x + \frac{\pi}{4}\right)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)}{\cos x} = -\sqrt{2} \cdot \frac{(1)(1)(1+1)}{\frac{1}{\sqrt{2}}}$$

$$= (-\sqrt{2})(2\sqrt{2}) = -4.$$

Example 18. *Evaluate the following limits :*

(i) $\lim_{x \to 0} \frac{\sin^{-1} 3x}{\sin 4x}$ (ii) $\lim_{x \to 0} \left\{\frac{1}{x} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$

(iii) $\lim_{x \to 0} \left\{\frac{1}{x} \sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}$ (iv) $\lim_{x \to 1} \frac{1-\sqrt{x}}{(\cos^{-1} x)^2}$

(v) $\lim_{x \to 1^-} \frac{1-x}{(\cos^{-1} x)^2}$.

Solution. (*i*) We have,

$$\lim_{x \to 0} \frac{\sin^{-1} 3x}{\sin 4x} = \lim_{x \to 0} \left[\frac{\frac{\sin^{-1} 3x}{3x} \cdot 3x}{\frac{\sin 4x}{4x} \cdot 4x}\right]$$

$$= \frac{3}{4} \frac{\lim_{x \to 0}\left(\frac{\sin^{-1} 3x}{3x}\right)}{\lim_{x \to 0}\left(\frac{\sin 4x}{4x}\right)}$$

$$= \frac{3}{4}\left(\frac{1}{1}\right) = \frac{3}{4}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin^{-1} \theta}{\theta} = 1, \ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

(*ii*) We have, $\lim_{x \to 0} \left(\frac{1}{x} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$

Let $x = \tan \theta \Rightarrow \theta \to 0$ as $x \to 0$

$$\therefore \lim_{x \to 0} \left(\frac{1}{x} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right) = \lim_{\theta \to 0} \left(\frac{1}{\tan \theta} \cos^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right)\right)$$

$$= \lim_{\theta \to 0} \left[\frac{1}{\tan \theta} \cos^{-1}(\cos 2\theta)\right] \qquad \left[\because \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$

$$= \lim_{\theta \to 0} \left(\frac{2\theta}{\tan\theta} \right) = 2 \lim_{\theta \to 0} \frac{1}{\left(\frac{\tan\theta}{\theta} \right)}$$

$$= 2\left(\frac{1}{1}\right) = 2. \qquad \left[\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

(*iii*) We have, $\lim_{x \to 0} \left[\frac{1}{x} \sin^{-1}\left(\frac{2x}{1+x^2} \right) \right]$

Let $x = \tan\theta \Rightarrow \theta \to 0$ as $x \to 0$

$$\therefore \lim_{x \to 0} \left[\frac{1}{x} \sin^{-1}\left(\frac{2x}{1+x^2} \right) \right] = \lim_{\theta \to 0} \left[\frac{1}{\tan\theta} . \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta} \right) \right]$$

$$= \lim_{\theta \to 0} \left[\frac{1}{\tan\theta} . \sin^{-1}(\sin 2\theta) \right] \qquad \left[\because \sin 2A = \frac{2\tan A}{1+\tan^2 A}\right]$$

$$= \lim_{\theta \to 0} \left(\frac{2\theta}{\tan\theta} \right) = 2 \lim_{\theta \to 0} \frac{1}{\left(\frac{\tan\theta}{\theta} \right)}$$

$$= 2(1) = 2. \qquad \left[\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

(*iv*) We have, $\lim_{x \to 1} \frac{1-\sqrt{x}}{(\cos^{-1} x)^2}$ **[Rationalisation]**

$$= \lim_{x \to 1} \frac{(1-\sqrt{x})}{(\cos^{-1} x)^2} \times \frac{(1+\sqrt{x})}{(1+(\sqrt{x})} = \lim_{x \to 1} \frac{1-x}{(\cos^{-1} x)^2 (1+\sqrt{x})}$$

Let $x = \cos\theta \Rightarrow \theta \to 0$ as $x \to 1$

$$\therefore \lim_{x \to 1} \frac{1-x}{(\cos^{-1} x)^2 (1+\sqrt{x})} = \lim_{\theta \to 0} \frac{1-\cos\theta}{[\cos^{-1}(\cos\theta)]^2 [1+\sqrt{\cos\theta}]}$$

$$= \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta^2 (1+\sqrt{\cos\theta})}$$

$$= \lim_{\theta \to 0} \frac{2\sin^2 \frac{\theta}{2}}{4 . \frac{\theta^2}{4}} . \frac{1}{1+\sqrt{\cos\theta}}$$

$$\left[\because 1-\cos 2A = 2\sin^2 A \Rightarrow 1-\cos A = 2\sin^2 \frac{A}{2}\right]$$

$$= \frac{1}{2} \lim_{\theta \to 0} \left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right)^2 \cdot \frac{1}{1+\sqrt{\cos \theta}}$$

$$= \frac{1}{2} (1)^2 \cdot \frac{1}{1+\sqrt{1}} = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right].$$

(*v*) We have, $\lim_{x \to 1^-} \frac{1-x}{(\cos^{-1} x)^2}$

Let $x = \cos \theta \Rightarrow \theta \to 0$ as $\theta \to 1^-$ $\qquad$ [$\because$ Whenever $0 < \theta < \pi, -1 \le x \le 1$]

$$\therefore \quad \lim_{x \to 1^-} \frac{1-x}{(\cos^{-1} x)^2} = \lim_{\theta \to 0} \frac{1-\cos \theta}{[\cos^{-1}(\cos \theta)]^2} = \lim_{\theta \to 0} \frac{1-\cos \theta}{\theta^2}$$

$$= \lim_{\theta \to 0} \frac{1-\cos \theta}{\theta^2} \times \frac{1+\cos \theta}{1+\cos \theta} \qquad \text{[Rationalisation]}$$

$$= \lim_{\theta \to 0} \frac{1-\cos^2 \theta}{\theta^2 (1+\cos \theta)} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2} \cdot \frac{1}{1+\cos \theta}$$

$$= \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right)^2 \cdot \lim_{\theta \to 0} \frac{1}{1+\cos \theta}$$

$$= (1)^2 \cdot \frac{1}{(1+1)} = \frac{1}{2}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

Example 19. *Evaluate the following limits :*

(*i*) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x}$ $\qquad$ (*ii*) $\lim_{x \to 1} \frac{1-x}{\pi - 2\sin^{-1} x}$

(*iii*) $\lim_{x \to 1/\sqrt{2}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$ $\qquad$ (*iv*) $\lim_{x \to 0} \frac{\sin^{-1} x + \tan^{-1} x}{7x - 5\sin^{-1} x}$

(*v*) $\lim_{x \to 0} \frac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}\,(\sin^{-1} x)^3}$.

Solution. (*i*) We have,

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x} = \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \qquad \text{[Rationalisation]}$$

$$= \lim_{x \to 0} \frac{(1+x)-(1-x)}{\sin^{-1}(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2x}{\sin^{-1} x} \cdot \frac{1}{(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{1}{x^3}\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) - \tan^{-1}(x)\right]$$

$$= \lim_{x \to 0} \frac{1}{x^3}\left[\tan^{-1}\left(\frac{\frac{x}{\sqrt{1-x^2}} - x}{1 + \frac{x^2}{\sqrt{1-x^2}}}\right)\right] \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x^3}\left[\tan^{-1}\left(\frac{x - x\sqrt{1-x^2}}{\sqrt{1-x^2}+x^2}\right)\right] = \lim_{x \to 0} \frac{1}{x^3}\left[\tan^{-1}\left(\frac{x(1-\sqrt{1-x^2})}{(x^2+\sqrt{1-x^2})}\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x^3}\left[\frac{\tan^{-1}\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}{\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)} \cdot \frac{x(1-\sqrt{1-x^2})}{(x^2+\sqrt{1-x^2})}\right]$$

$$= \lim_{x \to 0}\left[\frac{\tan^{-1}\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}{\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}\right] \cdot \left[\frac{(1-\sqrt{1-x^2})}{x^2} \cdot \frac{1}{(x^2+\sqrt{1-x^2})}\right]$$

$$= \lim_{x \to 0}\left[\frac{\tan^{-1}\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}{\frac{x(1-\sqrt{1-x^2})}{(x^2+\sqrt{1-x^2})}}\right]\left[\frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x^2(1+\sqrt{1-x^2})} \cdot \frac{1}{x^2+\sqrt{1-x^2}}\right]$$

[Rationalisation]

$$= \lim_{x \to 0}\left[\frac{\tan^{-1}\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}{\left(\frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}\right)}\right] \cdot \lim_{x \to 0}\left(\frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})}\right) \lim_{x \to 0}\left(\frac{1}{x^2+\sqrt{1-x^2}}\right)$$

$$= (1) \, . \lim_{x \to 0} \frac{x^2}{x^2 (1 + \sqrt{1 - x^2})} \, . \lim_{x \to 0} \frac{1}{(x^2 + \sqrt{1 - x^2})} \qquad \left[\because \lim_{\theta \to 0} \frac{\tan^{-1} \theta}{\theta} = 1\right]$$

$$= (1) \, . \left(\frac{1}{1+1}\right) . \left(\frac{1}{0+1}\right) = \frac{1}{2} .$$

(*ii*) We have,

$$\lim_{x \to 0} \frac{\sin^{-1} 3x}{\tan 7x} = \lim_{x \to 0} \left[\frac{\frac{\sin^{-1} 3x}{3x} . 3x}{\frac{\tan 7x}{7x} . 7x}\right] = \frac{3}{7} \frac{\lim_{x \to 0} \left(\frac{\sin^{-1} 3x}{3x}\right)}{\lim_{x \to 0} \left(\frac{\tan 7x}{7x}\right)}$$

$$= \frac{3}{7} \frac{(1)}{(1)} = \frac{3}{7} . \qquad \left[\because \lim_{\theta \to 0} \frac{\sin^{-1} \theta}{\theta} = 1 \quad \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

(*iii*) We have,

$$\lim_{x \to \infty} x \left(\tan^{-1} \frac{x+1}{x+4} - \frac{\pi}{4}\right) \qquad \left[\because \tan \frac{\pi}{4} = 1\right]$$

$$= \lim_{x \to \infty} x \left[\tan^{-1} \left(\frac{x+1}{x+4}\right) - \tan^{-1} (1)\right] \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy}\right)\right]$$

$$= \lim_{x \to \infty} x \tan^{-1} \left(\frac{\frac{x+1}{x+4} - 1}{1 + \frac{x+1}{x+4}}\right) = \lim_{x \to \infty} x \tan^{-1} \left(\frac{x+1-x-4}{x+4+x+1}\right)$$

$$= \lim_{x \to \infty} x \tan^{-1} \left(\frac{-3}{2x+5}\right) = \lim_{x \to \infty} x \left[\frac{\tan^{-1} \left(\frac{-3}{2x+5}\right)}{\left(\frac{-3}{2x+5}\right)} . \left(\frac{-3}{2x+5}\right)\right]$$

$$= \lim_{x \to \infty} \left(\frac{\tan^{-1} \left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right) . \lim_{x \to \infty} \left(\frac{-3x}{2x+5}\right)$$

$$= (1) \, . \lim_{x \to \infty} \left(\frac{-3}{2 + \frac{5}{x}}\right) = \left(\frac{-3}{2+0}\right) = \frac{-3}{2} .$$

(*iv*) We have, $\lim\limits_{x \to 0} \dfrac{\sin^{-1}(2x\sqrt{1-x^2})}{x}$

Let $x = \sin\theta \Rightarrow \theta \to 0$ as $x \to 0$

$\therefore \lim\limits_{x \to 0} \dfrac{\sin^{-1}(2x\sqrt{1-x^2})}{x} = \lim\limits_{\theta \to 0} \dfrac{\sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})}{\sin\theta}$ $[\because \sin^2 A + \cos^2 A = 1]$

$$= \lim_{\theta \to 0} \frac{\sin^{-1}(2\sin\theta\sqrt{\cos^2\theta})}{\sin\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin^{-1}(2\sin\theta\cos\theta)}{\sin\theta} \qquad [\because \sin 2A = 2\sin A\cos A]$$

$$= \lim_{\theta \to 0} \frac{\sin^{-1}(\sin 2\theta)}{\sin\theta} = \lim_{\theta \to 0}\left(\frac{2\theta}{\sin\theta}\right)$$

$$= 2\lim_{\theta \to 0}\left(\frac{\theta}{\sin\theta}\right) = 2(1) \qquad \left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 2.$$

(*v*) We have, $\lim\limits_{x \to 1} \dfrac{\cos^{-1}(4x^3 - 3x)}{\sqrt{1-x}}$

Let $x = \cos\theta \Rightarrow \theta \to 0$ as $x \to 1$

$\therefore \lim\limits_{x \to 1} \dfrac{\cos^{-1}(4x^3 - 3x)}{\sqrt{1-x}} = \lim\limits_{\theta \to 0} \dfrac{\cos^{-1}(4\cos^3\theta - 3\cos\theta)}{\sqrt{1-\cos\theta}}$

$[\because \cos 3A = 4\cos^3 A - 3\cos A]$

$$= \lim_{\theta \to 0} \frac{\cos^{-1}(\cos 3\theta)}{\sqrt{1-\cos\theta}} \qquad \left[\because 1 - \cos 2A = 2\sin^2 A \Rightarrow 1 - \cos A = 2\sin^2\frac{A}{2}\right]$$

$$= \lim_{\theta \to 0} \frac{3\theta}{\sqrt{2\sin^2\frac{\theta}{2}}} = \frac{3}{\sqrt{2}}\lim_{\theta \to 0}\frac{\frac{\theta}{2}.(2)}{\sin\frac{\theta}{2}}$$

$$= \frac{3.(2)}{\sqrt{2}}\lim_{\theta \to 0}\left(\frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = 3\sqrt{2}\,(1) \qquad \left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 3\sqrt{2}.$$

3.13 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

The following examples are based on the evaluation of limits involving exponential and logarithmic functions :

(*ii*) We have,

$$\lim_{x \to 0} (1+2x)^{1/x} = \lim_{x \to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^2$$

$$= \left[\lim_{x \to 0} (1+2x)^{\frac{1}{2x}}\right]^2 \qquad \left[\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$$= e^2.$$

(*iii*) We have,

$$\lim_{x \to 0}\left[\frac{(ab)^x - a^x - b^x + 1}{x^2}\right] = \lim_{x \to 0}\left[\frac{(a^x b^x - a^x - b^x + 1)}{x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{a^x(b^x - 1) - 1(b^x - 1)}{x^2}\right] = \lim_{x \to 0}\left[\frac{(a^x - 1)(b^x - 1)}{x^2}\right]$$

$$= \lim_{x \to 0}\left(\frac{a^x - 1}{x}\right) . \lim_{x \to 0}\left(\frac{b^x - 1}{x}\right)$$

$$= (\log a)(\log b). \qquad \left[\because \lim_{x \to 0}\frac{a^x - 1}{x} = \log a\right]$$

(*iv*) We have, $\lim_{x \to 0}\left[\frac{10^x - 2^x - 5^x + 1}{x^2}\right]$

Please try yourself.

[**Hint.** See part (*iii*) of the same example.] [**Ans.** (log 2) (log 5)]

(*v*) We have, $\lim_{x \to 2}\frac{x-2}{\log_a (x-1)}$

Let $x = 2 + h \Rightarrow h \to 0$ as $x \to 2$

$$\therefore \quad \lim_{x \to 2}\left[\frac{x-2}{\log_a (x-1)}\right] = \lim_{h \to 0}\frac{(2+h-2)}{\log_a [2+h-1]} = \lim_{h \to 0}\frac{h}{\log_a (h+1)}$$

$$= \lim_{h \to 0}\frac{h}{\log_e (1+h) . \log_a e}$$

$$= \frac{1}{\log_a e} . \frac{1}{\lim_{h \to 0}\left(\frac{\log (1+h)}{h}\right)} = \log_e a . \left(\frac{1}{1}\right) = \log a.$$

(*vi*) We have, $\lim_{x \to \infty} [x(4^{1/x} - 1)]$

Let $h = \frac{1}{x} \Rightarrow h \to 0$ as $x \to \infty$

$$\therefore \quad \lim_{x\to\infty} [x(4^{1/x}-1)] = \lim_{h\to 0} \frac{1}{h}(4^h - 1)$$

$$= \lim_{h\to 0}\left(\frac{4^h-1}{h}\right) \qquad \left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log a\right]$$

$$= \log 4.$$

Example 27. *Evaluate the following limits :*

(i) $\lim_{x\to 0} \frac{2^{3x}-3^x}{\sin 3x}$ (ii) $\lim_{x\to 0} \frac{\log(a+x)-\log a}{x}$

(iii) $\lim_{x\to 1} (x)^{\frac{1}{x-1}}$ (iv) $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x}$

(v) $\lim_{x\to 0} \frac{\log(1+x)}{x}$ (vi) $\lim_{x\to 0} \frac{\log(1+3x)}{3^x-1}$

(vii) $\lim_{x\to 0} \frac{x(e^{2+x}-e^2)}{1-\cos x}$.

Solution. (i) We have,

$$\lim_{x\to 0}\frac{2^{3x}-3^x}{\sin 3x} = \lim_{x\to 0}\left(\frac{2^{3x}-1+1-3^x}{\sin 3x}\right) \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \lim_{x\to 0}\left[\frac{(2^{3x}-1)-(3^x-1)}{\sin 3x}\right] = \lim_{x\to 0}\left[\frac{(2^{3x}-1)}{\sin 3x} - \frac{(3^x-1)}{\sin 3x}\right]$$

$$= \lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\cdot\frac{3x}{\sin 3x}\right) - \lim_{x\to 0}\left(\frac{3^x-1}{3x}\cdot\frac{3x}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\right)\cdot\lim_{x\to 0}\left(\frac{3x}{\sin 3x}\right) - \frac{1}{3}\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)\cdot\lim_{x\to 0}\left(\frac{3x}{\sin 3x}\right)$$

$$= (\log 2)(1) - \frac{1}{3}(\log 3)(1) \qquad \left[\begin{array}{l}\because \lim_{x\to 0}\frac{a^x-1}{x} = \log a \\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\end{array}\right]$$

$$= \log 2 - \frac{1}{3}\log 3.$$

(ii) We have,

$$\lim_{x\to 0}\frac{\log(a+x)-\log a}{x} = \lim_{x\to 0}\frac{\log\left(\frac{a+x}{a}\right)}{x} = \lim_{x\to 0}\frac{\log\left(1+\frac{x}{a}\right)}{a\cdot\frac{x}{a}}$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{a} \lim_{x \to 0} \left[\frac{\log\left(1 + \frac{x}{a}\right)}{\frac{x}{a}} \right]$$

$$= \frac{1}{a} \times 1 = \frac{1}{a}. \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \right]$$

(*iii*) We have, $\lim_{x \to 1} (x)^{\frac{1}{x-1}}$

Let $x = 1 + h \Rightarrow h \to 0$ as $x \to 1$

$$\therefore \quad \lim_{x \to 1} (x)^{\frac{1}{x-1}} = \lim_{h \to 0} (1+h)^{\frac{1}{1+h-1}} = \lim_{h \to 0} (1+h)^{\frac{1}{h}} = e. \qquad \left[\because \lim_{x \to 0} (1+x)^{1/x} = e \right]$$

(*iv*) We have,

$$\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x} = \lim_{x \to 0} \left[\frac{e^x - 1}{x} \cdot \frac{x^2}{2 \sin^2 \frac{x}{2}} \right] \qquad \left[\begin{array}{l} \because \ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow \ 1 - \cos A = 2 \sin^2 \frac{A}{2} \end{array} \right]$$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) \cdot \frac{1}{2} \lim_{x \to 0} \left(\frac{4 \cdot \frac{x^2}{4}}{\sin^2 \frac{x}{2}} \right)$$

$$= 1 \cdot \frac{4}{2} \lim_{x \to 0} \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 \qquad \left[\begin{array}{l} \because \ \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \\ \because \ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \end{array} \right]$$

$$= 1 \times 2 \times (1)^2 = 2.$$

(*v*) We have,

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} [\log(1+x)]$$

$$= \lim_{x \to 0} \log(1+x)^{1/x} \qquad [\because \ m \log n = \log n^m]$$

$$= \log \left[\lim_{x \to 0} (1+x)^{1/x} \right] \qquad \left[\because \ \lim_{x \to a} [\log f(x)] = \log \left[\lim_{x \to a} f(x) \right] \right]$$

$$= \log e \qquad \left[\because \ \lim_{x \to 0} (1+x)^{1/x} = e \right]$$

$$= 1.$$

(*vi*) We have,

$$\lim_{x\to 0}\frac{\log(1+3x)}{3^x-1}=\lim_{x\to 0}\left[\frac{\log(1+3x)}{3x}\cdot\frac{3x}{3^x-1}\right]$$

$$=\lim_{x\to 0}\frac{\log(1+3x)}{3x}\cdot\frac{3}{\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)}$$

$$=1\times\frac{3}{\log 3}=\frac{3}{\log 3}.\qquad\left[\because\ \lim_{x\to 0}\frac{\log(1+x)}{x}=1,\ \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$$

(*vii*) We have,

$$\lim_{x\to 0}\frac{x(e^{2+x}-e^2)}{1-\cos x}=\lim_{x\to 0}\frac{x(e^2.e^x-e^2)}{1-\cos x}$$

$$=\lim_{x\to 0}\left[\frac{e^2(e^x-1)}{2\sin^2\frac{x}{2}}.x\right]\qquad\left[\because\ 1-\cos 2A=2\sin^2 A\Rightarrow 1-\cos A=2\sin^2\frac{A}{2}\right]$$

$$=\frac{e^2}{2}\lim_{x\to 0}\left[\frac{e^x-1}{x}\cdot\frac{x^2}{\sin^2\frac{x}{2}}\right]=\frac{e^2}{2}\lim_{x\to 0}\left(\frac{e^x-1}{x}\right).\lim_{x\to 0}\left(\frac{4\frac{x^2}{4}}{\sin^2\frac{x}{2}}\right)$$

$$=\frac{e^2}{2}\times 1\times 4\lim_{x\to 0}\left(\frac{x/2}{\sin\frac{x}{2}}\right)^2\qquad\left[\because\ \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

$$=\frac{e^2}{2}\times 1\times 4\times(1)^2\qquad\left[\because\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$=2e^2.$$

Example 28. *Evaluate the following limits :*

(*i*) $\lim_{x\to 0}\frac{\log_3(1+7x)}{x}$

(*ii*) $\lim_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x},\ a>0$

(*iii*) $\lim_{x\to 0}(1+5x)^{1/2x}$

(*iv*) $\lim_{x\to\pi/2}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$

(*v*) $\lim_{x\to 0}\frac{a^{\sin x}-1}{\sin x}$.

Solution. (*i*) We have,

$$\lim_{x\to 0}\frac{\log_3(1+7x)}{x}=\lim_{x\to 0}\frac{\log_e(1+7x)}{(\log_e 3)\,x}\qquad\left[\because\ \log_b a=\frac{\log_e a}{\log_e b}\right]$$

$$=\lim_{x\to 0}\left(\frac{\log_e(1+7x)}{7x}\right)\cdot\left(\frac{7}{\log_e 3}\right)$$

$$=1.\frac{7}{\log_e 3}\qquad\left[\because\ \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$

$$=7\log_3 e.$$

(*ii*) We have, $\lim_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x}, a>0$

Let $x=1+h \Rightarrow h\to 0$ as $x\to 1$

$$\therefore\qquad \lim_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x}=\lim_{h\to 0}\left[\frac{a^{(1+h-1)}-1}{\sin\pi(1+h)}\right]$$

$$=\lim_{h\to 0}\left[\frac{a^h-1}{\sin(\pi+\pi h)}\right]=\lim_{h\to 0}\left(\frac{a^h-1}{-\sin\pi h}\right)$$

$$=-\lim_{h\to 0}\left(\frac{a^h-1}{h}.\frac{h}{\sin\pi h}\right)$$

$$=-\lim_{h\to 0}\left(\frac{a^h-1}{h}\right).\lim_{h\to 0}\frac{1}{\pi}\left(\frac{\pi h}{\sin\pi h}\right)$$

$$=-\frac{1}{\pi}(\log a).(1)\qquad\left[\because\ \lim_{x\to 0}\frac{a^x-1}{x}=\log a,\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$=-\frac{1}{\pi}\log a.$$

(*iii*) We have,

$$\lim_{x\to 0}(1+5x)^{1/2x}=\lim_{x\to 0}(1+5x)^{\left(\frac{1}{5x}\right)\left(\frac{5}{2}\right)}$$

$$=\lim_{x\to 0}[(1+5x)^{1/5x}]^{5/2}=(e)^{5/2}.\qquad\left[\because\ \lim_{x\to 0}(1+x)^{1/x}=e\right]$$

(*iv*) We have, $\lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)}$

Let $x = \frac{\pi}{2} + h \quad \Rightarrow \quad h \to 0$ as $x \to \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)} = \lim_{h \to 0} \frac{2^{-\cos\left(\frac{\pi}{2} + h\right)} - 1}{\left(\frac{\pi}{2} + h\right)\left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)}$$

$$= \lim_{h \to 0} \frac{2^{\sin h} - 1}{h\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \left[\frac{2^{\sin h} - 1}{\sin h} \cdot \frac{\sin h}{h} \cdot \frac{1}{\left(\frac{\pi}{2} + h\right)}\right]$$

$$= \lim_{h \to 0} \left(\frac{2^{\sin h} - 1}{\sin h}\right) \cdot \lim_{h \to 0} \left(\frac{\sin h}{h}\right) \cdot \frac{1}{\lim_{h \to 0} \left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{y \to 0} \left(\frac{2^y - 1}{y}\right) . 1 \times \frac{1}{\left(\frac{\pi}{2}\right)} \qquad \text{[where } y = \sin x\text{]}$$

$$= \log 2 \times 1 \times \frac{1}{\left(\frac{\pi}{2}\right)} \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a, \quad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{2}{\pi} \log 2.$$

(*v*) We have,

$$\lim_{x \to 0} \left(\frac{a^{\sin x} - 1}{\sin x}\right) = \lim_{y \to 0} \left(\frac{a^y - 1}{y}\right) \qquad \text{[where } y = \sin x\text{]}$$

$$= \log a. \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

Example 29. *Evaluate the following limits :*

(*i*) $\lim_{x \to 0} \frac{4^x - 1}{\sqrt{1 + x} - 1}$ (*ii*) $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx}, a \neq b$

(*iii*) $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1 + x} - 1}$ (*iv*) $\lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x}$.

$$= \lim_{h \to 0} \frac{\log\left(\frac{e+h}{e}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{e}\right)}{h}$$

$$\left[\because \ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{e}\right)}{\left(\frac{h}{e}\right) . e} = \frac{1}{e} \lim_{h \to 0} \left[\frac{\log\left(1 + \frac{h}{e}\right)}{\left(\frac{h}{e}\right)}\right]$$

$$= \frac{1}{e} \times 1 = \frac{1}{e} . \qquad \left[\because \ \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

(*iv*) We have,

$$\lim_{x \to 0} \left(\frac{e^x - e^{\sin x}}{x - \sin x}\right) = \lim_{x \to 0} \frac{e^{x - \sin x + \sin x} - e^{\sin x}}{x - \sin x} \qquad \text{[Note this step]}$$

$$= \lim_{x \to 0} \frac{e^{(x - \sin x)} . e^{\sin x} - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{e^{\sin x}(e^{x - \sin x} - 1)}{(x - \sin x)}$$

$$= \lim_{x \to 0} e^{\sin x} . \lim_{x \to 0} \left(\frac{e^{x - \sin x} - 1}{x - \sin x}\right)$$

$$= e^0 . \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right) \qquad \text{[where } y = (x - \sin x)\text{]}$$

$$= 1 \times 1 \qquad \left[\because \ \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

$$= 1.$$

(*v*) We have,

$$\lim_{x \to 0} \frac{\log(5+x) - \log(5-x)}{x} = \lim_{x \to 0} \frac{\log\left[5\left(1 + \frac{x}{5}\right)\right] - \log\left[5\left(1 - \frac{x}{5}\right)\right]}{x}$$

$$[\because \ \log(m.n) = \log m + \log n]$$

$$= \lim_{x \to 0} \frac{\left[\log(5) + \log\left(1 + \frac{x}{5}\right)\right] - \left[\log(5) + \log\left(1 - \frac{x}{5}\right)\right]}{x}$$

$$= \lim_{x \to 0} \frac{\log 5 + \log\left(1 + \frac{x}{5}\right) - \log 5 - \log\left(1 - \frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1 + \frac{x}{5}\right) - \log\left(1 - \frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1+\frac{x}{5}\right)}{5\frac{x}{5}} - \lim_{x \to 0} \frac{\log\left(1-\frac{x}{5}\right)}{(-5)\frac{x}{(-5)}}$$

$$= \frac{1}{5}\lim_{x \to 0}\left[\frac{\log\left(1+\frac{x}{5}\right)}{x/5}\right] + \frac{1}{5}\lim_{x \to 0}\left[\frac{\log\left(1-\frac{x}{5}\right)}{\left(-\frac{x}{5}\right)}\right]$$

$$= \frac{1}{5}(1) + \frac{1}{5}(1) \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{2}{5}.$$

Example 31. *Evaluate the following limits :*

(i) $\lim_{x \to 0} \frac{e^{3+x} - \sin x - e^3}{x}$ (ii) $\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$

(iii) $\lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{(e^{x^2} - 1)}$ (iv) $\lim_{x \to 0}\left(\frac{1}{x} - \frac{\log(1+x)}{x^2}\right)$

(v) $\lim_{x \to 0} \frac{e^{\tan x} - e^x}{(\tan x - x)}$.

Solution. (i) We have,

$$\lim_{x \to 0}\left(\frac{e^{3+x} - \sin x - e^3}{x}\right) = \lim_{x \to 0}\left(\frac{e^3 . e^x - e^3 - \sin x}{x}\right)$$

$$= \lim_{x \to 0}\left(\frac{e^3(e^x - 1) - \sin x}{x}\right) = \lim_{x \to 0}\left(e^3\frac{(e^x - 1)}{x} - \frac{\sin x}{x}\right)$$

$$= e^3 \lim_{x \to 0}\left(\frac{e^x - 1}{x}\right) - \lim_{x \to 0}\left(\frac{\sin x}{x}\right)$$

$$= e^3(1) - 1 \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1, \ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= e^3 - 1.$$

(ii) We have,

$$\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x} = \lim_{x \to 0}\left(\frac{\log(1+x^3)}{x^3} \cdot \frac{x^3}{\sin^3 x}\right)$$

$$= \lim_{x \to 0}\left(\frac{\log(1+x^3)}{x^3}\right) \cdot \lim_{x \to 0}\left(\frac{x}{\sin x}\right)^3$$

$$= 1 \times (1)^3 \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1, \ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= 1.$$

(*iii*) We have,

$$\lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{(e^{x^2} - 1)} = \lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{(e^{x^2} - 1)} \times \frac{(1 + \cos x\sqrt{\cos 2x})}{(1 + \cos x\sqrt{\cos 2x})} \qquad \text{[Rationalisation]}$$

$$= \lim_{x \to 0} \left(\frac{1}{\dfrac{e^{x^2} - 1}{x^2} . x^2}\right) . \lim_{x \to 0} \left(\frac{1 - \cos^2 x . \cos 2x}{1 + \cos x\sqrt{\cos 2x}}\right)$$

$$= \lim_{x \to 0} \left(\frac{1}{\dfrac{e^{x^2} - 1}{x^2}}\right) . \lim_{x \to 0} \left(\frac{1 - \cos^2 x\,(1 - 2\sin^2 x}{x^2\,(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$\left[\because 1 - \cos 2A = 2\sin^2 A \ \Rightarrow \ 1 - 2\sin^2 A = \cos 2A\right]$$

$$= 1 . \lim_{x \to 0} \frac{(1 - \cos^2 x + 2\sin^2 x\cos^2 x)}{x^2\,(1 + \cos x\sqrt{\cos 2x})} \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

$$= \lim_{x \to 0} \left(\frac{\sin^2 x + 2\sin^2 x\cos^2 x}{x^2\,(1 + \cos x\sqrt{\cos 2x})}\right) = \lim_{x \to 0} \frac{\sin^2 x\left(1 + 2\cos^2 x\right)}{x^2\,(1 + \cos x\sqrt{\cos 2x})}$$

$$[\because \ \sin^2 A + \cos^2 A = 1]$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 . \lim_{x \to 0} \left(\frac{1 + 2\cos^2 x}{1 + \cos x\sqrt{\cos 2x}}\right)$$

$$= (1)^2 \frac{(1+2)}{(1+1)} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{3}{2}.$$

(*iv*) We have,

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{\log(1+x)}{x^2}\right) = \lim_{x \to 0} \left(\frac{x - \log(1+x)}{x^2}\right)$$

$$= \lim_{x \to 0} \frac{x - \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} \ldots\ldots\right)}{x^2}$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots\ldots\right]$$

$$= \lim_{x \to 0} \frac{\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} \ldots\ldots\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{x^2\left[\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \frac{x^3}{5} \ldots\ldots\right]}{x^2}$$

$$= \lim_{x \to 0}\left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \frac{x^3}{5} \ldots\ldots\right) = \left(\frac{1}{2} - 0 + 0 - 0\right)$$

$$= \frac{1}{2}.$$

(v) We have,

$$\lim_{x \to 0}\left(\frac{e^{\tan x} - e^x}{\tan x - x}\right) = \lim_{x \to 0} \frac{e^x\left(e^{\tan x - x} - 1\right)}{(\tan x - x)}$$

$$= \lim_{x \to 0} e^x . \lim_{x \to 0}\left(\frac{e^{\tan x - x} - 1}{\tan x - x}\right) \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

$$= e^0 . \lim_{y \to 0}\left(\frac{e^y - 1}{y}\right) \qquad [\text{where } y = (\tan x - x)]$$

$$= 1 \times 1 = 1.$$

Example 32. *Evaluate the following limits :*

(*i*) $\lim_{x \to \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$ (*ii*) $\lim_{x \to \infty} \frac{x^2 + ax + b}{x^2 + px + q}$

(*iii*) $\lim_{x \to \infty} \frac{1}{(1-x)^2}$.

Solution. (*i*) We have,

$$\lim_{x \to \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \lim_{x \to \infty}\left(\frac{2x^2 + 5x + 3}{3x^2 + 10x + 8}\right)$$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \frac{2 + \frac{5}{x} + \frac{3}{x^2}}{3 + \frac{10}{x} + \frac{8}{x^2}} \qquad [\because \ x^2 \text{ is the highest power occurring in the fraction}]$$

$$= \left(\frac{2+0+0}{3+0+0}\right) = \frac{2}{3}.$$

(*ii*) We have, $\lim_{x \to \infty} \frac{x^2 + ax + b}{x^2 + px + q}$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \left(\frac{1 + \frac{a}{x} + \frac{b}{x^2}}{1 + \frac{p}{x} + \frac{q}{x^2}} \right) = \left(\frac{1+0+0}{1+0+0} \right) = 1.$$

[$\because$ x^2 is the highest power occurring in the fraction]

(*iii*) We have, $\lim_{x \to \infty} \frac{1}{(1-x)^2}$

Since, $x \to \infty \Rightarrow (x-1) \to \infty$

$\Rightarrow (x-1)^2 \to \infty \Rightarrow (1-x)^2 \to \infty$

$\Rightarrow \frac{1}{(1-x)^2} \to 0$

$\therefore \lim_{x \to \infty} \frac{1}{(1-x)^2} = 0.$

Example 33. *Evaluate the following limits :*

(*i*) $\lim_{x \to \infty} \frac{2x-1}{x+2}$ (*ii*) $\lim_{x \to \infty} \frac{3x^2 + x + 7}{x^2 - 2x + 5}$

(*iii*) $\lim_{x \to \infty} \frac{x^3 + 6x}{x^5 + 7}$ (*iv*) $\lim_{x \to \infty} \frac{(2x-1)^{10}(3x-1)^{30}}{(2x+1)^{50}}$.

Solution. (*i*) We have, $\lim_{x \to \infty} \left(\frac{2x-1}{x+2} \right)$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \left(\frac{2 - \frac{1}{x}}{1 + \frac{2}{x}} \right) = \left(\frac{2-0}{1+0} \right) = 2.$$

[$\because$ x is the highest power occurring in the fraction]

(*ii*) We have, $\lim_{x \to \infty} \frac{3x^2 + x + 7}{x^2 - 2x + 5}$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \left(\frac{3 + \frac{1}{x} + \frac{7}{x^2}}{1 - \frac{2}{x} + \frac{5}{x^2}} \right) = \left(\frac{3+0+0}{1-0+0} \right) = 3.$$

[$\because$ x^2 is the highest power occurring in the fraction]

Example 34. *Evaluate the following limits :*

(*i*) $\lim_{x \to \infty} \frac{x^8 - x^3 + 2}{-3x^4 + x + 7}$ (*ii*) $\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$

(*iii*) $\lim_{x \to \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$ (*iv*) $\lim_{x \to \infty} \frac{(2x-1)(x+97)(35-x)}{(x^2-5x+7)(2x+3)}$

(*v*) $\lim_{x \to \infty} \frac{\sqrt{3x^2-1} - \sqrt{2x^2-1}}{4x+3}$.

Solution. (*i*) We have, $\lim_{x \to \infty} \frac{x^8 - x^3 + 2}{-3x^4 + x + 7} = \lim_{x \to \infty} \frac{x^8\left(1 - \frac{x^3}{x^8} + \frac{2}{x^8}\right)}{-3x^4\left(1 - \frac{x}{3x^4} - \frac{7}{3x^4}\right)}$

$$= \lim_{x \to \infty} -\frac{x^4}{3} \frac{\left(1 - \frac{1}{x^5} + \frac{2}{x^8}\right)}{\left(1 - \frac{1}{3x^3} - \frac{7}{3x^4}\right)} = (-\infty)\left(\frac{1-0+0}{1-0-0}\right) = -\infty.$$

(*ii*) We have, $\lim_{x \to \infty} \frac{5x-6}{\sqrt{4x^2+9}}$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \left(\frac{\left(\frac{5}{1} - \frac{6}{x}\right)}{\sqrt{4 + \frac{9}{x^2}}} \right) = \frac{5-0}{\sqrt{4+0}} = \frac{5}{2}.$$

[$\because$ x is the highest power occurring in the fraction]

(*iii*) We have, $\lim_{x \to \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)} = \lim_{x \to \infty} \frac{x\left(3 - \frac{1}{x}\right).x\left(4 - \frac{2}{x}\right)}{x\left(1 + \frac{8}{x}\right).x\left(1 - \frac{1}{x}\right)} = \lim_{x \to \infty} \frac{\left(3 - \frac{1}{x}\right)\left(4 - \frac{2}{x}\right)}{\left(1 + \frac{8}{x}\right)\left(1 - \frac{1}{x}\right)}$

$$= \frac{(3-0)(4-0)}{(1+0)(1-0)} = 12.$$

(*iv*) We have, $\lim_{x \to \infty} \frac{(2x-1)(x+97)(35-x)}{(x^2-5x+7)(2x+3)}$

$$= \lim_{x \to \infty} \frac{x\left(2 - \frac{1}{x}\right).x\left(1 + \frac{97}{x}\right)x\left(\frac{35}{x} - 1\right)}{x^2\left(1 - \frac{5}{x} + \frac{7}{x^2}\right)x\left(2 + \frac{3}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\left(2 - \frac{1}{x}\right)\left(1 + \frac{97}{x}\right)\left(\frac{35}{x} - 1\right)}{\left(1 - \frac{5}{x} + \frac{7}{x^2}\right)\left(2 + \frac{3}{x}\right)}$$

$$= \frac{(2-0)(1+0)(0-1)}{(1-0+0)(2+0)} = \frac{-2}{2} = -1.$$

(*v*) We have, $\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} - \sqrt{2x^2 - 1}}{4x + 3}$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{3 - \frac{1}{x^2}} - \sqrt{2 - \frac{1}{x^2}}}{\left(4 + \frac{3}{x}\right)} = \frac{\sqrt{3} - \sqrt{2}}{(4+0)} = \left(\frac{\sqrt{3} - \sqrt{2}}{4}\right).$$

[$\because$ x is the highest power occurring in the fraction]

Example 35. *Evaluate the following limits :*

(*i*) $\lim_{x \to \infty} \frac{(2x-3)(3x-4)}{(4x-5)(5x-6)}$

(*ii*) $\lim_{x \to \infty} \frac{x^3 + x + 9}{x^2 + 1}$

(*iii*) $\lim_{x \to \infty} \frac{(2x-1)(3x+2)(4x-3)}{3x^3 + 7x - 8}$

(*iv*) $\lim_{x \to \infty} \sqrt{x}\left(\sqrt{x+c} - \sqrt{x}\right)$

(*v*) $\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 + 1} - 1}$

(*vi*) $\lim_{x \to \infty} \frac{x}{\sqrt{3x^2 + 1} - 1}$.

Solution. (*i*) We have,

$$\lim_{x \to \infty} \frac{(2x-3)(3x-4)}{(4x-5)(5x-6)} = \lim_{x \to \infty} \frac{x\left(2 - \frac{3}{x}\right) . x\left(3 - \frac{4}{x}\right)}{x\left(4 - \frac{5}{x}\right) . x\left(5 - \frac{6}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\left(2 - \frac{3}{x}\right)\left(3 - \frac{4}{x}\right)}{\left(4 - \frac{5}{x}\right)\left(5 - \frac{6}{x}\right)} = \frac{(2-0)(3-0)}{(4-0)(5-0)} = \frac{6}{20} = \frac{3}{10}.$$

Remark. Always remember that :

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \begin{cases} \dfrac{\text{leading co-efficient of } f(x)}{\text{leading co-efficient of } g(x)} ; \text{ if deg. } f(x) = \text{deg. } g(x) \\ 0 ; \text{ if deg. } f(x) < \text{deg. } g(x) \\ 1 ; \text{ if deg. } f(x) > \text{deg. } g(x) \end{cases}$$

where $f(x)$ and $g(x)$ are polynomial functions.

(*ii*) We have,

$$\lim_{x\to\infty}\frac{x^3+x+9}{x^2+1}=\lim_{x\to\infty}\frac{x^3\left(1+\frac{1}{x^2}+\frac{9}{x^3}\right)}{x^2\left(1+\frac{1}{x^2}\right)}$$

$$=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x^2}+\frac{9}{x^3}\right)}{\left(1+\frac{1}{x^2}\right)}=\frac{\infty(1+0+0)}{(1+0)}=\infty.$$

(*iii*) We have,

$$\lim_{x\to\infty}\frac{(2x-1)(3x+2)(4x-3)}{(3x^3+7x-8)}$$

Please try yourself.

[**Hint.** Dividing the numerator and denominator by x^3.] [**Ans.** 8.]

(*iv*) We have, $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$ [Rationalisation]

$$=\lim_{x\to\infty}\frac{\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)}{1}\times\frac{\left(\sqrt{x+c}+\sqrt{x}\right)}{\left(\sqrt{x+c}+\sqrt{x}\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}(x+c-x)}{\left(\sqrt{x+c}+\sqrt{x}\right)}=\lim_{x\to\infty}\frac{c\sqrt{x}}{\left(\sqrt{x+c}+\sqrt{x}\right)}$$

Dividing the numerator and denominator by $\sqrt{x}$, we get

$$\lim_{x\to\infty}\frac{c}{\sqrt{1+\frac{c}{x}}+1}=\frac{c}{\sqrt{1+0}+1}=\frac{c}{2}.$$

[$\because$ $\sqrt{x}$ is the highest power occurring in the fraction]

(*v*) We have, $\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+1}-1}$

Dividing the numerator and denominator by x, we get

$$\lim_{x\to\infty}\frac{1}{\sqrt{4+\frac{1}{x^2}}-\frac{1}{x}}=\frac{1}{\sqrt{4+0}-0}=\frac{1}{2}.$$

[$\because$ x is the highest power occurring in the fraction]

(*vi*) We have, $\lim_{x\to\infty}\frac{x}{\sqrt{3x^2+1}-1}$

Please try yourself.

[**Hint.** Dividing the numerator and denominator by x.] $\left[\textbf{Ans. }\frac{1}{\sqrt{3}}\right]$

Example 36. *Evaluate the following limits :*

(*i*) $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x - 4}{\sqrt{x^4 + 4}}$ (*ii*) $\lim\limits_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \dfrac{3}{n^2} + \ldots + \dfrac{(n-1)}{n^2}\right)$

(*iii*) $\lim\limits_{x \to \infty} \sqrt{x^2 + x + 1} - x$ (*iv*) $\lim\limits_{x \to \infty} (2x - \sqrt{4x^2 + x})$.

Solution. (*i*) We have, $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x - 4}{\sqrt{x^4 + 4}}$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \frac{2 - \frac{3}{x} - \frac{4}{x^2}}{\sqrt{1 + \frac{4}{x^4}}} = \frac{2 - 0 - 0}{\sqrt{1 + 0}} = 2.$$

[∵ x^2 is the highest power occurring in the fraction]

(*ii*) We have, $\lim\limits_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \dfrac{3}{n^2} + \ldots + \dfrac{n-1}{n^2}\right)$

$$= \lim_{n \to \infty} \frac{1 + 2 + 3 + \ldots + (n-1)}{n^2}$$

$$= \lim_{n \to \infty} \frac{n(n-1)}{2n^2} \qquad \left[\because\ 1 + 2 + 3 + \ldots + (n-1) = S_{n-1} = \frac{n(n-1)}{2}\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2} \lim_{n \to \infty}\left(1 - \frac{1}{n}\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}.$$

[As it is an A.P. series where $a = 1, d = 1$ and $n = n - 1$]

(*iii*) We have, $\lim\limits_{x \to \infty} (\sqrt{x^2 + x + 1} - x)$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2 + x + 1} - x}{1} \times \frac{(\sqrt{x^2 + x + 1} + x)}{(\sqrt{x^2 + x + 1} + x)} \qquad \text{[Rationalisation]}$$

$$= \lim_{x \to \infty} \frac{x^2 + x + 1 - x^2}{(\sqrt{x^2 + x + 1} + x)} = \lim_{x \to \infty} \frac{x + 1}{(\sqrt{x^2 + x + 1} + x)}$$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)}{\left(\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1\right)} = \frac{(1 + 0)}{(\sqrt{1 + 0 + 0} + 1)} = \frac{1}{2}$$

[∵ x is the highest power occurring in the fraction] (*iv*) We have, $\lim_{x \to \infty} (2x - \sqrt{4x^2 + x})$

$$= \lim_{x \to \infty} \frac{(2x - \sqrt{4x^2 + x})}{1} \times \frac{(2x + \sqrt{4x^2 + x})}{(2x + \sqrt{4x^2 + x})} \qquad \text{[Rationalisation]}$$

$$= \lim_{x \to \infty} \frac{4x^2 - (4x^2 + x)}{2x + \sqrt{4x^2 + x}} = \lim_{x \to \infty} \frac{-x}{2x + \sqrt{4x^2 + x}}$$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{-1}{2 + \sqrt{4 + \frac{1}{x}}} = \frac{-1}{2 + \sqrt{4 + 0}} = \frac{-1}{4}.$$

[$\because$ x is the highest power occurring in the fraction]

Example 37. *Evaluate the following limits :*

(*i*) $\lim_{x \to \infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1})$

(*ii*) $\lim_{x \to \infty} (x - \sqrt{x^2 + x})$

(*iii*) $\lim_{n \to \infty} \frac{\Sigma n^3}{n^4}$

(*iv*) $\lim_{x \to \infty} (\sqrt{x^2 - 8x} - x)$

(*v*) $\lim_{x \to \infty} \sqrt{x}(\sqrt{x + 5} - \sqrt{x})$

(*vi*) $\lim_{n \to \infty} \frac{\Sigma n^2}{n^3}$

(*vii*) $\lim_{n \to \infty} \left(\frac{\Sigma n^3}{2n^4 + 3n} \right)$.

Solution. (*i*) We have, $\lim_{x \to \infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1})$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}}{1} \times \frac{(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})}$$

[Rationalisation]

$$= \lim_{x \to \infty} \frac{(x^2 + x + 1) - (x^2 + 1)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}}$$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x^2}}}$$

[$\because$ x is the highest power occurring in the fraction]

$$= \frac{1}{\sqrt{1 + 0 + 0} + \sqrt{1 + 0}} = \frac{1}{2}.$$

(*ii*) We have, $\lim_{x \to \infty} (x - \sqrt{x^2 + x})$.

Please try yourself.

[**Hint.** (Rationalisation).] $\left[\textbf{Ans.} \left(-\frac{1}{2}\right)\right]$

(*iii*) We have, $\lim_{n \to \infty} \frac{\Sigma n^3}{n^4} = \lim_{n \to \infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4}$ $\left[\because \ \Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2\right]$

$$= \lim_{n \to \infty} \frac{n^2(n+1)^2}{4n^4} = \lim_{n \to \infty} \frac{(n+1)^2}{4n^2}$$

$$= \frac{1}{4} \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{4} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2$$

$$= \frac{1}{4}(1+0)^2 = \frac{1}{4}.$$

(*iv*) We have, $\lim_{x \to \infty} (\sqrt{x^2 - 8x} - x)$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 - 8x} - x)}{1} \times \frac{(\sqrt{x^2 - 8x} + x)}{(\sqrt{x^2 - 8x} + x)} \quad \text{[Rationalisation]}$$

$$= \lim_{x \to \infty} \frac{x^2 - 8x - x^2}{(\sqrt{x^2 - 8x} + x)} = \lim_{x \to \infty} \frac{-8x}{(\sqrt{x^2 - 8x} + x)}$$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{-8}{\left(\sqrt{1 - \frac{8}{x}} + 1\right)} = \frac{-8}{(\sqrt{1-0} + 1)} = \frac{-8}{2} = -4.$$

[$\because$ x is the highest power occurring in the fraction]

(*v*) We have, $\lim_{x \to \infty} \sqrt{x}(\sqrt{x+5} - \sqrt{x})$

$$= \lim_{x \to \infty} \frac{\sqrt{x}(\sqrt{x+5} - \sqrt{x})}{1} \times \frac{(\sqrt{x+5} + \sqrt{x})}{(\sqrt{x+5} + \sqrt{x})} \quad \text{[Rationalisation]}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}(x + 5 - x)}{(\sqrt{x+5} + \sqrt{x})} = \lim_{x \to \infty} \frac{5\sqrt{x}}{(\sqrt{x+5} + \sqrt{x})}$$

Dividing the numerator and denominator by $\sqrt{x}$, we get

$$\lim_{x \to \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1} = \frac{5}{\sqrt{1+0} + 1} = \frac{5}{2}.$$

[$\because$ $\sqrt{x}$ is the highest power occurring in the fraction]

(*vi*) We have,
$$\lim_{n\to\infty}\frac{\Sigma n^2}{n^3}$$
$$=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$
$$\left[\because\ \Sigma n^2 = 1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}\right]$$

Dividing the numerator and denominator by n^3, we get
$$\lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
[$\because$ n^3 is the highest power of n occurring in the fraction]
$$=\frac{1}{6}(1+0)(2+0)=\frac{2}{6}=\frac{1}{3}.$$

(*vii*) We have,
$$\lim_{n\to\infty}\frac{\Sigma n^3}{2n^4+3n}$$
$$\left[\because\ \Sigma n^3 = 1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2=\frac{n^2(n+1)^2}{4}\right]$$
$$=\lim_{n\to\infty}\left(\frac{\frac{n^2(n+1)^2}{4}}{2n^4+3n}\right)=\lim_{n\to\infty}\frac{n^2(n+1)^2}{4(2n^4+3n)}$$

Dividing the numerator and denominator by n^4, we get
$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{4\left(2+\frac{3}{n^3}\right)}=\frac{(1+0)^2}{4(2+0)}=\frac{1}{8}.$$
[$\because$ n^4 is the highest power of n occuring in the fraction]

Example 38. *Evaluate the following limits :*

(*i*) $\lim_{n\to\infty}\frac{1^3+2^3+3^3+\ldots+n^3}{n^4+7}$ (*ii*) $\lim_{n\to\infty}\frac{1+2+3+\ldots+n}{n^2}$

(*iii*) $\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

Solution. (*i*) We have,
$$\lim_{n\to\infty}\frac{1^3+2^3+3^3+\ldots+n^3}{n^4+7}\qquad\left[\because\ \Sigma n^3=\left(\frac{n(n+1)}{2}\right)^2\right]$$
$$=\lim_{n\to\infty}\frac{\Sigma n^3}{n^4+7}=\lim_{n\to\infty}\left[\frac{\frac{n^2(n+1)^2}{4}}{n^4+7}\right]$$

$$= \lim_{n\to\infty} \frac{n^2(n+1)^2}{4(n^4+7)}$$

Dividing the numerator and denominator by n^4, we get

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{4\left(1+\frac{7}{n^4}\right)} = \frac{(1+0)^2}{4(1+0)} = \frac{1}{4}.$$

[$\because$ n^4 is the highest power of n occurring in the fraction]

(*ii*) We have, $\lim_{n\to\infty} \frac{1+2+3+\ldots+n}{n^2}$ $\left[\because \ \Sigma n = \frac{n(n+1)}{2}\right]$

$$= \lim_{n\to\infty} \frac{\Sigma n}{n^2} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

Dividing the numerator and denominator by n^2, we get

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{2} = \left(\frac{1+0}{2}\right) = \frac{1}{2}.$$

[$\because$ n^2 is the highest power of n occurring in the fraction]

(*iii*) We have, $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

Dividing the numerator and denominator by $\sqrt{x}$, we get

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\sqrt{\frac{x}{x^2}+\sqrt{\frac{x}{x^4}}}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}$$

[$\because$ $\sqrt{x}$ is the highest power of x occurring in the fraction]

$$= \frac{1}{\sqrt{1+\sqrt{0+\sqrt{0}}}} = \frac{1}{1} = 1.$$

Example 39. *Prove that :* $\lim_{x\to\infty} (\sqrt{x^2+x+1}-x) \neq \lim_{x\to\infty} (\sqrt{x^2+1}-x)$.

Solution. We have,

$$\text{L.H.S.} = \lim_{x\to\infty} (\sqrt{x^2+x+1}-x)$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+1}-x)}{1} \times \frac{(\sqrt{x^2+x+1}+x)}{(\sqrt{x^2+x+1}+x)}$$ [Rationalisation]

$$= \lim_{x \to \infty} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x} = \lim_{x \to \infty} \left(\frac{x + 1}{\sqrt{x^2 + x + 1} + x} \right)$$

$$= \lim_{x \to \infty} \frac{x\left(1 + \frac{1}{x}\right)}{x\left[\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1\right]} = \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)}{\left[\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1\right]}$$

$$= \frac{1 + 0}{\sqrt{1 + 0 + 0} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

$$\text{R.H.S.} = \lim_{x \to \infty} (\sqrt{x^2 + 1} - x)$$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + 1} - x)}{1} \times \frac{(\sqrt{x^2 + 1} + x)}{(\sqrt{x^2 + 1} + x)} \qquad \text{[Rationalisation]}$$

$$= \lim_{x \to \infty} \frac{(x^2 + 1 - x^2)}{(\sqrt{x^2 + 1} + x)} = \lim_{x \to \infty} \frac{1}{\left[\sqrt{x^2 + 1} + x\right]}$$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + 1} = \frac{0}{\sqrt{1 + 0} + 1} = \frac{0}{2} = 0$$

[$\because$ x is the highest power occurring in the fraction]

Hence, L.H.S. $\neq$ R.H.S.

$$\therefore \quad \lim_{x \to \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \to \infty} (\sqrt{x^2 + 1} - x).$$

Example 40. *Evaluate the following limits :*

(*i*) $\displaystyle \lim_{n \to \infty} \frac{\Sigma n^2 + \Sigma n}{n^3 + 10n^2 + 2}$ (*ii*) $\displaystyle \lim_{n \to \infty} \frac{1 + 3 + 5 + \ldots + \text{to } n \text{ terms}}{n^2}$

(*iii*) *If* $\displaystyle \lim_{x \to \infty} \left\{ \frac{x^2 + 1}{x + 1} - (ax + b) \right\} = 0$*, find the values of a and b.*

Solution. (*i*) We have,

$$\lim_{n \to \infty} \frac{\Sigma n^2 + \Sigma n}{n^3 + 10n^2 + 2} = \lim_{n \to \infty} \frac{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}{(n^3 + 10n^2 + 2)}$$

$$\left[\because \ \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}, \Sigma n = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)+3n(n+1)}{6(n^3+10n^2+2)} = \lim_{n\to\infty} \frac{n(n+1)[2n+1+3]}{6(n^3+10n^2+2)}$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+4)}{6(n^3+10n^2+2)} = \lim_{n\to\infty} \frac{2n(n+1)(n+2)}{6(n^3+10n^2+2)}$$

Dividing the numerator and denominator by n^3, we get

$$= \lim_{n\to\infty} \frac{1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{3\left(1+\frac{10}{n}+\frac{2}{n^3}\right)} = \frac{(1+0)(1+0)}{3(1+0+0)} = \frac{1}{3}.$$

[$\because$ n^3 is the highest power of n occurring in the fraction]

(*ii*) We have,

$$\lim_{n\to\infty} \frac{1+3+5+\ldots \text{ to } n \text{ terms}}{n^2}$$

$$= \lim_{n\to\infty} \left[\frac{\frac{n}{2}[2(1)+(n-1)(2)]}{n^2}\right] = \lim_{n\to\infty} \frac{\frac{2n^2}{2}}{n^2}$$

$$\left[\because \text{ For an A.P. } S_n = \frac{n}{2}[2a+(n-1)d]. \text{ Here, } a=1, d=2\right]$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

(*iii*) We have, $\lim_{x\to\infty} \left\{\frac{x^2+1}{x+1} - (ax+b)\right\} = 0$

$$\Rightarrow \quad \lim_{x\to\infty} \left\{\frac{(x^2+1)-(x+1)(ax+b)}{x+1}\right\} = 0$$

$$\Rightarrow \quad \lim_{x\to\infty} \left\{\frac{(x^2+1)-(ax^2+bx+ax+b)}{x+1}\right\} = 0$$

$$\Rightarrow \quad \lim_{x\to\infty} \left\{\frac{x^2+1-ax^2-(a+b)x-b}{(x+1)}\right\} = 0$$

$$\Rightarrow \quad \lim_{x\to\infty} \left\{\frac{x^2(1-a)-x(a+b)+(1-b)}{x+1}\right\} = 0$$

Since the limit of the given expression is zero.

This is possible only if :

Co-efficient of $x^2 = 0$ and co-efficient of $x = 0$

$\therefore \quad (1-a) = 0$ and $-(a+b) = 0$

$\Rightarrow \quad a = 1$ and $-a - b = 0 \Rightarrow -b = a \Rightarrow b = -a$

$\Rightarrow \quad b = -1$

$\therefore \quad a = 1$ and $b = -1$.

Example 41. *Let* $f(x) = \dfrac{ax+b}{x+1}$, $\lim\limits_{x\to 0} f(x) = 2$ *and* $\lim\limits_{x\to\infty} f(x) = 1$.

Show that a = 1 and b = 2. Also find f(– 2).

Solution. We have, $\lim\limits_{x\to 0} f(x) = 2$

$$\Rightarrow \quad \lim_{x\to 0} \frac{ax+b}{x+1} = 2$$

$$\Rightarrow \quad \frac{a(0)+b}{0+1} = 2 \Rightarrow b = 2.$$

Also, $\lim\limits_{x\to\infty} f(x) = 1$

$$\Rightarrow \quad \lim_{x\to\infty} \frac{ax+b}{x+1} = 1$$

Dividing the numerator and denominator by x, we get

$$\lim_{x\to\infty} \left(\frac{a+\frac{b}{x}}{1+\frac{1}{x}} \right) = 1$$

[$\because$ x is the highest power occurring in the fraction]

$$\Rightarrow \quad \frac{a+0}{1+0} = 1 \Rightarrow a = 1$$

$$\therefore \quad f(x) = \frac{ax+b}{x+1} = \frac{1x+2}{x+1} = \frac{x+2}{x+1}$$

$$\therefore \quad f(-2) = \frac{-2+2}{-2+1} = \frac{0}{-1} = 0.$$

Example 42. *Evaluate the following limits :*

(i) If $\lim\limits_{x\to\infty} \left\{ \dfrac{x^2+1}{x+1} - (ax+b) \right\} = 2$, *find the values of a and b.*

(ii) $\lim\limits_{x\to 0} \left(1+\dfrac{2}{x}\right)^x$ *(iii)* $\lim\limits_{x\to\infty} \left(\dfrac{1}{x^2}+\dfrac{2}{x^2}+\ldots\ldots+\dfrac{x}{x^2}\right)$

(iv) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$ *(v)* $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x+1}\right)^x$

(vi) $\lim\limits_{x\to\infty} \left(\dfrac{x-1}{x+1}\right)^x$.

Solution. (*i*) We have, $$\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-(ax+b)\right\}=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left\{\frac{(x^2+1)-(x+1)(ax+b)}{(x+1)}\right\}=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left\{\frac{x^2+1-(ax^2+(a+b)x+b)}{(x+1)}\right\}=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left\{\frac{x^2+1-ax^2-(a+b)x-b)}{(x+1)}\right\}=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left\{\frac{(1-a)x^2-(a+b)x+(1-b)}{(x+1)}\right\}=2 \qquad ...(1)$$

Since the limit of the given expression is a finite non-zero number. Therefore, the degree of numerator and denominator must be same.

For this, we must have,

Co-efficient of $x^2 = 0$

$\therefore \quad (1-a)=0 \Rightarrow a=1$

Putting this value of a in equation (1), we have

$$\lim_{x\to\infty}\left\{\frac{(1-1)x^2-(1+b)x+(1-b)}{x+1}\right\}=2$$

$$\Rightarrow \quad \lim_{x\to\infty}\left\{\frac{-(1+b)(x+1)}{x+1}\right\}=2$$

$$\Rightarrow \quad -(1+b)=2 \Rightarrow -1-b=2 \Rightarrow b=-3$$

$$\therefore \quad a=1, b=-3.$$

(*ii*) We have, $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)^{x/2}\right]^2 = e^2.$ $\quad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\right]$

(*iii*) We have, $\lim_{x\to\infty}\left(\frac{1}{x^2}+\frac{2}{x^2}+......+\frac{x}{x^2}\right)$

$$= \lim_{x\to\infty}\frac{(1+2+......+x)}{x^2}$$

$$= \lim_{x\to\infty}\frac{\Sigma x}{x^2} \qquad \left[\because \Sigma x = \frac{x(x+1)}{2}\right]$$

$$= \lim_{x\to\infty}\frac{x(x+1)}{2x^2} = \lim_{x\to\infty}\left(\frac{x^2+x}{2x^2}\right)$$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \frac{\left(1+\frac{1}{x}\right)}{2} = \left(\frac{1+0}{2}\right) = \frac{1}{2}.$$

[$\because$ x^2 is the highest power of x occurring in the fraction]

(*iv*) We have, $\lim_{x \to \infty} \frac{\sqrt{x^2+1} - \sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$

Dividing the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{1+\frac{1}{x^2}} - \sqrt[3]{1+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}} - \sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}} = \frac{\sqrt{1+0} - \sqrt[3]{1+0}}{\sqrt[4]{1+0} - \sqrt[5]{0+0}} = \frac{1-1}{1-0} = 0.$$

[$\because$ x is the highest power occurring in the fraction]

(*v*) We have,

$$\lim_{x \to \infty} \left(1+\frac{1}{x+1}\right)^x = \lim_{x \to \infty} \left[\left(1+\frac{1}{x+1}\right)^{(x+1) \cdot \frac{x}{(x+1)}}\right]$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{1}{x+1}\right)^{x+1}\right]^{\left(\frac{x}{x+1}\right)}$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{1}{x+1}\right)^{x+1}\right]^{\left(\frac{1}{1+\frac{1}{x}}\right)} \qquad \left[\because \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e\right]$$

$$= e^{\frac{1}{1+0}} = e^1 = e.$$

(*vi*) We have,

$$\lim_{x \to \infty} \left(\frac{x-1}{x+1}\right)^x = \lim_{x \to \infty} \frac{\left(1-\frac{1}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x}$$

$$= \lim_{x \to \infty} \left[\frac{\left(1-\frac{1}{x}\right)^{(-x)(-1)}}{\left(1+\frac{1}{x}\right)^x}\right] = \frac{\lim_{x \to \infty} \left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}}{\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x}$$

$$= \frac{e^{-1}}{e} = \frac{1}{e^2}. \qquad \left[\because \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e\right]$$

Example 43. *Evaluate the following limits :*

(i) $\lim_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4}$ (ii) $\lim_{x \to -\infty} e^x$

(iii) $\lim_{x \to \infty} e^{-x}$ (iv) $\lim_{n \to \infty} \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right)$

(v) $\lim_{x \to \infty} \left(\frac{x+2}{x+3}\right)^{x+1}$ (vi) $\lim_{x \to \infty} 5^x \sin\left(\frac{a}{5^x}\right)$.

Solution. (i) We have,

$$\lim_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x \to \infty} \left(\frac{(x+1)+5}{x+1}\right)^{x+4}$$

$$= \lim_{x \to \infty} \left(1+\frac{5}{x+1}\right)^{x+4} = \lim_{x \to \infty} \left(1+\frac{5}{x+1}\right)^{(x+1)+3}$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{(x+1)} . \left(1+\frac{5}{x+1}\right)^3\right]$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{5.\left(\frac{x+1}{5}\right)} . \left(1+\frac{5}{x+1}\right)^3\right]$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{\frac{x+1}{5}}\right]^5 . \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)\right]^3$$

$$= e^5 . \left(1+\frac{5}{\infty}\right)^3 \qquad \left[\because \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e\right]$$

$$= e^5 (1+0)^3 = e^5.$$

(ii) We have, $\lim_{x \to -\infty} e^x$

Let $x = -y \Rightarrow y \to \infty$ as $x \to -\infty$

$$\therefore \quad \lim_{x \to -\infty} e^x = \lim_{y \to \infty} e^{-y} = \lim_{y \to \infty} \left(\frac{1}{e^y}\right) = \frac{1}{\infty} = 0.$$

(iii) We have, $\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \left(\frac{1}{e^x}\right) = \frac{1}{\infty} = 0.$

(iv) We have, $\lim_{n \to \infty} \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right)$

Let $\frac{2\pi}{n} = \theta \Rightarrow n = \frac{2\pi}{\theta}$

EXERCISE FOR PRACTICE

Evaluate the following limits : (Q. No. 1–15)

1. (i) $\lim_{x \to 0} \frac{\tan 8x}{\sin 2x}$ (ii) $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$

(iii) $\lim_{x \to 0} \frac{\tan 4x}{\tan 3x}$ (iv) $\lim_{x \to 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$

2. (i) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ (ii) $\lim_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$

(iii) $\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$ (iv) $\lim_{x \to 0} \frac{\sin 2x (1 - \cos 2x)}{x^3}$

3. (i) $\lim_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$ (ii) $\lim_{x \to 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x}$

(iii) $\lim_{x \to 0} \frac{\cos 2x - \operatorname{cosec} 2x}{x}$ (iv) $\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

4. (i) $\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$ (ii) $\lim_{x \to 0} \frac{3 \sin^2 x - 2 \sin x^2}{3x^2}$

5. (i) $\lim_{x \to 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$ (ii) $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$

6. (i) $\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2}$ (ii) $\lim_{x \to \pi/8} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

7. (i) $\lim_{x \to a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a}$ (ii) $\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

8. (i) $\lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$ (ii) $\lim_{x \to \pi/2} \frac{\cot x}{\frac{\pi}{2} - x}$

9. (i) $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$ (ii) $\lim_{x \to 0} \frac{\sin^{-1} x}{2x}$

10. (i) $\lim_{x \to 0} \frac{\sin^{-1} x + \tan^{-1} x}{7x - 5 \sin^{-1} x}$ (ii) $\lim_{x \to 0} \frac{\tan^{-1} 2x}{\sin 3x}$

11. (i) $\lim_{x \to 0} \frac{e^{\sin x} - 1}{\tan x}$ (ii) $\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$

12. (i) $\lim_{x \to 0} \frac{e^{2x} - e^x}{\sin 2x}$ (ii) $\lim_{x \to 5} \frac{e^x - e^5}{x - 5}$

13. (i) $\lim_{x \to 0} \frac{2x (e^x - 1)}{1 - \cos 3x}$ (ii) $\lim_{x \to e} \frac{\log x - 1}{x - e}$

14. (i) $\lim_{x \to 0} \frac{e^{\tan x} - 1}{x}$ (ii) $\lim_{x \to \infty} \frac{2x^2 + 5x + 9}{1 - 3x}$

15. (i) $\lim_{x \to \infty} \frac{2x + 7}{\sqrt{x^2 + 3} + \sqrt{x^2 + 9}}$ (ii) $\lim_{x \to \infty} (\sqrt{x^2 + 4x} - x)$.

Answers

1. (*i*) 4	(*ii*) $\frac{5}{3}$	(*iii*) $\frac{4}{3}$	(*iv*) $\frac{5}{3}$
2. (*i*) $\frac{1}{2}$	(*ii*) $\frac{8}{9}$	(*iii*) 3	(*iv*) 4
3. (*i*) $\frac{1}{3}$	(*ii*) $\frac{1}{15}$	(*iii*) -1	(*iv*) $\frac{3}{2}$
4. (*i*) 2	(*ii*) $\frac{1}{3}$	**5.** (*i*) $\frac{11}{6}$	(*ii*) -2
6. (*i*) $\frac{1}{2}$	(*ii*) $\frac{1}{16}$	**7.** (*i*) $\frac{1}{2\sqrt{a}} \cos \sqrt{a}$	(*ii*) 4
8. (*i*) $\frac{1}{2}$	(*ii*) 1	**9.** (*i*) $\frac{1}{4}$	(*ii*) $\frac{1}{2}$
10. (*i*) 1	(*ii*) $\frac{2}{3}$	**11.** (*i*) 1	(*ii*) (log 4) (log 2)
12. (*i*) $\frac{1}{2}$	(*ii*) e^5	**13.** (*i*) $\frac{4}{9}$	(*ii*) $\frac{1}{e}$
14. (*i*) 1	(*ii*) $-\infty$	**15.** (*i*) 1	(*ii*) 2.

4
Continuity

4.1 INTRODUCTION

In this chapter, we define the concept of a continuous function at a point and on an interval. The meaning of the word 'continuous' is same as we use in our daily life. The study of continuity of a function is based on limits. Here, we also prove the continuity of some of the well-known functions.

4.2 CONTINUITY AT A POINT

Let f be a real function and a be any point in the domain of f. Then, f is said to be continuous at a, if :

(*i*) $\lim_{x \to a} f(x)$ exist.

i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ both exist and are equal.

(*ii*) $\lim_{x \to a} f(x) = f(a)$.

4.2.1 Continuous Function. A function f is said to be a continuous function if it is continuous at every point of its domain.

4.2.2 Domain of Continuity. The set of all points where a function f is continuous is called the domain of continuity of f.

4.2.3 Left-Continuity at a Point. A function $f(x)$ is said to be left continuous or continuous from the left at $x = a$ if and only if :

$$\lim_{x \to a^-} f(x) \text{ exist and is equal to } f(a).$$

4.2.4 Right Continuity at a Point. A function $f(x)$ is said to be right continuous or continuous from the right at $x = a$ if and only if :

$$\lim_{x \to a^+} f(x) \text{ exist and is equal to } f(a).$$

Note. The concept of continuity at a point is defined only for those points, which are in the domain of the concerned function.

4.2.5 Continuity of a Function in an Open Interval. A function $f(x)$ is said to be continuous in an open interval (a, b) if and only if it is continuous at each point in (a, b).

4.2.6 Continuity of a Function in a Closed Interval. A function $f(x)$ is said to be continuous in the closed interval $[a, b]$ if and only if :

Proof. (*i*) Since f and g are continuous at $x = a$.

Therefore $\lim_{x \to a} f(x) = f(a)$ ...(1)

and $\lim_{x \to a} g(x) = g(a)$ ...(2)

We have, $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$= f(a) + g(a)$ [$\because$ By using (1) and (2)]

$= (f+g)(a)$.

$\therefore$ $f+g$ is continuous at $x = a$.

(*ii*) Since f and g are continuous at $x = a$.

Therefore, $\lim_{x \to a} f(x) = f(a)$ ...(1)

and $\lim_{x \to a} g(x) = g(a)$ ...(2)

We have, $\lim_{x \to a} (f-g)(x) = \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

$= f(a) - g(a)$ [$\because$ By using (1) and (2)]

$= (f-g)(a)$.

$\therefore$ $f-g$ is continuous at $x = a$.

(*iii*) Since f is continuous at $x = a$.

Therefore, $\lim_{x \to a} f(x) = f(a)$ ...(1)

We have, $\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

$= cf(a)$ [$\because$ By using (1)]

$\Rightarrow$ cf is continuous at $x = a$.

(*iv*) Since f and g are continuous at $x = a$,

Therefore, $\lim_{x \to a} f(x) = f(a)$ ...(1)

and $\lim_{x \to a} g(x) = g(a)$ ...(2)

We have, $\lim_{x \to a} (f\,.\,g)(x) = \lim_{x \to a} [f(x)\,.\,g(x)]$

$= \lim_{x \to a} f(x)\,.\,\lim_{x \to a} g(x)$

$= f(a)\,.\,g(a)$ [$\because$ By using (1) and (2)]

$= (fg)(a)$

$\therefore$ fg is continuous at $x = a$.

(*v*) Since f and g are continuous at $x = a$,

Therefore, $\lim_{x \to a} f(x) = f(a)$ ...(1)

and $\lim_{x \to a} g(x) = g(a)$ ...(2)

We have, $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

$= \frac{f(a)}{g(a)}$ [$\because$ By using (1) and (2), provided $g(a) \neq 0$]

$= \left(\frac{f}{g}\right)(a)$

$\therefore$ $\frac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$.

(*vi*) Since f is continuous at $x = a$.

Therefore, $\lim_{x \to a} f(x) = f(a)$ ...(1)

We have, $\lim_{x \to a} \left(\frac{1}{f}\right)(x) = \lim_{x \to a} \left[\frac{1}{f(x)}\right] = \frac{1}{\lim_{x \to a} f(x)}$ [$\because$ By using (1), provided $f(a) \neq 0$]

$= \frac{1}{f(a)} = \left(\frac{1}{f}\right)(a)$

$\therefore$ $\frac{1}{f}$ is continuous at $x = a$, provided $f(a) \neq 0$.

Theorem 2. *If f and g be continuous functions, then,*

(*i*) *$f + g$ is continuous* (*ii*) *$f - g$ is continuous*

(*iii*) *cf is continuous, where : c is a real number*

(*iv*) *fg is continuous*

(*v*) *$\frac{f}{g}$ is continuous at those points, where $g(x) \neq 0$.*

(*vi*) *$\frac{1}{f}$ is continuous at those points, where $f(x) \neq 0$.*

Proof. Let D_f and D_g be the domains of f and g respectively.

Then, domain of $(f + g)$ is $D_f \cap D_g$.

Let "a" be any arbitrary point in $D_f \cap D_g$

Then, $a \in D_f$ and $a \in D_g$

Since, f and g are continuous at every point of D_f and D_g respectively.

$\therefore$ $\lim_{x \to a} f(x) = f(a)$ ...(1)

and $\lim_{x \to a} g(x) = g(a)$ ...(2)

(*i*) We have,

$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$= f(a) + g(a)$ [$\because$ By using (1) and (2)]

$= (f + g)(a)$

$\therefore$ $f+g$ is continuous at a for all $a \in D_f \cap D_g$.

Hence, $f+g$ is continuous.

(*ii*) We have,

$$\lim_{x\to a} (f-g)(x) = \lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$= f(a) - g(a) \qquad [\because \text{ By using (1) and (2)}]$$

$$= (f-g)(a)$$

$\therefore$ $f-g$ is continuous at a for all $a \in D_f \cap D_g$.

Hence, $f-g$ is continuous.

(*iii*) We have,

$$\lim_{x\to a} (cf)(x) = \lim_{x\to a} c\,.\,f(x)$$

$$= c \lim_{x\to a} f(x) = cf(a) \qquad [\text{By using (1)}]$$

$$= (cf)(a)$$

$\therefore$ cf is continuous at a for all $a \in D_f \cap D_g$.

Hence, cf is continuous.

(*iv*) We have,

$$\lim_{x\to a} (fg)(x) = \lim_{x\to a} [f(x)\,.\,g(x)] = \lim_{x\to a} f(x)\,.\,\lim_{x\to a} g(x)$$

$$= f(a)\,g(a) \qquad [\because \text{ By using (1) and (2)}]$$

$$= (fg)(a)$$

$\therefore$ fg is continuous at a for all $a \in D_f \cap D_g$

Hence, fg is continuous.

(*v*) We have,

$$\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} \qquad [\because \text{ By using (1) and (2)}]$$

$$= \left(\frac{f}{g}\right)(a)$$

$\therefore$ $\dfrac{f}{g}$ is continuous at a for all $a \in D$.

where domain $\left(\dfrac{f}{g}\right) = [(D_f \cap D_g) - \{x : g(x) = 0\}] = D(\text{say})$.

Hence, $\left(\dfrac{f}{g}\right)$ is continuous.

(*vi*) We have,

$$\lim_{x\to a} \left(\frac{1}{f}\right)(x) = \lim_{x\to a} \left[\frac{1}{f(x)}\right] = \frac{1}{\lim_{x\to a} f(x)}$$

$$= \frac{1}{f(a)} \qquad [\because \text{ By using (1)}]$$

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This is the sum of a constant function a_0 (which is continuous) and the product of the identity function x (which is continuous) and the polynomial function $(a_1 + a_2x + \ldots\ldots + a_{n+1}x^n)$ of degree atmost n (which we assumed to be continuous).

Therefore, using algebra of continuous functions, it is continuous.

Thus, the continuity of a polynomial of degree n implies the continuity of a polynomial of degree $(n + 1)$.

Hence, by the principle of induction, every polynomial function is continuous.

Theorem 6. *Prove that every rational function is continuous.*

Proof. Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where : $p(x)$ and $q(x)$ are polynomial functions.

The domain of $f(x)$ is all real numbers except the points at which $q(x)$ is zero.

As $p(x)$ and $q(x)$ are polynomial functions, hence they are continuous.

Also, quotient of two continuous functions is continuous.

Hence, $f(x) = \dfrac{p(x)}{q(x)}$ is continuous in its domain.

Theorem 7. *Show that the modulus function given by $f(x) = |x|$, $x \in R$ is continuous at every point of R.*

Proof. We have, $$f(x) = \begin{cases} x & , \text{if } x > 0 \\ 0 & , \text{if } x = 0 \\ -x & , \text{if } x < 0 \end{cases}$$

Let c be any real number.

If $c = 0$, then,

$$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x)$$

$$\Rightarrow \qquad \lim_{x \to 0} f(x) = 0 \quad \text{and} \quad f(0) = 0.$$

$$\therefore \qquad \lim_{x \to 0} f(x) = f(0)$$

$\Rightarrow$ f is continuous at $c = 0$.

If $c > 0$, then,

$$\lim_{x \to c^+} f(x) = c = \lim_{x \to c^-} f(x)$$

$$\Rightarrow \qquad \lim_{x \to c} f(x) = c \quad \text{and} \quad f(c) = c \qquad [\because \ c > 0]$$

$$\therefore \qquad \lim_{x \to c} f(x) = f(c)$$

$\Rightarrow$ f is continuous at all $c > 0$.

If $c < 0$, then,

$$\lim_{x \to c^+} f(x) = -c = \lim_{x \to c^-} f(x)$$

$$\therefore \qquad \lim_{x \to c} f(x) = -c$$

and $$f(c) = -c \qquad [\because \; c < 0]$$

$$\Rightarrow \quad \lim_{x \to c} f(x) = f(c)$$

$\Rightarrow$ f is continuous at all $c < 0$.

Thus, f is continuous at any real number c.

Hence, $|x|$ is continuous for all x.

Theorem 8. *Let f and g be real functions such that fog is defined. If g is continuous at a and f is continuous at g(a), show that fog is continuous at a.*

Proof. Since, fog is defined.

$\therefore$ We have, Range $(g) \subseteq$ Domain (f).

As g is continuous at $x = a$,

$$\therefore \quad \lim_{x \to a} g(x) = g(a) \qquad ...(1)$$

As f is continuous at $x = g(a)$,

$$\therefore \quad \lim_{y \to g(a)} f(y) = f[g(a)] \qquad ...(2)$$

$$\therefore \quad \lim_{x \to a} (fog)(x) = \lim_{x \to a} f[g(x)]$$

$$= \lim_{g(x) \to g(a)} f[g(x)] \qquad \left[\begin{array}{l}\because \text{ By using (1)} \\ \therefore \text{ As } x \to a \Rightarrow g(x) \to g(a)\end{array}\right]$$

$$= \lim_{y \to g(a)} f(y)$$

$$= f[g(a)] \qquad \text{[By using (2)]}$$

$$\therefore \quad \lim_{x \to a} (fog)(x) = (fog)(a).$$

Thus, (fog) is continuous at a.

Theorem 9. *Show that the composition of two continuous functions is a continuous function.*

Proof. Let f and g be continuous functions such that gof is defined.

Then, range $(f) \subseteq$ domain (g).

Let a be an arbitrary point in the domain of f.

Then, $a \in$ domain $(f) \Rightarrow f(a) \in$ Range (f) $\qquad [\because$ range $(f) \subseteq$ domain $(g)]$

$\Rightarrow f(a) \in$ Domain (g)

Thus, f is continuous at a and g is continuous at $f(a)$.

$$\therefore \quad \lim_{x \to a} f(x) = f(a) \qquad ...(1)$$

and $$\lim_{y \to f(a)} g(y) = g[f(a)] \qquad ...(2)$$

$$\therefore \quad \lim_{x \to a} (gof)(x) = \lim_{x \to a} g[f(x)]$$

$$= \lim_{f(x) \to f(a)} g[f(x)] \qquad \left[\begin{array}{l}\because \text{ By using (1)} \\ \therefore \text{ As } x \to a \Rightarrow f(x) \to f(a)\end{array}\right]$$

$$= \lim_{y \to f(a)} g(y)$$

$$= g[f(a)] \qquad \text{[By using (2)]}$$

$$\Rightarrow \qquad \lim_{x \to a} (gof)(x) = (gof)(a)$$

Thus, *gof* is continuous at a for all $a \in$ domain (f).

Hence, *gof* is continuous.

Theorem 10. *Prove that the exponential function e^x is continuous.*

Proof. Let $f(x) = e^x$ and a be any real number.

As domain of e^x is R. [Putting $(x = a + h)$]

$$\therefore \qquad \lim_{x \to a} e^x = \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h$$

$$= e^a \cdot \lim_{h \to 0} e^h = e^a \cdot (1) = e^a \quad \text{and} \quad f(a) = e^a$$

$\therefore$ We have, $\lim_{x \to a} f(x) = f(a) = e^a$.

$\Rightarrow$ $f(x) = e^x$ is continuous at $x = a$ for all $a \in$ R.

Hence, $f(x) = e^x$ is continuous.

Theorem 11. *Prove that the logarithmic function is continuous.*

Proof. Let $f(x) = \log x$.

As the domain of f is the set of all positive real numbers.

Let a be any positive real number.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \log x$

$$= \lim_{x \to a} \left[\log\left(a \cdot \frac{x}{a}\right)\right] \qquad [\because \ \log m + \log n = \log (m \cdot n)]$$

$$= \lim_{x \to a} \left[\log a + \log \frac{x}{a}\right] = \log a + \lim_{x \to a} \log\left(\frac{x}{a}\right)$$

$$= \log a + 0 \qquad \left[\because \ \lim_{x \to a} \log\left(\frac{x}{a}\right) = \log 1 = 0\right]$$

$$= \log a$$

$$\therefore \qquad \lim_{x \to a} f(x) = f(a)$$

Thus, $f(x) = \log x$ is continuous at a for all $a \in R^+$.

Hence, $f(x) = \log x$ is continuous.

Theorem 12. *Prove that the following functions are continuous :*

(i) the sine function, sin x

(ii) the cosine function, cos x

(iii) the tangent function, tan x.

Proof. (*i*) Let $f(x) = \sin x$ and let a be any real number.

Clearly, domain (f) = R.

$$\therefore \qquad \lim_{x \to a} f(x) = \lim_{x \to a} \sin x$$

Put $x = a + h \Rightarrow h \to 0$ as $x \to a$

$$= \lim_{h \to 0} \sin(a + h) \quad [\because \sin(A + B) = \sin A \cos B + \cos A \sin B]$$

$$= \lim_{h \to 0} [\sin a \cos h + \cos a \sin h]$$

$$= \sin a \, . \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h$$

$$= \sin a\,(1) + \cos a\,(0) = \sin a$$

$\therefore \quad \lim_{x \to a} f(x) = f(a)$ for all $a \in R$.

Thus, $f(x) = \sin x$ is continuous at a for all $a \in R$.

Hence, $\sin x$ is continuous.

(*ii*) Let $f(x) = \cos x$ and let a be any real number, clearly, domain $(f) = R$.

$\therefore \quad \lim_{x \to a} f(x) = \lim_{x \to a} \cos x$

Put $x = a + h \Rightarrow h \to 0$ as $x \to a$

$$= \lim_{h \to 0} \cos(a + h) \quad [\because \cos(A + B) = \cos A \cos B - \sin A \sin B]$$

$$= \lim_{h \to 0} [\cos a \cos h - \sin a \sin h]$$

$$= \cos a \, . \lim_{h \to 0} \cos h - \sin a \lim_{h \to 0} \sin h$$

$$= \cos a \, . (1) - \sin a\,(0) = \cos a$$

$\therefore \quad \lim_{x \to a} f(x) = f(a)$ for all $a \in R$.

Thus, $f(x) = \cos x$ is continuous at a for all $a \in R$.

Hence, $\cos x$ is continuous.

(*iii*) Let $f(x) = \tan x$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}$$

where $f(x) = \sin x$ and $g(x) = \cos x$.

Since f and g are continuous for all x,

$\therefore \quad \frac{f}{g}$ is continuous, provided $\cos x \neq 0$.

$\therefore$ $\tan x$ is continuous at every point of its domain.

where Domain of $\tan x = \left(\frac{\sin x}{\cos x}\right) = [R - \{x : \cos x = 0\}]$.

SOME SOLVED EXAMPLES

Example 1. *(i) Show that the function*

$$f(x) = \begin{cases} x \ ; & \text{if } x \text{ is an integer} \\ 0 \ ; & \text{if } x \text{ is not an integer} \end{cases}$$

is discontinuous at each integral value of x.

(ii) If $$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}.$$

Find whether f(x) is continuous at x = 1.

Solution. *(i)* We have,

$$f(x) = \begin{cases} x \ ; & \text{if } x \text{ is an integer} \\ 0 \ ; & \text{if } x \text{ is not an integer} \end{cases}.$$

Let $x = a$, where : a is any integer.

$\therefore$ $$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = 0 \quad \left[\because (a-h) \text{ is not an integer}, \Rightarrow f(a-h) = 0\right]$$

and $$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = 0 \quad \left[\because (a+h) \text{ is not an integer}, \Rightarrow f(a+h) = 0\right]$$

$\therefore$ $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = 0$$

$\Rightarrow$ $$\lim_{x \to a} f(x) = 0$$

Also, $$f(a) = a \Rightarrow \lim_{x \to a} f(x) \neq f(a).$$

Hence, $f(x)$ is discontinuous at $x = a$.

(ii) We have, $$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$$

$\therefore$ $$\lim_{x \to 1} f(x) = \lim_{x \to 1}\left(\frac{x^2-1}{x-1}\right) = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$= \lim_{x \to 1} (x+1) = 1 + 1 = 2.$$

Also, it is given that, $f(1) = 2$.

$\therefore$ $$\lim_{x \to 1} f(x) = f(1) = 2.$$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Example 2. (*i*) *Test the continuity of the function f(x) at x = 2,*

$$f(x) = \begin{Bmatrix} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 0 & \text{for } x = 2 \end{Bmatrix}.$$

(*ii*) *Show that the function,*

$$f(x) = \begin{Bmatrix} \dfrac{|x|}{x} & ;\text{ when } x \neq 0 \\ 1 & ;\text{ when } x = 0 \end{Bmatrix}$$

is discontinuous at x = 0.

Solution. (*i*) We have,

$$f(x) = \begin{Bmatrix} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 0 & \text{for } x = 2 \end{Bmatrix}$$

$$\therefore \quad \lim_{x \to 2} f(x) = \lim_{x \to 2} \left(\frac{x^2-4}{x-2} \right) = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x+2) = 2 + 2 = 4.$$

Also, it is given that ; $f(2) = 0$

$$\therefore \quad \lim_{x \to 2} f(x) \neq f(2)$$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

(*ii*) We have,
$$f(x) = \begin{Bmatrix} \dfrac{|x|}{x} & ;\text{ when } x \neq 0 \\ 1 & ;\text{ when } x = 0 \end{Bmatrix}$$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1$$

And,
$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Hence, $f(x)$ is not continuous at $x = 0$.

Example 3. (*i*) *Discuss the continuity of*

$$f(x) = \begin{Bmatrix} x & ;\text{ if } x \geq 1 \\ x^2 & ;\text{ if } x < 1 \end{Bmatrix} \quad \text{at } x = 1.$$

(*ii*) *Discuss the continuity of*

$$f(x) = \begin{Bmatrix} \dfrac{x}{|x|} & ;\text{ if } x \neq 0 \\ 0 & ;\text{ if } x = 0 \end{Bmatrix} \quad \text{at } x = 0.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} x \;; & \text{if } x \geq 1 \\ x^2 \;; & \text{if } x < 1 \end{cases}$$

Let a be any real number.

Case I. When $a > 1$,

$\therefore \qquad \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h) = \lim_{h \to 0} (a - h) = a.$

And, $\qquad \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h) = \lim_{h \to 0} (a + h) = a$

Also, $\qquad f(a) = a.$

$\therefore \qquad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a) = a$

$\therefore$ $f(x)$ is continuous at each $a > 1$.

Case II. When $a < 1$,

$\therefore \qquad \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h) = \lim_{h \to 0} (a - h)^2$

$$= \lim_{h \to 0} (a^2 - 2ah + h^2) = a^2.$$

And, $\qquad \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$

$$= \lim_{h \to 0} (a + h)^2 = \lim_{h \to 0} (a^2 + 2ah + h^2) = a^2$$

Also, $\qquad f(a) = a^2$

$\therefore \qquad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a) = a^2$

$\therefore$ $f(x)$ is continuous at each $a < 1$.

Case III. When $a = 1$,

$\therefore \qquad \lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h) = \lim_{h \to 0} (1 - h)^2 = \lim_{h \to 0} (1 - 2h + h^2) = 1$

And, $\qquad \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} (1 + h) = 1.$

Also, $\qquad f(1) = 1$

$\therefore \qquad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1.$

$\therefore$ $f(x)$ is continuous at $a = 1$.

Thus, from all the three cases, it follows that $f(x)$ is continuous at $x = a$ for all $a \in$ R. Hence, $f(x)$ is continuous.

(*ii*) We have, $\qquad f(x) = \begin{cases} \dfrac{x}{|x|} \;; \text{if } x \neq 0 \\ 0 \quad ; \text{if } x = 0 \end{cases}.$

As we know that the identity function x is continuous and the modulus function $|x|$ is continuous.

(*ii*) We have, $$f(x) = \begin{cases} \dfrac{|x|}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\Rightarrow$ $f(x)$ is not continuous at $x = 0$.

Example 5. *Show that the function*

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}; & \text{for } x \neq 0 \\ 0 & ; \text{for } x = 0 \end{cases}$$ *is continuous at* $x = 0$.

Solution. We have, $$f(x) = \begin{cases} x \sin \dfrac{1}{x}; & \text{for } x \neq 0 \\ 0 & ; \text{for } x = 0 \end{cases}$$

$\therefore$ $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x \sin \frac{1}{x} \right)$$

$$= 0 \times \text{(a finite quantity between } -1 \text{ and } 1)$$

$$= 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0} f(x) = f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Aliter. $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= -h \sin \left(\frac{1}{-h} \right) = h \sin \frac{1}{h}$$

$$= 0 \times \text{(a finite quantity between } -1 \text{ and } 1)$$

$$= 0.$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= h \sin \left(\frac{1}{h} \right)$$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0.$

Also, $f(0) = 0$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0.$

$\therefore \quad f(x)$ is continuous at $x = 0$.

Example 6. (*i*) *Show that the function*

$$f(x) = \begin{cases} -x^2 & ; x \le 0 \\ x^2 & ; x > 0 \end{cases} \text{ is continuous at } x = 0.$$

(*ii*) *Show that the function* $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \ne 0 \\ 2 & , x = 0 \end{cases}$ *is continuous at* $x = 0$.

Solution. (*i*) We have, $f(x) = \begin{cases} -x^2 & ; x \le 0 \\ x^2 & ; x > 0 \end{cases}$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -(-h)^2 = 0$$

And, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} (h)^2 = 0.$$

Also, $f(0) = 0$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$

$\therefore \quad f(x)$ is continuous at $x = 0$.

(*ii*) We have, $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x ; & x \ne 0 \\ 2 & ; x = 0 \end{cases}$

$$\therefore \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{\sin x}{x} + \cos x \right) = \lim_{x \to 0} \left(\frac{\sin x}{x} \right) + \lim_{x \to 0} \cos x$$

$$= 1 + 1 = 2 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Also, $f(0) = 2$

$\therefore \quad \lim_{x \to 0} f(x) = f(0) = 2$

$\therefore \quad f(x)$ is continuous at $x = 0$.

Example 7. (*i*) *For what value of k, is the following function continuous at x = 0.*

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}.$$

(*ii*) *Find the point of discontinuity of the function*

$$f(x) = \begin{cases} \dfrac{x^4-16}{x-2} & ; \text{if } x \neq 2 \\ 16 & ; \text{if } x = 2 \end{cases}.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$$

$$\therefore \quad \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1-\cos 4x}{8x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 2x}{8x^2}\right) \qquad \left[\begin{array}{l} \because \; 1-\cos 2A = 2\sin^2 A \\ \Rightarrow \; 1-\cos 4A = 2\sin^2 2A \end{array}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 2x}{4x^2}\right) = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = (1)^2 = 1.$$

Since, the function $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad 1 = k \qquad [\because \; f(0) = k]$$

$\therefore$ The value of k for which the given function is continuous is $k = 1$.

(*ii*) We have, $$f(x) = \begin{cases} \dfrac{x^4-16}{x-2} & ; \text{if } x \neq 2 \\ 16 & ; \text{if } x = 2 \end{cases}$$

The function $f(x) = \dfrac{x^4-16}{x-2}$ being a rational function, it is continuous at all points of its domain, *i.e.*, for all real numbers except 2.

$\therefore$ Consider the given function at $x = 2$.

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{x^4-16}{x-2}\right) = \lim_{x\to 2^-} \frac{(x^2+4)(x^2-4)}{(x-2)}$$

$$= \lim_{x\to 2^-} \frac{(x^2+4)(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2^-} (x^2+4)(x+2)$$

$$= \lim_{h\to 0} [(2-h)^2+4]\,[2-h+2] \qquad [\text{Put } x = (2-h)]$$

Solution. (*i*) We have, $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2} & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$

$\therefore \quad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\dfrac{1-\cos 2x}{x^2}\right)$ $\quad$ [$\because$ $1 - \cos 2A = 2\sin^2 A$]

$$= \lim_{x\to 0}\left(\frac{2\sin^2 x}{x^2}\right) = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2.$$

Also, $\quad f(0) = 2$

$\therefore \quad \lim\limits_{x\to 0} f(x) = f(0) = 2.$

$\therefore$ $f(x)$ is continuous at $x = 0$.

(*ii*) We have, $\quad f(x) = \begin{Bmatrix} \dfrac{1-\cos 4x}{x^2} & ; x \neq 0 \\ 8 & ; x = 0 \end{Bmatrix}$

$\therefore \quad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\dfrac{1-\cos 4x}{x^2}\right)$ $\quad \begin{bmatrix} \because \ 1-\cos 2A = 2\sin^2 A \\ \Rightarrow 1-\cos 4A = 2\sin^2 2A \end{bmatrix}$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 2x}{x^2}\right) = 2\lim_{x\to 0}\frac{\sin^2 2x}{4x^2} \times 4$$

$$= 8\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8$$

Also, $\quad f(0) = 8$

$\therefore \quad \lim\limits_{x\to 0} f(x) = f(0) = 8$

$\therefore$ $f(x)$ is continuous at $x = 0$.

(*iii*) Please try yourself. [**Ans.** $f(x)$ is continuous at $x = 0$]

Example 13. (*i*) *Discuss the continuity of*

$$f(x) = \begin{Bmatrix} x^2 - 1 & ; x \leq 1 \\ x & ; x > 1 \end{Bmatrix} \text{ at } x = 1.$$

(*ii*) *Discuss the continuity of*

$$f(x) = \begin{Bmatrix} x\cos\dfrac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{Bmatrix} \text{ at } x = 0.$$

Solution. (*i*) We have, $\quad f(x) = \begin{Bmatrix} x^2 - 1 & ; x \leq 1 \\ x & ; x > 1 \end{Bmatrix}$

$\therefore \quad \lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h)$

$$= \lim_{h \to 0} [(1-h)^2 - 1] \qquad [\because \ f(x) = x^2 - 1 \text{ for } x < 1]$$

$$= \lim_{h \to 0} [1 + h^2 - 2h - 1] = \lim_{h \to 0} (h^2 - 2h) = 0$$

And, $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) \qquad [\because \ f(x) = x \text{ for } x > 1]$$

$$= \lim_{h \to 0} (1+h) = 1.$$

Also, $$f(1) = (1)^2 - 1 = 0$$

$\therefore$ $$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

(*ii*) We have, $$f(x) = \begin{cases} x \cos \dfrac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left\{ -h \cos\left(\frac{1}{-h}\right) \right\}$$

$$= -\lim_{h \to 0} h \cos \frac{1}{h} \qquad [\because \ \cos(-\theta) = \cos\theta]$$

= 0 × (a finite quantity between – 1 and 1)

= 0.

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \left(h \cos \frac{1}{h} \right)$$

= 0 × (a finite quantity between – 1 and 1)

= 0.

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Example 14. (*i*) *Show that* $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ *for* $x \neq 0$ *and* $f(0) = 0$ *is discontinuous at* $x = 0$.

(*ii*) *Discuss the continuity of the function*

$$f(x) = \begin{cases} \dfrac{1}{2} - x & ; 0 \le x < \dfrac{1}{2} \\ 1 & ; \quad x = \dfrac{1}{2} \\ \dfrac{3}{2} - x & ; \dfrac{1}{2} < x \le 1 \end{cases} \text{ at } x = \frac{1}{2}.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left(\frac{e^{\frac{1}{-h}} - 1}{e^{\frac{1}{-h}} + 1} \right) = \lim_{h \to 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1} \right)$$

$$= \frac{0-1}{0+1} = -1. \qquad \left[\because\ e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0\right]$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1} \right) \qquad \left[\text{Dividing the numerator and denominator by } e^{1/h}\right]$$

$$= \lim_{h \to 0} \left(\frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} \right) = \frac{1-0}{1+0} = 1$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

(*ii*) We have, $$f(x) = \begin{cases} \dfrac{1}{2} - x & ; \ 0 \leq x < \dfrac{1}{2} \\ 1 & ; \ x = \dfrac{1}{2} \\ \dfrac{3}{2} - x & ; \ \dfrac{1}{2} < x \leq 1 \end{cases}$$

$\therefore$ $$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right] \qquad \left[\because\ f(x) = \left(\frac{1}{2} - x\right) \text{ for } 0 \leq x < \frac{1}{2}\right]$$

$$= \lim_{h \to 0} (h) = 0$$

And, $$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

Example 16. (*i*) *Discuss the continuity of*

$$f(x) = \begin{cases} -1+x^2 & ; x<-1 \\ 1+\dfrac{x+x^2}{2} & ; x=-1 \\ 1+x^2 & ; x>-1 \end{cases} \text{ at } x = -1.$$

(*ii*) *Determine the value of k for which the function*

$$f(x) = \begin{cases} \dfrac{\sin 5x}{3x} & ; \text{if } x \neq 0 \\ k & ; \text{if } x = 0 \end{cases} \text{ is continuous at } x = 0.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} -1+x^2 & ; \ x<-1 \\ 1+\dfrac{x+x^2}{2} & ; \ x=-1 \\ 1+x^2 & ; \ x>-1 \end{cases}$$

$\therefore$ $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{h \to 0} f(-1-h)$

$= \lim\limits_{h \to 0} -1 + (-1-h)^2$ [$\because$ $f(x) = -1 + x^2$ for $x < -1$]

$= -1 + (-1-0)^2 = -1 + 1 = 0$

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{h \to 0} f(-1+h)$

$= \lim\limits_{h \to 0} 1 + (-1+h) = \lim\limits_{h \to 0} (h) = 0$ [$\because$ $f(x) = 1 + x$ for $x > -1$]

Also, $f(-1) = 1 + \dfrac{-1+(-1)^2}{2} = 1 + \dfrac{0}{2} = 1 + 0 = 1$

$\therefore$ $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x) = 0$

$\therefore$ $\lim\limits_{x \to -1} f(x) \neq f(-1)$

Hence, $f(x)$ is not continuous at $x = -1$.

(*ii*) We have, $f(x) = \begin{cases} \dfrac{\sin 5x}{3x} & ; \text{if } x \neq 0 \\ k & ; \text{if } x = 0 \end{cases}$

$\therefore$ $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left(\dfrac{\sin 5x}{3x}\right) = \lim\limits_{x \to 0} \dfrac{\sin 5x}{5x} \times \dfrac{5}{3}$

$= \dfrac{5}{3} \lim\limits_{x \to 0} \left(\dfrac{\sin 5x}{5x}\right) = \dfrac{5}{3}(1) = \dfrac{5}{3}$.

Since, $f(x)$ is continuous at $x = 0$,

$\therefore$ $\lim\limits_{x \to 0} f(x) = f(0)$

$$\Rightarrow \qquad \frac{5}{3} = k \quad \Rightarrow \quad k = \frac{5}{3}$$

Hence, the given function is continuous for $k = \frac{5}{3}$.

Example 17. (*i*) *Determine the value of the constant k so that the function*

$$f(x) = \begin{Bmatrix} kx^2 & ; x \le 2 \\ 3 & ; x > 2 \end{Bmatrix} \text{ is continuous.}$$

(*ii*) *If* $$f(x) = \begin{Bmatrix} 1 & ; x \le 3 \\ ax + b & ; 3 < x < 5 \\ 7 & ; x \ge 5 \end{Bmatrix}.$$

Find the values of a and b so that f(x) is continuous.

Solution. (*i*) We have, $f(x) = \begin{Bmatrix} kx^2 & ; x \le 2 \\ 3 & ; x > 2 \end{Bmatrix}$

When x ≤ 2, $f(x) = kx^2$, which being a polynomial function is continuous at each $x < 2$

When x > 2, $f(x) = 3$, which being a constant function is continuous at each $x > 2$.

So, let us discuss the continuity at $x = 2$.

$$\therefore \quad \lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h) \qquad [\because \ f(x) = kx^2 \text{ for } x \le 2]$$

$$= \lim_{h \to 0} k(2 - h)^2 = \lim_{h \to 0} k(4 + h^2 - 2h) = 4k$$

And, $$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 \qquad [\because \ f(x) = 3 \text{ for } x > 2]$$

$$= 3$$

Also, $$f(2) = k(2)^2 = 4k. \qquad [\because \ f(x) = kx^2 \text{ for } x = 2]$$

Since, $f(x)$ is continuous function.

$$\therefore \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \quad 4k = 3$$

$$\Rightarrow \quad k = \frac{3}{4}$$

Hence, the given function is continuous at $k = \frac{3}{4}$.

(*ii*) We have, $$f(x) = \begin{Bmatrix} 1 & ; x \le 3 \\ ax + b & ; 3 < x < 5 \\ 7 & ; 5 \le x \end{Bmatrix}$$

When x < 3, $f(x) = 1$, which being a constant function is continuous at each $x < 3$.

When 3 < x < 5, $f(x) = ax + b$, which being a polynomial function is continuous at each $x \in (3, 5)$.

When x > 5, $f(x) = 7$, which being a constant function is continuous at each $x > 5$.

When x = 3, let us discuss the continuity of the function at $x = 3$.

$\therefore \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 1 = 1$

And, $\quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (ax + b) = 3a + b$

Also, $\quad f(3) = 1.$

Since, $f(x)$ is continuous at $x = 3$,

$\therefore \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \quad 1 = 3a + b$

$\Rightarrow \quad 3a + b = 1.$ ...(1)

When x = 5, let us discuss the continuity at $x = 5$.

$\therefore \quad \lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (ax + b) = 5a + b$

And, $\quad \lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 7 = 7$

Also, $\quad f(5) = 7.$

Since, $f(x)$ is continuous at $x = 5$,

$\therefore \quad \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

$\Rightarrow \quad 5a + b = 7$ ...(2)

$\therefore$ From equations (1) and (2), we have

$$a = 3, b = -8.$$

Example 18. *Discuss the continuity of the function :*

$$f(x) = \begin{cases} 2 - x & ; x < 2 \\ 2 + x & ; x \geq 2 \end{cases} \text{ at } x = 2.$$

Solution. We have, $\quad f(x) = \begin{cases} 2 - x & ; x < 2 \\ 2 + x & ; x \geq 2 \end{cases}$

$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2 - h)$ $\quad [\because \; f(x) = 2 - x \text{ for } x < 2]$

$$= \lim_{h\to 0} 2 - (2 - h) = \lim_{h\to 0} h = 0$$

And, $\quad \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2 + h)$ $\quad [\because \; f(x) = 2 + x \text{ for } x \geq 2]$

$$= \lim_{h\to 0} (2 + 2 + h) = 4.$$

Also, $\quad f(2) = 2 + 2 = 4$

$\therefore \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore \quad \lim_{x\to 2} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 2$.

Example 20. (*i*) *Examine the continuity at $x = 1$ of the function*

$$f(x) = \begin{Bmatrix} 5x-4 & ; 0 < x \le 1 \\ 4x^2 - 3x & ; 1 < x < 2 \end{Bmatrix}.$$

(*ii*) *Discuss the continuity of the function*

$$f(x) = \begin{Bmatrix} x & ; & 0 \le x < 1 \\ 5 & ; & x = 1 \\ 2-x & ; & x > 1 \end{Bmatrix} \text{ at } x = 1.$$

Solution. (*i*) We have,

$$f(x) = \begin{Bmatrix} 5x-4 & ; 0 < x \le 1 \\ 4x^2 - 3x & ; 1 < x < 2 \end{Bmatrix}$$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} [5(1-h) - 4]$ $\quad [\because \ f(x) = 5x - 4 ; 0 < x \le 1]$

$= \lim_{h \to 0} (5 - 5h - 4) = \lim_{h \to 0} (1 - 5h) = 1.$

And, $\quad \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$ $\quad [\because \ f(x) = 4x^2 - 3x ; 1 < x < 2]$

$= \lim_{h \to 0} [4(1+h)^2 - 3(1+h)] = \lim_{h \to 0} [4(1 + h^2 + 2h) - 3 - 3h]$

$= \lim_{h \to 0} (4 + 4h^2 + 8h - 3 - 3h] = \lim_{h \to 0} (4h^2 + 5h + 1) = 1$

Also, $\quad f(1) = [5(1) - 4] = (5 - 4) = 1.$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1$

$\therefore \quad f(x)$ is continuous at $x = 1$.

(*ii*) We have, $\quad f(x) = \begin{Bmatrix} x & ; 0 \le x < 1 \\ 5 & ; x = 1 \\ 2-x & ; x > 1 \end{Bmatrix}$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$ $\quad [\because \ f(x) = x \text{ for } 0 \le x < 1]$

And, $\quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x)$ $\quad [\because \ f(x) = (2 - x) \text{ for } x > 1]$

$= (2 - 1) = 1$

Also, $\quad f(1) = 5.$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1$

$\therefore \quad \lim_{x \to 1} f(x)$ exists.

But, $\quad \lim_{x \to 1} f(x) \ne f(1)$

$\therefore \quad f(x)$ is discontinuous at $x = 1$.

Example 28. (*i*) *Show that the function defined by :*

$$f(x) = \begin{cases} \dfrac{[x]-1}{x-1} & ; x \neq 1 \\ -1 & ; x = 1 \end{cases}$$

is discontinuous at x = 1.

(*ii*) *Show that the function defined by :*

$$f(x) = \begin{cases} \dfrac{x}{|x|+2x^2}; & x \neq 0 \\ k & ; x = 0 \end{cases}$$

remains discontinuous at x = 0, regardless of the choice of k.

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{[x]-1}{x-1} & ; x \neq 1 \\ -1 & ; x = 1 \end{cases}$$

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{[1-h]-1}{(1-h-1)}$$

$$= \lim_{h \to 0} \frac{0-1}{-h} = \lim_{h \to 0} \left(\frac{1}{h}\right) = \frac{1}{0} = \infty.$$

And, $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \frac{[1+h]-1}{(1+h-1)}$$

$$= \lim_{h \to 0} \left(\frac{1-1}{h}\right) = \lim_{h \to 0} \left(\frac{0}{h}\right) = 0.$$

$$\therefore \quad \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore$ $\lim_{x \to 1} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x = 1$.

(*ii*) We have, $$f(x) = \begin{cases} \dfrac{x}{|x|+2x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{x}{|x|+2x^2}\right) \qquad [\because \ |x| = -x \text{ for } x < 0]$$

$$= \lim_{x \to 0^-} \left(\frac{x}{-x+2x^2}\right) = \lim_{x \to 0^-} \frac{x}{x(-1+2x)}$$

$$= \lim_{x \to 0^-} \left(\frac{1}{-1+2x}\right) = \left(\frac{1}{-1+0}\right) = -1$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{x}{|x|+2x^2}\right) = \lim_{x \to 0^+} \left(\frac{x}{x+2x^2}\right) \qquad [\because \ |x| = x \text{ for } x > 0]$$

Since, the given function is continuous at $x = -1$

$$\therefore \quad \lim_{x \to -1} f(x) = f(-1) \Rightarrow -4 = k$$

$$\Rightarrow k = -4.$$

$\therefore$ The function $f(x)$ is continuous at $x = -1$ for $k = -4$.

(*ii*) We have,
$$f(x) = \begin{cases} x & ; x < 1 \\ 1 + x & ; x > 1 \\ \dfrac{3}{2} & ; x = 1 \end{cases}$$

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1 \qquad [\because f(x) = x \text{ for } x < 1]$$

And,
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 + x) \qquad [\because f(x) = 1 + x \text{ for } x > 1]$$
$$= (1 + 1) = 2.$$

Also,
$$f(1) = \frac{3}{2}$$

$$\therefore \quad \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore$ $\lim_{x \to 1} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x = 1$.

Example 32. (*i*) *If the function f(x), defined by :*

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$$

is continuous at x = 0, find the value of k.

(*ii*) *If the function f(x) is defined by :*

$$f(x) = \begin{cases} \dfrac{1 - \cos kx}{x \sin x} & ; x \neq 0 \\ \dfrac{1}{2} & ; x = 0 \end{cases}$$

is continuous at x = 0, find the value of k.

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$$

$$\therefore \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{1 - \cos 2x}{2x^2} \right) \qquad [\because 1 - \cos 2A = 2 \sin^2 A]$$

$$= \lim_{x \to 0} \left(\frac{2 \sin^2 x}{2x^2} \right) = \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2 \qquad \left[\because \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right) = 1 \right]$$

$$= (1)^2 = 1$$

Also, $f(0) = k$.

Since, the given function $f(x)$ is continuous at $x = 0$.

$\therefore$ $\lim_{x \to 0} f(x) = f(0) \Rightarrow 1 = k \Rightarrow k = 1$

$\therefore$ The function $f(x)$ is continuous at $x = 0$ for $k = 1$.

(*ii*) We have
$$f(x) = \begin{cases} \dfrac{1 - \cos kx}{x \sin x} & ; x \neq 0 \\ \dfrac{1}{2} & ; x = 0 \end{cases}$$

$\therefore$
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{1 - \cos kx}{x \sin x} \right)$$

$$= \lim_{x \to 0} \frac{2 \sin^2 \left(\frac{kx}{2} \right)}{x} \times \frac{1}{\sin x} \qquad [\because 1 - \cos 2A = 2 \sin^2 A]$$

$$= 2 \lim_{x \to 0} \frac{\sin^2 \left(\frac{kx}{2} \right)}{\frac{k^2}{4} x^2} \times \frac{k^2}{4} \cdot \frac{x}{\sin x}$$

$$= 2 \times \frac{k^2}{4} \lim_{x \to 0} \left(\frac{\sin \left(\frac{kx}{2} \right)}{\frac{kx}{2}} \right)^2 \cdot \lim_{x \to 0} \left(\frac{x}{\sin x} \right)$$

$$= \frac{k^2}{2} \cdot (1)^2 \cdot (1) = \frac{k^2}{2}$$

Also, $f(0) = \frac{1}{2}$.

Since, the given function $f(x)$ is continuous at $x = 0$.

$\therefore$ $\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow$ $\frac{k^2}{2} = \frac{1}{2} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

Hence, the function $f(x)$ is continuous at $x = 0$ for $k = \pm 1$.

Example 33. (*i*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \dfrac{x - |x|}{x} & ; x \neq 0 \\ 2 & ; x = 0 \end{cases} \text{ at } x = 0.$$

(*ii*) *Show that the function* $f(x) = \begin{cases} |x| & ; x \leq 2 \\ [x] & ; x > 2 \end{cases}$ *is continuous on* $[0, 2]$.

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{x-|x|}{x} & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{x-|x|}{x}\right] = \lim_{x\to 0^-}\left[\frac{x-(-x)}{x}\right] = \lim_{x\to 0^-}\left(\frac{2x}{x}\right) = 2$$

And, $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{x-|x|}{x}\right) = \lim_{x\to 0^+}\left(\frac{x-x}{x}\right) = \lim_{x\to 0^+}\frac{0}{x} = 0$$

Also, $f(0) = 2$.

$\therefore$ $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

$\therefore$ $\lim_{x\to 0} f(x)$ does not exist.

$\therefore$ $f(x)$ is not continuous at $x = 0$.

(*ii*) We have, $$f(x) = \begin{cases} |x| & ; x \leq 2 \\ [x] & ; x > 2 \end{cases}$$

Here, $f(x) = |x| = x$ for $0 < x < 2$.

$\therefore$ $f(x)$ being a polynomial function is continuous. For all $x \in (0, 2)$

Now, let us discuss the continuity at $x = 0$ and at $x = 2$.

Continuity at x = 0 :

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} |-h| = \lim_{h\to 0} h = 0$$

And, $$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} |h| = \lim_{h\to 0} h = 0$$

Also, $f(0) = 0$.

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Continuity at x = 2 :

$\therefore$ $$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} |2-h| = \lim_{h\to 0} (2-h) = 2$$

And, $$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = \lim_{h\to 0} (2) = 2.$$

Also, $f(2) = 2$.

$\therefore$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 2$.

$\therefore$ $f(x)$ is continuous at $x = 2$.

Thus, the given function $f(x)$ is continuous on $[0, 2]$.

Example 34. *Discuss the continuity of the function $[1 - x] + [x - 1]$ at the points $x = -1$, 0 and 1.6.*

Solution. We have, $f(x) = [1 - x] + [x - 1]$.

Continuity at x = – 1 :

$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{h \to 0} f(-1-h) = \lim_{h \to 0} ([1-(-1-h)] + [(-1-h)-1])$

$= \lim_{h \to 0} ([2+h] + [-2-h]) = \lim_{h \to 0} (2 + (-3)) = -1$

And, $\quad \lim_{x \to -1^+} f(x) = \lim_{h \to 0} f(-1+h)$

$= \lim_{h \to 0} ([1-(-1+h)] + [(-1+h)-1])$

$= \lim_{h \to 0} ([2-h] + [-2+h]) = \lim_{h \to 0} (1 + (-2)) = -1.$

Also, $\quad f(-1) = [1-(-1)] + [-1-1] = [2] + [-2] = 2 - 2 = 0.$

$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = -1$

$\therefore \quad \lim_{x \to -1} f(x) = -1.$

$\therefore \quad \lim_{x \to -1} f(x) \neq f(-1)$

Hence, $f(x)$ is not continuous at $x = -1$.

Continuity at x = 0 :

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} ([1-(-h)] + [-h-1])$

$= \lim_{h \to 0} ([1+h] + [-1-h]) = \lim_{h \to 0} (1 + (-2)) = -1.$

And, $\quad \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} ([1-h] + [-1+h])$

$= \lim_{h \to 0} (0 + (-1)) = -1$

Also, $\quad f(0) = ([1-0] + [0-1]) = 1 + (-1) = 0$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -1$

$\therefore \quad \lim_{x \to 0} f(x) = -1$

$\therefore \quad \lim_{x \to 0} f(x) \neq f(0).$

Hence, $f(x)$ is not continuous at $x = 0$.

Continuity at x = 1.6 : Please try yourself. [**Ans.** $f(x)$ is continuous at $x = 1.6$].

Example 35. *(i) Discuss the continuity of the function :*

$$f(x) = \begin{Bmatrix} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & ; x \neq 0 \\ 1 & ; x = 0 \end{Bmatrix} \text{ at } x = 0.$$

(ii) Examine the continuity of the function :

$$f(x) = \begin{Bmatrix} \dfrac{xe^{1/x}}{1 + e^{1/x}} & ; x \neq 0 \\ 0 & ; x = 0 \end{Bmatrix} \text{ at } x = 0.$$

(*ii*) We have, $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & ; x < 0 \\ x + 2 & ; x \geq 0 \end{cases}$

Since, the function sin 2x and the identity function x are continuous functions.

$\therefore$ The quotient function $\dfrac{\sin 2x}{x}$ is continuous at each $x < 0$.

Also, the function $(x + 2)$, which being a polynomial function is continuous at each $x > 0$. So, let us discuss the continuity at $x = 0$.

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0}\left(\frac{\sin 2x}{x}\right) = \lim_{x \to 0}\left(\frac{\sin 2x}{2x}\right) \times 2 = 1 \times 2 = 2.$$

And, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} (x + 2) = 0 + 2 = 2$

Also, $f(0) = 0 + 2 = 2$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 2$$

Hence, $f(x)$ is continuous at $x = 0$.

Example 38. *For what value of k the following functions are continuous at x = 2.*

$$f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2} & ; x \neq 2 \\ k & ; x = 2 \end{cases}.$$

Solution. We have,

$$f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2} & ; x \neq 2 \\ k & ; x = 2 \end{cases}$$

$$\therefore \quad \lim_{x \to 2} f(x) = \lim_{x \to 2}\left(\frac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2}\right)$$

$$= \lim_{x \to 2}\left(\frac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2} \times \frac{\sqrt{5x+2} + \sqrt{4x+4}}{\sqrt{5x+2} + \sqrt{4x+4}}\right)$$

(Rationalization)

$$= \lim_{x \to 2}\left[\frac{(5x+2) - (4x+4)}{(x-2)\left(\sqrt{5x+2} + \sqrt{4x+4}\right)}\right]$$

$$= \lim_{x \to 2}\left[\frac{(5x+2-4x-4)}{(x-2)\left(\sqrt{5x+2} + \sqrt{4x+4}\right)}\right]$$

$$= \lim_{x \to 2} \left[\frac{(x-2)}{(x-2)\left(\sqrt{5x+2}+\sqrt{4x+4}\right)} \right]$$

$$= \lim_{x \to 2} \left[\frac{1}{\sqrt{5x+2}+\sqrt{4x+4}} \right] = \frac{1}{\sqrt{10+2}+\sqrt{8+4}}$$

$$= \frac{1}{\sqrt{12}+\sqrt{12}} = \frac{1}{2\sqrt{12}} = \frac{1}{4\sqrt{3}}$$

Also, $f(2) = k$.

Since, the given function $f(x)$ is continuous at $x = 2$.

$\therefore$ $\lim_{x \to 2} f(x) = f(2)$

$\Rightarrow$ $\frac{1}{4\sqrt{3}} = k \Rightarrow k = \frac{1}{4\sqrt{3}}$

Hence, $f(x)$ is continuous at $x = 2$ for $k = \frac{1}{4\sqrt{3}}$.

Example 39. (*i*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \frac{|x-a|}{x-a} & ; x \neq a \\ 1 & ; x = a \end{cases} \text{ at } x = a.$$

(*ii*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & ; x \neq 4 \\ 0 & ; x = 4 \end{cases}.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \frac{|x-a|}{x-a} & ; x \neq a \\ 1 & ; x = a \end{cases}$$

$\therefore$ $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \left[\frac{|a-h-a|}{a-h-a} \right]$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

And, $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left[\frac{|a+h-a|}{a+h-a} \right]$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore$ $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

$\Rightarrow \quad \lim_{x \to a} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = a$.

(*ii*) We have,
$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & ; x \neq 4 \\ 0 & ; x = 4 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{-(x-4)}{x-4} & ; x < 4 \\ \dfrac{(x-4)}{(x-4)} & ; x > 4 \\ 0 & ; x = 4 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -1 & ; x < 4 \\ 1 & ; x > 4 \\ 0 & ; x = 4 \end{cases}$$

$f(x) = -1$, which being a constant function, is continuous at each $x < 4$.
$f(x) = 1$, which being a constant function, is continuous at each $x > 4$.
Let us discuss the continuity at point $x = 4$.

$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (-1) = -1$

And, $\quad \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (1) = 1$

Also, $\quad f(4) = 0.$

$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

$\therefore \quad \lim_{x \to 4} f(x)$ exists.

But, $\quad \lim_{x \to 4} f(x) \neq f(4)$

$\therefore$ The function $f(x)$ is not continuous at $x = 4$.

Example 40. (*i*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \dfrac{x-1}{1+e^{\frac{1}{x-1}}} & ; x \neq 1 \\ 0 & ; x = 1 \end{cases} \text{ at } x = 1.$$

(*ii*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \text{ at } x = 0.$$

Also, $f(0) = 0.$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x = 0$.

Example 41. *If the function f(x) defined by :*

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a\,; & x < 4 \\ a + b\,; & x = 4 \\ \dfrac{x-4}{|x-4|} + b\,; & x > 4 \end{cases}$$

is continuous at $x = 4$, find the values of a and b.

Solution. We have,

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a\,; & x < 4 \\ a + b\,; & x = 4 \\ \dfrac{x-4}{|x-4|} + b\,; & x > 4 \end{cases}$$

$$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \left(\frac{x-4}{|x-4|} + a \right)$$

$$= \lim_{x \to 4^-} \left(\frac{x-4}{-(x-4)} + a \right)$$

$$= \lim_{x \to 4^-} (-1 + a) = (-1 + a).$$

And, $$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \left(\frac{x-4}{|x-4|} + b \right) = \lim_{x \to 4^+} \left(\frac{x-4}{x-4} + b \right)$$

$$= \lim_{x \to 4^+} (1 + b) = (1 + b)$$

Also, $f(4) = (a + b).$

Since, the function $f(x)$ is continuous at $x = 4$,

$$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4).$$

$\Rightarrow \quad (-1 + a) = (1 + b) = (a + b)$

$\Rightarrow \quad -1 + a = a + b \quad$ and $\quad (1 + b) = a + b$

$\Rightarrow \quad b = -1 \quad$ and $\quad a = 1$

$\therefore$ The function $f(x)$ is continuous at $x = 4$ for $a = 1$ and $b = -1$.

Example 42. (*i*) *Show that the function defined by :*

$$f(x) = \begin{cases} \dfrac{|x-2|}{x^2-4} & ; x \neq 2 \\ \dfrac{1}{4} & ; x = 2 \end{cases} \text{ is discontinuous at } x = 2.$$

(*ii*) *For what value of k the function defined by :*

$$f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & ; x \neq 4 \\ k & ; x = 4 \end{cases} \text{ is continuous at } x = 4.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{|x-2|}{x^2-4} & ; x \neq 2 \\ \dfrac{1}{4} & ; x = 2 \end{cases}$$

$\therefore$
$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \left(\frac{|2-h-2|}{(2-h)^2-4} \right)$$

$$= \lim_{h \to 0} \left(\frac{|-h|}{4+h^2-4h-4} \right) = \lim_{h \to 0} \left(\frac{h}{h^2-4h} \right) = \lim_{h \to 0} \frac{h}{h(h-4)}$$

$$= \lim_{h \to 0} \left(\frac{1}{h-4} \right) = \frac{1}{(0-4)} = \frac{-1}{4}.$$

And,
$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \left(\frac{|2+h-2|}{(2+h)^2-4} \right)$$

$$= \lim_{h \to 0} \left(\frac{|h|}{4+h^2+4h-4} \right) = \lim_{h \to 0} \left(\frac{h}{h^2+4h} \right) = \lim_{h \to 0} \frac{h}{h(h+4)}$$

$$= \lim_{h \to 0} \left(\frac{1}{h+4} \right) = \frac{1}{(0+4)} = \frac{1}{4}.$$

Also,
$$f(2) = \frac{1}{4}.$$

$\therefore$
$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 2$.

(*ii*) We have,
$$f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & ; x \neq 4 \\ k & ; x = 4 \end{cases}$$

$\therefore \quad \lim_{x \to 4} f(x) = \lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4} \right) = \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)}$

$= \lim_{x \to 4} (x + 4) = (4 + 4) = 8.$

Also, $f(4) = k.$

Since, the function $f(x)$ is continuous at $x = 4$.

$\Rightarrow \quad \lim_{x \to 4} f(x) = f(4)$

$\Rightarrow \quad 8 = k \Rightarrow k = 8.$

Hence, the function $f(x)$ is continuous at $x = 4$ for $k = 8$.

Example 43. (*i*) *Determine the value of constant m, so that the function :*

$$f(x) = \begin{cases} m(x^2 - 2x) & ; x < 0 \\ \cos x & ; x \geq 0 \end{cases} \text{ is continuous.}$$

(*ii*) *If the function f(x) defined by :*

$$f(x) = \begin{cases} 2x^2 + k & ; x \geq 0 \\ -2x^2 + k & ; x < 0 \end{cases}$$

is continuous at x = 0, find the value of k.

Solution. (*i*) We have,

$$f(x) = \begin{cases} m(x^2 - 2x) & ; x < 0 \\ \cos x & ; x \geq 0 \end{cases}$$

$f(x) = m(x^2 - 2x)$, which being a polynomial, is continuous at each $x < 0$

$f(x) = \cos x$, which being a cosine function, is continuous at each $x > 0$.

So, let us discuss the continuity at $x = 0$.

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} [m(x^2 - 2x)] \quad [\because f(x) = m(x^2 - 2x) \text{ for } x < 0]$

$= m(0 - 0) = 0.$

And, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\cos x) \quad [\because f(x) = \cos x \text{ for } x \geq 0]$

$= \cos 0 = 1$

Also, $f(0) = 1$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist.

Hence, the function $f(x)$ is not continuous at $x = 0$ for all values of m.

(*ii*) We have, $f(x) = \begin{cases} 2x^2 + k & ; x \geq 0 \\ -2x^2 + k & ; x < 0 \end{cases}$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x^2 + k) \quad [\because f(x) = (-2x^2 + k) \text{ for } x < 0]$

$= (0 + k) = k.$

$\therefore \qquad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (3x + 2) \qquad [\because \ f(x) = 3x + 2 \text{ for } 2 \le x \le 4]$

$= 3(4) + 2 = 12 + 2 = 14 \qquad ...(7)$

And, $\qquad \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (2ax + 5b) \qquad [\because \ f(x) = 2ax + 5b \text{ for } 4 \le x \le 8]$

$= 2a(4) + 5b = 8a + 5b \qquad ...(8)$

Also, $\qquad f(4) = 3(4) + 2 = 12 + 2 = 14 \qquad ...(9)$

Now, putting the values of equations (7), (8) and (9) in equation (6), we have

$8a + 5b = 14 \qquad ...(10)$

Solving equations (5) and (10), we get

$a = 3$ and $b = -2$.

EXERCISE FOR PRACTICE

1. If $\qquad f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & ; x \neq 1 \\ 2 & ; x = 1 \end{cases}$.

Find whether $f(x)$ is continuous at $x = 1$.

2. Examine the continuity of the function :

$$f(x) = \begin{cases} 2x - 1 & ; x < 2 \\ \dfrac{3x}{2} & ; x \ge 2 \end{cases} \text{ at } x = 2.$$

3. Examine the continuity of the function :

$$f(x) = \begin{cases} 5x - 4 & ; 0 < x \le 1 \\ 4x^3 - 3x & ; 1 < x < 2 \end{cases} \text{ at } x = 1.$$

4. Show that $\qquad f(x) = \begin{cases} \dfrac{x - |x|}{2} & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$ is discontinuous at $x = 0$.

5. For what value of k, the function :

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & ; x \neq 4 \\ k & ; x = 4 \end{cases} \text{ is continuous at } x = 4.$$

6. For what value of k, the function :

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & ; x \neq \dfrac{\pi}{2} \\ 3 & ; x = \dfrac{\pi}{2} \end{cases} \text{ is continuous at } x = \frac{\pi}{2}.$$

7. Discuss the continuity of the function :

$$f(x) = \begin{cases} (x - a) \sin\left(\dfrac{1}{x - a}\right) & ; x \neq a \\ 0 & ; x = a \end{cases} \text{ at } x = a.$$

8. Discuss the continuity of the function :

$$f(x) = \begin{cases} \dfrac{2|x| + x^2}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \text{ at } x = 0.$$

9. Determine the value of the constant k so that the function :

$$f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1} & ; x \neq 1 \\ k & ; x = 1 \end{cases} \text{ is continuous at } x = 1.$$

10. Show that the function : $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ when $x \neq 0$, $f(0) = 0$ is discontinuous at $x = 0$.

11. Find the point of discontinuity in the function $f(x)$ if any,

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x & ; \text{if } x \neq 0 \\ 5 & ; \text{if } x = 0 \end{cases}.$$

12. Show that the function :

$$f(x) = \begin{cases} 7x + 5 & ; \text{if } x \geq 0 \\ 5 - 3x & ; \text{if } x < 0 \end{cases} \text{ is continuous function.}$$

13. Discuss the continuity of the function $\dfrac{1+x}{1-x}$.

14. If the function $f(x)$ defined by :

$$f(x) = \begin{cases} 2 & ; x \leq 3 \\ ax + b & ; 3 < x < 5 \\ 9 & ; x \geq 5 \end{cases} \text{ is continuous, find the values of } a \text{ and } b.$$

15. Discuss the continuity of the function :

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & ; x < 0 \\ x + 2 & ; x \geq 0 \end{cases}.$$

16. Show that the function $f(x) = |x - 4|$ is a continuous function.

Answers

1. Continuous	2. Continuous	3. Continuous	5. $k = 8$
6. $k = 6$	7. Continuous	8. Discontinuous	9. $k = -1$
11. $x = 0$	13. Continuous	14. $a = \dfrac{7}{2}$, $b = -\dfrac{17}{2}$	15. Continuous.

5
Differentiability

5.1 INTRODUCTION

In the previous chapters, we studied the functions, limits and continuity. In this chapter, we will use the concept of limit to introduce the idea of differentiability. It help us to study rates at which physical quantities change.

5.2 DERIVABILITY OR DIFFERENTIABILITY AT A POINT

Let $f(x)$ be a real valued function and a be any point in its domain. Then, $f(x)$ is said to have a derivative at $x = a$ if and only if $f(x)$ is defined in some neighbourhood of a and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists finitely.}$$

where h be any small but arbitrary (positive or negative) number.

The value of this limit is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$.

i.e.,
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Now, $f(x)$ is differentiable at $x = a$, if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists finitely.

Also, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists if and only if $\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$

or
$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{-h} \text{ and } \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ both exist and are equal.}$$

5.3 LEFT AND RIGHT HAND DERIVATIVES AT A POINT

If the function $f(x)$ involves modulus function, bracket function and/or is defined by more than one rule, then $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ may depends upon the sign of increment h of x. In such cases, we calculate,

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ separately.}$$

These limits are called Left Hand Derivative of $f(x)$ at a and Right Hand Derivative of $f(x)$ at a and are denoted by $Lf'(a)$ and $Rf'(a)$ respectively.

$$= \lim_{h \to 0^-} \left(\frac{2-h-2}{+h} \right) = \lim_{h \to 0} \left(\frac{-h}{+h} \right) = -1.$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \left(\frac{f(0+h)-f(0)}{h} \right) \qquad [\because \ f(x) = 2 + x \text{ for } x \geq 0]$$

$$= \lim_{h \to 0^+} \left(\frac{(2+h)-(2+0)}{h} \right) = \lim_{h \to 0^+} \left(\frac{h}{h} \right) = 1$$

$\therefore$ $Lf'(0) \neq Rf'(0)$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Example 4. *Show that the function $f(x) = x^2$ for $x \leq 0$ and $f(x) = x$ for $x > 0$ is not derivable at $x = 0$.*

Solution. We have, $f(x) = \begin{Bmatrix} x^2 & ; x \leq 0 \\ x & ; x > 0 \end{Bmatrix}$

$$\therefore \quad \text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} \qquad [\because \ f(x) = x^2 \text{ for } x \leq 0]$$

$$= \lim_{h \to 0^-} \left(\frac{(h)^2 - 0}{h} \right) = \lim_{h \to 0^-} (h) = 0$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} \qquad [\because \ f(x) = x \text{ for } x > 0]$$

$$= \lim_{h \to 0^+} \left(\frac{h-0}{h} \right) = \lim_{h \to 0^+} 1 = 1$$

$\therefore$ $Lf'(0) \neq Rf'(0)$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Example 5. *Show that $f(x) = [x]$ is differentiable at $x = 1$.*

Solution. We have, $f(x) = [x]$.

$$\therefore \quad \text{L.H.D.} \quad Lf'(1) = \lim_{h \to 0^-} \left[\frac{f(1+h)-f(1)}{h} \right]$$

$$= \lim_{h \to 0^-} \left(\frac{[1+h]-[1]}{h} \right) \qquad \begin{bmatrix} \because \ [1+h] = 0 \text{ for } h \to 0^- \\ \text{and} \quad [1] = 1 \end{bmatrix}$$

$$= \lim_{h \to 0^-} \left(\frac{0-1}{h} \right) = -\lim_{h \to 0^-} \left(\frac{1}{h} \right) = \infty.$$

$$\text{R.H.D.} \quad Rf'(1) = \lim_{h \to 0^+} \left[\frac{f(1+h)-f(1)}{h} \right] = \lim_{h \to 0^+} \left[\frac{[1+h]-[1]}{h} \right]$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} \qquad \begin{bmatrix} \because \ [1+h] = 1 \text{ for } h \to 0^+ \\ [1] = 1 \end{bmatrix}$$

$$= \lim_{h \to 0^+} \left(\frac{0}{h}\right) = 0.$$

$\therefore \qquad Lf'(1) \neq Rf'(1).$

$\Rightarrow$ $f'(1)$ does not exist.

$\Rightarrow$ $f(x)$ is not differentiable at $x = 1$.

Example 6. *Show that the function f(x) defined by :*

$$f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

is continuous but not derivable at x = 0.

Solution. We have,

$$f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Continuity at x = 0 :

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -h \tan^{-1}\left(\frac{1}{-h}\right) = \lim_{h \to 0} h \tan^{-1}\left(\frac{1}{h}\right)$$

$$= 0 \times \tan^{-1} \infty = 0 \times \frac{\pi}{2} = 0.$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h \tan^{-1}\left(\frac{1}{h}\right)$$

$$= 0 \times \tan^{-1} \infty = 0 \times \frac{\pi}{2} = 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Derivability at x = 0 :

$$\text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \left(\frac{f(0+h) - f(0)}{h}\right) = \lim_{h \to 0^-} \left(\frac{h \tan^{-1}\left(\frac{1}{h}\right) - 0}{h}\right)$$

$$= \lim_{h \to 0^-} \left(\tan^{-1} \frac{1}{h}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}.$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \left(\frac{f(0+h) - f(0)}{h} \right)$$

$$= \lim_{h \to 0^+} \left(\frac{h \tan^{-1}\left(\frac{1}{h}\right) - 0}{h} \right) \qquad [\because \; h \to 0^+ \;\Rightarrow\; h > 0]$$

$$= \lim_{h \to 0^+} \left(\tan^{-1} \frac{1}{h} \right) = \tan^{-1} \infty = \frac{\pi}{2}$$

$\therefore \qquad Lf'(0) \neq Rf'(0)$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Example 7. *If $f(x)$ is differentiable at $x = a$, then evaluate :*

$$\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}.$$

Solution. Since $f(x)$ is differentiable at $x = a$,

$$\therefore \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad ...(1)$$

We have, $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$

Put $x = a + h \;\Rightarrow\; h \to 0$ as $x \to a$

$$\therefore \quad \lim_{x \to a} \left(\frac{x^2 f(a) - a^2 f(x)}{x - a} \right) = \lim_{h \to 0} \left[\frac{(a+h)^2 f(a) - a^2 f(a+h)}{a + h - a} \right]$$

$$= \lim_{h \to 0} \left(\frac{(a^2 + 2ah + h^2) f(a) - a^2 f(a+h)}{h} \right)$$

$$= \lim_{h \to 0} \left[\frac{a^2 f(a) + 2ahf(a) + h^2 f(a) - a^2 f(a+h)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{hf(a)(h + 2a) - a^2 [f(a+h) - f(a)]}{h} \right]$$

$$= \lim_{h \to 0} \left[\left(\frac{hf(a)(h+2a)}{h} \right) - a^2 \left(\frac{f(a+h) - f(a)}{h} \right) \right]$$

$$= \lim_{h \to 0} f(a)(h + 2a) - a^2 \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

$$= f(a)(0 + 2a) - a^2 f'(a) \qquad [\because \text{ By using (1)}]$$

$$= 2af(a) - a^2 f'(a).$$

Example 8. *Prove that the function f(x) defined by :*

$$f(x) = \begin{Bmatrix} x^2 + 1 & ; x \le 1 \\ 2x & ; x > 1 \end{Bmatrix} \text{ is differentiable at } x = 1.$$

Solution. We have,

$$f(x) = \begin{Bmatrix} x^2 + 1 & ; x \le 1 \\ 2x & ; x > 1 \end{Bmatrix}$$

$$\text{L.H.D. } Lf'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \qquad [\because \; f(x) = x^2 + 1 \text{ for } x \le 1]$$

$$= \lim_{h \to 0^-} \left[\frac{[(1+h)^2 + 1] - ((1)^2 + 1)}{h}\right] = \lim_{h \to 0^-} \left(\frac{1 + h^2 + 2h + 1 - 2}{h}\right)$$

$$= \lim_{h \to 0^-} \left(\frac{h(h+2)}{h}\right) = \lim_{h \to 0^-} (h + 2) = 2.$$

$$\text{R.H.D. } Rf'(1) = \lim_{h \to 0^+} \left(\frac{f(1+h) - f(1)}{h}\right) \qquad [\because \; f(x) = 2x \text{ for } x > 1]$$

$$= \lim_{h \to 0^+} \left(\frac{2(1+h) - 2(1)}{h}\right) = \lim_{h \to 0^+} \left(\frac{2 + 2h - 2}{h}\right)$$

$$= \lim_{h \to 0^+} \left(\frac{2h}{h}\right) = \lim_{h \to 0} (2) = 2.$$

$\therefore \qquad Lf'(1) = Rf'(1)$

$\therefore$ $f(x)$ is differentiable at $x = 1$.

Example 9. *Show that the function f(x) defined by :*

$$f(x) = \begin{Bmatrix} x^2 \sin\left(\frac{1}{x}\right) & ; x \ne 0 \\ 0 & ; x = 0 \end{Bmatrix}$$

is continuous and differentiable at x = 0.

Solution. We have,

$$f(x) = \begin{Bmatrix} x^2 \sin\left(\frac{1}{x}\right) & ; x \ne 0 \\ 0 & ; x = 0 \end{Bmatrix}$$

Continuity at x = 0 :

$$\therefore \qquad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h)^2 \sin\left(\frac{1}{-h}\right) = \lim_{h \to 0} h^2 \sin\left(\frac{1}{-h}\right)$$

$$= \lim_{h \to 0} - h^2 \sin\left(\frac{1}{h}\right) \qquad [\because \; \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0$

And, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} h^2 \sin\left(\frac{1}{h}\right)$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0$

Also, $f(0) = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Derivability at x = 0 :

$$\text{L.H.D. } Lf'(0) = \lim_{h \to 0^-}\left(\frac{f(0+h) - f(0)}{h}\right) = \lim_{h \to 0^-}\left(\frac{h^2 \sin \frac{1}{h} - 0}{h}\right) = \lim_{h \to 0^-}\left(h \sin \frac{1}{h}\right)$$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0.$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+}\left(\frac{f(0+h) - f(0)}{h}\right)$$

$$= \lim_{h \to 0^+}\left(\frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}\right) = \lim_{h \to 0^+}\left(h \sin\left(\frac{1}{h}\right)\right)$$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0$

$\therefore \quad Lf'(0) = Rf'(0)$

$\Rightarrow$ $f'(0)$ exist.

$\Rightarrow$ $f(x)$ is derivable at $x = 0$.

Hence, the given function $f(x)$ is continuous and differentiable at $x = 0$.

Example 10. *Show that the function $f(x) = |x - 1| + |x - 2|$ is not derivable at $x = 2$.*

Solution. We have, $f(x) = |x - 1| + |x - 2|$

$$\text{L.H.D. } Lf'(2) = \lim_{h \to 0^-}\left(\frac{f(2+h) - f(2)}{h}\right)$$

$$= \lim_{h \to 0^-}\left(\frac{(|2+h-1| + |2+h-2|) - (|2-1| + |2-2|)}{h}\right)$$

$$= \lim_{h \to 0^-}\left(\frac{|1+h| + |h| - |1| - 0}{h}\right)$$

$$= \lim_{h \to 0^-}\left(\frac{1 + h + (-h) - 1}{h}\right) \qquad [\because \ |h| = -h \text{ for } h < 0]$$

$$= \lim_{h \to 0^-}\left(\frac{0}{h}\right) = 0.$$

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$$= \lim_{h \to 0^+} \left[\frac{(0+h)\, e^{-\left(\frac{1}{|0+h|} + \frac{1}{(0+h)}\right)} - 0}{h} \right] = \lim_{h \to 0^+} \left[\frac{h\, e^{-\left(\frac{1}{|h|} + \frac{1}{h}\right)}}{h} \right]$$

$$= \lim_{h \to 0^+} \left[e^{-\left(\frac{1}{|h|} + \frac{1}{h}\right)} \right] = \lim_{h \to 0^+} \left[e^{-\left(\frac{1}{h} + \frac{1}{h}\right)} \right] \qquad [\because \ |h| = h \text{ for } h > 0]$$

$$= \lim_{h \to 0^+} e^{-2/h} = \lim_{h \to 0^+} \frac{1}{e^{2/h}}$$

$$= \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$\therefore$ $Lf'(0) \neq Rf'(0)$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not differentiable at $x = 0$.

Example 15. *Find the values of a and b, so that the function f(x) defined by :*

$$f(x) = \begin{cases} x^2 + 3x + a & ; x \le 1 \\ bx + 2 & ; x > 1 \end{cases}$$

is differentiable at each $x \in R$.

Solution. We have, $f(x) = \begin{cases} x^2 + 3x + a & ; x \le 1 \\ bx + 2 & ; x > 1 \end{cases}$

As it is given that, $f(x)$ is differentiable for all x.

$\therefore$ $f(x)$ is derivable at $x = 1$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$.

$\Rightarrow$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$.

$\Rightarrow$ $\lim_{x \to 1^-} (x^2 + 3x + a) = \lim_{x \to 1^+} (bx + 2) = 1 + 3 + a$ $\quad \left[\begin{array}{l} \because \ f(x) = x^2 + 3x + a \text{ for } x \le 1 \\ \text{and } f(x) = bx + 2 \text{ for } x > 1 \end{array}\right]$

$\Rightarrow$ $1 + 3 + a = b + 2 = 1 + 3 + a$

$\Rightarrow$ $4 + a = b + 2$ ...(1)

$\Rightarrow$ $a - b + 2 = 0$.

Now, $f(x)$ is differentiable at $x = 1$.

$\Rightarrow$ $Lf'(1) = Rf'(1)$ ...(2)

$\therefore$ L.H.D. $Lf'(1) = \lim_{h \to 0^-} \left(\frac{f(1+h) - f(1)}{h} \right)$

$$= \lim_{h \to 0^-} \left[\frac{[(1+h)^2 + 3(1+h) + a] - (1 + 3 + a)}{h} \right]$$

$$= \lim_{h \to 0^-} \left(\frac{1 + h^2 + 2h + 3 + 3h + a - 4 - a}{h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{h^2 + 5h}{h}\right) = \lim_{h \to 0^-} \left[\frac{h(h+5)}{h}\right] = \lim_{h \to 0^-} (h + 5) = 5.$$

$\Rightarrow \quad Lf'(1) = 5 \qquad ...(3)$

$$\text{R.H.D. } Rf'(1) = \lim_{h \to 0^+} \left(\frac{f(1+h) - f(1)}{h}\right)$$

$$= \lim_{h \to 0^+} \left(\frac{b(1+h) + 2 - (a+4)}{h}\right)$$

$$= \lim_{h \to 0^+} \left[\frac{b(1+h) + 2 - (b+2)}{h}\right] \qquad [\because \text{ By using (1)}]$$

$$= \lim_{h \to 0^+} \left(\frac{b + bh + 2 - b - 2}{h}\right)$$

$$= \lim_{h \to 0^+} \left(\frac{bh}{h}\right) = \lim_{h \to 0^+} b = b$$

$\Rightarrow \quad Rf'(1) = b \qquad ...(4)$

From equations, (2), (3) and (4), we get

$$\boxed{b = 5}$$

From equation (1), $4 + a = 5 + 2 \quad \Rightarrow \quad a = 7 - 4 = 3$

$$\Rightarrow \quad \boxed{a = 3}$$

Hence, $\quad a = 3$ and $b = 5$.

Example 16. *Discuss the continuity and derivability of*

$$f(x) = \begin{cases} x \cos \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \text{ at } x = 0.$$

Solution. We have, $f(x) = \begin{cases} x \cos \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Continuity at x = 0 :

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0^-} f(0 - h) = \lim_{h \to 0^-} f(-h)$$

$$= \lim_{h \to 0} -h \cos\left(\frac{1}{-h}\right) = \lim_{h \to 0} -h \cos\left(\frac{1}{h}\right) \qquad [\because \cos(-\theta) = \cos\theta]$$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0.$

And, $\quad \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \left(h \cos \frac{1}{h}\right)$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0.$

Also, $\quad f(0) = 0$

$\therefore \qquad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0.$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Derivability at x = 0 :

We know that, $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(0)}{h}$$

$$\Rightarrow \qquad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{h \cos \frac{1}{h} - 0}{h} \right) = \lim_{h \to 0} \left(\cos \frac{1}{h} \right)$$

which does not exist, because $\lim_{h \to 0} \left(\cos \frac{1}{h} \right)$ oscillates between -1 and $+1$.

$\therefore$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Hence, the function $f(x)$ is continuous at $x = 0$, but not differentiable at $x = 0$.

Example 17. *Show that the function :*

$$f(x) = \begin{cases} x^2 \cdot \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

is differentiable at $x = 0$.

Solution. Please try yourself.

[**Hint :** See Example 13.]

Example 18. *Show that the function $f(x) = | x - 1 | + | x + 1 |$ is not differentiable at $x = \pm 1$.*

Solution. Please try yourself.

[**Hint :** See Example 10.]

Example 19. *Discuss the differentiability of the function :*

$$f(x) = \begin{cases} x^3 + 2 & ; x \leq 1 \\ 3x & ; x > 1 \end{cases}$$

Solution. We have,

$$f(x) = \begin{cases} x^3 + 2 & ; x \leq 1 \\ 3x & ; x > 1 \end{cases}$$

Since, $f(x) = x^3 + 2$ for $x < 1$ and

$f(x) = 3x$ for $x > 1$, which being the polynomials are differentiable for all $x < 1$ as well as for all $x > 1$.

So, $f(x)$ is differentiable for all real values of x, if it is differentiable at $x = 1$.

$$\therefore \quad \text{L.H.D. } Lf'(1) = \lim_{h \to 0^-} \left(\frac{f(1+h) - f(1)}{h} \right)$$

Derivability at x = 1 :

$$\therefore \quad \text{L.H.D. } Lf'(1) = \lim_{h\to 0^-}\left[\frac{f(1+h)-f(1)}{h}\right]$$

$$= \lim_{h\to 0^-}\left[\frac{(1+h+a)-(a+1)}{h}\right] \qquad [\because \ f(x) = x + a \text{ for } x < 1]$$

$$= \lim_{h\to 0^-}\left(\frac{1+h+a-a-1}{h}\right) = \lim_{h\to 0^-}\left(\frac{h}{h}\right) = 1.$$

$$\text{R.H.D. } Rf'(1) = \lim_{h\to 0^+}\left[\frac{f(1+h)-f(1)}{h}\right]$$

$$= \lim_{h\to 0^+}\left[\frac{a(1+h)^2+1-(a+1)}{h}\right] \qquad [\because \ f(x) = ax^2 + 1 \text{ for } x \geq 1]$$

$$= \lim_{h\to 0^+}\left[\frac{a(1+h^2+2h)+1-a-1}{h}\right]$$

$$= \lim_{h\to 0^+}\left[\frac{a+ah^2+2ah-a}{h}\right] = \lim_{h\to 0^+}\left(\frac{ah^2+2ah}{h}\right)$$

$$= \lim_{h\to 0^+}\left[\frac{h(ah+2a)}{h}\right] = \lim_{h\to 0^+}(ah+2a) = 2a$$

For differentiability of $f(x)$ at $x = 1$, we must have,

$$Lf'(1) = Rf'(1) \ \Rightarrow \ 1 = 2a \ \Rightarrow \ a = \frac{1}{2}$$

$\therefore$ $f(x)$ is differentiable at $x = 1$ for $a = \frac{1}{2}$.

Example 22. *Show that the function $f(x) = |x-2|$ is continuous but not differentiable at $x = 2$.*

Solution. We have, $f(x) = |x-2|$

Continuity at x = 2 :

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} |2-h-2|$$

$$= \lim_{h\to 0} |-h| = \lim_{h\to 0} h = 0.$$

$$\text{And,} \quad \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} |2+h-2|$$

$$= \lim_{h\to 0} |h| = \lim_{h\to 0} h = 0.$$

$$\text{Also,} \quad f(2) = |2-2| = 0$$

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 0.$$

$\therefore$ $f(x)$ is continuous at $x = 2$.

Derivability at x = 2 :

$$\therefore \quad \text{L.H.D. } Lf'(2) = \lim_{h \to 0^-} \left[\frac{f(2+h) - f(2)}{h}\right] = \lim_{h \to 0^-} \left[\frac{|2+h-2| - |2-2|}{h}\right]$$

$$= \lim_{h \to 0^-} \left[\frac{|h| - 0}{h}\right] \qquad [\because \ |h| = -h \text{ for } h < 0]$$

$$= \lim_{h \to 0^-} \left(\frac{-h}{h}\right) = \lim_{h \to 0^-} (-1) = -1.$$

$$\text{R.H.D. } Rf'(2) = \lim_{h \to 0^+} \left[\frac{f(2+h) - f(2)}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{|2+h-2| - |2-2|}{h}\right] = \lim_{h \to 0^+} \left[\frac{|h| - 0}{h}\right]$$

$$= \lim_{h \to 0^+} \left(\frac{h}{h}\right) = \lim_{h \to 0^+} (1) = 1 \qquad [\because \ |h| = h \text{ for } h > 0]$$

$$\therefore \qquad Lf'(2) \neq Rf'(2)$$

$\Rightarrow$ $f'(2)$ does not exist.

$\Rightarrow$ $f(x)$ is not differentiable at $x = 2$.

Hence, the function $f(x)$ is continuous at $x = 2$, but not differentiable at $x = 2$.

Example 23. *If $f(2) = 4$ and $f'(2) = 1$, then find the value of* $\lim_{x \to 2} \left[\frac{xf(2) - 2f(x)}{x - 2}\right]$.

Solution. We have, $\lim_{x \to 2} \left[\frac{xf(2) - 2f(x)}{x - 2}\right]$

Put $x = h + 2 \ \Rightarrow \ h \to 0$ as $x \to 2$

$$\therefore \quad \lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2} = \lim_{h \to 0} \left[\frac{(2+h)f(2) - 2f(2+h)}{2 + h - 2}\right]$$

$$= \lim_{h \to 0} \left[\frac{2f(2) + hf(2) - 2f(2+h)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{hf(2) - 2[f(2+h) - f(2)]}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{hf(2)}{h} - \frac{2[f(2+h) - f(2)]}{h}\right]$$

$$= \lim_{h \to 0} f(2) - 2 \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= f(2) - 2f'(2)$$

$$= 4 - 2(1) = 2. \qquad [\because \text{ It is given that : } f(2) = 4 \text{ and } f'(2) = 1]$$

Example 24. *Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.*

Solution. We have, $f(x) = [x]$.

Continuity at x = 2 :

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2-h] = \lim_{h\to 0} 1 = 1.$$

And, $$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = \lim_{h\to 0} 2 = 2.$$

$$\therefore \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

$\therefore$ $\lim_{x\to 2} f(x)$ does not exist.

$\therefore$ $f(x)$ is not continuous at $x = 2$.

Derivability at x = 2 :

$$\therefore \quad \text{L.H.D. } Lf'(2) = \lim_{h\to 0^-}\left[\frac{f(2+h)-f(2)}{h}\right] = \lim_{h\to 0^-}\left(\frac{[2+h]-[2]}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{[2+h]-2}{h}\right) = \lim_{h\to 0^-}\left(\frac{1-2}{h}\right) \qquad [\because \quad [2+h] = 1 \text{ for } h < 0]$$

$$= -\lim_{h\to 0^-}\left(\frac{1}{h}\right) = \infty.$$

$$\text{R.H.D. } Rf'(2) = \lim_{h\to 0^+}\left[\frac{f(2+h)-f(2)}{h}\right]$$

$$= \lim_{h\to 0^+}\left(\frac{[2+h]-[2]}{h}\right) = \lim_{h\to 0^+}\left(\frac{[2+h]-2}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{2-2}{h}\right) = \lim_{h\to 0^+}\left(\frac{0}{h}\right) = 0 \qquad [\because \quad [2+h] = 2 \text{ for } h > 0]$$

$$\therefore \quad Lf'(2) \neq Rf'(2)$$

$\Rightarrow$ $f'(2)$ does not exist.

$\Rightarrow$ $f(x)$ is not differentiable at $x = 2$.

Hence, the function $f(x) = [x]$ is neither continuous nor differentiable at $x = 2$.

Example 25. *Show that the function f(x) defined by :*

$$f(x) = \begin{Bmatrix} x \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{Bmatrix}$$

is continuous but not differentiable at $x = 0$.

Solution. Please try yourself.

[**Hint :** See Example 9.]

Example 26. *For what choices of a, b and c, if any, does the function :*

$$f(x) = \begin{cases} ax^2 + bx + c & ; 0 \le x \le 1 \\ bx - c & ; 1 < x \le 2 \\ c & ; x > 2 \end{cases}$$

is differentiable at x = 1 and x = 2.

Solution. We have,

$$f(x) = \begin{cases} ax^2 + bx + c & ; 0 \le x \le 1 \\ bx - c & ; 1 < x \le 2 \\ c & ; x > 2 \end{cases}$$

Differentiability at x = 1 :

$\therefore$ L.H.D. $Lf'(1) = \lim_{h \to 0^-} \left[\frac{f(1+h) - f(1)}{h}\right]$ [$\because$ $f(x) = ax^2 + bx + c$ for $0 \le x \le 1$]

$$= \lim_{h \to 0^-} \left[\frac{\{a(1+h)^2 + b(1+h) + c\} - (a \cdot 1^2 + b \cdot 1 + c)}{h}\right]$$

$$= \lim_{h \to 0^-} \left(\frac{a(1+h^2+2h) + b + bh + c - a - b - c}{h}\right)$$

$$= \lim_{h \to 0^-} \left(\frac{a + ah^2 + 2ah + bh - a}{h}\right) = \lim_{h \to 0^-} \left[\frac{h(ah + 2a + b)}{h}\right]$$

$$= \lim_{h \to 0^-} (ah + 2a + b)$$

$\Rightarrow$ $Lf'(1) = 2a + b$ ...(1)

R.H.D. $Rf'(1) = \lim_{h \to 0^+} \left[\frac{f(1+h) - f(1)}{h}\right]$

$$= \lim_{h \to 0^+} \left[\frac{(b(1+h) - c) - (a.1^2 + b.1 + c)}{h}\right]$$

[$\because$ $f(x) = bx - c$ for $1 < x \le 2$]

$$= \lim_{h \to 0^+} \left[\frac{b + bh - c - a - b - c}{h}\right]$$

$$= \lim_{h \to 0^+} \left(\frac{bh - a - 2c}{h}\right) = \lim_{h \to 0^+} \left(\frac{bh}{h} - \frac{a + 2c}{h}\right)$$

$$Rf'(1) = \lim_{h \to 0^+} \left(b - \frac{a + 2c}{h}\right)$$

Since, $f'(1)$ exists, therefore we must have

$a + 2c = 0$

And, $b = 2a + b \Rightarrow 2a = 0 \Rightarrow a = 0$

Also, $a + 2c = 0 \Rightarrow c = 0$ [$\because$ $a = 0$]

$\therefore$ $a = 0$ and $c = 0$.

$$= \lim_{h \to 0^+} \left[\frac{h|h|}{h}\right] \qquad [\because \ |h| = h \text{ for } h > 0]$$

$$= \lim_{h \to 0^+} |h| = \lim_{h \to 0^+} h = 0$$

$\therefore \qquad Lf'(0) = Rf'(0)$

$\Rightarrow$ $f'(1)$ exists.

$\Rightarrow$ $f(x)$ is differentiable at $x = 0$.

Example 29. *If* $f(x) = \begin{Bmatrix} 2x+3 & ; x \le 1 \\ ax^2+bx & ; x > 1 \end{Bmatrix}$ *is differentiable everywhere, then, find the values of a and b.*

Solution. Please try yourself. **[Ans.** $a = -3$ and $b = 8$].

EXERCISE FOR PRACTICE

1. Examine the derivability of the function :

$$f(x) = \begin{Bmatrix} 3-2x & ; x < 4 \\ 2x-7 & ; x \ge 4 \end{Bmatrix} \text{ at } x = 4.$$

2. Find the value of p, if the function :

$$f(x) = \begin{Bmatrix} px^2+1 & ; x \ge 1 \\ x+p & ; x < 1 \end{Bmatrix} \text{ is differentiable at } x = 1.$$

3. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

4. Let $\qquad f(x) = \begin{Bmatrix} 2+x & ; x \ge 0 \\ 2-x & ; x < 0 \end{Bmatrix}.$

   Show that $f(x)$ is not derivable at $x = 0$.

5. Show that $\qquad f(x) = \begin{Bmatrix} x^2 \sin \frac{1}{x} & ; x \ne 0 \\ 0 & ; x = 0 \end{Bmatrix}$ is derivable at $x = 0$ and $f'(0) = 0$.

6. If $f(x)$ is derivable at $x = a$, then,

   Prove that : $\qquad \lim_{x \to a} \dfrac{xf(a) - af(x)}{x - a} = f(a) - af'(a).$

7. Show that $f(x) = |x - 2|$ is continuous but not derivable at $x = 2$.

8. Find the values of a and b so that the function :

$$f(x) = \begin{Bmatrix} x^2+3x+a & ; x \le 1 \\ bx+2 & ; x > 1 \end{Bmatrix} \text{ is differentiable at each } x \in \mathbb{R}.$$

Answers

1. Not derivable $\qquad$ 2. $p = \frac{1}{2}$ $\qquad$ 8. $a = 3$ and $b = 5$.

Note. The above result can also be extended to any finite number of differentiable functions as :

Let $$f(x) = c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x)$$

where $c_1, c_2, ... c_n$ are constants.

Then, By using theorems (4), and (5), we get

$$\frac{d}{dx}[f(x)] = f'(x) = \frac{d}{dx}[c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x)]$$

$$= \frac{d}{dx}[c_1 f_1(x)] + \frac{d}{dx}[c_2 f_2(x)] + ... + \frac{d}{dx}[c_n f_n(x)]$$

$$= c_1 f_1'(x) + c_2 f_2'(x) + ... + c_n f_n'(x)$$

Thus, the derivative of a linear sum of a finite number of functions is the linear sum of their derivatives.

Theorem 6. *The derivative of the difference of two functions is equal to the difference of their derivatives.*

i.e., $$\frac{d}{dx}(u - v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

$$= \frac{du}{dx} - \frac{dv}{dx}$$

where u and v are differentiable functions of x.

Proof. Proceed as in theorem 5.

Remark. (*i*) The results of theorems (5) and (6), can be summaries as :

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

$$= \frac{du}{dx} \pm \frac{dv}{dx}$$

(*ii*) In general, if $u_1, u_2, ... u_n$ are n differentiable functions of x, then, we have

$$\frac{d}{dx}(u_1 \pm u_2 \pm ...... \pm u_n) = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm ...... \pm \frac{du_n}{dx}.$$

6.7 PRODUCT RULE OF DIFFERENTIATION

Theorem 7. *If u and v are two differentiable functions of x, then, prove that :*

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

Proof. Let $u = f(x)$ and $v = g(x)$

Then, $$y = uv \quad ...(1)$$

Let δx be a small increment in x and δu, δv, δy are the corresponding increments of u, v, y respectively.

$$\therefore \quad y + \delta y = (u + \delta u)(v + \delta v)$$

$$= uv + u\delta v + v\delta u + \delta u \delta v \quad ...(2)$$

On subtracting (1) from (2), we get

$$\delta y = u\delta v + v\delta u + \delta u \delta v$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \frac{\delta u \delta v}{\delta x}$$ [Dividing both sides by δx]

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(*ii*) Let $y = \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = x^3 + \frac{1}{x} + x + \frac{1}{x^3}$

$\Rightarrow \quad y = x^3 + x^{-1} + x + x^{-3}$

$\therefore \quad \frac{dy}{dx} = 3x^2 - 1x^{-2} + 1 - 3x^{-4} = 3x^2 - \frac{1}{x^2} + 1 - \frac{3}{x^4}.$

$\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(*iii*) Let $y = (2x + 3)^2$

$= 4x^2 + 9 + 2(2x)(3)$ $\quad [\because (a + b)^2 = a^2 + b^2 + 2ab]$

$= 4x^2 + 9 + 12x$

$\therefore \quad \frac{dy}{dx} = 8x + 12 + 0 = 8x + 12.$ $\quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(*iv*) Let $y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = x\sqrt{x} + \frac{\sqrt{x}}{x} + \frac{x}{\sqrt{x}} + \frac{1}{x\sqrt{x}}$

$= x^{3/2} + \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{1}{x^{3/2}}$

$\Rightarrow \quad y = x^{3/2} + x^{-1/2} + x^{1/2} + x^{-3/2}$

$\therefore \quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} + \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} + \frac{1}{2}x^{\frac{1}{2}-1} + \left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$

$\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

$= \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-5/2}$

$= \frac{3}{2}\sqrt{x} - \frac{1}{2x^{3/2}} + \frac{1}{2x^{1/2}} - \frac{3}{2x^{5/2}}$

$= \frac{3}{2}\sqrt{x} - \frac{1}{2x\sqrt{x}} + \frac{1}{2\sqrt{x}} - \frac{3}{2x^2\sqrt{x}}$

$= \frac{1}{2}\left(3\sqrt{x} - \frac{1}{x\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{3}{x^2\sqrt{x}}\right).$

Example 7. *Differentiate the following functions w.r.t. x*

(*i*) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (*ii*) $\left(x^2 + \frac{1}{x^2}\right)^3$

(*iii*) $\frac{2x^2 + 3x + 4}{\sqrt{x}}$.

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$$= \frac{(x^3-1).2(2+5x)(5)-(2+5x)^2(3x^2)}{(x^3-1)^2}$$

$$= \frac{10(x^3-1)(2+5x)-(2+5x)^2(3x^2)}{(x^3-1)^2}$$

$$= \frac{(2+5x)[10x^3-10-3x^2(2+5x)]}{(x^3-1)^2} = \frac{(2+5x)[10x^3-10-6x^2-15x^3]}{(x^3-1)^2}$$

$$= \frac{(2+5x)(-5x^3-6x^2-10)}{(x^3-1)^2} = -\frac{(2+5x)(5x^3+6x^2+10)}{(x^3-1)^2}.$$

(*ii*) Let $y = \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

$\therefore$ $$\frac{dy}{dx} = \frac{(\sqrt{a}-\sqrt{x})\frac{d}{dx}(\sqrt{a}+\sqrt{x})-(\sqrt{a}+\sqrt{x})\frac{d}{dx}(\sqrt{a}-\sqrt{x})^{**}}{(\sqrt{a}-\sqrt{x})^2}$$

[By using Quotient Rule]

$$= \frac{(\sqrt{a}-\sqrt{x})\left(0+\frac{1}{2\sqrt{x}}\right)-(\sqrt{a}+\sqrt{x})\left(0-\frac{1}{2\sqrt{x}}\right)}{(\sqrt{a}-\sqrt{x})^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}\left[(\sqrt{a}-\sqrt{x})+(\sqrt{a}+\sqrt{x})\right]}{(\sqrt{a}-\sqrt{x})^2} = \frac{\frac{1}{2\sqrt{x}}.2\sqrt{a}}{(\sqrt{a}-\sqrt{x})^2} = \frac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}.$$

(*iii*) Let $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$

$\therefore$ $$\frac{dy}{dx} = \frac{(1+\sqrt{x})\frac{d}{dx}(1-\sqrt{x})-(1-\sqrt{x})\frac{d}{dx}(1+\sqrt{x})}{(1+\sqrt{x})^2}$$ [By using Quotient Rule]

$$= \frac{(1+\sqrt{x}).\left(0-\frac{1}{2\sqrt{x}}\right)-(1-\sqrt{x})\left(0+\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2}$$

****Remember.** $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$\therefore$ $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

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$$= n\left(x+\sqrt{x^2+a^2}\right)^{n-1}\cdot\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{n\left(x+\sqrt{x^2+a^2}\right)^n}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad \sqrt{x^2+a^2}\,\frac{dy}{dx}=n\left(x+\sqrt{x^2+a^2}\right)^n$$

$$\Rightarrow \quad \sqrt{x^2+a^2}\,\frac{dy}{dx}=ny. \qquad \left[\because\ y=\left(x+\sqrt{x^2+a^2}\right)^n\right]$$

(*iii*) Let $\qquad y=\left(x+\sqrt{x^2+a^2}\right)^{10}$

Please try yourself. **[Ans.** $\sqrt{x^2+a^2}\,\dfrac{dy}{dx}=10y$]

[**Hint.** Put $n = 10$ in the part (*ii*) of the same example].

(*iv*) Let $\qquad y=\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}=\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}$

$$\therefore \qquad \frac{dy}{dx}=\frac{\sqrt{a^2-x^2}\cdot\frac{d}{dx}\left(\sqrt{a^2+x^2}\right)-\left(\sqrt{a^2+x^2}\right)\cdot\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)}{\left(\sqrt{a^2-x^2}\right)^2}$$

[By using Quotient Rule]

$$=\frac{\sqrt{a^2-x^2}\cdot\frac{1}{2\sqrt{a^2+x^2}}\cdot\frac{d}{dx}(a^2+x^2)-\sqrt{a^2+x^2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\cdot\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)}$$

$$=\frac{\frac{\sqrt{a^2-x^2}}{2\sqrt{a^2+x^2}}(0+2x)-\frac{\sqrt{a^2+x^2}}{2\sqrt{a^2-x^2}}(0-2x)}{(a^2-x^2)}=\frac{\frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}+\frac{x\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}}{(a^2-x^2)}$$

$$=\frac{\frac{x(a^2-x^2)+x(a^2+x^2)}{\sqrt{a^2+x^2}\,\sqrt{a^2-x^2}}}{(a^2-x^2)}=\frac{2a^2x}{\sqrt{a^2+x^2}\,\sqrt{a^2-x^2}\,(a^2-x^2)}$$

$$=\frac{2a^2x}{\sqrt{a^2+x^2}\,.(a^2-x^2)^{3/2}}.$$

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Proof. Let $y = \tan x$ ...(1)

Let δx be a small increment in x and δy be the corresponding increment in y.

$\therefore \quad y + \delta y = \tan (x + \delta x)$ ...(2)

On subtracting (1) from (2), we get

$$\delta y = \tan (x + \delta x) - \tan x$$

$$= \frac{\sin (x + \delta x)}{\cos (x + \delta x)} - \frac{\sin x}{\cos x} = \frac{\sin (x + \delta x) \cos x - \cos (x + \delta x) \sin x}{\cos (x + \delta x) \cos x}$$

$$= \frac{\sin (x + \delta x - x)}{\cos (x + \delta x) \cos x} \qquad [\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$$

$$\Rightarrow \quad \delta y = \frac{\sin \delta x}{\cos (x + \delta x) \cos x}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos (x + \delta x) . \cos x} \qquad \text{[Dividing both sides by } \delta x]$$

Proceeding to the limits as $\delta x \to 0$:

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos (x + \delta x) . \cos x} \right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\sin \delta x}{\delta x} \right) \cdot \lim_{\delta x \to 0} \frac{1}{\cos (x + \delta x) . \cos x}$$

$$\therefore \quad \frac{dy}{dx} = 1 \cdot \frac{1}{\cos x . \cos x} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

Hence, $\frac{d}{dx} (\tan x) = \sec^2 x$.

7.2.4. Theorem. *By using definition, prove that :*

$$\frac{d}{dx} (\cot x) = - \text{cosec}^2 x.$$

Proof. Let $y = \cot x$ ...(1)

Let δx be a small increment in x and δy be the corresponding increment in y.

$\therefore \quad y + \delta y = \cot (x + \delta x)$ ...(2)

On subtracting (1) from (2), we get

$$\delta y = \cot (x + \delta x) - \cot x$$

$$= \frac{\cos (x + \delta x)}{\sin (x + \delta x)} - \frac{\cos x}{\sin x} = \frac{\sin x \cos (x + \delta x) - \cos x \sin (x + \delta x)}{\sin (x + \delta x) \sin x}$$

$$= \frac{\sin [x - (x + \delta x)]}{\sin (x + \delta x) \sin x} \qquad [\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$$

$$\Rightarrow \quad \delta y = \frac{\sin (-\delta x)}{\sin (x + \delta x) \sin x} = \frac{-\sin \delta x}{\sin (x + \delta x) \sin x} \qquad [\because \sin (-\theta) = -\sin \theta]$$

Proof. Let $y = \cot^{-1} x$

$\Rightarrow \quad x = \cot y \quad ...(1)$

Let δy be a small increment in y and δx be the corresponding increment in x.

$\therefore \quad x + \delta x = \cot (y + \delta y) \quad ...(2)$

On subtracting (1) from (2), we have

$$\delta x = \cot (y + \delta y) - \cot y = \frac{\cos (y+\delta y)}{\sin (y+\delta y)} - \frac{\cos y}{\sin y}$$

$$= \frac{\sin y \cos (y+\delta y) - \cos y \sin (y+\delta y)}{\sin (y+\delta y) \sin y}$$

$[\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{\sin [y - (y+\delta y)]}{\sin (y+\delta y) \sin y} = \frac{\sin (-\delta y)}{\sin (y+\delta y) \sin y}$$

$$\Rightarrow \quad \delta x = \frac{-\sin \delta y}{\sin (y+\delta y) \sin y} \quad [\because \sin (-\theta) = -\sin \theta]$$

$$\therefore \quad \frac{\delta x}{\delta y} = \frac{-1}{\sin (y+\delta y) \sin y} \cdot \frac{\sin \delta y}{\delta y} \quad \text{[Dividing both sides by } \delta y]$$

Proceeding to the limit, as $\delta y \to 0$, we have

$$\lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \lim_{\delta y \to 0} \left[\frac{-1}{\sin (y+\delta y) \sin y} \cdot \frac{\sin \delta y}{\delta y} \right]$$

$$\Rightarrow \quad \frac{dx}{dy} = - \lim_{\delta y \to 0} \frac{1}{\sin (y+\delta y) \sin y} \cdot \lim_{\delta y \to 0} \frac{\sin \delta y}{\delta y}$$

$$= \frac{-1}{\sin y . \sin y} . 1 \quad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-1}{\sin^2 y} = -\operatorname{cosec}^2 y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$$

$$= \frac{-1}{1 + \cot^2 y} \quad [\because \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{1+x^2} \quad [\because x = \cot y]$$

Hence, $\dfrac{d}{dx} (\cot^{-1} x) = \dfrac{-1}{1+x^2}$.

7.3.5. Theorem. *By using the first principles ; prove that :*

$$\frac{d}{dx} (\sec^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}} \; ; \; x \in R - (-1, 1).$$

Proof. Let $y = \sec^{-1} x$

$\Rightarrow \quad x = \sec y \quad ...(1)$

$$\delta y = 2\cos(2x + 1 + \delta x)\sin\delta x$$

$\therefore$ $$\frac{\delta y}{\delta x} = \frac{2\cos(2x+1+\delta x)\sin\delta x}{\delta x}$$ [Dividing both sides by δx]

Proceeding to the limits as $\delta x \to 0$

$\therefore$ $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[2\cos(2x+1+\delta x)\,.\,\frac{\sin\delta x}{\delta x}\right]$$

$\Rightarrow$ $$\frac{dy}{dx} = 2\lim_{\delta x \to 0}\cos(2x+1+\delta x)\,.\,\lim_{\delta x \to 0}\left(\frac{\sin\delta x}{\delta x}\right)$$

$$= 2\cos(2x + 1 + 0) \times 1$$ $\left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$

$\therefore$ $$\frac{dy}{dx} = 2\cos(2x + 1)$$

Hence, $\frac{d}{dx}(\sin(2x + 1)) = 2\cos(2x + 1)$.

(*ii*) Let $\quad y = \cos bx$ ...(1)

Let δx be a small increment in x and δy be the corresponding increment in y.

$\therefore$ $\quad y + \delta y = \cos b(x + \delta x)$ ...(2)

On subtracting (1) from (2), we have

$$\delta y = \cos(bx + b\delta x) - \cos bx$$

$$= -2\sin\left(\frac{bx + b\delta x + bx}{2}\right)\sin\left(\frac{bx + b\delta x - bx}{2}\right)$$

$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$

$$= -2\sin\left(\frac{2bx + b\delta x}{2}\right)\sin\left(\frac{b\delta x}{2}\right)$$

$\Rightarrow$ $$\delta y = -2\sin\left(bx + \frac{b\delta x}{2}\right)\sin\left(\frac{b\delta x}{2}\right)$$

$\therefore$ $$\frac{\delta y}{\delta x} = \frac{-2\sin\left(bx + \frac{b\delta x}{2}\right)\sin\left(\frac{b\delta x}{2}\right)}{\delta x}$$ [Dividing both sides by δx]

$\Rightarrow$ $$\frac{\delta y}{\delta x} = \frac{-2\sin\left(bx + \frac{b\delta x}{2}\right)\sin\left(\frac{b\delta x}{2}\right)}{\left(\frac{b\delta x}{2}\right)} \times \left(\frac{b}{2}\right)$$

$\left[\text{Multiply and divide the R.H.S. by } \frac{b}{2}\right]$

$$= \frac{2\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x.\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}.\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

$$\left[\text{Multiply and divide the R.H.S. by}\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)\right]$$

$$= 2\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}.\frac{(\sqrt{x+\delta x}-\sqrt{x})}{2\delta x}\times\frac{(\sqrt{x+\delta x}+\sqrt{x})}{(\sqrt{x+\delta x}+\sqrt{x})}$$

[Rationalisation]

$$= \cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}.\left[\frac{x+\delta x-x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}\right]$$

$$= \cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}.\frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$$

Proceeding to the limits as $\delta x \to 0$.

$$\therefore \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}.\frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\delta x\to 0}\left[\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}\right].\lim_{\delta x\to 0}\left[\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}\right]$$

$$= \cos\left(\frac{\sqrt{x}+\sqrt{x}}{2}\right).1.\frac{1}{(\sqrt{x}+\sqrt{x})} \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

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$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{2x} - \lim_{\delta y \to 0} \sin\left(y + \frac{\delta y}{2}\right) . \lim_{\delta y \to 0}\left[\frac{\sin\left(\frac{\delta y}{2}\right)}{\left(\frac{\delta y}{2}\right)}\right]$$

$$= \frac{-1}{2x} \sin y \,.\, 1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{-\sqrt{1-\cos^2 y}}{2x} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-\sqrt{1-x^4}}{2x} \Rightarrow \frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^4}} \qquad [\because x^2 = \cos y]$$

Hence, $\dfrac{d}{dx}(\cos^{-1} x^2) = \dfrac{-2x}{\sqrt{1-x^4}}$.

Example 9. *Differentiate the following function w.r.t. x using the first principle :*

(*i*) $\tan^{-1}\sqrt{x}$ (*ii*) $x \tan^{-1} x$

(*iii*) $\dfrac{\sin^{-1} x}{x}$.

Solution. (*i*) Let $y = \tan^{-1}\sqrt{x}$

$$\Rightarrow \quad \sqrt{x} = \tan y \qquad ...(1)$$

Let δx be a small increment in x and δy be the corresponding increment in y.

$$\therefore \quad \sqrt{x+\delta x} = \tan(y+\delta y) \qquad ...(2)$$

On subtracting (1) from (2), we have

$$\sqrt{x+\delta x} - \sqrt{x} = \tan(y+\delta y) - \tan y \qquad \text{[Rationalisation]}$$

$$= \frac{(\sqrt{x+\delta x} - \sqrt{x})}{1} \times \frac{(\sqrt{x+\delta x} + \sqrt{x})}{(\sqrt{x+\delta x} + \sqrt{x})} = \frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}$$

$$\Rightarrow \quad \frac{x+\delta x - x}{(\sqrt{x+\delta x} + \sqrt{x})} = \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\cos(y+\delta y)\cos y}$$

$$\Rightarrow \quad \frac{\delta x}{(\sqrt{x+\delta x} + \sqrt{x})} = \frac{\sin(y+\delta y - y)}{\cos(y+\delta y)\cos y} \qquad [\because \sin(A-B) = \sin A\cos B - \cos A \sin B]$$

$$\Rightarrow \quad \delta x = (\sqrt{x+\delta x} + \sqrt{x}) . \frac{\sin \delta y}{\cos(y+\delta y) . \cos y}$$

$$\therefore \quad \frac{\delta x}{\delta y} = (\sqrt{x+\delta x} + \sqrt{x}) . \frac{\sin \delta y}{\delta y . \cos(y+\delta y) . \cos y}$$

[Dividing both sides by δy]

Proceeding to the limit as $\delta x \to 0$, so that $\delta y \to 0$:

$$\therefore \quad \lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \lim_{\delta x \to 0} (\sqrt{x+\delta x} + \sqrt{x}) . \lim_{\delta y \to 0}\left[\frac{\sin \delta y}{\delta y . \cos(y+\delta y) . \cos y}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-\lim\limits_{\delta z \to 0}\left[z\cos\left(z+\frac{\delta z}{2}\right)\right].\lim\limits_{\delta z \to 0}\left[\frac{\sin(\delta z/2)}{(\delta z/2)}\right]+\lim\limits_{\delta z \to 0}\sin z}{\lim\limits_{\delta z \to 0}\left[\sin(z+\delta z)\sin z.\cos\left(z+\frac{\delta z}{2}\right)\right].\lim\limits_{\delta z \to 0}\left[\frac{\sin(\delta z/2)}{(\delta z/2)}\right]}$$

$$= \frac{-z\cos z \times 1 + \sin z}{\sin z.\sin z.\cos z \times 1} \qquad \left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{-z\sqrt{1-\sin^2 z} + \sin z}{\sin^2 z.\sqrt{1-\sin^2 z}} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{-\sin^{-1}x\sqrt{1-x^2} + x}{x^2\sqrt{1-x^2}} \qquad \text{[By using equation (4)]}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{x - \sqrt{1-x^2}\sin^{-1}x}{x^2\sqrt{1-x^2}}$$

Hence, $\frac{d}{dx}\left(\frac{\sin^{-1}x}{x}\right) = \frac{x-\sqrt{1-x^2}\sin^{-1}x}{x^2\sqrt{1-x^2}}$.

Example 10. *Differentiate the following functions w.r.t. x using the first principle :*

(*i*) $\cos^{-1}(4x^3 - 3x)$ (*ii*) $\cos^{-1}(2x + 3)$

(*iii*) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (*iv*) $\sec^{-1}\left(\frac{1}{x-1}\right)$.

Solution. (*i*) Let $y = \cos^{-1}(4x^3 - 3x)$

$$\Rightarrow \qquad 4x^3 - 3x = \cos y \qquad ...(1)$$

Let δx be a small increment in x and δy be the corresponding increment in y.

$$\therefore \qquad 4(x+\delta x)^3 - 3(x+\delta x) = \cos(y+\delta y) \qquad ...(2)$$

On subtracting (1) from (2), we have

$$4[x^3 + (\delta x)^3 + 3x^2\,\delta x + 3x(\delta x)^2] - 3(x+\delta x) - (4x^3 - 3x) = \cos(y+\delta y) - \cos y$$

$$\Rightarrow \quad 4x^3 + 4(\delta x)^3 + 12x^2\,\delta x + 12x\,(\delta x)^2 - 3x - 3\delta x - 4x^3 + 3x = \cos(y+\delta y) - \cos y$$

$$\Rightarrow \quad 4(\delta x)^3 + 12x^2\,\delta x + 12x\,(\delta x)^2 - 3\delta x = -2\sin\left(\frac{y+\delta y+y}{2}\right)\sin\left(\frac{y+\delta y-y}{2}\right)$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\Rightarrow \quad \delta x[4(\delta x)^2 + 12x^2 + 12x\,\delta x - 3] = -2\sin\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right) \quad \text{[Dividing both sides by } \delta y]$$

$$\therefore \qquad \frac{\delta x}{\delta y} = \frac{1}{[4(\delta x)^2 + 12x^2 + 12x\,\delta x - 3]}.\frac{-2\sin\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\delta y}$$

Proceeding to the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{\left[\dfrac{-\log x \sin\left(x + \dfrac{\delta x}{2}\right) \sin\left(\dfrac{\delta x}{2}\right)}{\left(\dfrac{\delta x}{2}\right)} \right] - \left\{ \dfrac{\cos x}{\delta x} . \log\left(1 + \dfrac{\delta x}{x}\right) \right\}}{\log(x + \delta x) . \log x} \right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\log x \lim_{\delta x \to 0} \sin\left(x + \dfrac{\delta x}{2}\right) . \lim_{\delta x \to 0} \left[\dfrac{\sin\left(\dfrac{\delta x}{2}\right)}{\left(\dfrac{\delta x}{2}\right)} \right] - \lim_{\delta x \to 0} \left[\dfrac{\cos x}{x} . \dfrac{x}{\delta x} \log\left(1 + \dfrac{\delta x}{x}\right) \right]}{\lim_{\delta x \to 0} [\log(x + \delta x) . \log x]}$$

$$\left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$[\because \ m \log n = \log n^m]$

$$= \frac{-\log x . \sin x . 1 - \dfrac{\cos x}{x} . \lim_{\delta x \to 0} \log\left(1 + \dfrac{\delta x}{x}\right)^{\frac{x}{\delta x}}}{(\log x)(\log x)}$$

$$\left[\begin{aligned} &\because \lim_{x \to 0} \log(1+x)^{1/x} \\ &= \log \lim_{x \to 0} (1+x)^{1/x} \\ &= \log e = 1 \end{aligned}\right]$$

$$= \frac{-\sin x \log x - \dfrac{\cos x}{x} \times 1}{(\log x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x \sin x \log x - \cos x}{x(\log x)^2}$$

Hence, $\dfrac{d}{dx}\left(\dfrac{\cos x}{\log x}\right) = \dfrac{-x \sin x \log x - \cos x}{x(\log x)^2}$.

(*iii*) Let $\quad y = a^{\sqrt{x}}$ ...(1)

Let δx be a small increment in x and δy be the corresponding increment in y.

$\therefore \quad y + \delta y = a^{\sqrt{x + \delta x}}$ ...(2)

On subtracting (1) from (2), we have

$$\delta y = a^{\sqrt{x+\delta x}} - a^{\sqrt{x}} = a^{\sqrt{x}} [a^{\sqrt{x+\delta x} - \sqrt{x}} - 1]$$

$$= a^{\sqrt{x}} \left[\frac{a^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \right] \times (\sqrt{x+\delta x} - \sqrt{x})$$

[Multiply and divide the R.H.S. by $(\sqrt{x+\delta x} - \sqrt{x})$]

$$\therefore \quad \frac{\delta y}{\delta x} = a^{\sqrt{x}} \left[\frac{a^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \right] \times \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x}$$

[Dividing both sides by δx]

Proceeding to the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[a^{\sqrt{x}} \left[\frac{a^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \right] \times \left(\frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x} \right) \right]$$

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$$= (\log 2)(1 + 1) \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= 2 \log 2.$$

(*iv*) We have,

$$\lim_{x \to 0} \left(\frac{e^{\tan x} - 1}{\tan x}\right) = \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right) \qquad [\text{where } y = \tan x]$$

$$= 1. \qquad \left[\because \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = 1\right]$$

Example 30. *Evaluate the following limits :*

(*i*) $\lim_{x \to 0} \frac{\sin 3x}{3^x - 1}$ (*ii*) $\lim_{x \to 0} (1 + 2x)^{1/x}$

(*iii*) $\lim_{x \to e} \left(\frac{\log x - 1}{x - e}\right)$ (*iv*) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

(*v*) $\lim_{x \to 0} \frac{\log(5 + x) - \log(5 - x)}{x}$.

Solution. (*i*) We have,

$$\lim_{x \to 0} \frac{\sin 3x}{3^x - 1} = \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3x}{3^x - 1}\right) = \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right) \frac{3}{\lim_{x \to 0} \left(\frac{3^x - 1}{x}\right)}$$

$$= (1) \cdot \frac{3}{\log 3} = \frac{3}{\log 3}. \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \ \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

(*ii*) We have,

$$\lim_{x \to 0} (1 + 2x)^{1/x} = \lim_{x \to 0} (1 + 2x)^{(1/2x)(2)}$$

$$= \lim_{x \to 0} [(1 + 2x)^{1/2x}]^2 \qquad \left[\because \lim_{x \to 0} (1 + x)^{1/x} = e\right]$$

$$= e^2.$$

(*iii*) We have, $\lim_{x \to e} \left(\frac{\log x - 1}{x - e}\right)$

Let $x = e + h \Rightarrow h \to 0$ as $x \to e$

$$\therefore \quad \lim_{x \to e} \left(\frac{\log x - 1}{x - e}\right) = \lim_{h \to 0} \left(\frac{\log(e + h) - 1}{e + h - e}\right)$$

$$= \lim_{h \to 0} \frac{\log(e + h) - \log e}{h} \qquad [\because \log e = 1]$$

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Example 43. *Evaluate the following limits :*

(*i*) $\lim\limits_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4}$ (*ii*) $\lim\limits_{x \to -\infty} e^x$

(*iii*) $\lim\limits_{x \to \infty} e^{-x}$ (*iv*) $\lim\limits_{n \to \infty} \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right)$

(*v*) $\lim\limits_{x \to \infty} \left(\frac{x+2}{x+3}\right)^{x+1}$ (*vi*) $\lim\limits_{x \to \infty} 5^x \sin\left(\frac{a}{5^x}\right)$.

Solution. (*i*) We have,

$$\lim_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x \to \infty} \left(\frac{(x+1)+5}{x+1}\right)^{x+4}$$

$$= \lim_{x \to \infty} \left(1+\frac{5}{x+1}\right)^{x+4} = \lim_{x \to \infty} \left(1+\frac{5}{x+1}\right)^{(x+1)+3}$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{(x+1)} \cdot \left(1+\frac{5}{x+1}\right)^3\right]$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{5 \cdot \left(\frac{x+1}{5}\right)} \cdot \left(1+\frac{5}{x+1}\right)^3\right]$$

$$= \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)^{\frac{x+1}{5}}\right]^5 \cdot \lim_{x \to \infty} \left[\left(1+\frac{5}{x+1}\right)\right]^3$$

$$= e^5 \cdot \left(1+\frac{5}{\infty}\right)^3 \qquad \left[\because \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e\right]$$

$$= e^5 (1+0)^3 = e^5.$$

(*ii*) We have, $\lim\limits_{x \to -\infty} e^x$

Let $x = -y \Rightarrow y \to \infty$ as $x \to -\infty$

$$\therefore \quad \lim_{x \to -\infty} e^x = \lim_{y \to \infty} e^{-y} = \lim_{y \to \infty} \left(\frac{1}{e^y}\right) = \frac{1}{\infty} = 0.$$

(*iii*) We have, $\lim\limits_{x \to \infty} e^{-x} = \lim\limits_{x \to \infty} \left(\frac{1}{e^x}\right) = \frac{1}{\infty} = 0.$

(*iv*) We have, $\lim\limits_{n \to \infty} \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right)$

Let $\frac{2\pi}{n} = \theta \Rightarrow n = \frac{2\pi}{\theta}$

$\therefore$ $f + g$ is continuous at a for all $a \in D_f \cap D_g$.

Hence, $f + g$ is continuous.

(*ii*) We have,

$$\lim_{x \to a} (f - g)(x) = \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$= f(a) - g(a) \quad [\because \text{ By using (1) and (2)}]$$

$$= (f - g)(a)$$

$\therefore$ $f - g$ is continuous at a for all $a \in D_f \cap D_g$.

Hence, $f - g$ is continuous.

(*iii*) We have,

$$\lim_{x \to a} (cf)(x) = \lim_{x \to a} c \, . \, f(x)$$

$$= c \lim_{x \to a} f(x) = cf(a) \quad [\text{By using (1)}]$$

$$= (cf)(a)$$

$\therefore$ cf is continuous at a for all $a \in D_f \cap D_g$.

Hence, cf is continuous.

(*iv*) We have,

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} [f(x) \, . \, g(x)] = \lim_{x \to a} f(x) \, . \, \lim_{x \to a} g(x)$$

$$= f(a) \, g(a) \quad [\because \text{ By using (1) and (2)}]$$

$$= (fg)(a)$$

$\therefore$ fg is continuous at a for all $a \in D_f \cap D_g$

Hence, fg is continuous.

(*v*) We have,

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} \quad [\because \text{ By using (1) and (2)}]$$

$$= \left(\frac{f}{g}\right)(a)$$

$\therefore$ $\dfrac{f}{g}$ is continuous at a for all $a \in D$.

where domain $\left(\dfrac{f}{g}\right) = [(D_f \cap D_g) - \{x : g(x) = 0\}] = D(\text{say})$.

Hence, $\left(\dfrac{f}{g}\right)$ is continuous.

(*vi*) We have,

$$\lim_{x \to a} \left(\frac{1}{f}\right)(x) = \lim_{x \to a} \left[\frac{1}{f(x)}\right] = \frac{1}{\lim_{x \to a} f(x)}$$

$$= \frac{1}{f(a)} \quad [\because \text{ By using (1)}]$$

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Example 7. (*i*) *For what value of k, is the following function continuous at x = 0.*

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}.$$

(*ii*) *Find the point of discontinuity of the function*

$$f(x) = \begin{cases} \dfrac{x^4-16}{x-2} & ; \text{if } x \neq 2 \\ 16 & ; \text{if } x = 2 \end{cases}.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$$

$$\therefore \quad \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1-\cos 4x}{8x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 2x}{8x^2}\right) \qquad \left[\begin{array}{l} \because \ 1-\cos 2A = 2\sin^2 A \\ \Rightarrow \ 1-\cos 4A = 2\sin^2 2A \end{array}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 2x}{4x^2}\right) = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = (1)^2 = 1.$$

Since, the function $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad 1 = k \qquad [\because \ f(0) = k]$$

$\therefore$ The value of k for which the given function is continuous is $k = 1$.

(*ii*) We have,
$$f(x) = \begin{cases} \dfrac{x^4-16}{x-2} & ; \text{if } x \neq 2 \\ 16 & ; \text{if } x = 2 \end{cases}$$

The function $f(x) = \dfrac{x^4-16}{x-2}$ being a rational function, it is continuous at all points of its domain, *i.e.*, for all real numbers except 2.

$\therefore$ Consider the given function at $x = 2$.

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{x^4-16}{x-2}\right) = \lim_{x\to 2^-}\frac{(x^2+4)(x^2-4)}{(x-2)}$$

$$= \lim_{x\to 2^-}\frac{(x^2+4)(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2^-}(x^2+4)(x+2)$$

$$= \lim_{h\to 0}[(2-h)^2+4][2-h+2] \qquad [\text{Put } x = (2-h)]$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left(\frac{e^{\frac{1}{-h}} - 1}{e^{\frac{1}{-h}} + 1} \right) = \lim_{h \to 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1} \right)$$

$$= \frac{0 - 1}{0 + 1} = -1. \qquad \left[\because\ e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \right]$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1} \right) \qquad \left[\begin{array}{l} \text{Dividing the numerator} \\ \text{and denominator by } e^{1/h} \end{array} \right]$$

$$= \lim_{h \to 0} \left(\frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} \right) = \frac{1 - 0}{1 + 0} = 1$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

(*ii*) We have,

$$f(x) = \begin{cases} \dfrac{1}{2} - x & ; & 0 \le x < \dfrac{1}{2} \\ 1 & ; & x = \dfrac{1}{2} \\ \dfrac{3}{2} - x & ; & \dfrac{1}{2} < x \le 1 \end{cases}$$

$\therefore$ $$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h \right) \right] \qquad \left[\because\ f(x) = \left(\frac{1}{2} - x \right) \text{ for } 0 \le x < \frac{1}{2} \right]$$

$$= \lim_{h \to 0} (h) = 0$$

And, $$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

Example 19. (*i*) *Discuss the continuity of the following functions at $x = 0$.*

$$f(x) = \begin{Bmatrix} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{Bmatrix}$$

(*ii*) $$f(x) = \begin{Bmatrix} \frac{|\sin x|}{x} & ; x \neq 0 \\ 1 & ; x = 0 \end{Bmatrix}.$$

Solution. (*i*) We have,

$$f(x) = \begin{Bmatrix} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{Bmatrix}$$

$\therefore$ $$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

$$= 0 \times \text{(a finite quantity between } -1 \text{ and } 1)$$

$$= 0$$

Also, $f(0) = 0.$

$\therefore$ $$\lim_{x \to 0} f(x) = f(0)$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

(*ii*) We have, $$f(x) = \begin{Bmatrix} \frac{|\sin x|}{x} & ; \text{if } x \neq 0 \\ 1 & ; \text{if } x = 0 \end{Bmatrix}$$

$\Rightarrow$ $$f(x) = \begin{Bmatrix} \frac{\sin x}{x} & ; \text{if } x > 0 \\ 1 & ; \text{if } x = 0 \\ -\frac{\sin x}{x} & ; \text{if } x < 0 \end{Bmatrix}$$

$\therefore$ $$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left[-\frac{\sin(-h)}{(-h)} \right] = \lim_{h \to 0} \left(-\frac{\sin h}{h} \right) \qquad [\because \sin(-\theta) = -\sin \theta]$$

$$= -\lim_{h \to 0} \left(\frac{\sin h}{h} \right) = -1$$

Also, $f(0) = 1$

$\therefore$ $$\lim_{x \to 0^-} f(x) \neq f(0)$$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

Solution. (*i*) We have,

$$f(x) = \begin{cases} \dfrac{\sin 3x}{\tan 2x} & ; x < 0 \\ \dfrac{3}{2} & ; x = 0 \\ \dfrac{\log(1+3x)}{e^{2x}-1} & ; x > 0 \end{cases}.$$

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin 3x}{\tan 2x}\right) = \lim_{x\to 0^-}\left[\frac{\sin 3x}{3x}\cdot 3x\cdot\frac{2x}{\tan 2x}\cdot\frac{1}{2x}\right]$$

$$= \lim_{x\to 0^-}\left(\frac{\sin 3x}{3x}\right)\cdot\frac{3}{2}\cdot\lim_{x\to 0}\left(\frac{2x}{\tan 2x}\right) = (1)\cdot\left(\frac{3}{2}\right)(1) = \frac{3}{2}$$

And, $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{\log(1+3x)}{e^{2x}-1}\right]$$

$$= \lim_{x\to 0^+}\left[\frac{\log(1+3x)}{3x}\cdot 3x\cdot\frac{2x}{e^{2x}-1}\cdot\frac{1}{2x}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\log(1+3x)}{3x}\right)\cdot\frac{3}{2}\lim_{x\to 0^+}\left(\frac{2x}{e^{2x}-1}\right) = (1)\cdot\left(\frac{3}{2}\right)\cdot(1) = \frac{3}{2}$$

Also, $$f(0) = \frac{3}{2}.$$

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = \frac{3}{2}$$

Hence, $f(x)$ is continuous at $x = 0$.

(*ii*) We have, $$f(x) = \begin{cases} \sin x & ; x < 0 \\ x & ; x \geq 0 \end{cases}$$

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0}\sin(-h) = \lim_{h\to 0}(-\sin h) = -\lim_{h\to 0}(\sin h) = 0.$$

And, $$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}(h) = 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

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$$= \lim_{h\to 0}\left(\frac{nh + \frac{n(n-1)}{2}h^2 + \ldots\ldots}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{h\left(n + \frac{n(n-1)}{2}h + \ldots\ldots\right)}{h}\right] = n.$$

Also, $f(1) = (n-1)$

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\therefore$ $\lim_{x\to 1} f(x)$ exist.

$\therefore$ $\lim_{x\to 1} f(x) \neq f(1)$

Hence, $f(x)$ is not continuous at $x = 1$.

Example 37. (*i*) *Find the value of a so that the function f(x) defined by :*

$$f(x) = \begin{cases} \frac{\sin^2 ax}{x^2} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$$

may be continuous at x = 0.

(*ii*) *Discuss the continuity of the function :*

$$f(x) = \begin{cases} \frac{\sin 2x}{x} & ; x < 0 \\ x + 2 & ; x \geq 0 \end{cases} \text{ at } x = 0.$$

Solution. (*i*) We have,

$$f(x) = \begin{cases} \frac{\sin^2 ax}{x^2} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$$

$\therefore$ $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin^2 ax}{x^2}\right) = \lim_{x\to 0}\frac{\sin^2 ax}{a^2x^2} \times a^2$$

$$= a^2 \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)^2 \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= a^2 \times 1 = a^2$$

Also, $f(0) = 1$

Since, the given function $f(x)$ is continuous at $x = 0$,

$\therefore$ $\lim_{x\to 0} f(x) = f(0) \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

Hence, the function $f(x)$ is continuous at $x = 0$ for $a = \pm 1$.

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$$= \lim_{h \to 0^-} \left(\frac{2-h-2}{+h} \right) = \lim_{h \to 0} \left(\frac{-h}{+h} \right) = -1.$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \left(\frac{f(0+h) - f(0)}{h} \right) \qquad [\because \ f(x) = 2 + x \text{ for } x \geq 0]$$

$$= \lim_{h \to 0^+} \left(\frac{(2+h) - (2+0)}{h} \right) = \lim_{h \to 0^+} \left(\frac{h}{h} \right) = 1$$

$\therefore$ $$Lf'(0) \neq Rf'(0)$$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Example 4. *Show that the function $f(x) = x^2$ for $x \leq 0$ and $f(x) = x$ for $x > 0$ is not derivable at $x = 0$.*

Solution. We have, $$f(x) = \begin{cases} x^2 & ; x \leq 0 \\ x & ; x > 0 \end{cases}$$

$\therefore$ $$\text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \qquad [\because \ f(x) = x^2 \text{ for } x \leq 0]$$

$$= \lim_{h \to 0^-} \left(\frac{(h)^2 - 0}{h} \right) = \lim_{h \to 0^-} (h) = 0$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \qquad [\because \ f(x) = x \text{ for } x > 0]$$

$$= \lim_{h \to 0^+} \left(\frac{h-0}{h} \right) = \lim_{h \to 0^+} 1 = 1$$

$\therefore$ $$Lf'(0) \neq Rf'(0)$$

$\Rightarrow$ $f'(0)$ does not exist.

$\Rightarrow$ $f(x)$ is not derivable at $x = 0$.

Example 5. *Show that $f(x) = [x]$ is differentiable at $x = 1$.*

Solution. We have, $f(x) = [x]$.

$\therefore$ L.H.D. $$Lf'(1) = \lim_{h \to 0^-} \left[\frac{f(1+h) - f(1)}{h} \right]$$

$$= \lim_{h \to 0^-} \left(\frac{[1+h] - [1]}{h} \right) \qquad \left[\begin{matrix} \because \ [1+h] = 0 \text{ for } h \to 0^- \\ \text{and} \quad [1] = 1 \end{matrix} \right]$$

$$= \lim_{h \to 0^-} \left(\frac{0-1}{h} \right) = -\lim_{h \to 0^-} \left(\frac{1}{h} \right) = \infty.$$

R.H.D. $$Rf'(1) = \lim_{h \to 0^+} \left[\frac{f(1+h) - f(1)}{h} \right] = \lim_{h \to 0^+} \left[\frac{[1+h] - [1]}{h} \right]$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} \qquad \left[\begin{matrix} \because \ [1+h] = 1 \text{ for } h \to 0^+ \\ [1] = 1 \end{matrix} \right]$$

$$= \lim_{h \to 0^-} \frac{h(h+2c)}{h}$$

$$= \lim_{h \to 0^-} (h + 2c) \Rightarrow Lf'(1) = 2c \qquad ...(2)$$

R.H.D. $$Rf'(c) = \lim_{h \to 0^+} \left[\frac{f(c+h) - f(c)}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{a(c+h) + b - c^2}{h}\right] \qquad [\because f(x) = ax + b \text{ for } x > c]$$

$$= \lim_{h \to 0^+} \left[\frac{ac + ah + b - c^2}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{ac + ah + b - (ac + b)}{h}\right] \qquad [\because \text{ By using (1)}]$$

$$= \lim_{h \to 0^+} \left(\frac{ah}{h}\right) = \lim_{h \to 0^+} a$$

$\Rightarrow$ $Rf'(c) = a$ ...(3)

Since, $f(x)$ is differentiable at $x = c$.

$\therefore$ $Lf'(c) = Rf'(c)$ [$\because$ By using (2) and (3)]

$\Rightarrow$ $2c = a$

$\Rightarrow$ $a = 2c$ ...(4)

Solving equations (1) and (4), we get

$b = -c^2$

Hence, $a = 2c, b = -c^2$.

Example 21. *Show that the function* $f(x) = \begin{cases} ax^2 + 1 & ; x \geq 1 \\ x + a & ; x < 1 \end{cases}$ *is continuous at $x = 1$. For what value of a, the function is differentiable at $x = 1$.*

Solution. We have,

$$f(x) = \begin{cases} ax^2 + 1 & ; x \geq 1 \\ x + a & ; x < 1 \end{cases}$$

Continuity at x = 1 :

$\therefore$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + a) = (1 + a)$

And, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2 + 1) = (a + 1)$

Also, $f(1) = a(1)^2 + 1 = (a + 1)$.

$\therefore$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = (a + 1)$

$\therefore$ $f(x)$ is continuous at $x = 1$ for all values of a.

$$= x^2 \cos\left(\frac{1}{x}\right) . \frac{d}{dx}\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) . 2x$$

$$= x^2 \cos\left(\frac{1}{x}\right) . \frac{d}{dx}(x^{-1}) + 2x \sin\frac{1}{x}$$

$$= x^2 \cos\frac{1}{x} . (-1x^{-2}) + 2x \sin\frac{1}{x} = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$$

$$= 2x \sin\frac{1}{x} - \cos\frac{1}{x} .$$

(*ii*) Let $y = x^5 \sin x (\tan x + \sec x)$.

$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}[x^5 \sin x (\tan x + \sec x)]$$

$$= x^5 \sin x . \frac{d}{dx}(\tan x + \sec x) + x^5 (\tan x + \sec x)\frac{d}{dx}(\sin x) + \sin x (\tan x + \sec x)\frac{d}{dx}(x^5)$$

$$= x^5 \sin x . (\sec^2 x + \sec x \tan x) + x^5 (\tan x + \sec x) . \cos x + \sin x (\tan x + \sec x) . 5x^4$$

$$= x^5 \sin x \sec x (\sec x + \tan x) + x^5 \cos x (\sec x + \tan x) + 5x^4 \sin x (\sec x + \tan x)$$

$$= x^4 (\sec x + \tan x) [x \sin x \sec x + x \cos x + 5 \sin x].$$

(*iii*) Let $y = x^3 . \dfrac{\sin x}{\cos x}$

$$y = x^3 \tan x$$

$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \tan x) = x^3 . \frac{d}{dx}(\tan x) + \tan x . \frac{d}{dx}(x^3)$$

$$= x^3 \sec^2 x + \tan x . (3x^2) = x^2(x \sec^2 x + 3 \tan x).$$

(*iv*) Let $y = \dfrac{1 - \cos x}{1 + \cos x}$

$$= \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \tan^2 \frac{x}{2}$$

$$\left[\begin{array}{l} \because\ 1 - \cos 2A = 2\sin^2 A \\ \Rightarrow\ 1 - \cos A = 2\sin^2 \frac{A}{2} \\ \quad 1 + \cos 2A = 2\cos^2 A \\ \Rightarrow\ 1 + \cos A = 2\cos^2 \frac{A}{2} \end{array}\right]$$

$\Rightarrow$
$$y = \left(\tan\frac{x}{2}\right)^2$$

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$$= \frac{(\sec^2 x)[1 - \tan x + 1 + \tan x]}{(1 - \tan x)^2} = \frac{2 \sec^2 x}{(1 - \tan x)^2}$$

$$= \frac{\frac{2}{\cos^2 x}}{\left(1 - \frac{\sin x}{\cos x}\right)^2} = \frac{\frac{2}{\cos^2 x}}{\frac{(\cos x - \sin x)^2}{\cos^2 x}} = \frac{2}{(\cos x - \sin x)^2}$$

$$= \frac{2}{\cos^2 x + \sin^2 x - 2 \sin x \cos x} \qquad \left[\begin{array}{l} \because \ \sin^2 A + \cos^2 A = 1 \\ 2 \sin A \cos A = \sin 2A \end{array}\right]$$

$$= \frac{2}{1 - \sin 2x}.$$

(*ii*) Let $\quad y = \frac{\sin \sqrt{x}}{\sqrt{x}}$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin \sqrt{x}}{\sqrt{x}}\right)$

$$= \frac{\sqrt{x} . \frac{d}{dx}(\sin \sqrt{x}) - (\sin \sqrt{x}) \frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x} . \cos \sqrt{x} . \frac{d}{dx}(\sqrt{x}) - (\sin \sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$\left[\begin{array}{l} \because \ \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) \\ = \frac{1}{2} x^{\frac{1}{2} - 1} \\ = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2x^{1/2}} \\ = \frac{1}{2\sqrt{x}} \end{array}\right]$$

$$= \frac{\sqrt{x} \cos \sqrt{x} . \frac{1}{2\sqrt{x}} - \frac{\sin \sqrt{x}}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x} \cos \sqrt{x} - \sin \sqrt{x}}{2x\sqrt{x}}.$$

(*iii*) Let $\quad y = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$

$$= \sqrt{\frac{1 + \sin x}{1 - \sin x} \times \frac{1 + \sin x}{1 + \sin x}} = \sqrt{\frac{(1 + \sin x)^2}{1 - \sin^2 x}} \qquad \text{[Rationalisation]}$$

$$= \frac{\sqrt{(1 + \sin x)^2}}{\sqrt{\cos^2 x}} = \frac{1 + \sin x}{\cos x} \qquad [\because \ \sin^2 A + \cos^2 A = 1]$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$$

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$\therefore \qquad \dfrac{dy}{dx} = \dfrac{3}{2}.$

(*vi*) Let $\qquad y = \cos^{-1}\left(\dfrac{a + b\cos x}{b + a\cos x}\right)$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right)\right]$$

$$= \frac{-1}{\sqrt{1 - \left(\dfrac{a + b\cos x}{b + a\cos x}\right)^2}} \cdot \frac{d}{dx}\left[\frac{a + b\cos x}{b + a\cos x}\right] \qquad \left[\because \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}\right]$$

$$= \frac{-1}{\sqrt{1 - \dfrac{(a + b\cos x)^2}{(b + a\cos x)^2}}} \cdot \left[\frac{(b + a\cos x)\,.\,\dfrac{d}{dx}(a + b\cos x) - (a + b\cos x)\,.\,\dfrac{d}{dx}(b + a\cos x)}{(b + a\cos x)^2}\right]$$

$$= \frac{-1}{\sqrt{\dfrac{(b + a\cos x)^2 - (a + b\cos x)^2}{(b + a\cos x)^2}}} \cdot \left[\frac{(b + a\cos x)(0 - b\sin x) - (a + b\cos x)(0 - a\sin x)}{(b + a\cos x)^2}\right]$$

$$= \frac{-(b + a\cos x)(-\sin x)(b^2 + ab\cos x - a^2 - ab\cos x)}{\sqrt{b^2 + a^2\cos^2 x + 2ab\cos x - a^2 - b^2\cos^2 x - 2ab\cos x}\ (b + a\cos x)^2}$$

$$= \frac{(\sin x)(b^2 - a^2)}{\sqrt{b^2(1 - \cos^2 x) - a^2(1 - \cos^2 x)}\ (b + a\cos x)} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{(b^2 - a^2)(\sin x)}{\sqrt{(b^2 - a^2)\sin^2 x}\ (b + a\cos x)} = \frac{\sqrt{b^2 - a^2}}{(b + a\cos x)}.$$

(*vii*) Let $\qquad y = \sin^{-1}\left(\dfrac{a + b\cos x}{b + a\cos x}\right)$

Please try yourself. $\qquad \left[\textbf{Ans. } \dfrac{-\sqrt{b^2 - a^2}}{b + a\cos x}\right]$

(*viii*) Let $\qquad y = \tan^{-1}\left(\dfrac{\cos x + \sin x}{\cos x - \sin x}\right)$

Please try yourself.

$\left[\textbf{Hint.}\text{ Dividing num. and denom. by } \cos x\text{, then apply } \tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \tan A}{1 - \tan A}.\right]$

[Ans. 1]

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(*vi*) Let $$y = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\times\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right]$$

[Rationalisation]

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(1+\sin x)-(1-\sin x)}\right]$$

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x})^2+(\sqrt{1-\sin x})^2+2\sqrt{1+\sin x}\,\sqrt{1-\sin x}}{1+\sin x-1+\sin x}\right]$$

$$= \cot^{-1}\left[\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{2\sin x}\right] \qquad [\because \sin^2 A+\cos^2 A=1]$$

$$= \cot^{-1}\left[\frac{2+2\sqrt{\cos^2 x}}{2\sin x}\right] = \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right]$$

$$= \cot^{-1}\left[\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right] \qquad \left[\begin{array}{l}\because 1+\cos 2A = 2\cos^2 A\\ \Rightarrow 1+\cos A = 2\cos^2\frac{A}{2}\\ \sin 2A = 2\sin A\cos A\\ \Rightarrow \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\end{array}\right]$$

$$y = \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{2}.$$

Example 42. *Differentiate the following functions w.r.t. x :*

(*i*) $\tan^{-1}\left(\frac{1-x}{1+x}\right)+\tan^{-1}\left(\frac{x+2}{1-2x}\right)$ (*ii*) $\operatorname{cosec}^{-1}\left(\frac{x^2+1}{x^2-1}\right)+\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

(*iii*) $\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$ (*iv*) $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)+\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

(*v*) $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$.

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Solution. (*i*) Let $y = \log [x + e^{\sqrt{x}}]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log (x + e^{\sqrt{x}})] = \frac{1}{x + e^{\sqrt{x}}} \cdot \frac{d}{dx}(x + e^{\sqrt{x}})$$

$$= \left(\frac{1}{x + e^{\sqrt{x}}}\right)\left[\frac{d}{dx}(x) + \frac{d}{dx}(e^{\sqrt{x}})\right] = \frac{1}{x + e^{\sqrt{x}}} \cdot \left[1 + e^{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})\right]$$

$$= \left(\frac{1}{x + e^{\sqrt{x}}}\right)\left[1 + e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\right] = \left(\frac{1}{x + e^{\sqrt{x}}}\right)\left[\frac{2\sqrt{x} + e^{\sqrt{x}}}{2\sqrt{x}}\right]$$

$$= \frac{2\sqrt{x} + e^{\sqrt{x}}}{2\sqrt{x}\,[x + e^{\sqrt{x}}]}.$$

(*ii*) Let $\quad y = \log [\log \{\log (x^3)\}]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log [\log \{\log (x^3)\}]] = \frac{1}{\log \{\log (x^3)\}} \cdot \frac{d}{dx}[\log \{\log (x^3)\}]$$

$$= \frac{1}{\log \{\log (x^3)\}} \cdot \frac{1}{\log (x^3)} \cdot \frac{d}{dx}[\log (x^3)]$$

$$= \frac{1}{\log \{\log (x^3)\}} \cdot \frac{1}{\log (x^3)} \cdot \frac{1}{x^3} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{\log (x^3) . \log \{\log (x^3)\}} \cdot \frac{1}{x^3}(3x^2)$$

$$= \frac{3}{x \log (x^3) . \log \{\log (x^3)\}}.$$

(*iii*) Let $\quad y = \log (x + 5)^6$

$$y = 6 \log (x + 5) \qquad [\because \log n^m = m \log n]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[6 \log (x + 5)] = 6 \cdot \frac{1}{(x+5)} \cdot \frac{d}{dx}(x + 5)$$

$$= \frac{6}{x+5}(1 + 0) = \frac{6}{x+5}.$$

(*iv*) Let $\quad y = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\left[\log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]\right) = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} \cdot \frac{d}{dx}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$$= \cot\left(\frac{\pi}{4} + \frac{x}{2}\right) . \sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$= 10^{10x} \cdot \log 10 \cdot \frac{d}{dx}(10x) = 10^{10x} \cdot \log 10 \cdot (10)$$

$$= 10 \cdot 10^{10x} \cdot \log 10.$$

Example 51. *Differentiate the following functions w.r.t. x :*

(*i*) $\log x \cdot e^{\tan x + x^2}$ (*ii*) $\log \sqrt{\dfrac{1 + \cos 2x}{1 - e^{2x}}}$

(*iii*) $e^{-ax^2} \cdot \sin(\log x)$ (*iv*) $\dfrac{e^x \log x}{x^2}$.

Solution. (*i*) Let $y = \log x \cdot e^{\tan x + x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log x \cdot e^{\tan x + x^2}] = \log x \cdot \frac{d}{dx}(e^{\tan x + x^2}) + e^{\tan x + x^2} \cdot \frac{d}{dx} \cdot (\log x)$$

$$= \log x \cdot e^{\tan x + x^2} \cdot \frac{d}{dx}(\tan x + x^2) + e^{\tan x + x^2} \cdot \frac{1}{x}$$

$$= e^{\tan x + x^2}\left[\log x\left(\frac{d}{dx}(\tan x) + \frac{d}{dx}(x^2)\right) + \frac{1}{x}\right]$$

$$= e^{\tan x + x^2}\left[\log x \cdot (\sec^2 x + 2x) + \frac{1}{x}\right].$$

(*ii*)
$$y = \log \sqrt{\frac{1 + \cos 2x}{1 - e^{2x}}} = \log\left(\frac{1 + \cos 2x}{1 - e^{2x}}\right)^{1/2}$$

$$= \frac{1}{2}\log\left(\frac{1 + \cos 2x}{1 - e^{2x}}\right) \qquad [\because \log m^n = n \log m]$$

$$y = \frac{1}{2}[\log(1 + \cos 2x) - \log(1 - e^{2x})] \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}[\log(1 + \cos 2x) - \log(1 - e^{2x})]$$

$$= \frac{1}{2}\left(\frac{d}{dx}[\log(1 + \cos 2x)] - \frac{d}{dx}[\log(1 - e^{2x})]\right)$$

$$= \frac{1}{2}\left[\frac{1}{(1 + \cos 2x)} \cdot \frac{d}{dx}(1 + \cos 2x) - \frac{1}{(1 - e^{2x})} \cdot \frac{d}{dx}(1 - e^{2x})\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1 + \cos 2x)} \cdot \left(0 - \sin 2x \frac{d}{dx}(2x)\right) - \frac{1}{(1 - e^{2x})}\left(0 - e^{2x} \cdot \frac{d}{dx}(2x)\right)\right]$$

$$= \frac{1}{2}\left[\frac{-2\sin 2x}{1 + \cos 2x} + \frac{2e^{2x}}{(1 - e^{2x})}\right] = \left[\frac{e^{2x}}{1 - e^{2x}} - \frac{\sin 2x}{1 + \cos 2x}\right]$$

8

Differentiation—III

8.1 INTRODUCTION

So far we have discussed the differentiation of algebraic functions, exponential functions, inverse trigonometric functions and logarithmic functions. In this chapter, we will be mainly discussing the differentiation of the implicit functions, logarithmic differentiation, differentiation of functions expressed in parametric form, differentiation of a function with respect to another and derivatives of higher order.

8.2 IMPLICIT FUNCTIONS

An equation of the form $f(x, y) = 0$ not expressing y explicitely in terms of x is called an implicit equation.

An implicit equation may determine one or more functions. A function determined by an implicit equation is called an Implicit Function.

e.g.,

The implicit equation $2x - 5y + 6 = 0$ determine one implicit function $y = \dfrac{2x + 6}{5}$.

8.2.1 Derivative of Implicit Functions. Most of the functions we have discussed so far have been explicitly defined by an algebraic equation *e.g.,*

the equation $$y = x^2 + 1$$

defines a function, where $f(x) = x^2 + 1$.

Not all functions are defined in such an explicit way.

e.g., An equation in x and y such as :

$$y + \sin y = x^2$$

is not easily solved for y in terms of x or x in terms of y, such a function is said to be defined implicitly by the given equation.

For the differentiation of such functions, we differentiate both sides w.r.t. x, but when we differentiate the term containing y, we shall apply chain rule of derivatives of composite functions.

e.g.,

$$\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}.$$

Note. The following derivatives will be found very useful in implicit differentiation :

(*i*) $\dfrac{d}{dx}(y^n) = ny^{n-1} \cdot \dfrac{dy}{dx}$ (*ii*) $\dfrac{d}{dx}(\log y) = \dfrac{1}{y} \cdot \dfrac{dy}{dx}$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left[n \cot nx - \frac{m}{\sqrt{1-x^2}}\right]$$

$$= e^{m\cos^{-1}x} \sin nx \left[n \cot nx - \frac{m}{\sqrt{1-x^2}}\right]. \qquad [\because \quad y = e^{m\cos^{-1}x} \sin nx]$$

Example 12. *Differentiate the following functions w.r.t. x*

(*i*) $(\sin x)^{\tan x} + (\cos x)^{\sec x}$ (*ii*) $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

(*iii*) $(\tan x)^{\tan x} + (\sin x)^{\sin x}$ (*iv*) $(\tan x)^{\sin x} + (\sin x)^{\tan x}$

(*v*) $(\tan x)^{\log x} + (\sin x)^{\cos x}$.

Solution. (*i*) Let $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$ ...(1)

Let $y = u + v$ $\left[\because \text{ Here, we cannot take logarithms directly as } \log(m+n) \neq \log m + \log n\right]$

$$\therefore \qquad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(2)$$

$\therefore$ $u = (\sin x)^{\tan x}$

Taking logarithms on both sides, we get

$$\log u = \log (\sin x)^{\tan x}$$

$$\Rightarrow \qquad \log u = \tan x \log (\sin x) \qquad [\because \quad \log m^n = n \log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log(\sin x) \cdot \sec^2 x$$

$$\Rightarrow \qquad \frac{du}{dx} = u\left[\tan x \cdot \frac{\cos x}{\sin x} + \log(\sin x) \cdot \sec^2 x\right]$$

$$= (\sin x)^{\tan x} [\tan x \cdot \cot x + \sec^2 x \log(\sin x)]$$

$$[\because \quad u = (\sin x)^{\tan x}]$$

$$\Rightarrow \qquad \frac{du}{dx} = (\sin x)^{\tan x} [1 + \sec^2 x \log(\sin x)] \qquad ...(3)$$

Let $v = (\cos x)^{\sec x}$

Taking logarithms on both sides, we get

$$\log v = \log (\cos x)^{\sec x}$$

$$\Rightarrow \qquad \log v = \sec x \log (\cos x) \qquad [\because \quad \log m^n = n \log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \sec x \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \log(\cos x) \cdot \sec x \tan x$$

$$\Rightarrow \qquad \frac{dv}{dx} = v\left[\sec x \cdot \frac{(-\sin x)}{\cos x} + \sec x \tan x \log(\cos x)\right]$$

$$= (\cos x)^{\sec x} [-\sec x \tan x + \sec x \tan x \log(\cos x)]$$

$$\Rightarrow \qquad \frac{dv}{dx} = (\cos x)^{\sec x} \cdot \sec x \tan x\, [\log(\cos x) - 1] \qquad ...(4)$$

Taking logarithms on both sides, we get

$$\log z = \log (x^x)$$

$$\Rightarrow \quad \log z = x \log x \qquad [\because \log m^n = n \log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{z} \cdot \frac{dz}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{dz}{dx} = z\,[1 + \log x] \qquad [\because z = x^x]$$

$$\Rightarrow \quad \frac{d(x^x)}{dx} = x^x\,[1 + \log x]$$

Putting this value of $\frac{d}{dx}(x^x)$ in equation (3), we have

$$\frac{du}{dx} = \cos(x^x) \cdot x^x (1 + \log x) \qquad ...(4)$$

Let $\quad v = (\sin x)^x \qquad$ [For solution see Ex. 8 (*iii*)]

$$\therefore \quad \frac{dv}{dx} = (\sin x)^x\,[x \cot x + \log(\sin x)] \qquad ...(5)$$

Substituting the values of equations (4) and (5) in equation (2), we have

$$\frac{dy}{dx} = x^x \cdot \cos(x^x)\,[1 + \log x] + (\sin x)^x\,[x \cot x + \log(\sin x)].$$

(*v*) Let $\quad y = x^{\tan x} + (\tan x)^{\cot x} \qquad ...(1)$

Let $\quad y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(2)$$

$\therefore \quad u = x^{\tan x}$

Taking logarithms on both sides, we get

$$\log u = \log (x^{\tan x})$$

$$\Rightarrow \quad \log u = \tan x \log x \qquad [\because \log m^n = n \log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = \tan x \cdot \frac{1}{x} + \log x \cdot (\sec^2 x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \sec^2 x \cdot \log x\right]$$

$$\frac{du}{dx} = x^{\tan x} \cdot \left[\frac{\tan x}{x} + \sec^2 x \cdot \log x\right] \qquad ...(3) \quad [\because u = x^{\tan x}]$$

Let $\quad v = (\tan x)^{\cot x}$

For solution see Example 9 (*vii*)

$$\therefore \quad \frac{dv}{dx} = (\tan x)^{\cot x} \cdot \text{cosec}^2 x\,[1 - \log(\tan x)] \qquad ...(4)$$

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About the Author

Deepak Bhardwaj is an M.Sc. M.Phil. in Mathematics. He is working as the Head of Department (Mathematics) in Parth Public Sr. Sec. School Gharaunda (Karnal), Haryana.

He has been teaching Post-Graduate, Graduate and Engineering students for the last eight years. He is the recipient of "Avantika Rajya Shikshak Samman-2004". He has written books on Real Analysis and Abstract Algebra for the Post-Graduate students of Kurukshetra University, Kurukshetra and Maharishi Dayanand University, Rohtak.

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