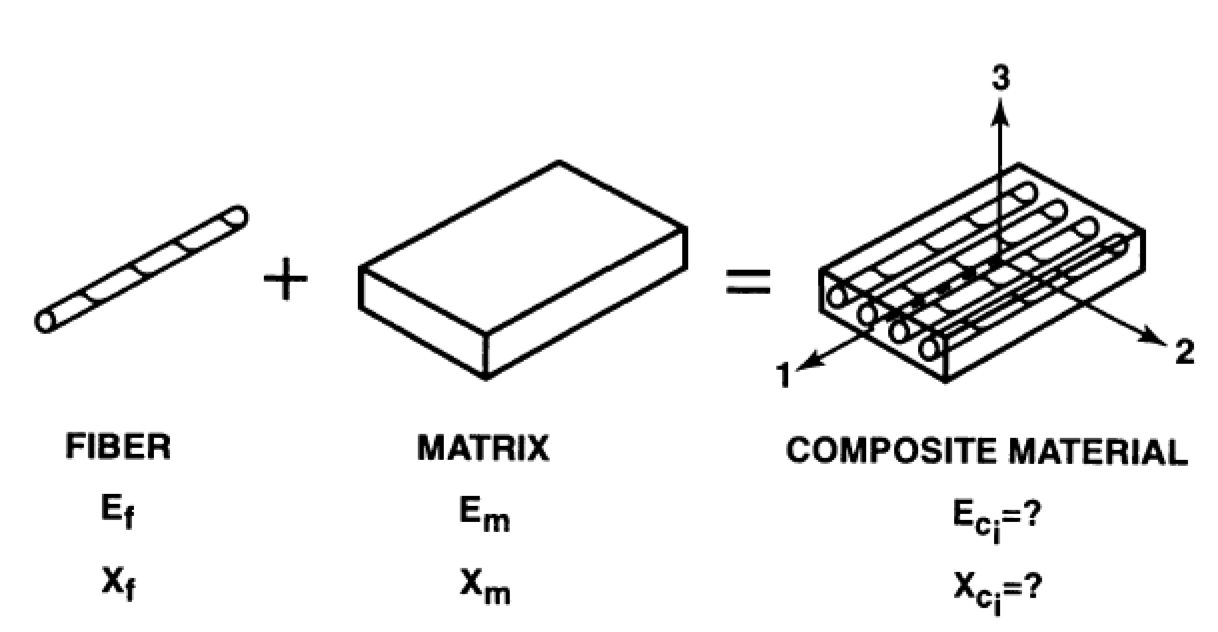
# MICROMECHANICS

## Rules of Mixture for Elastic Properties




Credits www.tech.plym.ac.uk,

'Rules of Mixtures' are mathematical expressions which give some property of the composite in terms of the properties, quantity and arrangement of its constituents.

They may be based on a number of simplifying assumptions, and their use in design should tempered with extreme caution!

# Density

For a general composite, total volume V, containing masses of constituents $M_a$, $M_b$, $M_c$,... the composite density is

$$\rho = \frac{M_a + M_b + M_c + \ldots}{V} = \frac{M_a}{V} + \frac{M_b}{V} + \ldots$$

In terms of the densities and volumes of the constituents:

$$\rho = \frac{V_a \rho_a}{V} + \frac{V_b \rho_b}{V} + \frac{V_c \rho_c}{V} + \ldots$$

# Density

But $v_a / V = V_a$ is the volume fraction of the constituent a, hence:

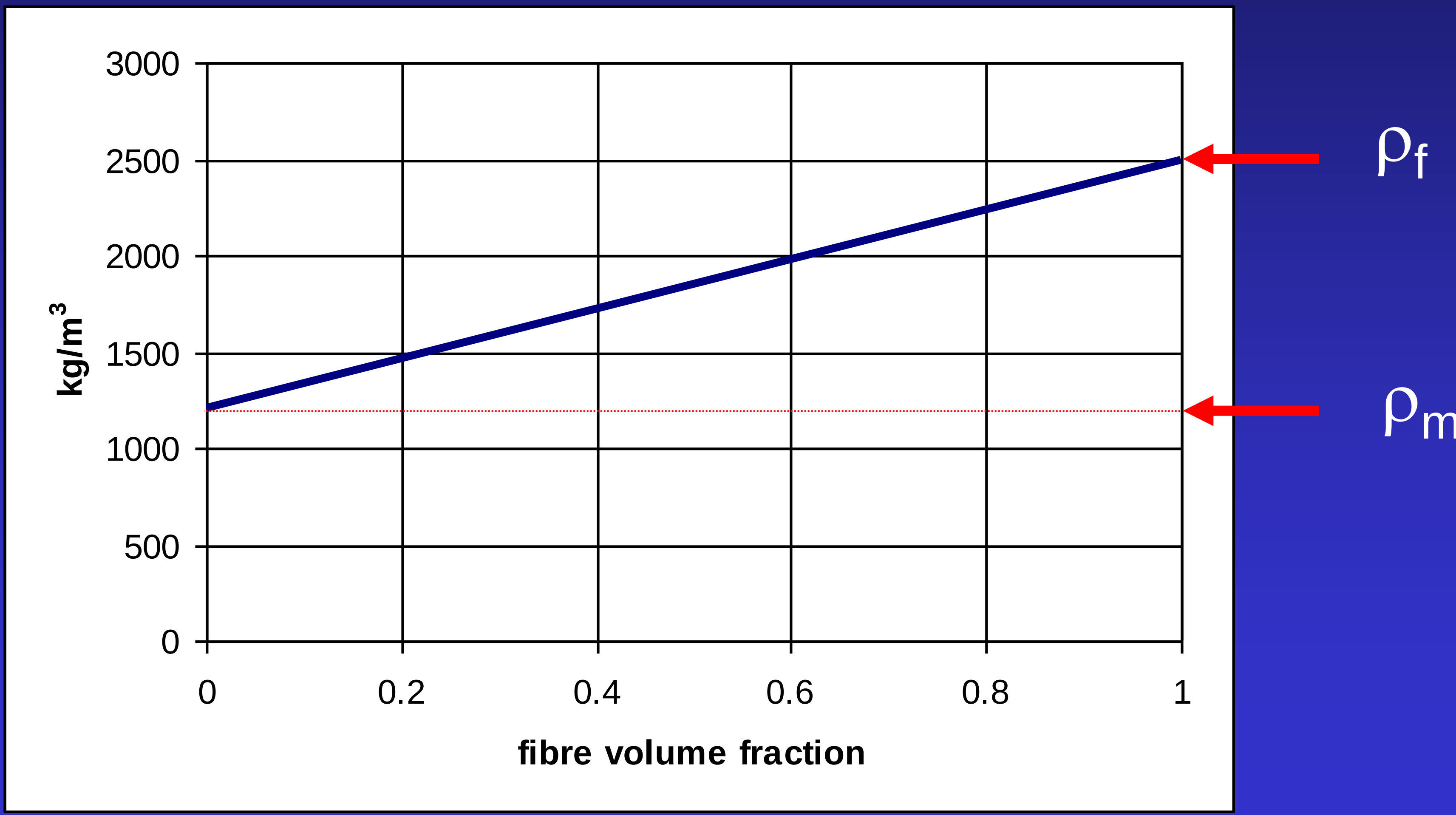
$$\rho = V_a \rho_a + V_b \rho_b + V_c \rho_c + \ldots$$

For the special case of a fibre-reinforced matrix:

$$\rho = V_f \rho_f + V_m \rho_m = V_f \rho_f + (1 - V_f) \rho_m = V_f (\rho_f - \rho_m) + \rho_m$$

since $V_f + V_m = 1$

# Rule of mixtures density for glass/epoxy composites



# Fibre Volume Fraction and Laminate Thickness

# How much reinforcement?

## ***Weight fraction***

Used in manufacture.

May refer to fibre or resin - 'GRP' manufacturers will specify a glass content of (e.g.) 25 wt%;  a prepreg supplier might give a resin content of 34 wt%.

## ***Volume fraction***

Used in design to calculate composite properties.
Almost always refers to *fibre* content.

# Weight fraction ↔ volume fraction conversion

We need to know the densities of each constituent in the composite ($\rho_a$, $\rho_b$, etc).

If we know the weight fractions ($W_a$, $W_b$,…), then the volume fraction of constituent a is:

$$V_a = \frac{W_a / \rho_a}{\sum W_i / \rho_i}$$

# Weight fraction ↔ volume fraction conversion

If we know the volume fractions ($V_a$, $V_b$,…), then the weight fraction of constituent a is:

$$W_a = \frac{V_a \rho_a}{\sum V_i \rho_i}$$

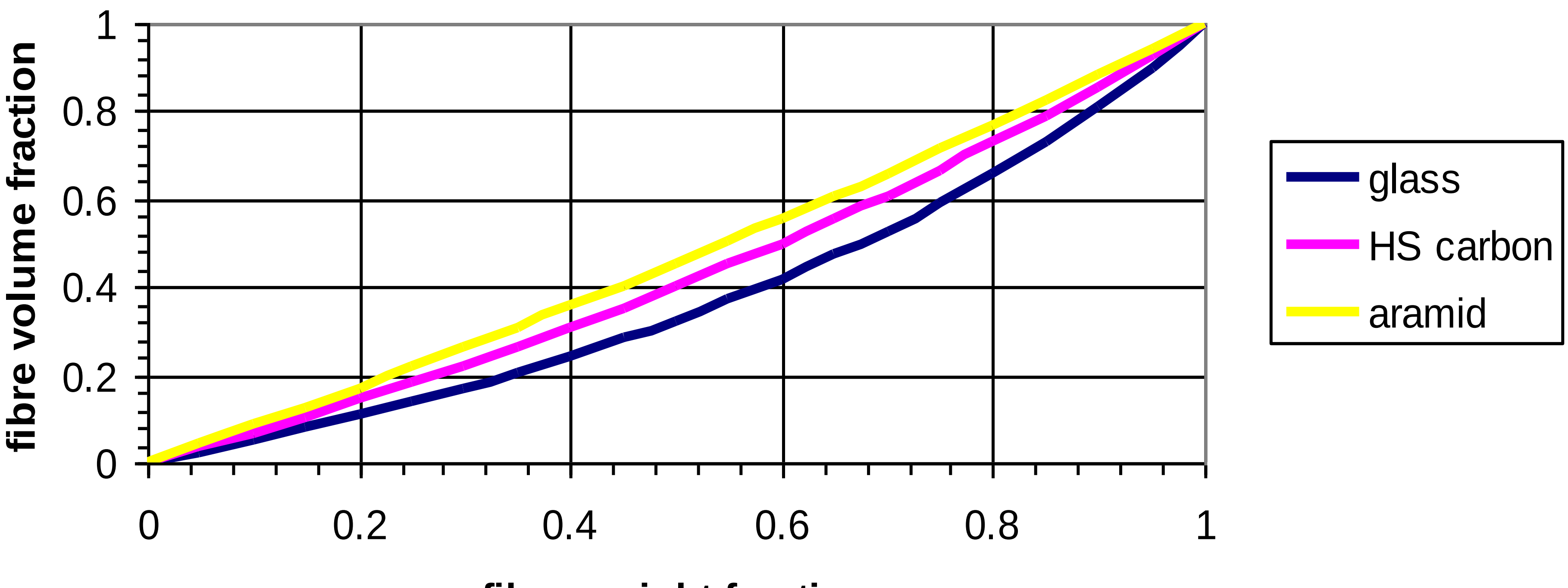
# Weight fraction ↔ volume fraction conversion

For the special case of a two-component composite (eg fibre and matrix):

$$V_f = \frac{W_f / \rho_f}{W_f / \rho_f + (1 - W_f) / \rho_m}$$

$$W_f = \frac{\rho_f V_f}{\rho_f V_f + \rho_m (1 - V_f)}$$

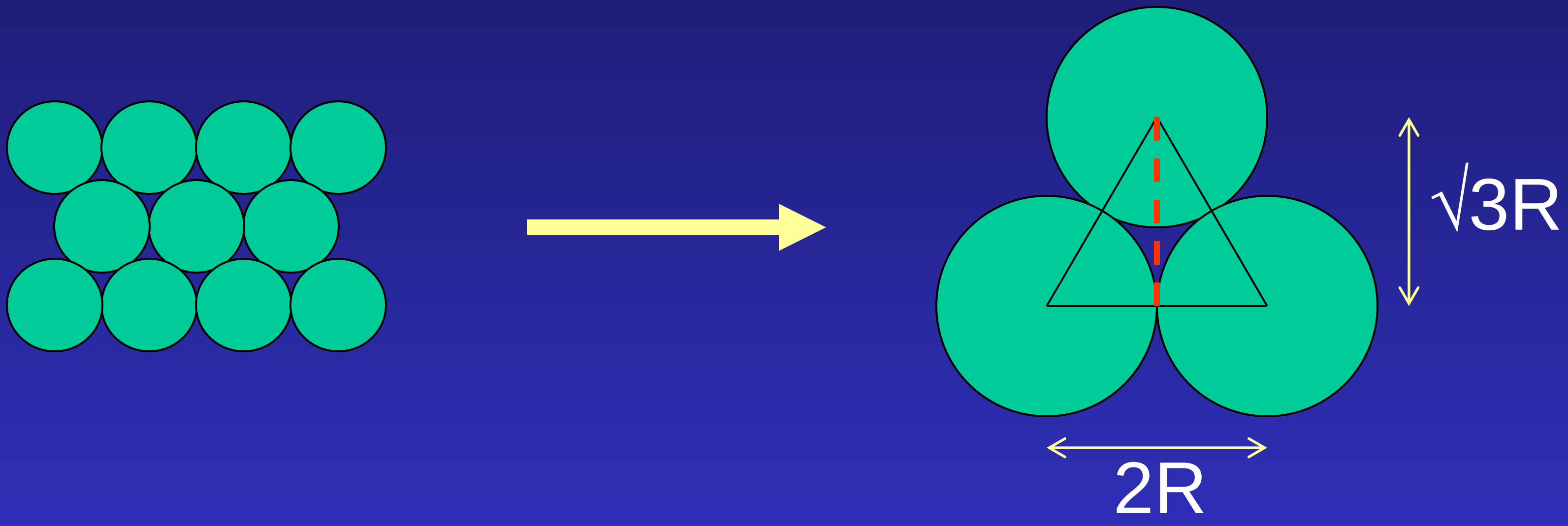
# Volume fraction - weight fraction conversion (epoxy resin matrix)



# Maximum fibre volume fraction

A composite cannot contain 100% fibre. Maximum volume fraction could be achieved only if unidirectional fibres are hexagonally ‘close packed’ - ie all fibres are touching.

# Maximum fibre volume fraction




The triangular unit cell has area $\sqrt{3}\ R^2$.

The unit cell contains an area of fibre (three 60º segments) equal to $\pi R^2 / 2$

# Maximum fibre volume fraction

In a unidirectional fibre composite, the fibre *area* fraction is the same as the fibre *volume* fraction, so:

$$V_f^{max} = \frac{\pi R^2 / 2}{\sqrt{3} R^2} = \frac{\pi}{2\sqrt{3}} = 0.908 \approx 91\%$$

# Maximum fibre volume fraction

In practice, perfect alignment is impossible. Maximum volume fraction depends on the method of manufacture, but for a unidirectional fibre composite is likely to be between 0.6 and 0.7.

# Maximum fibre volume fraction

For other forms of reinforcement, maximum volume fraction also depends on the detailed arrangement of the fibres.

The following values are typical:

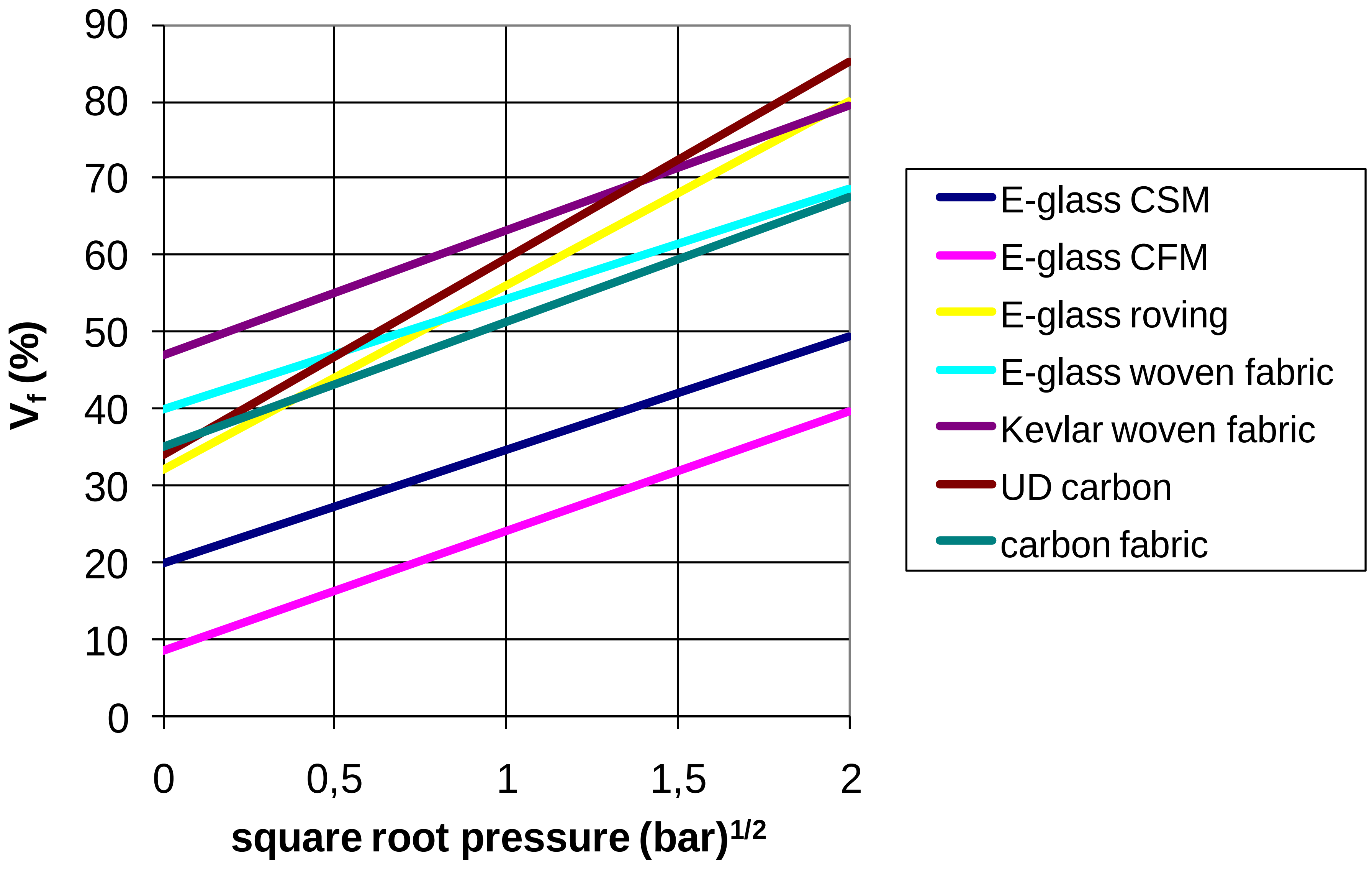
| | |
|---|---|
| stitched ‘non-crimp’ | 0.6 |
| woven fabric | 0.4 - 0.55 |
| random (chopped strand mat) | 0.15 - 0.25 |

# Compressibility of Reinforcement

All reinforcement types will reduce in thickness (in a non-linear way) if subjected to pressure.  For a given weight of reinforcement, the volume fraction will thus increase with pressure (P).

Empirically:  $V_f = a + b \sqrt{P}$, where a and b depend on fibre type and weave style.

# Compressed fibre volume fraction



# How much fibre?

Commercial reinforcements are characterised by their ***areal weight ($A_w$).*** This is simply the weight (usually given in g) of 1 $m^2$ of the reinforcement. $A_w$ depends on many factors - fibre density, tow or bundle size, weave style, etc.

$A_w$ may range from 50 $g/m^2$ or less (for lightweight surfacing tissues), up to more than 2000 $g/m^2$ for some heavyweight non-crimp fabrics.

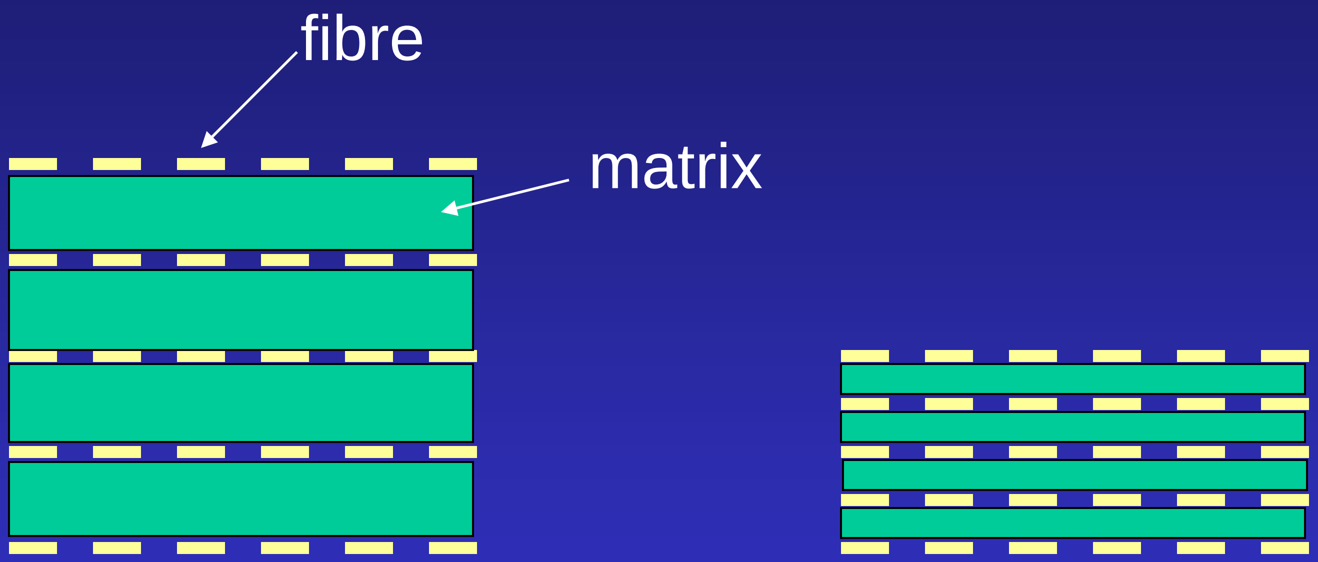
# Laminate thickness

The thickness of a composite laminate depends on the amount of reinforcement and the relative amount of resin which has been included.

For a given quantity of reinforcement, a laminate with a high fibre volume fraction will be thinner than one with a lower fibre volume fraction, since it will contain less resin.

# Laminate thickness

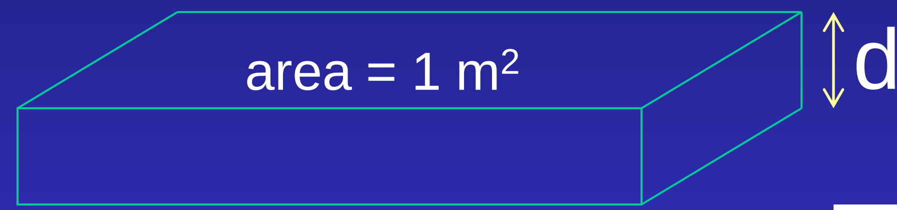
Two laminates, both containing 5 plies of reinforcement:




| | |
|---|---|
| high matrix content | low matrix content |
| low fibre content | high fibre content |
| = thick laminate | = thin laminate |

# Laminate thickness

Consider unit area of laminate, thickness d, containing n plies of reinforcement with areal weight $A_w$:




weight of fibre = $nA_w$

volume of fibre = $\dfrac{nA_w}{\rho_f}$

fibre volume fraction: $$V_f = \frac{nA_w}{\rho_f d}$$

# Laminate thickness

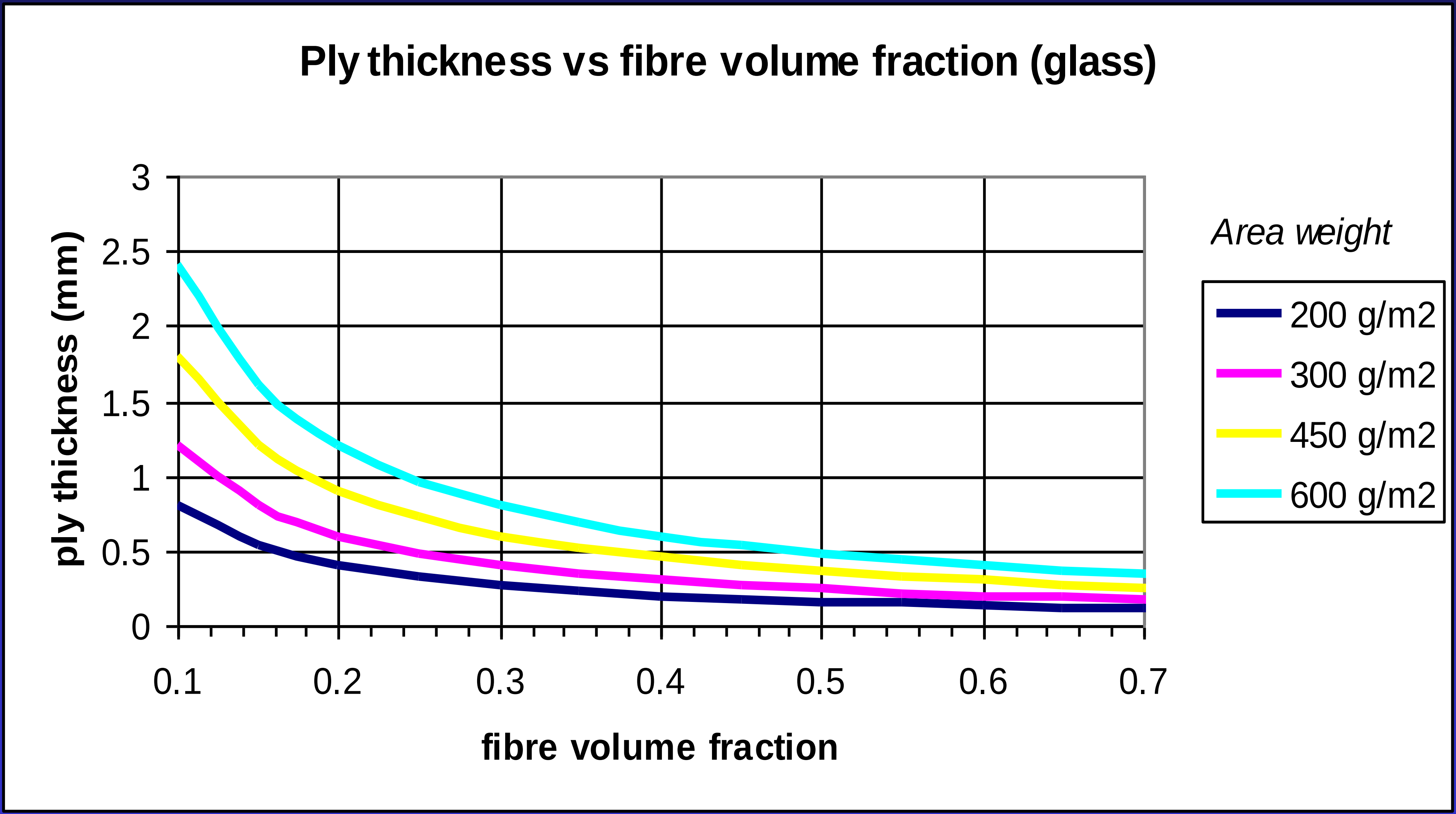
Fibre volume fraction is thus inversely proportional to laminate thickness.

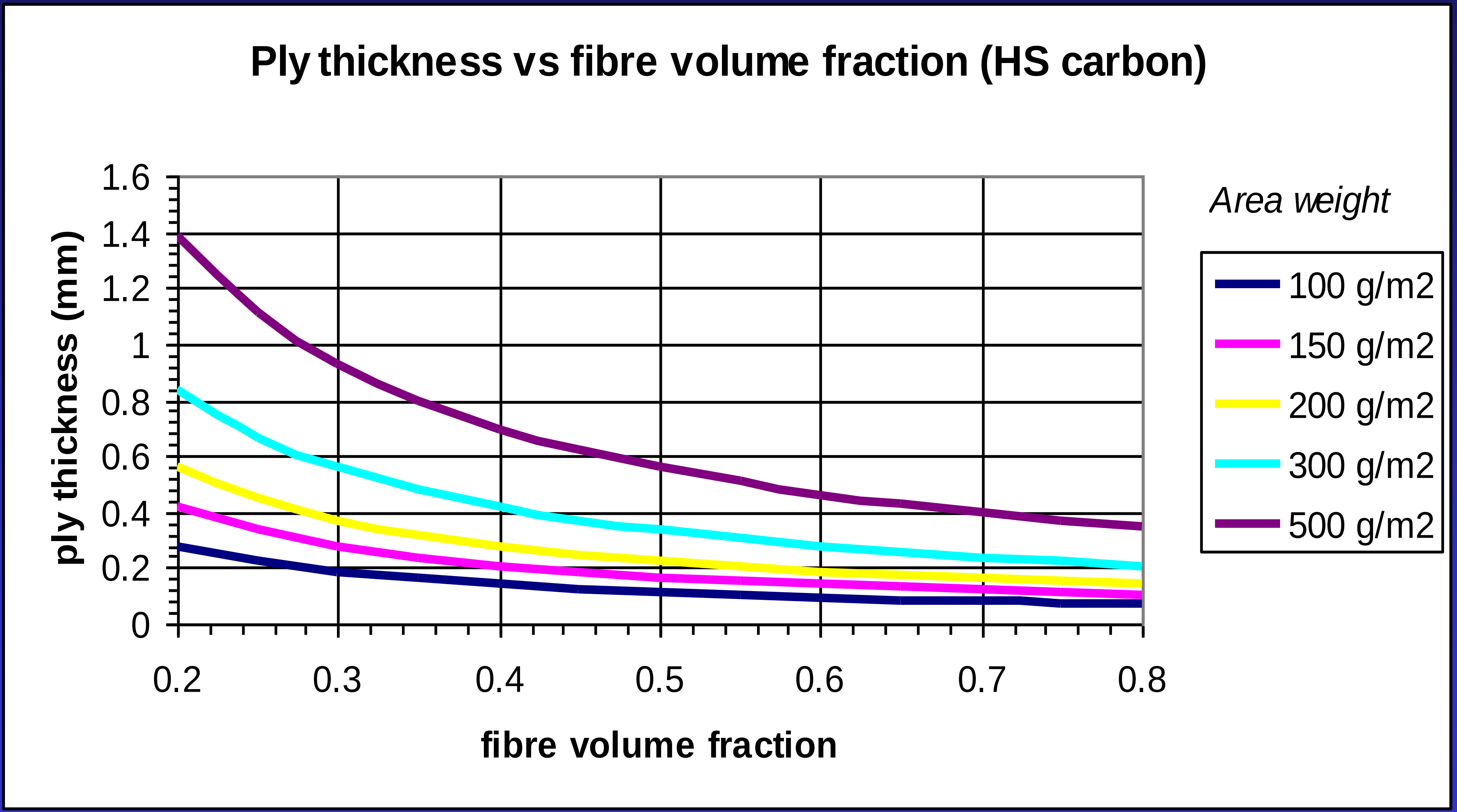
If the fibre content and laminate thickness are defined, we can calculate the fibre volume fraction:

$$V_f = \frac{nA_w}{\rho_f d}$$

If the fibre content and volume fraction are defined, we can calculate the laminate thickness :

$$d = \frac{nA_w}{\rho_f V_f}$$


Ply thickness vs fibre volume fraction (glass)
ply thickness (mm)
3
2.5
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
fibre volume fraction
Area weight
200 g/m2
300 g/m2
450 g/m2
600 g/m2



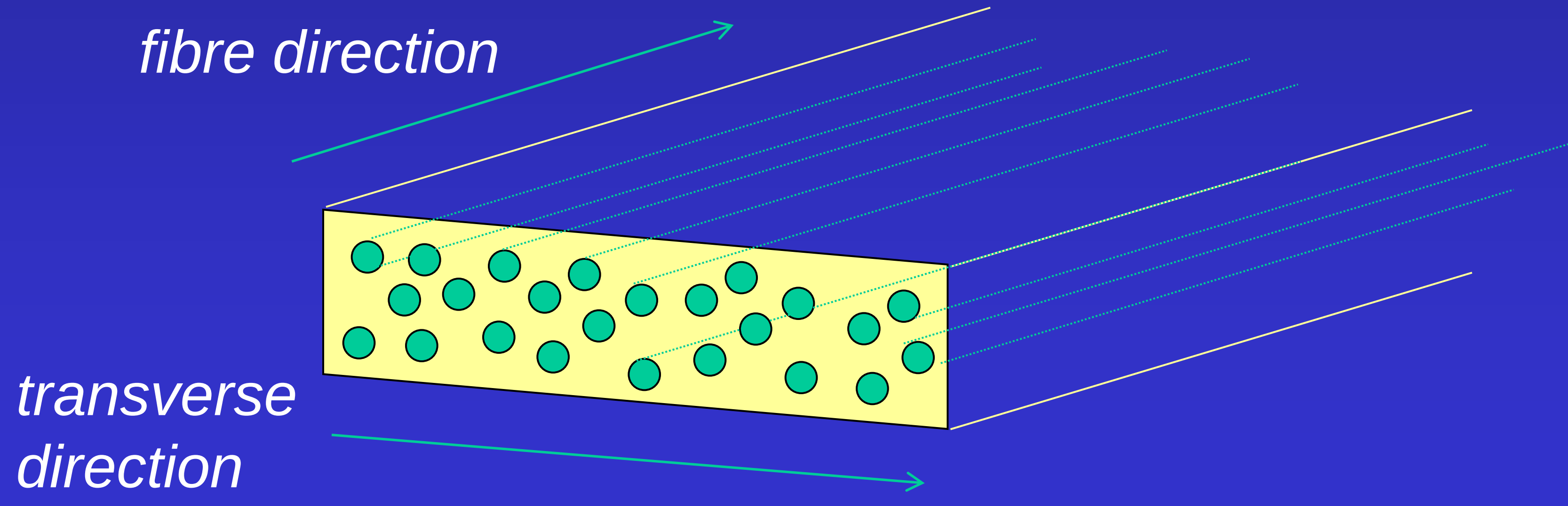
Ply thickness vs fibre volume fraction (HS carbon)
ply thickness (mm)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
fibre volume fraction
Area weight
100 g/m2
150 g/m2
200 g/m2
300 g/m2
500 g/m2

# Micromechanical models for stiffness

# Unidirectional ply

Unidirectional fibres are the simplest arrangement of fibres to analyse.

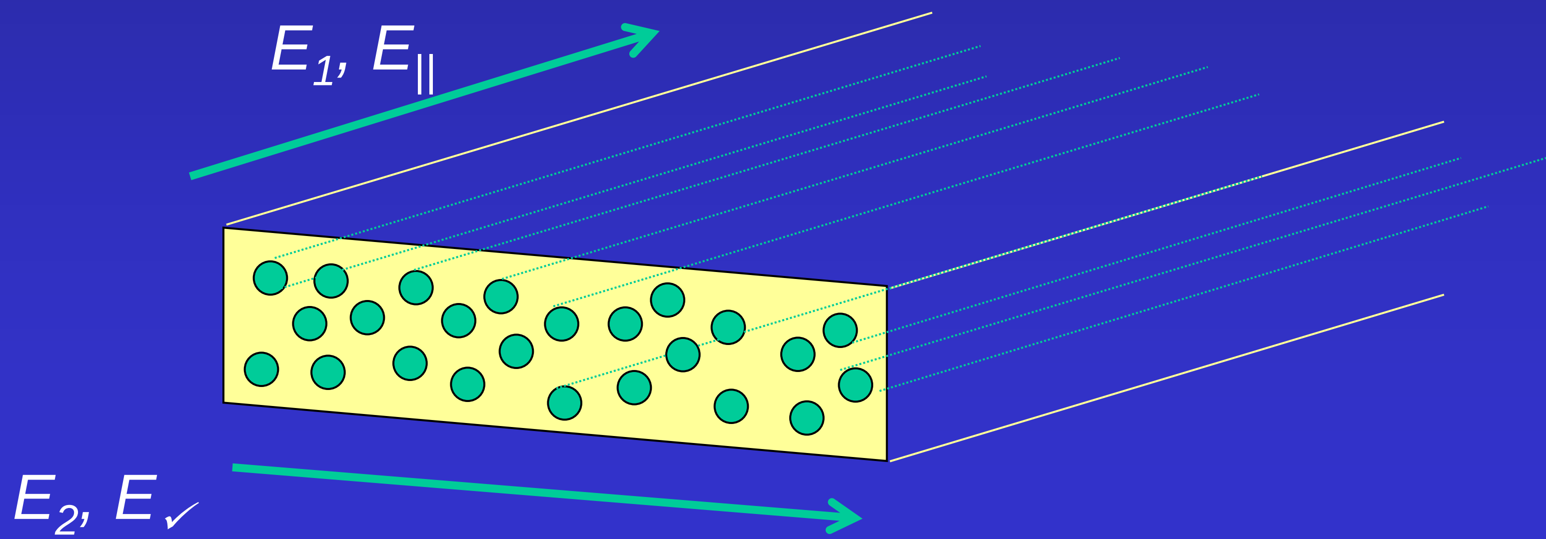
They provide maximum properties in the fibre direction, but minimum properties in the transverse direction.



# Unidirectional ply

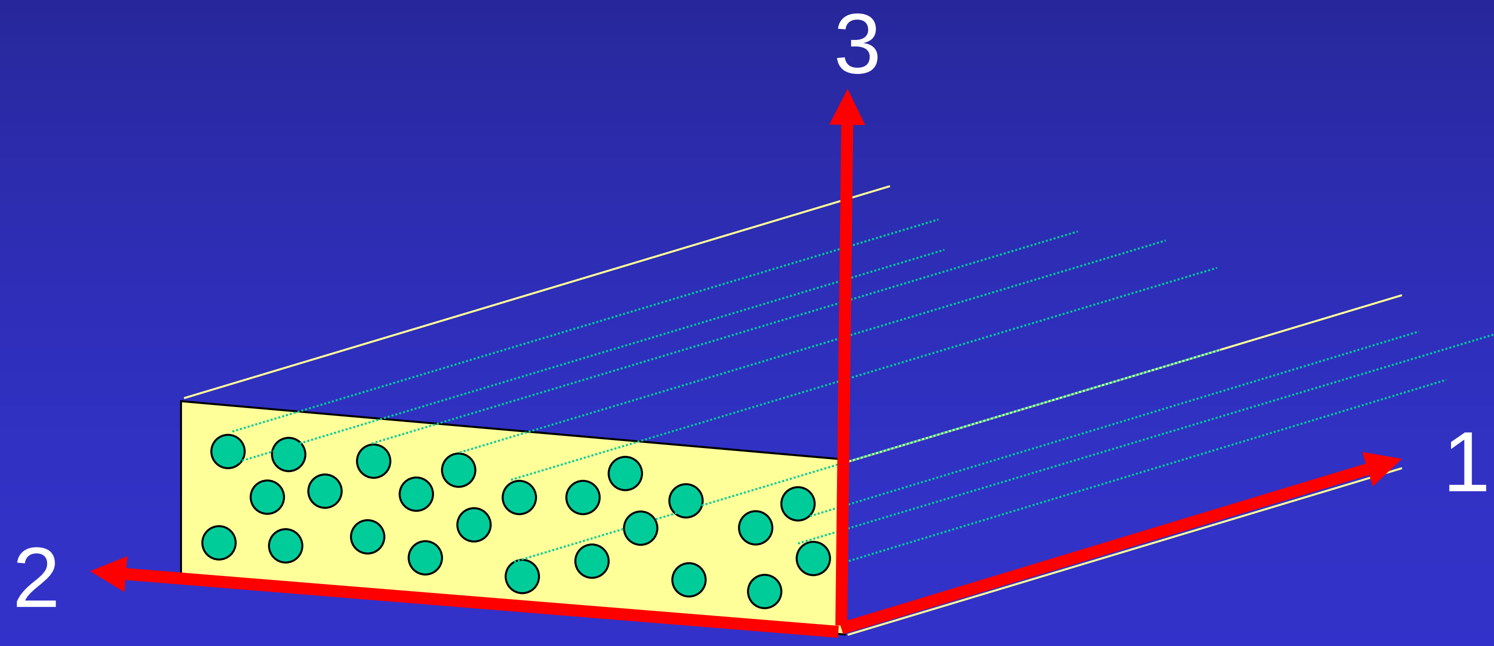
We expect the unidirectional composite to have different tensile moduli in different directions.

These properties may be labelled in several different ways:



# Unidirectional ply

By convention, the principal axes of the ply are labelled ‘1, 2, 3’. This is used to denote the fact that ply may be aligned differently from the cartesian axes x, y, z.



# Unidirectional ply - longitudinal tensile modulus

We make the following assumptions in developing a rule of mixtures:

- Fibres are uniform, parallel and continuous.
- Perfect bonding between fibre and matrix.
- Longitudinal load produces equal strain in fibre and matrix.

# Unidirectional ply - longitudinal tensile modulus

- A load applied in the fibre direction is shared between fibre and matrix:

  $$F_1 = F_f + F_m$$

- The stresses depend on the cross-sectional areas of fibre and matrix:

  $$\sigma_1 A = \sigma_f A_f + \sigma_m A_m$$

  where $A$ $(= A_f + A_m)$ is the total cross-sectional area of the ply

# Unidirectional ply - longitudinal tensile modulus

- Applying Hooke's law:
  $E_1 \varepsilon_1 A = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$
  where Poisson contraction has been ignored

- But the strain in fibre, matrix and composite are the same, so
  $\varepsilon_1 = \varepsilon_f = \varepsilon_m$, and:

  $E_1 A = E_f A_f + E_m A_m$

# Unidirectional ply - longitudinal tensile modulus

Dividing through by area A:

$$E_1 = E_f\,(A_f / A) + E_m\,(A_m / A)$$

But for the unidirectional ply, ($A_f$ / $A$) and ($A_m$ / $A$) are the same as volume fractions $V_f$ and $V_m = 1\text{-}V_f$. Hence:

$$\boldsymbol{E_1 = E_f\,V_f + E_m\,(1\text{-}V_f)}$$

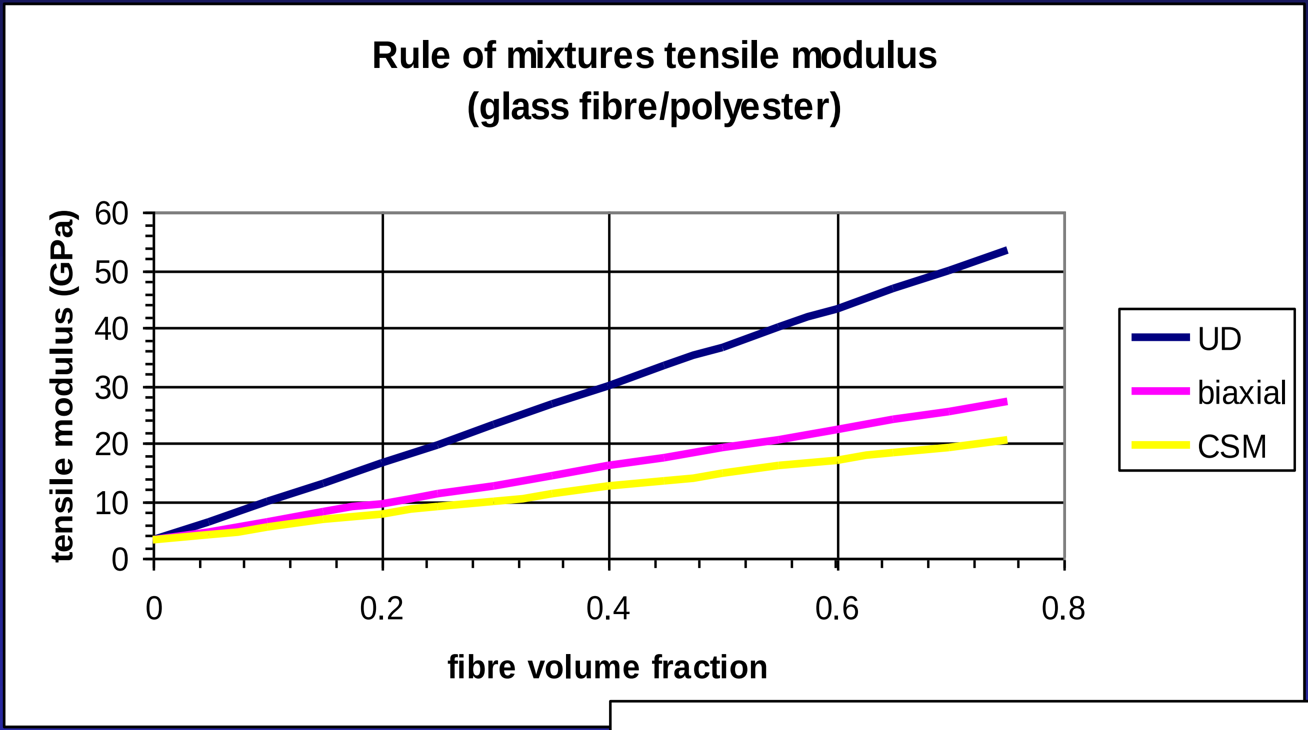
# Unidirectional ply - longitudinal tensile modulus

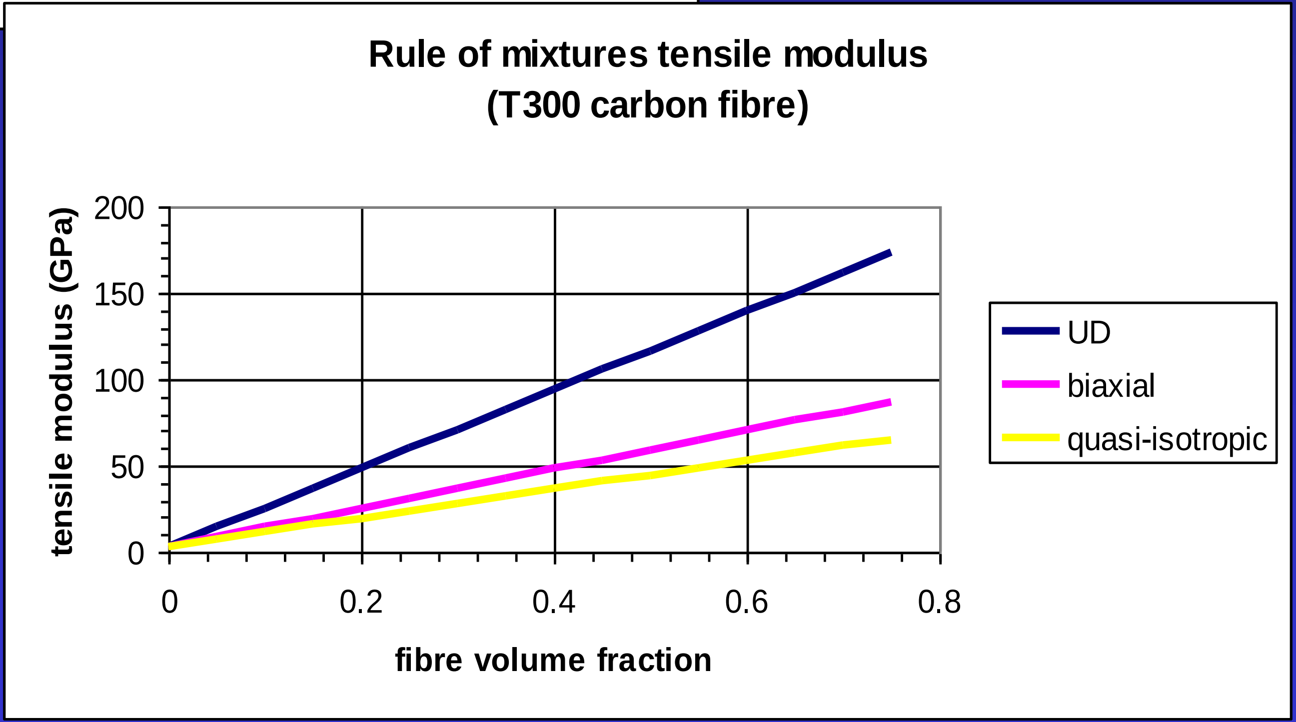
$$E_1 = E_f V_f + E_m (1 - V_f)$$

Note the similarity to the rules of mixture expression for density.

In polymer composites, $E_f >> E_m$, so

$$E_1 \approx E_f V_f$$

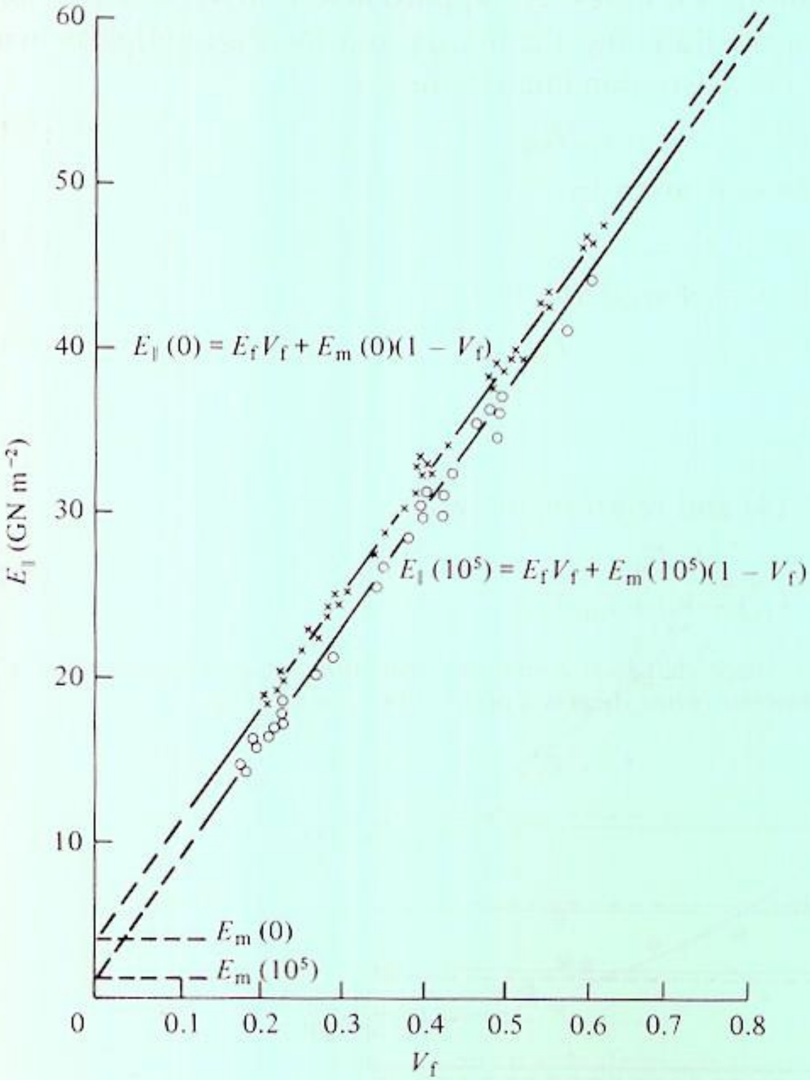

Rule of mixtures tensile modulus
(glass fibre/polyester)
tensile modulus (GPa)
60
50
40
30
20
10
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
CSM


Rule of mixtures tensile modulus
(T300 carbon fibre)
tensile modulus (GPa)
200
150
100
50
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
quasi-isotropic

This rule of mixtures is a good fit to experimental data

(source: Hull, Introduction to Composite Materials, CUP)

Fig. 5.2. Elastic moduli measured parallel to fibres of undirectional laminae of glass fibres and polyester resin with different $V_f$. (From Brintrup Dr-Ing thesis 1975, Technischen Hochschule, Aachen.)

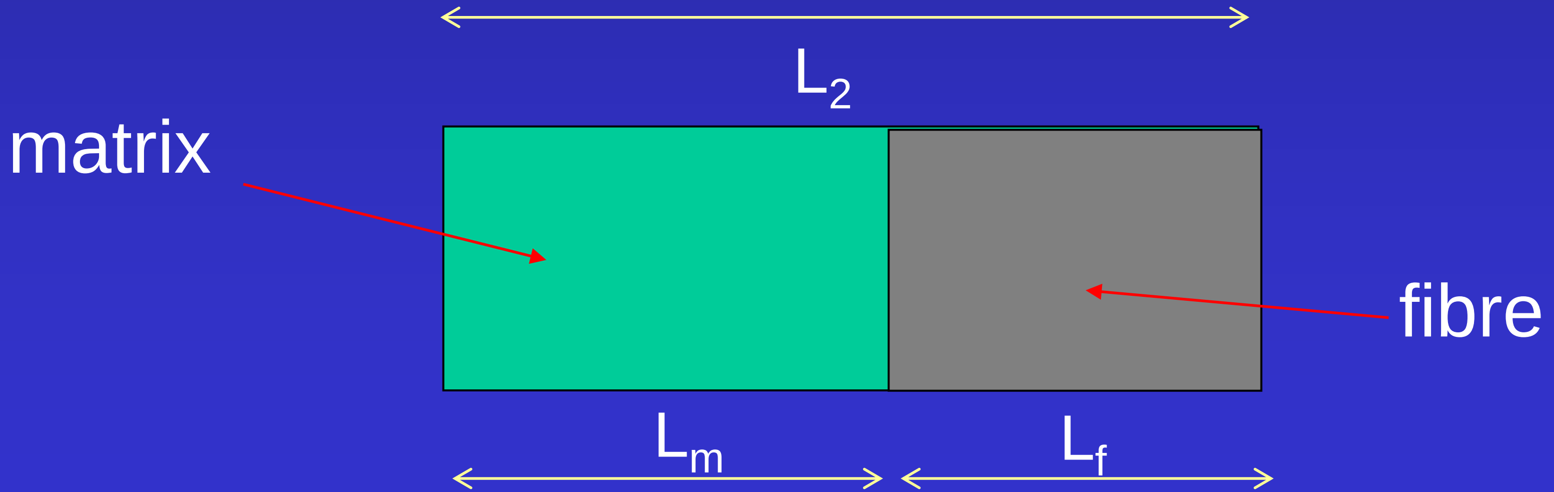



† Notation – for a unidirectional lamina $E_1 = E_{\parallel}$, $E_2 = E_{\perp}$, $G_{12} = G_{\parallel\perp}$ and $\nu_{12} = \nu_{\parallel\perp}$.

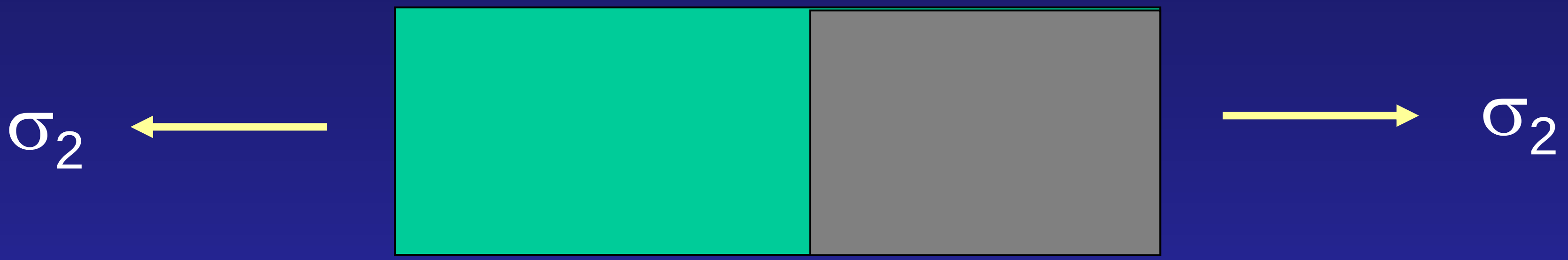
# Unidirectional ply - transverse tensile modulus

For the transverse stiffness, a load is applied at right angles to the fibres.

The model is very much simplified, and the fibres are lumped together:



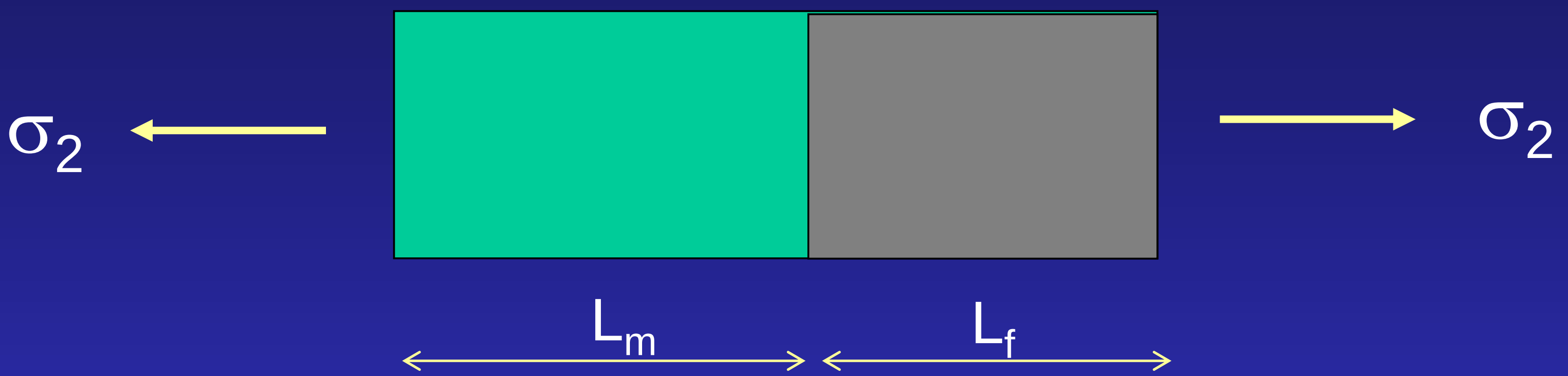
# Unidirectional ply - transverse tensile modulus




It is assumed that the stress is the same in each component ($\sigma_2 = \sigma_f = \sigma_m$).

Poisson contraction effects are ignored.

# Unidirectional ply - transverse tensile modulus

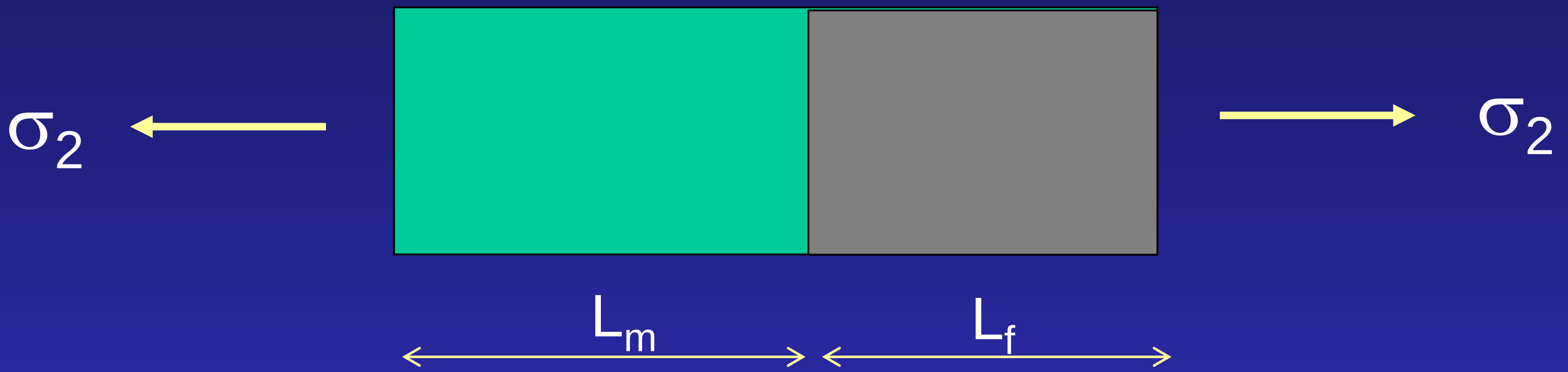



The total extension is $\delta_2 = \delta_f + \delta_m$, so the strain is given by:

$$\varepsilon_2 L_2 = \varepsilon_f L_f + \varepsilon_m L_m$$

so that $\varepsilon_2 = \varepsilon_f\,(L_f\,/\,L_2) + \varepsilon_m\,(L_m\,/\,L_2)$

# Unidirectional ply - transverse tensile modulus

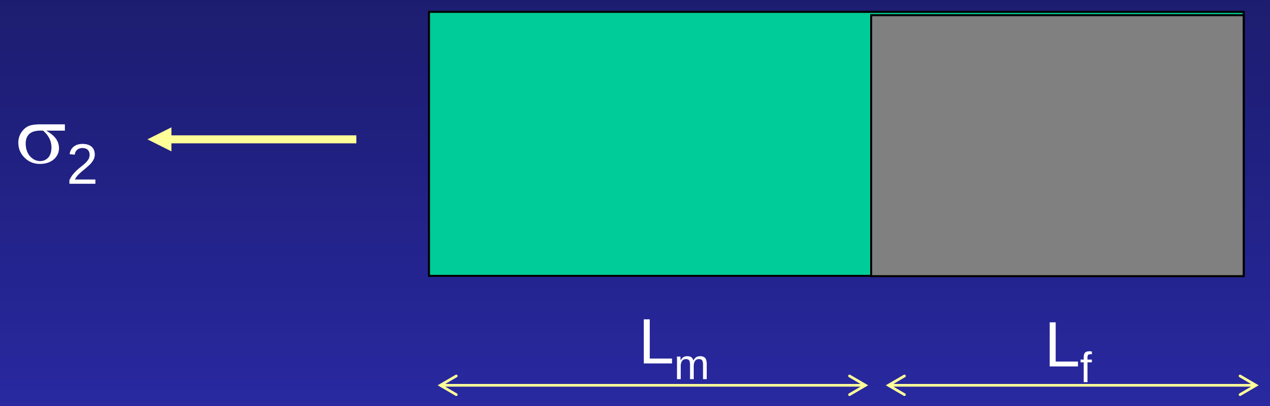



But $L_f / L_2 = V_f$ and $L_m / L_2 = V_m = 1\text{-}V_f$

So $\varepsilon_2 = \varepsilon_f V_f + \varepsilon_m (1\text{-}V_f)$

and $\sigma_2 / E_2 = \sigma_f V_f / E_f + \sigma_m (1\text{-}V_f) / E_m$

# Unidirectional ply - transverse tensile modulus




But $\sigma_2 = \sigma_f = \sigma_m$, so that:

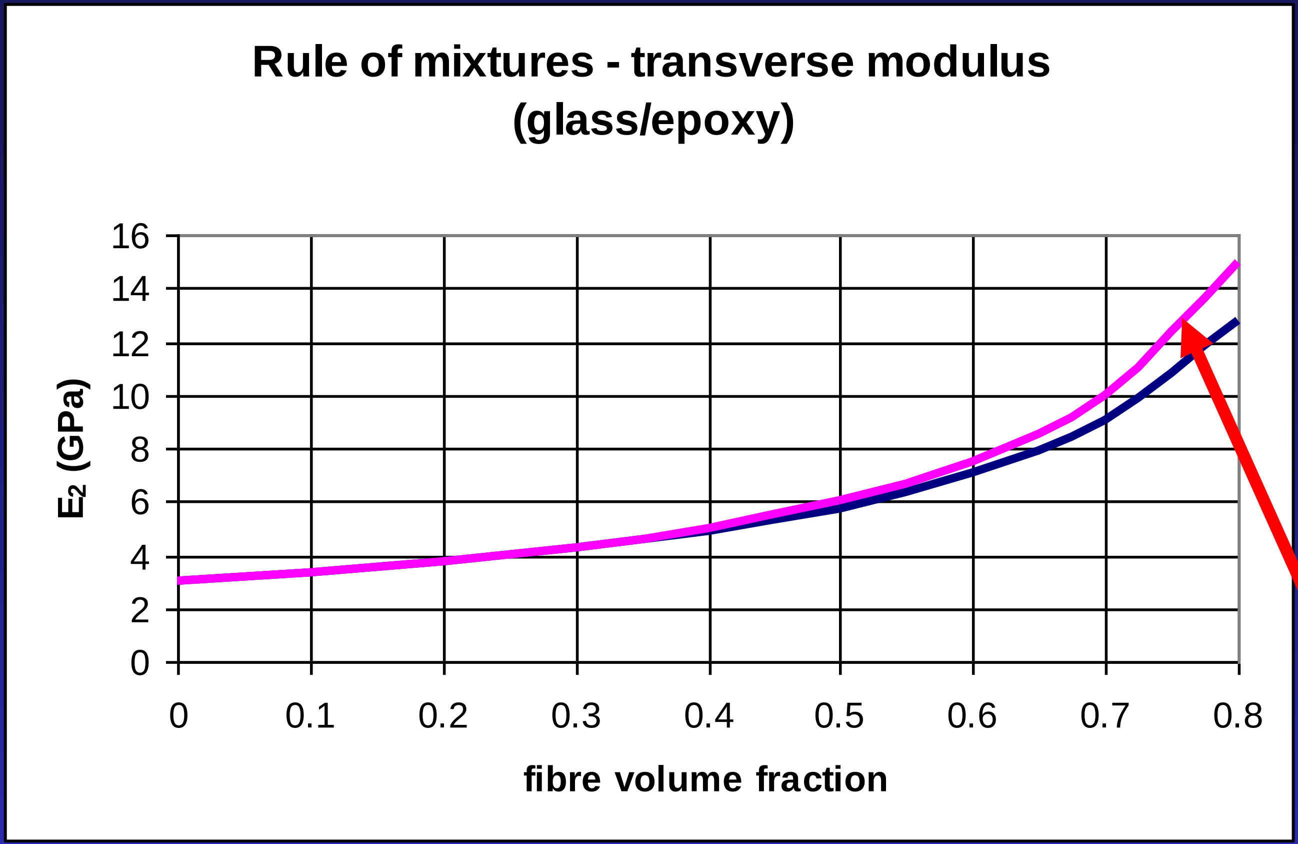
$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{(1-V_f)}{E_m}$$

or

$$E_2 = \frac{E_f E_m}{E_m V_f + E_f (1-V_f)}$$

$$\frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{V^m}{G^m}$$

$$\nu_{12} = \nu_{12}^f V^f + \nu^m V^m$$

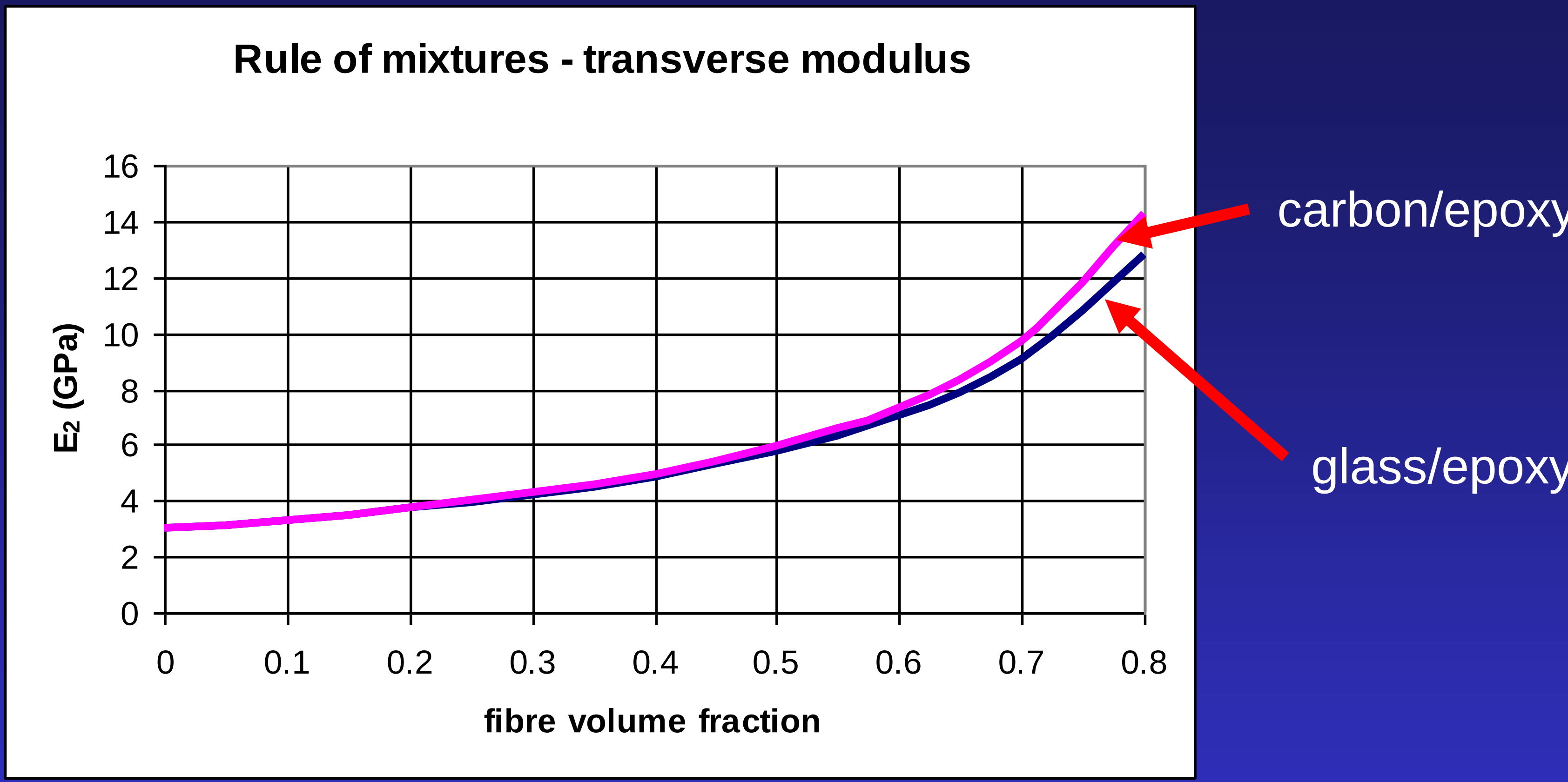



If $E_f >> E_m$,

$E_2 \approx E_m / (1-V_f)$

Note that $E_2$ is not particularly sensitive to $V_f$.

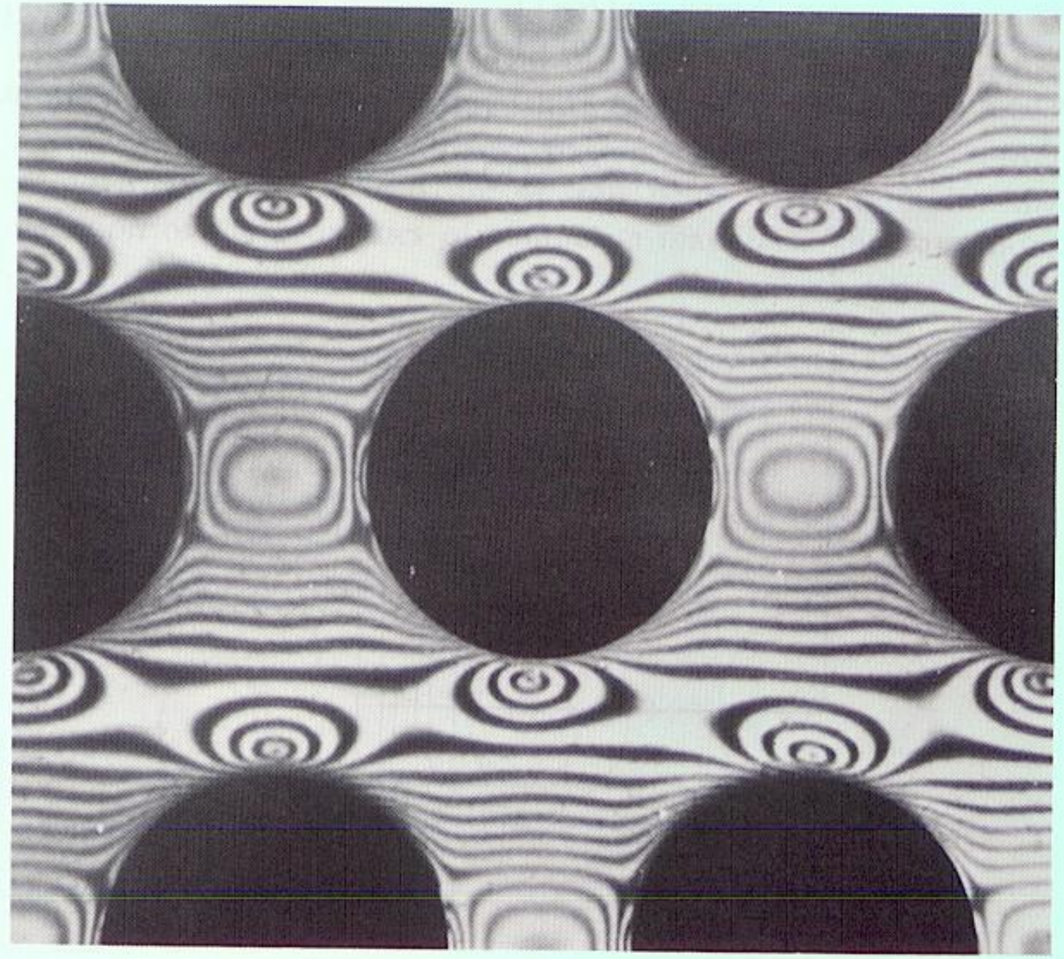
If $E_f >> E_m$, $E_2$ is almost independent of fibre property:




The transverse modulus is dominated by the matrix, and is virtually independent of the reinforcement.

The transverse rule of mixtures is not particularly accurate, due to the simplifications made - Poisson effects are not negligible, and the strain distribution is not uniform:

Fig. 5.8. Isochromatic fringes in a macromodel composite material loaded in transverse tension. From Puck (1967).



(source:  Hull, Introduction to Composite Materials, CUP)

# Underestimation of E2

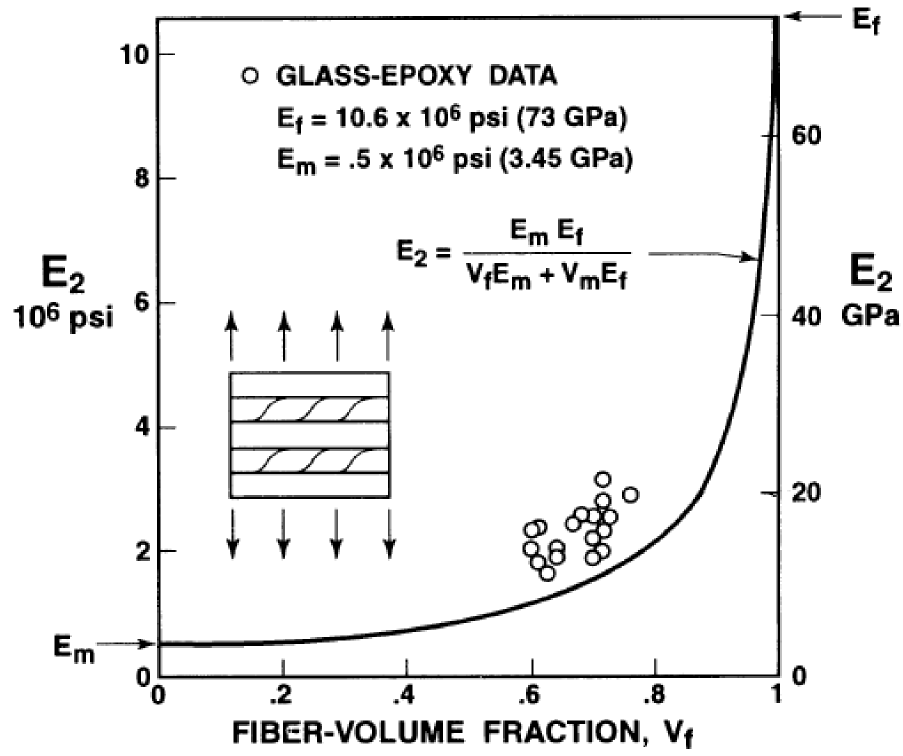



Figure 3-12 Predicted versus Measured $E_2$ *(Data from Tsai [3-1])*

# Unidirectional ply - transverse tensile modulus

Many theoretical studies have been undertaken to develop better micromechanical models (eg the semi-empirical Halpin-Tsai equations).

A simple improvement for transverse modulus is (isotropic fiber):

$$E_2 = \frac{E_f E'_m}{E'_m V_f + E_f (1 - V_f)}$$

where

$$E'_m = \frac{E_m}{1 - \nu_m^2}$$

# semi-empirical Halpin-Tsai equations

$$E_1 \cong E_f V_f + E_m V_m$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

$$\frac{M}{M_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

$$\eta = \frac{(M_f / M_m) - 1}{(M_f / M_m) + \xi}$$

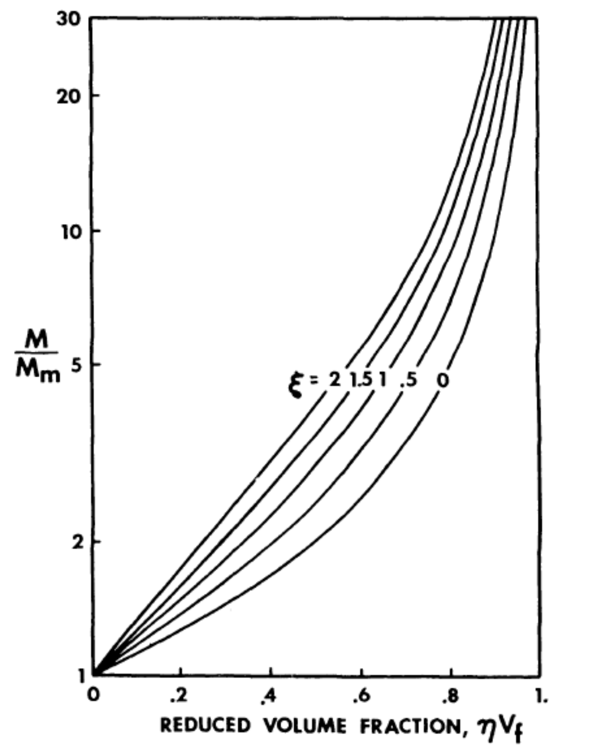



Figure 3-39 Master $M/M_m$ Curves for Various $\xi$

$M$ = composite material modulus $E_2$, $G_{12}$, or $\nu_{23}$
$M_f$ = corresponding fiber modulus $E_f$, $G_f$, or $\nu_f$
$M_m$ = corresponding matrix modulus $E_m$, $G_m$, or $\nu_m$

# Some improvements

$$\frac{1}{E_2} = \frac{\eta^f V^f}{E_2^f} + \frac{\eta^m V^m}{E^m}$$

$$\eta^f = \frac{E_1^f V^f + \left[\left(1 - \nu_{12}^f \nu_{21}^f\right) E^m + \nu^m \nu_{21}^f E_1^f\right] V^m}{E_1^f V^f + E^m V^m}$$

$$\eta^m = \frac{\left[\left(1 - {\nu^m}^2\right) E_1^f - \left(1 - \nu^m \nu_{12}^f\right) E^m\right] V^f + E^m V^m}{E_1^f V^f + E^m V^m}$$

# Generalised rule of mixtures for tensile modulus

$$E = \eta_L \eta_o E_f V_f + E_m (1-V_f)$$

$\eta_L$ is a length correction factor. Typically, $\eta_L \approx 1$ for fibres longer than about 10 mm.

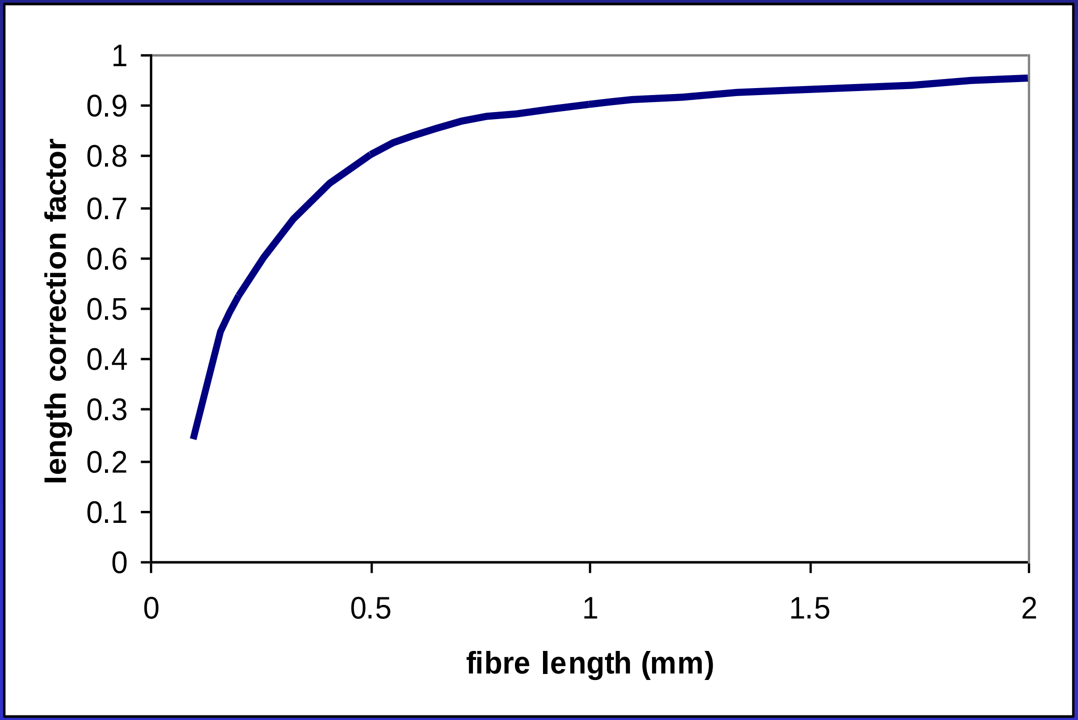
$\eta_o$ corrects for non-unidirectional reinforcement:

| | $\eta_o$ |
|---|---|
| unidirectional | 1.0 |
| biaxial | 0.5 |
| biaxial at $\pm 45^o$ | 0.25 |
| random (in-plane) | 0.375 |
| random (3D) | 0.2 |

# Theoretical length correction factor

$$\eta_L = 1 - \frac{\tanh(\beta L/2)}{(\beta L/2)}$$

$$\beta = \sqrt{\frac{8G_m}{E_f D^2 \ln(2R/D)}}$$




Theoretical length correction factor for glass fibre/epoxy, assuming inter-fibre separation of 20 D.

# Stiffness of short fibre composites

For aligned short fibre composites (difficult to achieve in polymers!), the rule of mixtures for modulus in the fibre direction is:

$$E = \eta_L E_f V_f + E_m (1 - V_f)$$

The length correction factor ($\eta_L$) can be derived theoretically. Provided L > 1 mm, $\eta_L$ > 0.9

For composites in which fibres are not perfectly aligned the full rule of mixtures expression is used, incorporating both $\eta_L$ *and* $\eta_o$.

In short fibre-reinforced thermosetting polymer composites, it is reasonable to assume that the fibres are always well above their critical length, and that the elastic properties are determined primarily by orientation effects.

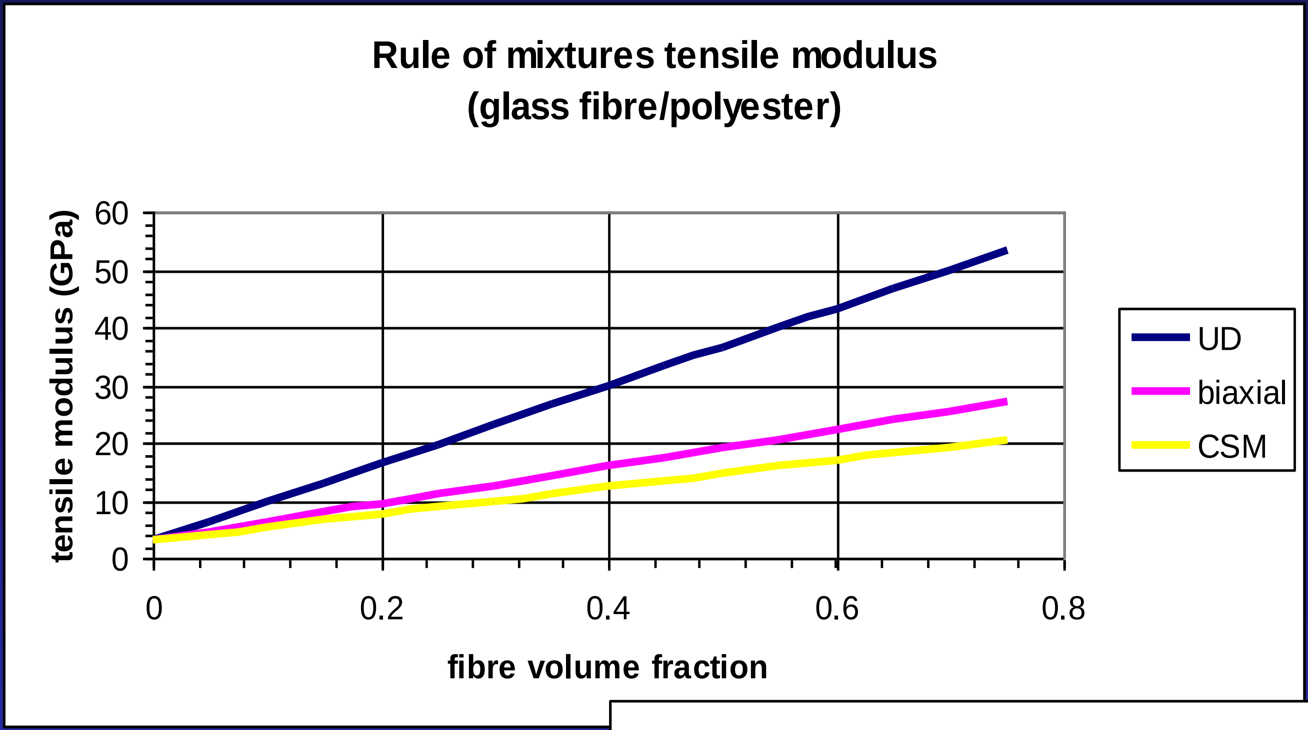
The following equations give reasonably accurate estimates for the isotropic in-plane elastic constants:

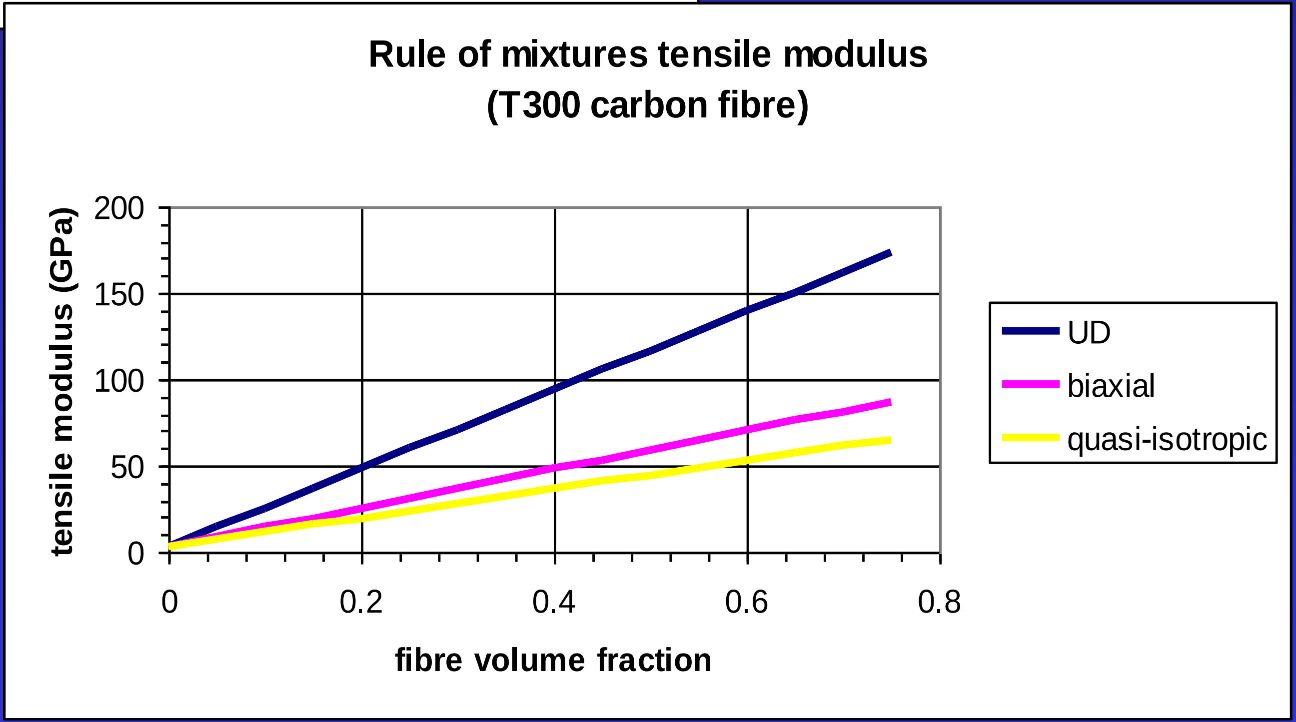
$$E = \frac{3}{8}E_1 + \frac{5}{8}E_2$$

$$G = \frac{1}{8}E_1 + \frac{1}{4}E_2$$

$$\nu = \frac{E}{2G} - 1$$

where $E_1$ and $E_2$ are the ‘UD’ values calculated earlier


Rule of mixtures tensile modulus
(glass fibre/polyester)
tensile modulus (GPa)
60
50
40
30
20
10
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
CSM


Rule of mixtures tensile modulus
(T300 carbon fibre)
tensile modulus (GPa)
200
150
100
50
0
0
0.2
0.4
0.6
0.8
fibre volume fraction
UD
biaxial
quasi-isotropic

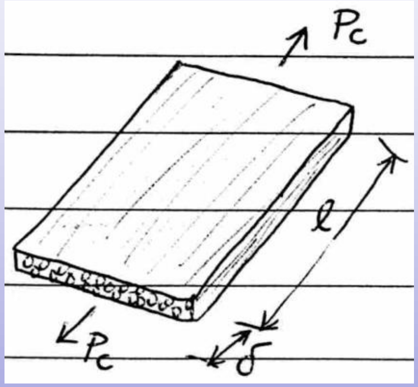
# Other rules of mixtures

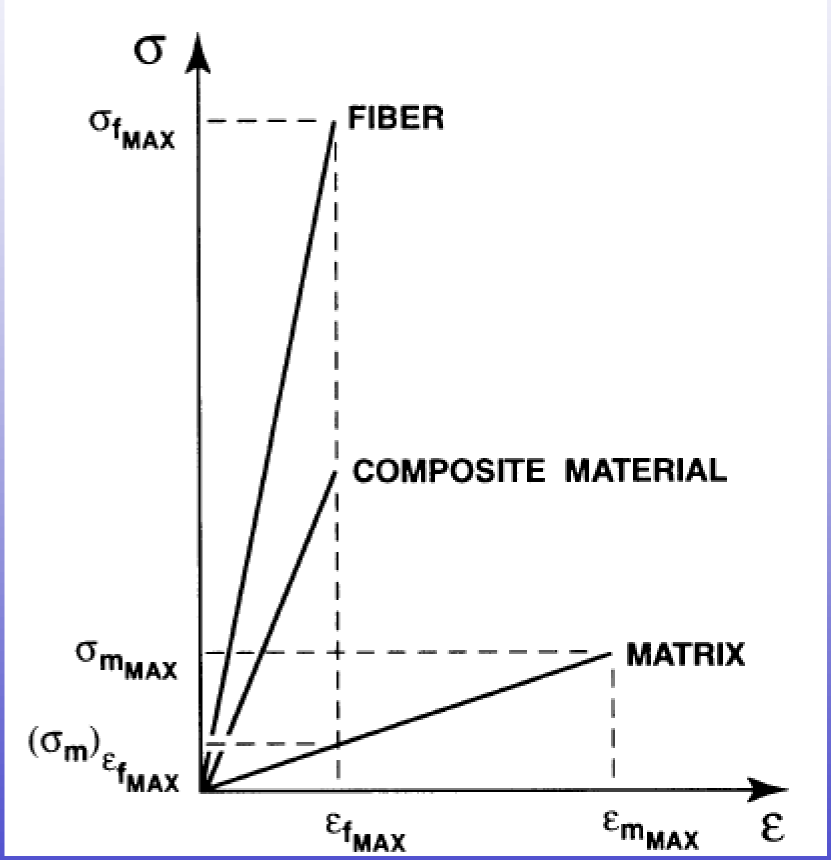
- Thermal expansion:

$$\alpha_1 = \frac{\alpha_1^f E_1^f V^f + \alpha^m E^m V^m}{E_1^f V^f + E^m V^m}$$

$$\alpha_2 = \left[\alpha_2^f - \left(\frac{E^m}{E_1}\right)\nu_1^f(\alpha^m - \alpha_1^f)V^m\right]V^f + \left[\alpha^m + \left(\frac{E_1^f}{E_1}\right)\nu^m(\alpha^m - \alpha_1^f)V^f\right]V^m$$

# Mechanics of materials: Longitudinal Strength



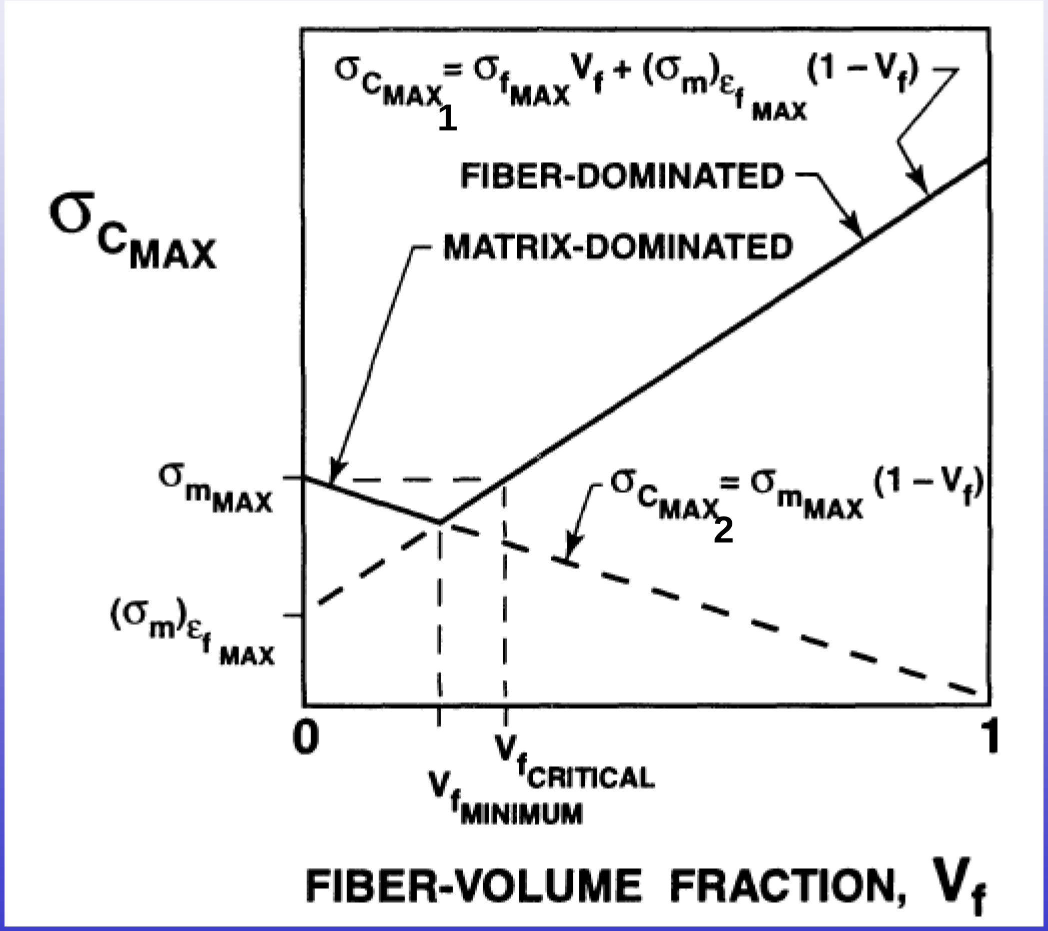

$$\sigma_c = \sigma_f v_f + \sigma_m v_m$$
$$= \sigma_f v_f + \sigma_m \left(1 - v_f\right)$$

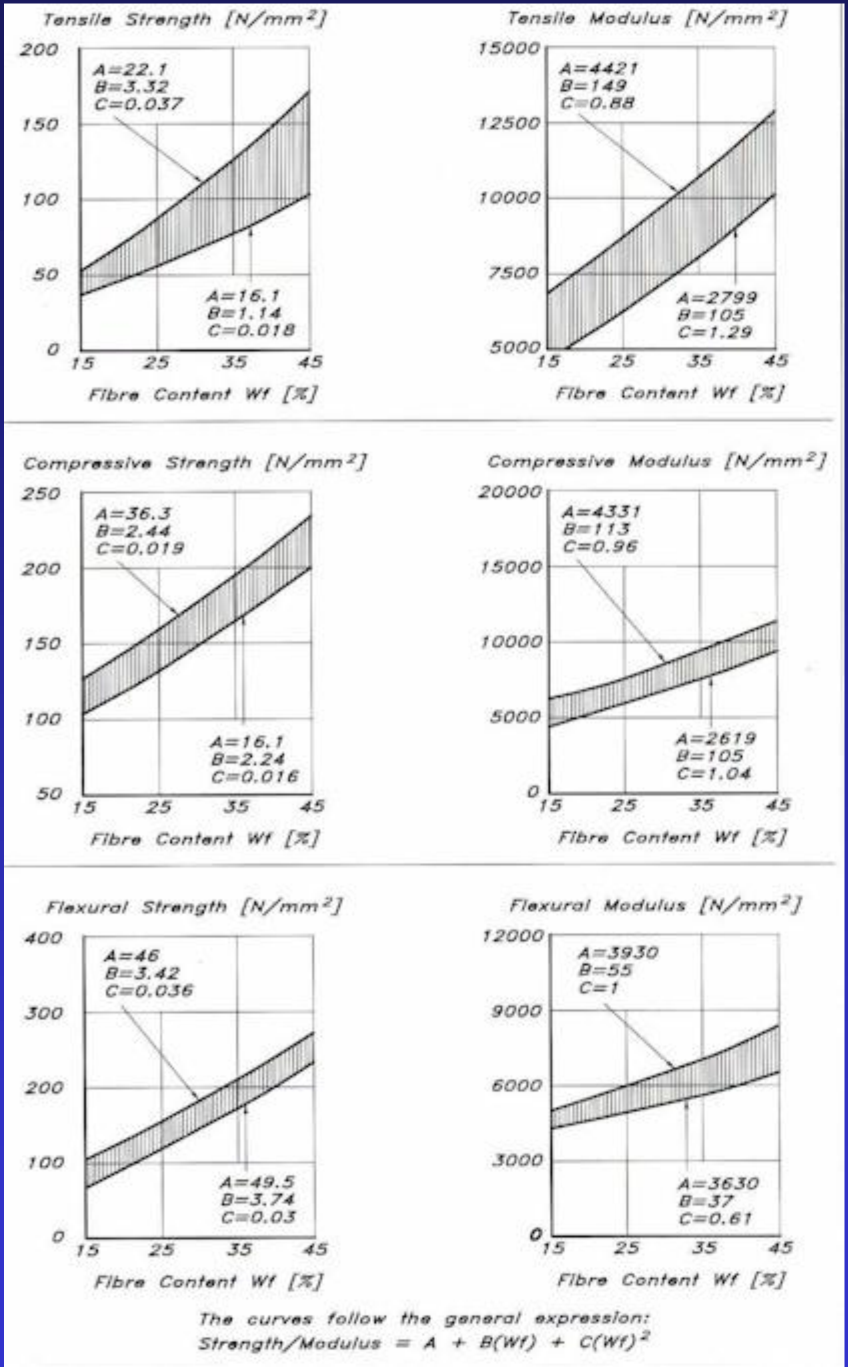
# Modes of Failure

$$\sigma_c = \max(\sigma_{c\max 1}, \sigma_{c\max 2})$$

fiber-controlled failure:

matrix-controlled failure:



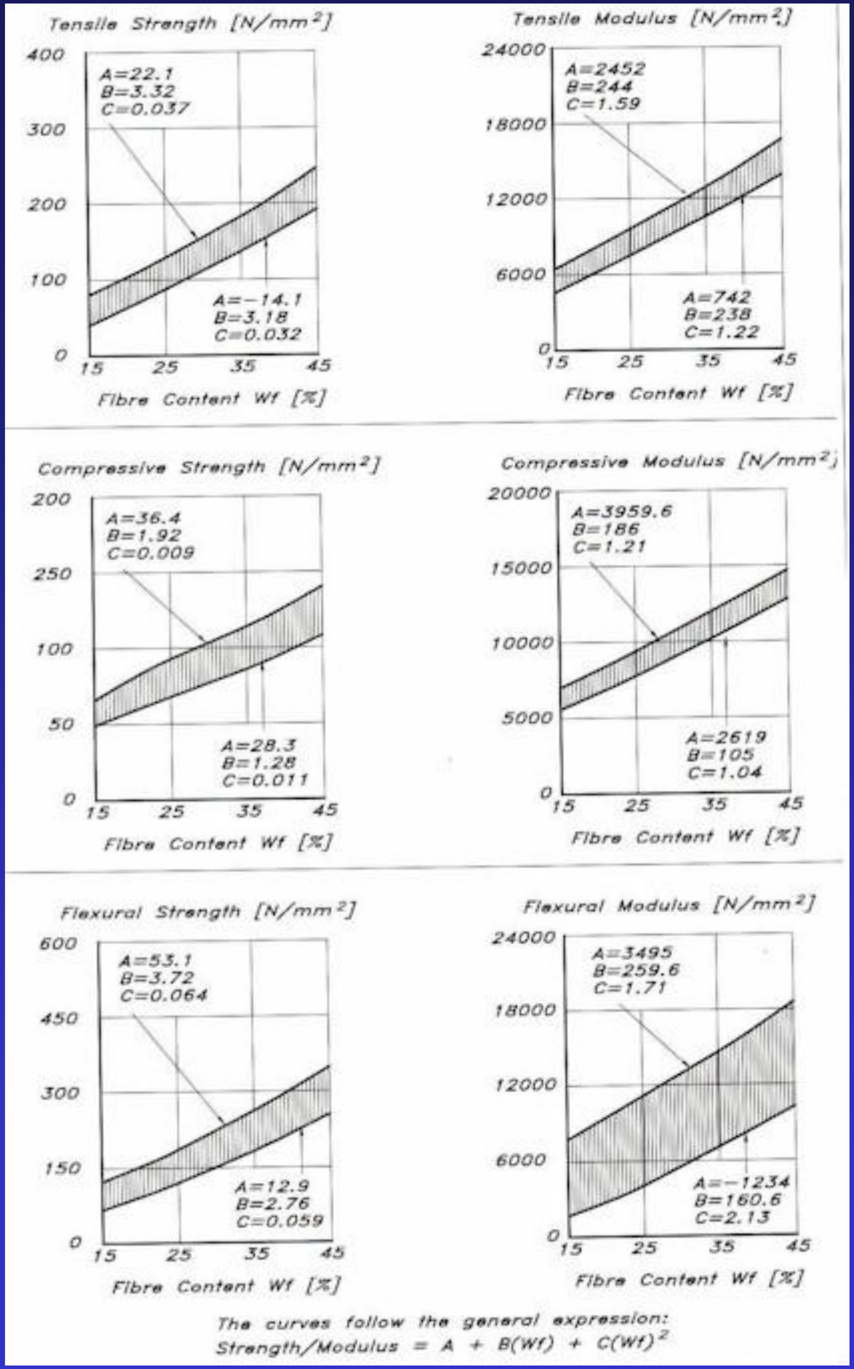



# Rules of mixture properties for CSM-polyester laminates

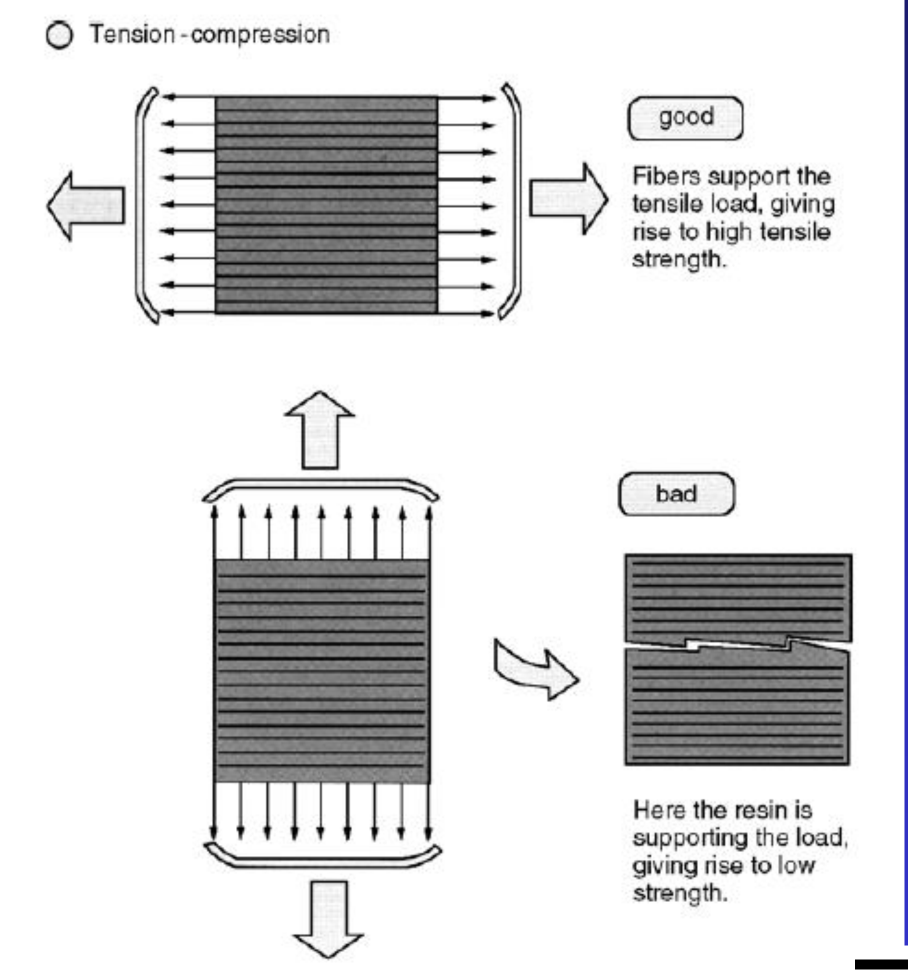
*Larsson & Eliasson, Principles of Yacht Design*

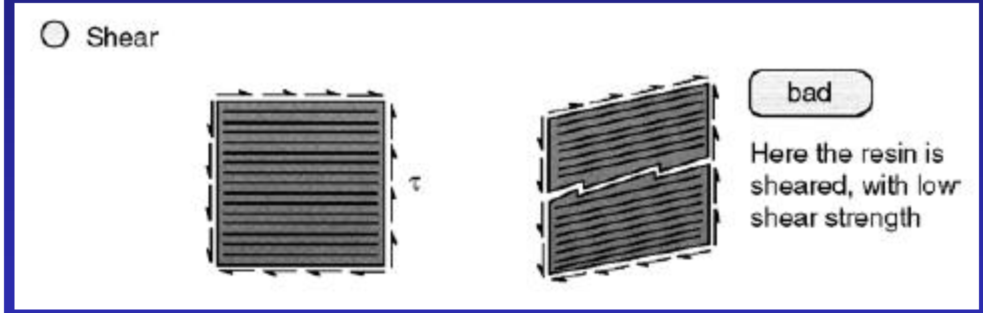
Rules of mixture properties for glass woven roving-polyester laminates

*Larsson & Eliasson, Principles of Yacht Design*

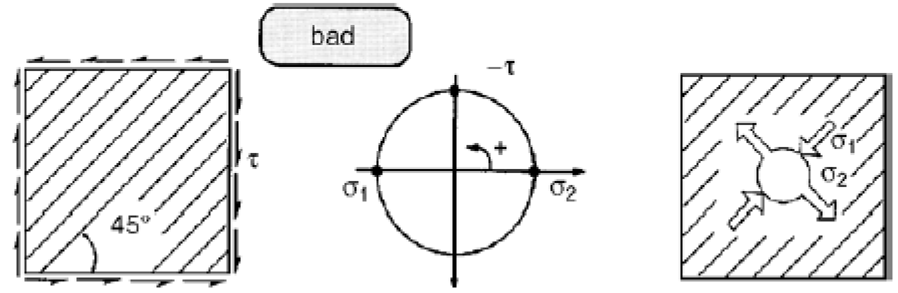


# Design

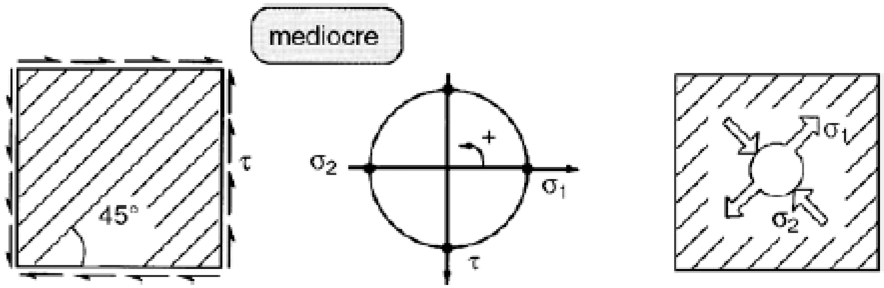


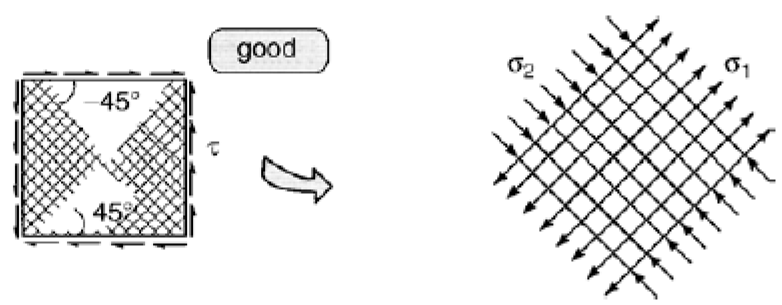

Effect of Ply Orientation




**Bad Design**

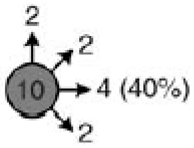


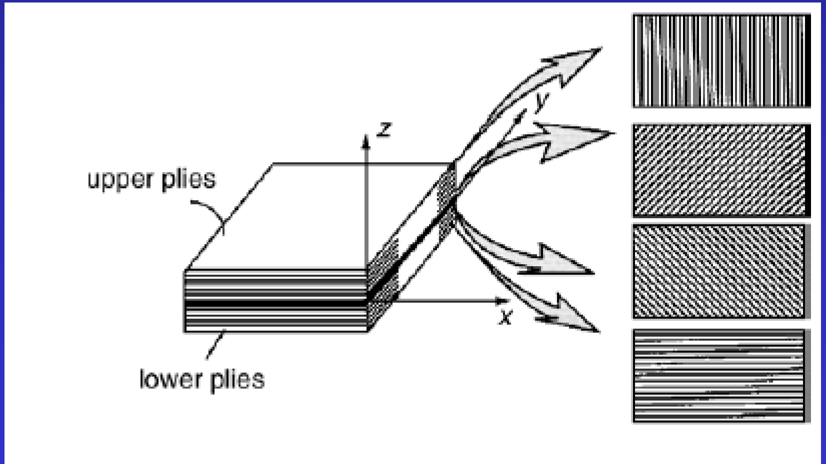

**Mediocre Design**




**Good Design**

# Analysis of Laminated Composites

| PLY NUMBER | ORIENTATION | CONVENTIONAL NOTATION | SYMBOL |
|---|---|---|---|
| 10 | 90° | | |
| 9 | 0° | | |
| 8 | 0° | | |
| 7 | − 45° | | |
| 6 | + 45° | | |
| mid plane | | $[90/0_2/-45/45]_s$ |  |
| 5 | + 45° | | |
| 4 | − 45° | | |
| 3 | 0° | | |
| 2 | 0° | | |
| 1 | 90° | | |




Laminate and its Middle Plane x–y