AKSORN

Work-Textbook

KEY

MASTERING MATHEMATICS

- Yeap Ban Har, Ph.D.
- Joseph Yeo Boon Wooi, Ph.D.
- Teh Keng Seng

Secondary

3A

Work-Textbook

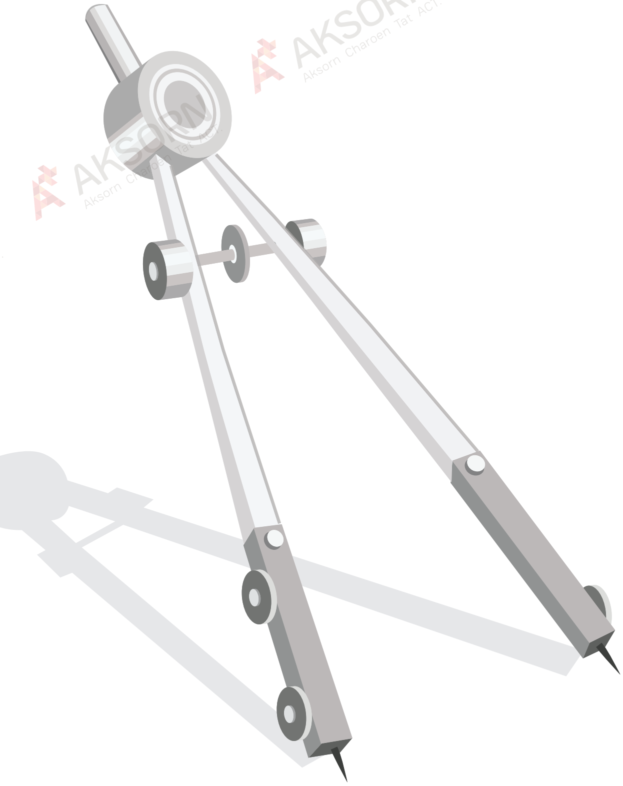
Mastering Mathematics

Secondary 3A

Secondary
3A

Subject Area: Mathematics

Based on Thailand's newly revised curriculum of B.E. 2560 (A.D. 2017)



KEY

Authors
Yeap Ban Har, Ph.D.
Joseph Yeo Boon Wooi, Ph.D.
Teh Keng Seng

Editors
Poranee Tatsanasaengsoon
Tipsuda Buakaew
Sathawut Anuraksakulwet
Kanjana Heenpooh

www.aksorn.com
Published and distributed in Thailand by:
Aksorn Charoen Tat ACT. Co., Ltd.
142 Tanao Road, Phra Nakhon, Bangkok, Thailand 10200
Tel./Fax. +66 2622 2999 (auto phone switch 20 lines)
Printed at: **Thai Romklao Co., Ltd.** Tel. +66 2903 9101-6

This Secondary Mathematics Series, adapted from the New Syllabus Mathematics series, is published in collaboration with Shing Lee Publishers Pte Ltd, Singapore.

SL Education

Preface

Mastering Mathematics is a series of Work-Textbooks written based on the B.E. 2560 (A.D. 2017) revised version of Thailand's Basic Education Core Curriculum B.E. 2551 (A.D. 2008). This series is adapted and developed from the New Syllabus Mathematics series in collaboration with Shing Lee Publishers Pte Ltd, Singapore aiming to enhance skills needed in the 21st Century, which include analytical skill, problem-solving skill, creativity, ICT skills and collaboration skill. These skills help students to act on economic, social, cultural and environmental changes potentially such that they are able to compete and live harmoniously with the world community.

Key Features

Worked Example

This shows students how to apply what they have learned to solved related problems and how to present their working clearly.

Practice Now

At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

Similar Questions

A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

Exercise

The questions are classified into three levels of difficulty—Basic, Intermediate and Advanced.

Summary

At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learned.

Review Exercise

This is included at the end of each chapter for the consolidation of learning of concepts.

Challenge Yourself

Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

Active Learning Activities

Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.

Class Discussion

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

Thinking Time

Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.

Journal Writing

Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.

Performance Task

Mini projects are designed to develop research and presentation skills in the students.

Marginal Notes

ATTENTION

This contains important information that students should know.

PROBLEM SOLVING TIP

This guides students on how to approach a problem.

INFORMATION

This includes information that may be of interest to students.

RECALL

This contains certain mathematical concepts or rules that students have learned previously.

JUST FOR FUN

This contains puzzles, fascinating facts and interesting stories as enrichment for students.

INTERNET RESOURCES

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

Contents

Interactive 3D on pages 155, 168, 172, 180

KEY

Table of Learning Standards and Indicators for Mastering Mathematics Secondary 3A

Learning Standard and Indicators / Chapter/Topic	Strand 1						Strand 2					Strand 3	
	Standard M 1.1	Standard M 1.2		Standard M 1.3			Standard M 2.1		Standard M 2.2			Standard M 3.1	Standard M 3.2
	Indicator	Indicator		Indicator			Indicator		Indicator			Indicator	Indicator
	-	1	2	1	2	3	1	2	1	2	3	1	1
Chapter 1 Linear Inequalities in One Variable	Taught in other levels												
1.1 Linear Inequalities in One Variable				✓									
1.2 Real-life Applications of Linear Inequalities in One Variable				✓									
1.3 Solving Simultaneous Linear Inequalities				✓									
Chapter 2 Quadratic Equations in One Variable													
2.1 Solving Quadratic Equations in One Variable by Factorization					✓								
2.2 Solving Quadratic Equations in One Variable by Completing the Square					✓								
2.3 Solving Quadratic Equations in One Variable by Using Formula					✓								
2.4 Real-life Applications of Quadratic Equations					✓								
Chapter 3 Quadratic Functions													
3.1 Graphs of Functions			✓										
3.2 Graphs of Quadratic Functions in the Form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$			✓										
3.3 Graphs of Quadratic Functions in the Form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$			✓										
Chapter 4 Pyramids, Cones and Spheres													
4.1 Pyramids							✓	✓					
4.2 Cones							✓	✓					
4.3 Spheres							✓	✓					
4.4 Composite Solids							✓	✓					
Chapter 5 Statistics													
5.1 Mean, Quartiles, Range and Interquartile Range												✓	
5.2 Box-and-Whisker Plots												✓	
Chapter 6 Probability													
6.1 Introduction to Probability													✓
6.2 Sample Space													✓
6.3 Probability of Single Events													✓
6.4 Real-life Applications of Probability													✓
6.5 Probability of Simple Combined Events													✓

Chapter 1

Linear Inequalities in One Variable

Domestic shipping rates may vary depending on the weight of a parcel and the shipping time. If we want to deliver a parcel weighing up to 500 grams to reach its intended destination within 3 days, should we send it via registered mail or express mail service?

Indicator

- Understand and apply the inequality properties of linear inequalities in one variable to analyze and solve relating problems. **(MA 1.3 G. 9/1)**

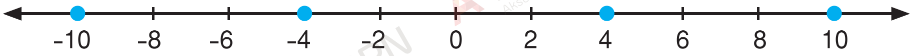
Compulsory Details

- Linear inequalities in one variable
- Solving linear inequalities in one variable
- Real-life applications of linear inequalities in one variable

1.1

Linear Inequalities in One Variable

We have learned how to represent numbers on a number line. A number on the number line is more than any number on its left and less than any number on its right.



Since the number 10 is to the right of the number 4, 10 is more than 4 (we write $10 > 4$). Similarly, -4 is to the right of -10, so -4 is more than -10 (we write $-4 > -10$). Alternatively, we can say that -10 is less than -4 (we write $-10 < -4$). $10 > 4$ and $-10 < -4$ are known as **inequalities**.

Notation	Meaning
$a > b$	a is more than b.
$a < b$	a is less than b.
$a \geq b$	a is more than or equal to b.
$a \leq b$	a is less than or equal to b.
$a \neq b$	a is not equal to b.

Inequalities are the notations that represent the relationships between numbers by using signs $>$, $<$, $\leq$, $\geq$ or $\neq$.

Similar Questions

Exercise 1A Question 1

Investigation

Consider the inequality 10 > 6 and complete the table.

Cases	Working	Inequality	Is the inequality sign reversed?	Conclusion
Multiplication by a positive number on both sides of the inequality 10 > 6	$10 \times 5 > 6 \times 5$	$50 > 30$	No	If $x > y$ and $c > 0$, then $cx > cy$.
Division by a positive number on both sides of the inequality 10 > 6	$10 \div 5 > 6 \div 5$	$2 > 1.2$	No	If $x > y$ and $c > 0$, then $\frac{x}{c} \boxed{>} \frac{y}{c}$
Multiplication by a negative number on both sides of the inequality 10 > 6	$10 \times (-5) > 6 \times (-5)$	$-50 < -30$	Yes	If $x > y$ and $c < 0$, then $cx \boxed{<} cy$
Division by a negative number on both sides of the inequality 10 > 6	$10 \div (-5) > 6 \div (-5)$	$-2 < -1.2$	No	If $x > y$ and $c < 0$, then $\frac{x}{c} \boxed{<} \frac{y}{c}$

From **Investigation**, we can conclude that:

We can multiply or divide both sides of an inequality by a positive number without having to reverse the inequality sign. However, if we multiply or divide both sides of an inequality by a negative number, we will have to reverse the inequality sign.

Let x, y and c be any real numbers.

If $x \geq y$ and $c > 0$, then $cx \geq cy$ and $\frac{x}{c} \geq \frac{y}{c}$.

If $x \geq y$ and $c < 0$, then $cx \leq cy$ and $\frac{x}{c} \leq \frac{y}{c}$.

Solving Linear Inequalities in One Variable

Let x be the number of students attending a workshop.

- If there are 100 students, we write $x = 100$.

  This is a **linear equation in variable**. It has only one answer, i.e. only one value of x satisfies the equation.

- If there are less than 100 students, we write $x < 100$.

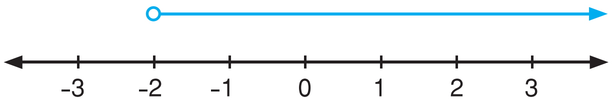
  This is a **linear inequality in variable**. It has many answers, i.e. many values of x satisfy the inequality. x can take on integer values ranging from 0 to 99 inclusive. In this context, non-integer and negative values of x have no meaning.

For a linear inequality in variable, all values that satisfy the inequality are called the **solutions of the inequality**. We can use a number line to illustrate these solutions. For example:

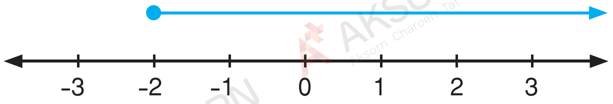
ATTENTION

On a number line, a circle ○ is used to indicate that x cannot take on a particular value whereas a dot ● is used to indicate that x can take on the particular value.

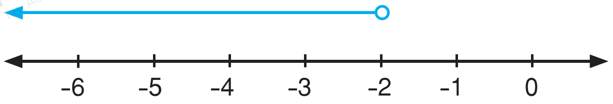
- some of the solutions of the inequality $x > -2$ are the real numbers more than -2, e.g. -1, 0, 1.5, …



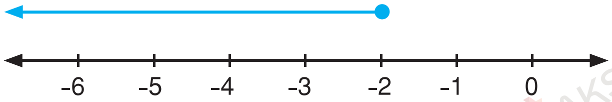
- some of the solutions of the inequality $x \geq -2$ are the real numbers more than or equal to -2, e.g. -2, -1, 1.5, …



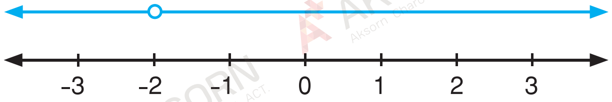
- some of the solutions of the inequality $x < -2$ are the real numbers less than -2, e.g. -3, -4, -5.25, …



- some of the solutions of the inequality $x \leq -2$ are the real numbers less than or equal to -2, e.g. -2, -4, -5.25, …



- some of the solutions of the inequality $x \neq -2$ are the real numbers not equal to -2, e.g. -3, -1, 1.5, …



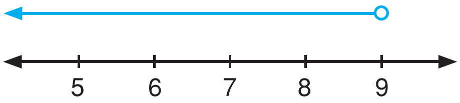
Worked Example 1

Solve each of the following inequalities and illustrate the solutions on a number line.

1) $3x < 27$ 2) $-2x \geq 4$ 3) $x \neq 8$

Solution:

1)
$$\begin{aligned} 3x &< 27 \\ x &< \frac{27}{3} \\ x &< 9 \end{aligned}$$

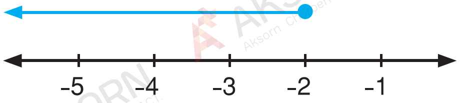


Verify the solution

When we substitute x with the real numbers less than 9, e.g. 8, 7, 6.5 in the inequality $3x < 27$, the inequality will always be true.

Therefore, the solutions of the inequality $3x < 27$ are the real numbers less than 9.

2)
$$\begin{aligned} -2x &\geq 4 \\ (-2x) \times (-1) &\leq 4 \times (-1) \\ 2x &\leq -4 \\ x &\leq \frac{-4}{2} \\ x &\leq -2 \end{aligned}$$

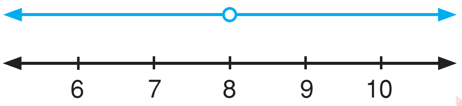


Verify the solution

When we substitute x with the real numbers less than or equal to -2, e.g. -2, -2.5, -4 in the inequality $-2x \geq 4$, the inequality will always be true.

Therefore, the solutions of the inequality $-2x \geq 4$ are the real numbers less than or equal to -2.

3) $x \neq 8$



Verify the solution

When we substitute x with the real numbers not equal to 8, e.g. 6, 7, 9.5 in the inequality $x \neq 8$, the inequality will always be true.

Therefore, the solutions of the inequality $x \neq 8$ are the real numbers not equal to 8.

Practice Now

Similar Questions

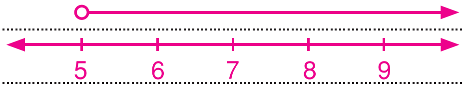
Exercise 1A Questions 2, 7-9, 13

Solve each of the following inequalities and illustrate the solutions on a number line.

1) $15x > 75$

$x > \frac{75}{15}$

$x > 5$



Verify the solution

When we substitute x with the real numbers more than 5, e.g. 6, 7, 8.5 in the inequality $15x > 75$, the inequality will always be true.

Therefore, the solutions of the inequality $15x > 75$ are the real numbers more than 5.

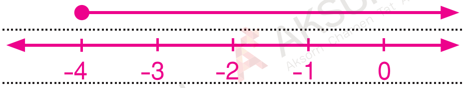
2) $-4x \leq 16$

$(-4x) \times (-1) \geq 16 \times (-1)$

$4x \geq -16$

$x \geq \frac{-16}{4}$

$x \geq -4$



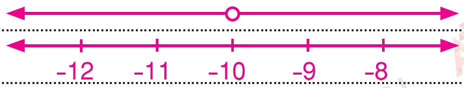
Verify the solution

When we substitute x with the real numbers more than or equal to -4, e.g. -3, -2, -1.25 in the inequality $-4x \leq 16$, the inequality will always be true.

Therefore, the solutions of the inequality $-4x \leq 16$ are the real numbers more than or equal to -4.

3) $x \neq -10$

$x \neq -10$



Verify the solution

When we substitute x with the real numbers not equal to -10, e.g. -8, -9.5, -11 in the inequality $x \neq -10$, the inequality will always be true.

Therefore, the solutions of the inequality $x \neq -10$ are the real numbers not equal to -10.

Similar Questions

Exercise 1A Question 3

Investigation

Fill in each blank with '>' or '<' and answer the questions.

1. 1) $6 < 12$ $\qquad 6 + 2 \boxed{<} 12 + 2$ $\qquad 6 - 4 \boxed{<} 12 - 4$

2) If $6 < 12$ and a is a real number, then $6 + a \boxed{<} 12 + a$ and $6 - a \boxed{<} 12 - a$.

3) If $12 > 6$ and a is a real number, then $12 + a \boxed{>} 6 + a$ and $12 - a \boxed{>} 6 - a$.

2. 1) $-6 \boxed{<} 12$ $\qquad -6 + 2 \boxed{<} 12 + 2$ $\qquad -6 - 4 \boxed{<} 12 - 4$

2) If $-6 < 12$ and a is a real number, then $-6 + a \boxed{<} 12 + a$ and $-6 - a \boxed{<} 12 - a$.

3) If $12 > -6$ and a is a real number, then $12 + a \boxed{>} -6 + a$ and $12 - a \boxed{>} -6 - a$.

3. 1) $6 \boxed{>} -12$ $\qquad 6 + 2 \boxed{>} -12 + 2$ $\qquad 6 - 4 \boxed{>} -12 - 4$

2) What do you observe about your answers in Question 1?

When we add or subtract a positive or a negative number to and from both sides of an inequality, the inequality sign does not change.

4. Do the conclusions which you have drawn in questions 1-3 apply to $6 \leq 12$?

What about $12 \geq 6$?

Yes, the conclusion applies.

KEY

From **Investigation**, we observe that when we add or subtract a positive or a negative number from both sides of an inequality, the inequality sign does not change.

Let x, y and a be any real numbers.
If $x > y$, then $x + a > y + a$ and $x - a > y - a$.
If $x \geq y$, then $x + a \geq y + a$ and $x - a \geq y - a$.
If $x < y$, then $x + a < y + a$ and $x - a < y - a$.
If $x \leq y$, then $x + a \leq y + a$ and $x - a \leq y - a$.

Journal Writing

One example of real-life applications of inequalities is the speed limit of vehicles traveling on expressways. Give one real-life application of inequalities.

Worked Example 2

Solve each of the following inequalities and illustrate the solutions on a number line.

1) $x + 4 < 3$

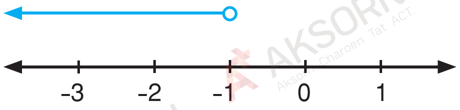
2) $-4y - 5 \geq 11$

Solution:

1) $x + 4 < 3$

$x + 4 - 4 < 3 - 4$ *(subtract 4 from both sides)*

$x < -1$



Verify the solution

When we substitute x with the real numbers less than -1, e.g. -2, -3.5, -4 in the inequality $x + 4 < 3$, the inequality will always be true.

Therefore, the solutions of the inequality $x + 4 < 3$ are the real numbers less than -1.

2) $-4y - 5 \geq 11$

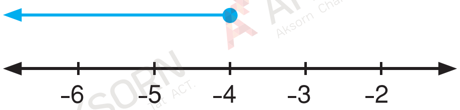
$-4y - 5 + 5 \geq 11 + 5$ *(add 5 to both sides)*

$-4y \geq 16$

$4y \leq -16$ *(multiply both sides by -1)*

$\frac{4y}{4} \leq \frac{-16}{4}$ *(divide by 4 on both sides)*

$y \leq -4$



Verify the solution

When we substitute y with the real numbers less than or equal to -4, e.g. -4, -5.25, -7 in the inequality $-4y - 5 \geq 11$, the inequality will always be true.

Therefore, the solutions of the inequality $-4y - 5 \geq 11$ are the real numbers less than or equal to -4.

Practice Now

Similar Questions

Exercise 1A Questions 4(1)-(4)

KEY

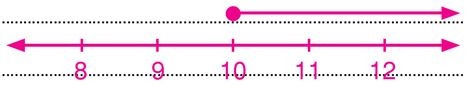
Solve each of the following inequalities and illustrate the solutions on a number line.

1) $x - 3 \geq 7$

Add 3 to both sides.

$x - 3 + 3 \geq 7 + 3$

$x \geq 10$



2) $-2y + 4 > 3$

Subtract 4 from both sides.

$-2y + 4 - 4 > 3 - 4$

$-2y > -1$

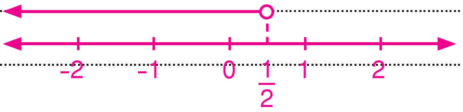
Multiply both sides by -1.

$2y < 1$

Divide both sides by 2.

$\frac{2y}{2} < \frac{1}{2}$

$y < \frac{1}{2}$



Thinking Time

1. Given an equation in the form $ax + b = c$ where a, b and c are constants and $a > 0$, list the steps to find the value of x. Would the steps change if $a < 0$?

   $ax + b = c$ where a, b and c are constants and $a > 0$

   Step 1: Arrange the terms such that the constants are all on one side of the equation, i.e.

   $ax = c - b$.

   Step 2: Divide both sides by a to solve for x.

   The steps will be the same if $a < 0$.

2. Given an inequality in the form $ax + b > c$ where a, b and c are constants and $a > 0$, list the steps to find the range of values of x. How would the steps change if $a < 0$?

   $ax + b = c$ where a, b and c are constants and $a > 0$

   Step 1: Arrange the terms such that the constants are all on one side of the inequality, i.e.

   $ax > c - b$.

   Step 2: Divide both sides by a to solve for x. The inequality sign (>) remains.

   The steps will change if $a < 0$ such that the inequality sign will change to <.

3. Given an inequality in the form $ax + b \geq c$ where a, b and c are constants and $a > 0$, list the steps to find the range of values of x. How would the steps change if $a < 0$?

   $ax + b \geq c$ where a, b and c are constants and $a > 0$

   Step 1: Arrange the terms such that the constants are all on one side of the inequality, i.e.

   $ax \geq c - b$.

   Step 2: Divide both sides by a to solve for x. The inequality sign (≥) remains.

   The steps will change if $a < 0$ such that the inequality sign will change to ≤.

4. For the inequalities $ax + b < c$, $ax + b \leq c$, $ax + b > c$ and $ax + b \geq c$, we say that the corresponding linear equation is $ax + b = c$. How is the solution of each inequality related to that of the corresponding linear equation?

   The solutions of $ax + b > c$ and $ax + b < c$ do not include the solution of its corresponding linear equation $ax + b = c$.

   The solutions of $ax + b \geq c$ and $ax + b \leq c$ include the solution of its corresponding linear equation $ax + b = c$.

Worked Example 3

Solve the inequality $8 - x > 3$, illustrate the solution on a number line and answer the questions.

1) Find the largest prime value of x that satisfies the inequality.

2) Find the positive integer values of x that satisfy the inequality.

Solution:

ATTENTION

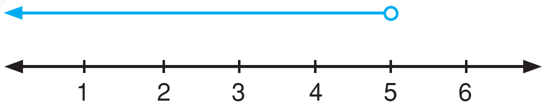
We should always justify the solutions of inequalities.

$8 - x > 3$

$8 - x - 8 > 3 - 8$ *(subtract 8 from both sides)*

$-x > -5$

$x < 5$ *(multiply both sides by -1)*



1) The largest prime value of x that satisfies the inequality is 3.

2) The positive integer values of x that satisfy the inequality are 1, 2, 3 and 4.

Practice Now

Similar Questions

Exercise 1A Question 5

Solve the inequality $5 - x < -9$, illustrate the solution on a number line and answer the questions.

1) Find the smallest prime value of x that satisfies the inequality.

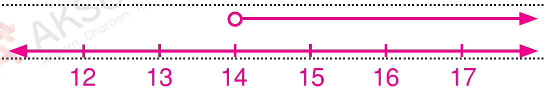
2) Find the smallest perfect cube value of x that satisfies the inequality.

$5 - x < -9$

$5 - x - 5 < -9 - 5$ *(subtract 5 from both sides)*

$-x < -14$

$x > 14$ *(multiply both sides by -1)*



1) The smallest prime value of x that satisfies the inequality is 17.

2) The smallest perfect cube value of x that satisfies the inequality is $27 = 3^3$

Worked Example 4

Solve each of the following inequalities.

1) $3x - 2 > 2(1 - x)$

2) $\frac{y}{4} \leq \frac{y + 1}{7}$

Solution:

1)
$$\begin{aligned} 3x - 2 &> 2(1 - x) \\ 3x - 2 &> 2 - 2x \\ 3x - 2 + 2x &> 2 - 2x + 2x && \textit{(add 2x to both sides)} \\ 5x - 2 &> 2 \\ 5x - 2 + 2 &> 2 + 2 && \textit{(add 2 to both sides)} \\ 5x &> 4 \\ x &> \frac{4}{5} && \textit{(divide by 5 on both sides)} \end{aligned}$$

2)
$$\begin{aligned} \frac{y}{4} &\leq \frac{y + 1}{7} \\ \frac{y}{4} \times 4 \times 7 &\leq \frac{y + 1}{7} \times 4 \times 7 && \textit{(multiply by 4 × 7 on both sides)} \\ 7y &\leq 4(y + 1) \\ 7y &\leq 4y + 4 \\ 7y - 4y &\leq 4y + 4 - 4y && \textit{(subtract 4y from both sides)} \\ 3y &\leq 4 \\ y &\leq \frac{4}{3} && \textit{(divide by 3 on both sides)} \\ y &\leq 1\frac{1}{3} \end{aligned}$$

PROBLEM SOLVING TIP

The LCM of 4 and 7 is 4 × 7.

Practice Now

Similar Questions

Exercise 1A Questions 4(5)-(8), 6, 10-12, 14

Solve each of the following inequalities.

1) $15x + 1 < 5(3 + x)$

$$\begin{aligned} 15x + 1 &< 15 + 5x \\ 15x + 1 - 5x &< 15 + 5x - 5x && \textit{(subtract 5x from both sides)} \\ 10x + 1 &< 15 \\ 10x + 1 - 1 &< 15 - 1 && \textit{(subtract 1 from both sides)} \\ 10x &< 14 \\ x &< 1\frac{2}{5} && \textit{(divide by 10 on both sides)} \end{aligned}$$

2) $\frac{16y}{3} \geq \frac{y + 1}{2}$

$\frac{16y}{3} \times 3 \times 2 \geq \frac{y + 1}{2} \times 3 \times 2$ *(multiply by 3 × 2 on both sides)*

$32y \geq 3(y + 1)$

$32y \geq 3y + 3$

$32y - 3y \geq 3y + 3 - 3y$ *(subtract 3y from both sides)*

$29y \geq 3$

$y \geq \frac{3}{29}$ *(divide by 29 on both sides)*

Exercise 1A

Basic Level

1. **Fill in each box with <, >, ≤ or ≥.**

1) If $x > y$, then $5x \boxed{>} 5y$.

2) If $x < y$, then $\frac{x}{-20} \boxed{>} \frac{y}{-20}$.

3) If $x \geq y$, then $-3x \boxed{\leq} -3y$.

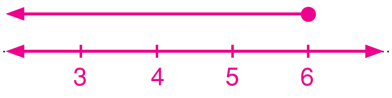
4) If $x \leq y$, then $\frac{x}{10} \boxed{\leq} \frac{y}{10}$.

2. **Solve each of the following inequalities and illustrate the solutions on a number line.**

1) $3x \leq 18$

$x \leq \frac{18}{3}$

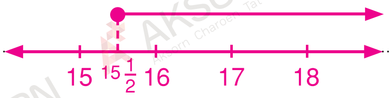
$x \leq 6$



2) $4x \geq 62$

$x \geq \frac{62}{4}$

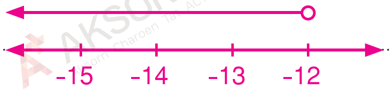
$x \geq 15\frac{1}{2}$



3) $3y < -36$

$y < \frac{-36}{3}$

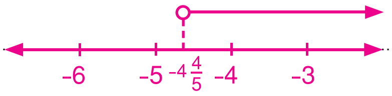
$y < -12$



4) $5y > -24$

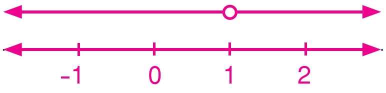
$y > \frac{-24}{5}$

$y > -4\frac{4}{5}$



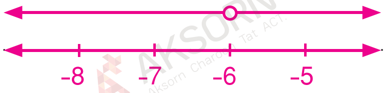
5) $x \neq 1$

$x \neq 1$



6) $x \neq -6$

$x \neq -6$



3. **Fill in each box with <, >, ≤ or ≥.**

1) $5 + h \boxed{<} 7 + h$ where h is a real number.

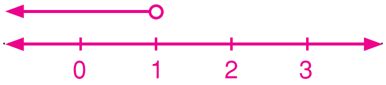
2) $5 - k \boxed{<} 7 - k$ where k is a real number.

4. **Solve each of the following inequalities and illustrate the solutions on a number line.**

1) $a + 2 < 3$

$a + 2 - 2 < 3 - 2$

$a < 1$

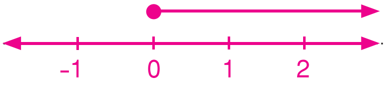


2) $4 - b \leq 4$

$4 - b - 4 \leq 4 - 4$

$-b \leq 0$

$b \geq 0$



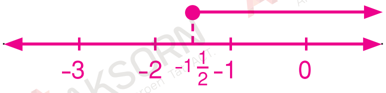
3) $-2c - 1 \leq 2$

$-2c - 1 + 1 \leq 2 + 1$

$-2c \leq 3$

$c \geq \frac{3}{-2}$

$c \geq -1\frac{1}{2}$

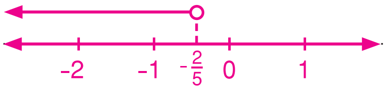


4) $2 + 5d < 0$

$2 + 5d - 2 < 0 - 2$

$5d < -2$

$d < -\frac{2}{5}$



5) $e - 7 \geq 1 - e$

$e - 7 + e \geq 1 - e + e$

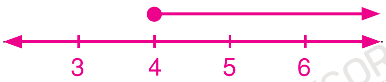
$2e - 7 \geq 1$

$2e - 7 + 7 \geq 1 + 7$

$2e \geq 8$

$e \geq \frac{8}{2}$

$e \geq 4$

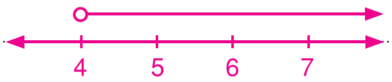


6) $5f > 4(f + 1)$

$5f > 4f + 4$

$5f - 4f > 4f + 4 - 4f$

$f > 4$



7) $4g + 5 \geq 2(-2g)$

$4g + 5 \geq -4g$

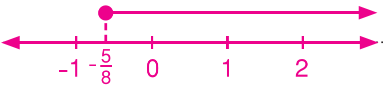
$4g + 5 + 4g \geq -4g + 4g$

$8g + 5 \geq 0$

$8g + 5 - 5 \geq 0 - 5$

$8g \geq -5$

$g \geq -\frac{5}{8}$



8) $3(1 - 4h) > 8 - 7h$

$3 - 12h > 8 - 7h$

$3 - 12h + 7h > 8 - 7h + 7h$

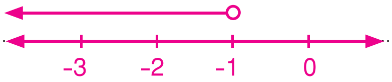
$3 - 5h > 8$

$3 - 5h - 3 > 8 - 3$

$-5h > 5$

$h < \frac{5}{-5}$

$h < -1$



5. **Solve the inequality $7 + 2x \leq 16$, illustrate the solution on a number line and answer the questions.**

1) Find the largest integer value of x that satisfies the inequality.

2) Find the largest perfect square value of x that satisfies the inequality.

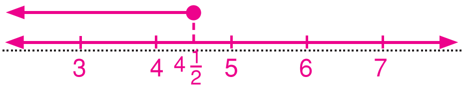
$7 + 2x \leq 16$

$7 + 2x - 7 \leq 16 - 7$

$2x \leq 9$

$x \leq \frac{9}{2}$

$x \leq 4\frac{1}{2}$



1) The largest integer value of x that satisfies the inequality is 4.

2) The largest perfect square value of x that satisfies the inequality is 4.

6. **Solve the inequality $3 - 4x > 3x - 18$, illustrate the solution on a number line and answer the questions.**

1) Find the largest prime value(s) of x that satisfies the inequality.

2) Does $x = 0$ satisfy the inequality? Explain your answer.

$3 - 4x > 3x - 18$

$3 - 4x - 3 > 3x - 18 - 3$

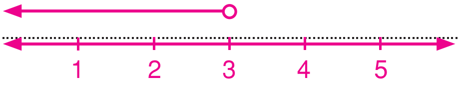
$-4x > 3x - 21$

$-4x - 3x > 3x - 21 - 3x$

$-7x > -21$

$x < \frac{-21}{-7}$

$x < 3$



1) The prime value of x that satisfies the inequality is 2.

2) Yes, $x = 0$ is less than 3.

Intermediate Level

7. **Find the smallest rational value of y that satisfies the inequality $8 \leq 7y$.**

$8 \leq 7y$

$7y \geq 8$

$y \geq \frac{8}{7}$

$y \geq 1\frac{1}{7}$

Therefore, the smallest rational value of y that satisfies the inequality $8 \leq 7y$ is $1\frac{1}{7}$.

8. **Find the smallest prime value of x that satisfies the inequality $20x > 33$.**

$20x > 33$

$x > \frac{33}{20}$

$x > 1\frac{13}{20}$

Therefore, the smallest prime value of x that satisfies the inequality $20x > 33$ is 2.

9. Find the greatest odd integer value of x that satisfies the inequality $-3x > -105$.

$-3x > -105$

$3x < 105$

$x < \frac{105}{3}$

$x < 35$

Therefore, the greatest odd integer value of x that satisfies the inequality $-3x > -105$ is 33.

10. Solve each of the following inequalities and illustrate each solution on a number line.

1) $4(p + 1) < -3(p - 4)$

$4p + 4 < -3p + 12$

$4p + 4 + 3p < -3p + 12 + 3p$

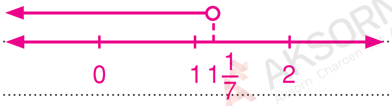
$7p + 4 < 12$

$7p + 4 - 4 < 12 - 4$

$7p < 8$

$p < \frac{8}{7}$

$p < 1\frac{1}{7}$



2) $6 - (1 - 2q) \geq 3(5q - 2)$

$6 - 1 + 2q \geq 15q - 6$

$5 + 2q \geq 15q - 6$

$5 + 2q - 15q \geq 15q - 6 - 15q$

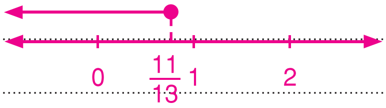
$5 - 13q \geq -6$

$5 - 13q - 5 \geq -6 - 5$

$-13q \geq -11$

$q \leq \frac{-11}{-13}$

$q \leq \frac{11}{13}$



11. Solve each of the following inequalities.

1) $\frac{1}{4}(a - 2) + \frac{2}{3} < \frac{1}{6}(a - 4)$

$$\begin{aligned}
\left[\frac{1}{4}(a - 2) + \frac{2}{3}\right] \times 12 &< \frac{1}{6}(a - 4) \times 12 \\
3(a - 2) + 8 &< 2(a - 4) \\
3a - 6 + 8 &< 2a - 8 \\
3a + 2 &< 2a - 8 \\
3a + 2 - 2a &< 2a - 8 - 2a \\
a + 2 &< -8 \\
a + 2 - 2 &< -8 - 2 \\
a &< -10
\end{aligned}$$

2) $\frac{b + 1}{2} + \frac{3b + 1}{4} \leq \frac{3b - 1}{4} + 2$

$$\begin{aligned}
\left(\frac{b + 1}{2} + \frac{3b + 1}{4}\right) \times 4 &\leq \left(\frac{3b - 1}{4} + 2\right) \times 4 \\
2(b + 1) + (3b + 1) &\leq (3b - 1) + 8 \\
2b + 2 + 3b + 1 &\leq 3b - 1 + 8 \\
5b + 3 &\leq 3b + 7 \\
5b + 3 - 3b &\leq 3b + 7 - 3b \\
2b + 3 - 3 &\leq 7 - 3 \\
2b &\leq 4 \\
b &\leq \frac{4}{2} \\
b &\leq 2
\end{aligned}$$

3) $\frac{1}{5}(3c + 4) - \frac{1}{3}(c + 1) \geq 1 - \frac{1}{3}(c + 5)$

$$\left[\frac{1}{5}(3c + 4) - \frac{1}{3}(c + 1)\right] \times 5 \times 3 \geq \left[1 - \frac{1}{3}(c + 5)\right] \times 5 \times 3$$

$$3(3c + 4) - 5(c + 1) \geq 15 - 5(c + 5)$$

$$9c + 12 - 5c - 5 \geq 15 - 5c - 25$$

$$4c + 7 \geq -10 - 5c$$

$$4c + 7 + 5c \geq -10 - 5c + 5c$$

$$9c + 7 \geq -10$$

$$9c + 7 - 7 \geq -10 - 7$$

$$9c \geq -17$$

$$c \geq \frac{-17}{9}$$

$$c \geq -1\frac{8}{9}$$

4) $4\left(\frac{d}{3} + \frac{3}{4}\right) > 3\left(\frac{d}{2} - 5\right)$

$$\frac{4}{3}d + \frac{12}{4} > \frac{3}{2}d - 15$$

$$\frac{4}{3}d + 3 - \frac{3}{2}d > \frac{3}{2}d - 15 - \frac{3}{2}d$$

$$-\frac{1}{6}d + 3 > -15$$

$$-\frac{1}{6}d + 3 - 3 > -15 - 3$$

$$-\frac{1}{6}d > -18$$

$$-\frac{1}{6}d \times 6 > -18 \times 6$$

$$-d > -108$$

$$d < 108$$

12. Find the largest possible value of p that satisfies the inequality $\frac{1}{6}(2 - p) - 3 \geq \frac{p}{10}$.

$$\frac{1}{6}(2 - p) - 3 \geq \frac{p}{10}$$
$$\left[\frac{1}{6}(2 - p) - 3\right] \times 6 \times 10 \geq \frac{p}{10} \times 6 \times 10$$
$$(2 - p)(10) - 180 \geq 6p$$
$$20 - 10p - 180 \geq 6p$$
$$-10p - 160 \geq 6p$$
$$-10p - 160 - 6p \geq 6p - 6p$$

$$-16p - 160 \geq 0$$
$$-16p - 160 + 160 \geq 0 + 160$$
$$-16p \geq 160$$
$$p \leq \frac{160}{-16}$$
$$p \leq -10$$

Therefore, the largest possible value of p is -10.

Advanced Level

13. Find all the integer values of y that satisfies the inequalities $5y < 35$ and $-2y \leq -6$.

$5y < 35$ and $-2y \leq -6$

$y < \frac{35}{5}$ and $2y \geq 6$

$y < 7$ and $y \geq \frac{6}{2}$

$y \geq 3$

Therefore, the integer values of y are 3, 4, 5 and 6.

14. If a satisfies the inequality $\frac{1}{3}(2x - 7) \leq \frac{3x + 2}{2}$, then find the smallest possible value of a^2.

$$\frac{1}{3}(2x - 7) \leq \frac{3x + 2}{2}$$
$$\left[\frac{1}{3}(2x - 7)\right] \times 3 \times 2 \leq \left(\frac{3x + 2}{2}\right) \times 3 \times 2$$
$$(2x - 7)(2) \leq (3x + 2)(3)$$
$$4x - 14 \leq 9x + 6$$
$$4x - 14 - 9x \leq 9x + 6 - 9x$$
$$-5x - 14 \leq 6$$
$$-5x - 14 + 14 \leq 6 + 14$$
$$-5x \leq 20$$
$$x \geq \frac{20}{-5}$$
$$x \geq -4$$

Therefore, the smallest possible value of a^2 is $0^2 = 0$.

1.2 Real-life Applications of Linear Inequalities in One Variable

In this section, we shall take a look at how inequalities are used to solve problems.

Worked Example 5

A curry puff costs 9 baht. By setting up an inequality, find the maximum number of curry puffs that can be bought with 200 baht.

Solution:

Let the number of curry puffs that can be bought with 200 baht be x.

We get
$$\begin{aligned} 9x &\leq 200 \\ x &\leq \frac{200}{9} \\ x &\leq 22\frac{2}{9}. \end{aligned}$$

Therefore, the maximum number of curry puffs that can be bought with 200 baht is 22.

KEY

Practice Now

Similar Questions

Exercise 1B Question 1

A bus can ferry a maximum of 45 people. By setting up an inequality, find the minimum number of buses that are needed to furry 520 people.

Let the number of buses that are needed to furry 520 people be x.

We get
$$\begin{aligned} 45x &\geq 520 \\ x &\geq \frac{520}{45} \\ x &\geq 11\frac{5}{9}. \end{aligned}$$

Therefore, the minimum number of buses that are needed to furry 520 people is 12.

Worked Example 6

Darcy scored 66 marks for his first class test and 72 marks for his second class test. What is the maximum mark he must score for his third class test to meet his target of obtaining an average of 75 marks or more for the three test?

Solution:

Let x be the marks scored by Darcy in his third class test.

We get
$$\begin{aligned}
\frac{66 + 72 + x}{3} &\geq 75 \\
\frac{66 + 72 + x}{3} \times 3 &\geq 75 \times 3 \\
66 + 72 + x &\geq 225 \\
138 + x &\geq 225 \\
138 + x - 138 &\geq 225 - 138 \\
x &\geq 87.
\end{aligned}$$

Therefore, Darcy must score at least 87 marks for his third class test.

Practice Now

Similar Questions

Exercise 1B Questions 2, 7-8

The minimum mark to obtain a Grade A is 75. Cheryn managed to achieve an average of Grade A for three of her English quizzes. What is the minimum mark she scored in her first quiz if she scored 76 and 89 marks in her second and third quiz, respectively?

Let x be the marks scored by Cheryn in her first quiz.

We get
$$\begin{aligned}
\frac{x + 76 + 89}{3} &\geq 75 \\
\frac{x + 76 + 89}{3} \times 3 &\geq 75 \times 3 \\
x + 76 + 89 &\geq 225 \\
x + 165 &\geq 225 \\
x + 165 - 165 &\geq 225 - 165 \\
x &\geq 60.
\end{aligned}$$

Therefore, Cheryn must score at least 60 marks for her first quiz.

Worked Example 7

An IQ test consists of 20 multiple choice questions. 3 points are awarded for a correct answer, and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Rosie attempted a total of 19 questions and her total score for the IQ test was above 32. Find the minimum number of correct answers she obtained.

Solution:

Let x be the number of correct answers.

Then, the number of incorrect answers is equal to $19 - x$.

Since Rosie scored more than 32 marks,

we get

$$\begin{aligned} 3x - (19 - x) &> 32 \\ 3x - 19 + x &> 32 \\ 4x - 19 + 19 &> 32 + 19 \\ 4x &> 51 \\ x &> \frac{51}{4} \\ x &> 12.75. \end{aligned}$$

Therefore, Rosie obtained at least 13 correct answers.

Similar Questions

Exercise 1B Questions 3, 9-10

Practice Now

Paul has 12 pieces of 100 baht and 50 baht in his wallet. If the total value of all the notes is less than 950 baht, what is the maximum number of 100 baht notes that he has?

Let x be the number of 100 baht notes.

Then, the number of 50 baht notes is $12 - x$.

Since the total value of the notes is less than 950 baht,

we get

$$\begin{aligned} 100x + 50(12 - x) &< 950 \\ 10x + 5(12 - x) &< 95 \\ 10x + 60 - 5x &< 95 \\ 5x + 60 &< 95 \\ 5x + 60 - 60 &< 95 - 60 \end{aligned}$$

$$\begin{aligned} 5x &< 35 \\ x &< \frac{35}{5} \\ x &< 7. \end{aligned}$$

Therefore, the maximum number of 100 baht notes that Paul has is 6.

1.3 Solving Simultaneous Linear Inequalities

To solve linear inequalities simultaneously, we find the solution(s) to each inequality separately, then we consider only the common solutions of the inequalities.

For example, given that $x \geq 5$ and $x \leq 8$, then the range of values of x which satisfies both inequalities is $5 \leq x \leq 8$.

Does $x = 1$ satisfy both $3x \leq x + 6$ and $2x + 4 < 3x + 6$? Does $x = -3$ satisfy both $3x \leq x + 6$ and $2x + 4 < 3x + 6$?

ATTENTION

To check if $x = 1$ satisfies the inequality $3x \leq x + 6$, substitute $x = 1$ into the inequality and check if LHS $\leq$ RHS.

Worked Example 8

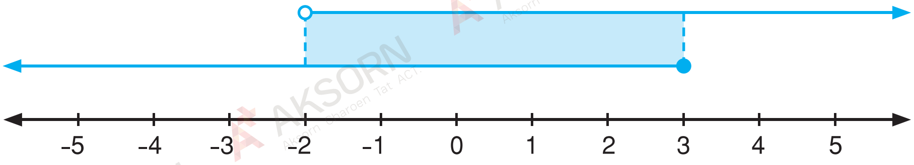
Find the range of values of x for which $3x \leq x + 6$ and $2x + 4 < 3x + 6$.

Solution:

$$\begin{array}{rcl}
3x & \leq & x + 6 \\
3x - x & \leq & x + 6 - x \\
2x & \leq & 6 \\
x & \leq & \frac{6}{2} \\
x & \leq & 3
\end{array}
\qquad \text{and} \qquad
\begin{array}{rcl}
2x + 4 & < & 3x + 6 \\
2x + 4 - 3x & < & 3x + 6 - 3x \\
-x + 4 & < & 6 \\
-x + 4 - 4 & < & 6 - 4 \\
-x & < & 2 \\
x & > & -2
\end{array}$$

ATTENTION

-2 is not a solution to the inequality.



Therefore, the solutions satisfying both inequalities $3x \leq x + 6$ and $2x + 4 < 3x + 6$ are $-2 < x \leq 3$.

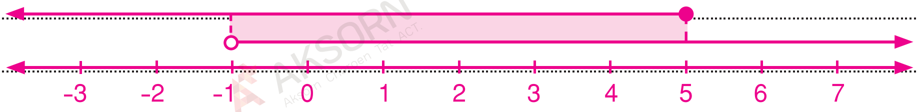
Practice Now

Similar Questions

Exercise 1B Questions 4-5, 11

Find the range of values of x for which $2x - 3 \leq 7$ and $2x + 1 > -3x - 4$.

$$\begin{aligned} 2x - 3 &\leq 7 \\ 2x - 3 + 3 &\leq 7 + 3 \\ 2x &\leq 10 \\ x &\leq \frac{10}{2} \\ x &\leq 5 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x + 1 &> -3x - 4 \\ 2x + 1 + 3x &> -3x - 4 + 3x \\ 5x + 1 &> -4 \\ 5x &> -5 \\ x &> -1 \end{aligned}$$



Therefore, the solutions satisfying both inequalities are $-1 < x \leq 5$.

Worked Example 9

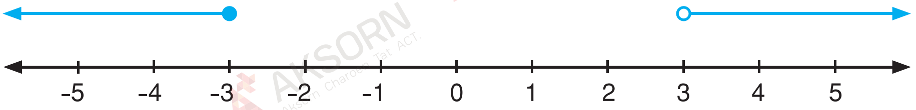
Solve the inequality $4x+14 \leq x+5 < 3x-1$ and illustrate the solution on a number line.

Solution:

Write the inequality $4x + 14 \leq x + 5 < 3x - 1$ in the form of $4x + 14 \leq x + 5$ and $x + 5 < 3x - 1$.

Then, solve the inequalities $4x + 14 \leq x + 5$ and $x + 5 < 3x - 1$.

$$\begin{aligned} 4x + 14 &\leq x + 5 \\ 4x + 14 - x &\leq x + 5 - x \\ 3x + 14 &\leq 5 \\ 3x + 14 - 14 &\leq 5 - 14 \\ 3x &\leq -9 \\ x &\leq -3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 5 &< 3x - 1 \\ x + 5 - 3x &< 3x - 1 - 3x \\ -2x + 5 &< -1 \\ -2x + 5 - 5 &< -1 - 5 \\ -2x &< -6 \\ 2x &> 6 \\ x &> 3 \end{aligned}$$



From the number line, the graphs of the inequalities $4x + 14 \leq x + 5$ and $x + 5 < 3x - 1$ have no overlapping region.

Therefore, there is no solution that satisfies both inequalities.

Practice Now

Similar Questions

Exercise 1B Questions 6, 13-18

Solve the inequality $8x + 13 \le 4x - 3 < 5x - 11$ and illustrate the solution on a number line.

Write the inequality $8x + 13 \le 4x - 3 < 5x - 11$ in the form $8x + 13 \le 4x - 3$ and $4x - 3 < 5x - 11$.

Then, solve the inequalities $8x + 13 \le 4x - 3$ and $4x - 3 < 5x - 11$.

$8x + 13 \le 4x - 3$ and $4x - 3 < 5x - 11$

$8x + 13 - 4x \le 4x - 3 - 4x$ $\quad$ $4x - 3 - 5x < 5x - 11 - 5x$

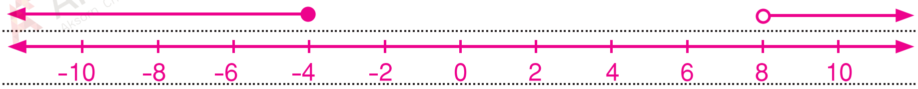
$4x + 13 \le -3$ $\quad$ $-x - 3 < -11$

$4x + 13 - 13 \le -3 - 13$ $\quad$ $-x - 3 + 3 < -11 + 3$

$4x \le -16$ $\quad$ $-x < -8$

$x \le \frac{-16}{4}$ $\quad$ $x > 8$

$x \le -4$



From the number line, the graphs of the inequalities $8x + 13 \le 4x - 3$ and $4x - 3 < 5x - 11$ have no overlapping region.

Therefore, there is no solution that satisfies both inequalities.

Performance Task

The table shows the postage rates for packages from Thailand to Singapore by a local company.

Mass (x g)	Postage (baht)
$0 < x \le 250$	720
$250 < x \le 500$	760
$500 < x \le 1{,}000$	860
$1{,}000 < x \le 1{,}500$	970
$1{,}500 < x \le 2{,}000$	1,070

Search on the Internet for the postage rates for parcels to Malaysia, New Zealand and Finland, displaying your findings in a table similar to the above.

Exercise 1B

Basic Level

1. A van can ferry a maximum of 12 people. By setting up an inequality, find the minimum number of vans that are needed to ferry 80 people.

Let the number of vans that are needed to ferry 80 people be x.

We get $12x \geq 80$

$x \geq \frac{80}{12}$

$x \geq 6\frac{2}{3}$.

Therefore, the minimum number of vans that are needed to ferry 80 people is 7.

2. On weekends, a movie ticket costs 150 baht. Form an inequality and solve it to find the maximum number of tickets Kate can buy with 3,500 baht.

Let the number of tickets Kate can buy be x.

We get $150x \leq 3{,}500$

$x \leq \frac{3{,}500}{150}$

$x \leq 23\frac{1}{3}$.

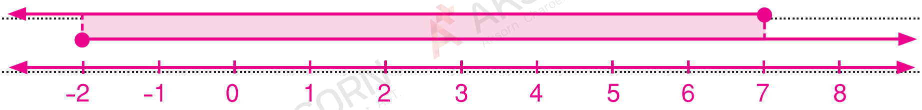
Therefore, the maximum number of tickets Kate can buy is 23.

3. Find the range of values of x for which $x - 4 \leq 3$ and $3x \geq -6$.

$x - 4 \leq 3$ and $3x \geq -6$

$x - 4 + 4 \leq 3 + 4$ $\qquad x \geq \frac{-6}{3}$

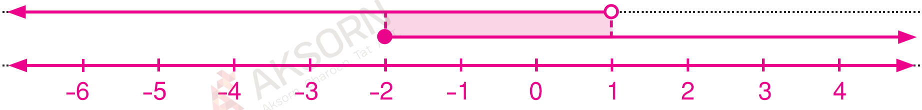
$x \leq 7$ $\qquad x \geq -2$



Therefore, the solution satisfying both inequalities $x - 4 \leq 3$ and $3x \geq -6$ is $-2 \leq x \leq 7$.

4. Find the integer values of x for which $5x - 1 < 4$ and $3x + 5 \geq x + 1$.

$5x - 1 < 4$	and	$3x + 5 \geq x + 1$
$5x - 1 + 1 < 4 + 1$		$3x + 5 - x \geq x + 1 - x$
$5x < 5$		$2x + 5 \geq 1$
$x < 1$		$2x + 5 - 5 \geq 1 - 5$
		$2x \geq -4$
		$x \geq -2$



From the number line, the values of x that satisfy both inequalities $5x - 1 < 4$ and $3x + 5 \geq x + 1$ are $-2 \leq x < 1$.

Therefore, the integer values of x that satisfy both inequalities $5x - 1 < 4$ and $3x + 5 \geq x + 1$ are -2, -1 and 0.

KEY

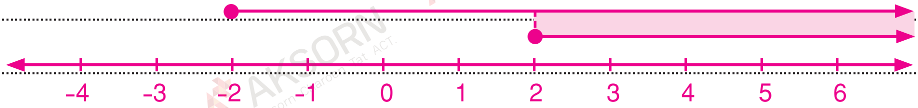
5. Solve each of the following pairs of inequalities and illustrate each solution on a number line.

1) $-4 \leq 2x \leq 3x - 2$

Write the inequality $-4 \leq 2x \leq 3x - 2$ in the form $-4 \leq 2x$ and $2x \leq 3x - 2$.

Then, solve the inequalities $-4 \leq 2x$ and $2x \leq 3x - 2$.

$-4 \leq 2x$	and	$2x \leq 3x - 2$
$2x \geq -4$		$2x - 3x \leq 3x - 2 - 3x$
$x \geq -2$		$-x \leq -2$
		$x \geq \frac{-2}{-1}$
		$x \geq 2$



Therefore, the solution satisfying the inequality $-4 \leq 2x \leq 3x - 2$ is $x \geq 2$.

2) $1 - x < -2 \le 3 - x$

Write the inequality $1 - x < -2 \le 3 - x$ in the form $1 - x < -2$ and $-2 \le 3 - x$.

Then, solve the inequalities $1 - x < -2$ and $-2 \le 3 - x$.

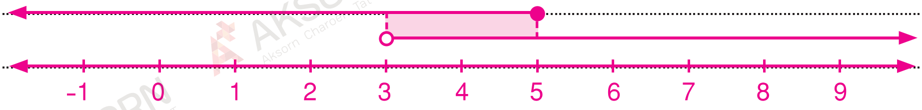
$1 - x < -2$ and $-2 \le 3 - x$

$1 - x - 1 < -2 - 1$ $\qquad$ $-2 + x \le 3 - x + x$

$-x < -3$ $\qquad$ $-2 + x \le 3$

$x > \frac{-3}{-1}$ $\qquad$ $-2 + x + 2 \le 3 + 2$

$x > 3$ $\qquad$ $x \le 5$



Therefore, the solutions satisfying the inequality $1 - x < -2 \le 3 - x$ is $3 < x \le 5$.

3) $3x - 3 < x - 9 < 2x$

Write the inequality $3x - 3 < x - 9 < 2x$ in the form of $3x - 3 < x - 9$ and $x - 9 < 2x$.

Then, solve the inequalities $3x - 3 < x - 9$ and $x - 9 < 2x$.

$3x - 3 < x - 9$ and $x - 9 < 2x$

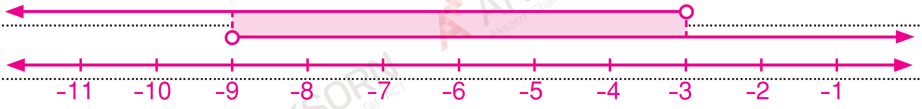
$3x - 3 - x < x - 9 - x$ $\qquad$ $x - 9 - 2x < 2x - 2x$

$2x - 3 < -9$ $\qquad$ $-x - 9 < 0$

$2x - 3 + 3 < -9 + 3$ $\qquad$ $-x - 9 + 9 < 0 + 9$

$2x < -6$ $\qquad$ $-x < 9$

$x < -3$ $\qquad$ $x > -9$



Therefore, the solutions satisfying the inequality $3x - 3 < x - 9 < 2x$ is $-9 < x < -3$.

4) $2x \leq x + 6 < 3x + 5$

Write the inequality $2x \leq x + 6 < 3x + 5$ in the form $2x \leq x + 6$ and $x + 6 < 3x + 5$.

Then, solve the inequalities $2x \leq x + 6$ and $x + 6 < 3x + 5$.

$2x \leq x + 6$ and $x + 6 < 3x + 5$

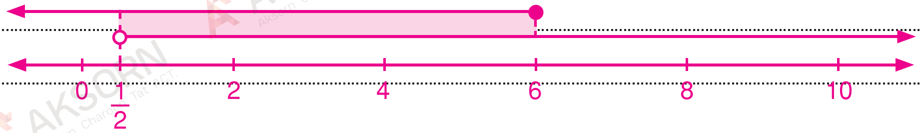
$2x - x \leq x + 6 - x$ $\qquad$ $x + 6 - 3x < 3x + 5 - 3x$

$x \leq 6$ $\qquad$ $-2x + 6 < 5$

$-2x + 6 - 6 < 5 - 6$

$-2x < -1$

$x > \frac{1}{2}$



Therefore, the solutions satisfying the inequality $2x \leq x + 6 < 3x + 5$ is $\frac{1}{2} < x \leq 6$.

Intermediate Level

6. Ceasar's car consumes gasoline at an average rate of 8 liters daily. Before he begins his journey, he tops up the gasoline in his car to 100 liters. Given that he will next top up the gasoline in his car when there are 20 liters left, form an inequality and solve it to find the maximum number of days he can travel before he has to top up the gasoline in his car.

Let the number of days be x.

We get $\qquad 8x + 20 \leq 100$

$8x + 20 - 20 \leq 100 - 20$

$8x \leq 80$

$x \leq 10.$

Therefore, the maximum number of days is 10.

7. If the sum of three consecutive integers is less than 75, find the cube of the largest possible integer.

Let the three consecutive integers be x, $(x + 1)$ and $(x + 2)$.

We get $x + (x + 1) + (x + 2) < 75$

$3x + 3 < 75$

$3x + 3 - 3 < 75 - 3$

$3x < 72$

$x < 24$.

The largest possible integer value of $x = 23$.

The largest possible integer $= 23 + 2 = 25$.

Therefore, the cube of the largest possible integer $= 25^3 = 15{,}625$.

8. In a math quiz, 5 points are awarded for a correct answer, and 2 points are deducted for a wrong answer, or if a question is left unanswered. Aurora attempted all 30 questions and her total score for the quiz was not more than 66. Find the maximum number of correct answers she obtained.

Let the number of correct answers obtained be x.

We get $5x - 2(30 - x) \leq 66$

$5x - 60 + 2x \leq 66$

$7x - 60 \leq 66$

$7x \leq 126$

$x \leq 18$.

Therefore, the number of correct answers she obtained is 18.

9. Kaira opened her coin bank to find 50 5-baht and 2-baht coins. If the total value of all the coins is more than 132 baht, find the minimum number of 5-baht coins she has.

Let x be the number of 5-baht coins.

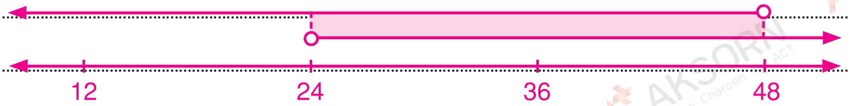
Then, the number of 2-baht coins is equal to $50 - x$.

We get
$$\begin{aligned} 5x + 2(50 - x) &> 132 \\ 5x + 100 - 2x &> 132 \\ 3x + 100 &> 132 \\ 3x &> 32 \\ x &> 10\tfrac{2}{3}. \end{aligned}$$

Therefore, the minimum number of 5-baht coins she has is 11.

10. Given that x is a prime number, find the values of x for which $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$.

$$\begin{aligned} \tfrac{1}{2}x - 4 &> \tfrac{1}{3}x \\ \tfrac{1}{6}x - 4 &> 0 \\ \tfrac{1}{6}x &> 4 \\ x &> 24 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tfrac{1}{6}x + 1 &< \tfrac{1}{8}x + 3 \\ \tfrac{1}{24}x + 1 &< 3 \\ \tfrac{1}{24}x &< 2 \\ x &< 48 \end{aligned}$$



From the number line, the values of x that satisfy both inequalities $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$ is $24 < x < 48$.

Therefore, the prime values of x that satisfy both inequalities $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$ is 29, 31, 37, 41, 43 and 47.

11. Given that x is a prime number, find the values of x for which $3x-2 \geq 10 \geq x+4$.

Write the inequality $3x - 2 \geq 10 \geq x + 4$ in the form $3x - 2 \geq 10$ and $10 \geq x + 4$.

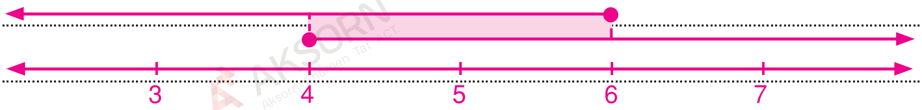
Then, solve the inequalities $3x - 2 \geq 10$ and $10 \geq x + 4$.

$3x - 2 \geq 10$ and $10 \geq x + 4$

$3x \geq 12$ $\qquad$ $10 - x \geq 4$

$x \geq 4$ $\qquad$ $-x \geq -6$

$x \leq 6$



From the number line, the values of x that satisfy the inequality $3x - 2 \geq 10 \geq x + 4$ is $4 \leq x \leq 6$.

Therefore, the prime value of x that satisfies the inequality $3x - 2 \geq 10 \geq x + 4$ is 5.

12. Given that x is an integer, find the values of x for which $x + 2 < 5\sqrt{17} < x + 3$.

Write the inequality $x + 2 < 5\sqrt{17} < x + 3$ in the form $x + 2 < 5\sqrt{17}$ and $5\sqrt{17} < x + 3$.

Then, solve the inequalities $x + 2 < 5\sqrt{17}$ and $5\sqrt{17} < x + 3$.

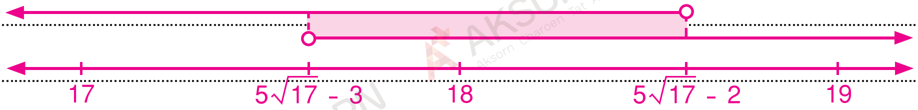
$x + 2 < 5\sqrt{17}$ and $5\sqrt{17} < x + 3$

$x < 5\sqrt{17} - 2$ $\qquad$ $5\sqrt{17} - x < 3$

$-x < 3 - 5\sqrt{17}$

$x > -(3 - 5\sqrt{17})$

$x > 5\sqrt{17} - 3$



Therefore, the integer value of x that satisfies the inequality $x + 2 < 5\sqrt{17} < x + 3$ is 18.

13. Solve each of the following inequalities and illustrate the solution on a number line.

1) $3 - a \leq a - 4 \leq 9 - 2a$

Write the inequality $3 - a \leq a - 4 \leq 9 - 2a$ in the form $3 - a \leq a - 4$ and $a - 4 \leq 9 - 2a$.

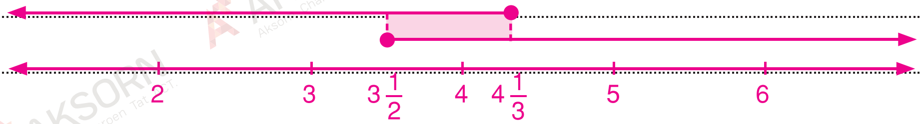
Then, solve the inequalities $3 - a \leq a - 4$ and $a - 4 \leq 9 - 2a$.

$3 - a \leq a - 4$ and $a - 4 \leq 9 - 2a$

$3 - 2a \leq -4$ $\quad$ $3a - 4 \leq 9$

$-2a \leq -7$ $\quad$ $3a \leq 13$

$a \geq 3\frac{1}{2}$ $\quad$ $a \leq 4\frac{1}{3}$



Therefore, the solutions satisfying the inequality $3 - a \leq a - 4 \leq 9 - 2a$ is $3\frac{1}{2} \leq a \leq 4\frac{1}{3}$.

2) $1 - b < b - 1 < 11 - 2b$

Write the inequality $1 - b < b - 1 < 11 - 2b$ in the form $1 - b < b - 1$ and $b - 1 < 11 - 2b$.

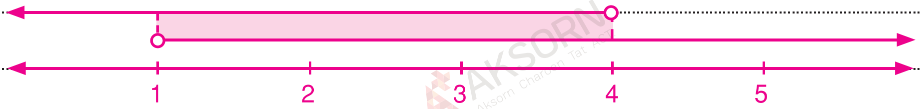
Then, solve the inequalities $1 - b < b - 1$ and $b - 1 < 11 - 2b$.

$1 - b < b - 1$ and $b - 1 < 11 - 2b$

$1 - 2b < -1$ $\quad$ $3b - 1 < 11$

$-2b < -2$ $\quad$ $3b < 12$

$b > 1$ $\quad$ $b < 4$



Therefore, the solutions satisfying the inequality $1 - b < b - 1 < 11 - 2b$ is $1 < b < 4$.

14. Solve each of the following inequalities and illustrate the solution on a number line.

1) $2(1 - a) > a - 1 \geq \frac{a - 2}{7}$

Write the inequality $2(1 - a) > a - 1 \geq \frac{a - 2}{7}$ in the form $2(1 - a) > a - 1$ and $a - 1 \geq \frac{a - 2}{7}$.

Then, solve the inequalities $2(1 - a) > a - 1$ and $a - 1 \geq \frac{a - 2}{7}$.

$2(1 - a) > a - 1$ and $a - 1 \geq \frac{a - 2}{7}$

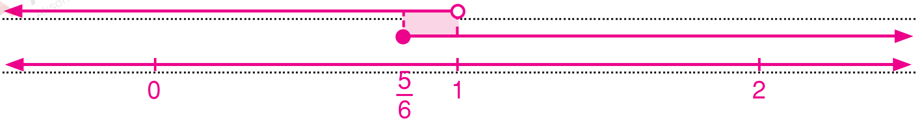
$2 - 2a > a - 1$ $\qquad$ $7(a - 1) \geq a - 2$

$2 - 3a > -1$ $\qquad$ $7a - 7 \geq a - 2$

$-3a > -3$ $\qquad$ $6a - 7 \geq -2$

$a < 1$ $\qquad$ $6a \geq 5$

$a \geq \frac{5}{6}$



Therefore, the solutions satisfying the inequality $2(1 - a) > a - 1 \geq \frac{a - 2}{7}$ is $\frac{5}{6} \leq a < 1$.

2) $b - 5 < \frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$

Write the inequality $b - 5 < \frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$ in the form $b - 5 < \frac{2b}{5}$ and $\frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$.

Then, solve the inequalities $b - 5 < \frac{2b}{5}$ and $\frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$.

$b - 5 < \frac{2b}{5}$ and $\frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$

$5(b - 5) < 2b$ $\qquad$ $2b \leq 5\left(\frac{b}{2} + \frac{1}{5}\right)$

$5b - 25 < 2b$ $\qquad$ $2b \leq \frac{5b}{2} + 1$

$3b - 25 < 0$ $\qquad$ $-\frac{b}{2} \leq 1$

$3b < 25$ $\qquad$ $-b \leq 2$

$b < 8\frac{1}{3}$ $\qquad$ $b \geq -2$

Therefore, the solution satisfying the inequality $b - 5 < \frac{2b}{5} \leq \frac{b}{2} + \frac{1}{5}$ is $-2 \leq b < 8\frac{1}{3}$.

15. Find the integer values of x that satisfy each of the following inequalities.

1) $3x + 2 < 19 < 5x - 4$

Write the inequality $3x + 2 < 19 < 5x - 4$ in the form $3x + 2 < 19$ and $19 < 5x - 4$.

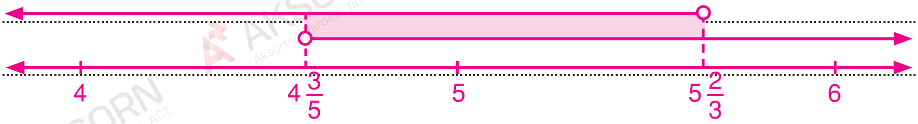
Then, solve the inequalities $3x + 2 < 19$ and $19 < 5x - 4$.

$3x + 2 < 19$ and $19 < 5x - 4$

$3x < 17$ $\qquad$ $19 - 5x < -4$

$x < 5\frac{2}{3}$ $\qquad$ $-5x < -23$

$x > 4\frac{3}{5}$



From the number line, the values of x that satisfy the inequality $3x + 2 < 19 < 5x - 4$ are $4\frac{3}{5} < x < 5\frac{2}{3}$.

Therefore, the integer value of x that satisfies the inequality $3x + 2 < 19 < 5x - 4$ is 5.

2) $-10 < 7 - 2x \leq -1$

Write the inequality $-10 < 7 - 2x \leq -1$ in the form $-10 < 7 - 2x$ and $7 - 2x \leq -1$.

Then, solve the inequalities $-10 < 7 - 2x$ and $7 - 2x \leq -1$.

$-10 < 7 - 2x$ and $7 - 2x \leq -1$

$-10 + 2x < 7$ $\qquad$ $-2x \leq -8$

$2x < 17$ $\qquad$ $x \geq 4$

$x < 8\frac{1}{2}$

From the number line, the values of x that satisfy the inequality $-10 < 7 - 2x \leq -1$ are $4 \leq x < 8\frac{1}{2}$.

Therefore, the integer values of x that satisfy the inequality $-10 < 7 - 2x \leq -1$ are 4, 5, 6, 7 and 8.

16. **Given that $0 \leq x \leq 7$ and $1 \leq y \leq 5$, find the following.**

1) The largest value of $x + y$

The largest value of $x + y$

is $7 + 5 = 12$.

2) The smallest value of $x - y$

The smallest value $x - y$

is $0 - 5 = -5$.

3) The largest value of xy

The largest value of xy

is $7 \times 5 = 35$.

4) The smallest value of $\frac{x}{y}$

The smallest value $\frac{x}{y}$

is $\frac{0}{5} = 0$.

17. **Given that $-4 \leq a \leq -1$ and $-6 \leq b \leq -2$, find the following.**

1) The smallest value of $a + b$

The smallest value of $a + b$

is $-4 + (-6) = -10$.

2) The largest value of $a - b$

The largest value of $a - b$

is $-1 - (-6) = 5$.

3) The smallest value of ab

The smallest value of ab

is $(-1)(-2) = 2$.

4) The largest value of $\frac{a}{b}$

The largest value of $\frac{a}{b}$

is $\frac{-4}{-2} = 2$.

5) The largest and smallest values of a^2

The largest value of a^2 is $(-4)^2 = 16$.

The smallest value of a^2 is $(-1)^2 = 1$.

6) The largest value of $b^2 - a$

The largest value of $b^2 - a$

is $(-6)^2 - (-4) = 40$.

Advanced Level

18. **State whether each of the following statements is true or false. If your answer is 'false', offer an explanation to support your case.**

1) If $a > b$ and both a and b are negative, then $\frac{a}{b} > 1$.

False. If $a = -1$ and $b = -2$, then $\frac{-1}{-2} = \frac{1}{2}$, which is less than 1.

2) If $a > b$ and both a and b are negative, then $a^3 > b^3$.

True

3) If $a > b$ and both a and b are negative, then $\frac{b}{a} - \frac{a}{b} > 0$.

True

Summary

1. Linear inequality in one variable

There is only one variable in a linear inequality in one variable.

2. Solutions of a linear inequality in one variable

Solutions of a linear inequality in one variable are any real numbers that substitute variables and make the inequality true.

3. Solving a linear inequality in one variable

1) Multiply or divide both sides by a positive number without having to reverse the inequality sign.

Case	Multiplying both sides of the inequality by a positive number *c*	Dividing both sides of the inequality by a positive number *c*
$x > y$	$cx > cy$	$\frac{x}{c} > \frac{y}{c}$
$x \geq y$	$cx \geq cy$	$\frac{x}{c} \geq \frac{y}{c}$
$x < y$	$cx < cy$	$\frac{x}{c} < \frac{y}{c}$
$x \leq y$	$cx \leq cy$	$\frac{x}{c} \leq \frac{y}{c}$

2) Add or subtract a positive or negative number to or from both sides without having to reverse the inequality sign.

Case	Adding a number to both sides of the inequality	Subtracting a number from both sides of the inequality
$x > y$	$x + a > y + a$	$x - a > y - a$
$x \geq y$	$x + a \geq y + a$	$x - a \geq y - a$
$x < y$	$x + a < y + a$	$x - a < y - a$
$x \leq y$	$x + a \leq y + a$	$x - a \leq y - a$

Reverse the inequality sign if we multiply or divide both sides by a negative number.

Case	Multiplying both sides of the inequality by a negative number *d*	Dividing both sides of the inequality by a negative number *d*
$x > y$	$dx < dy$	$\frac{x}{d} < \frac{y}{d}$
$x \geq y$	$dx \leq dy$	$\frac{x}{d} \leq \frac{y}{d}$
$x < y$	$dx > dy$	$\frac{x}{d} > \frac{y}{d}$
$x \leq y$	$dx \geq dy$	$\frac{x}{d} \geq \frac{y}{d}$

4. Solving a simultaneous linear inequality

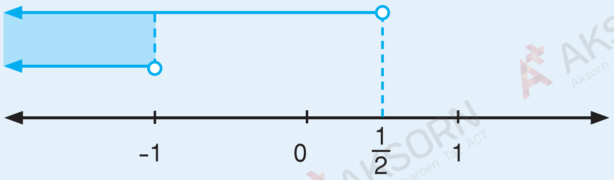
When solving a pair of simultaneous linear inequalities, we solve each of the inequalities first, i.e. solutions of simultaneous linear inequalities are the real numbers that substitute variables and make all the inequalities true.

For example, to solve $2x + 1 < x < 1 - x$, we write it in the form $2x + 1 < x$ and $x < 1 - x$. Then, we solve the inequalities $2x + 1 < x$ and $x < 1 - x$.

$$2x + 1 < x \qquad \text{and} \qquad x < 1 - x$$

$$x < -1 \qquad\qquad 2x < 1$$

$$x < \frac{1}{2}$$



Therefore, the solution satisfying the inequality $2x + 1 < x < 1 - x$ is $x < -1$.

Review Exercise 1

1. Solve each of the following inequalities.

1) $18x < -25$

$x < \frac{-25}{18}$

$x < -1\frac{7}{18}$

2) $-10y \geq -24$

$10y \leq 24$

$y \leq \frac{24}{10}$

$y \leq 2\frac{2}{5}$

2. Find the greatest integer value of y that satisfies the inequality $-3y > 24$.

$-3y > 24$

$3y < -24$

$y < \frac{-24}{3}$

$y < -8$

Therefore, the greatest integer value of y is -9.

3. Given that x satisfies the inequality $5x < 125$, find the greatest value of x if x is divisible by 12.

$5x < 125$

$x < \frac{125}{5}$

$x < 25$

Therefore, the greatest value of x if x is divisible by 12 is 24.

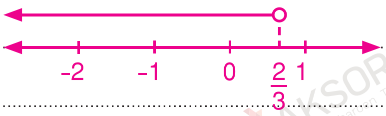
4. Solve each of the following inequalities and illustrate each solution on a number line.

1) $2a + 1 < 5 - 4a$

$6a + 1 < 5$

$6a < 4$

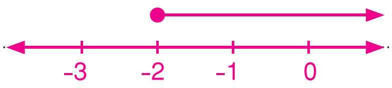
$a < \frac{2}{3}$



2) $b \geq \frac{1}{2}b - 1$

$\frac{1}{2}b \geq -1$

$b \geq -2$

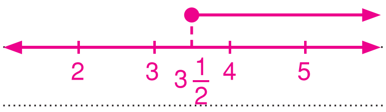


3) $2(c - 3) \geq 1$

$2c - 6 \geq 1$

$2c \geq 7$

$c \geq 3\frac{1}{2}$

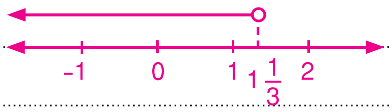


4) $-3 - d > 2d - 7$

$-3 - 3d > -7$

$-3d > -4$

$d < 1\frac{1}{3}$



5. Solve each of the following inequalities.

1) $3 + \frac{a}{4} > 5 + \frac{a}{3}$

$36 + 3a > 60 + 4a$

$36 - a > 60$

$-a > 24$

$a < -24$

2) $\frac{4b}{9} - 5 < 3 - \frac{2b}{3}$

$4b - 45 < 27 - 6b$

$10b - 45 < 27$

$10b < 72$

$b < 7\frac{1}{5}$

3) $\frac{4c}{9} - \frac{3}{4} \geq c - \frac{1}{2}$

$16c - 27 \geq 36c - 18$

$-20c - 27 \geq -18$

$-20c \geq 9$

$c \leq -\frac{9}{20}$

4) $\frac{d-2}{3} < \frac{2d+3}{5} + \frac{5}{8}$

$\frac{d-2}{3} - \frac{2d+3}{5} < \frac{5}{8}$

$5(d-2) - 3(2d+3) < \frac{75}{8}$

$5d - 10 - 6d - 9 < \frac{75}{8}$

$-d - 19 < \frac{75}{8}$

$-d < 28\frac{3}{8}$

$d > -28\frac{3}{8}$

5) $\frac{1}{3}(e+2) \geq \frac{2}{3} + \frac{1}{4}(e-1)$

$\frac{1}{3}(e+2) - \frac{1}{4}(e-1) \geq \frac{2}{3}$

$4(e+2) - 3(e-1) \geq 8$

$4e + 8 - 3e + 3 \geq 8$

$e + 11 \geq 8$

$e \geq -3$

6) $5 - \frac{2f-5}{6} \leq \frac{f+3}{2} + \frac{2(f+1)}{3}$

$30 - (2f-5) \leq 3(f+3) + 4(f+1)$

$30 - 2f + 5 \leq 3f + 9 + 4f + 4$

$35 - 2f \leq 7f + 13$

$35 - 9f \leq 13$

$-9f \leq -22$

$f \geq 2\frac{4}{9}$

KEY

6. Solve each of the following inequalities.

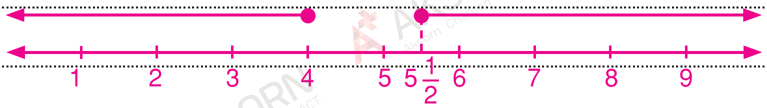
1) $5 - a \leq a - 6 \leq 10 - 3a$

$5 - a \leq a - 6$ and $a - 6 \leq 10 - 3a$

$5 - 2a \leq -6$ $\quad$ $4a - 6 \leq 10$

$-2a \leq -11$ $\quad$ $4a \leq 16$

$a \geq 5\frac{1}{2}$ $\quad$ $a \leq 4$



From the number line, the graph of the inequality $5 - a \leq a - 6 \leq 10 - 3a$ has no overlapping region.

Therefore, there is no solution that satisfies the inequality $5 - a \leq a - 6 \leq 10 - 3a$.

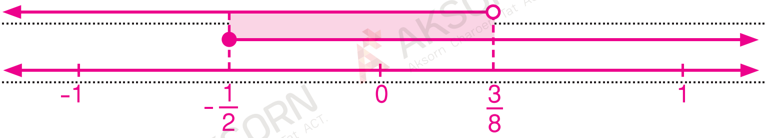
2) $4b - 1 < \frac{1}{2} \le 3b + 2$

$4b - 1 < \frac{1}{2}$ and $\frac{1}{2} \le 3b + 2$

$4b < \frac{3}{2}$ $\qquad$ $\frac{1}{2} - 3b \le 2$

$b < \frac{3}{8}$ $\qquad$ $-3b \le \frac{3}{2}$

$b \ge -\frac{1}{2}$



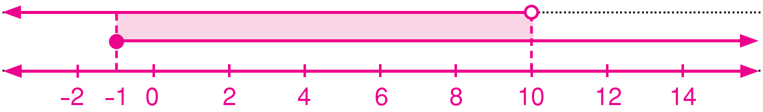
Therefore, the solution satisfying the inequality $4b - 1 < \frac{1}{2} \le 3b + 2$ is $-\frac{1}{2} \le b < \frac{3}{8}$.

3) $2c + 1 \ge c > 3c - 20$

$2c + 1 \ge c$ and $c > 3c - 20$

$c + 1 \ge 0$ $\qquad$ $-2c > -20$

$c \ge -1$ $\qquad$ $c < 10$



Therefore, the solution satisfying the inequality $2c + 1 \ge c > 3c - 20$ is $-1 \le c < 10$.

7. **Find the smallest integer value of x that satisfies the inequality $27 - 2x \le 8$.**

$27 - 2x \le 8$

$-2x \le -19$

$x \ge 9\frac{1}{2}$

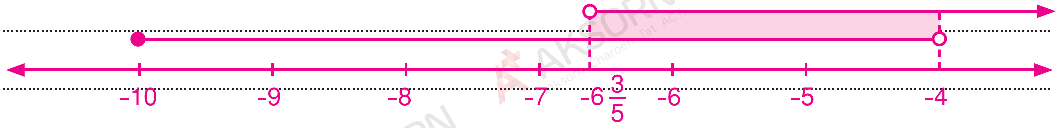
Therefore, the smallest integer value of x is 10.

8. Find the integer values of x that satisfy the inequalities $-10 \leq x < -4$ and $2 - 5x < 35$.

$-10 \leq x < -4$ and $2 - 5x < 35$

$-5x < 33$

$x > -6\frac{3}{5}$



From the number line, the values of x that satisfy the inequalities $-10 \leq x < -4$ and $2 - 5x < 35$ are $-6\frac{3}{5} < x < -4$.

Therefore, the integer values of x that satisfy both inequalities $-10 \leq x < -4$ and $2 - 5x < 35$ are -6 and -5.

9. Given that $-1 \leq x \leq 5$ and $2 \leq y \leq 6$, find the following.

1) Find the largest and the smallest possible values of $y - x$.

Let m be the largest possible value of $y - x$.

We get $m = 6 - (-1) = 7$.

Let n be the smallest possible value of $y - x$.

We get $n = 2 - 5 = -3$.

2) Find the largest and the smallest possible values of $\frac{x}{y}$.

Let m be the largest possible value of $\frac{x}{y}$.

We get $m = \frac{x}{y} = \frac{5}{2} = 2\frac{1}{2}$.

Let n be the smallest possible value of $\frac{x}{y}$.

We get $n = \frac{x}{y} = \frac{-1}{2} = -\frac{1}{2}$.

3) Find the largest and the smallest possible values of $\frac{x^2}{y}$.

Let m be the largest possible value of $\frac{x^2}{y}$.

We get $m = \frac{x^2}{y} = \frac{5^2}{2} = 12\frac{1}{2}$.

Let n be the smallest possible value of $\frac{x^2}{y}$.

We get $n = \frac{x^2}{y} = \frac{0}{6} = 0$.

10. Given that $-3 \leq x \leq 7$ and $4 \leq y \leq 10$, find the following.

1) Find the smallest possible value of $x - y$.

Let n be the smallest possible value of $x - y$.

We get $n = -3 - 10 = -13$.

2) Find the largest possible value of $\frac{x}{y}$.

Let m be the largest possible value of $\frac{x}{y}$.

We get $m = \frac{7}{4} = 1\frac{3}{4}$.

3) Find the largest possible value of $x^2 - y^2$.

Let m be the largest possible value of $x^2 - y^2$.

We get $m = 7^2 - 4^2 = 49 - 16 = 33$.

4) Find the smallest possible value of $x^3 + y^3$.

Let n be the smallest possible value of $x^3 + y^3$.

We get $n = (-3)^3 + 4^3 = -27 + 64 = 37$.

11. **The perimeter of a square is at most 81 cm. What is the greatest possible area of the square?**

Let the side of a square be x cm.

We get $4x \leq 81$

$x \leq 20\frac{1}{4}$.

The greatest possible value of $x = 20\frac{1}{4}$.

Therefore, the greatest possible area of the square $= 20\frac{1}{4} \times 20\frac{1}{4}$

$\approx 410.1 \text{ cm}^2$.

12. **The masses of a sheet of writing paper and an envelope are 3 grams and 5 grams, respectively. It costs 32 baht to send a letter with a mass not exceeding 20 grams. Mike has 32 baht worth of stamps. If x is the number of sheets of writing paper, form an inequality in x and find the maximum number of sheets of writing paper that he can use.**

Let the number of sheets of writing paper be x.

We get $3x + 5 \leq 20$

$x \leq 5$.

Therefore, the maximum number of sheets of writing paper that he can use is 5.

13. Alex is 3 years younger than Fay. If the sum of their ages is at most 50 years, find the maximum possible age of Alex 5 years ago.

Let the age of Fay now be x years.

Then, the age of Alex is $(x - 3)$ years.

We get $x + (x - 3) \leq 50$

$2x - 3 \leq 50$

$2x \leq 53$

$x \leq 26\frac{1}{2}$.

The maximum possible age of Fay now is 26 years.

The maximum possible age of Alex now is 23 years.

Therefore, the maximum possible age of Alex 5 years ago is 18 years.

14. Juno has 16 1-baht and 25-satang coins in her pocket. Given that the total value of the coins in her pocket is at most 22 baht, find the maximum number of 25-satang coins that she has.

Let x be the number of 25-satang coins.

We get $(16 \times 1) + 0.25x \leq 22$

$16 + 0.25x \leq 22$

$0.25x \leq 6$

$x \leq 24$.

Therefore, the maximum number of 25-satang coins that Juno has is 24.

15. In a set of 20 True/False questions, 2 points are awarded for a correct answer, and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Meena answered 15 questions and left the remaining questions unanswered. If her total score is greater than 24, find the maximum number of questions she answered wrongly.

Let x be the number of questions Meena answered wrongly.

We get $[2(15 - x)] - (1 \times x) > 24$

$30 - 2x - x > 24$

$30 - 3x > 24$

$-3x > -6$

$x < 2$.

Therefore, the maximum number of questions Meena answered wrongly is 1.

Challenge Yourself

1. Given that $6 \leq x \leq 8$ and $0.2 \leq y \leq 0.5$, find all the possible solutions of $\frac{x}{y}$.

From $6 \leq x \leq 8$ and $0.2 \leq y \leq 0.5$,

we get the largest possible value of $\frac{x}{y} = \frac{8}{0.2} = 40$,

and the smallest value of $\frac{x}{y} = \frac{6}{0.5} = 12$.

Therefore, all the possible solutions satisfying $\frac{x}{y}$ are $12 \leq \frac{x}{y} \leq 40$.

2. Find the range of values of x for which $\frac{3x - 5}{x^2 - 14x + 49} > 0$, $x \neq 7$.

$$\frac{3x - 5}{x^2 - 14x + 49} > 0$$

$$\frac{3x - 5}{(x - 7)^2} > 0$$

Since $(x - 7)^2 > 0$,

we get $3x - 5 > 0$

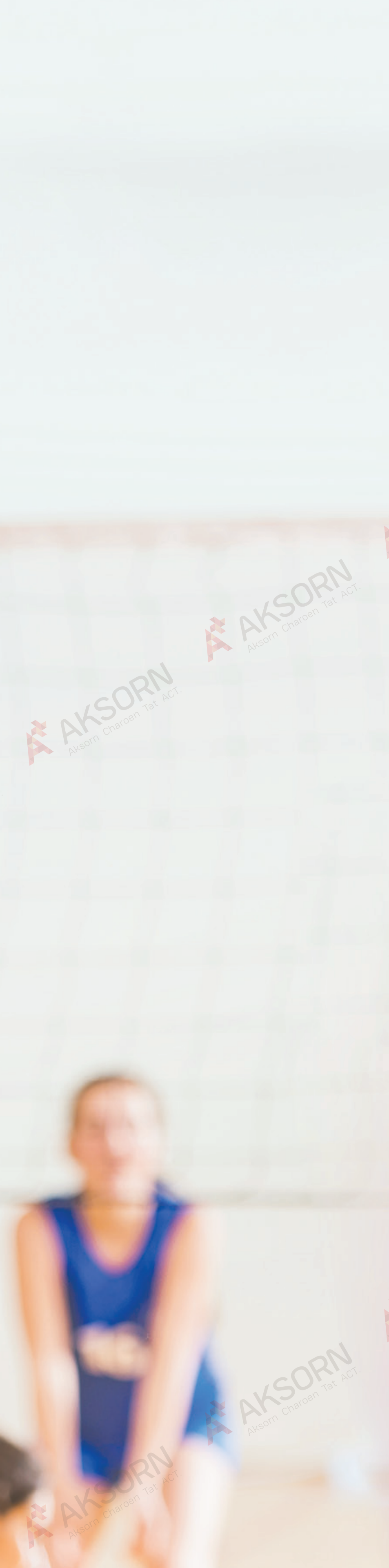
$3x > 5$

$x > 1\frac{2}{3}$.

Therefore, the possible values of x that satisfy the inequality $\frac{3x - 5}{x^2 - 14x + 49} > 0$

where $x \neq 7$ are $1\frac{2}{3} < x < 7$ and $x > 7$.

KEY



Chapter 2

Quadratic Equations in One Variable

A volleyball player did an overhand serve, in which the direction of the ball can be explained by the equation $h = ut + \frac{1}{2}at^2$. It is used for finding the ball's height above the ground after t sec when the ball was released from the player.

Indicator

- Understand and apply quadratic equations in one variable to solving mathematical problems. **(MA 1.3 G. 9/2)**

Compulsory Details

- Quadratic equations in one variable
- Solving quadratic equations in one variable
- Real-life applications of quadratic equations in one variable

2.1 Solving Quadratic Equations in One Variable by Factorization

In arithmetic, we have learned that the product of any number and zero is equal to zero. For example, $2 \times 0 = 0$, $-6 \times 0 = 0$, and $0 \times (-7) = 0$.

Similarly, in algebra, if two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0. We shall use this principle to solve quadratic equations in one variable of the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$.

ATTENTION

If two factors P and Q are such that $P \times Q = R$ where $R \neq 0$, we cannot conclude that either $P = R$ or $Q = R$ or both P and Q are equal to R.

Worked Example 1

KEY

Solve each of the following equations.

1) $x(x - 1) = 0$

2) $2x(x + 1) = 0$

3) $(x - 2)(x + 3) = 0$

4) $(2x - 3)(3x + 5) = 0$

Solution:

1)
$$\begin{aligned} x(x - 1) &= 0 \\ x &= 0 \quad \text{or} \quad x - 1 = 0 \\ \text{Therefore,} \quad x &= 0 \quad \text{or} \quad x = 1. \end{aligned}$$

2)
$$\begin{aligned} 2x(x + 1) &= 0 \\ 2x &= 0 \quad \text{or} \quad x + 1 = 0 \\ \text{Therefore,} \quad x &= 0 \quad \text{or} \quad x = -1. \end{aligned}$$

3)
$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ x - 2 &= 0 \quad \text{or} \quad x + 3 = 0 \\ \text{Therefore,} \quad x &= 2 \quad \text{or} \quad x = -3. \end{aligned}$$

4)
$$\begin{aligned} (2x - 3)(3x + 5) &= 0 \\ 2x - 3 &= 0 \quad \text{or} \quad 3x + 5 = 0 \\ \text{Therefore,} \quad x &= 1\tfrac{1}{2} \quad \text{or} \quad x = -1\tfrac{2}{3}. \end{aligned}$$

RECALL

Solutions of equations are any real numbers that substitute variables and satisfy the equations.

Practice Now

Similar Questions

Exercise 2A Questions 1, 2

Solve each of the following equations.

1) $x(x + 2) = 0$

$x = 0$ or $x + 2 = 0$

Therefore, $x = 0$ or $x = -2.$

2) $3x(x - 1) = 0$

$3x = 0$ or $x - 1 = 0$

Therefore, $x = 0$ or $x = 1.$

3) $(x + 5)(x - 7) = 0$

$x + 5 = 0$ or $x - 7 = 0$

Therefore, $x = -5$ or $x = 7.$

4) $(3x + 2)(4x - 5) = 0$

$3x + 2 = 0$ or $4x - 5 = 0$

Therefore, $x = -\frac{2}{3}$ or $x = 1\frac{1}{4}.$

Worked Example 2

Solve each of the following equations.

1) $2x^2 + 6x = 0$

2) $9x^2 - 4 = 0$

3) $x^2 - 3x - 28 = 0$

4) $2x^2 + 5x - 12 = 0$

Solution:

1)
$$\begin{aligned} 2x^2 + 6x &= 0 \\ 2x(x + 3) &= 0 && \textit{(factorize by extracting the common factor } 2x\textit{)} \\ 2x &= 0 \quad \text{or} \quad x + 3 = 0 \\ \text{Therefore,} \quad x &= 0 \quad \text{or} \quad x = -3. \end{aligned}$$

2)
$$\begin{aligned} 9x^2 - 4 &= 0 \\ (3x)^2 - 2^2 &= 0 \\ (3x + 2)(3x - 2) &= 0 && \textit{(factorize by using } a^2 - b^2 = (a + b)(a - b)\textit{)} \\ 3x + 2 &= 0 \quad \text{or} \quad 3x - 2 = 0 \\ \text{Therefore,} \quad x &= -\frac{2}{3} \quad \text{or} \quad x = \frac{2}{3}. \end{aligned}$$

3)
$$\begin{aligned} x^2 - 3x - 28 &= 0 \\ (x + 4)(x - 7) &= 0 && \textit{(factorize by using the multiplication frame)} \\ x + 4 &= 0 \quad \text{or} \quad x - 7 = 0 \\ \text{Therefore,} \quad x &= -4 \quad \text{or} \quad x = 7. \end{aligned}$$

4) $2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$ *(factorize by using the multiplication frame)*

$2x - 3 = 0$ or $x + 4 = 0$

Therefore, $x = 1\frac{1}{2}$ or $x = -4.$

Practice Now

Similar Questions

Exercise 2A Questions 3-7, 9-11, 13

Solve each of the following equations.

1) $3x^2 - 15x = 0$

$3x(x - 5) = 0$

$3x = 0$ or $x - 5 = 0$

Therefore, $x = 0$ or $x = 5.$

2) $x^2 + 8x + 16 = 0$

$x^2 + 2(x)(4) + 4^2 = 0$

$(x + 4)^2 = 0$

$x + 4 = 0$

Therefore, $x = -4.$

3) $x^2 + 5x + 4 = 0$

$(x + 1)(x + 4) = 0$

$x + 1 = 0$ or $x + 4 = 0$

Therefore, $x = -1$ or $x = -4.$

4) $3x^2 - 17x + 10 = 0$

$(3x - 2)(x - 5) = 0$

$3x - 2 = 0$ or $x - 5 = 0$

Therefore, $x = \frac{2}{3}$ or $x = 5.$

Worked Example 3

Solve each of the following equations.

1) $x(x + 6) = -5$

2) $2y(8y + 3) = 1$

Solution:

1) $x(x + 6) = -5$

$x^2 + 6x = -5$

$x^2 + 6x + 5 = 0$

$(x + 1)(x + 5) = 0$ *(factorize by using the multiplication frame)*

$x + 1 = 0$ or $x + 5 = 0$

Therefore, $x = -1$ or $x = -5.$

2) $2y(8y + 3) = 1$

$16y^2 + 6y = 1$

$16y^2 + 6y - 1 = 0$

$(8y - 1)(2y + 1) = 0$ *(factorize by using the multiplication frame)*

$8y - 1 = 0$ or $2y + 1 = 0$

Therefore, $y = \frac{1}{8}$ or $y = -\frac{1}{2}$.

Practice Now

Similar Questions

Exercise 2A Questions 8, 12

Solve each of the following equations.

1) $x(x + 1) = 6$

$x^2 + x - 6 = 0$

$(x - 2)(x + 3) = 0$

$x - 2 = 0$ or $x + 3 = 0$

Therefore, $x = 2$ or $x = -3$.

2) $9y(1 - y) = 2$

$9y - 9y^2 - 2 = 0$

$9y^2 - 9y + 2 = 0$

$(3y - 1)(3y - 2) = 0$

$3y - 1 = 0$ or $3y - 2 = 0$

Therefore, $y = \frac{1}{3}$ or $y = \frac{2}{3}$.

Thinking Time

Assume $x = y$

Multiply by x on both sides: $x^2 = xy$

Subtract y^2 from both sides: $x^2 - y^2 = xy - y^2$

Factorize both sides: $(x + y)(x - y) = y(x - y)$

Divide by $x - y$ on both sides: $x + y = y$

Since $x = y$, then $2y = y$

Divide by y on both sides: $2 = 1$

Since $2 \neq 1$, what went wrong in the steps above?

Since we assume that $x = y$, then $x - y = 0$.

Therefore, the step 'Divide by $x - y$ on both sides' is an invalid step, as we are dividing both sides by 0.

Exercise 2A

Basic Level

1. Solve each of the following equations.

1) $a(a - 9) = 0$

$a = 0$ or $a - 9 = 0$

Therefore, $a = 0$ or $a = 9$.

2) $b(b + 7) = 0$

$b = 0$ or $b + 7 = 0$

Therefore, $b = 0$ or $b = -7$.

3) $5c(c + 1) = 0$

$5c = 0$ or $c + 1 = 0$

Therefore, $c = 0$ or $c = -1$.

4) $2d(d - 6) = 0$

$2d = 0$ or $d - 6 = 0$

Therefore, $d = 0$ or $d = 6$.

5) $-f(3f - 7) = 0$

$-f = 0$ or $3f - 7 = 0$

Therefore, $f = 0$ or $f = 2\frac{1}{3}$.

6) $-\frac{1}{2}h(2h + 3) = 0$

$-\frac{1}{2}h = 0$ or $2h + 3 = 0$

Therefore, $h = 0$ or $h = -1\frac{1}{2}$.

KEY

2. Solve each of the following equations.

1) $(k - 4)(k - 9) = 0$

$k - 4 = 0$ or $k - 9 = 0$

Therefore, $k = 4$ or $k = 9$.

2) $(m - 3)(m + 5) = 0$

$m - 3 = 0$ or $m + 5 = 0$

Therefore, $m = 3$ or $m = -5$.

3) $(n + 4)(n - 11) = 0$

$n + 4 = 0$ or $n - 11 = 0$

Therefore, $n = -4$ or $n = 11$.

4) $(p + 1)(p + 2) = 0$

$p + 1 = 0$ or $p + 2 = 0$

Therefore, $p = -1$ or $p = -2$.

5) $(7q - 6)(4q - 5) = 0$

$7q - 6 = 0$ or $4q - 5 = 0$

Therefore, $q = \frac{6}{7}$ or $q = 1\frac{1}{4}$.

6) $(3r - 5)(2r + 1) = 0$

$3r - 5 = 0$ or $2r + 1 = 0$

Therefore, $r = 1\frac{2}{3}$ or $r = -\frac{1}{2}$.

7) $(5s + 3)(2 - s) = 0$

$5s + 3 = 0$ or $2 - s = 0$

Therefore, $s = -\frac{3}{5}$ or $s = 2$.

8) $(-2t - 5)(8t - 5) = 0$

$-2t - 5 = 0$ or $8t - 5 = 0$

Therefore, $t = -2\frac{1}{2}$ or $t = \frac{5}{8}$.

3. Solve each of the following equations.

1) $a^2 + 9a = 0$

$a(a + 9) = 0$

$a = 0$ or $a + 9 = 0$

Therefore, $a = 0$ or $a = -9$.

2) $3b^2 - 4b = 0$

$b(3b - 4) = 0$

$b = 0$ or $3b - 4 = 0$

Therefore, $b = 0$ or $b = 1\frac{1}{3}$.

3) $3c - 81c^2 = 0$

$3c(1 - 27c) = 0$

$3c = 0$ or $1 - 27c = 0$

Therefore, $c = 0$ or $c = \frac{1}{27}$.

4) $-4d^2 - 16d = 0$

$-4d(d + 4) = 0$

$-4d = 0$ or $d + 4 = 0$

Therefore, $d = 0$ or $d = -4$.

4. Solve each of the following equations.

1) $k^2 + 12k + 36 = 0$

$k^2 + 2(k)(6) + 6^2 = 0$

$(k + 6)^2 = 0$

$k + 6 = 0$

Therefore, $k = -6$.

2) $m^2 - 16m + 64 = 0$

$m^2 - 2(m)(8) + 8^2 = 0$

$(m - 8)^2 = 0$

$m - 8 = 0$

Therefore, $m = 8$.

3) $n^2 - 16 = 0$

$n^2 - 4^2 = 0$

$(n + 4)(n - 4) = 0$

$n + 4 = 0$ or $n - 4 = 0$

Therefore, $n = -4$ or $n = 4$.

4) $25p^2 + 70p + 49 = 0$

$(5p)^2 + 2(5p)(7) + 7^2 = 0$

$(5p + 7)^2 = 0$

$5p + 7 = 0$

Therefore, $p = -1\frac{2}{5}$.

5) $4q^2 - 12q + 9 = 0$

$(2q)^2 - 2(2q)(3) + 3^2 = 0$

$(2q - 3)^2 = 0$

$2q - 3 = 0$

Therefore, $q = 1\frac{1}{2}$.

6) $4r^2 - 100 = 0$

$(2r)^2 - 10^2 = 0$

$(2r + 10)(2r - 10) = 0$

$2r + 10 = 0$ or $2r - 10 = 0$

Therefore, $r = -5$ or $r = 5$.

KEY

5. Solve each of the following equations.

1) $s^2 + 10s + 21 = 0$

$(s + 3)(s + 7) = 0$

$s + 3 = 0$ or $s + 7 = 0$

Therefore, $s = -3$ or $s = -7$.

2) $t^2 - 16t + 63 = 0$

$(t - 7)(t - 9) = 0$

$t - 7 = 0$ or $t - 9 = 0$

Therefore, $t = 7$ or $t = 9$.

3) $u^2 + 6u - 27 = 0$

$(u - 3)(u + 9) = 0$

$u - 3 = 0$ or $u + 9 = 0$

Therefore, $u = 3$ or $u = -9$.

4) $v^2 - 5v - 24 = 0$

$(v + 3)(v - 8) = 0$

$v + 3 = 0$ or $v - 8 = 0$

Therefore, $v = -3$ or $v = 8$.

5) $3w^2 + 49w + 60 = 0$

$(3w + 4)(w + 15) = 0$

$3w + 4 = 0$ or $w + 15 = 0$

Therefore, $w = -1\frac{1}{3}$ or $w = -15$.

6) $6x^2 - 29x + 20 = 0$

$(6x - 5)(x - 4) = 0$

$6x - 5 = 0$ or $x - 4 = 0$

Therefore, $x = \frac{5}{6}$ or $x = 4$.

Intermediate Level

6. Solve each of the following equations.

1) $121 - a^2 = 0$

$11^2 - a^2 = 0$

$(11 + a)(11 - a) = 0$

$11 + a = 0$ or $11 - a = 0$

Therefore, $a = -11$ or $a = 11$.

2) $128 - 2b^2 = 0$

$64 - b^2 = 0$

$8^2 - b^2 = 0$

$(8 + b)(8 - b) = 0$

$8 + b = 0$ or $8 - b = 0$

Therefore, $b = -8$ or $b = 8$.

3) $c^2 - \frac{1}{4} = 0$

$c^2 - \left(\frac{1}{2}\right)^2 = 0$

$\left(c + \frac{1}{2}\right)\left(c - \frac{1}{2}\right) = 0$

$c + \frac{1}{2} = 0$ or $c - \frac{1}{2} = 0$

Therefore, $c = -\frac{1}{2}$ or $c = \frac{1}{2}$.

4) $\frac{4}{9} - \frac{d^2}{25} = 0$

$\left(\frac{2}{3}\right)^2 - \left(\frac{d}{5}\right)^2 = 0$

$\left(\frac{2}{3} + \frac{d}{5}\right)\left(\frac{2}{3} - \frac{d}{5}\right) = 0$

$\frac{2}{3} + \frac{d}{5} = 0$ or $\frac{2}{3} - \frac{d}{5} = 0$

Therefore, $d = -3\frac{1}{3}$ or $d = 3\frac{1}{3}$.

7. Solve each of the following equations.

1) $7f + f^2 = 60$

$f^2 + 7f - 60 = 0$

$(f - 5)(f + 12) = 0$

$f - 5 = 0$ or $f + 12 = 0$

Therefore, $f = 5$ or $f = -12$.

2) $15 = 8h^2 - 2h$

$8h^2 - 2h - 15 = 0$

$(4h + 5)(2h - 3) = 0$

$4h + 5 = 0$ or $2h - 3 = 0$

Therefore, $h = -1\frac{1}{4}$ or $h = 1\frac{1}{2}$.

8. Solve each of the following equations.

1) $k(2k + 5) = 3$

$2k^2 + 5k = 3$

$2k^2 + 5k - 3 = 0$

$(2k - 1)(k + 3) = 0$

$2k - 1 = 0$ or $k + 3 = 0$

Therefore, $k = \frac{1}{2}$ or $k = -3$.

2) $(m - 2)(m + 4) = 27$

$m^2 + 4m - 2m - 8 = 27$

$m^2 + 2m - 35 = 0$

$(m - 5)(m + 7) = 0$

$m - 5 = 0$ or $m + 7 = 0$

Therefore, $m = 5$ or $m = -7$.

3) $3n^2 - 5(n + 1) = 7n + 58$

$3n^2 - 5n - 5 = 7n + 58$

$3n^2 - 12n - 63 = 0$

$n^2 - 4n - 21 = 0$

$(n + 3)(n - 7) = 0$

$n + 3 = 0$ or $n - 7 = 0$

Therefore, $n = -3$ or $n = 7$.

4) $(3p + 1)(p - 4) = -5(p - 1)$

$3p^2 - 12p + p - 4 = -5p + 5$

$3p^2 - 6p - 9 = 0$

$p^2 - 2p - 3 = 0$

$(p + 1)(p - 3) = 0$

$p + 1 = 0$ or $p - 3 = 0$

Therefore, $p = -1$ or $p = 3$.

9. Solve the equation $6(y - 3)^2 - (y - 3) - 15 = 0$

Let $x = y - 3$.

We get $6x^2 - x - 15 = 0$

$(3x - 5)(2x + 3) = 0$

$3x - 5 = 0$ or $2x + 3 = 0$

$x = 1\frac{2}{3}$ or $x = -1\frac{1}{2}$.

We get $y - 3 = 1\frac{2}{3}$ or $y - 3 = -1\frac{1}{2}$.

Therefore, $y = 4\frac{2}{3}$ or $y = 1\frac{1}{2}$.

Advanced Level

10. Solve each of the following equations.

1) $\frac{1}{2}x^2 - \frac{11}{4}x + \frac{5}{4} = 0$

$2x^2 - 11x + 5 = 0$

$(2x - 1)(x - 5) = 0$

$2x - 1 = 0$ or $x - 5 = 0$

Therefore, $x = \frac{1}{2}$ or $x = 5.$

2) $2 - 3.5y - 9.75y^2 = 0$

$8 - 14y - 39y^2 = 0$

$(4 - 13y)(2 + 3y) = 0$

$4 - 13y = 0$ or $2 + 3y = 0$

Therefore, $y = \frac{4}{13}$ or $y = -\frac{2}{3}.$

11. Solve the equation $9x^2y^2 - 12xy + 4 = 0$, expressing y in terms of x.

$9x^2y^2 - 12xy + 4 = 0$

$(3xy)^2 - 2(3xy)(2) + 2^2 = 0$

$(3xy - 2)^2 = 0$

$3xy - 2 = 0$

Therefore, $y = \frac{2}{3x}.$

12. Solve the equation $x - (2x - 3)^2 = -6(x^2 + x - 2)$.

$x - (2x - 3)^2 = -6(x^2 + x - 2)$

$x - (4x^2 - 12x + 9) = -6x^2 - 6x + 12$

$x - 4x^2 + 12x - 9 = -6x^2 - 6x + 12$

$2x^2 + 19x - 21 = 0$

$(2x + 21)(x - 1) = 0$

$2x + 21 = 0$ or $x - 1 = 0$

Therefore, $x = -10\frac{1}{2}$ or $x = 1.$

13. Answer each of the following questions.

1) If $x = 5$ is a solution of the equation $x^2 - qx + 10 = 0$, find the value of q.

Since $x = 5.$

we get $5^2 - q(5) + 10 = 0$

$25 - 5q + 10 = 0$

$5q = 35.$

Therefore, $q = 7.$

2) Find another solution of the equation in 1.

$x^2 - 7x + 10 = 0$

$(x - 2)(x - 5) = 0$

$x - 2 = 0$ or $x - 5 = 0$

$x = 2$ or $x = 5$

Therefore, another solution of the equation is $x = 2.$

2.2 Solving Quadratic Equations in One Variable by Completing the Square

1. Solving Quadratic Equations in One Variable of the Form $(x + a)^2 = b$

In the previous section, we solved the equation by factorization. However, the solutions of some quadratic equations in one variable cannot be obtained by factorization. An example of this type of quadratic equation is $x^2 + 6x - 5 = 0$. If this equation can be written in the form $(x + a)^2 = b$ where a and b are real numbers, then it can be solved easily by taking the square roots on both sides of the equation to obtain the solution.

Worked Example 4

Solve the equation $(x + 3)^2 = 14$, giving your answers correct to 2 decimal places.

Solution:

$$(x + 3)^2 = 14$$

$$x + 3 = \pm\sqrt{14} \quad \textit{(take the square roots on both sides)}$$

$$x + 3 = \sqrt{14} \quad \text{or} \quad x + 3 = -\sqrt{14}$$

$$\text{Therefore, } x = \sqrt{14} - 3 \quad \text{or} \quad x = -\sqrt{14} - 3$$

$$\approx 0.74 \qquad \approx -6.74.$$

Practice Now

Similar Questions
Exercise 2B Question 1

Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $(x + 7)^2 = 100$

$x + 7 = \pm\sqrt{100}$

$x + 7 = \pm 10$

$x + 7 = 10$ or $x + 7 = -10$

Therefore, $x = 3$ $\quad x = -17.$

2) $(y - 5)^2 = 11$

$y - 5 = \pm\sqrt{11}$

$y - 5 = \sqrt{11}$ or $y - 5 = -\sqrt{11}$

Therefore, $y = \sqrt{11} + 5$ $\quad y = -\sqrt{11} + 5$

≈ 8.32 $\quad \approx 1.68.$

KEY

2. Completing the Square for a Quadratic Expression

To express a quadratic expression of the form $x^2 + px + q = 0$ in the form $(x + a)^2 = b$, we first need to learn how to complete the square for a quadratic expression $x^2 + px$.

Let us consider the expansion of $(x + 3)^2$. We can use algebra discs to represent the expansion $(x + 3)^2 = x^2 + 6x + 9$ in the form of a square array or a multiplication frame.

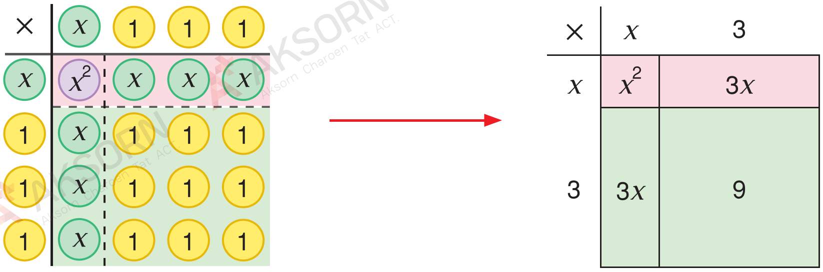


Figure 2.1

We observe that in the square array:

- the x^2 disc is in the top left-hand corner,
- the nine 1 discs are arranged as a 3 by 3 square at the bottom right-hand corner,
- the six x discs are divided equally into 2 parts, i.e. $6x$ is divided into 2 parts of $3x$.

Quadratic expressions of the form $(x + a)^2$ can be arranged into a multiplication frame similar to the example in the figure.

However, not all quadratic expressions can be expressed in the form $(x + a)^2$. For example, the expression $x^2 + 6x$ can be arranged as shown in figure 2.2.

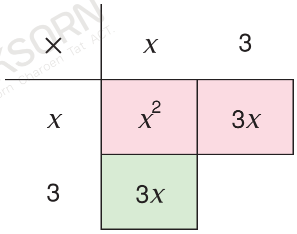


Figure 2.2

Comparing figures 2.1 and 2.2, what number must be added to complete the square? We observe that 9 must be added to $x^2 + 6x$ to make it into $(x + 3)^2$. However, $x^2 + 6x \neq (x + 3)^2$.

×	x	3
x	x^2	$3x$
3	$3x$	

$\neq$

×	x	3
x	x^2	$3x$
3	$3x$	9

Figure 2.3

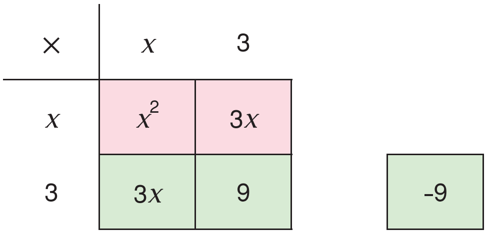
Since we add 9 to $x^2 + 6x$, we must subtract 9 as follows:

$$\begin{aligned} x^2 + 6x &= x^2 + 6x + 9 - 9 \\ &= (x + 3)^2 - 9 \end{aligned}$$

Pictorially, it looks like:

ATTENTION

Essentially, we add $9 - 9 = 0$ to $x^2 + 6x$ so that the equality will still hold.



×	x	3
x	x^2	$3x$
3	$3x$	9

Figure 2.4

Investigation

Completing the square for quadratic expressions of the form $x^2 + px$

To make a quadratic expression of the form $x^2 + px$ into a perfect square $(x + a)^2$, we have to add a number, b, to $x^2 + px$. In this investigation, we will find a relationship between b and p.

Complete the table and answer the questions. The second one has been done for you.

No.	Quadratic expression $(x^2 + px)$	Number that must be added to complete the square, (b)	$\frac{1}{2} \times$ coefficient of x, $\left(\frac{p}{2}\right)$	Quadratic expression of the form $(x + a)^2 - b$
a.	$x^2 + 4x$ $\begin{array}{ccc} \times & x & 2 \\ \hline x & x^2 & 2x \\ 2 & 2x & \end{array}$	$2^2 = 4$	$\frac{4}{2} = 2$	$x^2 + 4x$ $= (x^2 + 4x + 2^2) - 2^2$ $= (x + 2)^2 - 4$ $\begin{array}{ccc} \times & x & 2 \\ \hline x & x^2 & 2x \\ 2 & 2x & 4 \end{array}$ $\boxed{-4}$
b.	$x^2 + 6x$ $\begin{array}{ccc} \times & x & 3 \\ \hline x & x^2 & 3x \\ 3 & 3x & \end{array}$	$3^2 = 9$	$\frac{6}{2} = 3$	$x^2 + 6x$ $= (x^2 + 6x + 3^2) - 3^2$ $= (x + 3)^2 - 9$ $\begin{array}{ccc} \times & x & 3 \\ \hline x & x^2 & 3x \\ 3 & 3x & 9 \end{array}$ $\boxed{-9}$
c.	$x^2 + 8x$ $\begin{array}{ccc} \times & x & 4 \\ \hline x & x^2 & 4x \\ 4 & 4x & \end{array}$	$4^2 = 16$	$\frac{8}{2} = 4$	$x^2 + 8x$ $= (x^2 + 8x + 4^2) - 4^2$ $= (x + 4)^2 - 16$ $\begin{array}{ccc} \times & x & 4 \\ \hline x & x^2 & 4x \\ 4 & 4x & 16 \end{array}$ $\boxed{-16}$

KEY

No.	Quadratic expression $(x^2 + px)$	Number that must be added to complete the square, (b)	$\frac{1}{2}$ × coefficient of x, $\left(\frac{p}{2}\right)$	Quadratic expression of the form $(x + a)^2 - b$
d.	$x^2 + 10x$ × x 5 x x^2 $5x$ 5 $5x$	$5^2 = 25$	$\frac{10}{2} = 5$	$x^2 + 10$ $= x^2 + 10x + 5^2 - 5^2$ $= (x + 5)^2 - 25$ × x 5 x x^2 $5x$ 5 $5x$ 25 -25

1. What is the relationship between b and p?

$b = \left(\frac{p}{2}\right)^2$

2. To express $x^2 + px$ in the form $(x + a)^2 - b$, write down an expression of a and b in terms of p.

$a = \frac{p}{2}$, $b = \left(\frac{p}{2}\right)^2$

From **Investigation**, we can conclude that:

If $x^2 + px = (x + a)^2 - b$, then $a = \frac{p}{2}$ and $b = \left(\frac{p}{2}\right)^2$.

Therefore, $x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$.

For a quadratic expression of the form $x^2 + px + q$, we can express it as follows:

$$
\begin{aligned}
x^2 + px + q &= (x^2 + px) + q \\
&= \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q
\end{aligned}
$$

Worked Example 5

Express each of the following expressions in the form $(x + a)^2 + b$.

1) $x^2 + 10x$ 2) $x^2 - 5x$ 3) $x^2 + 2x + 3$

Solution:

1) The coefficient of x is 10. Half of this is 5.

Therefore, $x^2 + 10x = [x^2 + 10x + 5^2] - 5^2$

$= (x + 5)^2 - 25.$

2) The coefficient of x is -5. Half of this is $-\frac{5}{2}$.

Therefore, $x^2 - 5x = \left[x^2 - 5x + \left(-\frac{5}{2}\right)^2\right] - \left(-\frac{5}{2}\right)^2$

$= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4}.$

3) $x^2 + 2x + 3 = (x^2 + 2x) + 3$

The coefficient of x is 2. Half of this is 1.

Therefore, $x^2 + 2x + 3 = [x^2 + 2x + 1^2] - 1^2 + 3$

$= (x + 1)^2 + 2.$

Similar Questions

Exercise 2B Question 2

Practice Now

Express each of the following expressions in the form $(x + a)^2 + b$.

1) $x^2 + 20x$

The coefficient of x is 20.

Half of this is 10.

Therefore, $x^2 + 20x$

$= (x^2 + 20x + 10^2) - 10^2$

$= (x + 10)^2 - 100.$

2) $x^2 - 7x$

The coefficient of x is -7.

Half of this is $-\frac{7}{2}$.

Therefore, $x^2 - 7x$

$= \left[x^2 - 7x + \left(-\frac{7}{2}\right)^2\right] - \left(-\frac{7}{2}\right)^2$

$= \left(x - \frac{7}{2}\right)^2 - \frac{49}{4}.$

3) $x^2 + \frac{1}{5}x$

The coefficient of x is $\frac{1}{5}$.

Half of this is $\frac{1}{10}$.

Therefore, $x^2 + \frac{1}{5}x$

$= \left[x^2 + \frac{1}{5}x + \left(\frac{1}{10}\right)^2\right] - \left(\frac{1}{10}\right)^2$

$= \left(x + \frac{1}{10}\right)^2 - \frac{1}{100}.$

4) $x^2 + 6x - 9$

$x^2 + 6x - 9 = (x^2 + 6x) - 9$

The coefficient of x is 6.

Half of this is 3.

Therefore, $x^2 + 6x - 9$

$= (x^2 + 6x + 3^2) - 3^2 - 9$

$= (x + 3)^2 - 18.$

Worked Example 6

Solve the equation $x^2 + 4x - 3 = 0$, giving your answers correct to 2 decimal places.

Solution:

As $x^2 + 4x - 3$ cannot be easily factorized, we need to transform the equation $x^2 + 4x - 3 = 0$ into the form $(x + a)^2 = b$ as follows:

$x^2 + 4x - 3 = 0$

$x^2 + 4x = 3$ *(rewrite the equation such that the constant term is on the RHS of the equation)*

$x^2 + 4x + \left(\frac{4}{2}\right)^2 = 3 + \left(\frac{4}{2}\right)^2$ *(add $\left(\frac{4}{2}\right)^2$ to both sides of the equation to complete the square for the LHS)*

$x^2 + 4x + 2^2 = 3 + 2^2$

$(x + 2)^2 = 7$ *(factorize the expression on the LHS and simplify the RHS)*

$x + 2 = \pm\sqrt{7}$ *(take the square roots on both sides)*

$x + 2 = \sqrt{7}$ or $x + 2 = -\sqrt{7}$

Therefore, $x = \sqrt{7} - 2$ or $x = -\sqrt{7} - 2$

≈ 0.65 $\approx -4.65.$

Similar Questions

Exercise 2B Questions 3-5

Practice Now

1. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $x^2 + 6x - 4 = 0$

$x^2 + 6x = 4$

$x^2 + 6x + \left(\frac{6}{2}\right)^2 = 4 + \left(\frac{6}{2}\right)^2$

$x^2 + 6x + 3^2 = 4 + 3^2$

$(x + 3)^2 = 13$

$x + 3 = \pm\sqrt{13}$

$x + 3 = \sqrt{13}$ or $x + 3 = -\sqrt{13}$

Therefore, $x = \sqrt{13} - 3$ $x = -\sqrt{13} - 3$

≈ 0.61 $\approx -6.61.$

2) $x^2 - x - 1 = 0$

$x^2 - x = 1$

$x^2 - x + \left(-\frac{1}{2}\right)^2 = 1 + \left(-\frac{1}{2}\right)^2$

$\left(x - \frac{1}{2}\right)^2 = 1 + \frac{1}{4}$

$x - \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$

$x - \frac{1}{2} = \sqrt{\frac{5}{4}}$ or $x - \frac{1}{2} = -\sqrt{\frac{5}{4}}$

Therefore, $x = \sqrt{\frac{5}{4}} + \frac{1}{2}$ $x = -\sqrt{\frac{5}{4}} + \frac{1}{2}$

≈ 1.62 $\approx -0.62.$

2. Solve the equation $(x + 4)(x - 3) = 15$, giving your answers correct to 2 decimal places.

$$(x + 4)(x - 3) = 15$$

$$x^2 + x - 12 = 15$$

$$x^2 + x = 27$$

$$x^2 + x + \left(\frac{1}{2}\right)^2 = 27 + \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{1}{2}\right)^2 = 27 + \frac{1}{4}$$

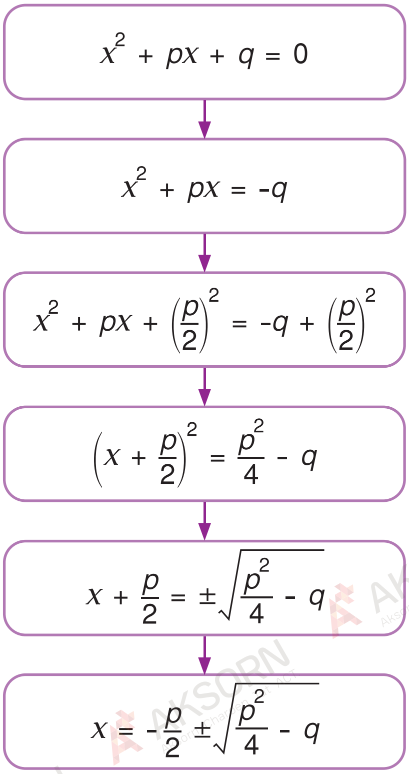
$$x + \frac{1}{2} = \pm\sqrt{\frac{109}{4}}$$

$$x + \frac{1}{2} = \sqrt{\frac{109}{4}} \quad \text{or} \quad x + \frac{1}{2} = -\sqrt{\frac{109}{4}}$$

Therefore, $x = \sqrt{\frac{109}{4}} - \frac{1}{2}$ $\qquad$ $x = -\sqrt{\frac{109}{4}} - \frac{1}{2}$

≈ 4.72 $\qquad$ $\approx -5.72.$

From **Worked Example 6**, in general, the steps taken to solve a quadratic equation $x^2 + px + q = 0$ where p and q are real numbers by completing the square are as follows:



$$x^2 + px + q = 0$$

$$x^2 + px = -q$$

(add -q to both sides of the equation)

$$x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$$

(add $\left(\frac{p}{2}\right)^2$ to both sides of the equation)

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q$$

(factorize the expression on the LHS and simplify the RHS)

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}$$

(take the square roots on both sides)

$$x = -\frac{p}{2} \pm\sqrt{\frac{p^2}{4} - q}$$

(add $-\frac{p}{2}$ to both sides of the equation)

Exercise 2B

Basic Level

1. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $(x + 1)^2 = 9$

$x + 1 = \pm\sqrt{9}$

$x + 1 = \pm 3$

$x + 1 = 3$ or $x + 1 = -3$

Therefore, $x = 2$ $x = -4.$

2) $(5x - 4)^2 = 81$

$5x - 4 = \pm\sqrt{81}$

$5x - 4 = \pm 9$

$5x - 4 = 9$ or $5x - 4 = -9$

Therefore, $x = 2\frac{3}{5}$ $x = -1.$

3) $(7 - 3x)^2 = \frac{9}{16}$

$7 - 3x = \pm\sqrt{\frac{9}{16}}$

$7 - 3x = \frac{3}{4}$ or $7 - 3x = -\frac{3}{4}$

$3x = 6\frac{1}{4}$ $3x = 7\frac{3}{4}$

Therefore, $x = 2\frac{1}{12}$ $x = 2\frac{7}{12}.$

4) $(x + 3)^2 = 11$

$x + 3 = \pm\sqrt{11}$

$x + 3 = \sqrt{11}$ or $x + 3 = -\sqrt{11}$

Therefore, $x = \sqrt{11} - 3$ $x = -\sqrt{11} - 3$

≈ 0.32 $\approx -6.32.$

5) $(2x - 3)^2 = 23$

$2x - 3 = \pm\sqrt{23}$

$2x - 3 = \sqrt{23}$ or $2x - 3 = -\sqrt{23}$

$2x = \sqrt{23} + 3$ $2x = -\sqrt{23} + 3$

Therefore, $x \approx 3.90$ $x \approx -0.90.$

6) $\left(\frac{1}{2} - x\right)^2 = 10$

$\frac{1}{2} - x = \pm\sqrt{10}$

$\frac{1}{2} - x = \sqrt{10}$ or $\frac{1}{2} - x = -\sqrt{10}$

Therefore, $x = \frac{1}{2} - \sqrt{10}$ $x = \frac{1}{2} + \sqrt{10}$

≈ -2.66 $\approx 3.66.$

Intermediate Level

2. Express each of the following expressions in the form $(x + a)^2 + b$.

1) $x^2 + 12x$

The coefficient of x is 12.

Half of this is 6.

Therefore, $x^2 + 12x = [x^2 + 12x + 6^2] - 6^2$

$= (x + 6)^2 - 36.$

2) $x^2 + 3x - 2$

The coefficient of x is 3.

Half of this is $\frac{3}{2}$

Therefore,

$x^2 + 3x - 2 = \left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] - \left(\frac{3}{2}\right)^2 - 2$

$= \left(x + \frac{3}{2}\right)^2 - \frac{17}{4}.$

3) $x^2 + \frac{1}{2}x$

The coefficient of x is $\frac{1}{2}$.

Half of this is $\frac{1}{4}$.

Therefore,

$$x^2 + \frac{1}{2}x = \left[x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right] - \left(\frac{1}{4}\right)^2$$
$$= \left(x + \frac{1}{4}\right)^2 - \frac{1}{16}.$$

4) $x^2 - 1.4x$

The coefficient of x is -1.4.

Half of this is -0.7.

Therefore,

$$x^2 - 1.4x = [x^2 - 1.4x + (-0.7)^2] - (-0.7)^2$$
$$= (x - 0.7)^2 - 0.49.$$

3. **Solve each of the following equations, giving your answers correct to 2 decimal places.**

1) $x^2 + 17x - 30 = 0$

$$x^2 + 17x = 30$$
$$x^2 + 17x + \left(\frac{17}{2}\right)^2 = 30 + \left(\frac{17}{2}\right)^2$$
$$\left(x + \frac{17}{2}\right)^2 = \frac{409}{4}$$
$$x + \frac{17}{2} = \pm\sqrt{\frac{409}{4}}$$
$$x + \frac{17}{2} = \sqrt{\frac{409}{4}} \text{ or } x + \frac{17}{2} = -\sqrt{\frac{409}{4}}$$

Therefore, $x = \sqrt{\frac{409}{4}} - \frac{17}{2}$ $x = -\sqrt{\frac{409}{4}} - \frac{17}{2}$

≈ 1.61 $\approx -18.61.$

2) $x^2 - 5x - 5 = 0$

$$x^2 - 5x = 5$$
$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = 5 + \left(-\frac{5}{2}\right)^2$$
$$\left(x - \frac{5}{2}\right)^2 = \frac{45}{4}$$
$$x - \frac{5}{2} = \pm\sqrt{\frac{45}{4}}$$
$$x - \frac{5}{2} = \sqrt{\frac{45}{4}} \text{ or } x - \frac{5}{2} = -\sqrt{\frac{45}{4}}$$

Therefore, $x = \sqrt{\frac{45}{4}} + \frac{5}{2}$ $x = -\sqrt{\frac{45}{4}} + \frac{5}{2}$

≈ 5.85 $\approx -0.85.$

3) $x^2 + \frac{1}{4}x - 3 = 0$

$$x^2 + \frac{1}{4}x = 3$$
$$x^2 + \frac{1}{4}x + \left(\frac{1}{8}\right)^2 = 3 + \left(\frac{1}{8}\right)^2$$
$$\left(x + \frac{1}{8}\right)^2 = \frac{193}{64}$$
$$x + \frac{1}{8} = \pm\sqrt{\frac{193}{64}}$$
$$x + \frac{1}{8} = \sqrt{\frac{193}{64}} \text{ or } x + \frac{1}{8} = -\sqrt{\frac{193}{64}}$$

Therefore, $x = \sqrt{\frac{193}{64}} - \frac{1}{8}$ $x = -\sqrt{\frac{193}{64}} - \frac{1}{8}$

≈ 1.61 $\approx -1.86.$

4) $x^2 - 4.8x + 2 = 0$

$$x^2 - 4.8x = -2$$
$$x^2 - 4.8x + (-2.4)^2 = -2 + (-2.4)^2$$
$$(x - 2.4)^2 = 3.76$$
$$x - 2.4 = \pm\sqrt{3.76}$$
$$x - 2.4 = \sqrt{3.76} \text{ or } x - 2.4 = -\sqrt{3.76}$$

Therefore, $x = \sqrt{3.76} + 2.4$ $x = -\sqrt{3.76} + 2.4$

≈ 4.34 $\approx 0.46.$

KEY

4. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $x(x - 3) = 5x + 1$

$$x^2 - 3x = 5x + 1$$
$$x^2 - 8x = 1$$
$$x^2 - 8x + (-4)^2 = 1 + (-4)^2$$
$$(x - 4)^2 = 17$$
$$x - 4 = \pm\sqrt{17}$$
$$x - 4 = \sqrt{17} \quad \text{or} \quad x - 4 = -\sqrt{17}$$

Therefore, $x = \sqrt{17} + 4$, $x = -\sqrt{17} + 4$

≈ 8.12, $\approx -0.12.$

2) $(x + 1)^2 = 7x$

$$x^2 + 2x + 1 = 7x$$
$$x^2 - 5x = -1$$
$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -1 + \left(-\frac{5}{2}\right)^2$$
$$\left(x - \frac{5}{2}\right)^2 = \frac{21}{4}$$
$$x - \frac{5}{2} = \pm\sqrt{\frac{21}{4}}$$
$$x - \frac{5}{2} = \sqrt{\frac{21}{4}} \quad \text{or} \quad x - \frac{5}{2} = -\sqrt{\frac{21}{4}}$$

Therefore, $x = \sqrt{\frac{21}{4}} + \frac{5}{2}$, $x = -\sqrt{\frac{21}{4}} + \frac{5}{2}$

≈ 4.79, $\approx 0.21.$

3) $(x + 2)(x - 5) = 4x$

$$x^2 - 3x - 10 = 4x$$
$$x^2 - 7x = 10$$
$$x^2 - 7x + \left(-\frac{7}{2}\right)^2 = 10 + \left(-\frac{7}{2}\right)^2$$
$$\left(x - \frac{7}{2}\right)^2 = \frac{89}{4}$$
$$x - \frac{7}{2} = \pm\sqrt{\frac{89}{4}}$$
$$x - \frac{7}{2} = \sqrt{\frac{89}{4}} \quad \text{or} \quad x - \frac{7}{2} = -\sqrt{\frac{89}{4}}$$

Therefore, $x = \sqrt{\frac{89}{4}} + \frac{7}{2}$, $x = -\sqrt{\frac{89}{4}} + \frac{7}{2}$

≈ 8.22, $\approx -1.22.$

4) $x(x - 4) = 2(x + 7)$

$$x^2 - 4x = 2x + 14$$
$$x^2 - 6x = 14$$
$$x^2 - 6x + (-3)^2 = 14 + (-3)^2$$
$$(x - 3)^2 = 23$$
$$x - 3 = \pm\sqrt{23}$$
$$x - 3 = \sqrt{23} \quad \text{or} \quad x - 3 = -\sqrt{23}$$

Therefore, $x = \sqrt{23} + 3$, $x = -\sqrt{23} + 3$

≈ 7.80, $\approx -1.80.$

Advanced Level

5. Given the equation $y^2 - ay - 6 = 0$ where a is a constant, find the expressions for y in terms of a.

$$y^2 - ay - 6 = 0$$
$$y^2 - ay = 6$$
$$y^2 - ay + \left(-\frac{a}{2}\right)^2 = 6 + \left(-\frac{a}{2}\right)^2$$
$$\left(y - \frac{a}{2}\right)^2 = 6 + \frac{a^2}{4}$$
$$y - \frac{a}{2} = \pm\sqrt{6 + \frac{a^2}{4}}$$
$$y - \frac{a}{2} = \pm\sqrt{\frac{a^2 + 24}{4}}$$
$$y - \frac{a}{2} = \pm\frac{\sqrt{a^2 + 24}}{2}$$
$$y = \frac{a}{2} \pm \frac{\sqrt{a^2 + 24}}{2}$$
$$y = \frac{a \pm \sqrt{a^2 + 24}}{2}.$$

2.3

Solving Quadratic Equations in One Variable by Using Formula

The general form of a quadratic equation is $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$. Now, we shall use the method of completing the square to derive a formula for the solution to all quadratic equations.

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \textit{(divide throughout by a)} \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} && \textit{(rewrite the equation such that the constant term is on the RHS of the equation)} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \textit{(add } \left(\frac{b}{2a}\right)^2 \textit{ to both sides of the equation to make the LHS a perfect square)} \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} && \textit{(factorize the expression on the LHS and simplify the RHS)} \\
&= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \textit{(take the square roots on both sides)} \\
&= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Therefore, $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

From the above working, we can conclude that:

> If $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$, then
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The above formula for solving quadratic equations is usually used when the quadratic expression cannot be factorized easily.

Worked Example 7

Solve the equation $3x^2 + 4x - 5 = 0$, giving your answers correct to 2 decimal places.

Solution:

Comparing $3x^2 + 4x - 5 = 0$ with $ax^2 + bx + c = 0$, we get $a = 3$, $b = 4$, and $c = -5$.

$$
\begin{aligned}
\text{Therefore, } x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)} \\
&= \frac{-4 \pm \sqrt{16 - (-60)}}{6} \\
&= \frac{-4 \pm \sqrt{16 + 60}}{6} \\
&= \frac{-4 \pm \sqrt{76}}{6} \\
&\approx 0.79 \text{ or } -2.12.
\end{aligned}
$$

ATTENTION

Always ensure that the equation is in the form $ax^2 + bx + c = 0$ before substituting the values of a, b and c into the formula.

Practice Now

Similar Questions

Exercise 2C Questions 1-3

Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $2x^2 + 3x - 7 = 0$

Comparing $2x^2 + 3x - 7 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = 3$, and $c = -7$.

$$
\begin{aligned}
\text{Therefore, } x &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-7)}}{2(2)} \\
&= \frac{-3 \pm \sqrt{65}}{4} \\
&\approx 1.27 \text{ or } -2.77.
\end{aligned}
$$

2) $(x - 1)^2 = 4x - 5$

$x^2 - 2x + 1 = 4x - 5$

$x^2 - 6x + 6 = 0$

Comparing $x^2 - 6x + 6 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = -6$, and $c = 6$.

$$
\begin{aligned}
\text{Therefore, } x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)} \\
&= \frac{6 \pm \sqrt{12}}{2} \\
&\approx 4.73 \text{ or } 1.27.
\end{aligned}
$$

Similar Questions

Exercise 2C Question 4

Class Discussion

Solutions of quadratic equations

Work in pairs and consider each of the following equations.

$4x^2 - 12x + 9 = 0$ $\qquad$ $2x^2 + 5x + 8 = 0$ $\qquad$ $3x^2 + 5x - 4 = 0$

1) Find the value of $b^2 - 4ac$.
2) Use the quadratic formula to solve the equation. Are there any real solutions? Explain your answer.
3) What can you say about the sign of $b^2 - 4ac$ and the number of real solutions of a quadratic equation?

1. $4x^2 - 12x + 9 = 0$.

Comparing $4x^2 - 12x + 9 = 0$ with $ax^2 + bx + c = 0$, we get $a = 4$, $b = -12$ and $c = 9$.

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{0}}{2(4)} = 1.5$$

Therefore, when $b^2 - 4ac = 0$, the equation has one real solution.

2. $2x^2 + 5x + 8 = 0$

Comparing $2x^2 + 5x + 8 = 0$ with $ax^2 + bx + c = 0$, we get $a = 2$, $b = 5$ and $c = 8$.

$$b^2 - 4ac = 5^2 - 4(2)(8) = -39$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{-39}}{2(2)}$$

Therefore, when $b^2 - 4ac < 0$, the equation has no real solutions.

3. $3x^2 + 5x - 4 = 0$

Comparing $3x^2 + 5x - 4 = 0$ with $ax^2 + bx + c = 0$, we get $a = 3$, $b = 5$ and $c = -4$.

$$b^2 - 4ac = 5^2 - 4(3)(-4) = 73$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{73}}{2(3)}$$

$$\approx 0.59 \text{ or } -2.26$$

Therefore, when $b^2 - 4ac > 0$, the equation has two real solutions.

From **Class Discussion**, we observe that:

For a quadratic equation $ax^2 + bx + c = 0$,

- if $b^2 - 4ac > 0$, the equation has **two real solutions;**
- if $b^2 - 4ac = 0$, the equation has **one real solution;**
- if $b^2 - 4ac < 0$, the equation has **no real solutions.**

Exercise 2C

Basic Level

1. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $x^2 + 4x + 1 = 0$

Comparing $x^2 + 4x + 1 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 4$ and $c = 1$.

Therefore, $x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{12}}{2}$

≈ -0.27 or -3.73.

2) $3x^2 + 6x - 1 = 0$

Comparing $3x^2 + 6x - 1 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 3$, $b = 6$ and $c = -1$.

Therefore, $x = \dfrac{-6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)}$

$= \dfrac{-6 \pm \sqrt{48}}{6}$

≈ 0.15 or -2.15.

3) $3x^2 - 5x - 17 = 0$

Comparing $3x^2 - 5x - 17 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 3$, $b = -5$ and $c = -17$.

Therefore, $x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-17)}}{2(3)}$

$= \dfrac{5 \pm \sqrt{229}}{6}$

≈ 3.36 or -1.69.

4) $-5x^2 + 10x - 2 = 0$

Comparing $-5x^2 + 10x - 2 = 0$

with $ax^2 + bx + c = 0$,

we get $a = -5$, $b = 10$ and $c = -2$.

Therefore, $x = \dfrac{-10 \pm \sqrt{10^2 - 4(-5)(-2)}}{2(-5)}$

$= \dfrac{-10 \pm \sqrt{60}}{-10}$

≈ 0.23 or 1.77.

2. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $x^2 + 5x = 21$

$x^2 + 5x - 21 = 0$

Comparing $x^2 + 5x - 21 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 5$ and $c = -21$.

$$\text{Therefore, } x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-21)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{109}}{2}$$

$$\approx 2.72, -7.72.$$

2) $10x^2 - 12x = 15$

$10x^2 - 12x - 15 = 0$

Comparing $10x^2 - 12x - 15 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 10$, $b = -12$ and $c = -15$.

Therefore,

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(10)(-15)}}{2(10)}$$

$$= \frac{12 \pm \sqrt{744}}{20}$$

$$\approx 1.96 \text{ or } -0.76.$$

3) $8x^2 = 3x + 6$

$8x^2 - 3x - 6 = 0$

Comparing $8x^2 - 3x - 6 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 8$, $b = -3$ and $c = -6$.

$$\text{Therefore, } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(8)(-6)}}{2(8)}$$

$$= \frac{3 \pm \sqrt{201}}{16}$$

$$\approx 1.07 \text{ or } -0.70.$$

4) $16x - 61 = x^2$

$-x^2 + 16x - 61 = 0$

Comparing $-x^2 + 16x - 61 = 0$

with $ax^2 + bx + c = 0$,

we get $a = -1$, $b = 16$ and $c = -61$.

$$\text{Therefore, } x = \frac{-16 \pm \sqrt{16^2 - 4(-1)(-61)}}{2(-1)}$$

$$= \frac{-16 \pm \sqrt{12}}{-2}$$

$$\approx 6.27 \text{ or } 9.73.$$

Intermediate Level

3. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $x(x + 1) = 1$

$x^2 + x - 1 = 0$

Comparing $x^2 + x - 1 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 1$ and $c = -1$.

$$\text{Therefore, } x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$\approx 0.62 \text{ or } -1.62.$$

2) $x(x - 5) = 7 - 2x$

$x^2 - 5x = 7 - 2x$

$x^2 - 3x - 7 = 0$

Comparing $x^2 - 3x - 7 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = -3$ and $c = -7$.

$$\text{Therefore, } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{37}}{2}$$

$$\approx 4.54 \text{ or } -1.54.$$

3) $(2x + 3)(x - 1) - x(x + 2) = 0$

$2x^2 - 2x + 3x - 3 - x^2 - 2x = 0$

$x^2 - x - 3 = 0$

Comparing $x^2 - x - 3 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = -1$ and $c = -3$.

Therefore, $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$

$= \frac{1 \pm \sqrt{13}}{2}$

≈ 2.30 or -1.30.

4) $(4x - 3)^2 + (4x + 3)^2 = 25$

$16x^2 - 24x + 9 + 16x^2 + 24x + 9 = 25$

$32x^2 - 7 = 0$

Comparing $32x^2 - 7 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 32$, $b = 0$ and $c = -7$.

Therefore, $x = \frac{0 \pm \sqrt{0^2 - 4(32)(-7)}}{2(32)}$

$= \frac{0 \pm \sqrt{896}}{64}$

≈ 0.47 or -0.47.

4. Solve each of the following equations if possible.

1) $0.5(x^2 + 1) = x$

$0.5x^2 + 0.5 = x$

$0.5x^2 - x + 0.5 = 0$

Comparing $0.5x^2 - x + 0.5 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 0.5$, $b = -1$ and $c = 0.5$.

$b^2 - 4ac = (-1)^2 - 4(0.5)(0.5) = 0$

Since $b^2 - 4ac = 0$,

the equation has one real solution.

Therefore, $x = \frac{-(-1) \pm \sqrt{0}}{2(0.5)} = 1$.

2) $3x - 4 = (4x - 3)^2$

$3x - 4 = 16x^2 - 24x + 9$

$16x^2 - 27x + 13 = 0$

Comparing $16x^2 - 27x + 13 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 16$, $b = -27$ and $c = 13$.

$b^2 - 4ac = (-27)^2 - 4(16)(13) = -103$

Since $b^2 - 4ac < 0$,

the equation has no real solutions.

2.4 Real-life Applications of Quadratic Equations

In this section, we will take a look at how mathematical and real-life problems can be solved using quadratic equations.

Worked Example 8

The area of a circle, *A*, increases with its radius, *r* cm. This can be represented by the formula $A = \pi r^2$. Find the diameter of the circle when the area of the circle is 56 cm^2. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Solution:

$$\pi r^2 = 56$$

$$r2 = \frac{56}{\pi}$$

$$r = \sqrt{\frac{56}{\pi}}$$

$$\approx \sqrt{\frac{56}{\frac{22}{7}}}$$

$$\approx 4.22 \text{ cm}$$

Diameter of circle $\approx 4.22 \times 2$

$= 8.44$ cm

Therefore, when the area of the circle is 56 cm^2, its diameter is 8.44 cm.

Practice Now

Similar Questions

Exercise 2D Question 1

The volume of a cylinder, *V*, increases with its height and its base radius, *r* cm. This can be represented by the formula $V = \pi r^2 h$. Find the diameter of a cylindrical glass when the volume of the glass is 770 cm^3 and the height is 20 cm. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

$$\pi r^2 h = 770$$

$$r^2 = \frac{770}{\pi h}$$

$$r = \sqrt{\frac{770}{20\pi}}$$

$$\approx \sqrt{\frac{770}{20\left(\frac{22}{7}\right)}}$$

$$= 3.5 \text{ cm}$$

Diameter of circle $\approx 3.5 \times 2$

$= 7$ cm

Therefore, when the volume of the cylindrical glass is 770 cm^3, and the height is 20 cm, its diameter is 7 cm.

Worked Example 9

Two consecutive positive odd numbers are such that the sum of their squares is 130. Find the two numbers.

Solution:

Let the smaller number be x.

Then, the next consecutive odd number is $x + 2$.

$$\begin{aligned}
x^2 + (x+2)^2 &= 130 \\
x^2 + x^2 + 4x + 4 &= 130 \\
2x^2 + 4x - 126 &= 0 \\
x^2 + 2x - 63 &= 0 \\
(x-7)(x+9) &= 0 \\
x - 7 = 0 \quad &\text{or} \quad x + 9 = 0 \\
x = 7 \quad &\text{or} \quad x = -9
\end{aligned}$$

Since x is a positive odd number, we get $x = 7$.

$$\begin{aligned}
x + 2 &= 7 + 2 \\
&= 9
\end{aligned}$$

Therefore, the two consecutive positive odd numbers are 7 and 9.

Practice Now

Similar Questions

Exercise 2D Questions 2-5

Two consecutive positive even numbers are such that the sum of their squares is 164. Find the two numbers.

Let the smaller number be x.

Then, the next consecutive even number is $x + 2$.

$$\begin{aligned}
x^2 + (x+2)^2 &= 164 \\
x^2 + x^2 + 4x + 4 &= 164 \\
2x^2 + 4x - 160 &= 0 \\
x^2 + 2x - 80 &= 0 \\
(x-8)(x+10) &= 0
\end{aligned}$$

$$\begin{aligned}
x - 8 = 0 \quad &\text{or} \quad x + 10 = 0 \\
x = 8 \quad &\text{or} \quad x = -10
\end{aligned}$$

Since x is a positive even number, we get $x = 8$.

$$\begin{aligned}
x + 2 &= 8 + 2 \\
&= 10.
\end{aligned}$$

Therefore, the two consecutive positive even numbers are 8 and 10.

Worked Example 10

The perimeter of a rectangular field is 50 m, and its area is 150 m^2. Calculate the length and the width of the field.

Solution:

Let the length of the rectangular field be x m.

Then, the width of the rectangular field

$$\text{is} = \frac{50 - 2x}{2} = 25 - x \text{ m.}$$

$$x(25 - x) = 150$$

$$25x - x^2 = 150$$

$$x^2 - 25x + 150 = 0$$

$$(x - 10)(x - 15) = 0$$

$$x - 10 = 0 \quad \text{or} \quad x - 15 = 0$$

$$x = 10 \quad \text{or} \quad x = 15$$

When $x = 10$,

the width of the field

$= 25 - 10 = 15$ m.

When $x = 15$,

the width of the field

$= 25 - 15 = 10$ m.

The length of a rectangle usually refers to the longer side.

Therefore, the field is 10 m wide and 15 m long.

Similar Questions

Exercise 2D Questions 6-10

Practice Now

The perimeter of a rectangle is 20 cm, and its area is 24 cm^2. Find the length and the width of the rectangle.

Let the length of the rectangle be x cm.

Then, the width of the rectangle

$$= \frac{20 - 2x}{2} = 10 - x \text{ cm.}$$

$$x(10 - x) = 24$$

$$10x - x^2 = 24$$

$$x^2 - 10x + 24 = 0$$

$$(x - 4)(x - 6) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = 4 \quad \text{or} \quad x = 6$$

When $x = 4$,

the width of the rectangle

$= 10 - 4 = 6$ cm.

When x = 6, the width of the rectangle

$= 10 - 6 = 4$ cm.

The length of a rectangle usually refers to the longer side.

Therefore, the rectangle is 4 cm wide and 6 cm long.

Worked Example 11

The height, y m, of an object projected directly upward from the ground can be modeled by $y = 45t - 5t^2$ where t is the time in seconds after it leaves the ground.

1) Calculate the height of the object 5 sec after it leaves the ground.

2) At what time will the object strike the ground again?

Solution:

1) When $t = 5$

$$y = 45(5) - 5(5)^2$$

$$= 100$$

Therefore, the height of the object 5 sec after it leaves the ground = 100 m.

2) When the object strikes the ground, $y = 0$.

$$45t - 5t^2 = 0$$

$$5t(9 - t) = 0$$

$$5t = 0 \quad \text{or} \quad 9 - t = 0$$

$$t = 0 \quad \text{or} \quad t = 9$$

Therefore, the object will strike the ground again 9 sec after it leaves the ground.

Similar Questions

Exercise 2D Question 13

Practice Now

The height, y m, of a model rocket launched directly upward from the ground can be modeled by $y = 64t - 4t^2$ where t is the time in seconds after it leaves the ground.

1) Calculate the height of the rocket 12 sec after it leaves the ground.

When $t = 12$,

$$y = 64(12) - 4(12)^2$$

$$= 192.$$

Therefore, the height of the rocket 12 sec after it leaves the ground = 192 m.

2) At what time will the rocket strike the ground?

When the rocket strikes the ground, $y = 0$.

$$64t - 4t^2 = 0$$

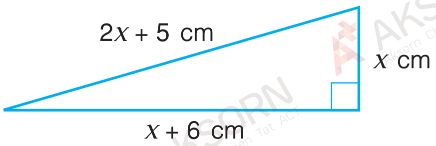
$$4t(16 - t) = 0$$

$$4t = 0 \quad \text{or} \quad 16 - t = 0$$

$$t = 0 \quad \text{or} \quad t = 16$$

Therefore, the rocket will strike the ground 16 sec after it leaves the ground.

Worked Example 12



On a map, a piece of land is in the shape of a right triangle with sides of length x cm, $x + 6$ cm and $2x + 5$ cm.

1) From the information given, formulate an equation and show that it simplifies to $2x^2 + 8x - 11 = 0$.
2) Solve the equation $2x^2 + 8x - 11 = 0$, giving both answers correct to 3 decimal places.
3) Find the perimeter of the triangle.

Solution:

1) By using the Pythagorean theorem,

$$x^2 + (x + 6)^2 = (2x + 5)^2$$

$$x^2 + x^2 + 12x + 36 = 4x^2 + 20x + 25.$$

Therefore, $2x^2 + 8x - 11 = 0$.

2) Comparing $2x^2 + 8x - 11 = 0$ with $ax^2 + bx + c = 0$, we get $a = 2$, $b = 8$ and $c = -11$.

$$\text{Therefore, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(-11)}}{2(2)}$$

$$= \frac{-8 \pm \sqrt{64 + 88}}{4}$$

$$= \frac{-8 \pm \sqrt{152}}{4}$$

$$\approx 1.082 \text{ or } -5.082.$$

3) The perimeter of the triangle $= x + (x + 6) + (2x + 5)$

$$= x + x + 6 + 2x + 5$$

$$= 4x + 11 \text{ cm}$$

Since the length of a triangle cannot be a negative value, $x \approx 1.082$.

Therefore, the perimeter of the triangle $\approx 4(1.082) + 11$

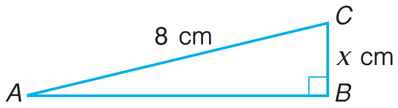
$$= 15.328 \text{ cm.}$$

Practice Now

Similar Questions

Exercise 2D Questions 11-12

The figure shows a right triangle *ABC* with dimensions as shown.



1) If the perimeter of the triangle is 17 cm, write down an expression, in terms of x, for the length of AB.

$AB = 17 - 8 - x$

$= 9 - x$ cm.

2) Formulate an equation in x and show that it simplifies to $2x^2 - 18x + 17 = 0$

By using the Pythagorean theorem,

$AC^2 = AB^2 + BC^2$

$8^2 = (9 - x)^2 + x^2$

$64 = 81 - 18x + x^2 + x^2$

$0 = 17 - 18x + 2x^2$.

Therefore, $2x^2 - 18x + 17 = 0$.

3) Solve the equation $2x^2 - 18x + 17 = 0$, giving both answers correct to 3 decimal places.

Comparing $2x^2 - 18x + 17 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = -18$ and $c = 17$.

Therefore, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-18) \pm \sqrt{(-18)^2 - 4(2)(17)}}{2(2)}$

$= \dfrac{18 \pm \sqrt{188}}{4}$

≈ 7.928 or 1.072.

4) Find the area of the triangle.

$BC \approx 1.072$ cm.

$AB \approx 9 - 1.072 = 7.928$ cm.

The area of the triangle

$= \frac{1}{2} \times AB \times BC$

$\approx \frac{1}{2} \times 7.928 \times 1.072$

≈ 4.25 cm^2.

Exercise 2D

Basic Level

1. The distance, s m, through which a heavy object falls from rest increases with the time taken, t sec. This can be represented by the equation $s = 5t^2$. Find the time taken by the object to fall 20 m.

When $s = 20$,

$20 = 5t^2$

$t^2 = \frac{20}{5}$

$= 4$

$t = \sqrt{4}$

$= 2$ sec.

Therefore, the time taken is 2 sec.

2. The sum of a counting number and twice the square of the number is 10. Find the number.

Let the number be x.

$x + 2x^2 = 10$

$2x^2 + x - 10 = 0$

$(2x + 5)(x - 2) = 0$

$2x + 5 = 0$ or $x - 2 = 0$

$x = -2\frac{1}{2}$ or $x = 2$

A counting number is a positive integer.

Therefore, the number is 2.

3. If four times a counting number is subtracted from three times the square of the number, the result 15 is obtained. Find the number.

Let the number be x.

$3x^2 - 4x = 15$

$3x^2 - 4x - 15 = 0$

$(3x + 5)(x - 3) = 0$

$3x + 5 = 0$ or $x - 3 = 0$

$x = -1\frac{2}{3}$ or $x = 3$

A counting number is a positive integer.

Therefore, the number is 3.

4. Two consecutive positive numbers are such that the sum of their squares is 113. Find the two numbers.

Let the smaller number be x.

Then, the next consecutive positive number is $x + 1$.

$x^2 + (x + 1)^2 = 113$

$x^2 + x^2 + 2x + 1 = 113$

$2x^2 + 2x - 112 = 0$

$x^2 + x - 56 = 0$

$(x - 7)(x + 8) = 0$

$x - 7 = 0$ or $x + 8 = 0$

$x = 7$ or $x = -8$

Since x is a positive integer, we get $x = 7$.

$x + 1 = 7 + 1 = 8$

Therefore, the two numbers are 7 and 8.

5. The difference between two positive numbers is 7, and the square of their sum is 289. Find the two numbers.

Let the smaller number be x.

Then, the greater positive number is $x + 7$.

$[x + (x + 7)]^2 = 289$

$(2x + 7)^2 = 289$

$4x^2 + 28x + 49 = 289$

$4x^2 + 28x - 240 = 0$

$(x - 5)(x + 12) = 0$

$x - 5 = 0$ or $x + 12 = 0$

$x = 5$ or $x = -12$

Since x is a positive integer, we get $x = 5$.

$x + 7 = 5 + 7 = 12$

Therefore, the two numbers are 5 and 12.

Intermediate Level

6. The perimeter of a rectangular campsite is 64 m, and its area is 207 m^2. Find the length and the width of the campsite.

Let the length of the rectangular campsite be x m.

Then, the width of the rectangular campsite

$= \frac{64 - 2x}{2} = 32 - x$ m.

$x(32 - x) = 207$

$32x - x^2 = 207$

$x^2 - 32x + 207 = 0$

$(x - 9)(x - 23) = 0$

$x - 9 = 0$ or $x - 23 = 0$

$x = 9$ or $x = 23$

When $x = 9$, the width of the rectangular campsite $= 32 - 9$

$= 23$ m.

When $x = 23$, the width of the rectangular campsite $= 32 - 23$

$= 9$ m.

The length of a rectangle usually refers to the longer side.

Therefore, the rectangular campsite is 9 m wide and 23 m long.

K
E
Y

7. A rectangular field, 70 m long and 50 m wide, is surrounded by a concrete path of uniform width. Given that the area of the path is 1,024 m^2, find the width of the path.

Let the width of the path be x m.

Then, the width of the field (including the path)

$= 2x + 50$ m.

The length of the field (including the path)

$= 2x + 70$ m.

The area of the field (excluding the path)

$= 70 \times 50 = 3{,}500$ m.

$(2x + 70)(2x + 50) - 3{,}500 = 1{,}024$

$4x^2 + 100x + 140x + 3{,}500 - 3{,}500 = 1{,}024$

$4x^2 + 240x - 1{,}024 = 0$

$x^2 + 60x - 256 = 0$

$(x - 4)(x + 64) = 0$

$x - 4 = 0$ or $x + 64 = 0$

$x = 4$ or $x = -64$

The width is always a positive value.

Therefore, the width of the path is 4 m.

8. A piece of wire 44 cm long is cut into two parts. Each part is bent to form a square. Given that the total area of the two squares is 65 cm^2, find the perimeter of each square.

Let the perimeter of the smaller square be x cm.Then, the perimeter of the larger square

$= 44 - x$ cm.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{44 - x}{4}\right)^2 = 65$$

$$\frac{x^2}{16} + \frac{1,936 - 88x + x^2}{16} = 65$$

$$x^2 + 1,936 - 88x + x^2 = 1,040$$

$$2x^2 - 88x + 896 = 0$$

$$x^2 - 44x + 448 = 0$$

$$(x - 16)(x - 28) = 0$$

$x - 16 = 0$ or $x - 28 = 0$

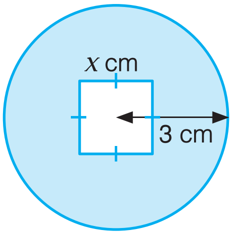
$x = 16$ or $x = 28$

When $x = 16$, the perimeter of the larger square $= 44 - 16 = 28$ cm.

When $x = 28$, the perimeter of the larger square $= 44 - 28 = 16$ cm.

Therefore, the perimeter of each square is 16 cm and 28 cm.

9. The figure shows an ancient coin, which was once used in China. The coin is in the shape of a circle of radius 3 cm with a square of sides x cm removed from its center. The area of each face of the coin is 7π cm^2.



1) Form an equation in x and show that it reduces to $2\pi - x^2 = 0$.

Area of circle $= \pi(3)^2 = 9\pi$ cm^2

Area of circle removed from circle $= x^2$ cm^2

$$9\pi - x^2 = 7\pi$$

$$2\pi - x^2 = 0$$

2) Solve the equation $2\pi - x^2 = 0$, giving your answers correct to 2 decimal places. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

$2\pi - x^2 = 0$

$x^2 = 2\pi$

$x = \pm\sqrt{2\pi}$

$\approx \pm\sqrt{2\left(\frac{22}{7}\right)} \approx \pm 2.51$

3) Find the perimeter of the square.

Since the length is always a positive value, we get $x \approx 2.51$.

Therefore, the perimeter of the square $\approx 4(2.51) \approx 10$ cm.

10. **Amy walks at an average speed of $x + 1$ km/h for x hours and cycles at an average speed of $2x + 5$ km/h for $x - 1$ hours. She covers a total distance of 90 km.**

1) Form an equation in x and show that it reduces to $3x^2 + 4x - 95 = 0$.

Distance Amy walks $= (x + 1)x$

$= x^2 + x$ km

Distance Amy cycles $= (2x + 5)(x - 1)$

$= 2x^2 - 2x + 5x - 5$

$= 2x^2 + 3x - 5$ km

Therefore, $(x^2 + x) + (2x^2 + 3x - 5) = 90$

$3x^2 + 4x - 5 = 90$

$3x^2 + 4x - 95 = 0.$

2) Solve the equation $3x^2 + 4x - 95 = 0$.

$3x^2 + 4x - 95 = 0$

$(3x + 19)(x - 5) = 0$

$3x + 19 = 0$ or $x - 5 = 0$

Therefore, $x = -6\frac{1}{3}$ or $x = 5$.

3) Find the time taken for her entire journey.

Since the distance is always a positive value, we get $x = 5$.

Therefore, the time taken for her entire journey $= x + (x - 1)$

$= 5 + (5 - 1)$

$= 9$ hours.

11. The perimeter of a rectangle is 112 cm, and its width is x cm.

1) Find, in terms of x, an expression for the length of the rectangle.

Length of rectangle $= \dfrac{112 - 2x}{2}$

$= 56 - x$ cm

2) Given that the area of the rectangle is 597 cm^2, form an equation in x and show that it reduces to $x^2 - 56x + 597 = 0$.

Area of rectangle $=$ 597 cm^2

$x(56 - x) = 597$

$56x - x^2 = 597$

$56x - x^2 - 597 = 0$

$x^2 - 56x + 597 = 0$

3) Solve the equation $x^2 - 56x + 597 = 0$, giving both answers correct to 2 decimal places.

Comparing $x^2 - 56x + 597 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = -56$ and $c = 597$.

Therefore, $x = \dfrac{-(-56) \pm \sqrt{(-56)^2 - 4(1)(597)}}{2(1)}$

$= \dfrac{56 \pm \sqrt{748}}{2}$

≈ 41.67 or 14.33.

4) Find the length of the diagonal of the rectangle.

The length of a rectangle usually refers to the longer side.

When $x \approx 14.33$,

the width of the rectangle $\approx$ 14.33 cm,

and the length of the rectangle

$\approx 56 - 14.33 = 41.67$ cm.

By using the Pythagorean theorem,

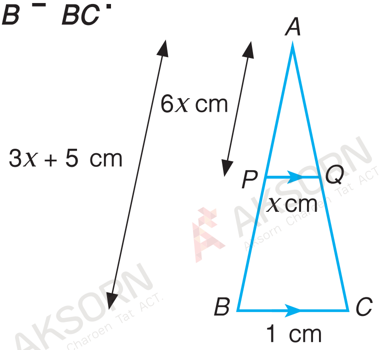
Length of diagonal^2 = Length^2 + Width^2

$\approx (41.67)^2 + (14.33)^2$

Length of diagonal $\approx \sqrt{(41.67)^2 + (14.33)^2}$

$\approx$ 44.1 cm.

12. The figure shows a triangle ABC in which $AP = 6x$ cm, $AB = 3x + 5$ cm, $PQ = x$ cm and $BC = 1$ cm. P and Q are two points on the lines $\overline{AB}$ and $\overline{AC}$ respectively such that $\frac{AP}{AB} = \frac{PQ}{BC}$.



1) Form an equation in x and show that it reduces to $3x^2 - x = 0$.

Given $\frac{AP}{AB} = \frac{PQ}{BC}$,

we get $\frac{6x}{3x + 5} = \frac{x}{1}$

$$\frac{6x}{3x + 5} = x.$$

$$\frac{6x}{3x + 5} \times (3x + 5) = x \times (3x + 5)$$

$$6x = x(3x + 5)$$

$$6x = 3x^2 + 5x$$

$$0 = 3x^2 - x$$

$$3x^2 - x = 0$$

2) Solve the equation $3x^2 - x = 0$.

$$3x^2 - x = 0$$

$$x(3x - 1) = 0$$

Therefore, $x = 0$ or $3x - 1 = 0$

$$3x = 1$$

$$x = \frac{1}{3}.$$

3) Find the length of $\overline{PB}$.

$$PB = AB - AP$$

$$= \left[\left(3 \times \frac{1}{3}\right) + 5\right] - 6\left(\frac{1}{3}\right)$$

$$= 6 - 2$$

$$= 4 \text{ cm}$$

Advanced Level

13. **The height, h m, of a ball projected directly upward from the ground can be modeled by $h = 56t - 7t^2$ where t is the time in seconds after it leaves the ground.**

1) Find the height of the ball 3.5 sec after it leaves the ground.

When $t = 3.5$,

$h = 56(3.5) - 7(3.5)^2$

$= 110.25$.

Therefore, the height of the ball 3.5 sec after it leaves the ground is 110.25 m.

2) At what time will the ball strike the ground?

When the ball strikes the ground, $h = 0$.

$56t - 7t^2 = 0$

$7t(8 - t) = 0$

$7t = 0$ or $8 - t = 0$

$t = 0$ or $t = 8$

Therefore, the ball will strike the ground after 8 sec.

3) When will the ball be 49 m above the ground?

Briefly explain why there are two possible answers.

When $h = 49$,

$56t - 7t^2 = 49$.

$7t^2 - 56t + 49 = 0$

$t^2 - 8t + 7 = 0$

$(t - 1)(t - 7) = 0$

$t - 1 = 0$ or $t - 7 = 0$

$t = 1$ or $t = 7$

Therefore, the ball will be 49 m above the ground after 1 sec and 7 sec. The first time refers to when the ball is going upward, while the second time refers to when the ball is coming down. There are two possible answers.

Summary

A quadratic equation in one variable in the form of $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$ can be solved by:

1) factorization
2) completing the square
3) using formula

Factorization	To solve a quadratic equation in one variable in the form of $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$ by factorization: • Apply the principle: If two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$, or both P and Q are equal to 0. • Factorize the expression, then set out each factor to be equal to 0, and solve each equation for the solution(s) of the equation.
Completing the square	To solve a quadratic equation in one variable in the form of $x^2 + px + q = 0$ by **completing the square**: • Rewrite the equation such that the constant term is on the RHS of the equation, i.e. $x^2 + px = -q$. • Add $\left(\frac{p}{2}\right)^2$ to both sides of the equation to form $\left(x + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$. • Take the square roots on both sides of the equation to solve for x.
Using formula	The formula for solving a quadratic equation in the form $ax^2 + bx + c = 0$ is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Note that when $b^2 - 4ac < 0$, the equation has no real solutions.

Review Exercise 2

1. Solve each of the following equations.

1) $(4a + 11)(3a - 7) = 0$

$(4a + 11)(3a - 7) = 0$

$4a + 11 = 0$ or $3a - 7 = 0$

Therefore, $a = -2\frac{3}{4}$ or $a = 2\frac{1}{3}$.

2) $4b^2 = 7b$

$4b^2 - 7b = 0$

$b(4b - 7) = 0$

$b = 0$ or $4b - 7 = 0$

Therefore, $b = 0$ or $b = 1\frac{3}{4}$.

3) $49c^2 - 140c + 100 = 0$

$(7c)^2 - 2(7c)(10) + 10^2 = 0$

$(7c - 10)^2 = 0$

$7c - 10 = 0$

Therefore, $c = 1\frac{3}{7}$.

4) $147 - 3d^2 = 0$

$-3(d^2 - 49) = 0$

$d^2 - 7^2 = 0$

$(d + 7)(d - 7) = 0$

$d + 7 = 0$ or $d - 7 = 0$

Therefore, $d = -7$ or $d = 7$.

2. Solve each of the following equations.

1) $6k^2 + 11k - 10 = 0$

$(3k - 2)(2k + 5) = 0$

$3k - 2 = 0$ or $2k + 5 = 0$

Therefore, $k = \frac{2}{3}$ or $k = -2\frac{1}{2}$.

2) $3m(8 - 3m) = 16$

$9m^2 - 24m + 16 = 0$

$(3m)^2 - 2(3m)(4) + 4^2 = 0$

$(3m - 4)^2 = 0$

$3m - 4 = 0$

Therefore, $m = 1\frac{1}{3}$.

3) $(3n - 1)^2 = 12n + 8$

$9n^2 - 6n + 1 = 12n + 8$

$9n^2 - 18n - 7 = 0$

$(3n + 1)(3n - 7) = 0$

$3n + 1 = 0$ or $3n - 7 = 0$

Therefore, $n = -\frac{1}{3}$ or $n = 2\frac{1}{3}$.

4) $(5p + 1)(p + 4) = 2(7p + 5)$

$5p^2 + 20p + p + 4 = 14p + 10$

$5p^2 + 7p - 6 = 0$

$(5p - 3)(p + 2) = 0$

$5p - 3 = 0$ or $p + 2 = 0$

Therefore, $p = \frac{3}{5}$ or $p = -2$.

3. **Solve each of the following equations.**

1) $2x^2 - 11x - 21 = 0$

$(2x + 3)(x - 7) = 0$

$2x + 3 = 0$ or $x - 7 = 0$

Therefore, $x = -1\frac{1}{2}$ or $x = 7.$

2) $2(y + 2)^2 - 11(y + 2) - 21 = 0$

From question 1, let $x = y + 2.$

$y + 2 = -1\frac{1}{2}$ or $y + 2 = 7$

Therefore, $y = -3\frac{1}{2}$ or $y = 5.$

4. **Answer each of the following questions.**

1) If $x = 3$ is a solution of the equation $2x^2 - 5x + k = 0$, find the value of k.

When $x = 3.$

$2(3)^2 - 5(3) + k = 0$

$18 - 15 + k = 0.$

Therefore, $k = -3.$

2) Find the other solution of the equation. $2x^2 - 5x + k = 0$

$2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$2x + 1 = 0$ or $x - 3 = 0$

$x = -\frac{1}{2}$ or $x = 3$

Therefore, the other solution.

$2x^2 - 5x + k = 0$ is $x = -\frac{1}{2}.$

5. **Solve each of the following equations by completing the square, giving your answers correct to 2 decimal places.**

1) $x^2 + 8x + 5 = 0$

$x^2 + 8x = -5$

$x^2 + 8x + \left(\frac{8}{2}\right)^2 = -5 + \left(\frac{8}{2}\right)^2$

$(x + 4)^2 = 11$

$x + 4 = \pm\sqrt{11}$

$x + 4 = \sqrt{11}$ or $x + 4 = -\sqrt{11}$

Therefore, $x = \sqrt{11} - 4$ or $x = -\sqrt{11} - 4$

≈ -0.68 $\approx -7.32.$

2) $x^2 + 7x - 3 = 0$

$x^2 + 7x = 3$

$x^2 + 7x + \left(\frac{7}{2}\right)^2 = 3 + \left(\frac{7}{2}\right)^2$

$\left(x + \frac{7}{2}\right)^2 = \frac{61}{4}$

$x + \frac{7}{2} = \pm\sqrt{\frac{61}{4}}$

$x + \frac{7}{2} = \sqrt{\frac{61}{4}}$ or $x + \frac{7}{2} = -\sqrt{\frac{61}{4}}$

Therefore, $x = \sqrt{\frac{61}{4}} - \frac{7}{2}$ or $x = -\sqrt{\frac{61}{4}} - \frac{7}{2}$

≈ 0.41 $\approx -7.41.$

3) $x^2 - 11x - 7 = 0$

$$x^2 - 11x = 7$$

$$x^2 - 11x + \left(-\frac{11}{2}\right)^2 = 7 + \left(-\frac{11}{2}\right)^2$$

$$\left(x - \frac{11}{2}\right)^2 = \frac{149}{4}$$

$$x - \frac{11}{2} = \pm\sqrt{\frac{149}{4}}$$

$$x - \frac{11}{2} = \sqrt{\frac{149}{4}} \text{ or } x - \frac{11}{2} = -\sqrt{\frac{149}{4}}$$

Therefore, $x = \sqrt{\frac{149}{4}} + \frac{11}{2}$ or $x = -\sqrt{\frac{149}{4}} + \frac{11}{2}$

≈ 11.06 $\approx -0.60.$

4) $x^2 + 1.2x = 1$

$$x^2 + 1.2x + \left(\frac{1.2}{2}\right)^2 = 1 + \left(\frac{1.2}{2}\right)^2$$

$$(x + 0.6)^2 = 1.36$$

$$x + 0.6 = \pm\sqrt{1.36}$$

$$x + 0.6 = \sqrt{1.36} \text{ or } x + 0.6 = -\sqrt{1.36}$$

Therefore, $x = \sqrt{1.36} - 0.6$ or $x = -\sqrt{1.36} - 0.6$

≈ 0.57 $\approx -1.77.$

6. By using the quadratic formula, solve each of the following equations, giving your answers correct to 2 decimal places.

1) $-4x^2 + x + 5 = 0$

Comparing $-4x^2 + x + 5 = 0$

with $ax^2 + bx + c = 0,$

we get $a = -4,\ b = 1$ and $c = 5.$

$$\text{Therefore, } x = \frac{-1 \pm \sqrt{1^2 - 4(-4)(5)}}{2(-4)}$$

$$= \frac{-1 \pm \sqrt{81}}{-8}$$

$$= -1 \text{ or } 1\frac{1}{4}.$$

2) $3x^2 = 5x + 1$

$3x^2 - 5x - 1 = 0$

Comparing $3x^2 - 5x - 1 = 0$

with $ax^2 + bx + c = 0,$

we get $a = 3,\ b = -5$ and $c = -1.$

$$\text{Therefore, } x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{37}}{6}$$

$$\approx 1.85 \text{ or } -0.18.$$

7. Solve each of the following equations, giving your answers correct to 2 decimal places.

1) $(x - 3)^2 = \frac{4}{25}$

$$x - 3 = \pm\sqrt{\frac{4}{25}}$$

$$x - 3 = \sqrt{\frac{4}{25}} \text{ or } x - 3 = -\sqrt{\frac{4}{25}}$$

Therefore, $x = \sqrt{\frac{4}{25}} + 3$ or $x = -\sqrt{\frac{4}{25}} + 3$

$= \frac{2}{5} + 3$ $= -\frac{2}{5} + 3$

$= 3\frac{2}{5}$ $= 2\frac{3}{5}.$

2) $(4 - x)^2 = 12$

$$4 - x = \pm\sqrt{12}$$

$$4 - x = \sqrt{12} \text{ or } 4 - x = -\sqrt{12}$$

Therefore, $x = 4 - \sqrt{12}$ or $x = 4 + \sqrt{12}$

≈ 0.54 $\approx 7.46.$

3) $(x - 1)(x + 3) = 9$

$x^2 + 3x - x - 3 = 9$

$x^2 + 2x - 12 = 0$

Comparing $x^2 + 2x - 12 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 2$ and $c = -12$.

Therefore, $x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-12)}}{2(1)}$

$= \frac{-2 \pm \sqrt{52}}{2}$

≈ 2.61 or -4.61.

4) $x(x + 4) = 17$

$x^2 + 4x = 17$

$x^2 + 4x - 17 = 0$

Comparing $x^2 + 4x - 17 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 4$ and $c = -17$.

Therefore, $x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-17)}}{2(1)}$

$= \frac{-4 \pm \sqrt{84}}{2}$

≈ 2.58 or -6.58.

8. Answer each of the following questions.

1) Solve the equation $2x^2 - 7x + 4 = 0$, giving your answers correct to 2 decimal places.

Comparing $2x^2 - 7x + 4 = 0$

with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = -7$ and $c = 4$.

Therefore, $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(4)}}{2(2)}$

$= \frac{7 \pm \sqrt{17}}{4}$

≈ 2.78 or 0.72.

2) Find the values of y that satisfy the equation $2(y - 1)^2 - 7(y - 1) + 4 = 0$.

Let $y - 1 = x$.

We get $2x^2 - 7x + 4 = 0$.

Since $x \approx 2.78$ or 0.72,

$y - 1 \approx 2.78$ or $y - 1 \approx 0.72$

$y = 3.78$ $y = 1.72$.

9. Form a quadratic equation in the form $ax^2 + bx + c = 0$ where a, b and c are integers, given each of the following solutions.

1) $x = 2$, $x = \frac{6}{7}$

$(x - 2)\left(x - \frac{6}{7}\right) = 0$

$x^2 - \frac{6}{7}x - 2x + \frac{12}{7} = 0$

$x^2 - \frac{20}{7}x + \frac{12}{7} = 0$

$7x^2 - 20x + 12 = 0$ where $a = 7$, $b = -20$, and $c = 12$.

2) $x = -\frac{1}{2}$, $x = -\frac{2}{3}$

$\left(x + \frac{1}{2}\right)\left(x + \frac{2}{3}\right) = 0$

$x^2 + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{3} = 0$

$x^2 + \frac{7}{6}x + \frac{1}{3} = 0$

$6x^2 + 7x + 2 = 0$ where $a = 6$, $b = 7$, and $c = 2$.

10. **The difference between two numbers is 3. If the square of the smaller number is equal to four times the larger number, find the two numbers.**

Let the smaller number be x.

Then, the greater number is $x + 3$.

$$x^2 = 4(x + 3)$$

$$= 4x + 12$$

$$x^2 - 4x - 12 = 0$$

$$(x + 2)(x - 6) = 0$$

$$x + 2 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = -2 \quad \text{or} \quad x = 6$$

When $x = -2$,

$$x + 3 = -2 + 3$$

$$= 1.$$

When $x = 6$,

$$x + 3 = 6 + 3$$

$$= 9.$$

Therefore, the two numbers are -2 and 1, or 6 and 9.

11. **Ian's father is x^2 years old while Ian is x years old. In $4x$ years' time, Ian's father will be twice as old as him.**

1) Form an equation in x and show that it reduces to $x^2 - 6x = 0$.

$$x^2 + 4x = 2(x + 4x)$$

$$= 2(5x)$$

$$= 10x$$

Therefore, $x^2 - 6x = 0$.

2) Solve the equation $x^2 - 6x = 0$.

$$x^2 - 6x = 0$$

$$x(x - 6) = 0$$

Therefore, $x = 0$ or $x - 6 = 0$

$$x = 0 \quad \text{or} \quad x = 6.$$

3) Find the age of Ian's father when Ian was born.

Since age is always a positive value, we get $x = 6$.

Age of Ian's father when Ian was born $= x^2 - x$

$$= 6^2 - 6$$

$$= 30 \text{ years.}$$

12. The length and the width of a rectangle are $2x + 5$ cm and $2x - 1$ cm, respectively. The area of the rectangle is three times the area of a square of sides $x + 1$ cm.

1) Form an equation in x and show that it reduces to $x^2 + 2x - 8 = 0$.

$(2x + 5)(2x - 1) = 3(x + 1)^2$

$4x^2 - 2x + 10x - 5 = 3(x^2 + 2x + 1)$

$4x^2 + 8x - 5 = 3x^2 + 6x + 3$

Therefore, $x^2 + 2x - 8 = 0$

2) Solve the equation $x^2 + 2x - 8 = 0$.

$x^2 + 2x - 8 = 0$

$(x - 2)(x + 4) = 0$

$x - 2 = 0$ or $x + 4 = 0$

Therefore, $x = 2$ or $x = -4$.

3) Find the perimeter of the rectangle.

When $x = -4$,

$2x - 1 = 2(-4) - 1 = -9 < 0$.

Since the length is always a positive value, we get $x = 2$.

The perimeter of the rectangle

$= 2[(2x + 5) + (2x - 1)]$

$= 2(4x + 4)$

$= 2[4(2) + 4]$

$= 24$ cm.

13. If each student in a class sends a New Year greeting card to each of his/her classmates, a total of 870 cards will be sent. Find the number of students in the class.

Let the number of students in the class be x.

Each student will send out

$x - 1$ cards.

$x(x - 1) = 870$

$x^2 - x - 870 = 0$

$(x - 30)(x + 29) = 0$

$x - 30 = 0$ or $x + 29 = 0$

$x = 30$ or $x = -29$

The number of students is always a positive value.

Therefore, the number of students is 30.

14. **The height, y m, of an object projected directly upward from the ground can be modeled by $y = 20t - 5t^2$ where t is the time in seconds after it leaves the ground.**

1) Find the height of the object 2 sec after it leaves the ground.

When $t = 2$,

$y = 20(2) - 5(2)^2$

$= 20$.

Therefore, the height of the object 2 sec after it leaves the ground is 20 m.

2) When will the object be 15 m above the ground?

When $y = 15$,

$20t - 5t^2 = 15$

$t^2 - 4t + 3 = 0$

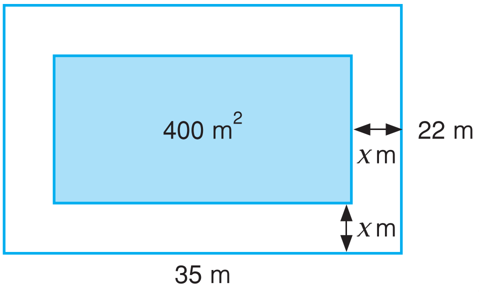
$(t - 1)(t - 3) = 0$

$t - 1 = 0$ or $t - 3 = 0$

$t = 1$ $t = 3$

Therefore, the object will be 15 m above the ground after 1 sec and 3 sec.

15. **A rectangular function room has dimensions 35 m by 22 m. Part of the floor is covered with ceramic tiles, as shown by the shaded rectangle in the figure.**



1) Given that the part of the floor which is not covered by the tiles has a uniform width of x m, write down an expression, in terms of x, for the length and the width of the floor covered by the tiles.

Length of the floor covered by the tiles = $35 - 2x$ m

Width of the floor covered by the tiles = $22 - 2x$ m

2) Given that the floor area covered by the tiles is 400 m^2, form an equation in x and show that it reduces to $2x^2 - 57x + 185 = 0$.

The floor area covered by the tiles = 400 m^2

$$(35 - 2x)(22 - 2x) = 400$$

$$770 - 70x - 44x + 4x^2 = 400$$

$$4x^2 - 114x + 770 = 400$$

$$4x^2 - 114x + 370 = 0$$

$$2x^2 - 57x + 185 = 0$$

3) Solve the equation $2x^2 - 57x + 185 = 0$, giving both answers correct to 2 decimal places.

$2x^2 - 57x + 185 = 0$

Comparing $2x^2 -57x + 185 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = -57$ and $c = 185$.

Therefore, $x = \frac{-(-57) \pm \sqrt{(-57)^2 - 4(2)(185)}}{2(2)}$

$$= \frac{57 \pm \sqrt{1,769}}{4}$$

$$\approx 24.76 \text{ or } 3.74.$$

4) State the width of the floor that is not covered by the tiles.

The width of the floor that is not covered by the tiles = 3.74 m.

Challenge Yourself

Given that the difference between the solutions of the equation $2x^2 - 6x - k = 0$ is 5, find the value of k.

Let the smaller solution of $2x^2 - 6x - k = 0$ be n. That is, $x = n$ or $x - n = 0$.

Then, the other larger solution is $x = n + 5$ or $x - n - 5 = 0$.

$$
\begin{aligned}
(x - n)(x - n - 5) &= 0\\
x^2 - nx - 5x - nx + n^2 + 5n &= 0\\
x^2 - 2nx - 5x + n^2 + 5n &= 0\\
x^2 - (2n + 5)x + n(n + 5) &= 0\\
2x^2 - 2(2n + 5)x + 2n(n + 5) &= 0
\end{aligned}
$$

Comparing terms with $2x^2 - 6x - k = 0$,

we get

$$
\begin{aligned}
2(2n + 5) &= 6\\
2n + 5 &= 3\\
2n &= -2\\
n &= -1.
\end{aligned}
$$

When $n = -1$,

$$
\begin{aligned}
-k &= 2n(n + 5)\\
&= 2(-1)(-1 + 5)\\
&= -8.
\end{aligned}
$$

Therefore, $k = 8$.

KEY

Chapter 3

Quadratic Functions

Many real-life problems can be modeled by quadratic functions, and related problems can then be solved using quadratic functions. For example, in physics, quadratic models are used to solve problems involving the motion of objects, such as a rocket that is projected directly upward, and in finance, the formulation of quadratic functions to solve problems involving maximization of profit and minimization of cost helps businesspersons make informed decisions.

Indicator

- Understand and apply the knowledge of quadratic functions to solving mathematical and real-life problems. **(MA 1.2 G. 9/2)**

Compulsory Details

- Graphs of quadratic functions
- Real-life applications of quadratic functions

3.1 Graphs of Functions

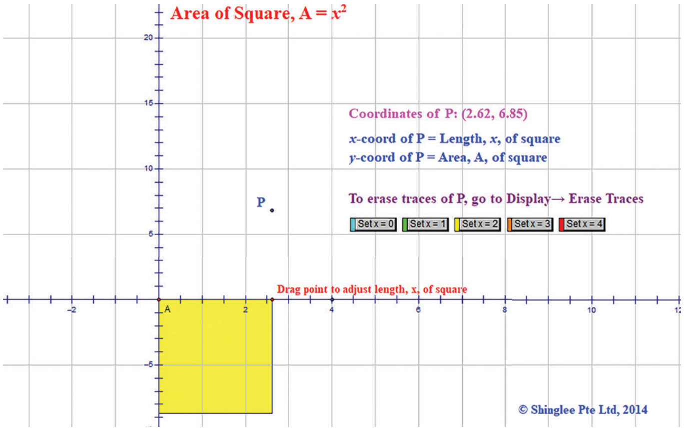
1. Quadratic Functions

Investigation

Relationship between the area of a square and the length of its side

Work in pairs to do the following activity and answer the questions.

1. Go to https://www.shinglee.com.sg/StudentResources/ and click on NSM2/ ⟶ Area of Square.
2. Open Area of Square.



Area of Square, $A = x^2$

Coordinates of P: (2.62, 6.85)

x-coord of P = Length, x, of square

y-coord of P = Area, A, of square

To erase traces of P, go to Display→ Erase Traces

Drag point to adjust length, x, of square

© Shinglee Pte Ltd, 2014

3. According to the Geometer's Sketchpad, the X-coordinate of Point P represents the length, i.e. x units, of a square while the Y-coordinate of Point P represents its area, i.e. A square units. Hence, Point P will trace out the graph of $A = x^2$. Click on each of the buttons: Set $x = 0$, Set $x = 1$, etc.
4. For each value of x, how many corresponding values of A are there? And is $A = x^2$ the equation of a function?

   For each value of x, there is the exactly one corresponding value of A, and $A = x^2$ is the equation of a function.

5. After clicking on the button to set the value of x, are the obtained graphs straight lines? Explain.

   No, the obtained graphs are hyperbolas.

From **Investigation**, when we substitute x as a real number into $A = x^2$, there will be the only value of A that corresponds to x. We call this type of relationship a **function**. The graph of $A = x^2$ is a curve, and we normally call $A = x^2$ a **quadratic function**.

On this topic, we will learn about graphs of quadratic functions, which are in the form of curves called **parabolas**.

Investigation

Graphs of $y = x^2$ and $y = -x^2$

Work in pairs to do the following activity and answer the questions.

1. Draw graphs of $y = x^2$ and $y = -x^2$ by using dynamic mathematic programs such as the Geometer's Sketchpad and GeoGebra.
2. Study the graphs of $y = x^2$ and $y = -x^2$ and answer the following questions:
   1) Both graphs pass through a particular point on the coordinate axes. What are the coordinates of this point?

      The graphs of $y = x^2$ and $y = -x^2$ pass through (0, 0).

   2) State the lowest or highest point of each graph.

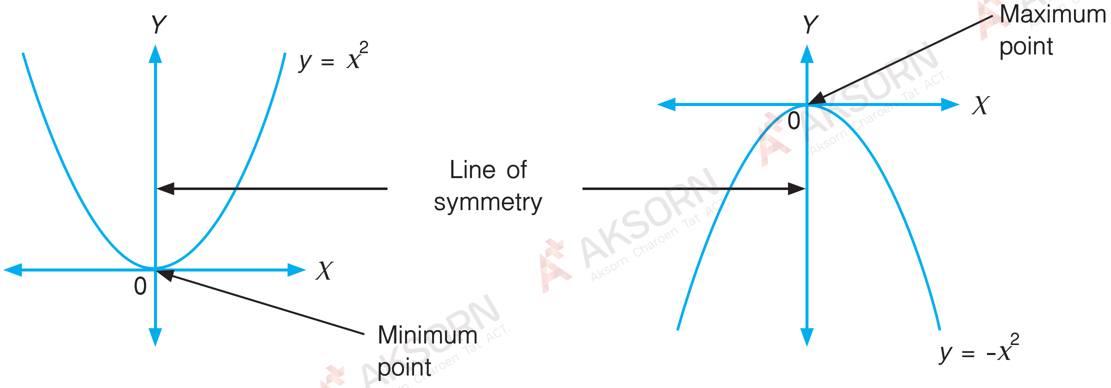
      The minimum point of the graph of $y = x^2$ is (0, 0), and there is no maximum point.

      The maximum point of the graph of $y = -x^2$ is (0, 0), and there is no minimum point.

   3) Both graphs are symmetrical about one of the axes. Name the axis.

      The graphs of $y = x^2$ and $y = -x^2$ are symmetrical about the Y-axis, i.e. the equation of the line of symmetry of the graphs is $x = 0$.

From **Investigation**, we can conclude that:



1) The graphs of $y = x^2$ and $y = -x^2$ pass through the origin.
2) The graph of $y = x^2$ is a parabola that opens upward, and the minimum point is (0, 0).

   The graph of $y = -x^2$ is a parabola that opens downward, and the maximum point is (0, 0).
3) The graphs of $y = x^2$ and $y = -x^2$ have the Y-axis or $x = 0$ as the line of symmetry.

Definition The general form of the equation of a quadratic function is given by $y = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$.

When $a = 1$, $b = 0$ and $c = 0$, we have a quadratic function which has an equation $y = x^2$.

When $a = -1$, $b = 0$ and $c = 0$, we have a quadratic function which has an equation $y = -x^2$.

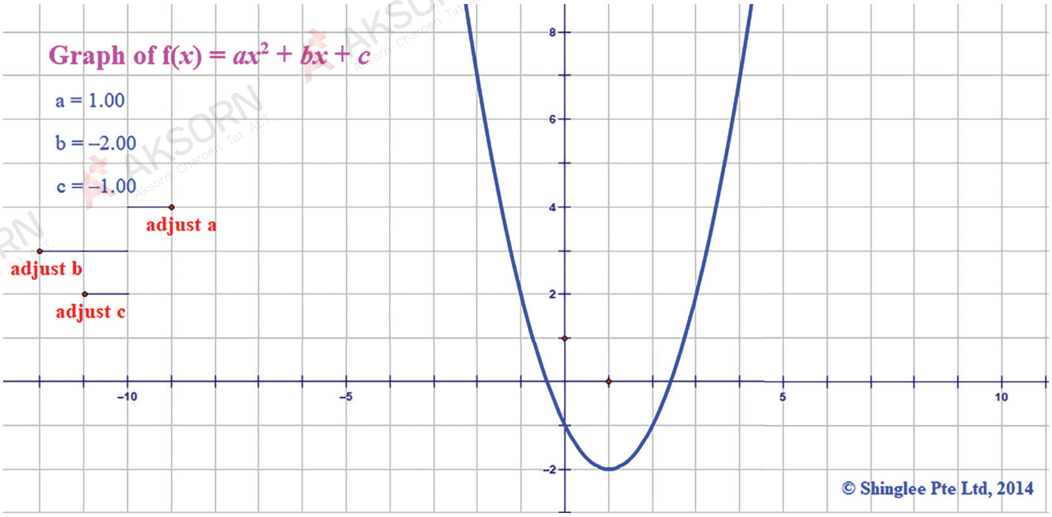
Do all graphs of quadratic functions have the same shapes and properties as the graphs of $y = x^2$ and $y = -x^2$? We shall now take a look at the graphs of other quadratic functions.

Investigation

Graphs of $y = ax^2 + bx + c$ where $a \neq 0$

Work in pairs to do the following activity and answer the questions.

1. Go to https://www.shinglee.com.sg/StudentResources/ and click on NSM2/ ⟶ Graphs of Quadratic Functions.
2. Open Graphs of Quadratic Functions.



© Shinglee Pte Ltd, 2014

Part 1: Effect of Value of *a*

1. With the Geometer's Sketchpad, adjust $b = 0$ and $c = 0$.
2. Increase the value of a where $a > 0$. What do you notice about the shape of the graph?

   As the value of a increases, the shape of the graph becomes narrower.

3. Decrease the value of a where $a > 0$. What do you notice about the shape of the graph?

   As the value of a decreases, the shape of the graph becomes wider.

4. Decrease the value of a until it becomes negative. What do you notice about the shape of the graph?

   As the value of a decreases until it becomes negative, the shape of the graph changes from one that opens upward to one that opens downward.

5. How does the value of a affect the shape of the graph? What happens when $a > 0$ and $a < 0$?

The value of a determines whether the shape of the graph becomes narrower or wider. When $a > 0$, the curve opens upward indefinitely and when $a < 0$, the curve opens downward indefinitely.

Part 2: Effect of Value of c

1. With the Geometer's Sketchpad, adjust $a = 1$ and $b = 1$.
2. Increase the value of c. What do you notice about the shape of the graph?

   As the value of c increases, the position of the graph changes by shifting upward along the Y-axis.
3. Decrease the value of c. What do you notice about the shape of the graph?

   As the value of c decreases, the position of the graph changes by shifting downward along the Y-axis.
4. How does the value of c affect the shape of the graph?

   The value of c affects the distance the graph is from the X-axis, i.e. the minimum point or maximum point, and determines whether the graph is above the X-axis or below the Y-axis.

Part 3: Effect of Value of b

1. With the Geometer's Sketchpad, adjust $a = 1$, $b = -2$ and $c = 0$. Explain the shape and position of the graph when the value of b increases.

   The shape of the graph opens upward indefinitely. As the value of b increases, the graph shifts from right to left along the X-axis.
2. With the Geometer's Sketchpad, adjust $a = -1$, $b = -2$ and $c = 0$. Explain the shape and position of the graph when the value of b increases.

   The shape of the graph opens downward indefinitely. As the value of b increases, the graph shifts from left to right along the X-axis.
3. How does the value of b affect the shape of the graph?

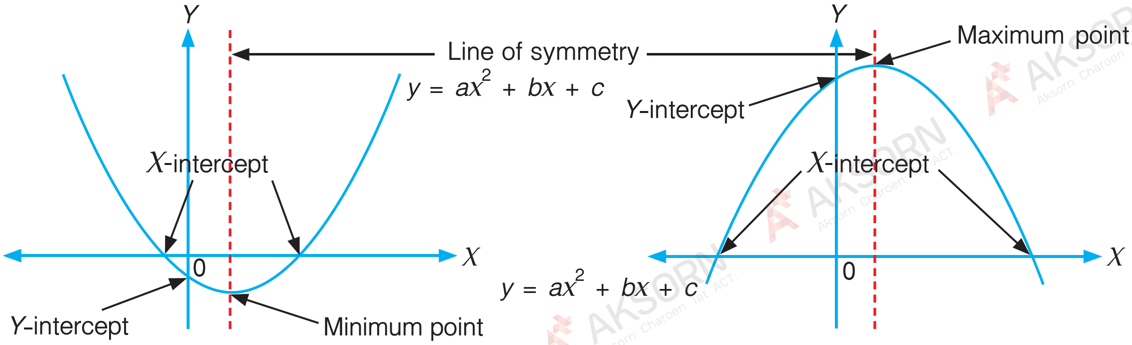
   The value of b affects the distance the graph is from the Y-axis, i.e. the minimum point or maximum point, and determines whether the graph is on the left or right of the Y-axis.

Part 4: Graphs of Quadratic Functions

Complete the table.

No.	Quadratic functions	Coefficient of x^2	Opening upward/ downward	Coordinates of minimum/ maximum point	Line of symmetry	X-intercept(s)	Y-intercept
1	$y = x^2 - 4x + 3$	1	Upward	(2, 1)	$x = 2$	1, 3	3
2	$y = -x^2 - 2x + 3$	-1	Downward	(-1, 4)	$x = -1$	-3, 1	3
3	$y = x^2 - 4x + 4$	1	Upward	(2, 0)	$x = 2$	2	4
4	$y = -4x^2 + 12x - 9$	-4	Downward	(1.5, 0)	$x = 1.5$	1.5	-9
5	$y = 2x^2 + 2x + 1$	2	Upward	(-0.5, 0.5)	$x = -0.5$	None	1
6	$y = -3x^2 + x - 4$	-3	Downward	$\left(\frac{1}{6}, 3\frac{11}{12}\right)$	$x = \frac{1}{6}$	None	-4

From **Investigation**, we can conclude that:



1) For $a > 0$, the graph opens upward indefinitely and has a minimum point.

   For $a < 0$, the graph opens downward indefinitely and has a maximum point.

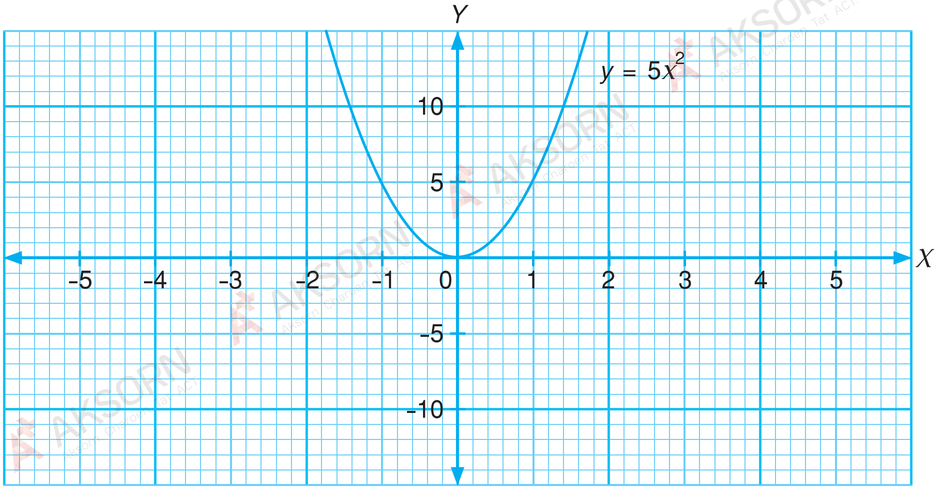
2) The smaller $|a|$, the wider the graph opens.

3) The line of symmetry of the graph passes through its minimum or maximum point.

4) The graph may have 0, 1 or 2 X-intercept(s), but it has only 1 Y-intercept.

Worked Example 1

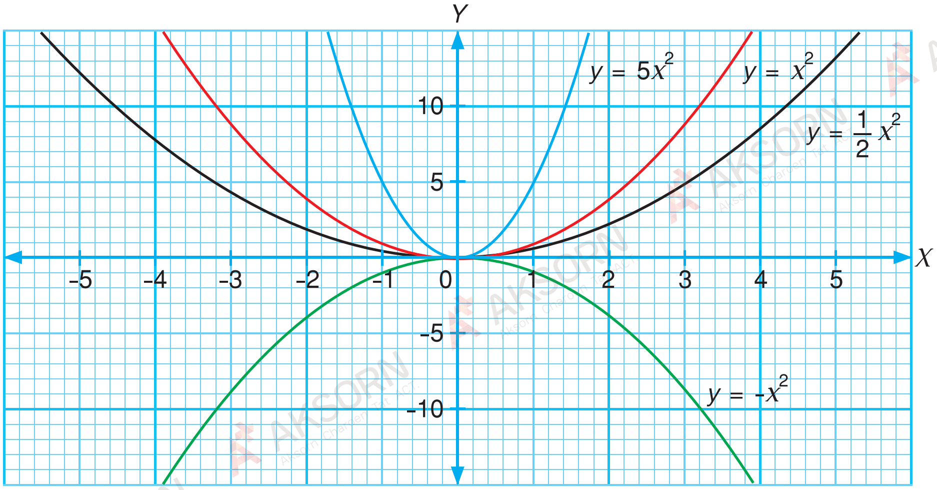
Determine the graph of $y = 5x^2$.



On the same grid, sketch the graphs of the following:

1) $y = x^2$ 2) $y = -x^2$ 3) $y = \frac{1}{2}x^2$

Solution:

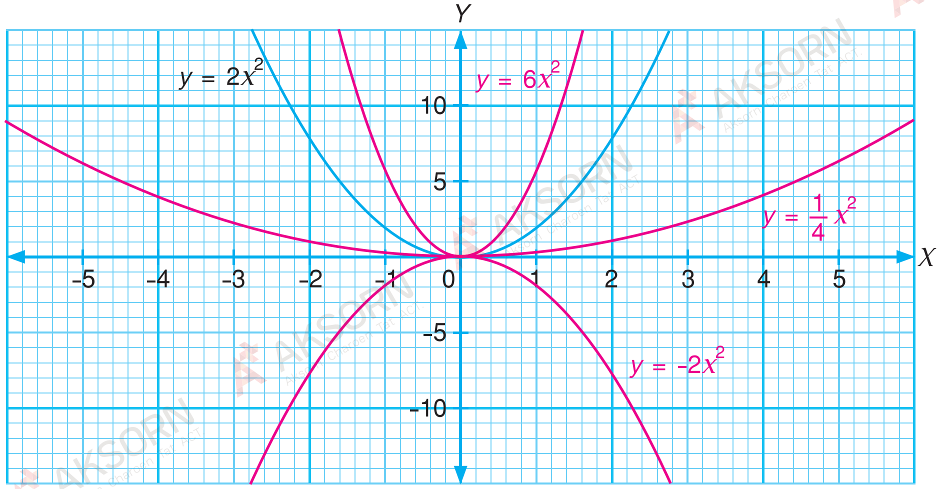


Practice Now

Similar Questions

Exercise 3A Question 1

Determine the graph of $y = 2x^2$.



On the same grid, sketch the graphs of the following:

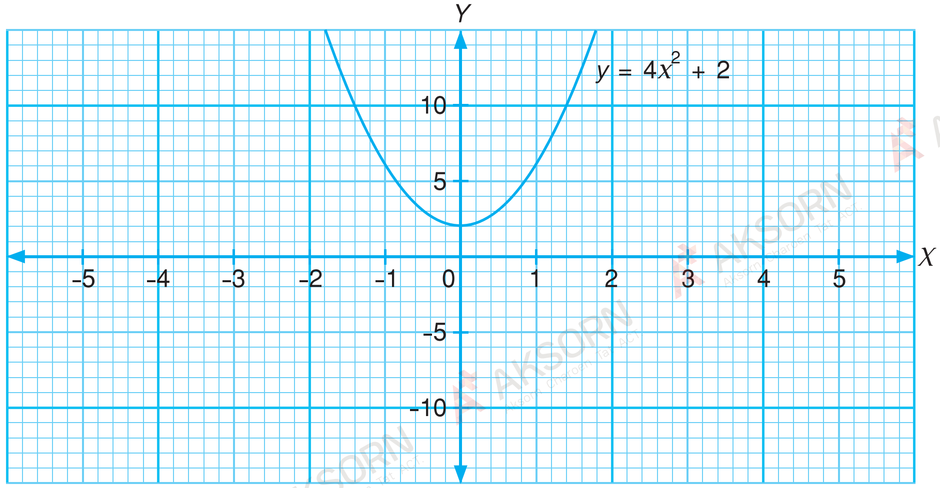
1) $y = 6x^2$

2) $y = -2x^2$

3) $y = \frac{1}{4}x^2$

Worked Example 2

Determine the graph of $y = 4x^2 + 2$.



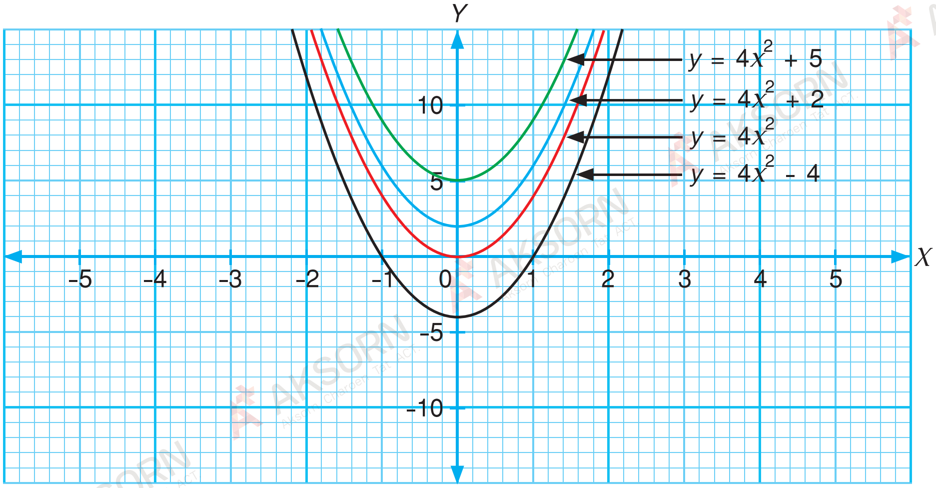
On the same grid, sketch the graphs of the following:

1) $y = 4x^2$

2) $y = 4x^2 + 5$

3) $y = 4x^2 - 4$

Solution:

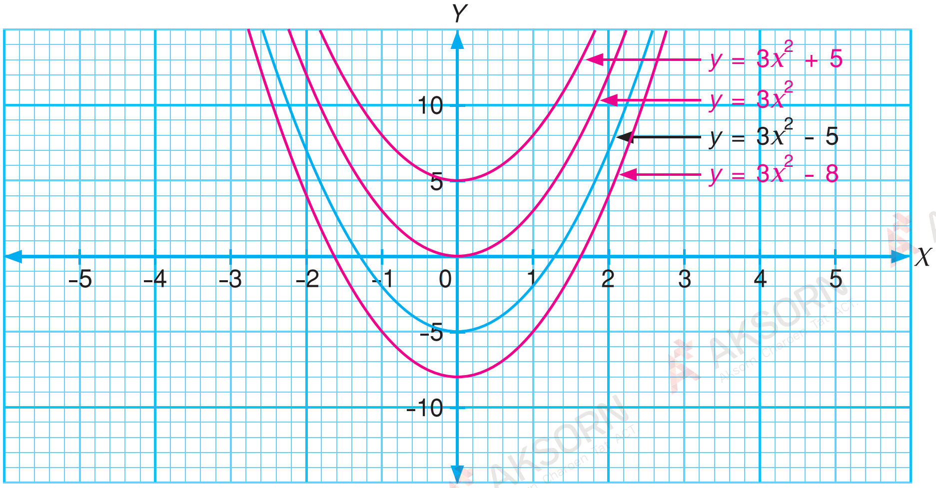


Practice Now

Similar Questions

Exercise 3A Questions 2-4

Determine the graph of $y = 3x^2 - 5$.



On the same grid, sketch the graphs of the following:

1) $y = 3x^2$　　2) $y = 3x^2 + 5$　　3) $y = 3x^2 - 8$

2. Applications of Quadratic Functions to Solving Problems

On the previous topic, we have learned about the characteristics of quadratic functions. Next, we will apply this knowledge to solving real-life problems, as in the following example:

Worked Example 3

William kicked a ball upward. If $h = 27t - 6t^2$ is the relationship between height (h) in meters and time (t) in seconds while the ball was being above the ground, find the following:

1) The graph of $h = 27t - 6t^2$
2) The duration in seconds and the height in meters of the ball at the maximum point after it was kicked upward
3) The duration in seconds that the ball would fall to the ground

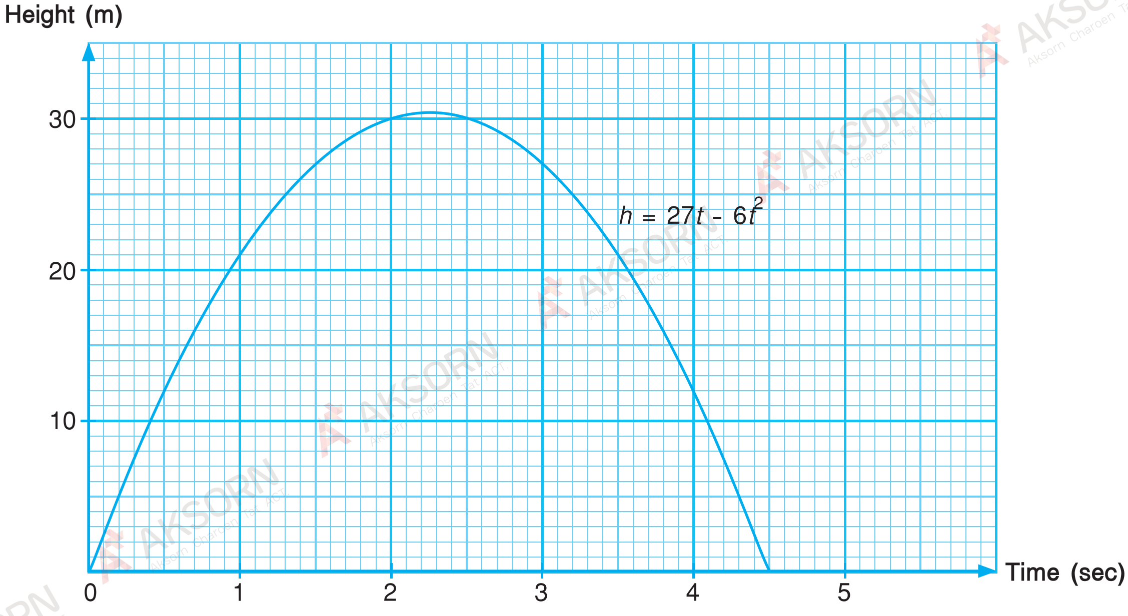
ATTENTION

According to the relationship that $h = 27t - 6t^2$, we can find the height (h) of the ball when $h \geq 0$.

Solution:

1)

t	0	0.5	1	2	3	4	4.5
h	0	12	21	30	27	12	0



2) The ball was at the maximum point above the ground after about 2.25 seconds once it was kicked upward, and it was approximately 30.5 m high from the ground.

3) The ball would fall to the ground after it was kicked upward for 4.5 sec.

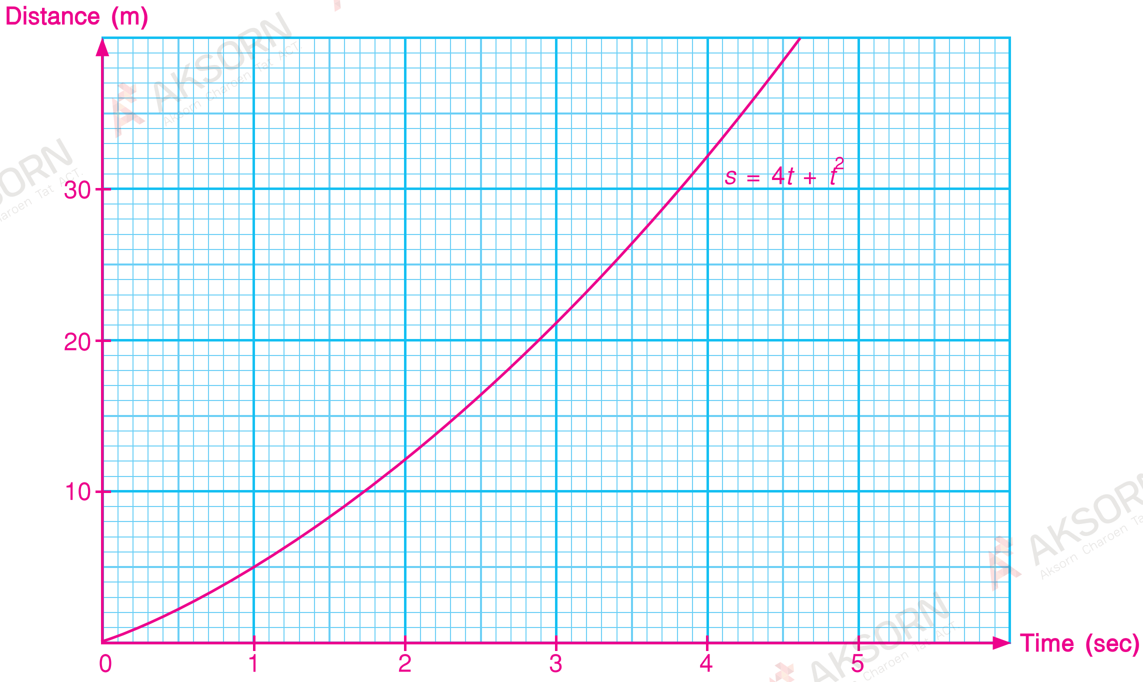
Similar Questions

Exercise 3A Questions 5-6

Practice Now

An object is released through a curved rail. If $s = 4t + t^2$ is the relationship between distance (s) in meters and time (t) in seconds after the object was released, find the following:

1) The graph of $s = 4t + t^2$



2) The distance in meters of the object after it was released through the curved rail for 2.6 sec

The distance of the object after it was released through the curved rail for 2.6 sec is 17.25 m.

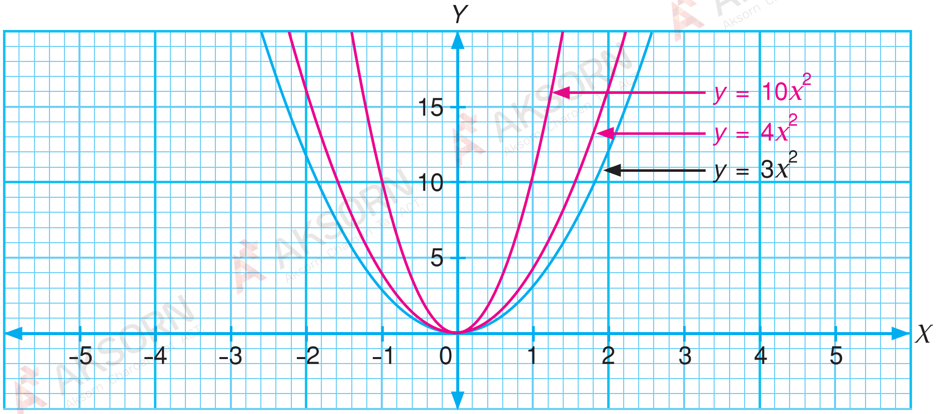
3) The duration in seconds of the object after it was released through the curved rail for 30 m

The duration of the object after it was released through the curved rail for 30 m is 3.85 sec.

Exercise 3A

Basic Level

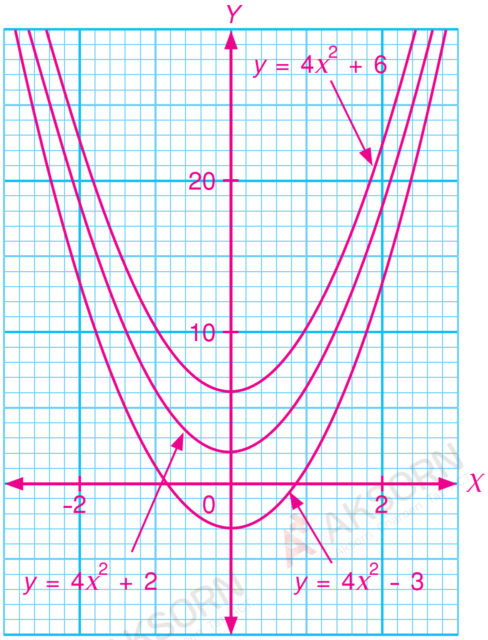
1. On the same grid as $y = 3x^2$, sketch the graph of $y = 4x^2$ and $y = 10x^2$.

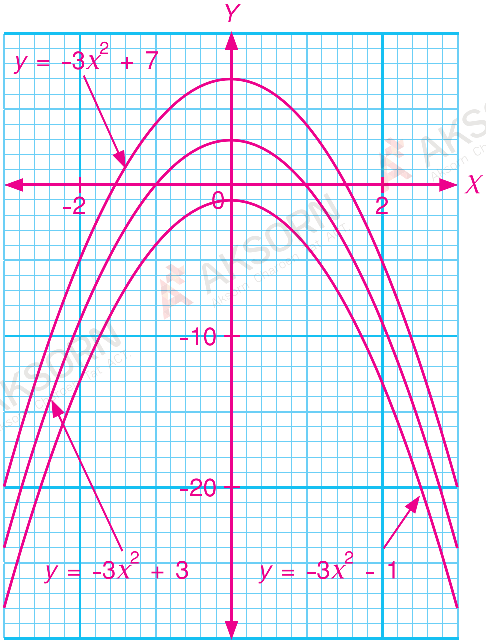


2. Sketch the graphs of the following quadratic functions.

1) $y = 4x^2 - 3$, $y = 4x^2 + 2$ and $y = 4x^2 + 6$

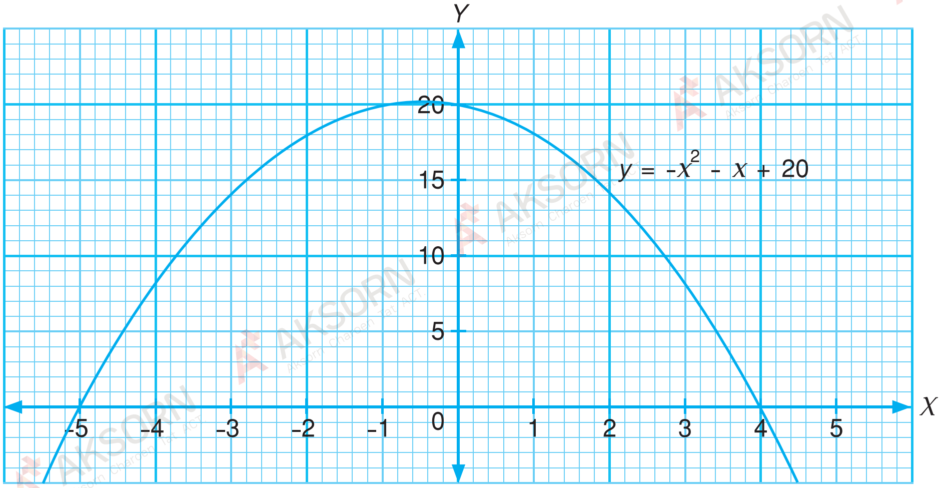
2) $y = -3x^2 - 1$, $y = -3x^2 + 3$ and $y = -3x^2 + 7$





Intermediate Level

3. Determine the graph of $y = -x^2 - x + 20$ and answer the following questions.



1) Find the X-intercept(s) and the Y-intercept.

Let $y = 0$.

We get $-x^2 - x + 20 = 0$

$x^2 + x - 20 = 0$

$(x + 5)(x - 4) = 0$

$x + 5 = 0$ or $x - 4 = 0$

$x = -5$ or $x = 4$.

Let $x = 0$.

We get $y = -0^2 - 0 + 20$

$y = 20$.

Therefore, the graph of $y = -x^2 - x + 20$ passes through the X-axis at (-5, 0) and (4, 0) and through the Y-axis at (0, 20).

2) If (3, h) is on the graph of $y = -x^2 - x + 20$, then what is the value of h?

Since (3, h) is on the graph of $y = -x^2 - x + 20$,

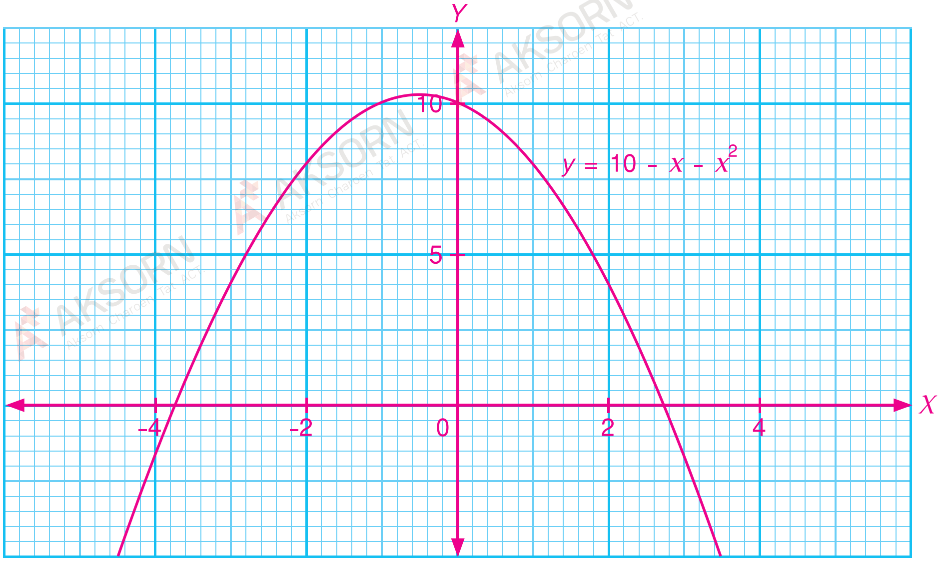
we get: $h = -3^2 - 3 + 20$

$h = 8$.

4. The following table displays the values of x and y that correspond to $y = 10 - x - x^2$.

x	-4	-3	-2	-1	0	1	2	3
y	-2	4	8	10	10	8	4	-2

1) Sketch the graph of $y = 10 - x - x^2$.



2) If $x = -1.5$, then what is the value of y?

The value of y is approximately 9.3.

3) Find the maximum value of the function and the value of x that corresponds to it.

The maximum value of the function is approximately 10.3, and the value of x that corresponds to it is -0.5.

4) Solve the equation $10 - x - x^2 = 1.6$ by using a graph.

The graph of $y = 1.6$, which is a straight line, passes through the graph of $y = 10 - x - x^2$ at (-3.45, 1.6) and (2.45, 1.6).

Therefore, the solution to the equation $10 - x - x^2 = 1.6$ is $x = -3.45, 2.45$.

5. **A triangle has its base of $x + 2$ cm long and $7 - x$ cm high.**

1) Demonstrate that $A = 7 + \frac{5}{2}x - \frac{1}{2}x^2$ when A is the area of the triangle.

Since A is the area of the triangle,

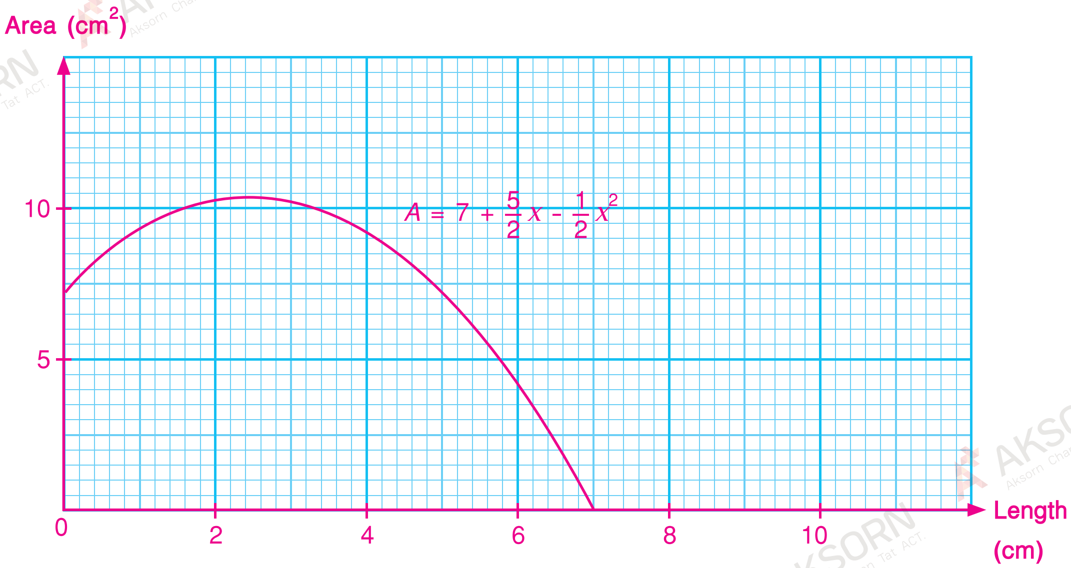
we get $= \frac{1}{2}(x + 2)(7 - x)$

$= \frac{1}{2}(7x - x^2 + 14 - 2x)$

$= \frac{1}{2}(14 + 5x - x^2)$

$= 7 + \frac{5}{2}x - \frac{1}{2}x^2.$

2) Sketch the graph of $A = 7 + \frac{5}{2}x - \frac{1}{2}x^2$.



3) Find the base length and the height that make this triangle have the most area.

Since the maximum point of $A = 7 + \frac{5}{2}x - \frac{1}{2}x^2$ is (2.5, 10.125).

then the value of x that makes this triangle have the most area is $x = 2.5$.

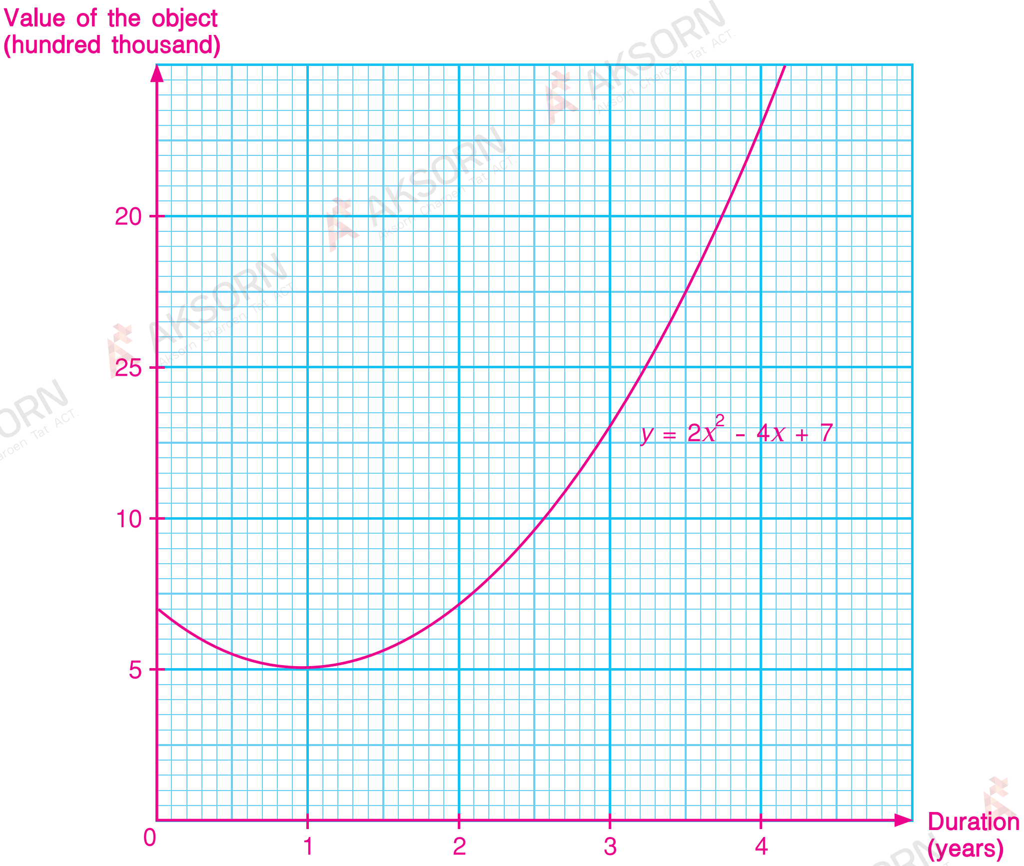
Therefore, the base length that makes this triangle have the most area is 2.5 + 2 = 4.5 cm,

and the height that makes this triangle have the most area is 7 - 2.5 = 4.5 cm.

Advanced Level

6. If $y = 2x^2 - 4x + 7$ is the relationship between the value (hundred thousand) of an object and the duration (years) that has passed since Peter bought the object.

1) Sketch the graph of $y = 2x^2 - 4x + 7$.



2) How many years does it take for the object to have the least value? And how much is it in Thai baht?

The object has the least value after Peter bought it for 1 year, which is 500,000 baht.

3.2

Graphs of Quadratic Functions in the Form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$

Investigation

Graphs of quadratic functions in the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$

1. Construct the graph of $y = (x - 3)(x - k)$ where $k = -2, -1, 0, 1$ and 2 by using dynamic mathematical programs such as the Geometer's Sketchpad and GeoGebra.
2. Does the graph open upward or downward?

   The graph opens upward.

3. Write down the coordinates of the point(s) where the graph cuts the X-axis, i.e. the X-intercept.

   For $k = -2$, the graph of the function cuts the X-axis at $(-2, 0)$ and $(3, 0)$.

   For $k = -1$, the graph of the function cuts the X-axis at $(-1, 0)$ and $(3, 0)$.

   For $k = 0$, the graph of the function cuts the X-axis at $(0, 0)$ and $(3, 0)$.

   For $k = 1$, the graph of the function cuts the X-axis at $(1, 0)$ and $(3, 0)$.

   For $k = 2$, the graph of the function cuts the X-axis at $(2, 0)$ and $(3, 0)$.

4. Write down the coordinates of the point(s) where the graph cuts the Y-axis, i.e. the Y-intercept.

   For $k = -2$, the graph of the function cuts the Y-axis at $(0, -6)$.

   For $k = -1$, the graph of the function cuts the Y-axis at $(0, -3)$.

   For $k = 0$, the graph of the function cuts the Y-axis at $(0, 0)$.

   For $k = 1$, the graph of the function cuts the Y-axis at $(0, 3)$.

   For $k = 2$, the graph of the function cuts the Y-axis at $(0, 6)$.

5. What is the relationship between the X-intercepts and the line of symmetry?

   The line of symmetry is halfway between the X-intercepts.

6. State the equation of the line of symmetry of the graph.

For $k = -2$, the vertical line $x = \frac{1}{2}$ of the graph is the line of symmetry.

For $k = -1$, the vertical line $x = 1$ of the graph is the line of symmetry.

For $k = 0$, the vertical line $x = 1\frac{1}{2}$ of the graph is the line of symmetry.

For $k = 1$, the vertical line $x = 2$ of the graph is the line of symmetry.

For $k = 2$, the vertical line $x = 2\frac{1}{2}$ of the graph is the line of symmetry.

7. Write down the coordinates of the maximum or the minimum point of the graph.

For $k = -2$, the minimum point of the graph is $\left(\frac{1}{2}, -6\frac{1}{4}\right)$.

For $k = -1$, the minimum point of the graph is $(1, -4)$.

For $k = 0$, the minimum point of the graph is $\left(1\frac{1}{2}, -2\frac{1}{4}\right)$.

For $k = 1$, the minimum point of the graph is $(2, -1)$.

For $k = 2$, the minimum point of the graph is $\left(2\frac{1}{2}, -\frac{1}{4}\right)$.

8. Repeat the steps 1-7 by adjusting $y = (x - 3)(x - k)$ to $y = -(x - 3)(x - k)$, $y = (x - 5)(x - k)$ and $y = -(x - 5)(x - \mathrm{k})$, respectively.

9. By looking at the equation of each graph, how do you determine if it opens upward or downward?

The graph of $y = (x - h)(x - k)$ where $k = -2, -1, 0, 1$ and 2 opens upward.

The graph of $y = -(x - h)(x - k)$ where $k = -2, -1, 0, 1$ and 2 opens downward.

10. By looking at the equation of each graph, how do you determine the coordinates of the points where the graph cuts the X-axis?

The graph of $y = (x - h)(x - \mathrm{k})$ where $\mathrm{k} = -2, -1, 0, 1$ and 2 cuts the X-axis at $(h, 0)$ and $(k, 0)$.

The graph of $y = -(x - h)(x - \mathrm{k})$ where $\mathrm{k} = -2, -1, 0, 1$ and 2 cuts the X-axis at $(h, 0)$ and $(k, 0)$.

11. What can you say about the line of symmetry of each graph?

The graph of $y = (x - h)(x - k)$ is symmetrical about the vertical line that is parallel to the Y-axis and passes through the minimum point.

The graph of $y = -(x - h)(x - k)$ is symmetrical about the vertical line that is parallel to the Y-axis and passes through the maximum point.

From **Investigation**, we can conclude that:

1) For the equation $y = (x - h)(x - k)$, the graph opens upward. The graph cuts the X-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that is parallel to the Y-axis and passes through the minimum point.
2) For the equation $y = -(x - h)(x - k)$, the graph opens downward. The graph cuts the X-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that is parallel to the Y-axis and passes through the maximum point.

Worked Example 4

Sketch the graph of $y = (x - 1)(x - 5)$.

Solution:

Since the coefficient of x^2 is 1, the graph opens upward.

When $y = 0$,

we get $(x - 1)(x - 5) = 0$

$$x - 1 = 0 \quad \text{or} \quad x - 5 = 0$$

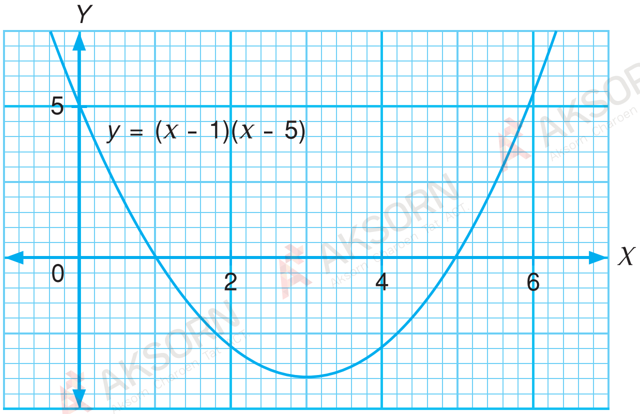
$$x = 1 \quad \text{or} \quad x = 5.$$

When $x = 0$,

we get $y = (-1)(-5)$

$$y = 5.$$

Therefore, the graph cuts the X-axis at $(1, 0)$ and $(5, 0)$ and the Y-axis at $(0, 5)$.



PROBLEM SOLVING TIP

Step 1: State the coefficient of x^2 to determine if the graph opens upward or downward.

Step 2: Obtain the X-intercepts by substituting $y = 0$ into the equation.

Step 3: Obtain the Y-intercept by substituting $x = 0$ into the equation.

Step 4: Sketch the graph.

ATTENTION

- The line of symmetry is halfway between the X-intercepts.
- How do we find the X-coordinates of the midpoint? In Worked Example 4, it is (3, -4).
- With this information, how can we find the coordinates of the minimum point?

Practice Now

Similar Questions

Exercise 3B Questions 1-5

Sketch the graph of $y = (3 - x)(x + 5)$.

Since the coefficient of x^2 is -1, the graph opens downward.

When $y = 0$,

we get $(3 - x)(x + 5) = 0$

$3 - x = 0$ or $x + 5 = 0$

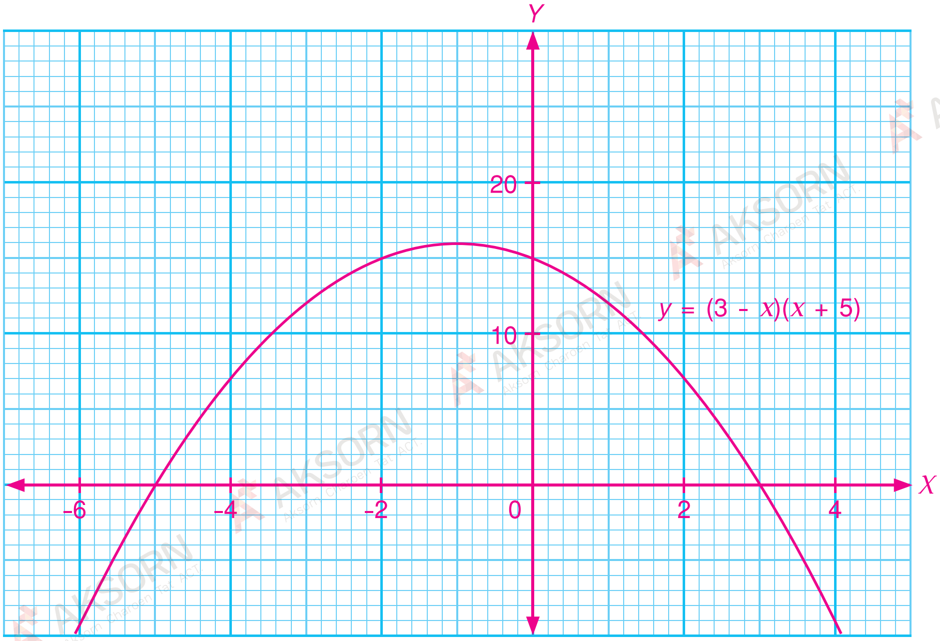
$x = 3$ or $x = -5$.

When $x = 0$,

we get $y = (3)(5)$

$y = 15$.

Therefore, the graph cuts the X-axis at (-5, 0) and (3, 0) and the Y-axis at (0, 15).



Exercise 3B

Basic Level

1. Sketch the graph of each of the following quadratic functions.

1) $y = (x + 1)(x + 3)$

Since the coefficient of x^2 is 1, the graph opens upward.

When $y = 0$,

we get $(x + 1)(x + 3) = 0$

$x + 1 = 0$ or $x + 3 = 0$

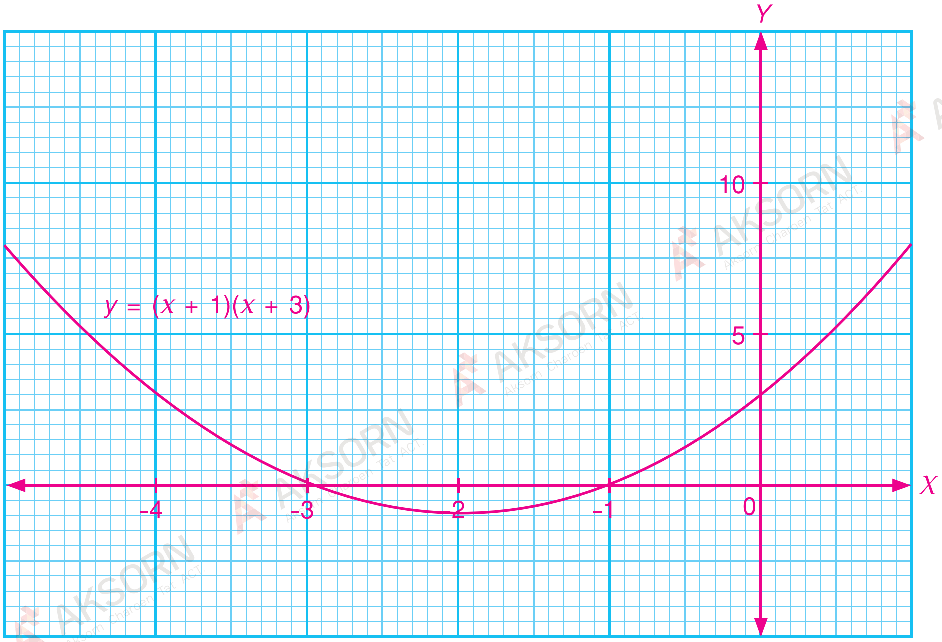
$x = -1$ or $x = -3$.

When $x = 0$,

we get $y = (1)(3)$

$y = 3$.

Therefore, the graph cuts the X-axis at (-1, 0) and (-3, 0) and the Y-axis at (0, 3).



2) $y = -(x - 1)(x + 6)$

Since the coefficient of x^2 is -1, the graph opens downward.

When $y = 0$,

we get $-(x - 1)(x + 6) = 0$

$x - 1 = 0$ or $x + 6 = 0$

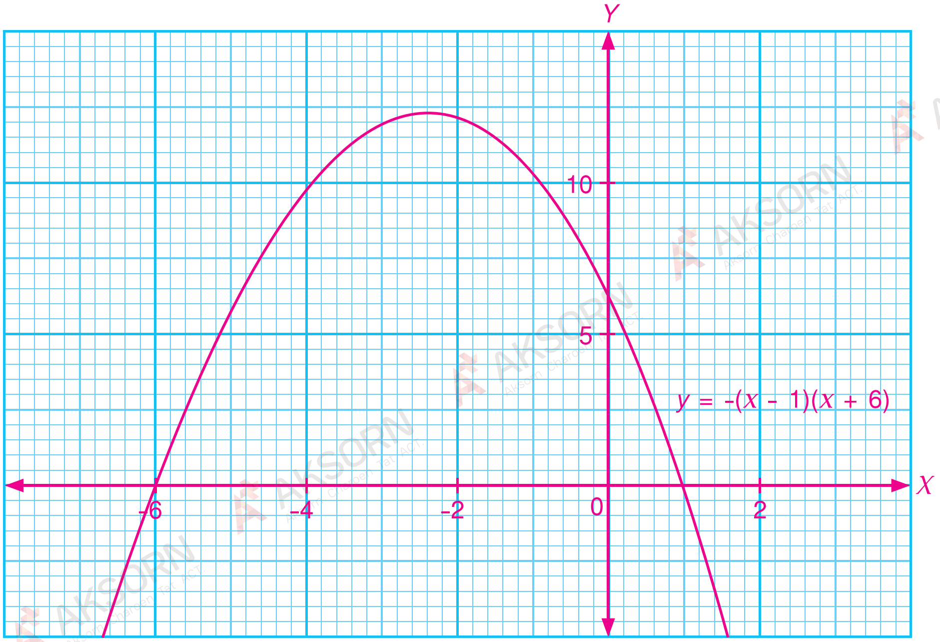
$x = 1$ or $x = -6$.

When $x = 0$,

we get $y = -(-1)(6)$

$y = 6$.

Therefore, the graph cuts the X-axis at (1, 0) and (-6, 0) and the Y-axis at (0, 6).



Intermediate Level

2. Answer each of the following questions.

1) Factorize $x^2 + \frac{3}{4}x$.

$x^2 + \frac{3}{4}x = x\left(x + \frac{3}{4}\right)$

2) Sketch the graph of $y = x^2 + \frac{3}{4}x$.

Since $y = x^2 + \frac{3}{4}x = x\left(x + \frac{3}{4}\right)$

and the coefficient of x^2 is 1,

the graph opens upward.

When $y = 0$,

we get $x\left(x + \frac{3}{4}\right) = 0$

$x = 0$ or $x = -\frac{3}{4}$.

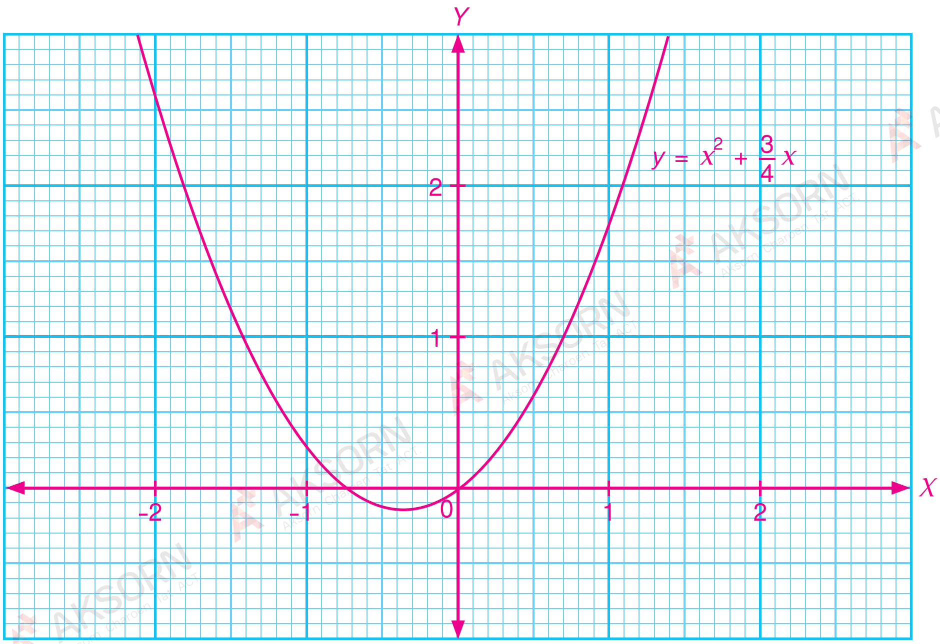
When $x = 0$,

we get $y = 0\left(0 + \frac{3}{4}\right)$

$y = 0$.

Therefore, the graph cuts the X-axis at

(0, 0) and $\left(-\frac{3}{4}, 0\right)$ and the Y-axis at (0, 0).



3. Sketch the graph of $y = -(x^2 - x)$.

Since the coefficient of x^2 is -1, the graph opens downward.

When $y = 0$,

we get $-(x^2 - x) = 0$

$-x(x - 1) = 0$

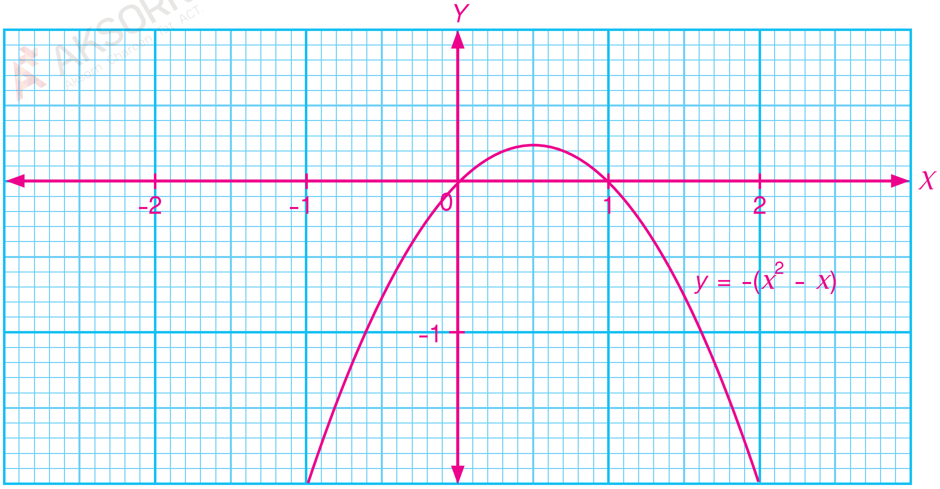
$x = 0$ or $x = 1$.

When $x = 0$,

we get $y = -(0^2 - 0)$

$y = 0$.

Therefore, the graph cuts the X-axis at (0, 0) and (1, 0) and the Y-axis at (0, 0).



4. Sketch the graph of $y = x^2 + x - 6$.

Since the coefficient of x^2 is 1, the graph opens upward.

When $y = 0$,

we get $x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

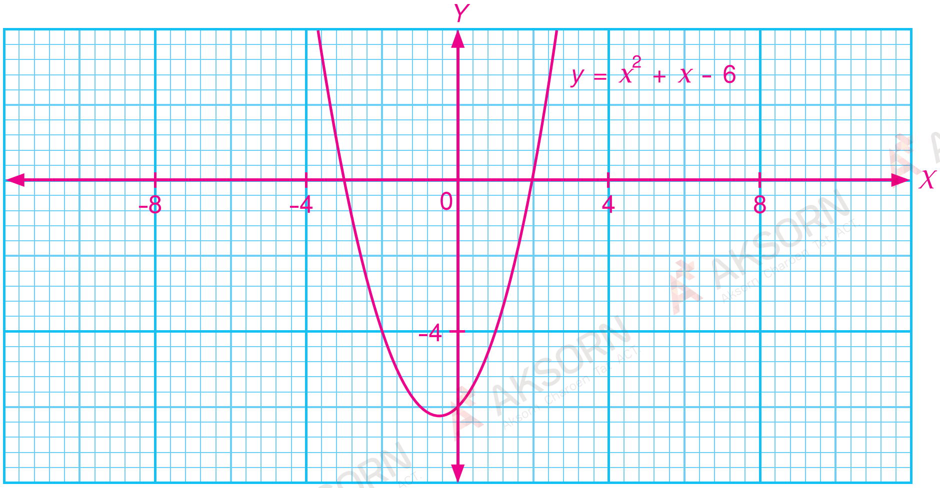
$x = -3$ or $x = 2$.

When $x = 0$,

we get $y = 0^2 + 0 - 6$

$y = -6$.

Therefore, the graph cuts the X-axis at (-3, 0) and (2, 0) and the Y-axis at (0, -6).



Advanced Level

5. Sketch the graph of $y = x^2 - 4x + 3$.

Since the coefficient of x^2 is 1, the graph opens upward.

When $y = 0$,

we get $x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

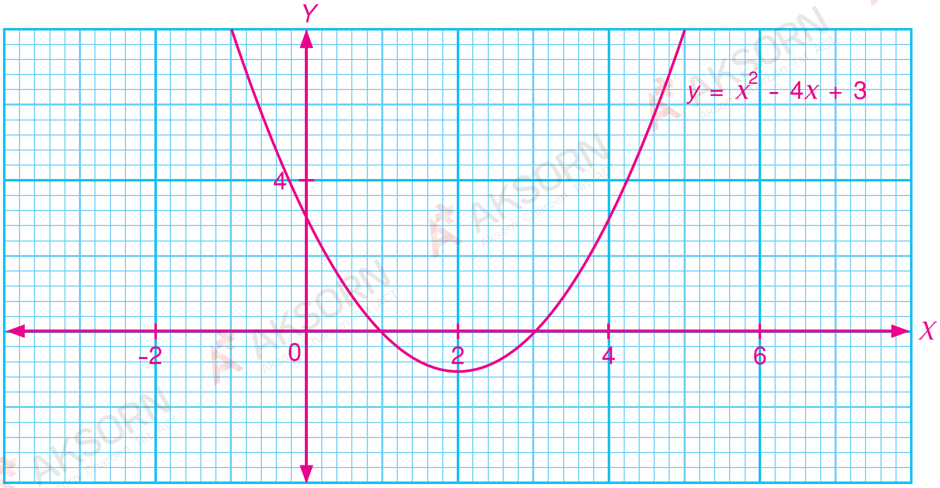
$x = 1$ or $x = 3$.

When $x = 0$,

we get $y = 0^2 - 4(0) + 3$

$y = 3$.

Therefore, the graph cuts the X-axis at (1, 0) and (3, 0) and the Y-axis at (0, 3).



3.3 Graphs of Quadratic Functions in the Form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$

Investigation

Graphs of $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$.

1. Plot the graph of $y = (x - 2)^2 + q$ where $q = -4, -1, 0, 1$ and 4 by using dynamic mathematical programs such as the Geometer's Sketchpad and GeoGebra.

2. Does the graph open upward or downward?

   The graph opens upward.

3. Write down the coordinates of the point(s) where the graph cuts the X-axis, i.e. the X-intercepts.

   For $q = -4$, the graph cuts the X-axis at (0, 0) and (4, 0).

   For $q = -1$, the graph cuts the X-axis at (1, 0) and (3, 0).

   For $q = 0$, the graph cuts the X-axis at (2, 0).

   For $q = 1$, the graph does not cut the X-axis.

   For $q = 4$, the graph does not cut the X-axis.

4. Write down the coordinates of the point(s) where the graph cuts the Y-axis, i.e. the Y-intercepts.

   For $q = -4$, the graph cuts the Y-axis at (0, 0).

   For $q = -1$, the graph cuts the Y-axis at (0, 3).

   For $q = 0$, the graph cuts the Y-axis at (0, 4).

   For $q = 1$, the graph cuts the Y-axis at (0, 5).

   For $q = 4$, the graph cuts the Y-axis at (0, 8).

5. State the equation of line of symmetry of the graph.

For $q = -4$, the vertical line $x = 2$ is the line of symmetry.

For $q = -1$, the vertical line $x = 2$ is the line of symmetry.

For $q = 0$, the vertical line $x = 2$ is the line of symmetry.

For $q = 1$, the vertical line $x = 2$ is the line of symmetry.

For $q = 4$, the vertical line $x = 2$ is the line of symmetry.

6. Write down the coordinates of the maximum or the minimum point of the graph.

For $q = -4$, the minimum point of the graph is (2, -4).

For $q = -1$, the minimum point of the graph is (2, -1).

For $q = 0$, the minimum point of the graph is (2, 0).

For $q = 1$, the minimum point of the graph is (2, 1).

For $q = 4$, the minimum point of the graph is (2, 4).

7. Repeat the steps 1-6 by adjusting $y = (x - 2)^2 + q$ to $y = -(x -2)^2 + q$, $y = (x + 3)^2 + q$ and $y = -(x + 3)^2 + q$, respectively.

8. By looking at the equation of each graph, how do you determine if it opens upward or downward?

The graph of $y = (x - p)^2 + q$ where $q = -4, -1, 0, 1$ and 4 opens upward.

The graph of $y = -(x - p)^2 + q$ where $q = -4, -1, 0, 1$ and 4 opens downward.

9. By looking at the equation of each graph, how do you determine the coordinates of the maximum or the minimum point?

For $y = (x - p)^2 + q$ where $q = -4, -1, 0, 1$ and 4, the minimum point is (p, q).

For $y = -(x - p)^2 + q$ where $q = -4, -1, 0, 1$ and 4, the maximum point is (p, q).

10. What can you say about the line of symmetry of each graph?

The graph of $y = (x - p)^2 + q$ is symmetrical about the vertical line that is parallel to the Y-axis at $x = p$ and passes through the minimum point.

The graph of $y = -(x - p)^2 + q$ is symmetrical about the vertical line that is parallel to the Y-axis at $x = p$ and passes through the maximum point.

From **Investigation**, we can conclude that:

1) For the equation $y = (x - p)^2 + q$, the graph opens upward. The coordinates of the minimum point of the graph are (p, q), and the graph is symmetrical about the line that is parallel to the Y-axis at $x = p$.

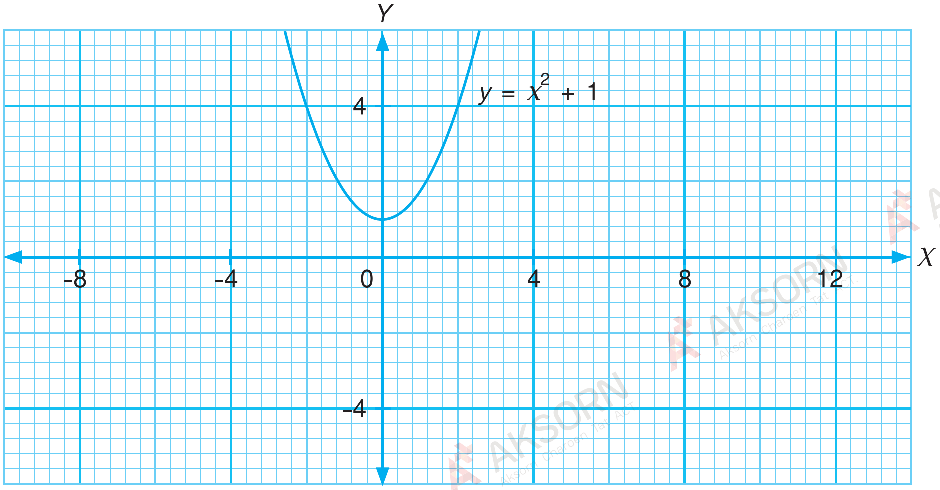
2) For the equation $y = -(x - p)^2 + q$, the graph opens downward. The coordinates of the maximum point of the graph are (p, q), and the graph is symmetrical about the line that is parallel to the Y-axis at $x = p$.

INFORMATION

The graph of a quadratic function is a parabola. When it opens upward, we say it is concave upward.

Worked Example 5

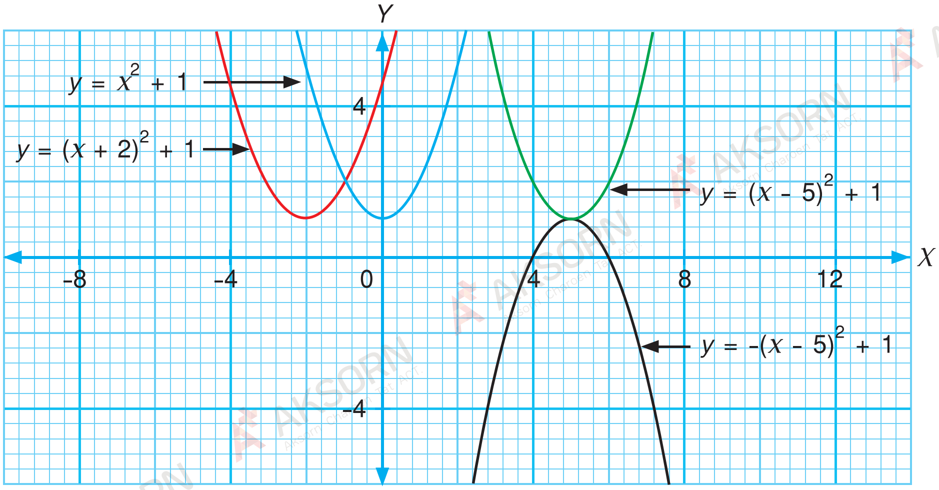
Determine the graph of $y = x^2 + 1$.



On the same grid, sketch the graphs of the following:

1) $y = (x + 2)^2 + 1$ 2) $y = (x - 5)^2 + 1$ 3) $y = -(x - 5)^2 + 1$

Solution:

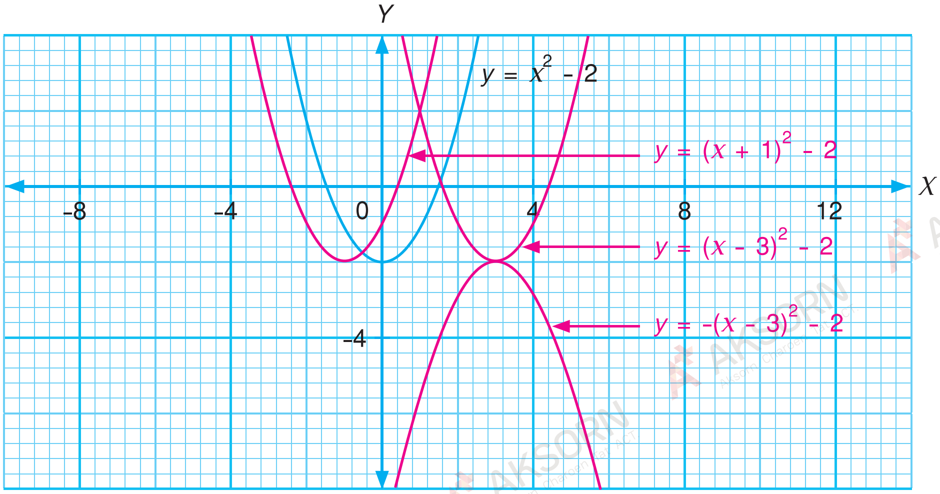


Practice Now

Similar Questions

Exercise 3C Question 1

Determine the graph of $y = x^2 - 2$.

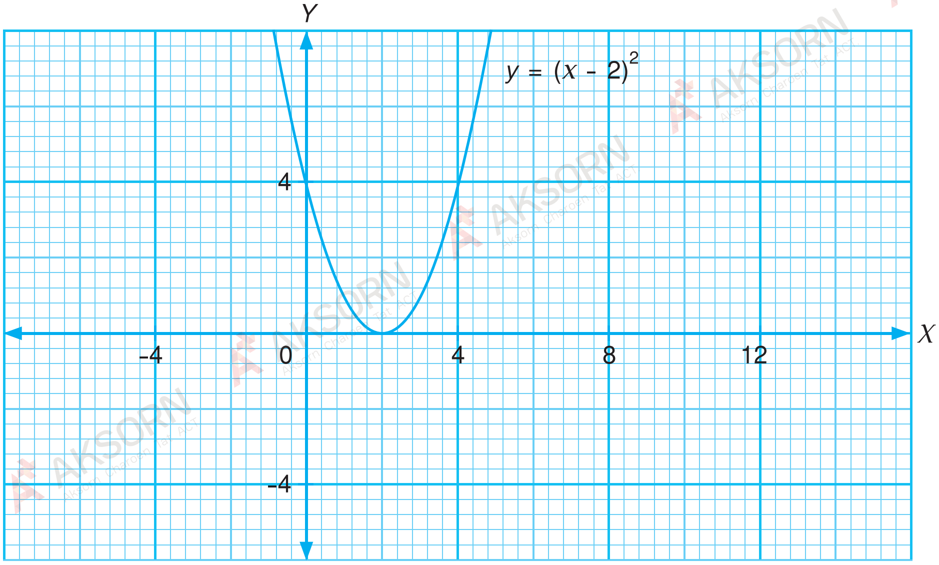


On the grid, sketch the graphs of the following:

1) $y = (x + 1)^2 - 2$
2) $y = (x - 3)^2 - 2$
3) $y = -(x - 3)^2 - 2$

Worked Example 6

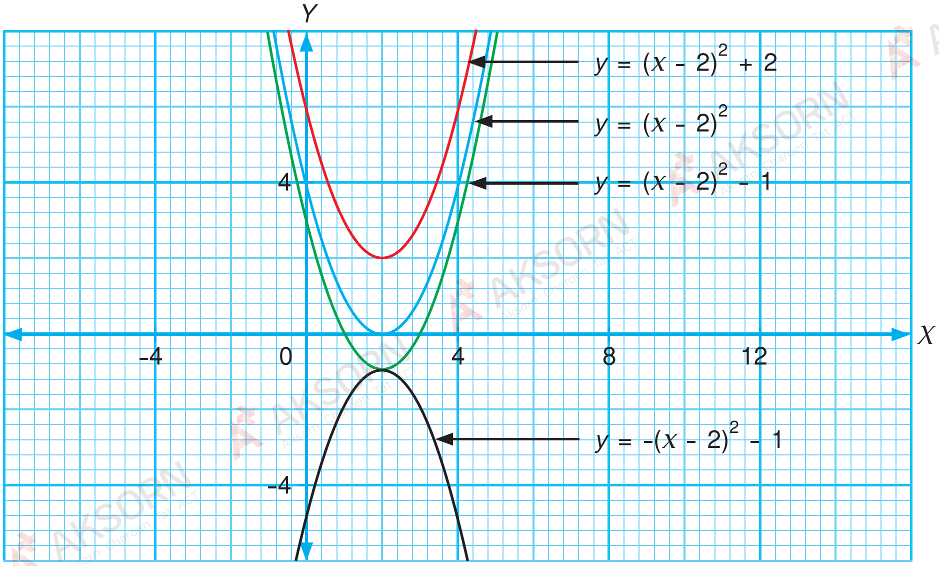
Determine the graph of $y = (x - 2)^2$.



On the same grid, sketch the graphs of the following:

1) $y = (x - 2)^2 + 2$ 2) $y = (x - 2)^2 - 1$ 3) $y = -(x - 2)^2 - 1$

Solution:

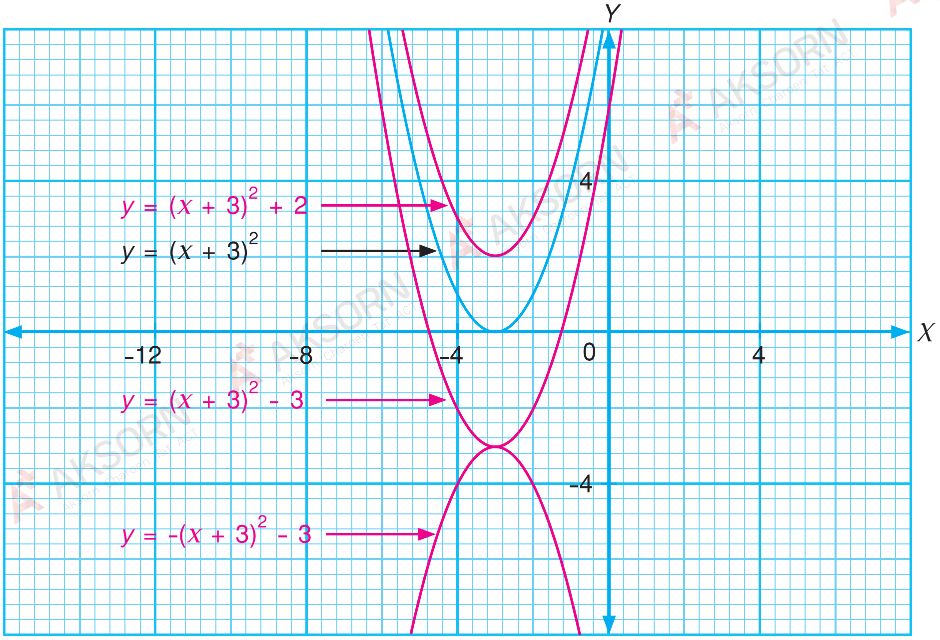


Practice Now

Similar Questions

Exercise 3C Question 2

Determine the graph of $y = (x + 3)^2$.



On the grid, sketch the graphs of the following:

1) $y = (x + 3)^2 + 2$ 2) $y = (x + 3)^2 - 3$ 3) $y = -(x + 3)^2 - 3$

Worked Example 7

Given the quadratic function $y = -(x - 1)^2 + 4$, answer the following questions.

1) Find the coordinates of the X- and Y-intercepts.
2) Write down the coordinates of the maximum point of the graph.
3) Sketch the graph.
4) State the equation of the line of symmetry of the graph.

PROBLEM SOLVING TIP

Step 1: State the coefficient of x^2 to determine if the graph opens upward or downward.

Step 2: Since the equation is of the form $y = -(x - p)^2 + q$, the coordinates of the maximum point are (p, q).

Step 3: Obtain the X-intercepts by substituting $y = 0$ into the equation.

Step 4: Obtain the Y-intercepts by substituting $x = 0$ into the equation.

Step 5: Sketch the graph.

Solution:

1) Since the coefficient of x^2 is -1, the graph opens downward.

When $y = 0$,

we get $-(x - 1)^2 + 4 = 0$

$$-(x - 1)^2 = -4$$

$$(x - 1)^2 = 4$$

$x - 1 = 2$ or $x - 1 = -2$

$x = 3$ or $x = -1.$

When $x = 0$,

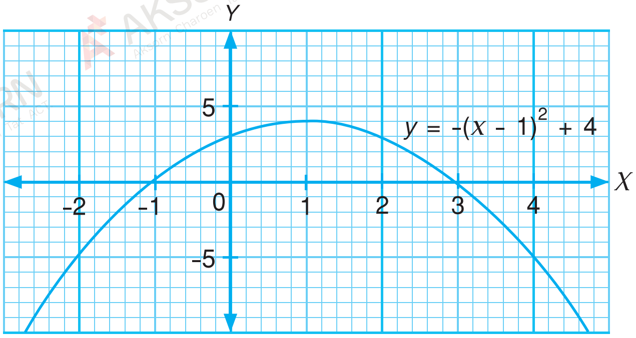
we get $y = -(-1)^2 + 4$

$= 3.$

Therefore, the graph cuts the X-axis at (3, 0) and (-1, 0) and the Y-axis at (0, 3).

2) The coordinates of the maximum point are (1, 4).

3)



4) The equation of the line of symmetry is $x = 1$.

Practice Now

Similar Questions

Exercise 3C Question 3

Given the quadratic function $y = (x + 1)^2 - 1$, answer each of the following questions.

1) Find the coordinates of the X- and Y-intercepts.

Since the coefficient of x^2 is 1, the graph opens upward.

When $y = 0$,

we get $(x + 1)^2 - 1 = 0$

$$(x + 1)^2 = 1$$

$x + 1 = 1$ or $x + 1 = -1$

$x = 0$ or $x = -2.$

When $x = 0$,

we get $y = 1^2 - 1$

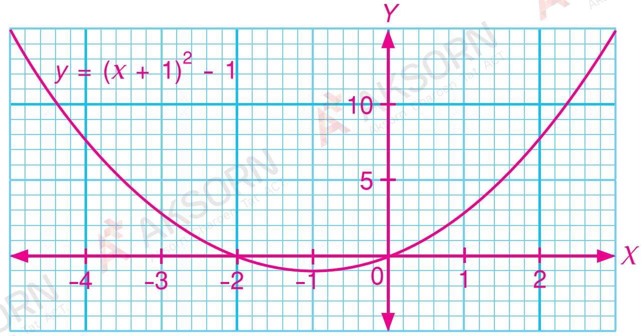
$= 0.$

Therefore, the graph cuts the X-axis at (0, 0) and (-2, 0) and the Y-axis at (0, 0).

2) Write down the coordinates of the minimum point of the graph.

The coordinates of the minimum point are (-1, -1).

3) Sketch the graph.



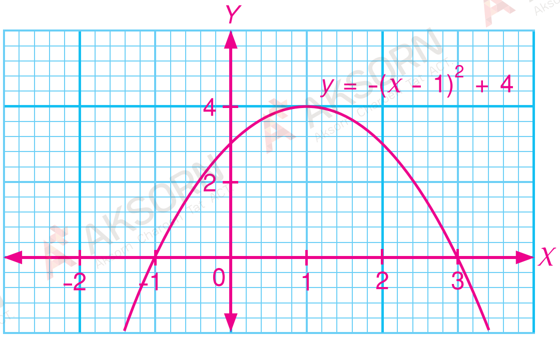
4) State the equation of the line of symmetry of the graph.

The equation of the line of symmetry is $x = -1$.

Thinking Time

1. Express $y = -(x - 1)^2 + 4$ in the factorized form $y = -(x - h)(x - k)$, and hence sketch the graph.

The quadratic function $y = -(x - 1)^2 + 4$ in the factorized form $y = -(x - h)(x - k)$ can be expressed as $y = -(x + 1)(x - 3)$.



2. Express a quadratic function in the form $y = (x - a)^2 + b$ where the graph cuts the X-axis at (1, 0) and (3, 0) and the coordinates of the minimum point are (2, -1).

For the equation $y = (x - a)^2 + b$, the graph opens upward. The coordinates of the minimum point of the graph are (a, b), and the graph is symmetrical about the line $x = a$. Therefore, the quadratic function in the form $y = (x - a)^2 + b$ where the minimum point is (2, -1) is $y = (x - 2)^2 - 1$.

Worked Example 8

Determine $y = x^2 - 4x + 2$ and answer each of the following questions.

1) Express $x^2 - 4x + 2$ in the form $(x - p)^2 + q$.
2) Write down the coordinates of the minimum point of the graph.
3) Sketch the graph of $y = x^2 - 4x + 2$.
4) State the equation of the line of symmetry of the graph.

PROBLEM SOLVING TIP

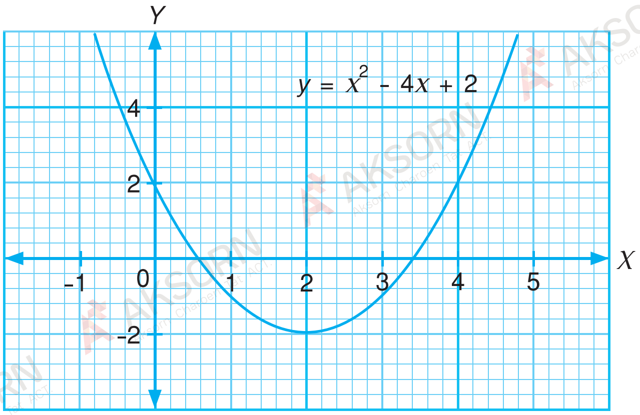
Since the equation is of the form $y = (x - p)^2 + q$, the coordinates of the minimum point are (p, q).

Solution:

1)
$$\begin{aligned} x^2 - 4x + 2 &= \left[x^2 - 4x + \left(-\frac{4}{2}\right)^2\right] - \left(-\frac{4}{2}\right)^2 + 2 \\ &= (x - 2)^2 - 2 \end{aligned}$$

2) The coordinates of the minimum point are (2, -2).

3)



4) The equation of the line of symmetry is $x = 2$.

Practice Now

Similar Questions

Exercise 3C Questions 4-5

Determine $y = x^2 + x + 1$ and answer each of the following questions.

1) Express $x^2 + x + 1$ in the form $(x - p)^2 + q$.

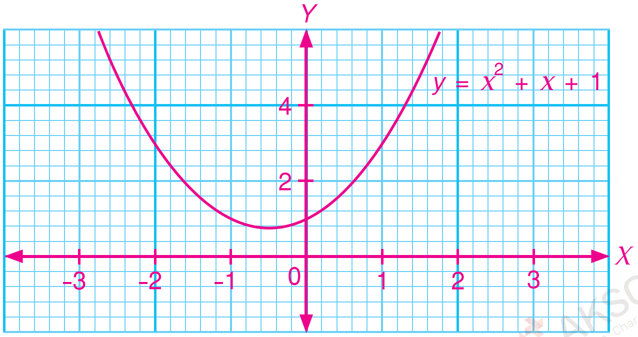
$$x^2 + x + 1 = \left[x^2 + x + \left(\frac{1}{2}\right)^2\right] - \left(\frac{1}{2}\right)^2 + 1$$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

2) Write down the coordinates of the minimum point of the graph.

The coordinates of the minimum point are $\left(-\frac{1}{2}, \frac{3}{4}\right)$.

3) Sketch the graph of $y = x^2 + x + 1$.



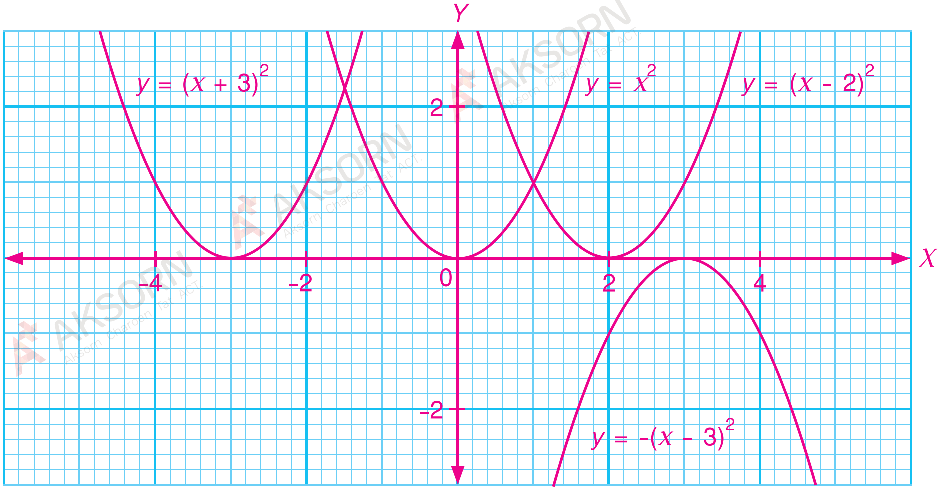
4) State the equation of the line of symmetry of the graph.

The equation of the line of symmetry is $x = -\frac{1}{2}$.

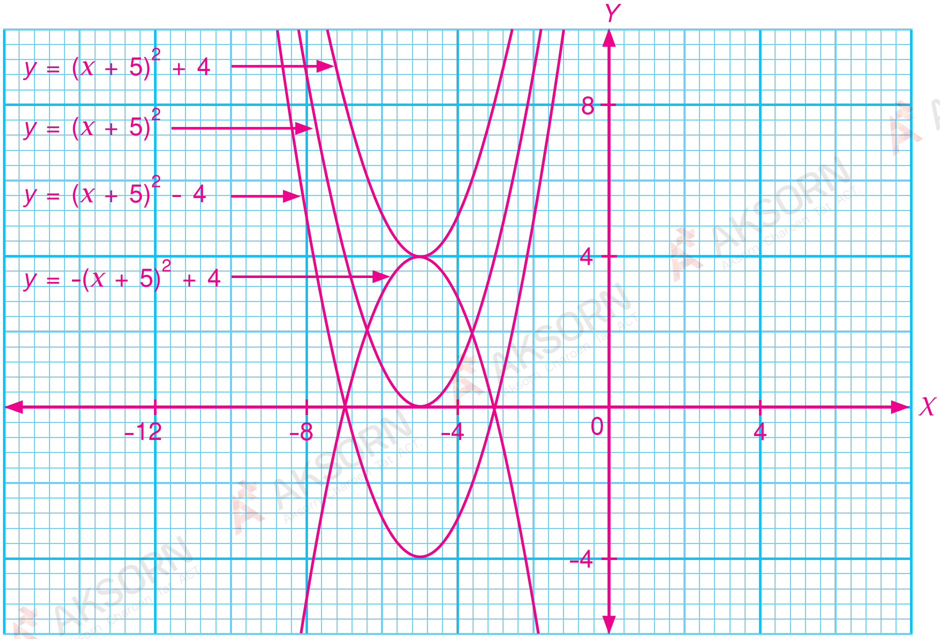
Exercise 3C

Basic Level

1. On the same grid, sketch the graphs of $y = x^2$, $y = (x + 3)^2$, $y = (x - 2)^2$ and $y = -(x - 3)^2$.



2. On the same grid, sketch the graphs of $y = (x + 5)^2$, $y = (x + 5)^2 + 4$, $y = (x + 5)^2 - 4$ and $y = -(x + 5)^2 + 4$.



3. Sketch the graphs of the following functions, stating the coordinates of the maximum or the minimum point and the equation of the line of symmetry.

1) $y = (x - 3)^2 + 1$

Since the coefficient of x^2 is 1, the graph

opens upward.

When $y = 0$,

we get $(x - 3)^2 + 1 = 0$

$(x - 3)^2 = -1$.

When $x = 0$,

we get $y = (0 - 3)^2 + 1 = 10$.

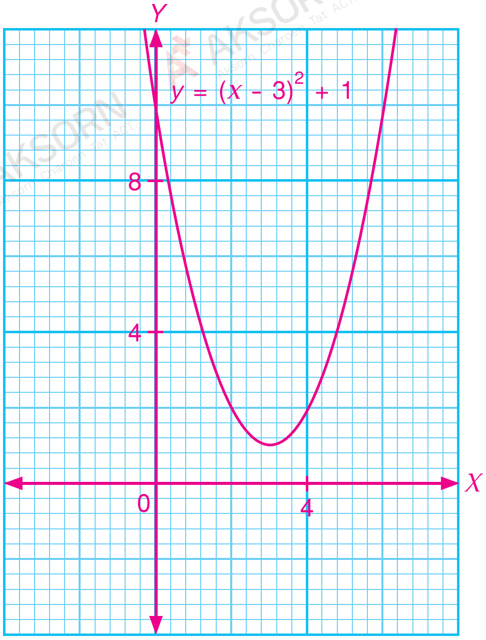
Therefore, the graph does not cut the

X-axis, but it cuts the Y-axis at (0, 10).

The coordinates of the minimum point are

(3, 1). The equation of the line of symmetry

is $x = 3$.



2) $y = -(x - 4)^2 - 1$

Since the coefficient of x^2 is -1, the graph

opens downward.

When $y = 0$,

we get $-(x - 4)^2 - 1 = 0$

$(x - 4)^2 = -1$.

When $x = 0$,

we get $y = -(0 - 4)^2 - 1 = -17$.

Therefore, the graph does not cut the X-axis,

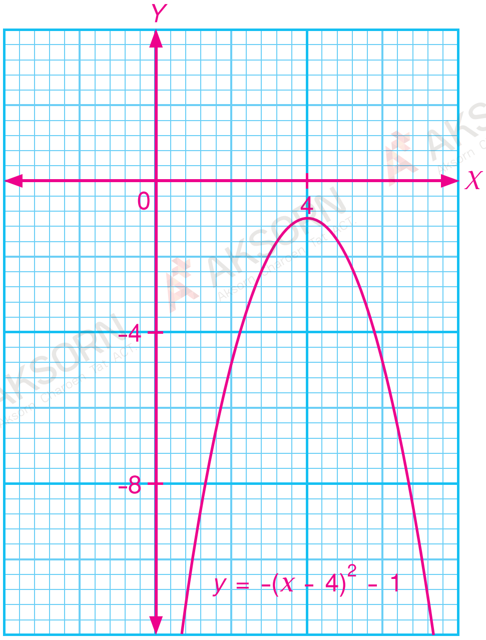
but it cuts the Y-axis at (0, -17).

The coordinates of the maximum point are

(4, -1).

The equation of the line of symmetry is

$x = 4$.



Intermediate Level

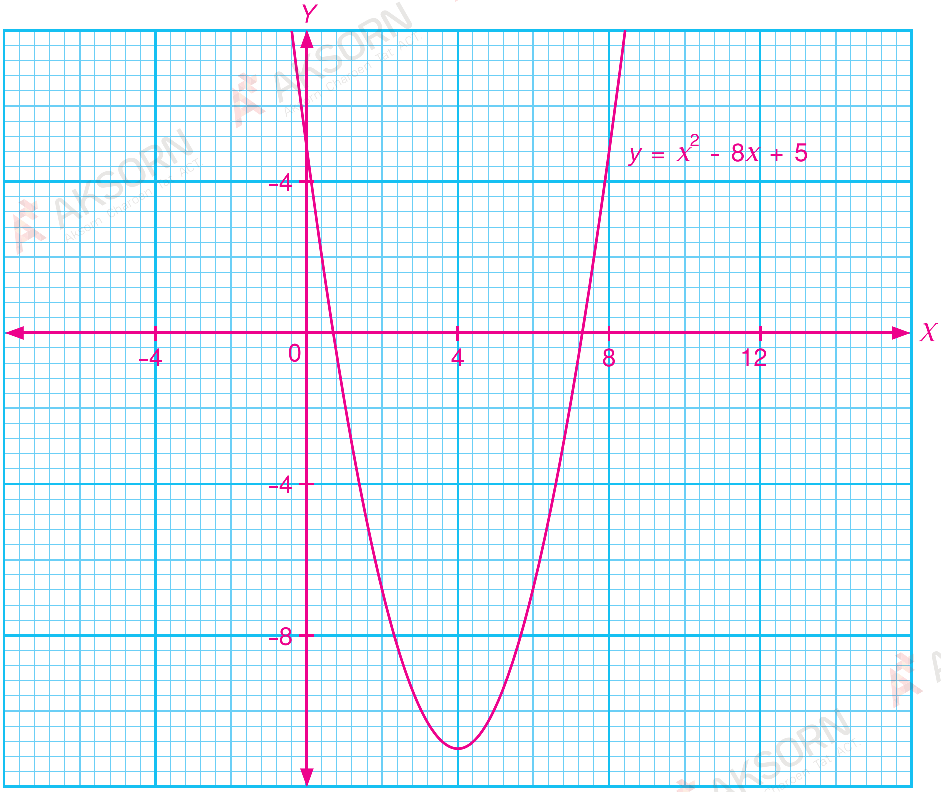
4. **Determine $y = x^2 - 8x + 5$ and answer each of the following questions.**

1) Express $x^2 - 8x + 5$ in the form $(x - p)^2 + q$.

$$x^2 - 8x + 5 = \left[x^2 - 8x + \left(-\frac{8}{2}\right)^2\right] - \left(-\frac{8}{2}\right)^2 + 5$$

$$= (x - 4)^2 - 11$$

2) Sketch the graph of $y = x^2 - 8x + 5$.



3) Write down the coordinates of the minimum point of the graph.

The coordinates of the minimum point are (4, -11).

4) State the equation of the line of symmetry of the graph.

The equation of the line of symmetry is $x = 4$.

Advanced Level

5. Given that $-x^2 + 10x - 4$ can be expressed in the form $-(x - p)^2 + q$.

1) State the value of p and of q.

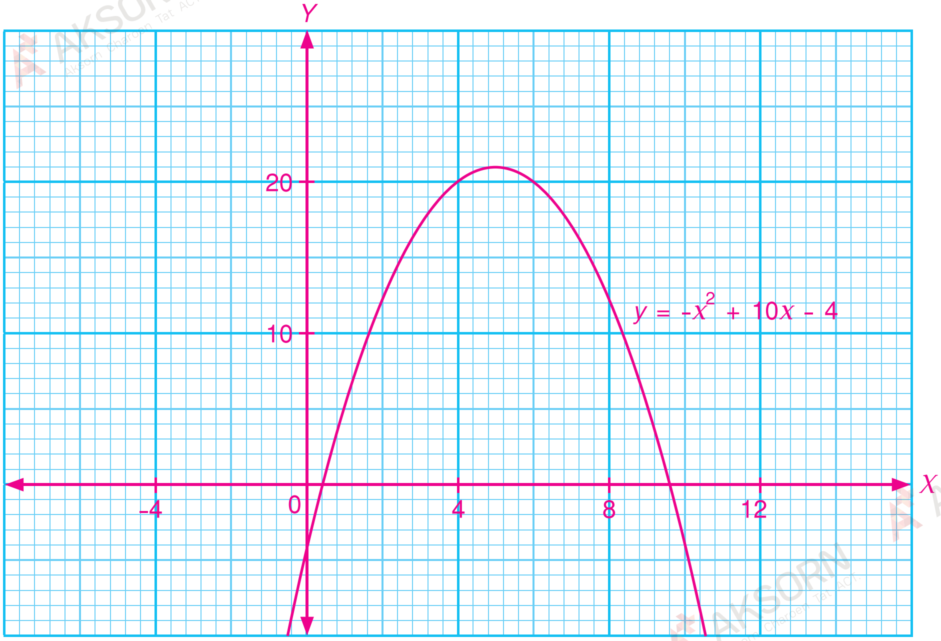
$$-x^2 + 10x - 4 = -(x^2 - 10x + 4)$$

$$= -\left[\left(x^2 - 10x + \left(-\frac{10}{2}\right)^2\right) - \left(-\frac{10}{2}\right)^2 + 4\right]$$

$$= -(x - 5)^2 + 21$$

Therefore, $p = 5$ and $q = 21$.

2) Sketch the graph of $y = -x^2 + 10x - 4$.

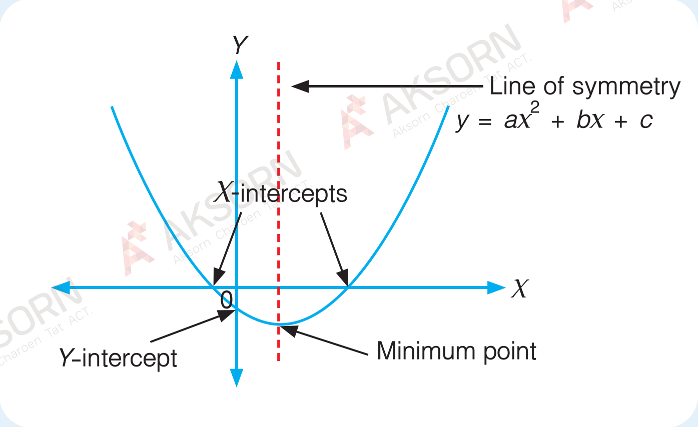


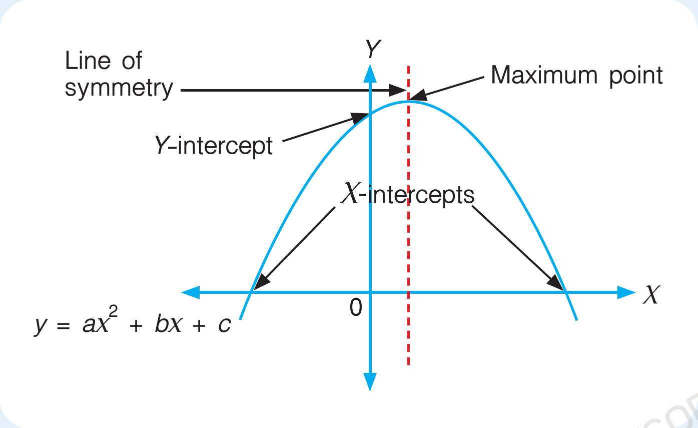
3) Write down the coordinates of the maximum point of the graph.

The coordinates of the maximum point are (5, 21).

Summary

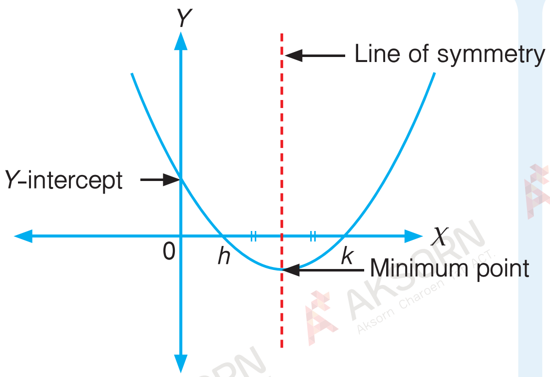
1. The general form of the equation of a quadratic function is $y = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$.



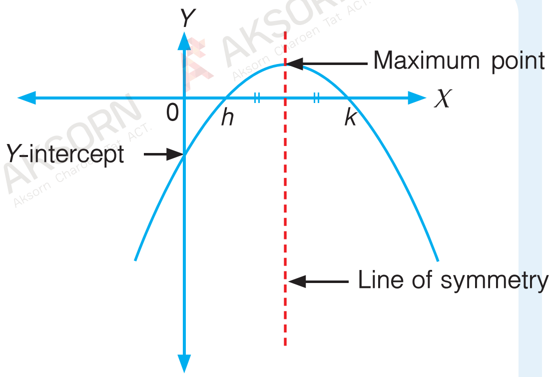


- For $a > 0$, the graph opens upward indefinitely and has a minimum point.
  For $a < 0$, the graph opens downward indefinitely and has a maximum point.
- The smaller $|a|$, the wider the graph opens.
- The line of symmetry of the graph passes through its minimum or maximum point.
- The graph may have 0, 1 or 2 X-intercepts, but it has only 1 Y-intercept.

2. Graphs of quadratic functions of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$



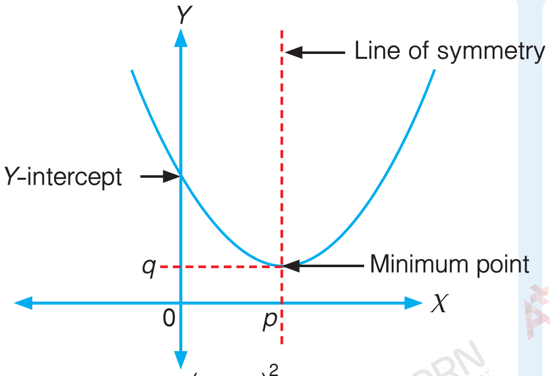
$y = (x - h)(x - k)$



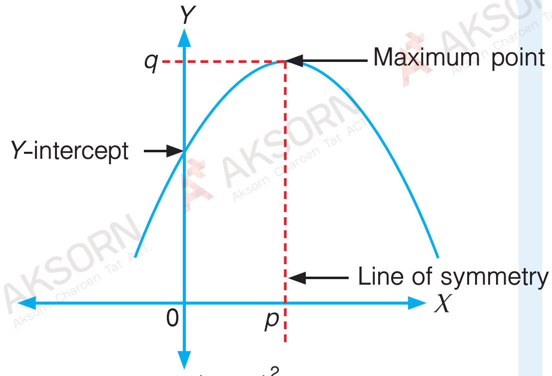
$y = -(x - h)(x - k)$

The line of symmetry passes through the midpoint of the X-intercepts.

3. Graph of quadratic function of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$



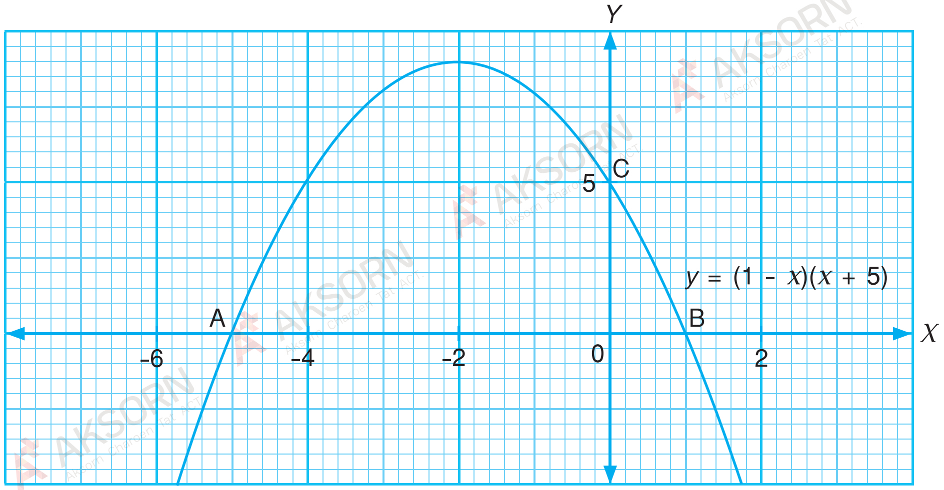
$y = (x - p)^2 + q$



$y = -(x - p)^2 + q$

Review Exercise 3

1. Determine the graph of $y = (1 - x)(x + 5)$.



Find the coordinates of A, B and C.

Let $y = 0$.

We get $(1 - x)(x + 5) = 0$

$1 - x = 0$ or $x + 5 = 0$

$x = 1$ or $x = -5$.

Let $x = 0$.

We get $y = (1 - 0)(0 + 5)$

$= 5$.

According to this, the graph of the function cuts the X-axis at (-5, 0) and (1, 0) and the Y-axis at (0, 5).

Therefore, the coordinates of A, B and C are (-5, 0), (1, 0) and (0, 5), respectively.

2. The variables x and y are connected by the equation $y = x^2 + 3x - 4$. Some values of x and the corresponding values of y are given in the table.

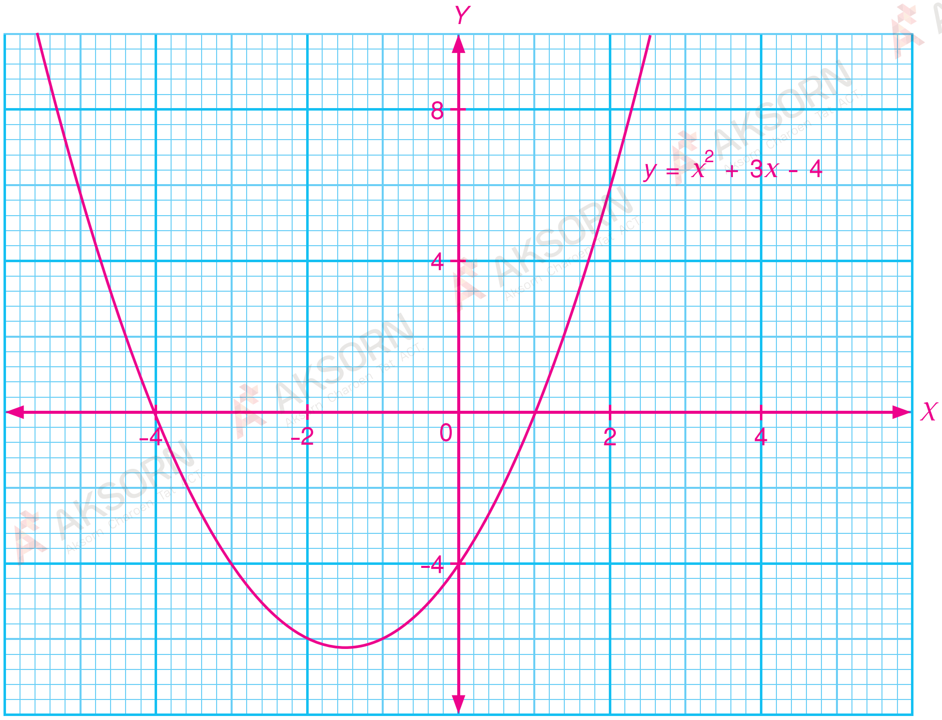
x	-5	-4	-3	-2	-1	0	1	2
y	p	0	-4	-6	-6	-4	0	6

1) Find the value of p.

When $x = -5$, $y = p$.

We get $p = (-5)^2 + 3(-5) - 4 = 6$.

2) Sketch the graph of $y = x^2 + 3x - 4$.



3) Find the value of y when $x = -2.7$.

When $x = -2.7$, we get $y = 4.8$.

4) Find the value of x when $y = 5$.

When $y = 5$, we get $x \approx -4.85, 1.85$.

5) Find the minimum point of the function.

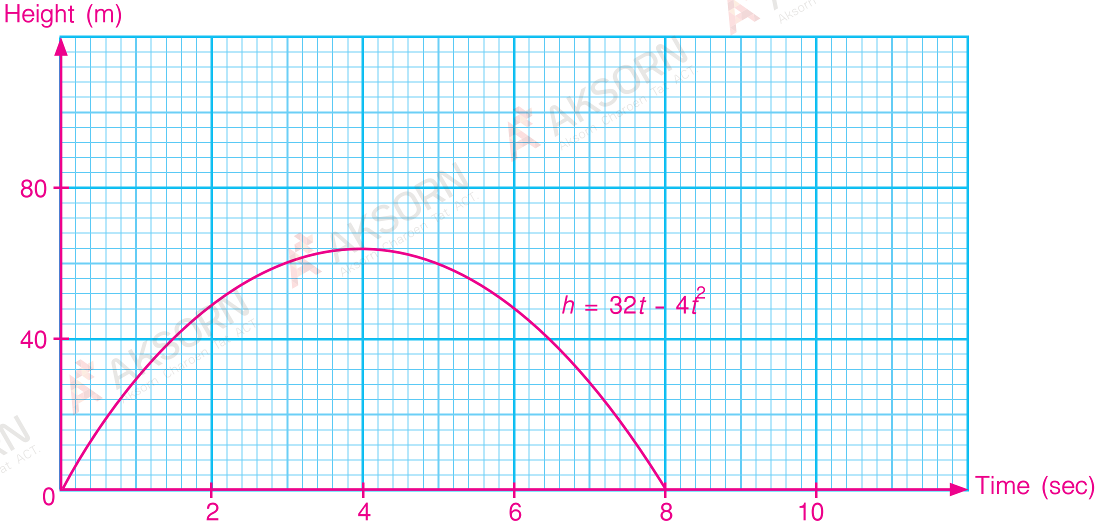
The minimum point of the function is $y = -6.3$.

6) Find the line of symmetry of the graph.

The line of symmetry is $x = -1.5$.

3. **Mike strikes a golf ball into the air from the ground. The height, h meters, of the ball can be modelled by $h = 32t - 4t^2$ where t is the time in seconds after it leaves the ground.**

1) Sketch the graph of $h = 32t - 4t^2$.

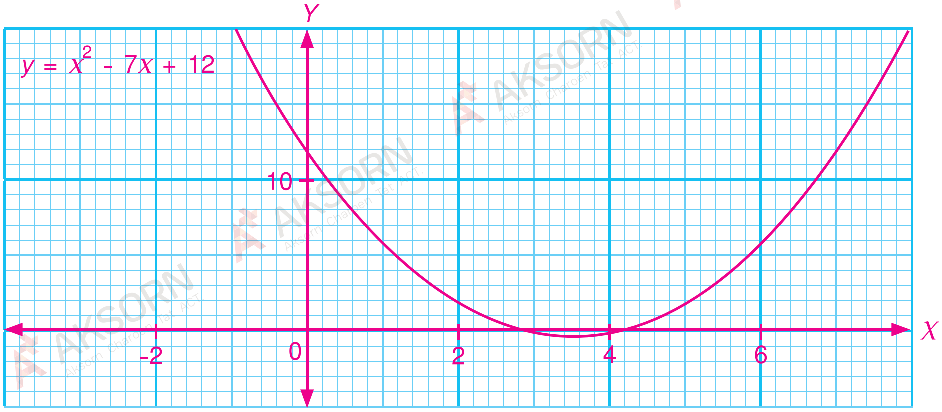


2) Find the maximum height of the ball above the ground and the time at which this occurs.

The maximum height of the ball above the ground is 64 m, and the time at the maximum height is 4 sec.

4. **Express $y = x^2 - 7x + 12$ in the form of $y = (x - h)(x - k)$ and sketch the graph of the function.**

$y = x^2 - 7x + 12$ in the form of $y = (x - h)(x - k)$ is $y = (x - 3)(x - 4)$.



5. Express $y = -x^2 + 5x - 4$ in the form of $y = -(x - p)^2 + q$ and sketch the graph of the function.

$$y = -x^2 + 5x - 4$$

$$= -(x^2 - 5x + 4)$$

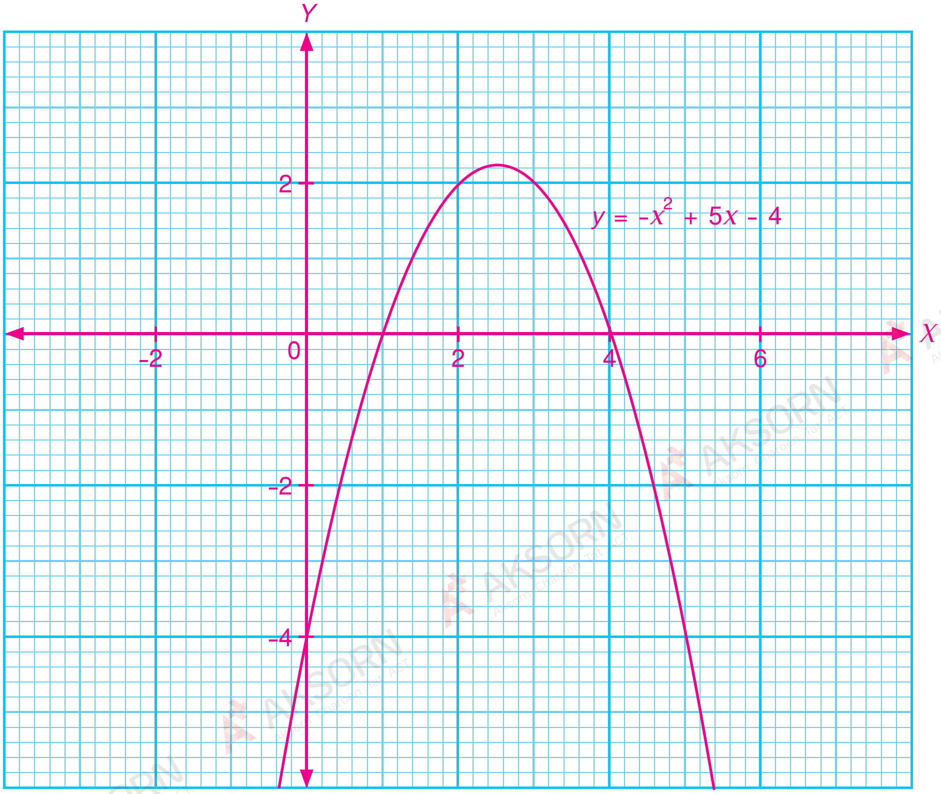
$$= -\left[\left(x^2 - 5x + \left(-\frac{5}{2}\right)^2\right) - \left(-\frac{5}{2}\right)^2 + 4\right]$$

$$= -\left[\left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 4\right]$$

$$= -\left[\left(x - \frac{5}{2}\right)^2 - \frac{9}{4}\right]$$

$$= -\left(x - \frac{5}{2}\right)^2 + \frac{9}{4}$$

Therefore, $y = x^2 + 5x - 4$ in the form of $y = -(x - p)^2 + q$ is $-\left(x - \frac{5}{2}\right)^2 + \frac{9}{4}$.



6. A stone was thrown from the top of a vertical tower. Its position during the flight is represented by the equation $h = 60 + 25s - s^2$ where h m is the height of the stone and s m is the horizontal distance from the foot of the tower.

1) Solve the equation $h = 60 + 25s - s^2$.

$60 + 25s - s^2 = 0$

$-s^2 + 25s + 60 = 0$

Comparing $-s^2 + 25s + 60 = 0$ with $ax^2 + bx + c = 0$,

we get $a = -1$, $b = 25$ and $c = 60$.

From $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

we get $s = \dfrac{-25 \pm \sqrt{25^2 - 4(-1)(60)}}{2(-1)}$

$= \dfrac{-25 \pm \sqrt{865}}{-2}$

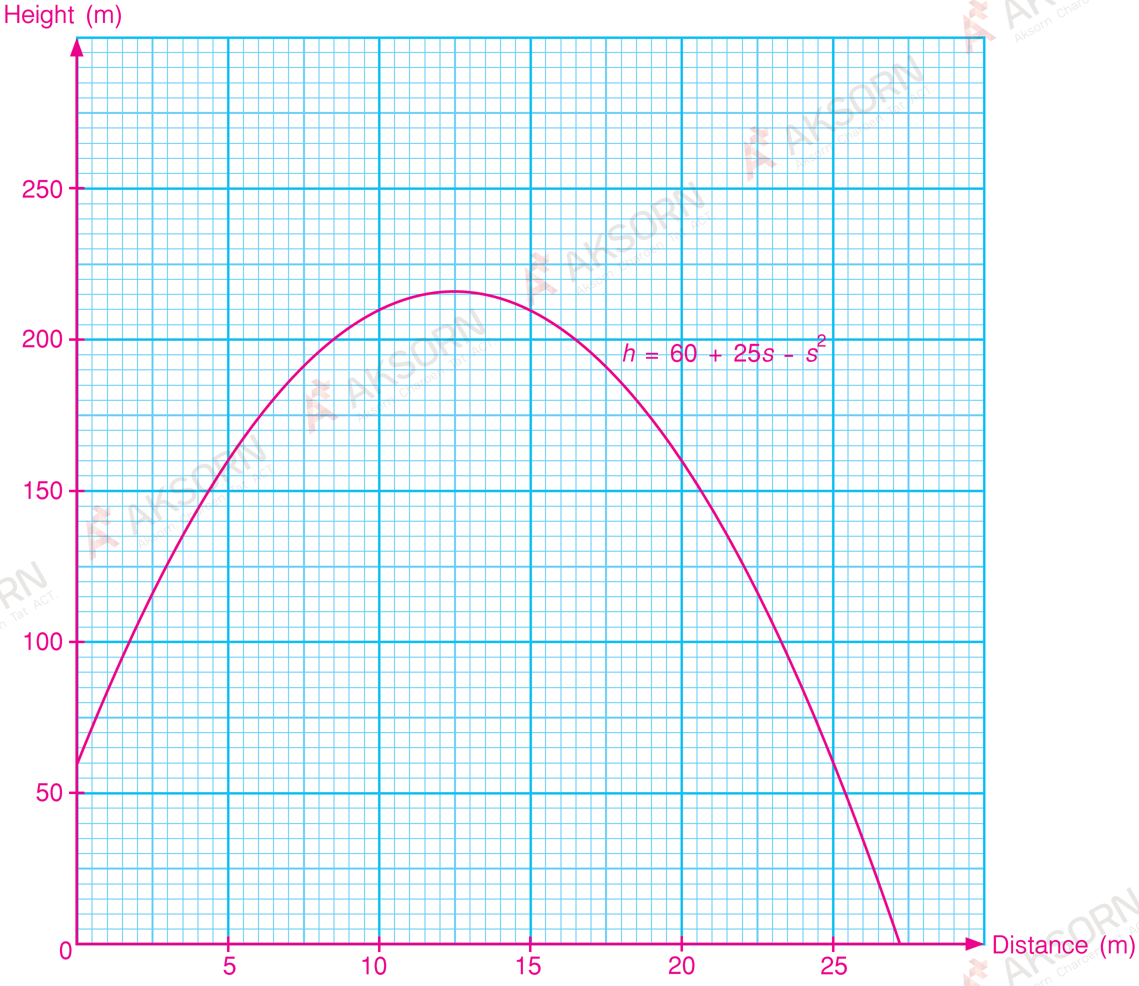
$\approx -2.2, 27.2.$

2) Explain briefly what the positive solution in Question 1 represents.

The positive solution in Question 1 represents the horizontal distance of the stone from the foot of the tower, in which the distance must be a positive real number. Therefore, the solution in Question 1 must be a positive value.

KEY

3) Sketch the graph of $h = 60 + 25s - s^2$.



4) Find the greatest height reached by the stone.

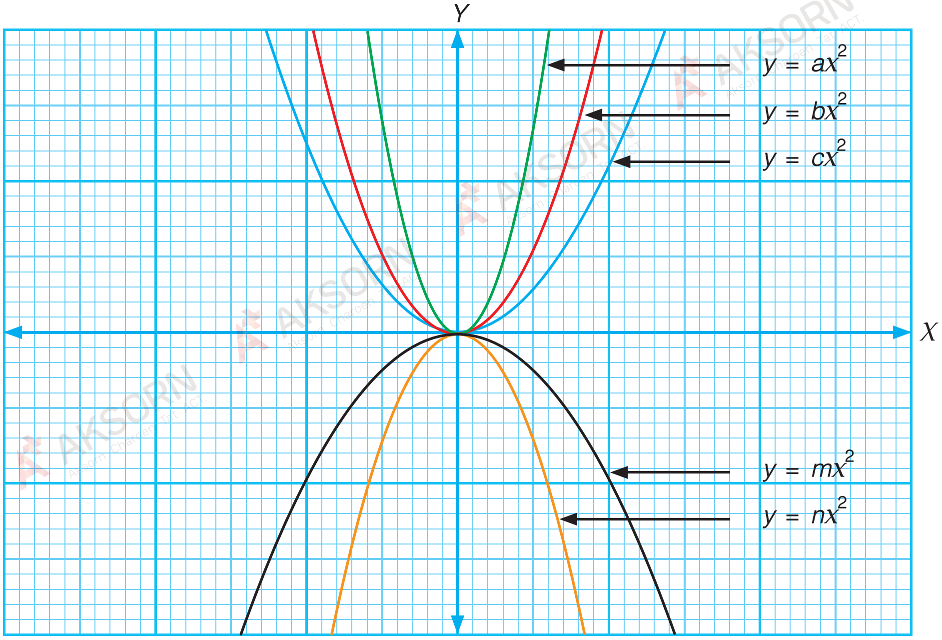
From the graph, the greatest height reached by the stone is 216 m.

5) Find the horizontal distance from the foot of the tower when the stone is 180 m above the ground.

When the stone is 180 m above the ground, h = 180. The horizontal distance from the foot of the tower is 6.5 m or 18.5 m.

Challenge Yourself

Determine the graph of $y = ax^2$, $y = bx^2$, $y = cx^2$, $y = mx^2$ and $y = nx^2$.



Identify whether each of the following cases is true.

1) $a > b > c$

True

2) $a < b < c$

False

3) $c > b$, $a > b$

False

4) $c < b$, $a < b$

False

5) $m > 0$, $n > 0$, $m > n$

False

6) $m > 0$, $n > 0$, $m < n$

False

7) $m < 0$, $n < 0$, $m > n$

False

8) $m < 0$, $n < 0$, $m < n$

True

KEY

Chapter 4

Pyramids, Cones and Spheres

The photo shows some ice cream cones. How does the manufacturers determine the volume of ice cream needed to fill each cone completely?

Indicators

- Understand and apply the surface area of pyramids, cones and spheres to solve real-life and mathematical problems. **(MA 2.1 G. 9/1)**
- Understand and apply the volume of pyramids, cones and spheres to solve real-life and mathematical problems. **(MA 2.1 G. 9/2)**

Compulsory Details

- Calculation of the surface area of pyramids, cones and spheres
- Applications of the surface area of pyramids, cones and spheres to solve problems
- Calculation of the volume of pyramids, cones and spheres
- Applications of the volume of pyramids, cones and spheres to solve problems

4.1 Pyramids

In primary level, we have learned that a pyramid is a solid in which one of the faces is a polygonal base and the other slanted faces are triangles joined to the edges (or sides) of the base. The vertex where all the slanted faces meet is called the apex, which is opposite to the base, and the apex is not in the same plane as the base. The other parts of a pyramid are as follows:

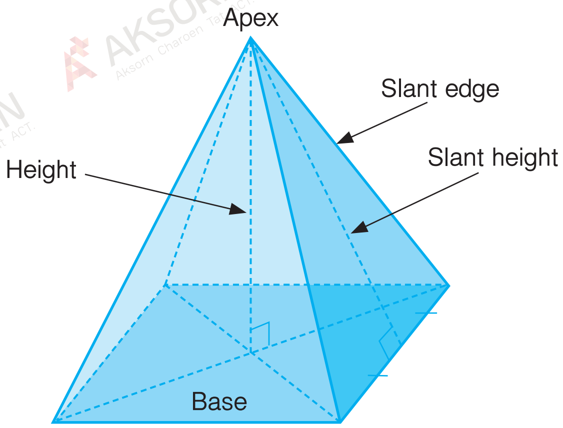
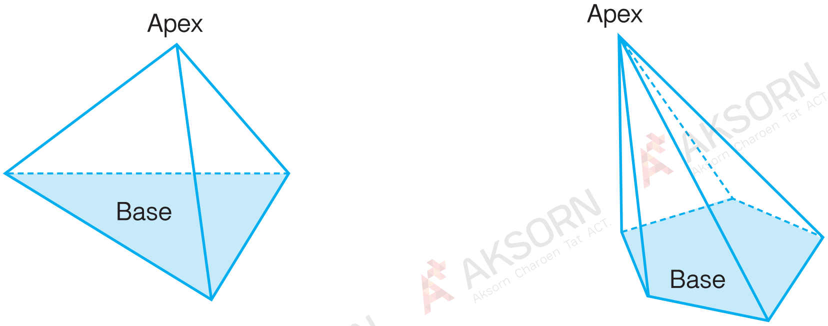


Fig. 4.1

A pyramid is also named after its polygonal base.



(a) Triangular Pyramid

(b) Pentagonal Pyramid

Fig. 4.2

1. Volume of Pyramids

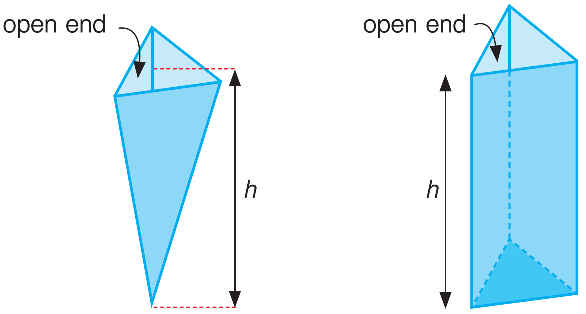
We have learned that:

Volume of prism = area of cross-section × height
= base area × height

We will learn now how to find the volume of a pyramid.

Investigation

Consider an open triangular pyramid and an open triangular prism with the same base and the same height (see Fig. 4.3). If we fill the entire pyramid with sand before pouring it into the prism, how many times will it take to fill the prism completely?



(a) Triangular pyramid (b) Triangular prism

Fig. 4.3

Fig. 4.4 shows the net of a triangular pyramid which can be photocopied and pasted on a piece of cardboard before cutting it out and folding along the dotted lines to obtain an open pyramid as shown in Fig. 4.3(a).

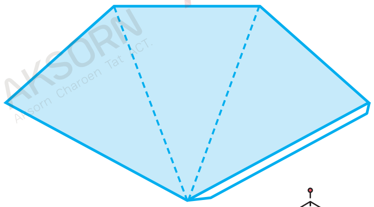


Fig. 4.4

Nets of pyramids

www.aksorn.com/interactive3D/RS942

Similarly, use the net of a triangular prism in Fig. 4.5 to make an open prism as shown in Fig. 4.3(b).

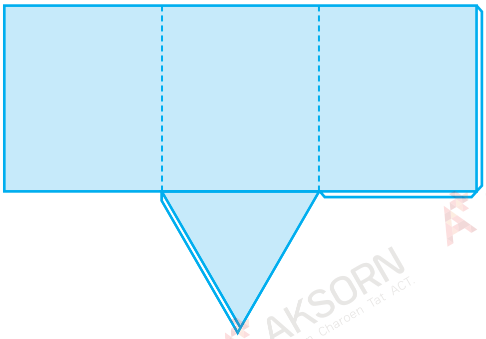


Fig. 4.5

ATTENTION

- The cardboard is necessary. If we use only paper, the pyramid will be distorted due to the mass of the sand and the result will be inaccurate.
- The tabs are not part of the net as they are for gluing purposes.

Notice that both the pyramid in Fig. 4.4 and the prism in Fig. 4.5 have the same base and the same height.

Fill the entire pyramid with sand such that the sand is level. Pour the sand into the prism. How many times must you do this until the prism is completely filled? Do you get the same result as your classmates? This suggests that:

Volume of pyramid = $\frac{1}{3}$ × volume of corresponding prism

You may want to repeat the experiment in **Investigation** for different pyramids and their corresponding prisms (i.e. prisms with the same base and height). Fig. 4.6 shows a series of photos where sand is poured from a square pyramid into a square prism with the same base and the same height. The process is repeated until the prism is completely full. It shows that it takes 3 times the volume of a pyramid to fill the prism completely.

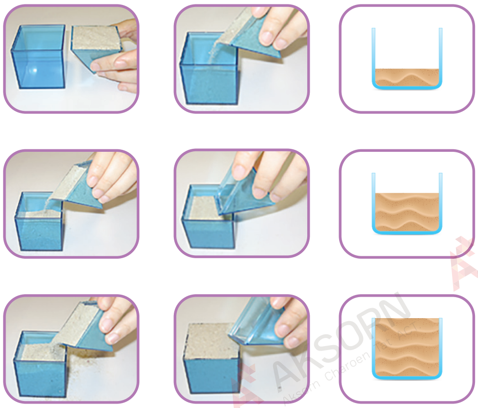


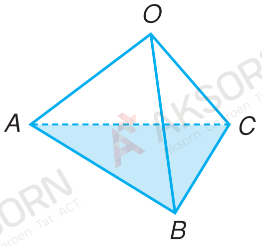
Fig. 4.6

From **Investigation**, we can conclude that:

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{volume of corresponding prism}$$
$$= \frac{1}{3} \times \text{base area} \times \text{height}$$

Worked Example 1

OABC is a triangular pyramid with a base area of 25 cm^2 and a height of 8 cm. Find the volume of the triangular pyramid.



Solution:

$$\text{Volume of triangular pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times 25 \times 8$$
$$= 66\frac{2}{3} \text{ cm}^3$$

Practice Now

Similar Questions

Exercise 4A Questions 1-3, 8

A triangular pyramid has a base area of 36 cm^2 and a height of 7 cm. Find the volume of the triangular pyramid.

$$\text{Volume of triangular pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times 36 \times 7$$
$$= 84 \text{ cm}^3$$

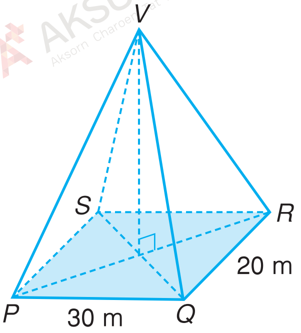
Worked Example 2

VPQRS is a rectangular pyramid where *PQ* = 30 m and *QR* = 20 m. Given that the volume of the pyramid is 7,000 m^3, find its height.

Solution:

$$\begin{aligned}\text{Volume of pyramid} &= \frac{1}{3} \times \text{base area} \times \text{height}\\ 7{,}000 &= \frac{1}{3} \times (30 \times 20) \times \text{height}\\ 7{,}000 &= 200 \times \text{height}\\ \text{Height} &= 35\end{aligned}$$

Therefore, the height of the pyramid is 35 m.



Practice Now

Similar Questions

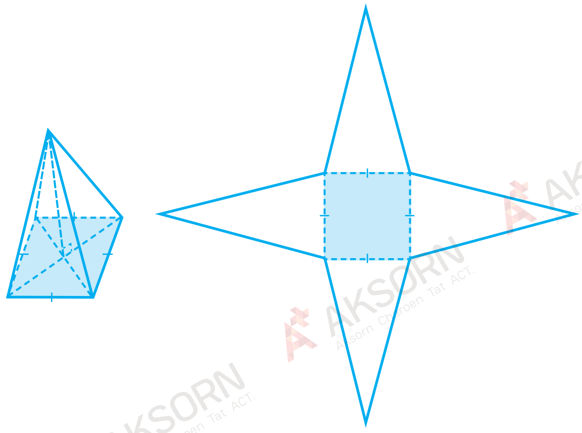
Exercise 4A Questions 4-6, 9-10

The volume of a pyramid with a square base of length 5 m is 75 m^3. Find its height.

$$\begin{aligned}\text{Volume of pyramid} &= \frac{1}{3} \times \text{base area} \times \text{height}\\ 75 &= \frac{1}{3} \times (5 \times 5) \times \text{height}\\ 75 &= \frac{25}{3} \times \text{height}\\ \text{Height} &= 9\end{aligned}$$

Therefore, the height of the pyramid is 9 m.

2. Surface Area of Pyramids



(a) (b)

Fig. 4.7

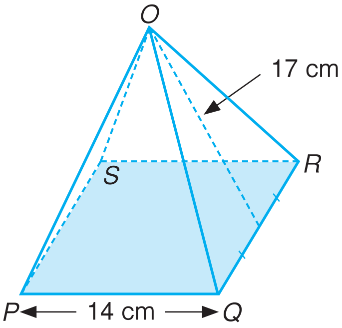
Fig. 4.7(a) shows a pyramid with a square base and Fig. 4.7(b) shows its net. If we fold the net along the dotted lines, we will obtain the pyramid. How do we find the total surface area of the pyramid?

From the net, it can be seen that:

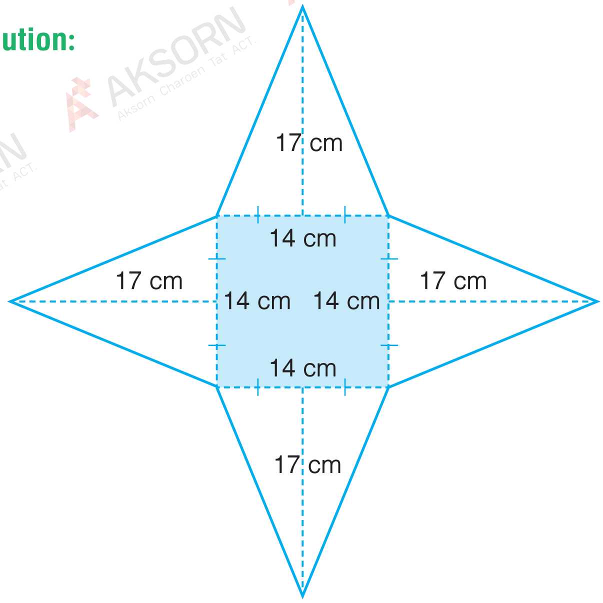
Total surface area of pyramid = total area of all faces

Worked Example 3

OPQRS is a pyramid with a square base of length 14 cm. Given that the slant height of the pyramid is 17 cm, draw its net and hence, find its total surface area.



Solution:



ATTENTION

When we are required to draw the net of a solid, we do not draw the tabs as they are for gluing purposes (see Fig. 4.4 and Fig. 4.5). We just draw the net without the tabs (see Fig. 4.7(b)).

$$\begin{aligned}
\text{Area of each triangular face} &= \frac{1}{2} \times 14 \times 17 \\
&= 119 \text{ cm}^2 \\
\text{Area of square base} &= 14 \times 14 \\
&= 196 \text{ cm}^2 \\
\text{Total surface area of pyramid} &= (4 \times \text{area of each triangular face}) + \text{area of square base} \\
&= (4 \times 119) + 196 \\
&= 672 \text{ cm}^2
\end{aligned}$$

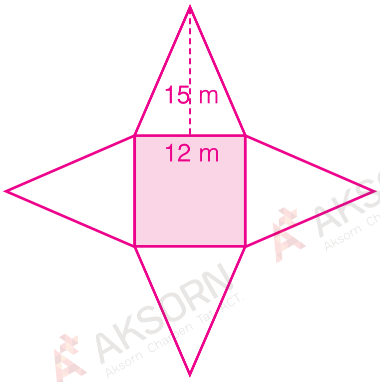
KEY

Similar Questions

Exercise 4A Question 7

Practice Now

A pyramid has a square base of length of 12 m. Given that the slant height of the pyramid is 15 m, draw its net and hence, find its total surface area.



Area of each triangular face

$= \frac{1}{2} \times 12 \times 15$

$= 90 \text{ m}^2$

Area of square base

$= 12 \times 12$

$= 144 \text{ m}^2$

Total surface area of pyramid

= (4 × area of each triangular face)

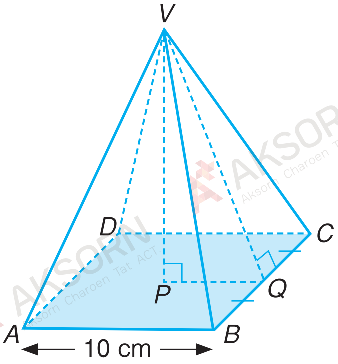
+ area of square base

$= (4 \times 90) + 144$

$= 504 \text{ m}^2$

Worked Example 4

A pyramid has a square base of length of 10 cm and a total surface area of 272 cm^2. Find the volume of the pyramid.



Solution:

Total surface area of pyramid

= (4 × area of each triangular face)

+ area of square base

Area of each triangular face

$$= \frac{\text{total surface area of pyramid} - \text{area of square base}}{4}$$

$$= \frac{272 - (10 \times 10)}{4}$$

$$= \frac{172}{4}$$

$$= 43 \text{ cm}^2$$

$$\text{Area of } \Delta VBC = \frac{1}{2} \times 10 \times VQ$$
$$5 \times VQ = 43$$
$$VQ = 8.6 \text{ cm}$$
$$PQ = \frac{1}{2} \times AB$$
$$= \frac{1}{2} \times 10$$
$$= 5 \text{ cm}$$

We have $\angle VPQ = 90^\circ$.

By using the Pythagorean theorem,

$$VQ^2 = VP^2 + PQ^2$$
$$(8.6)^2 = VP^2 + 5^2$$
$$VP^2 = (8.6)^2 - 5^2$$
$$= 73.96 - 25$$
$$= 48.96.$$

Since VP is the height, the value is always positive.

We get $VP = \sqrt{48.96}$

≈ 7 cm.

Volume of pyramid

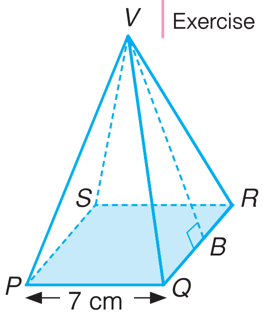
$$= \frac{1}{3} \times \text{base area} \times \text{height}$$
$$\approx \frac{1}{3} \times 10 \times 10 \times 7$$
$$\approx 233.33 \text{ cm}^3$$

Similar Questions

Exercise 4A Questions 11-14

Practice Now

$VPQRS$ is a pyramid where the length of each side of its square base is 7 cm. Given that the total surface area of the pyramid is 161 cm², find the following.



1) Slant height VB

Total surface area of pyramid = (4 × area of each triangular face) + area of square base

$$\text{Area of each triangular face} = \frac{\text{total surface area of pyramid - area of square base}}{4}$$
$$= \frac{161 - (7 \times 7)}{4}$$
$$= \frac{112}{4}$$
$$= 28 \text{ cm}^2$$
$$\text{Area of } \Delta VQR = \frac{1}{2} \times 7 \times VB$$
$$28 = \frac{1}{2} \times 7 \times VB$$
$$VB = 8 \text{ cm}$$

2) The volume of the pyramid

Let the height of the pyramid be VT.

We get $TB = \frac{1}{2} \times PQ$

$= \frac{1}{2} \times 7$

$= 3.5$ cm.

We have $\angle VTB = 90^\circ$.

By using the Pythagorean theorem,

$VB^2 = VT^2 + TB^2$

$8^2 = VT^2 + (3.5)^2$

$VT^2 = 8^2 - (3.5)^2$

$= 64 - 12.25$

$= 51.75$.

Since VT is the height, the value is always positive.

We have $VT = \sqrt{51.75}$ cm.

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

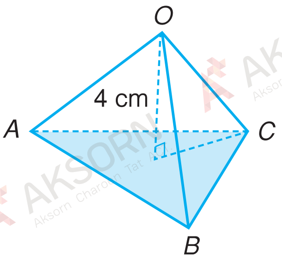
$\approx \frac{1}{3} \times 7 \times 7 \times \sqrt{51.75}$

≈ 117.50 cm^3

Exercise 4A

Basic Level

1. *OABC* is a triangular pyramid with a base area of 15 cm^2 and a height of 4 cm. Find the volume of the triangular pyramid.

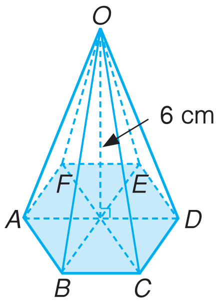


Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 15 \times 4$

$= 20$ cm^3

2. *OABCDEF* is a hexagonal pyramid with a base area of 23 cm^2 and a height of 6 cm. Find the volume of the pyramid.

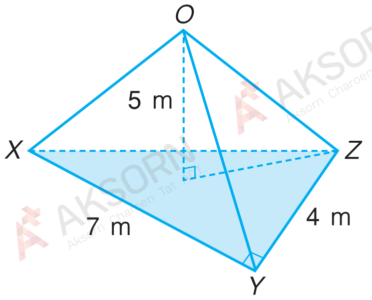


Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 23 \times 6$

$= 46$ cm^3

3. *OXYZ* is a pyramid whose base is a right triangle where *XY* = 7 m and *YZ* = 4 m. Given that the height of the pyramid is 5 m, find its volume.



Base area of pyramid $= \frac{1}{2} \times 7 \times 4$

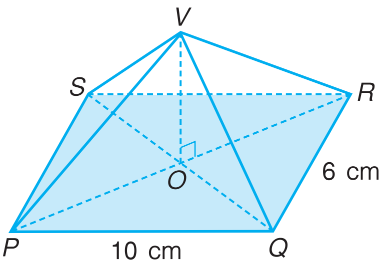
$= 14 \text{ m}^2$

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 14 \times 5$

$= 23\frac{1}{3} \text{ m}^3$

4. *VPQRS* is a rectangular pyramid where *PQ* = 10 cm and *QR* = 6 cm. Given that the volume of the pyramid is 100 cm^3, find its height *VO*.



Base area of pyramid $= 10 \times 6$

$= 60 \text{ cm}^2$

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$100 = \frac{1}{3} \times 60 \times$ height

$100 = 20 \times$ height

Height $= 5$

Therefore, the height of the pyramid is 5 cm.

5. A pyramid with a triangular base has a volume of 50 cm^3. If the base and the height of the triangular base are 5 cm and 8 cm respectively, find the height of the pyramid.

Base area of pyramid $= \frac{1}{2} \times 5 \times 8$

$= 20 \text{ cm}^2$

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$50 = \frac{1}{3} \times 20 \times$ height

$50 = \frac{20}{3} \times$ height

Height $= 7.5$

Therefore, the height of the pyramid is 7.5 cm.

6. The volume of a square pyramid with a height of 12 m is 100 m^3. Find the length of its square base.

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$100 = \frac{1}{3} \times$ base area $\times 12$

$100 = 4 \times$ base area

Base area $= 25\ m^2$

$x^2 = 25$

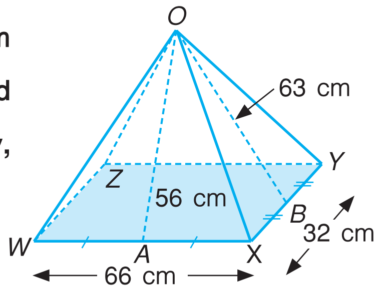
Since x is the side length, the value is always positive.

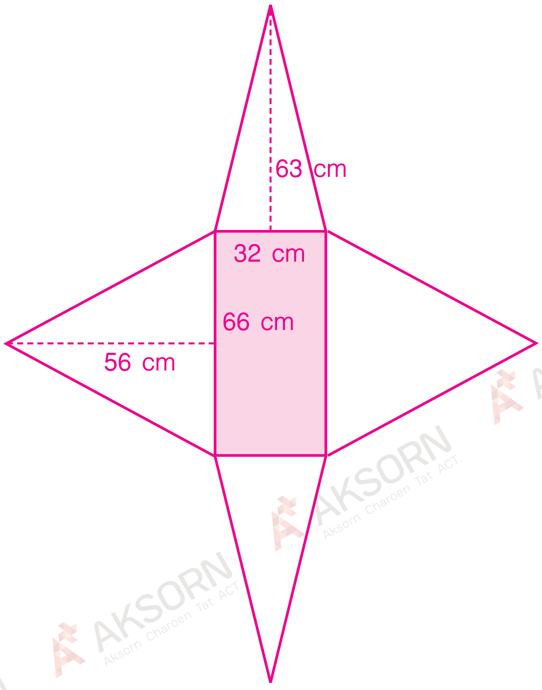
We get $x = \sqrt{25}$

$= 5$ m.

Therefore, the length of its square base is 5 m.

7. *OWXYZ* is a rectangular pyramid where *WX* = 66 cm and *XY* = 32 cm. Given that the slant heights *OA* and *OB* of the pyramid are 56 cm and 63 cm respectively, draw its net and hence find its total surface area.





Area of Δ *OWX* $= \frac{1}{2} \times 66 \times 56$

$= 1{,}848\ cm^2$

Area of Δ *OXY* $= \frac{1}{2} \times 32 \times 63$

$= 1{,}008\ cm^2$

Base area $= 66 \times 32$

$= 2{,}112\ cm^2$

Total surface area of pyramid

$=$ (2 × area Δ *OWX*)

$+$ (2 × area of Δ *OXY*) + base area

$= 2 \times 1{,}848 + 2 \times 1{,}008 + 2{,}112$

$= 7{,}824\ cm^2$

Intermediate Level

8. A glass paperweight is in the shape of a solid pyramid with a square base of length 6 cm and a height of 7 cm. Given that the density of the glass is 3.1 g/cm^3, find the mass of four identical paperweights.

(Density = $\frac{\text{Mass}}{\text{Volume}}$)

Volume of paperweight $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 6 \times 6 \times 7 = 84$ cm^3

Mass of 4 paperweights = volume of 4 paperweights × density

$= 4 \times 84 \times 3.1 = 1{,}041.6$ g

9. A solid pentagonal pyramid has a mass of 500 g. It is made of a material with a density of 6 g/cm^3. Given that the base area of the pyramid is 30 cm^2, find its height. (Density = $\frac{\text{Mass}}{\text{Volume}}$)

Volume of pyramid = mass of pyramid ÷ density

$= 500 \div 6 = 83\frac{1}{3}$ cm^3

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

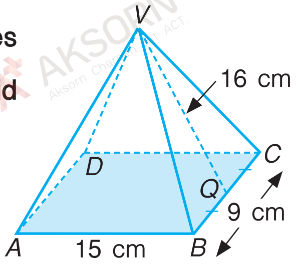
$83\frac{1}{3} = \frac{1}{3} \times 30 \times$ height

$83\frac{1}{3} = 10 \times$ height

Height $= 8\frac{1}{3}$

Therefore, the height of the pyramid is $8\frac{1}{3}$ cm.

10. *VABCD* is a pyramid with a rectangular base of sides 15 cm by 9 cm. Given that the slant height of the pyramid is 16 cm, find the following.



1) The height of the pyramid

Let the height of the pyramid be *VP*.

$PQ = 15 \div 2 = 7.5$ cm

By using the Pythagorean theorem, $VP = \sqrt{16^2 - (7.5)^2} \approx 14.13$ cm.

Therefore, the height of the pyramid is approximately 14.13 cm.

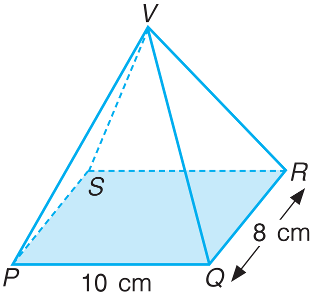
2) The volume of the pyramid

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$\approx \frac{1}{3} \times 15 \times 9 \times 14.13$

$= 635.85\ \text{cm}^3$

11. ***VPQRS*** **is a pyramid with a rectangular base of sides 10 cm by 8 cm. Given that the volume of the pyramid is 180 cm^3, find the following.**



1) The height of the pyramid

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$180 = \frac{1}{3} \times 10 \times 8 \times \text{height}$

$180 = \frac{80}{3} \times \text{height}$

Height $= 6.75$

Therefore, the height of the pyramid is 6.75 cm.

2) The surface area of the pyramid

Let the slant height from V to PQ be ℓ_1 cm, and

the slant height from V to QR be ℓ_2 cm.

By using the Pythagorean theorem,

$\ell_1 = \sqrt{(6.75)^2 + 4^2}$

≈ 7.846 cm

$\ell_2 = \sqrt{(6.75)^2 + 5^2}$

≈ 8.4 cm.

Total surface area of pyramid $=$ area of all triangular faces $+$ area of square base

$\approx 2 \times \left(\frac{1}{2} \times 10 \times 7.846 + \frac{1}{2} \times 8 \times 8.4\right) + 10 \times 8$

$= 2(39.23 + 33.6) + 80$

$= 225.66\ \text{cm}^2$

Advanced Level

12. A solid pyramid has a rectangular base of sides 15 cm by 10 cm and a height of 20 cm. It is placed inside an open cubical tank of sides 30 cm. The tank is then completely filled with water. If the pyramid is removed, what will be depth of the remaining water in the tank?

Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3} \times 15 \times 10 \times 20 = 1{,}000 \text{ cm}^3$

Volume of cubical tank $=$ length^3

$= 30^3 = 27{,}000 \text{ cm}^3$

Volume of water left in tank after pyramid is removed $= 27{,}000 - 1{,}000$

$= 26{,}000 \text{ cm}^3$

Let the depth of the remaining water in the tank be d cm.

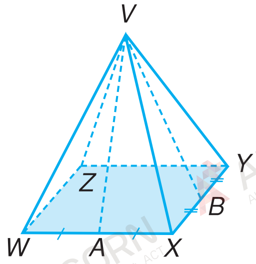
$$30 \times 30 \times d = 26{,}000$$

$$900d = 26{,}000$$

$$d = 28\frac{8}{9}$$

Therefore, the depth of the remaining water is $28\frac{8}{9}$ cm.

13. *VWXYZ* is a rectangular pyramid where *WX* is longer than *XY*. Is the slant height *VA* longer or shorter than the slant height *VB*? Explain your answer.



Let WX be a, XY be b and the height of the pyramid be h, i.e. $a > b$.

By using the Pythagorean theorem,

$$VA = \sqrt{h^2 + \left(\frac{b}{2}\right)^2} = \sqrt{h^2 + \frac{b^2}{4}}$$

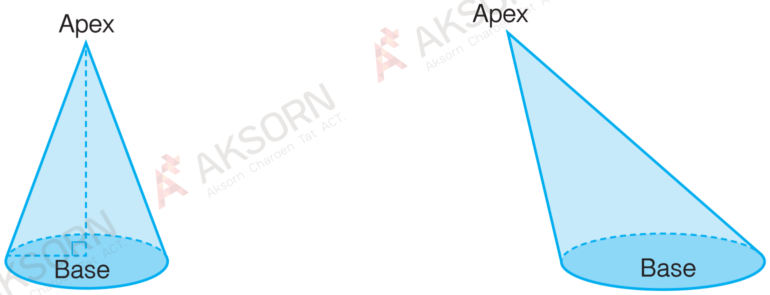
$$VB = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} = \sqrt{h^2 + \frac{a^2}{4}}.$$

Since $a > b$, then $VB > VA$.

Therefore, the slant height VA is shorter than VB.

4.2 Cones

In primary level, we have learned that a cone is a solid with the circular base and the curved surface tapers into the apex, but the apex is not in the same plane as the base.



(a) Circular Cone

(b) Elliptical Cone

Fig. 4.8

The perpendicular height (or simply the height) of a cone is the perpendicular distance from the apex to the base of the cone. The slant height of a right circular cone is the distance from the apex to the circumference of the base. And, the radius of the circular base is the base radius of the cone.

Cones

www.aksorn.com/interactive3D/RS944

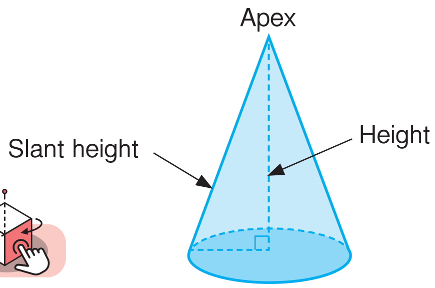


Fig. 4.9

Journal Writing

One of your classmates is absent and he does not understand what a cone is. Write in your journal how you would explain to him some features of a cone and how a cone is different from a cylinder or a pyramid.

1. Volume of Cones

We have learned that:

Volume of cylinder = area of cross-section × height

= base area × height

= $\pi r^2 h$

where r is the base radius and h is the height of the cylinder.

Now, we will learn how to find the volume of a cone by comparing a cone with a pyramid that has a regular polygonal base.

Investigation

Comparison between a cone and a pyramid

1. Fig. 4.10 shows (a) a regular pentagon, (b) a regular hexagon, (c) a regular 12-gon, and (d) a regular 16-gon inside a circle, respectively. If the number of sides of a regular polygon is increased indefinitely, what will the polygon become?

   The polygon will become a circle.

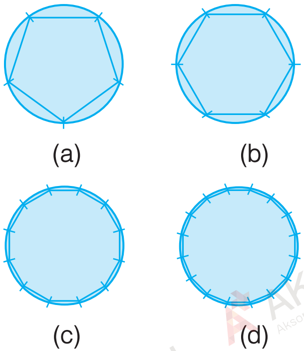


Fig. 4.10

2. Fig. 4.11 shows a sequence of regular pyramids, i.e. right pyramids with regular polygonal bases. If the number of sides of the regular polygonal base of a pyramid is increased indefinitely, what will the pyramid become?

   The pyramid will become a cone.

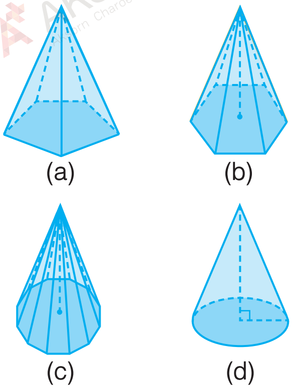


Fig. 4.11

From **Investigation**, in many ways, a cone is like a pyramid. However, a cone is not a pyramid because the base of a pyramid must be a polygon, but the base of a cone is a circle. Although a regular polygon can become a circle if its number of sides is increased indefinitely, a polygon must have a finite number of sides, and so a circle is not a polygon.

$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3} \times \text{base area} \times \text{height}\\ &= \frac{1}{3}\pi r^2 h\end{aligned}$$

where r is the base radius and h is the height of the cone.

Thinking Time

If a cone and a cylinder have the same base and the same height, what is the relationship between the volume of the cone and the volume of this corresponding cylinder?

Volume of a cone = $\frac{1}{3}\pi r^2 h$

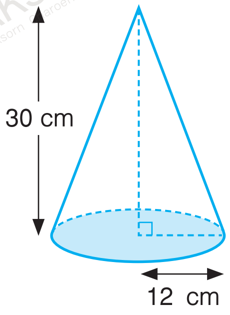
Volume of a cylinder = $\pi r^2 h$

Since the cone and cylinder have the same base and same height,

then the volume of the cone = $\frac{1}{3}$ × volume of a cylinder.

Worked Example 5

A cone has a circular base of radius 12 cm and a height of 30 cm. Find the volume of the cone. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Solution:

$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3}\pi r^2 h\\ &= \frac{1}{3} \times \pi \times 12^2 \times 30\\ &= 1{,}440\pi\\ &\approx 4{,}525.71 \text{ cm}^3\end{aligned}$$

Practice Now

Similar Questions

Exercise 4B Questions 1-3, 6

A cone has a circular base of radius 8 cm and a height of 17 cm. Find the volume of the cone. $\left(\text{Take } \pi \approx \dfrac{22}{7}.\right)$

Volume of cone $= \frac{1}{3}\pi r^2 h$

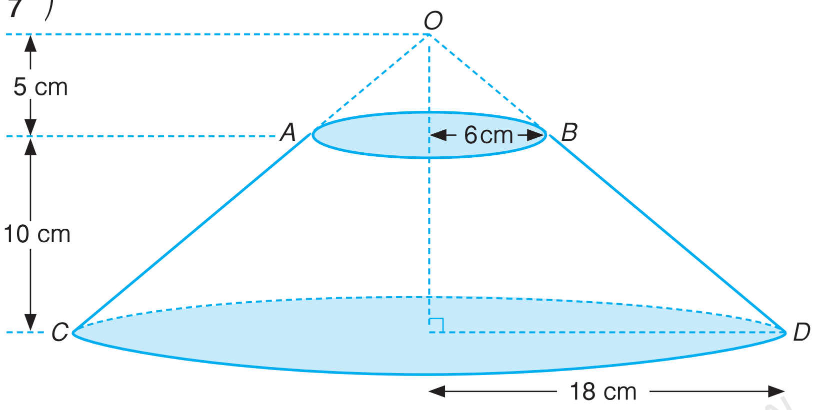
$= \frac{1}{3} \times \pi \times 8^2 \times 17$

$= 362\frac{2}{3}\pi$

$\approx$ 1,139.81 cm^3

Worked Example 6

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 6 cm and a height of 5 cm from the bigger cone *OCD* that has a base radius of 18 cm. Given that the height of the frustum is 10 cm, find its volume. $\left(\text{Take } \pi \approx \dfrac{22}{7}.\right)$



Solution:

Height of cone *OCD* = 5 + 10 = 15 cm

Volume of frustum = volume of cone *OCD* - volume of cone *OAB*

$= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(R^2 H - r^2 h)$

$= \frac{1}{3}\pi(18^2 \times 15 - 6^2 \times 5)$

$= \frac{1}{3}\pi(4{,}680)$

$= 1{,}560\pi$

$\approx 4{,}902.86 \text{ cm}^3$

Practice Now

Similar Questions

Exercise 4B Question 9

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 5 cm and a height of 4 cm from the bigger cone *OCD* that has a base radius of 20 cm. Given that the height of the frustum is 12 cm, find its volume. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Height of cone *OCD* $= 4 + 12 = 16$ cm

$$\begin{aligned}
\text{Volume of frustum} &= \text{volume of cone } OCD - \text{volume of cone } OAB \\
&= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi(R^2 H - r^2 h) \\
&= \frac{1}{3}\pi(20^2 \times 16 - 5^2 \times 4) \\
&= \frac{1}{3}\pi(6{,}300) \\
&= 2{,}100\pi \\
&\approx 6{,}600 \text{ cm}^3
\end{aligned}$$

KEY

2. Surface Area of Cones

As a cone has a curved surface, it is difficult to use the flat slant faces of a pyramid as an analogy to find a formula for the curved surface area of a cone. Instead, we will use the same approach as finding the formula for the area of a circle.

Investigation

Curved surface area of cones

Consider the cone as shown in Fig. 4.12(a).

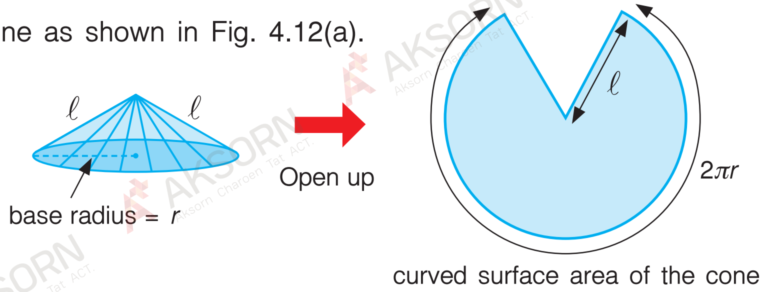


Fig. 4.12(a)

Net of cones

www.aksorn.com/interactive3D/RS941

Unfold the curved surface of the cone to become the sector shown in Fig. 4.12(a). Divide the sector into 44 smaller sectors and arrange them to form the shape as shown in Fig. 4.12(b).

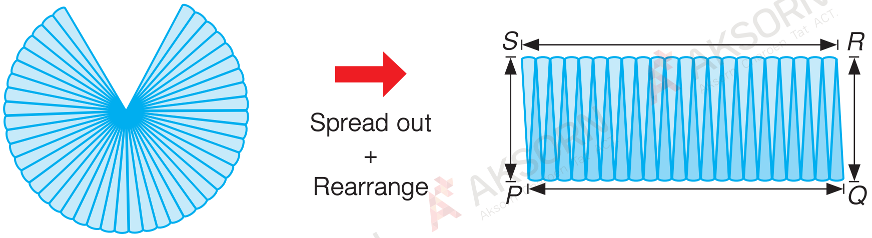


Fig. 4.12(b)

If the number of sectors is increased indefinitely, then the shape in Fig. 4.12(b) will become a ....rectangle........ *PQRS*.

Since *PQ* + *RS* = circumference of the base circle in Fig. 4.12(a), then the length of the rectangle is ......πr.......

Since PS = slant height of the cone in Fig. 4.12(a), then the width of the rectangle is PS ......ℓ.......

Curved surface area of cone = area of rectangle *PQRS*

= ...*PQ*... × ...*PS*...

= ...$\pi r\ell$...

From **Investigation**, we have:

$$\text{Curved surface area of cone} = \pi r\ell$$

where r is the base radius and ℓ is the slant height of the cone.

Thinking Time

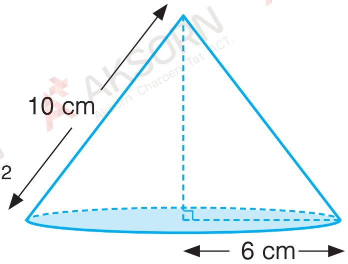
What is the total surface area of a solid cone?

Total surface area of a solid cone = curved surface area of cone + base area of cone

$= \pi r\ell + \pi r^2$

Worked Example 7

A cone has a circular base of radius 6 cm and a slant height of 10 cm. Find the total surface area of the cone. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Solution:

$$
\begin{aligned}
\text{Total surface area of cone} &= \pi r\ell + \pi r^2 \\
&= \pi \times 6 \times 10 + \pi \times 6^2 \\
&= 60\pi + 36\pi \\
&= 96\pi \\
&\approx 301.71 \text{ cm}^2
\end{aligned}
$$

Practice Now

Similar Questions

Exercise 4B Questions 4-5, 7

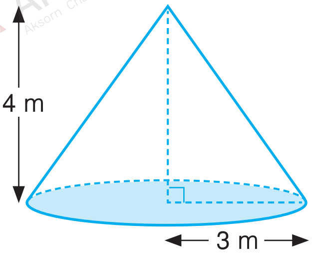
A cone has a circular base of radius 9 cm and a slant height of 5 cm. Find the total surface area of the cone. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

KEY

$$
\begin{aligned}
\text{Total surface area of cone} &= \pi r\ell + \pi r^2 \\
&= \pi \times 9 \times 5 + \pi \times 9^2 \\
&= 45\pi + 81\pi \\
&= 126\pi \\
&\approx 396 \text{ cm}^2
\end{aligned}
$$

Worked Example 8

A cone has a circular base of radius 3 m and a height of 4 m. Find the curved surface area of the cone. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Solution:

Let the slant height of the cone be ℓ m.

By using the Pythagorean theorem, $\ell = \sqrt{3^2 + 4^2} = 5$ m.

$$
\begin{aligned}
\text{Curved surface area of cone} &= \pi r\ell \\
&= \pi \times 3 \times 5 \\
&= 15\pi \\
&\approx 47.14 \text{ m}^2
\end{aligned}
$$

Practice Now

Similar Questions

Exercise 4B Question 8

A cone has a circular base of radius 8 m and a height of 15 m. Find the curved surface area of the cone. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Let the slant height of the cone be ℓ m.

By using the Pythagorean theorem, $\ell = \sqrt{8^2 + 15^2} = 17$ m.

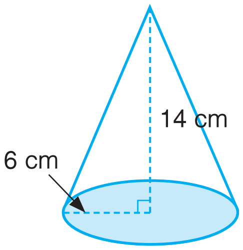
$$\begin{aligned}\text{Curved surface area of cone} &= \pi r \ell \\ &= \pi \times 8 \times 17 \\ &= 136\pi \\ &\approx 427.43 \text{ m}^2\end{aligned}$$

Exercise 4B

Basic Level

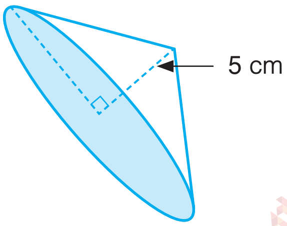
1. Find the volume of each of the following cones. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1)



$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \pi \times 6^2 \times 14 \\ &= 168\pi \\ &\approx 528 \text{ cm}^3\end{aligned}$$

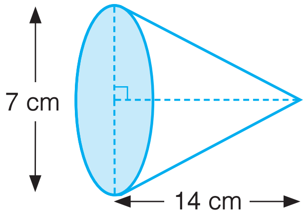
2)



Area base = 154 cm^2

$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3} \times \text{base area} \times \text{height} \\ &= \frac{1}{3} \times 154 \times 5 \\ &= 256\frac{2}{3} \text{ cm}^3\end{aligned}$$

3)



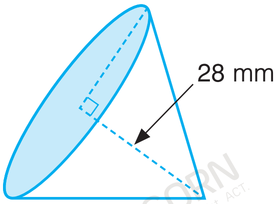
Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times \left(\frac{7}{2}\right)^2 \times 14$

$= 57\frac{1}{6}\pi$

$\approx 179.67 \text{ cm}^3$

4)



Circumference $= 132$

$2\pi r = 132$

$r = \frac{132}{2\pi}$

$= \frac{66}{\pi}$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times \left(\frac{66}{\pi}\right)^2 \times 28$

$= \frac{40{,}656}{\pi}$

$\approx 12{,}936 \text{ mm}^3$

2. **Find the height of each of the following cones.** $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) The volume of a cone with a circular base of radius 8 cm is 320π cm^3.

Volume of cone $= \frac{1}{3}\pi r^2 h$

$320\pi = \frac{1}{3} \times \pi \times 8^2 \times h$

$320\pi = 21\frac{1}{3}\pi h$

$h = 15$ cm

2) A cone has a base area of 20 m^2 and a volume of 160 m^3.

Volume of cone $= \frac{1}{3} \times$ base area $\times$ height

$160 = \frac{1}{3} \times 20 \times$ height

$160 = \frac{20}{3} \times$ height

Height $= 24$ m

3. **A cone has a height of 14 cm and a volume of 132 cm^3. Find the radius of the circular base.** $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$132 = \frac{1}{3} \times \frac{22}{7} \times r^2 \times 14$

$132 = \frac{44}{3} \times r^2$

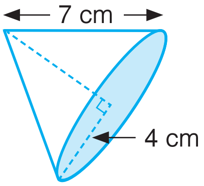
$r^2 = 9$

$r = \sqrt{9}$ (Since r is the length, its value is always more than 0.)

$= 3$ cm

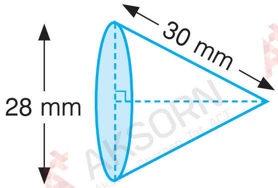
4. Find the total surface area of each of the following cones. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1)



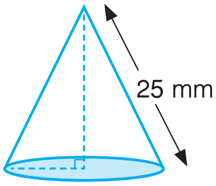
$$\text{Total surface area of cone} = \pi r\ell + \pi r^2$$
$$= \pi \times 4 \times 7 + \pi \times 4^2$$
$$= 28\pi + 16\pi$$
$$= 44\pi$$
$$\approx 138.29 \text{ cm}^2$$

2)



$$\text{Radius of cone} = 28 \div 2 = 14 \text{ mm}$$
$$\text{Total surface area of cone} = \pi r\ell + \pi r^2$$
$$= \pi \times 14 \times 30 + \pi \times 14^2$$
$$= 420\pi + 196\pi$$
$$= 616\pi$$
$$\approx 1{,}936 \text{ mm}^2$$

3)



Circumference of base = 132 mm

$$\text{Circumference} = 132$$
$$2\pi r = 132$$
$$r = \frac{66}{\pi}$$

$$\text{Total surface area of cone} = \pi r\ell + \pi r^2$$
$$= \pi \times \frac{66}{\pi} \times 25 + \pi \times \left(\frac{66}{\pi}\right)^2$$
$$= 1{,}650 + \frac{4{,}356}{\pi}$$
$$\approx 3{,}036 \text{ mm}^2$$

5. A cone has a circular base of radius 6 mm. Given that the curved surface area of the cone is 84π mm^2, find its slant height.

$$\text{Curved surface area of cone} = 84\pi \text{ mm}^2$$
$$\pi(6)\ell = 84\pi$$
$$6\pi\ell = 84\pi$$
$$\ell = \frac{84\pi}{6\pi}$$
$$= 14$$

Therefore, the slant height of the cone is 14 mm.

Intermediate Level

6. A conical funnel of diameter 23.2 cm and depth 42 cm contains water filled to the brim. The water is poured into a cylindrical tin of diameter 16.2 cm. If the tin must contain all the water, find its least possible height.

Radius of conical funnel = 23.2 ÷ 2 = 11.6 cm

Volume of conical funnel

$= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times (11.6)^2 \times 42$

$= 1{,}883.84\pi \text{ cm}^3$

Radius of cylindrical tin = 16.2 ÷ 2

= 8.1 cm

Volume of cylindrical tin $= 1{,}883.84\pi \text{ cm}^3$

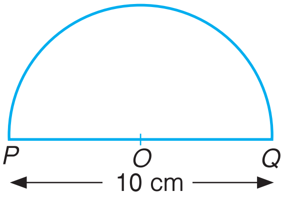
$\pi(8.1)^2 h = 1{,}883.84\pi$

$65.61\pi h = 1{,}883.84\pi$

$h = \frac{1{,}883.84\pi}{65.61\pi}$

≈ 28.71 cm

7. The semicircle shown is folded to form a right circular cone so that arc *PQ* becomes the circumference of the base. Given that the diameter of the semicircle, *PQ*, is 10 cm, find the following. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



1) The diameter of the base

Let the diameter of the semicircle be d_1 cm, and the diameter of the base of the cone be d_2 cm.

Circumference of semicircle $= \frac{1}{2}\pi d_1$

$= \frac{1}{2} \times \pi \times 10$

$= 5\pi$ cm

Circumference of base of cone $= 5\pi$ cm

$\pi d_2 = 5\pi$

$d_2 = 5$

Therefore, the diameter of the base of the cone is 5 cm.

2) The curved surface area of the cone

Radius of the cone = 5 ÷ 2

= 2.5 cm

Slant height of the cone = 10 ÷ 2

= 5 cm

Curved surface area of the cone $= \pi r \ell$

$= \pi \times 2.5 \times 5$

$= 12.5\pi$

$\approx 39.29 \text{ cm}^2$

8. A circular cone has a height of 17 mm and a slant height of 21 mm, find the following. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) The volume

By using the Pythagorean theorem,

$r^2 = 21^2 - 17^2$

$\quad = 152.$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 152 \times 17$

$= 861\frac{1}{3}\pi$

$\approx 2{,}707.05 \text{ mm}^3$

2) The total surface area of the cone

Total surface area of cone $= \pi r\ell + \pi r^2$

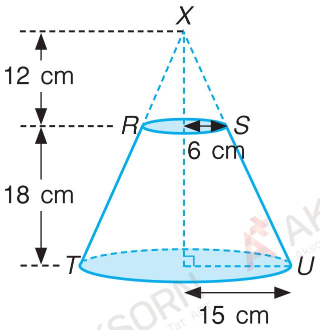
$= \pi \times \sqrt{152} \times 21 + \pi \times 152$

$= 21\sqrt{152}\,\pi + 152\pi$

$\approx 1{,}291.42 \text{ mm}^2$

Advanced Level

9. The figure shows a frustum which is obtained by removing the smaller cone *XRS* with a base radius of 6 cm and a height of 12 cm from the bigger cone *XTU* that has a base radius of 15 cm. Given that the height of the frustum is 18 cm, find its volume. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Height of cone *XTU* $= 18 + 12 = 30$ cm

Volume of frustum = volume of cone *XTU* – volume of cone *XRS*

$= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(R^2 H - r^2 h)$

$= \frac{1}{3}\pi(15^2 \times 30 - 6^2 \times 12)$

$= \frac{1}{3}\pi(6{,}318)$

$= 2{,}016\pi$

$\approx 6{,}618.86 \text{ cm}^3$

4.3 Spheres

In primary level, we have learned that a sphere is in curved shape where each point on its surface has an equal distance from the center. Fig. 4.13 shows some real-life examples of spheres.


(a) Soccer Ball (b) Crystal Ball (c) Globe

Fig. 4.13

INTERNET RESOURCES

Map projection is a method of representing the surface of a sphere on a plane. Is it possible to draw the map of all the countries in the world on a plane accurately without distorting the shape or the area? Search on the Internet to find out more about conformal (or area-preserving) map projections.

Spheres

www.aksorn.com/interactive3D/RS943

1. Volume of Spheres

Class Discussion

Is the king’s crown made of pure gold?

Archimedes is one of the three greatest mathematicians of all time. He lived from 287 B.C. to 212 B.C. in Greece. One day, the King asked Archimedes to find out whether his crown was made of pure gold or whether the goldsmith had cheated. The real problem lies in finding the volume of the crown. Archimedes thought for a long time, but still had no idea. He then decided to take a break by taking a bath. As he stepped into the bathtub, the water overflowed. Archimedes was so excited by this discovery that he dashed out into the street shouting ‘Eureka’ which means ‘I have found it!’, but had forgotten that he was unclothed.

Archimedes realized that a sinking solid displaces an amount of water that is equal to the volume of the solid. To find the volume of the crown, we can fill up the Eureka can (named as a result of the above story) as shown in Fig. 4.14(a) with water until it overflows before putting in the crown. The volume of water displaced into the container as shown in Fig. 4.14(b) is equal to the volume of the crown.

(a) Before (b) After

Fig. 4.14

Suppose Archimedes found out that the volume of water displaced was 714 cm^3, and the mass of the crown was 11.6 kg. Given that the density of gold is 19.3 g/cm^3, determine whether the crown was made of pure gold. Do you arrive at the same conclusion as your classmates?

$\left(\boxed{\text{Hint}} : \text{Density of object } (g/cm^3) = \dfrac{\text{Mass (g)}}{\text{Volume } (cm^3)} \right)$

$\text{Density of the crown} = \dfrac{11.6 \text{ kg}}{714 \text{ cm}^3}$

$= \dfrac{(11.6 \times 1{,}000) \text{ g}}{714 \text{ cm}^3}$

$\approx 16.2 \text{ g/cm}^3$

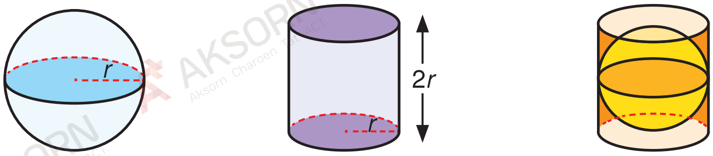
Since 16.2 g/cm^3 ≠ 19.3 g/cm^3, the crown was not made of pure gold.

Investigation

Volume of spheres

Part 1: Archimedes' discovery of the volume of a sphere

Using the displacement method mentioned in **Class Discussion**, Archimedes discovered a formula to find the volume of a sphere. Fig. 4.15(a) shows a sphere of radius r and Fig. 4.15(b) shows its related circular cylinder of base radius r and height $2r$. Archimedes filled the cylinder with water and placed the sphere inside it. See Fig. 4.15(c).



(a) Sphere (b) Related cylinder (c) Sphere inside related cylinder

Fig. 4.15

Archimedes found out that the volume of water displaced was equal to $\frac{2}{3}$ of the volume of the cylinder.

Part 2: Work out of the volume of a sphere

$$\begin{aligned}\text{Volume of cylinder} &= \pi r^2 h \\ &= \pi \times r^2 \times 2r \\ &= 2\pi r^3 \\ \text{Volume of sphere} &= \frac{2}{3} \times \text{volume of cylinder} \\ &= \frac{2}{3} \times 2\pi r^3 \\ &= \frac{4}{3}\pi r^3\end{aligned}$$

From **Investigation**, we have:

$$\boxed{\text{Volume of sphere} = \frac{4}{3}\pi r^3}$$

where r is the radius of the sphere.

Worked Example 9

A ball bearing (which is spherical in shape) has a radius of 0.3 cm. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$ Find the following.

1) The volume of the ball bearing
2) The mass of 6,000 identical ball bearings if they are made of steel of density 7.85 g/cm^3.

Solution:

1) Volume of ball bearing

$$\begin{aligned}&= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times (0.3)^3 \\ &= 0.036\pi \approx 0.113 \text{ cm}^3\end{aligned}$$

2) Mass of 6,000 ball bearings

$$\begin{aligned}&= \text{volume of 6,000 ball bearings} \times \text{density} \\ &= 6{,}000 \times 0.036\pi \times 7.85 \\ &\approx 5{,}329.03 \text{ g}\end{aligned}$$

Practice Now

Similar Questions

Exercise 4C Questions 1-2, 6-8, 10

A lead ball bearing has a diameter of 0.4 cm. Find the mass of 5,000 identical lead ball bearings if the mass of 1 cm^3 of the lead is 11.3 g. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Radius of ball bearing $= 0.4 \div 2 = 0.2$ cm

Volume of ball bearing $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (0.2)^3$

$= \frac{4\pi}{375}\ \text{cm}^3$

Mass of 5,000 ball bearings = volume of 5,000 ball bearings × density

$= 5{,}000 \times \frac{4\pi}{375} \times 11.3$

$\approx 1{,}894.10$ g

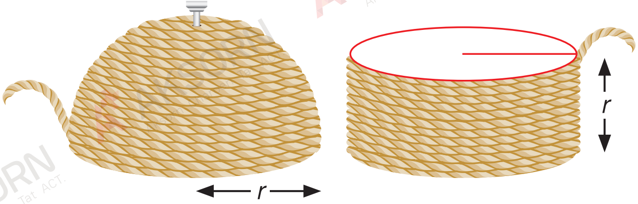
2. Surface Area of Spheres

Investigation

Surface area of spheres

Part 1: Archimedes' discovery of the surface area of a sphere

Archimedes also discovered a formula to find the surface area of a sphere. Fig. 4.16(a) shows a hemisphere of radius *r*. One end of a piece of twine is stuck in the center of the curved surface of the hemisphere by a pin before the twine is coiled around the curved surface completely. Fig. 4.16(b) shows a circular cylinder of base radius *r* and height *r*. One end of a piece of twine is stuck at the bottom of the curved surface of the cylinder by a pin before the twine is coiled around the curved surface completely.



(a) Hemisphere (b) Cylinder

Fig. 4.16

Archimedes found out that the two pieces of twine were of the same length.

Length of second piece of twine $= 2\pi rh$

$= 2\pi \times r \times r$

$= 2\pi r^2$

Curved surface area of sphere = 2 × length of first piece of twine

= 2 × length of second piece of twine

$= 2 \times 2\pi r^2$

$= 4\pi r^2$

Part 2: Simple experiment to find the surface area of a sphere

1. Take an orange and cut it into halves. Place one half flat on a sheet of paper and draw a circle around it. The area of the circle is πr^2 where r is the radius of the orange, assuming that the orange is spherical.
2. Repeat Step 1 and draw 4 more circles.
3. Peel the skin of both halves of the orange, tear into small pieces, and cover as many circles as completely as possible with the skin.
4. How many circles are covered completely with the orange skin?

   4 circles are covered completely with the orange skin.
5. What do you think is the surface area of the orange (spherical in shape)?

   Surface area of the orange $= 4\pi r^2$

From **Investigation**, we can conclude that:

$$\boxed{\text{Surface area of sphere} = 4\pi r^2}$$

where r is the radius of the sphere.

Thinking Time

What is the total surface area of a solid hemisphere?

Total surface area of a solid hemisphere

$= $ curved surface area of hemisphere + base area of hemisphere

$= \frac{1}{2} \times 4\pi r^2 + \pi r^2$

$= 2\pi r^2 + \pi r^2$

$= 3\pi r^2$

Worked Example 10

A solid sphere has a diameter of 14 cm. Find its surface area. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Solution:

Radius of sphere $= 14 \div 2$

$= 7$ cm

Surface area of sphere $= 4\pi r^2$

$= 4 \times \pi \times 7^2$

$= 196\pi$

≈ 616 cm^2

Similar Questions

Exercise 4C Questions 3-4

Practice Now

A solid sphere has a diameter of 25 cm. Find its surface area. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Radius of sphere $= 25 \div 2$

$= 12.5$ cm

Surface area of sphere $= 4\pi r^2$

$= 4 \times \pi \times (12.5)^2$

$= 625\pi$

$\approx 1{,}964.29$ cm^2

Worked Example 11

A solid hemisphere has a curved surface area of 175 cm^2. Find its radius. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Solution:

$$\begin{aligned}
\text{Curved surface area of solid hemisphere} = \frac{1}{2} \times 4\pi r^2 = 2\pi r^2 &= 175 \\
r^2 &= \frac{175}{2\pi} \\
r &= \sqrt{\frac{175}{2\pi}} \text{ (since } r > 0) \\
&\approx 5.28 \text{ cm}
\end{aligned}$$

Similar Questions

Exercise 4C Questions 5, 9

Practice Now

A hemisphere has a curved surface area of 200 cm^2. Find its radius. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

$$\begin{aligned}
\text{Curved surface area of hemisphere} = \frac{1}{2} \times 4\pi r^2 = 2\pi r^2 &= 200 \\
r^2 &= \frac{200}{2\pi} \\
r &= \sqrt{\frac{100}{\pi}} \text{ (since } r > 0) \\
&\approx 5.64 \text{ cm}
\end{aligned}$$

KEY

Exercise 4C

Basic Level

1. Find the volume of each of the following spheres with the given radius. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) 14 mm

$$\begin{aligned}
\text{Volume of sphere} &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3} \times \pi \times 14^3 \\
&= 3{,}658\frac{2}{3}\pi \\
&\approx 11{,}498.67 \text{ mm}^3
\end{aligned}$$

2) 4 m

$$\begin{aligned}
\text{Volume of sphere} &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3} \times \pi \times 4^3 \\
&= 85\frac{1}{3}\pi \\
&\approx 268.19 \text{ m}^3
\end{aligned}$$

2. Find the radius of each of the following spheres with the given volume. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) 1,416 cm^3

$$\text{Volume of sphere} = 1{,}416$$
$$\frac{4}{3}\pi r^3 = 1{,}416$$
$$r^3 = \frac{1{,}062}{\pi}$$
$$r = \sqrt[3]{\frac{1{,}062}{\pi}}$$
$$\approx 6.97 \text{ cm}$$

2) 780 m^3

$$\text{Volume of sphere} = 780$$
$$\frac{4}{3}\pi r^3 = 780$$
$$r^3 = \frac{585}{\pi}$$
$$r = \sqrt[3]{\frac{585}{\pi}}$$
$$\approx 5.71 \text{ m}$$

3) 498π mm^3

$$\text{Volume of sphere} = 498\pi$$
$$\frac{4}{3}\pi r^3 = 498\pi$$
$$r^3 = 373\frac{1}{2}$$
$$r = \sqrt[3]{373\frac{1}{2}}$$
$$\approx 7.20 \text{ mm}$$

4) $15\frac{3}{16}\pi$ m^3

$$\text{Volume of sphere} = 15\frac{3}{16}\pi$$
$$\frac{4}{3}\pi r^3 = 15\frac{3}{16}\pi$$
$$r^3 = \frac{729}{64}$$
$$r = \sqrt[3]{\frac{729}{64}}$$
$$= 2.25 \text{ m}$$

3. Find the surface area of each of the following spheres with the given radius. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) 12 cm

$$\text{Surface area of sphere} = 4\pi r^2$$
$$= 4 \times \pi \times 12^2$$
$$= 576\pi$$
$$\approx 1{,}810.29 \text{ cm}^2$$

2) 3 m

$$\text{Surface area of sphere} = 4\pi r^2$$
$$= 4 \times \pi \times 3^2$$
$$= 36\pi$$
$$\approx 113.14 \text{ m}^2$$

4. Find the total surface area of a hemisphere of radius 7 cm. (Take $\pi \approx 3.14$.)

$$\text{Total surface area of hemisphere} = \frac{1}{2} \times 4\pi r^2 + \pi r^2$$
$$= 3\pi r^2$$
$$= 3 \times \pi \times 7^2$$
$$= 147\pi$$
$$\approx 147 \times 3.14$$
$$= 461.58 \text{ cm}^2$$

5. Find the radius of each of the following spheres with the given surface area. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) 210 cm^2

Surface area of sphere $= 210$

$4\pi r^2 = 210$

$r^2 = \frac{210}{4\pi}$

$r = \sqrt{\frac{210}{4\pi}} \quad (r > 0)$

≈ 4.09 cm

2) $3{,}163 \text{ m}^2$

Surface area of sphere $= 3{,}163$

$4\pi r^2 = 3{,}163$

$r^2 = \frac{3{,}163}{4\pi}$

$r = \sqrt{\frac{3{,}163}{4\pi}} \quad (r > 0)$

≈ 15.86 m

3) $911\pi \text{ mm}^2$

Surface area of sphere $= 911\pi$

$4\pi r^2 = 911\pi$

$r^2 = \frac{911}{4}$

$r = \sqrt{\frac{911}{4}} \quad (r > 0)$

≈ 15.09 mm

4) $49\pi \text{ m}^2$

Surface area of sphere $= 49\pi$

$4\pi r^2 = 49\pi$

$r^2 = \frac{49}{4}$

$r = \sqrt{\frac{49}{4}} \quad (r > 0)$

$= 3.5$ m

Intermediate Level

6. Find the number of steel ball bearings, each of diameter 0.7 cm, which can be made from 1 kg of steel, given that 1 cm^3 of steel has a mass of 7.85 g. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Radius of each ball bearing $= 0.7 \div 2$

$= 0.35$ cm

Volume of each ball bearing $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (0.35)^3$

$\approx 0.1796 \text{ cm}^3$

Mass of each ball bearing $= 0.1796 \times 7.85$

$= 1.40986$ g

Number of ball bearings $= \frac{1{,}000}{1.40986}$

≈ 709

7. Fifty-four solid hemispheres, each of diameter 2 cm, are melted to form a single sphere. Find the radius of the sphere.

Radius of hemisphere $= 2 \div 2$

$= 1$ cm

Volume of hemisphere $= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{2}{3}\pi r^3$

$= \frac{2}{3} \times \pi \times 1^3$

$= \frac{2}{3}\pi$ cm^3

Volume of sphere $= \frac{2}{3}\pi \times 54$

$\frac{4}{3}\pi R^3 = 36\pi$

$R^3 = 27$

$R = \sqrt[3]{27}$

$= 3$ cm

8. A cylindrical tin has an internal diameter of 18 cm. It contains water to a depth of 13.2 cm. A heavy spherical ball bearing of diameter 9.3 cm is dropped into the tin. Find the new height of water in the tin.

Radius of cylindrical tin $= 18 \div 2 = 9$ cm

Volume of water in the cylindrical tin

$= \pi r^2 h$

$= \pi \times 9^2 \times 13.2$

$= 1{,}069.2\pi$ cm^3

Radius of spherical ball bearing

$= 9.3 \div 2 = 4.65$ cm

Volume of spherical ball bearing $= \frac{4}{3}\pi R^3$

$= \frac{4}{3} \times \pi \times (4.65)^3$

$= 134.0595\pi$ cm^3

Volume of water and spherical ball bearing

$= 1{,}069.2\pi + 134.0595\pi$

$= 1{,}203.2595\pi$ cm^3

Volume in the cylindrical tin $= 1{,}203.2595\pi$

$\pi \times 9^2 \times H = 1{,}203.2595\pi$

$81\pi H = 1{,}203.2595\pi$

$H = \frac{1{,}203.2595}{81}$

≈ 14.86 cm

9. A basketball has a surface area of 1,810 cm^2. Find its volume. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Surface area of basketball $= 1{,}810$ cm^3

$4\pi r^2 = 1{,}810$

$r^2 = \frac{1{,}810}{4\pi}$

$r = \sqrt{\frac{1{,}810}{4\pi}}$

≈ 12 cm

Volume of basketball $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 12^3$

$= 2{,}304\pi$

$\approx 7{,}241.14$ cm^3

Advanced Level

10. A cylindrical can has a base radius of 3.4 cm. It contains a certain amount of water such that when a sphere is placed inside the can, the water just covers the sphere. If the sphere fits exactly inside the can, find the following. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) The surface area of the can that is in contact with the water when the sphere is inside the can

Radius of sphere = radius of cylindrical can = 3.4 cm

Diameter of sphere = $3.4 \times 2 = 6.8$ cm

Depth of water in the can when the sphere was placed inside = 6.8 cm

Surface area of can in contact with water $= \pi r^2 + 2\pi rh$

$= \pi \times (3.4)^2 + 2 \times \pi \times 3.4 \times 6.8$

$= 11.56\pi + 46.24\pi$

$= 57.8\pi$

$\approx 181.66 \text{ cm}^2$

2) The depth of water in the can before the sphere was placed inside the can

Volume of water in the can before the sphere was placed inside

$= \pi r^2 h - \frac{4}{3}\pi r^3$

$= \pi \times (3.4)^2 \times 6.8 - \frac{4}{3} \times \pi \times (3.4)^3$

$= 3.4^2\pi\left(6.8 - 4\frac{8}{15}\right)$

$= (3.4)^2 \times 2\frac{4}{15} \times \pi \text{ cm}^3$

Let the depth of water in the can before the sphere was placed inside be d cm.

Volume of water $= (3.4)^2 \times 2\frac{4}{15} \times \pi \text{ cm}^3$

$\pi r^2 d = (3.4)^2 \times 2\frac{4}{15} \times \pi$

$\pi(3.4)^2 d = (3.4)^2 \times 2\frac{4}{15} \times \pi$

$d = \dfrac{(3.4)^2 \times 2\frac{4}{15} \times \pi}{(3.4)^2\pi}$

$= 2\frac{4}{15}$

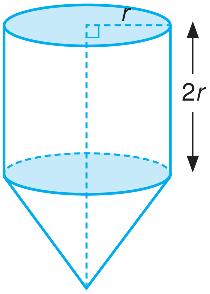
Therefore, the depth of water in the can was $2\frac{4}{15}$ cm.

4.4 Composite Solids

In this section, we will learn how to solve problems involving the volume and the surface area of composite solids as in the following example.

Worked Example 12

A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $2r$ cm. Given that the height of the cone is two-thirds of the height of the cylinder and 5 liters of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in liters.



Solution:

RECALL

$1\ \ell = 1{,}000\ \text{cm}^3$

$$\begin{aligned}\text{Height of cone} &= \frac{2}{3} \times \text{height of cylinder}\\ &= \frac{2}{3} \times 2r\\ &= \frac{4}{3}r\\ \text{Volume of cone} &= \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right)\\ &= \frac{4}{9}\pi r^3\end{aligned}$$

Since volume of cone $= 5\ \ell = 5{,}000\ \text{cm}^3$,

$$\begin{aligned}\text{then} \quad \frac{4}{9}\pi r^3 &= 5{,}000\\ r^3 &= \frac{5{,}000 \times 9}{4\pi}\\ &= \frac{11{,}250}{\pi}.\end{aligned}$$

$$\begin{aligned}\text{Volume of cylinder} &= \pi r^2(2r)\\ &= 2\pi r^3\\ &= 2\pi \times \frac{11{,}250}{\pi}\\ &= 22{,}500\ \text{cm}^3 = 22.5\ \ell\end{aligned}$$

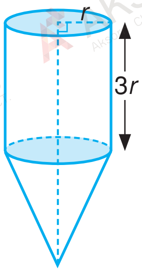
Therefore, the amount of water needed to fill the container completely is $22.5 + 5 = 27.5\ \ell$.

Practice Now

Similar Questions

Exercise 4D Questions 1-2, 4-5

A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $3r$ cm. Given that the height of the cone is three-quarters of the height of the cylinder and 10 liters of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in liters.



Height of cone $= \frac{3}{4} \times$ height of cylinder

$= \frac{3}{4} \times 3r$

$= \frac{9}{4}r$

Volume of cone $= \frac{1}{3}\pi r^2\left(\frac{9}{4}r\right)$

$= \frac{3}{4}\pi r^3$

Since volume of cone $= 10\ \ell$

$= 10{,}000\ \text{cm}^3$,

then $\frac{3}{4}\pi r^3 = 10{,}000$

$r^3 = \frac{10{,}000 \times 4}{3\pi}$

$= \frac{40{,}000}{3\pi}$.

Volume of cylinder $= \pi r^2(3r)$

$= 3\pi r^3$

$= 3\pi \times \frac{40{,}000}{3\pi}$

$= 40{,}000\ \text{cm}^3$

$= 40\ \ell$

Therefore, the amount of water needed to fill the container completely

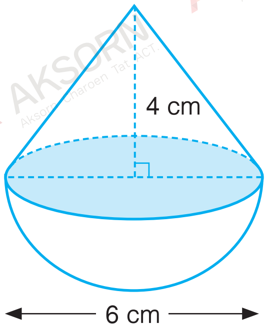
$= 40 + 10$

$= 50\ \ell$.

Worked Example 13

A solid consists of a cone and a hemisphere which share a common base. The cone has a height of 4 cm and a base diameter of 6 cm. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) Find the volume and the total surface area.
2) The solid is melted and recast to form a solid cylinder with a height of 4 cm. Find the radius of the cylinder.
3) If 1,000 identical cylinders are to be painted and each tin of paint is enough to paint an area of 5 m^2, find the number of tins of paint needed.



Solution:

1) Radius of cone $= $ radius of hemisphere

$= 6 \div 2$

$= 3$ cm

Volume of solid $=$ volume of cone $+$ volume of hemisphere

$$= \frac{1}{3} \times \pi \times 3^2 \times 4 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 3^3$$

$$= 12\pi + 18\pi$$

$$= 30\pi$$

$$\approx 94.3 \text{ cm}^3$$

By using the Pythagorean theorem,

slant height of cone $= \sqrt{3^2 + 4^2}$

$= 5$ cm.

Total surface area of solid $=$ curved surface area of cone $+$ curved surface area of hemisphere

$$= \pi \times 3 \times 5 + 2 \times \pi \times 3^2$$

$$= 15\pi + 18\pi$$

$$= 33\pi$$

$$\approx 103.71 \text{ cm}^2$$

2) Volume of cylinder $= \pi r^2(4) = 30\pi$

$$4\pi r^2 = 30\pi$$

$$r^2 = \frac{30}{4}$$

$$r = \sqrt{7.5}$$

$$\approx 2.74 \text{ cm}$$

3) Surface area of one cylinder $= 2 \times \pi \times (\sqrt{7.5})^2 + 2 \times \pi \times 2.74 \times 4$

$$= 15\pi + 21.92\pi$$

$$= 36.92\pi$$

$$\approx 116 \text{ cm}^2$$

Surface area of 1,000 cylinders $\approx 116{,}000 \text{ cm}^2$

$= 11.6 \text{ m}^2$

Since $\frac{11.6}{5} = 2.32$,

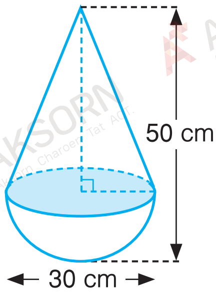
then 3 tins of paint are needed to paint 1,000 cylinders.

Practice Now

Similar Questions

Exercise 4D Questions 3, 6-9

A solid consists of a cone and a hemisphere which share a common base. The solid has a height of 50 cm, and the hemisphere has a diameter of 30 cm. Find the following. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



1) The volume and the total surface area of the solid

Radius of hemisphere = 30 ÷ 2 = 15 cm

Height of cone = 50 - 15 = 35 cm

Volume of solid

$= \text{volume of cone} + \text{volume of hemisphere}$

$= \frac{1}{3} \times \pi \times 15^2 \times 35 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 15^3$

$= 2{,}625\pi + 2{,}250\pi$

$= 4{,}875\pi$

$\approx 15{,}321.43 \text{ cm}^3$

By using the Pythagorean theorem,

slant height of cone $= \sqrt{15^2 + 35^2}$

$\approx 38.08 \text{ cm}.$

Total surface area of solid

$= \text{curved surface area of cone} + \text{curved surface area of hemisphere}$

$= \pi \times 15 \times 38.08 + 2 \times \pi \times 15^2$

$= 571.2\pi + 450\pi$

$= 1{,}021.2\pi$

$\approx 3{,}209.49 \text{ cm}^2$

2) The solid is melted and recast to form a solid cylinder with a radius of 12.5 cm. Find the height and the surface area of the cylinder.

Volume of cylinder $= 4{,}875\pi$

$\pi(12.5^2)h = 4{,}875\pi$

$h = \frac{4{,}875\pi}{156.25\pi}$

$= 31.2 \text{ cm}$

Therefore, the height of the cylinder is 31.2 cm.

Surface area of the cylinder $= 2\pi r^2 + 2\pi rh$

$= 2 \times \pi \times (12.5)^2 + 2 \times \pi \times 12.5 \times 31.2$

$= 312.5\pi + 780\pi$

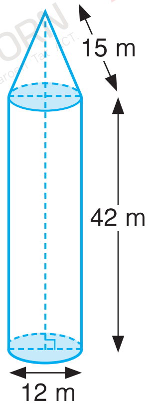
$= 1{,}092.5\pi$

$\approx 3{,}433.57 \text{ cm}^2$

Exercise 4D

Basic Level

1. A rocket in the shape of a cone is attached to a cylinder with the same base radius. The cone has a slant height of 15 m. The cylinder has a base diameter of 12 m and a height of 42 m. Find the total surface area of the rocket. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Radius of cylinder = 2 ÷ 2 = 6 m

Total surface area of rocket = flat surface of cylinder + curved surface area of cylinder + curved surface area of cone

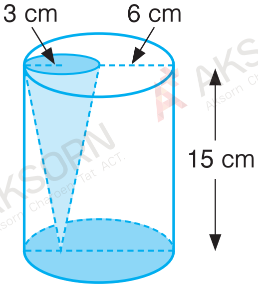
$= \pi \times 6^2 + 2 \times \pi \times 6 \times 42 + \pi \times 6 \times 15$

$= 36\pi + 504\pi + 90\pi$

$= 630\pi$

$\approx 1{,}980 \text{ m}^2$

2. A wooden cylinder has a radius of 6 cm and a height of 15 cm. A hole in the shape of a cone is bored into one of its end. If the cone has a radius of 3 cm as shown, find the volume of the remaining solid. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Volume of solid = volume of cylinder - volume of cone

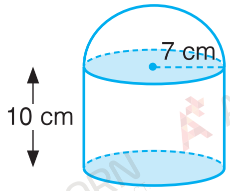
$= \pi \times 6^2 \times 15 - \frac{1}{3} \times \pi \times 3^2 \times 15$

$= 540\pi - 45\pi$

$= 495\pi$

$\approx 1{,}555.71 \text{ cm}^3$

3. A solid consists of a hemisphere and a cylinder which share a common base. The cylinder has a base radius of 7 cm and a height of 10 cm. Find the volume and the total surface area of the solid. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Volume of solid

$= $ volume of hemisphere + volume of cylinder

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 7^3 + \pi \times 7^2 \times 10$

$= 228\frac{2}{3}\pi + 490\pi$

$= 718\frac{2}{3}\pi$

$\approx 2{,}258.67 \text{ cm}^3$

Total surface area of solid

$= $ flat surface area of cylinder + curved surface area of cylinder + curved surface area of hemisphere

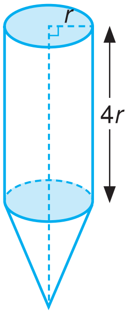
$= \pi \times 7^2 + 2 \times \pi \times 7 \times 10 + \frac{1}{2} \times 4 \times \pi \times 7^2$

$= 49\pi + 140\pi + 98\pi$

$= 287\pi \approx 902 \text{ cm}^2$

Intermediate Level

4. A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $4r$ cm. Given that the height of the cone is three-fifths of the height of the cylinder and 7 liters of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in liters.



Height of cone $= \frac{3}{5} \times$ height of cylinder

$= \frac{3}{5} \times 4r$

$= \frac{12}{5}r$

Volume of cone $= \frac{1}{3} \times \pi r^2 \left(\frac{12}{5}r\right)$

$= \frac{4}{5}\pi r^3$

Since volume of cone $= 7\ \ell$

$= 7{,}000 \text{ cm}^3,$

then $\frac{4}{5}\pi r^3 = 7{,}000$

$r^3 = \frac{7{,}000 \times 5}{4\pi}$

$= \frac{8{,}750}{\pi}.$

Volume of cylindrical container

$= \pi r^2(4r)$

$= 4\pi r^3$

$= 4\pi \times \frac{8{,}750}{\pi}$

$= 35{,}000 \text{ cm}^3$

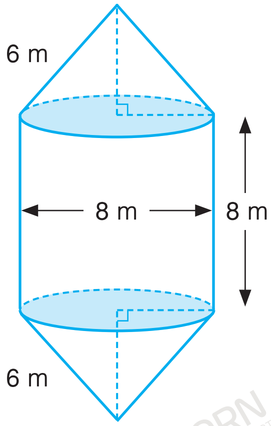
$= 35\ \ell$

Therefore, the amount of water needed

$= 35 + 7$

$= 42\ \ell.$

5. Find the total surface area and the volume of the solid cylinder with conical ends as shown. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Radius of cylinder = 8 ÷ 2 = 4 m

Total surface area of solid cylinder with conical ends

= 2 × curved surface area of cone + curved surface area of cylinder

$= 2 \times \pi \times 4 \times 6 + 2 \times \pi \times 4 \times 8$

$= 48\pi + 64\pi = 112\pi \approx 352 \text{ m}^2$

By using the Pythagorean theorem,

$h = \sqrt{6^2 - 4^2} \approx 4.47 \text{ m.}$

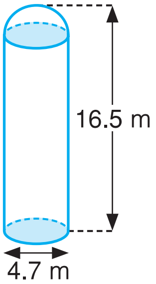
Volume of the solid cylinder with conical ends

= 2 × volume of cone + volume of cylinder

$= 2 \times \frac{1}{3} \times \pi \times 4^2 \times 4.47 + \pi \times 4^2 \times 8$

$\approx 552.14 \text{ m}^3$

6. A storage tank consists of a hemisphere and a cylinder which share a common base. The tank has a height of 16.5 m, and the cylinder has a base diameter of 4.7 m. Find the capacity of the tank. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Radius of cylinder = 4.7 ÷ 2 = 2.35 m

Height of cylinder = 16.5 - 2.35 = 14.15 m

Capacity of tank = volume of hemisphere + volume of cylinder

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times (2.35)^3 + \pi \times (2.35)^2 \times 14.15$

$\approx 272.79 \text{ m}^3$

7. A solid metal ball of radius 3 cm is melted and recast to form a solid circular cone of radius 4 cm. Find the height of the cone.

Volume of cone = volume of ball

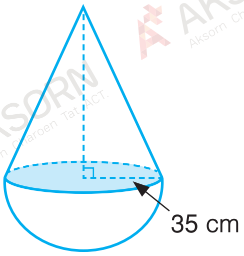
$\frac{1}{3} \times \pi \times 4^2 \times h = \frac{4}{3} \times \pi \times 3^3$

$\frac{16}{3}\pi h = 36\pi$

$h = 36 \times \frac{3}{16} = 6.75 \text{ cm}$

Advanced Level

8. A steel solid consists of a cone and a hemisphere which share a common base. The cone has a base radius of 35 cm. Given that the volume of the cone is equal to $1\frac{1}{5}$ of the volume of the hemisphere, find the following.



1) The height of the cone

$$\begin{aligned}
\text{Volume of cone} &= 1\frac{1}{5} \times \text{volume of hemisphere} \\
\frac{1}{3} \times \pi \times 35^2 \times h &= 1\frac{1}{5} \times \frac{1}{2} \times \frac{4}{3} \times \pi \times 35^3 \\
408\frac{1}{3}\pi h &= 34{,}300\pi \\
h &= 34{,}300 \div 408\frac{1}{3} \\
&= 84 \text{ cm}
\end{aligned}$$

2) The total surface area of the solid, leaving your answer in terms of π

By using the Pythagorean theorem, $\ell = \sqrt{84^2 + 35^2} = 91$ cm.

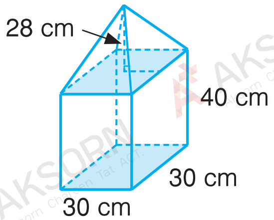
Total surface area of the solid

= curved surface area of cone + curved surface area of hemisphere

$= \pi \times 35 \times 91 + \frac{1}{2} \times \pi \times 4 \times 35^2$

$= 3{,}185\pi + 2{,}450\pi = 5{,}635\pi \text{ cm}^2$

9. A solid consists of a pyramid of height 28 cm attached to a cuboid with a square base of sides 30 cm and a height of 40 cm. Find the volume and the total surface area of the solid.



$$\begin{aligned}
\text{Volume of solid} &= \text{volume of pyramid} + \text{volume of cuboid} \\
&= \frac{1}{3} \times 30 \times 30 \times 28 + 30 \times 30 \times 40 \\
&= 44{,}400 \text{ cm}^3
\end{aligned}$$

By using the Pythagorean theorem, slant height of pyramid $= \sqrt{28^2 + 15^2} \approx 31.76$ cm.

$$\begin{aligned}
\text{Total surface area of solid} &= \text{total surface area of visible sides of cuboid} + \\
&\quad\ \text{total surface area of all triangular faces of pyramid} \\
&= (30 \times 30 + 4 \times 30 \times 40) + \left(4 \times \frac{1}{2} \times 30 \times 31.76\right) \\
&\approx 7{,}605.6 \text{ cm}^2
\end{aligned}$$

Summary

1. **A pyramid and its corresponding prism with the same base and the same height**

   Volume of solid $= \frac{1}{3} \times$ volume of corresponding prism

   $= \frac{1}{3} \times$ base area $\times$ height

2. **Total surface area of pyramid = total area of all faces**

3. **A cone and its corresponding cylinder with the same base radius r and the same height h**

   Volume of cone $= \frac{1}{3} \times$ volume of corresponding cylinder

   $= \frac{1}{3} \times$ base area $\times$ height

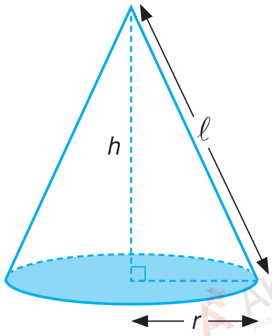
   $= \frac{1}{3}\pi r^2 h$

4. **A cone with base radius r and slant height ℓ**

   Curved surface area of cone $= \pi r\ell$

   Total surface area of cone $= \pi r\ell + \pi r^2$

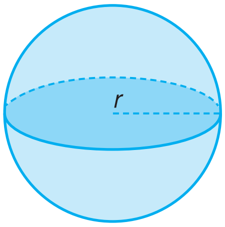
   $= \pi r(\ell + r)$



5. **A sphere with radius r**

   Volume of sphere $= \frac{4}{3}\pi r^3$

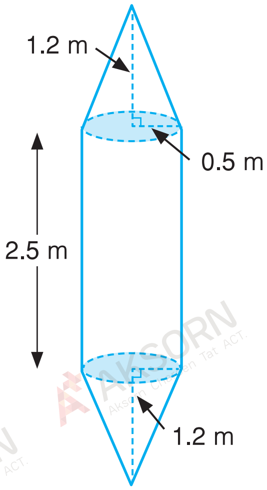
   Surface area of sphere $= 4\pi r^2$



Review Exercise 4

1. For each of the following solids, find the volume and the total surface area of the solid. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1)



Volume of solid $= 2 \times$ volume of cone $+$ volume of cylinder

$= 2 \times \frac{1}{3} \times \pi \times (0.5)^2 \times 1.2 + \pi \times (0.5)^2 \times 2.5$

$= \frac{1}{5}\pi + \frac{5}{8}\pi$

$= \frac{33}{40}\pi$

$\approx 2.59 \text{ m}^3$

By using the Pythagorean theorem,

slant height of cone $= \sqrt{1.2^2 + 0.5^2} = 1.3$ m.

Total surface area of solid

$=$ 2 × curved surface area of cone + curved surface area of cylinder

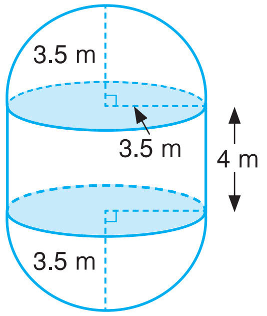
$= 2 \times \pi \times 0.5 \times 1.3 + 2 \times \pi \times 0.5 \times 2.5$

$= 1.3\pi + 2.5\pi$

$= 3.8\pi$

$\approx 11.94 \text{ m}^2$

2)



Volume of solid $= 2 \times$ volume of hemisphere $+$ volume of cylinder

$= 2 \times \frac{1}{2} \times \frac{4}{3} \times \pi \times (3.5)^3 + \pi \times (3.5)^2 \times 4$

$\approx 333.67 \text{ m}^3$

Total surface area of solid

$=$ 2 × curved surface area of hemisphere + curved surface area of cylinder

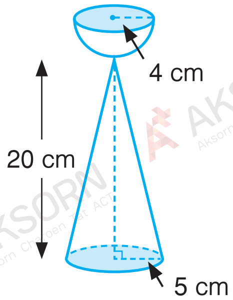
$= 2 \times \frac{1}{2} \times 4 \times \pi \times (3.5)^2 + 2 \times \pi \times 3.5 \times 4$

$= 49\pi + 28\pi$

$= 77\pi$

$\approx 242 \text{ m}^2$

2. The figure shows a decorative structure made up of a solid cone and a solid hemisphere. The hemisphere has a radius of 4 cm. The cone has a base radius of 5 cm and a height of 20 cm. Find the volume and the total surface area of the structure. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$



Volume of structure

$= $ volume of cone + volume of hemisphere

$= \frac{1}{3} \times \pi \times 5^2 \times 20 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 4^3$

$= 166\frac{2}{3}\pi + 42\frac{2}{3}\pi$

$= 209\frac{1}{3}\pi$

$\approx 658 \text{ cm}^3$

By using the Pythagorean theorem,

slant height $= \sqrt{20^2 + 5^2}$

≈ 20.62 cm.

Total surface area of structure

$=$ total surface area of cone + total surface area of hemisphere

$= (\pi \times 5^2 + \pi \times 5 \times 20.62)$

$+ \left(\pi \times 4^2 + \frac{1}{2} \times 4 \times \pi \times 4^2\right)$

$= 128.1\pi + 48\pi$

$= 176.1\pi$

$\approx 553.46 \text{ cm}^2$

3. A solid metal model of a rocket is made up of a cone attached to a cylinder with the same base radius. The cone has a height of 49 cm. The cylinder has a base radius of 18 cm and a height of 192 cm. Given that the mass of the model rocket is 2,145 kg, find the density, in kg/m^3, of the metal which the model rocket is made of. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$ $\left(\text{Density} = \frac{\text{Mass}}{\text{Volume}}\right)$

Volume of rocket $=$ volume of cone + volume of cylinder

$= \frac{1}{3} \times \pi \times 18^2 \times 49 + \pi \times 18^2 \times 192$

$= 5{,}292\pi + 62{,}208\pi$

$= 67{,}500\pi \text{ cm}^3$

$= 0.0675\pi \text{ m}^3$

Density of metal $= \frac{2{,}145}{0.0675\pi}$

$\approx 10{,}111.11 \text{ kg/m}^3$

4. Two solid spheres have surface areas of 144π and 256π cm^2, respectively. They are melted and recast to form a larger sphere. Find the surface area of the larger sphere in square centimeters. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Surface area of first sphere $= 144\pi$ cm^2

$4\pi r_1^2 = 144\pi$

$r_1^2 = 36$

$r_1 = \sqrt{36}\ (r_1 > 0)$

$= 6$ cm

Volume of first sphere

$= \frac{4}{3}\pi r_1^3 = \frac{4}{3} \times \pi \times 6^3 = 288\pi$ cm^3

Surface area of second sphere $= 256\pi$ cm^2

$4\pi r_2^2 = 256\pi$

$r_2^2 = 64$

$r_2 = \sqrt{64}\ (r_2 > 0)$

$= 8$ cm

Volume of second sphere $= \frac{4}{3}\pi r_2^3$

$= \frac{4}{3} \times \pi \times 8^3$

$= 682\frac{2}{3}\pi$ cm^3

Volume of larger sphere $= 288\pi + 682\frac{2}{3}\pi$

$\frac{4}{3}\pi R^3 = 970\frac{2}{3}\pi$

$R^3 = 728$

$R = \sqrt[3]{728}$

≈ 8.996 cm

Surface area of larger sphere $= 4\pi R^2$

$= 4 \times \pi \times (8.996)^2$

$\approx 1{,}017.38$ cm^2

5. A hollow metal sphere has an external diameter of 12 cm and a thickness of 2 cm. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) Given that the mass of 1 cm^3 of the metal is 5.4 g, find the mass of the hollow sphere in kilograms.

External radius $= 12 \div 2 = 6$ cm

Internal diameter $= 12 - 2 - 2 = 8$ cm

Internal radius $= 8 \div 2 = 4$ cm

Volume of hollow sphere $= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$

$= \left(\frac{4}{3} \times \pi \times 6^3\right)$

$- \left(\frac{4}{3} \times \pi \times 4^3\right)$

$= 288\pi - 85\frac{1}{3}\pi$

$= 202\frac{2}{3}\pi$ cm^3

Mass of hollow sphere $= 202\frac{2}{3}\pi \times 5.4$

$= 3{,}439.54$ g

≈ 3.44 kg

2) The hollow sphere is melted and recast to form a solid sphere. Find the radius of the solid sphere.

Volume of solid sphere $= 202\frac{2}{3}\pi$ cm^3

$\frac{4}{3} \times \pi \times r^3 = 202\frac{2}{3}\pi$

$r^3 = 152$

$r = \sqrt[3]{152} \approx 5.34$ cm

6. A house has a hemisphere roof of 10 m in diameter. Find the cost of painting the curved surface of the roof at 42 baht per square meter. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

Radius of hemisphere roof = 10 ÷ 2 = 5 m

Curved surface area of hemisphere roof

$= \frac{1}{2} \times 4\pi r^2 = 2 \times \pi \times 5^2$

$= 50\pi \text{ m}^2$

Cost of painting $= 50\pi \times 42$

$= 6{,}600$ baht

7. A metal hemisphere bowl has an external diameter of 50.8 cm and a thickness of 2.54 cm. $\left(\text{Take } \pi \approx \frac{22}{7}.\right)$

1) Given that the empty bowl weighs 97.9 kg, find the density, in kg/m^3, of the metal which the bowl is made of.

External radius = 50.8 ÷ 2 = 25.4 cm

Internal diameter = 50.8 - 2.54 - 2.54 = 45.72 cm

Internal radius = 45.72 ÷ 2 = 22.86 cm

Volume of metal hemispherical bowl $= \frac{1}{2} \times \frac{4}{3}\pi R^3 - \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times (25.4)^3 - \frac{1}{2} \times \frac{4}{3} \times \pi \times (22.86)^3$

$\approx 9{,}304.73 \text{ cm}^3$

$\approx 0.0093 \text{ m}^3$

Density of metal $= \frac{97.9}{0.0093} \approx 10{,}526.88 \text{ kg/m}^3$

2) If the bowl is completely filled with a liquid of density 31.75 kg/m^3, find the mass of the liquid in grams.

Volume of liquid in the bowl $= \frac{1}{2} \times \frac{4}{3}\pi R^3$

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times (22.86)^3$

$\approx 25{,}030 \text{ cm}^3$

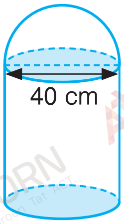
$= 0.02503 \text{ m}^3$

Mass of liquid = 31.75 × 0.02503

≈ 0.795 kg

= 795 g

8. The figure shows a solid cylindrical stone pillar with a hemisphere top. The pillar has a diameter of 40 cm. If the pillar has the same mass as a solid stone sphere of the same material of radius 40 cm, find the height of the pillar.



Radius of pillar = 40 ÷ 2 = 20 cm

Since the pillar has the same mass as a solid stone sphere of the same material, then the pillar has the same volume as the solid stone sphere.

$$\text{Volume of solid stone sphere} = \frac{4}{3}\pi R^3$$

$$= \frac{4}{3} \times \pi \times 40^3$$

$$= \frac{256{,}000}{3}\pi \text{ cm}^3.$$

$$\text{Volume of pillar} = \frac{256{,}000}{3}\pi \text{ cm}^3$$

$$\pi \times 20^2 \times h + \frac{1}{2} \times \frac{4}{3} \times \pi \times 20^3 = \frac{256{,}000}{3}\pi$$

$$400\pi h + \frac{16{,}000}{3}\pi = \frac{256{,}000}{3}\pi$$

$$400\pi h = 80{,}000\pi$$

$$h = 200 \text{ cm}$$

9. A cylinder and a cone are of the same height $2r$ and of the same base diameter $2r$. A sphere has a diameter of $2r$. Find the ratio of the volume of the cylinder to that of the cone and to that of the sphere.

Radius of cylinder and cone = $2r \div 2 = r$ units

Radius of sphere = $2r \div 2 = r$ units

Volume of cylinder = $\pi \times r^2 \times 2r = 2\pi r^3$ units^3

Volume of cone = $\frac{1}{3} \times \pi \times r^2 \times 2r = \frac{2}{3}\pi r^3$ units^3

Volume of sphere = $\frac{4}{3} \times \pi \times r^3 = \frac{4}{3}\pi r^3$ units^3

Ratio of volume of cylinder to volume of cone to volume of sphere

$= 2\pi r^3 : \frac{2}{3}\pi r^3 : \frac{4}{3}\pi r^3$

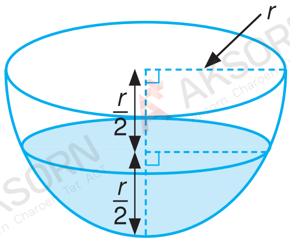
$= 2 : \frac{2}{3} : \frac{4}{3}$

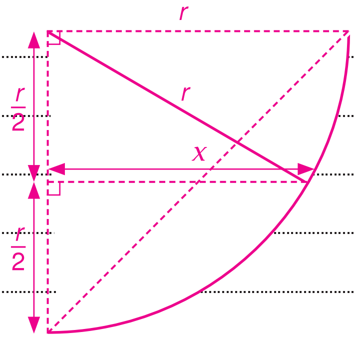
$= 6 : 2 : 4$

$= 3 : 1 : 2$

Challenge Yourself

The figure shows a bowl in the shape of a hemisphere with a radius of r cm. It is filled with water to a depth of $\frac{r}{2}$ cm. Show that the volume of the water is more than $\frac{1}{8}$ of the volume of the bowl.





By using the Pythagorean theorem,

$$x = \sqrt{r^2 - \left(\frac{r}{2}\right)^2}$$

$$= \frac{\sqrt{3}}{2}r.$$

Since $x = \frac{\sqrt{3}}{2}r > \frac{r}{2}$, then the water in the bowl is not in the shape of a hemisphere.

However, the volume of water is more than that of the hemisphere with a radius of $\frac{r}{2}$ cm.

The volume of water > the volume of the hemisphere with a radius of $\frac{r}{2}$ cm

$$= \frac{1}{2} \times \frac{4}{3} \times \pi \times \left(\frac{r}{2}\right)^3$$

$$= \frac{1}{2} \times \frac{4}{3} \times \pi \times \frac{r^3}{8}$$

$$= \frac{1}{8} \times \frac{1}{2} \times \frac{4}{3} \times \pi \times r^3$$

$$= \frac{1}{8} \times \text{volume of bowl}$$

Therefore, the volume of water in the bowl is $\frac{1}{8}$ times that of the bowl.

Chapter 5

Statistics

In Thailand, the daily mean maximum temperature was 27.9°C and 28.1°C in 2015 and 2016, respectively. Based on these statistics, can we conclude that the temperature for both years were about the same? To have a better understanding of the temperatures of both years, we should also know how the temperatures were spread throughout the year. In this chapter, we will learn how to measure the spread of a set of data using box-and-whisker plots.

Indicator

- Understand and apply statistics to displaying and analyzing data from box-and-whisker plots, including interpreting results, and to solve real-life problems with a suitable technology. **(MA 3.1 G. 9/1)**

Compulsory Details

- Data and data analysis:
  - box-and-whisker plots
- Interpretation of results
- Real-life applications of statistics

5.1 Mean, Quartiles, Range and Interquartile Range

In Secondary 2, we have learned how to find the median of a set of data. The median is a measure of the average and is the middle value when the data are arranged in an ascending order. In this section, we will learn how to find the quartiles, range and interquartile range for discrete data.

Discrete data refers to a set of data which only takes on distinct values. For example, a data set showing the number of phone calls received in a day can only take on distinct values, e.g. 1, 5, and 12, but not 1.5, $4\frac{2}{3}$, etc.

Consider the following set of distinct data arranged in ascending order:

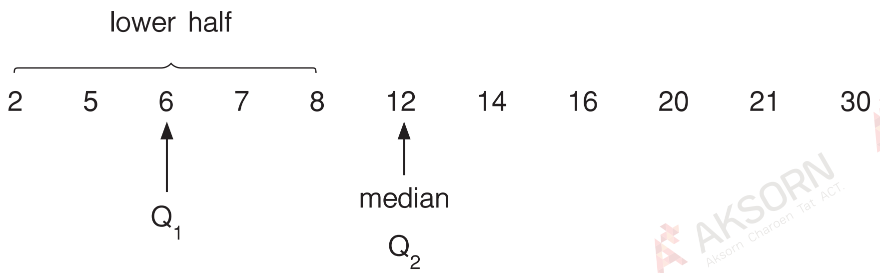
Set A: 2 5 6 7 8 12 14 16 20 21 30

The total number of data values is 11, i.e. n = 11.

We have learned that for a set of discrete data, the median is the value of the data in the middle position, i.e. the 6^{th} position as follows:

$$2 \quad 5 \quad 6 \quad 7 \quad 8 \quad \underset{\substack{\uparrow \\ \text{median}}}{\boxed{12}} \quad 14 \quad 16 \quad 20 \quad 21 \quad 30$$

We see that the median 12 divides the data in 2 equal halves, with 5 values on each side of the median. We consider the 5 values on the left of the median. The middle value of these 5 values is 6, and it is called the lower quartile or **the first quartile (Q_1)**.



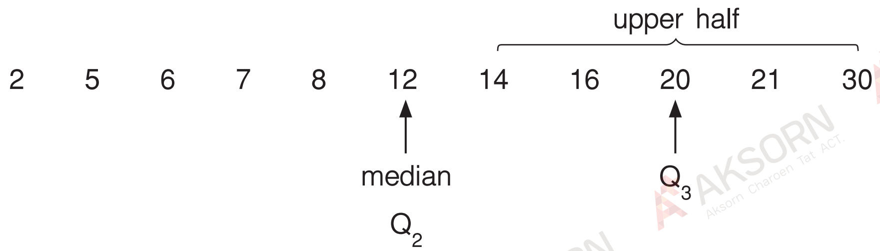
The first quartile (Q_1) can be considered as the first-quarter value. 25% (or one quarter) of the data is less than or equal to this value.

Since the median is the middle value or second-quarter value, the median is also called **the second quartile (Q_2)**. 50% (or half) of the data is less than or equal to this value.

Similarly, we consider the 5 values in the right of the median. The middle value of the 5 values is 20, and it is called the upper quartile or **the third quartile (Q_3)**. 75% (or three quarters) of the data is less than or equal to this value.

INTERNET RESOURCES

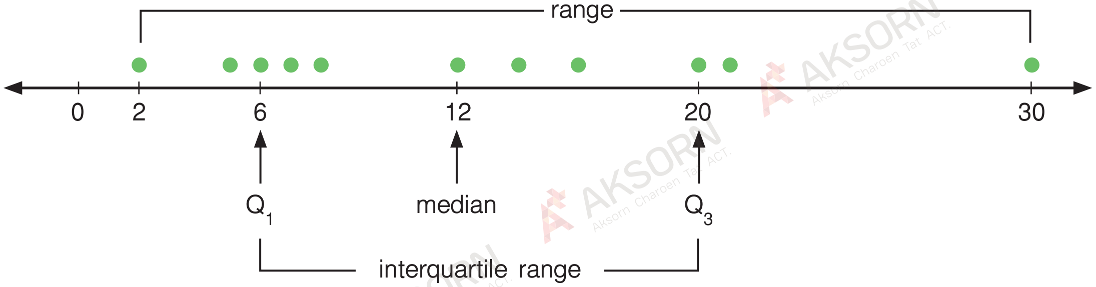
There are other formulas and methods to find the lower quartile, Q_1 and the upper quartile, Q_3. Search on the Internet for more information. But, we will use the method shown in the textbook.



We see that the quartiles obtained by the above method divide the data, which is arranged in ascending order into 4 roughly equal parts.

Now that we have learned how to find the median and quartiles for a given set of data, we shall learn how to measure the spread of the data by using the **range** and the **interquartile range**.

According to Set A, the median, Q_1, Q_3, range and the interquartile range are indicated in the dot diagram as shown.



These measures of spread show the degree of variation or how spread out the data values are, which use the range and the interquartile range, i.e. the more range and the more interquartile range the data have, the more the data spread out.

For Set A, we can find the range and the interquartile range as follows:

Range = largest value - smallest value

= 30 - 2

= 28

Interquartile range = $Q_3 - Q_1$

= 20 - 6

= 14

ATTENTION

The interquartile range is the range of the middle 50% of the data.

The interquartile range is a better measure of the spread of the data than the range because it tells us how the middle 50% of the data are distributed. The range only consists of the difference between the largest and the smallest values of the set of data.

The interquartile range is not affected by extreme values as it does not consider the behavior of the data that is lower than Q_1 (the lowest 25%) and higher than Q_3 (the highest 25%).

Worked Example 1

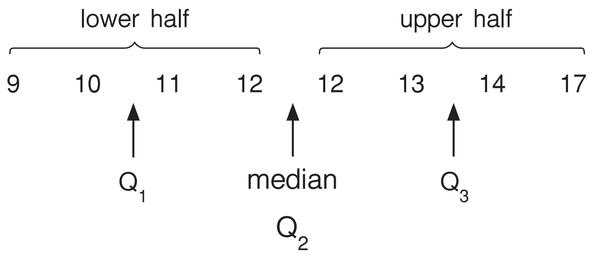
The following set of data shows the number of marks scored by 8 students where there are 20 questions on the exam paper.

10	12	12	13	9	17	11	14

1) For the given set of data, find Q_1, Q_2 and Q_3.
2) Find the range.
3) Find the interquartile range.

Solution:

1) Arrange the given data in ascending order.



We get n = 8.

$Q_2 = \frac{12 + 12}{2} = 12$ marks *(When n is an even number, the median will be equal to the average of the two middle values.)*

$Q_1 = \frac{10 + 11}{2} = 10.5$ marks *(When the number of data that is less than or equal to the median is even, Q_1 is the average of the two middle values.)*

$Q_3 = \frac{13 + 14}{2} = 13.5$ marks *(When the number of data that is more than or equal to the median is even, Q_3 is the average of the two middle values.)*

2) Range = 17 - 9
= 8 marks

3) Interquartile range = $Q_3 - Q_1$
= 13.5 - 10.5
= 3 marks

Similar Questions

Exercise 5A Questions 1-5

Practice Now

The following set of data shows the number of sit-ups done by 10 students during a physical fitness test.

12	22	36	10	14	45	59	44	38	25

1) For the given set of data, find Q_1, Q_2 and Q_3.

Arrange the given data in ascending order.

$$\overbrace{10 \quad 12 \quad \underset{\underset{Q_1}{\uparrow}}{14} \quad 22 \quad 25}^{\text{lower half}} \quad \underset{\underset{Q_2}{\uparrow}}{} \quad \overbrace{36 \quad 38 \quad \underset{\underset{Q_3}{\uparrow}}{44} \quad 45 \quad 59}^{\text{upper half}}$$

For the given data, $n = 10$.

Therefore, $Q_2 = \frac{25 + 36}{2} = 30.5$

$Q_1 = 14$

$Q_3 = 44$

2) Find the range.

Range $= 59 - 10$

$= 49$

3) Find the interquartile range.

Interquartile range $= Q_3 - Q_1$

$= 44 - 14$

$= 30$

Exercise 5A

Basic Level

1. Find Q_1, Q_2, Q_3, range and interquartile range for the following sets of data.

1) 7, 6, 4, 8, 2, 5, 10

Arrange the given data in ascending order.

$$\overbrace{2\quad 4\quad 5}^{\text{lower half}}\quad 6\quad \overbrace{7\quad 8\quad 10}^{\text{upper half}}$$

$$\underset{Q_1}{\uparrow}\ \text{at } 4,\qquad \underset{Q_2}{\uparrow}\ \text{at } 6,\qquad \underset{Q_3}{\uparrow}\ \text{at } 8$$

For the given data, $n = 7$.

Therefore, $Q_2 = 6$

$Q_1 = 4$

$Q_3 = 8$

Range $= 10 - 2$

$= 8$

Interquartile range $= Q_3 - Q_1$

$= 8 - 4$

$= 4$

2) 63, 80, 54, 70, 51, 72, 64, 66

Arrange the given data in ascending order.

$$\overbrace{51\quad 54\quad 63\quad 64}^{\text{lower half}}\quad \overbrace{66\quad 70\quad 72\quad 80}^{\text{upper half}}$$

$$51\quad 54\ \underset{Q_1}{\uparrow}\ 63\quad 64\ \underset{Q_2}{\uparrow}\ 66\quad 70\ \underset{Q_3}{\uparrow}\ 72\quad 80$$

For the given data, $n = 8$.

Therefore, $Q_2 = \frac{64 + 66}{2} = 65$

$Q_1 = \frac{54 + 63}{2} = 58.5$

$Q_3 = \frac{70 + 72}{2} = 71$

Range $= 80 - 51$

$= 29$

Interquartile range $= Q_3 - Q_1$

$= 71 - 58.5$

$= 12.5$

3) 14, 18, 22, 10, 27, 32, 40, 16, 9

Arrange the given data in ascending order.

$$\overbrace{9 \quad 10 \quad \underset{\underset{Q_1}{\uparrow}}{} \quad 14 \quad 16}^{\text{lower half}} \quad \underset{\underset{Q_2}{\uparrow}}{18} \quad \overbrace{22 \quad 27 \quad \underset{\underset{Q_3}{\uparrow}}{} \quad 32 \quad 40}^{\text{upper half}}$$

For the given data, $n = 9$.

Therefore, $Q_2 = 18$

$Q_1 = \frac{10 + 14}{2} = 12$

$Q_3 = \frac{27 + 32}{2} = 29.5$

Range = 40 - 9 = 31

Interquartile range = $Q_3 - Q_1$

= 29.5 - 12

= 17.5

4) 138, 164, 250, 184, 102, 244, 168, 207, 98, 86

Arrange the given data in ascending order.

$$\overbrace{86 \quad 98 \quad \underset{\underset{Q_1}{\uparrow}}{102} \quad 138 \quad 164}^{\text{lower half}} \quad \underset{\underset{Q_2}{\uparrow}}{} \quad \overbrace{168 \quad 184 \quad \underset{\underset{Q_3}{\uparrow}}{207} \quad 244 \quad 250}^{\text{upper half}}$$

For the given data, $n = 10$.

Therefore, $Q_2 = \frac{164 + 168}{2} = 166$

$Q_1 = 102$

$Q_3 = 207$

Range = 250 - 86 = 164

Interquartile range = $Q_3 - Q_1$

= 207 - 102

= 105

5) 10.4, 8.5, 13.1, 11.8, 6.7, 22.4, 4.9, 2.7, 15.1

Arrange the given data in ascending order.

$$\overbrace{2.7 \quad 4.9 \quad \underset{\underset{Q_1}{\uparrow}}{} \quad 6.7 \quad 8.5}^{\text{lower half}} \quad \underset{\underset{Q_2}{\uparrow}}{10.4} \quad \overbrace{11.8 \quad 13.1 \quad \underset{\underset{Q_3}{\uparrow}}{} \quad 15.1 \quad 22.4}^{\text{upper half}}$$

For the given data, $n = 9$.

Therefore, $Q_2 = 10.4$

$Q_1 = \frac{4.9 + 6.7}{2} = 5.8$

$Q_3 = \frac{13.1 + 15.1}{2} = 14.1$

Range = 22.4 - 2.7 = 19.7

Interquartile range = $Q_3 - Q_1$

= 14.1 - 5.8

= 8.3

2. **The following set of data shows the number of distinctions scored by 10 classes for a particular examination. Each class has 40 students.**

0	1	6	9	24	0	27	6	9	29

From the given data, find the following:

1) Find Q_1, Q_2 and Q_3.

Arrange the given data in ascending order.

$$\overbrace{0 \quad 0 \quad \underset{\substack{\uparrow \\ Q_1}}{1} \quad 6 \quad 6}^{\text{lower half}} \quad \underset{\substack{\uparrow \\ Q_2}}{} \quad \overbrace{9 \quad 9 \quad \underset{\substack{\uparrow \\ Q_3}}{24} \quad 27 \quad 29}^{\text{upper half}}$$

For the given data, $n = 10$.

Therefore, $Q_2 = \frac{6 + 9}{2} = 7.5$

$Q_1 = 1$

$Q_3 = 24$

2) Find the range and the interquartile range.

Range = 29 - 0 = 29 people

Interquartile range $= Q_3 - Q_1$

$= 24 - 1$

$= 23$ people

3. **The stem-and-leaf diagram below shows the math quiz marks of 20 students.**

Stem	Leaf
0	9
1	2
2	1 2 8
3	0
4	0 1 1 2 8 9
6	0
7	2 3 9
8	7 7 8
9	5

Key: 0 | 9 means 9 marks.

Find the following:

1) The median mark

Median mark $= \frac{42 + 48}{2}$

$= 45$

2) The range

Range = 95 - 9

= 86

3) The interquartile range

$Q_3 = \frac{73 + 79}{2} = 76$

$Q_1 = \frac{28 + 30}{2} = 29$

Therefore, the interquartile range of the math quiz marks = 76 - 29 = 47.

Intermediate Level

4. **The Pollutant Standard Index (PSI) is a measure of the air pollution.**

[Note: 0-50 Good, 51-100 Moderate, 101-200 Unhealthy.]

City A

76	68	153	122	91
112	124	89	110	167

City B

15	44	55	77	51
80	31	29	24	19

1) For each city, find the range, the median and the interquartile range of the PSI.

For City A:

Arrange the given data in ascending order.

$$\overbrace{68\quad 76\quad \underset{\underset{Q_1}{\uparrow}}{89}\quad 91\quad 110}^{\text{lower half}}\ \underset{\underset{Q_2}{\uparrow}}{\,}\ \overbrace{112\quad 122\quad \underset{\underset{Q_3}{\uparrow}}{124}\quad 153\quad 167}^{\text{upper half}}$$

For the given data, n = 10.

Therefore, $Q_2 = \frac{110 + 112}{2}$

$= 111$ PSI

$Q_1 = 89$ PSI

$Q_3 = 124$ PSI

Range = 167 - 68 = 99 PSI

Median = 111 PSI

Interquartile range = 124 - 89 = 35 PSI

For City B:

Arrange the given data in ascending order.

$$\overbrace{15\quad 19\quad \underset{\underset{Q_1}{\uparrow}}{24}\quad 29\quad 31}^{\text{lower half}}\ \underset{\underset{Q_2}{\uparrow}}{\,}\ \overbrace{44\quad 51\quad \underset{\underset{Q_3}{\uparrow}}{55}\quad 77\quad 80}^{\text{upper half}}$$

For the given data, n = 10.

Therefore, $Q_2 = \frac{31 + 44}{2}$

$= 37.5$ PSI

$Q_1 = 24$ PSI

$Q_3 = 55$ PSI

Range = 80 - 15 = 65 PSI

Median = 37.5 PSI

Interquartile range = 55 - 24 = 31 PSI

2) Which data set shows a greater spread?

City A shows a greater spread.

3) Compare briefly the level of air pollution in the two cities.

The air pollution of City A is worse than City B since the median PSI for City A is much higher than that of City B.

Advanced Level

5. The stem-and-leaf diagram below represents the science quiz marks of 24 students.

Stem	Leaf
0	7
1	2
2	4 8
3	0 4 5 9
4	3 5 6 6 6 7 9
5	0 1
6	2 2 6
7	8
8	3 5 7

Key: 0 | 7 means 7 marks.

1) Find the median mark, the range and the interquartile range.

Median mark = $\frac{46 + 46}{2}$ = 46 marks

Range = 87 - 7 = 80 marks

Q_1 = $\frac{34 + 35}{2}$ = 34.5 marks

Q_3 = $\frac{62 + 62}{2}$ = 62 marks

Interquartile range = 62 - 34.5

= 27.5 marks

2) It was discovered that the marks were tallied incorrectly. The correct marks were all 3 marks more than those recorded. Explain how the median is affected by this error.

The correct median is now 46 + 3 = 49 marks

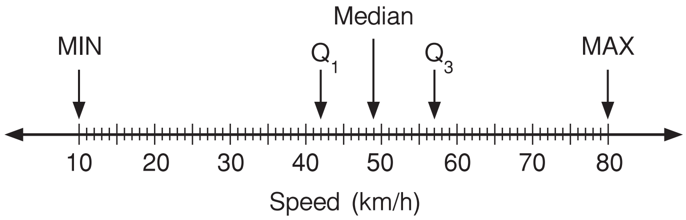
5.2 Box-and-Whisker Plots

In this section, we will learn how to draw and interpret a box-and-whisker plot, which is another way to show the distribution of a set of data.

Let us look at the following information for the speeds of 100 motor vehicles.

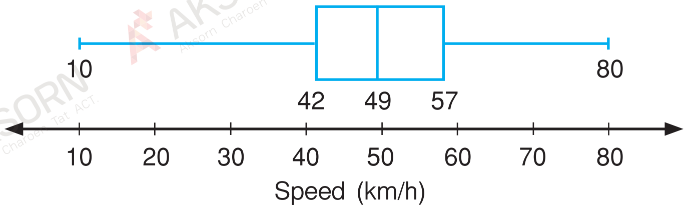
- The maximum speed is 80 km/h.
- The minimum speed is 10 km/h.
- The median speed is 49 km/h.
- The lower quartile (Q_1) is 42 km/h.
- The upper quartile (Q_3) is 57 km/h.

We can present this information on a box-and-whisker plot. To begin, we draw a horizontal number line using a suitable scale. The number line must be long enough to contain all the data points. On top of the number line, the positions of the MIN (minimum speed), the MAX (maximum speed) and the quartiles are indicated as shown.



A rectangular box is drawn above the number line, with the left side at the lower quartile (Q_1) and the right side at the upper quartile (Q_3). A vertical line is then drawn inside the box to indicate the median. This rectangular box represents **the box** of a box-and-whisker plot.

Above the number line, the MIN and the MAX are marked. Two line segments are then drawn to connect the MIN and the MAX to the sides of the box. These two line segments represent **the whiskers** of a **box-and-whisker plot** as shown below:



From the box-and-whisker plot, we have:

Range = MAX - MIN and Interquartile range = $Q_3 - Q_1$

= 80 - 10 = 57 - 42

= 70 km/h = 15 km/h

A box-and-whisker plot is a way of summarizing a set of data. If we are interested in only the five values (i.e. min, max, Q_1, Q_2 and Q_3), then we use the box-and-whisker plot. When comparing two sets of data, it is easier to use the box-and-whisker plot because we will usually compare only the medians and the interquartile ranges.

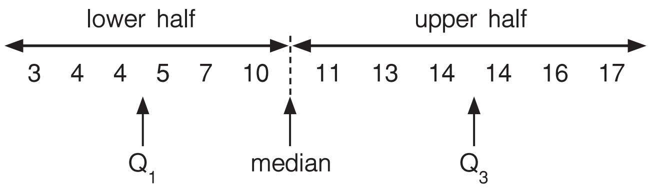
Worked Example 2

Draw a box-and-whisker plot for the given set of data.

10 4 3 16 14 13 4 7 11 5 17 14

Solution:

Arranging the given data in ascending order:



For the given data, n = 12, MIN = 3 and MAX = 17.

Therefore, the median = $\frac{10 + 11}{2}$ = 10.5.

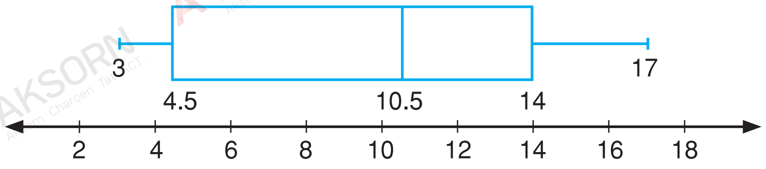
$Q_1 = \frac{4 + 5}{2} = 4.5$ *(When the number of data that is less than or equal to the median is even, Q_1 is the average of the two middle values.)*

$Q_3 = \frac{14 + 14}{2} = 14$ *(When the number of data that is more than or equal to the median is even, Q_3 is the average of the two middle values.)*

INTERNET RESOURCES

Search on the Internet for how to draw a box-and-whisker plot using Microsoft Excel. Then, try drawing a box-and-whisker plot based on the given data in the example.

The box-and-whisker plot is drawn below.



Similar Questions
Exercise 5B Question 1

Practice Now

Draw a box-and-whisker plot for the given set of data.

20 14 23 9 7 13 29 9 16

Arrange the given data in ascending order.

$$\begin{array}{ccccccccccccccccc} 7 & & 9 & & 9 & & 13 & & 14 & & 16 & & 20 & & 23 & & 29 \\ & & & \uparrow & & & & & \uparrow & & & & & \uparrow & & & \\ & & & Q_1 & & & & & Q_2 & & & & & Q_3 & & & \end{array}$$

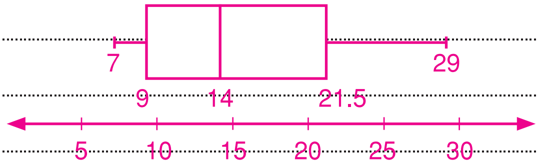
For the given data, $n = 9$.

Therefore, $Q_2 = 14$

$$Q_1 = \frac{9 + 9}{2} = 9$$

$$Q_3 = \frac{20 + 23}{2} = 21.5$$

The box-and-whisker plot is drawn below.



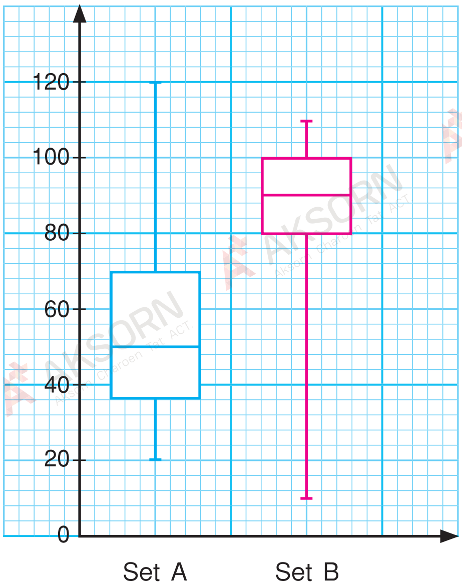
Class Discussion

Vertical box-and-whisker plots

Box-and-whisker plots can also be drawn vertically. The following table shows the summary statistics for two sets of data, A and B.

	Set A	Set B
MIN	20	10
MAX	120	110
Q_1	36	80
Median	50	90
Q_3	70	100

The table shows the box-and-whisker plot, which is drawn vertically for the data in Set A.



1) On the square grid and scale given, draw a vertical box-and-whisker plot for the data in Set B.

2) What do the heights of the rectangular boxes represent? Compare the heights of the two rectangular boxes corresponding to the data in Set A and Set B.

The heights of the rectangular boxes represent the interquartile range of the data, in which the heights of the boxes in Set A are more than those in Set B.

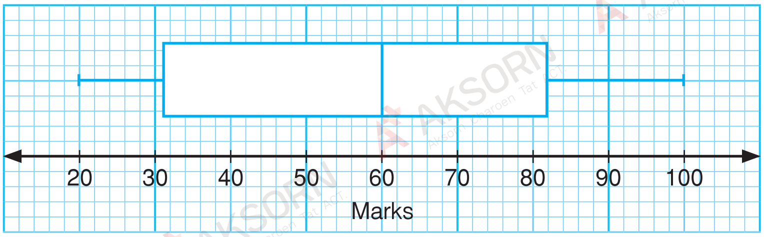
3) From the heights of the rectangular boxes, what can we infer about the spread of the data in Set A and Set B?

Since the heights of the boxes in Set A are more than those in Set B, the interquartile range of Set A is more than that of Set B, i.e. the data in Set A spreads out more.

From **Class Discussion**, we have learned that box-and-whisker plots can also be drawn vertically. Box-and-whisker plots give us a visualization of the spread of a set of data and also facilitate comparisons between two or more sets of data.

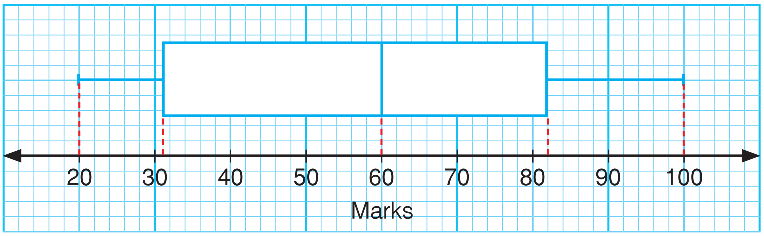
Worked Example 3

A class of students took an English proficiency test. The results are represented by a box-and-whisker plot as shown below.



1) State the median mark.
2) Find the range of the marks of the class.
3) Find the interquartile range of the marks.

Solution:



From the box-and-whisker plot:

1) The median score is 60 marks.

2) Range = MAX - MIN

= 100 - 20

= 80 marks

3) Interquartile range = $Q_3 - Q_1$

= 82 - 31

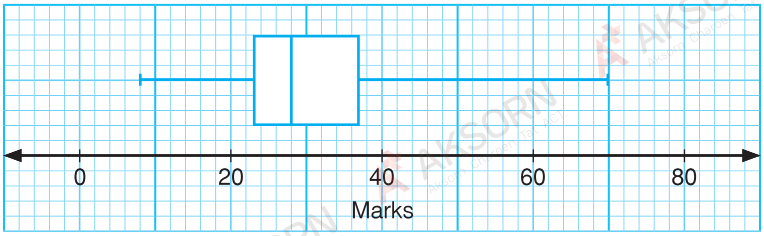
= 51 marks

Practice Now

Similar Questions

Exercise 5B Questions 2-5

A class of 50 students took a social studies test. The results are represented by a box-and-whisker plot as shown below. The maximum mark of the test is 80.



1) Find the median mark.

The median mark is 28.

2) Find the range.

Range = MAX - MIN

= 70 - 8

= 62 marks

3) Find the interquartile range.

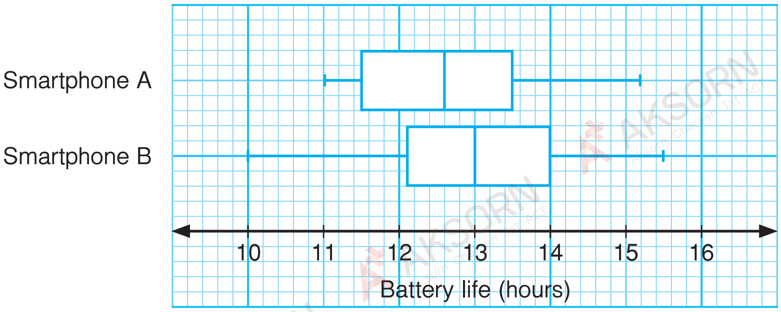
Interquartile range = $Q_3 - Q_1$

= 37 - 23

= 14 marks

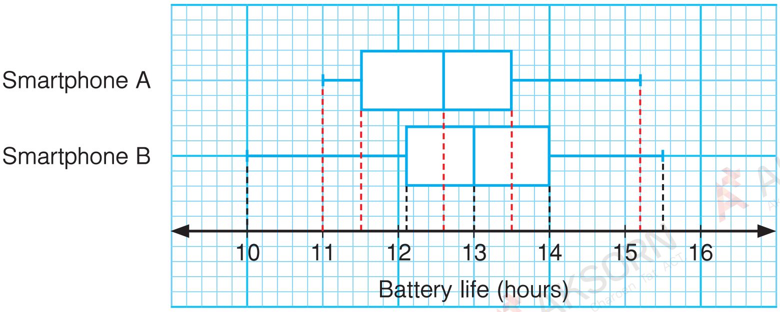
Worked Example 4

The box-and-whisker plots show the distribution of the battery life (hours) of two brands of smartphones, Smartphone A and Smartphone B. 150 smartphones of each type were fully charged and tested for their battery lives.



1) For Smartphone A, use the diagram to find the range, the median and the interquartile range.
2) For Smartphone B, use the diagram to find the range, the median and the interquartile range.
3) Which brand of smartphone has a longer battery life on average? State a reason.

Solution:



1) For Smartphone A,

Range = MAX - MIN

= 15.2 - 11

= 4.2 hours

Median = 12.6 hours

Interquartile range = $Q_3 - Q_1$

= 13.5 - 11.5

= 2 hours

2) For Smartphone B,

Range = MAX - MIN

= 15.5 - 10

= 5.5 hours

Median = 13 hours

Interquartile range = $Q_3 - Q_1$

= 14 - 12.1

= 1.9 hours

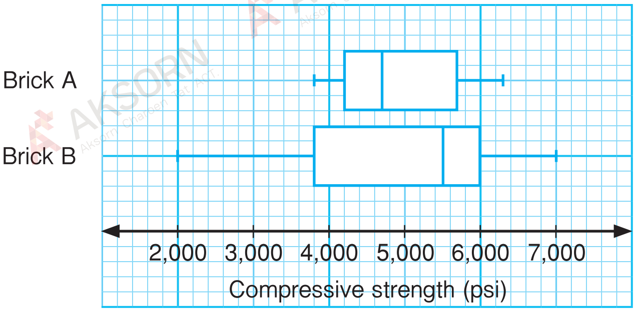
3) Smartphone B has a longer median battery life because the median of Smartphone B is more than that of Smartphone A.

Similar Questions

Exercise 5B Questions 6-9

Practice Now

A developer can choose between two different types of bricks for the construction of a new shopping complex. The box-and-whisker plots show the results of tests on the compressive strength of 200 bricks, measured in pounds per square inch (psi) of the two types of bricks. The higher the value of the psi, the stronger the brick.



1) For Brick A, find the range, the median and the interquartile range.

For Brick A,

Range = MAX - MIN

= 6,300 - 3,800

= 2,500 psi

Median = 4,700 psi

Interquartile range = $Q_3 - Q_1$

= 5,700 - 4,200

= 1,500 psi

2) For Brick B, find the range, the median and the interquartile range.

For Brick B,

Range = MAX - MIN

= 7,000 - 2,000

= 5,000 psi

Median = 5,500 psi

Interquartile range = $Q_3 - Q_1$

= 6,000 - 3,800

= 2,200 psi

3) On average, which type of brick is stronger? State a reason to support your answer.

Brick B because its median compressive strength (psi) is higher as compared to that of Brick A.

Exercise 5B

Basic Level

1. Draw a box-and-whisker plot for each of the following sets of data.

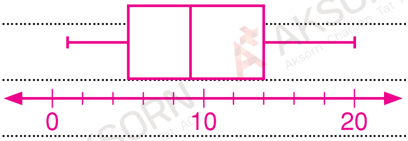
1) 1, 14, 9, 8, 20, 11, 5

Arrange the given data in ascending order.

1 5 8 9 11 14 20

5 ↑ Q_1, 9 ↑ Q_2, 14 ↑ Q_3

The box-and-whisker plot is drawn below.



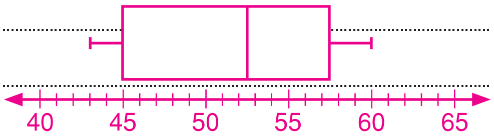
2) 45, 51, 57, 43, 45, 60, 58, 54

Arrange the given data in ascending order.

43 45 45 51 54 57 58 60

Q_1 between 45 and 45, Q_2 between 51 and 54, Q_3 between 57 and 58

The box-and-whisker plot is drawn below.



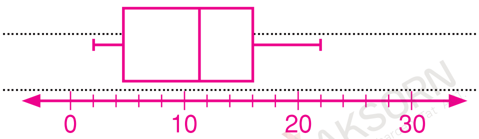
3) 3, 6, 11, 2, 17, 22, 15, 8, 21, 3, 15, 12

Arrange the given data in ascending order.

2 3 3 6 8 11 12 15 15 17 21 22

Q_1 Q_2 Q_3

The box-and-whisker plot is drawn below.



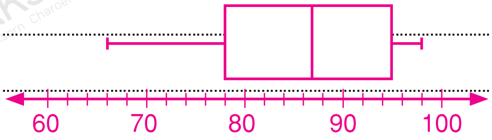
4) 79, 87, 66, 96, 98, 87, 82, 77, 93

Arrange the given data in ascending order.

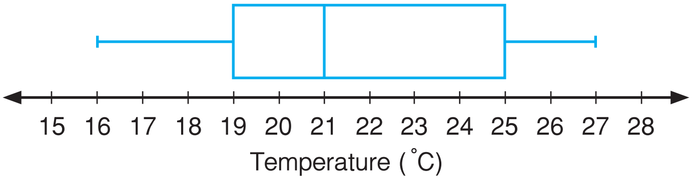
66 77 79 82 87 87 93 96 98

Q_1 Q_2 Q_3

The box-and-whisker plot is drawn below.



2. **The following diagram shows the box-and-whisker plot for the daily temperature (°C) from June 1 to 30 in a city.**



1) Find Q_1, Q_2 and Q_3 of the temperature.

Q_1 = 19°C

Q_2 = 21°C

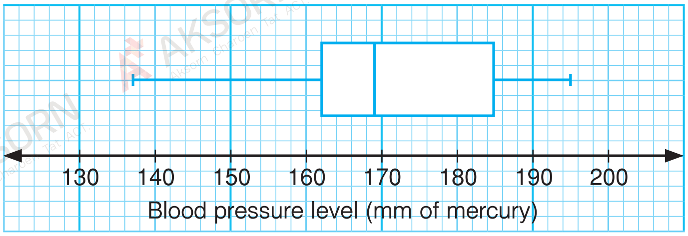
Q_3 = 25°C

2) Find the range of the temperature in June.

Range = 27 - 16

= 11°C

3. **The box-and-whisker plot below shows the blood pressure level (in mm of mercury) of patients who have taken a certain prescription drug.**



1) Find the median blood pressure level of the patients.

Median blood pressure level of the patients = 169 mm of mercury

2) Find the interquartile range.

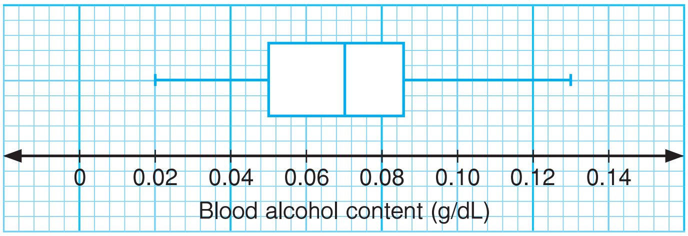
Q_1 = 162 mm of mercury

Q_3 = 185 mm of mercury

Interquartile range = 185 - 162

= 23 mm of mercury

4. **The following diagram shows the box-and-whisker plot for the alcohol content in the blood (grams per deciliter of blood) of drivers who were given breathalyzer tests.**



1) Find Q_1, Q_2 and Q_3 of the alcohol content of drivers.

Q_1 = 0.05 g/dL

Q_2 = 0.07 g/dL

Q_3 = 0.086 g/dL

2) Compare the spread of the alcohol content between the highest 25% (values higher than Q_3) and the lowest 25% (values lower than Q_1) of the drivers.

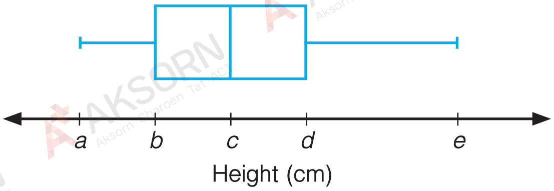
The highest 25% (values higher than Q_3) of the drivers has a larger spread of alcohol content as compared to that of the lowest 25% (values lower than Q_1) of the drivers.

Intermediate Level

5. The heights of basketball players (cm) in an NBA team are given below.

168	180	185	192	192	195
195	196	198	200	205	213

The data can be represented in the box-and-whisker plot below.



1) Find the values of *a*, *b*, *c*, *d* and *e*.

Arrange the given data in ascending order.

168 180 185 192 192 195 195 196 198 200 205 213

$\uparrow Q_1$ $\uparrow Q_2$ $\uparrow Q_3$

$a = 168$

$b = Q_1 = \frac{185 + 192}{2} = 188.5$

$c = Q_2 = \frac{195 + 195}{2} = 195$

$d = Q_3 = \frac{198 + 200}{2} = 199$

$e = 213$

2) Calculate *d* - *b*. What does it represent?

$d - b = 199 - 188.5$

$= 10.5$

It represents the interquartile range of the data.

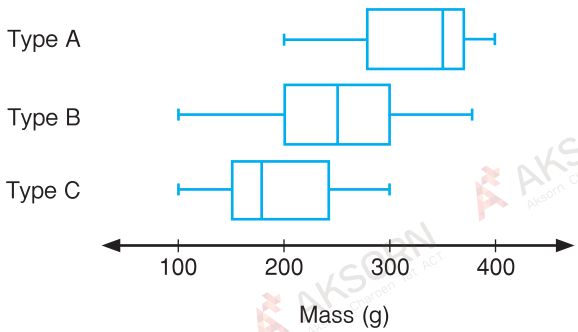
3) Calculate *e* - *a*. What does it represent?

$e - a = 213 - 168$

$= 45$

It represents the range of the data.

6. **The following box-and-whisker plots show the masses (g) of three types of apples.**



1) Which type of apples has the highest median mass and which type has the lowest median mass?

Type A has the highest median mass, and Type C has the lowest median mass.

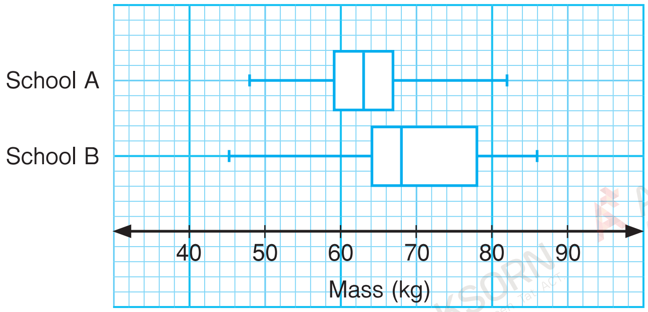
2) Which type of apple has masses which are more evenly distributed?

Type B

3) Which type of apples has masses which have a greater spread?

Type B

7. **The box-and-whisker plots show the masses (kg) of Secondary 3 students from School A and School B.**



1) For School A, find the range, the median and the interquartile range.

Range = 82 - 48 = 34 kg

Median = 63 kg

Interquartile range = $Q_3 - Q_1$

= 67 - 59 = 8 kg

2) For School B, find the range, the median and the interquartile range.

Range = 86 - 45

= 41 kg

Median = 68 kg

Interquartile range = $Q_3 - Q_1$

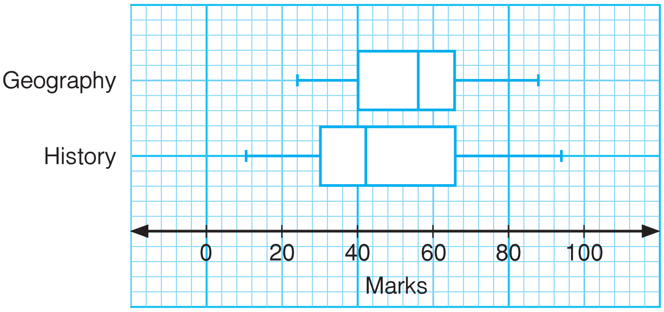
= 78 - 64

= 14 kg

3) “Students from School B are generally heavier than students from School A.” Do you agree with this statement? State a reason to support your answer.

Agree. School B has a higher median mass as compared to that of School A.

8. **The box-and-whisker plots show the marks obtained by some students in the history and geography exams. The maximum mark for both exams is 100.**



1) From the geography exam, find the range, the median and the interquartile range.

Range = 88 - 24

= 64 marks

Median = 56 marks

Interquartile range = $Q_3 - Q_1$

= 66 - 40

= 26 marks

2) From the history exam, find the range, the median and the interquartile range.

Range = 94 - 10

= 84 marks

Median = 42 marks

Interquartile range = $Q_3 - Q_1$

= 66 - 30

= 36 marks

3) Nyra said that the geography exam is easier than the history exam.

Do you agree with Nyra? State two reasons to support your answer.

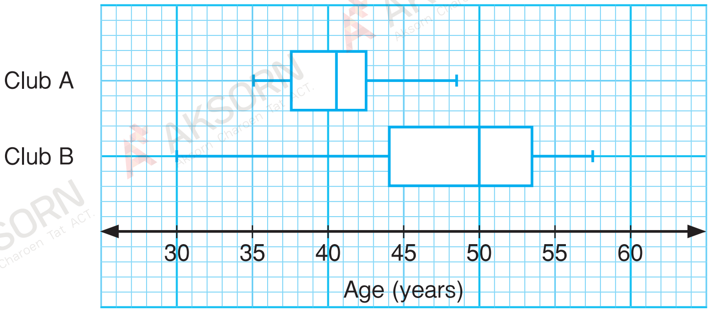
Agree. The lower quartile and the median mark for the geography exam are higher than those of the history exam, while the upper quartile for the geography exam is equal to the upper quartile for the history exam.

4) Which exam has a wider spread of marks? State a reason to support your answer.

History exam. Its interquartile range is larger than that of the geography exam.

Advanced Level

9. **The box-and-whisker plots show the distribution of the ages (in years) of 60 members from Club A and Club B.**



1) For Club A, find the median age and the interquartile range.

Median age = 40.5 years

Q_1 = 37.5 years

Q_3 = 42.5 years

Interquartile range = 42.5 - 37.5

= 5 years

2) For Club B, find the median age and the interquartile range.

Median age = 50 years

Q_1 = 44 years

Q_3 = 53.5 years

Interquartile range = 53.5 - 44

= 9.5 years

3) For the box-and-whisker plot for Club B, the left whisker is much longer than the right whisker. Explain what this means.

The youngest 25% of the members has a larger spread of ages as compared to that of the oldest 25% of the members.

4) Which club shows a greater spread of ages?

Club B because its interquartile range is higher as compared to that of Club A.

5) Comment briefly on the distribution of ages between the members in Club A and Club B.

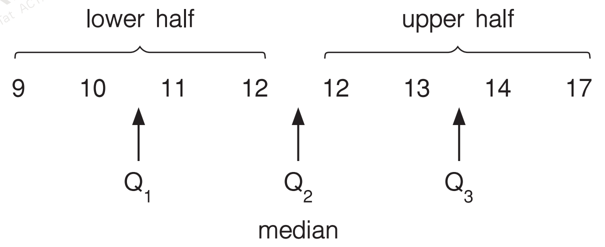
The median age of Club B is higher than the median age of Club A, so the members from Club B are generally older in age as compared to those of the members from Club A. Also, the interquartile range of the ages of Club B members is larger than that of Club A members, which indicates the wider spread of ages of Club B members.

Summary

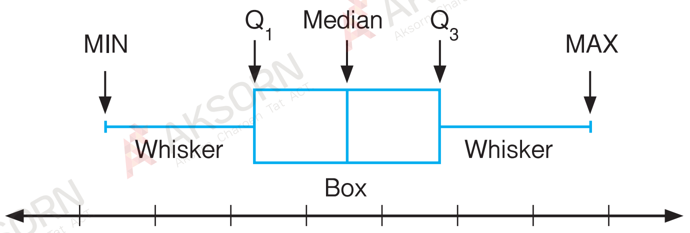
1. **The first quartile (Q_1)** is defined as the value that is less than or equal to that value, which takes up one-fourth or 25% of the data.

   The second quartile (Q_2) is defined as the value that is less than or equal to that value, which is half or 50% of the data, or defined as the middle value when arranged in ascending order, i.e. the median is equal to the second quartile.

   The third quartile (Q_3) is defined as the value that is less than or equal to that value, which takes up three-fourths or 75% of the data.



2. The range of set of data is the difference between the largest value and the smallest value.
3. The interquartile range is the difference between the upper quartile (Q_3) and the lower quartile (Q_1). It measures the spread of the middle 50% of the data.
4. A box-and-whisker plot illustrates the range, lower quartile (Q_1), median and upper quartile (Q_3) of a frequency distribution.



Review Exercise 5

1. Find the Q_1, Q_2, Q_3, range and interquartile range of the following sets of data.

1) 10, 7, 4, 3, 6, 9, 2

Arrange the given data in ascending order.

$$\overbrace{2\quad \underset{\substack{\uparrow\\ Q_1}}{3}\quad 4}^{\text{lower half}}\quad \underset{\substack{\uparrow\\ Q_2}}{6}\quad \overbrace{7\quad \underset{\substack{\uparrow\\ Q_3}}{9}\quad 10}^{\text{upper half}}$$

For the given data, $n = 7$.

Therefore, $Q_2 = 6$

$Q_1 = 3$

$Q_3 = 9$

Range = 10 - 2 = 8

Interquartile range = 9 - 3 = 6

2) 95, 60, 66, 48, 76, 87, 82, 55

Arrange the given data in ascending order.

$$\overbrace{48\quad 55\quad \underset{\substack{\uparrow\\ Q_1}}{}\quad 60\quad 66}^{\text{lower half}}\quad \underset{\substack{\uparrow\\ Q_2}}{}\quad \overbrace{76\quad 82\quad \underset{\substack{\uparrow\\ Q_3}}{}\quad 87\quad 95}^{\text{upper half}}$$

For the given data, $n = 8$.

Therefore, $Q_2 = \frac{66 + 76}{2} = 71$

$Q_1 = \frac{55 + 60}{2} = 57.5$

$Q_3 = \frac{82 + 87}{2} = 84.5$

Range = 95 - 48 = 47

Interquartile range = 84.5 - 57.5 = 27

3) 170, 219, 152, 208, 140, 167, 96, 223, 181

Arrange the given data in ascending order.

$$\overbrace{96 \quad 140 \underset{\overset{\uparrow}{Q_1}}{\quad} 152 \quad 167}^{\text{lower half}} \quad \underset{\overset{\uparrow}{Q_2}}{170} \quad \overbrace{181 \quad 208 \underset{\overset{\uparrow}{Q_3}}{\quad} 219 \quad 223}^{\text{upper half}}$$

For the given data, $n = 9$.

Therefore, $Q_2 = 170$

$Q_1 = \frac{140 + 152}{2} = 146$

$Q_3 = \frac{208 + 219}{2} = 213.5$

Range $= 223 - 96 = 127$

Interquartile range $= 213.5 - 146 = 67.5$

2. The daily amounts of money (in dollars) spent by Kate over a period of 14 days are recorded as follows:

10	120	20	5	9	12	30
15	13	23	19	20	84	9

Find the median, interquartile range and mean of the data.

Arrange the given data in ascending order.

$$5 \quad 9 \quad 9 \quad \underset{\overset{\uparrow}{Q_1}}{10} \quad 12 \quad 13 \quad 15 \underset{\overset{\uparrow}{Q_2}}{\quad} 19 \quad 20 \quad 20 \quad \underset{\overset{\uparrow}{Q_3}}{23} \quad 30 \quad 84 \quad 120$$

For the given data, $n = 14$.

Therefore, $Q_2 = \frac{15 + 19}{2} = 17$ dollars

$Q_1 = 10$ dollars

$Q_3 = 23$ dollars

Median $(Q_2) = 17$ dollars

Interquartile range $= 23 - 10$

$= 13$ dollars

Average

$= (5 + 9 + 9 + 10 + 12 + 13 + 15 + 19$

$+ 20 + 20 + 23 + 30 + 84 + 120) \div 14$

≈ 27.79 dollars

3. **The stem-and-leaf diagram below represents the number of people visiting an art exhibition each day for the first 20 days.**

Stem	Leaf
15	5
16	4 4 5 9
17	0 2 7
18	4 6 8
19	0 3 6 6 7
20	5 8
21	
22	1
23	6

Key: 15 | 5 means 155 people.

Find the following:

1) The median

Median = $\frac{186 + 188}{2}$ = 187 people

2) The range

Range = 236 - 155 = 81 people

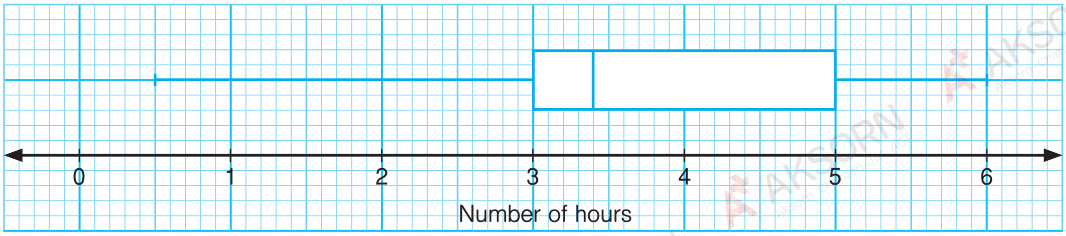
3) The interquartile range

Q_1 = $\frac{169 + 170}{2}$ = 169.5 people

Q_3 = $\frac{196 + 197}{2}$ = 196.5 people

Interquartile range = 196.5 - 169.5 = 27 people

4. **The number of hours a group of students surfed the Internet on a particular day was represented by a box-and-whisker plot as shown.**



Find the median, range of the number of hours and interquartile range.

Median = 3.4 hours

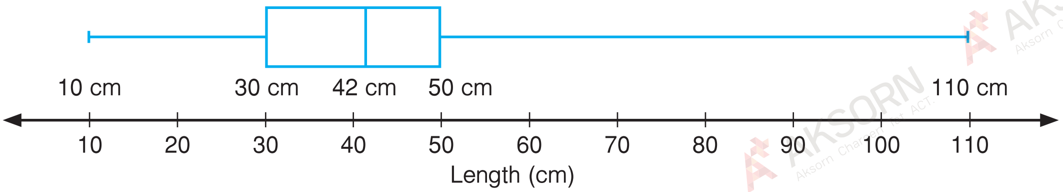
Range = 6 - 0.5 = 5.5 hours

Q_1 = 3 hours

Q_3 = 5 hours

Interquartile range = 5 - 3 = 2 hours

5. **The box-and-whisker plot below shows the lengths of objects found in a storeroom.**



1) Consider and describe the distribution of the lengths of the objects.

The minimum length is 10 cm, and the maximum length is 110 cm. The lengths of the objects have a lower quartile of 30 cm, a median of 42 cm and an upper quartile of 50 cm.

2) Find the range of the data.

Range = 110 - 10

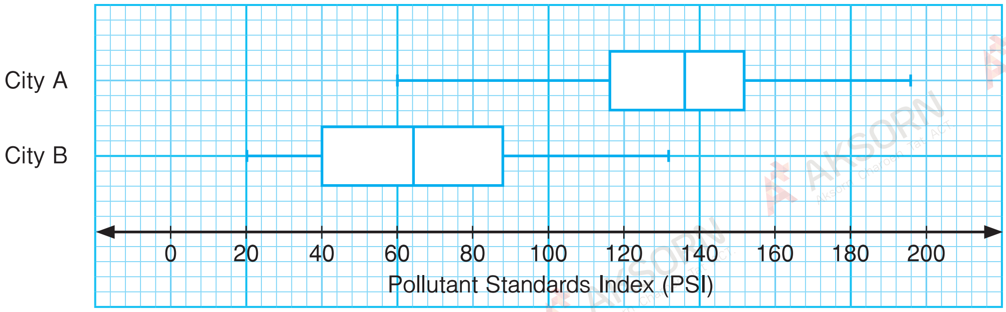
= 100 cm

3) Find the interquartile range of the data.

Interquartile range = 50 - 30

= 20 cm

6. **The following diagram shows the box-and-whisker plots of the air pollution of City A and City B measured by the Pollutant Standards Index (PSI) in a certain month. [Note: 0-50 Good, 51-100 Moderate, 101-200 Unhealthy.]**



1) Write down Q_1, Q_2 and Q_3 for each city.

For City A,

Q_1 = 116 PSI

Q_2 = 136 PSI

Q_3 = 152 PSI

For City B,

Q_1 = 40 PSI

Q_2 = 64 PSI

Q_3 = 88 PSI

2) Find the interquartile range of each city.

For City A, interquartile range = 152 - 116

= 36 PSI

For City B, interquartile range = 88 - 40

= 48 PSI

3) Which data set shows a greater spread?

City B shows a greater spread.

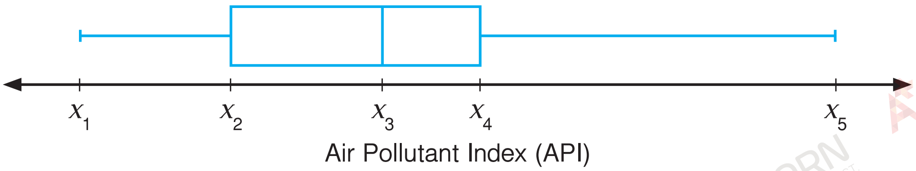
4) Compare briefly these two distributions.

City A has worse air pollution than City B since the median PSI for City A is much higher than that of City B.

7. **In a country, the Air Pollutant Index (API) in 10 cities at a certain time was recorded as follows. [Note: 0-25 Good, 26-50 Moderate, 51-100 Unhealthy.]**

80	67	20	91	17
27	33	46	50	56

The data is represented by the following box-and-whisker plot.



1) State the values of x_1, x_2, x_3, x_4 and x_5.

Arrange the given data in ascending order.

$$\overbrace{17\quad 20\quad \underset{\underset{Q_1}{\uparrow}}{27}\quad 33\quad 46}^{\text{lower half}}\ \underset{\underset{Q_2}{\uparrow}}{}\ \overbrace{50\quad 56\quad \underset{\underset{Q_3}{\uparrow}}{67}\quad 80\quad 91}^{\text{upper half}}$$

For the given data, $n = 10$.

We get $Q_2 = \frac{46 + 50}{2} = 48$ API

$Q_1 = 27$ API

$Q_3 = 67$ API

Therefore, $x_1 = 17$, $x_2 = 27$, $x_3 = 48$, $x_4 = 67$ and $x_5 = 91$

2) Find the range of the data.

Range = 91 - 17

= 74 API

3) Find the interquartile range of the data.

Interquartile range = 67 - 27

= 40 API

4) Find the percentage of the cities whose API is considered as unhealthy.

Percentage of cities whose API is considered as unhealthy $= \frac{4}{10} \times 100$

$= 40\%$

8. The stem-and-leaf diagram shows the masses in kg of a group of children.

Stem	Leaf									
30	2	3								
40	0	1	2	6	6	7	9	9		
50	3	3	3	4	6	6	8	8	9	9

Key: 30 | 2 means 30.2 kg.

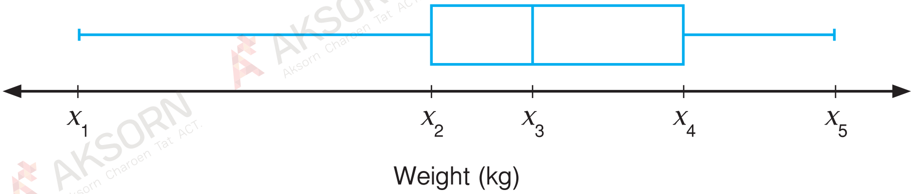
1) How many children were weighed?

From the diagram, 20 children were weighted.

2) Write down the mass of the lightest child.

Mass of lightest child = 30.2 kg

3) A box-and-whisker plot is drawn to represent the data.



(1) Find the values of x_1, x_2, x_3, x_4 and x_5.

MIN = 30.2 kg

Q_1 = $\frac{40.2 + 40.6}{2}$ = 40.4 kg

Q_3 = $\frac{50.6 + 50.6}{2}$ = 50.6 kg

Median = $\frac{40.9 + 50.3}{2}$ = 45.6 kg

MAX = 50.9 kg

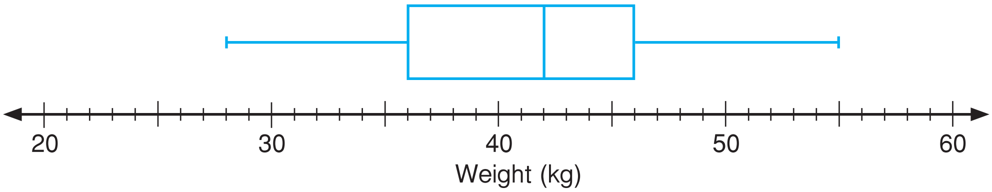
Therefore, x_1 = 30.2, x_2 = 40.4, x_3 = 45.6, x_4 = 50.6 and x_5 = 50.9.

(2) Find the value of $x_4 - x_2$. What does it represent?

$x_4 - x_2$ = 50.6 - 40.4 = 10.2

It represents the interquartile range.

4) The children were encouraged to exercise more to reduce their masses to less than or equal to 40.5 kg. Children with masses exceeding 40.5 kg were considered overweight. Three months later, the same group of children was again weighted. The masses (in kg) are shown in the box-and-whisker plot below:



(1) What is the decrease in median mass after three months?

Median after three months = 42 kg

Decrease in median = 45.6 - 42 = 3.6 kg

(2) What is the decrease in Q_3 after three months?

Q_3 after three months = 46 kg

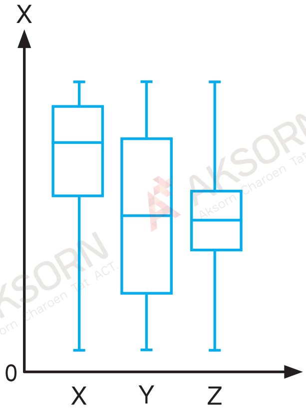
Decrease in Q_3 = 50.6 - 46 = 4.6 kg

(3) Compare briefly, the masses of the children before and after they were encouraged to exercise more.

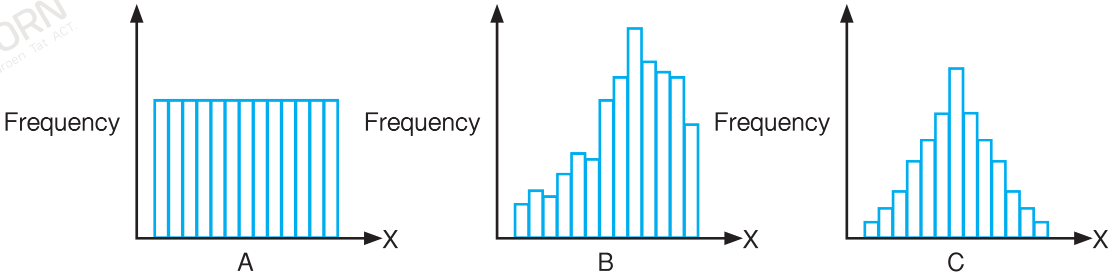
The masses of the children decreased after being encouraged to exercise as indicated in the decrease in median and Q_3.

Challenge Yourself

Box-and-whisker diagrams for three sets of data, X, Y and Z are shown below.



The histograms below (A, B and C) show the frequency distributions for three sets of data.



Match each of the three data sets to their respective histograms. State reasons to support your answer.

Data X matches Histogram B because the lower whisker is much longer than the upper whisker, which matches the relatively many more shorter bars on the left side of the histogram than the right side of the histogram.

Data Y matched Histogram A because the whiskers are of equal length and the median is at the center of the box, which matches the equal distribution of the bars of the histogram.

Data Z matches Histogram C because the box is short with the median at the center of the box, indicating that the distribution is concentrated at the center as shown in Histogram C. The whiskers are of equal length, which corresponds to the balanced and symmetrical distribution on the left and right sides of the histogram.

Chapter 6

Probability

We normally draw presents to celebrate New Year, in which presents are brought one for each attendee and then each of them is labeled with a number. There are two sets of numbers that will be written on small pieces of paper, i.e. one set attached to the presents and the other one used for drawing. The paper can be put in a jar or folded to hind the numbers inside. When we draw a particular number at random, we will get the present that matches the number we have drawn.

Indicator

- Understand random experiments and use the outcomes to find the probability of events. **(MA 3.2 G. 9/1)**

Compulsory Details

- Outcomes obtained in probability experiments
- Probability
- Real-life applications of probability

6.1 Introduction to Probability

We often make statements such as:

- There is a 50:50 **chance** of our school winning the National Inter-School Basketball Championship.
- It cannot **predict** whether I will obtain a 'six' in the next roll of a die.
- It **will probably** rain today.

We make such statements because we are uncertain whether an event will occur. For an uncertain event, we can discuss about its chance of occurrence.

Thinking Time

1. The following are some events which we may come across in our everyday life.


Event A: The sun will rise from the east every day.


Event B: Your father will win the lottery this year.

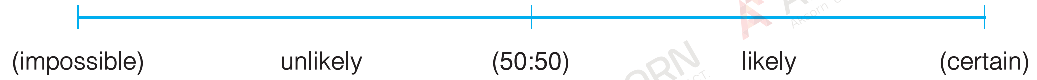

Event C: You obtain a 'tail' when you toss a coin.


Event D: It will snow in Thailand at least once a year.


Event E: Babies drink milk every day.

Each of these events may or may not happen. Mark these events A to E on the line to show the likelihood they will occur.

(Answers can vary depending on the teacher's discretion.)



2. **We can use values between 0 and 1 inclusive to measure the chance of an event occurring where an impossible event takes on the value 0 and a certain event takes on the value 1. If there is a 50:50 chance that an event will occur, what value does it take?**

0.5

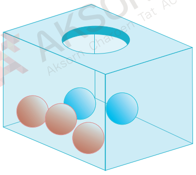
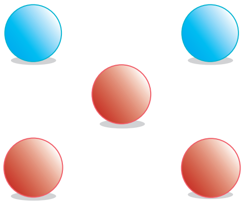
3. **Write down an event that corresponds to each of the five categories above. Mark out each event on the number line based on the estimated chance of occurrence.**

(Answers can vary depending on the teacher's discretion.)

In our everyday life, we use words, such as 'unlikely', 'likely' or 'certain' to describe the chance of an event occurring. The measure of chance, which takes on values between 0 and 1 inclusive, is known as **probability**. We will learn about probability in this chapter.

6.2 Sample Space

When we perform a scientific experiment, we will obtain a certain result or outcome. However, in probability, the result or the outcome is not certain—it depends on chance. The following table shows some examples of probability experiments and their possible outcomes.

Probability experiment	Possible outcomes
 Tossing a coin	 Head  Tail
 Rolling a die	
Ten identical cards numbered 11, 12, 13,..., 20 are placed in a box. One card is drawn at random from the box. 	
Two blue balls and three red balls of the same size are placed in a box. One ball is drawn at random from the box. 	

Consider the experiment where a coin is tossed. The results are either getting a 'head' or a 'tail'. These results are referred to as the **random experiment**.

The collection of all the possible outcomes of a probability experiment is called the **sample space**. In the case of tossing a coin, the sample space is a 'head' and a 'tail'. What is the sample space when a die is rolled?

Worked Example 1

A fair die is rolled. Write down the sample space and state the total number of possible outcomes.

Solution:

A die has the numbers 1, 2, 3, 4, 5 and 6 and its six faces,

i.e. the sample space consists of the numbers 1, 2, 3, 4, 5 and 6.

Total number of possible outcomes = 6

Similar Questions

Exercise 6A Question 1

KEY

Practice Now

Write down the sample space for choosing an even number from 1 to 17 at random. State the total number of possible outcomes.

Even numbers from 1 to 17 include 2, 4, 6, 8, 10, 12, 14 and 16.

i.e. the sample space consists of the numbers 2, 4, 6, 8, 10, 12, 14 and 16.

Total number of possible outcomes = 8

Worked Example 2

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

1) Picking a shirt at random from a wardrobe containing 3 identical white shirts and 4 identical blue shirts.

2) Choosing a two-digit number greater than 65 from a list of numbers running from 1 to 100.

Solution:

1) Let W_1, W_2 and W_3 represent the 3 white shirts; B_1, B_2, B_3 and B_4 represent the 4 blue shirts. The sample space consists of W_1, W_2, W_3, B_1, B_2, B_3 and B_4.

   Total number of possible outcomes = 7

2) The sample space consists of the integers 66, 67, 68, …, 100.

   Total number of possible outcomes = cards numbered 1 to 100 - cards numbered 1 to 65

   = 100 - 65 = 35

Similar Questions

Exercise 6A Question 2

Practice Now

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

1) Drawing a marble at random from a bag containing 5 identical blue marbles and 4 identical red marbles

   Let B_1, B_2, B_3, B_4 and B_5 represent the 5 blue marbles; R_1, R_2, R_3 and R_4 represent the 4 red marbles.

   The sample space consists of B_1, B_2, B_3, B_4, B_5, R_1, R_2, R_3 and R_4.

   Total number of possible outcomes = 9

2) Picking a letter at random from a box containing identical cards with letters that spell the word 'NATIONAL'

   The sample space consists of N_1, A_1, T, I, O, N_2, A_2 and L.

   Total number of possible outcomes = 8

3) Selecting a receipt at random from a receipt book with running serial numbers from 357 to 389.

   The sample space consists of the serial numbers

   Total number of possible outcomes

   = first 389 receipts - first 356 receipts

   = 389 - 356 = 33

6.3 Probability of Single Events

In Section 6.1, we have learned that probability is a measure of chance.

Investigation

Do the following activity and answer the questions.

1. When tossing a coin, are we able to state beforehand with certainty whether the outcome is a 'head' or a 'tail'?
2. Toss a coin 20 times. Record the outcome of each toss in the following table.

Outcome	Tally	Number of 'heads' or 'tails' for 20 tosses	Fraction of obtaining a 'head' or a 'tail'
Head			
Tail	(Answers can vary depending on the outcomes.)		

Compare your results with those of your classmates. Are they the same? What can you deduce about the results of tossing a coin?

3. In groups of 5, record each of the member's results obtained in 2 in the following table.

Outcome	Total number of 'heads' or 'tails'	Fraction of obtaining a 'head' or a 'tail'
Head		
Tail	(Answers can vary depending on the outcomes.)	

4. As a class, record everyone's results in the following table.

Outcome	Total number of 'heads' or 'tails'	Fraction of obtaining a 'head' or a 'tail'
Head		
Tail	(Answers can vary depending on the outcomes.)	

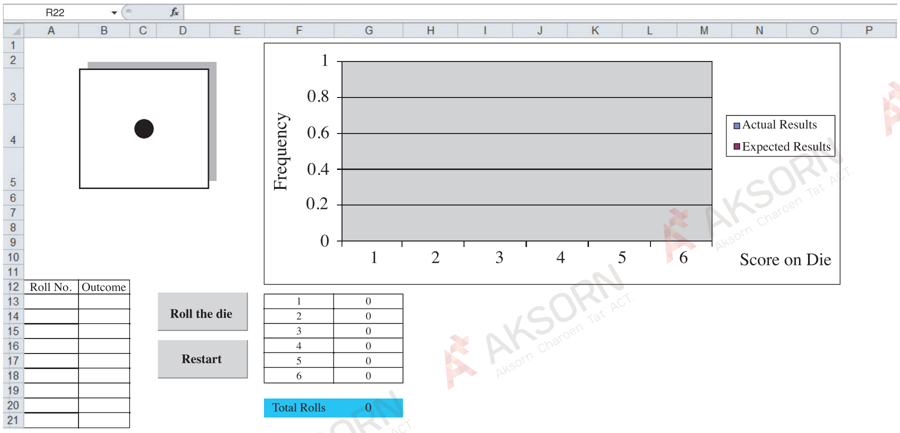
5. Look at the last column in the three tables. Do you notice that the probabilities of obtaining a 'head' or a 'tail' approach $\frac{1}{2}$ when there are more tosses?
6. If we toss a coin 1,000 times, would we expect to obtain exactly 500 'heads' and exactly 500 'tails'? Explain your answer.

When a coin is tossed, if the chance of obtaining a 'head' is the same as the chance of obtaining a 'tail', we say that the coin is fair or unbiased. This means that for a fair coin, there are two equally likely outcomes, i.e. obtaining a 'head' and obtaining a 'tail'. Thus, the chance of obtaining a 'head' is 1 out of 2. We say that the probability of obtaining a 'head' is $\frac{1}{2}$ or 50%. What is the probability of obtaining a 'tail'?

Investigation

Do the following activity and answer the questions.

1. Go to https://www.shinglee.com.sg/StudentResources/ And click on NSM2/ ⟶ Rolling a Die
2. Open the spreadsheet 'Rolling a Die'.



Roll No.	Outcome

1	0
2	0
3	0
4	0
5	0
6	0

Total Rolls 0

3. Click on the button 'Roll the die' and it will roll the die once. Repeat for a total of 20 rolls. Record the number of '1', '2', '3', '4', '5' and '6' obtained in the following table. Compute the fraction of obtaining each outcome.

Outcome	Number of corresponding outcomes for 20 rolls	Fraction of obtaining each corresponding outcome for 20 rolls
1		
2		
3	(Answers can vary depending on the outcomes.)	
4		
5		
6		

4. As a class, add and record the total number of ‘1’, ‘2’, ‘3’, ‘4’, ‘5’ and ‘6’ obtained by all students. Compute the fraction of obtaining each outcome.

Outcome for ……… rolls	Total number of corresponding outcomes	Fraction of obtaining each corresponding outcome
1		
2		
3	(Answers can vary depending on the outcomes.)	
4		
5		
6		

5. Look at the last column in the two tables. Do you notice that the probabilities of obtaining any one of the six outcomes approach $\frac{1}{6}$ when there are more rolls?
6. If we roll a die 600 times, would we expect to obtained exactly 100 ‘6’? Explain your answer.

When a die is rolled, there are six possible outcomes, i.e. 1, 2, 3, 4, 5 and 6. If the die is fair, then each of the six outcomes is equally likely to occur. Thus, the chance of obtaining a ‘six’ is 1 out of 6. We say that the probability of obtaining a ‘six’ is $\frac{1}{6}$.

In general, in a probability experiment with *m* equally likely outcomes, if *k* of these outcomes favor the occurrence of an event E, then the probability, P(E), of the event happening is given by:

$$P(E) = \frac{\text{Number of favorable outcomes for event E}}{\text{Total number of possible outcomes}} = \frac{k}{m}$$

This is known as **theoretical probability**, the probability that is obtained based on mathematical theory.

In **Investigation**, you conducted a probability experiment to determine the chance of obtaining a 'head' or a 'tail' when a coin is tossed. Based on theoretical probability, the probability of obtaining a 'head' is $\frac{1}{2}$, i.e. 10 'heads' in 20 tosses. However, it is unlikely that all our classmates obtained 10 'heads' in 20 tosses. The probability that you obtained in the experiment is known as **experimental probability**. Thus, if you obtained 11 'heads' in 20 tosses, your experimental probability of getting a 'head' is $\frac{11}{20}$.

From **Investigation**, we can conclude that as the number of trials increases, the experimental probability of an outcome occurring tends toward the theoretical probability of the outcome happening.

Worked Example 3

A card is drawn at random from a box containing 15 cards numbered 1, 2, 3, ..., 15. Find the probability of drawing the following:

1) a '9'
2) an odd number
3) a negative number
4) a number less than 20

Solution:

Total number of possible outcomes = 15

1) P(drawing a '9') = $\frac{1}{15}$

2) There are 8 odd numbers from 1 to 15, i.e. 1, 3, 5, 7, 9, 11, 13 and 15.

P(drawing an odd number) = $\frac{8}{15}$

3) There are no negative numbers from 1 to 15.

P(drawing a negative number) = $\frac{0}{15}$ = 0

4) All the 15 numbers from 1 to 15 are less than 20.

P(drawing a number less than 20) = $\frac{15}{15}$ = 1

Similar Questions

Exercise 6A Questions 3, 10

Practice Now

A ball is drawn at random from a bag containing some balls numbered 25, 26, 27, …, 40. Find the probability of drawing the following:

1) a '30'

Total number of possible outcomes = 40 - 24 = 16

P(drawing a '30') = $\frac{1}{16}$

2) an even number

There are 8 even numbers from 25 to 40, i.e. 26, 28, 30, 32, 34, 36, 38 and 40.

P(drawing an even number) = $\frac{8}{16}$ = $\frac{1}{2}$

3) a two-digit number

There are 16 two-digit numbers from 25 to 40, i.e. 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39 and 40.

P(drawing a two-digit number) = $\frac{16}{16}$ = 1

4) a number greater than 40

There are no numbers greater than 40 from 25 to 40.

P(drawing a number greater than 40) = $\frac{0}{16}$ = 0

KEY

In **Worked Example 3**, we observe that the probability of drawing a negative number from 15 cards numbered 1, 2, 3, …, 15 is 0. This means that we will never be able to draw a negative number from the 15 cards. In the same worked example, we also notice that the probability of drawing a number less than 20 from the 15 cards is 1. This means that we will definitely be able to draw a number less than 20 from the 15 cards.

Thinking Time

1. **In Thinking Time on page 244, the event D 'It will snow in Thailand at least once a year.' is an impossible event, i.e. it will never occur. What can we say about the probability of D occurring?**

   The probability of D is equal to 0 since the event will never happen.

2. **In Thinking Time on page 244, the event A 'The sun will rise from the east every day.' is a certain event, i.e. it will definitely occur. What can we say about the probability of A occurring?**

   The probability of A is equal to 1 since the event will definitely happen.

3. **Is it possible that the probability of an event occurring is less than 0 or greater than 1?**

   It is impossible.

From **Thinking Time**, we can conclude that:

For any event E, $0 \le P(E) \le 1$.

- $P(E) = 0$ if and only if E is an impossible event, i.e. it will never occur.
- $P(E) = 1$ if and only if E is a certain event, i.e. it will definitely occur.

Worked Example 4

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing the following:

1) a heart

2) a card which is not a heart

Solution:

Total number of possible outcomes = 52

1) There are 13 hearts in the pack.

   P(drawing a heart) = $\frac{13}{52} = \frac{1}{4}$

2) Since there are 13 hearts in the pack, there are 52 - 13 = 39 cards which are not hearts.

   P(drawing a card which is not a heart) = $\frac{39}{52} = \frac{3}{4}$

Notice: P(drawing a card which is not a heart)

= 1 - P(drawing a heart).

ATTENTION

There are 4 suits in a standard pack of 52 playing cards, i.e. club ♣, diamond ♦, heart ♥ and spade ♠. Each suit has 13 cards, i.e. Ace, 2, 3, …, 10, Jack, Queen and King.

- All the clubs and spades are black in color.
- All the diamonds and hearts are red in color.

KEY

Practice Now

Similar Questions

Exercise 6A Questions 4, 7

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing the following:

1) the Queen of spades

   There is only one Queen of spades in the pack.

   P(drawing the Queen of spades) = $\frac{1}{52}$

2) a card which is not the Queen of spades

   Since there is only one Queen of spades in the pack, there are 52 - 1 = 51 cards which are not the Queen of spades.

   P(drawing a card which is not the Queen of spades) = $\frac{51}{52}$

Worked Example 5

A letter is chosen at random from the word 'BANGKOK'. Find the probability that the letter is the following:

1) a 'K'

2) a vowel

3) not a vowel

Solution:

Total number of letters = 7

1) There are 2 'K's.

$$P(\text{a 'K' is chosen}) = \frac{2}{7}$$

2) There are 2 vowels, i.e. 1 'A' and 1 'O'.

$$P(\text{letter chosen is a vowel}) = \frac{2}{7}$$

3) **Method 1:** There are 5 consonants, i.e. B, G, N, K_1 and K_2.

$$P(\text{letter chosen is not a vowel}) = \frac{5}{7}$$

Method 2:

$$P(\text{letter chosen is not a vowel}) = 1 - P(\text{letter chosen is a vowel})$$

$$= 1 - \frac{2}{7} = \frac{5}{7}$$

JUST FOR FUN

The first three children of a couple are boys. So, what is the possibility that their next child will be a girl?

Similar Questions

Exercise 6A Questions 5, 6, 8, 9

Practice Now

A letter is chosen at random from the word 'STUDENTS'. Find the probability that the letter is the following:

1) an 'N'

Total number of letters = 8; there is 1 'N'.

$$P(\text{an 'N' is chosen}) = \frac{1}{8}$$

2) a consonant

There are 6 consonants, i.e. 2 'S's, 2 'T's, 1 'D' and 1 'N'.

$$P(\text{letter chosen is a consonant}) = \frac{6}{8} = \frac{3}{4}$$

3) not a consonant

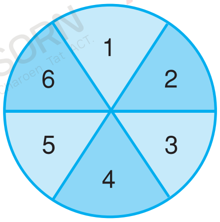
$$P(\text{letter chosen is not a consonant}) = 1 - P(\text{letter chosen is a consonant})$$

$$= 1 - \frac{3}{4} = \frac{1}{4}$$

Exercise 6A

Basic Level

1. A dart board is divided into 6 equal sectors. When a dart lands on it, the number of the sector on which it lands is noted. Write down the sample space and state the total number of possible outcomes.



The dart board has the numbers 1, 2, 3, 4, 5 and 6, i.e. the sample space consists of the numbers 1, 2, 3, 4, 5 and 6.

Total number of possible outcomes = 6

2. For each of the following experiments, write down the sample space and state the total number of possible outcomes.

1) Drawing a ping pong ball at random from a bag containing 5 identical red ping pong balls, 3 identical blue ping pong balls and 2 identical green ping pong balls.

Let R_1, R_2, R_3, R_4 and R_5 represent the 5 red ping pong balls;

B_1, B_2 and B_3 represent the 3 blue ping pong balls;

G_1 and G_2 represent the 2 green ping pong balls.

The sample space consists of R_1, R_2, R_3, R_4, R_5, B_1, B_2, B_3, G_1 and G_2.

Total number of possible outcomes = w10

2) Picking a letter at random from a box containing identical cards with letters that spell the word ‘TEACHER’

The sample space consists T, E_1, A, C, H, E_2 and R.

Total number of possible outcomes = 7

3) Choosing a three-digit number at random.

The sample space consists 100, 101, 102, ..., 999.

Total number of possible outcomes

= first 999 numbers - first 99 numbers

= 999 - 99 = 900

3. **A card is drawn at random from a box containing cards numbered 35, 36, 37 …, 55. Find the probability of drawing the following:**

1) a prime number

Total number of possible outcomes = 55 - 34 = 21

There are 5 prime numbers between 35 and 55, i.e. 37, 41, 43, 47 and 53.

P(drawing a prime number) = $\frac{5}{21}$

2) a number greater than 55

There are no numbers greater than 55.

P(drawing a number greater than 55) = 0

3) a number that is divisible by 7

There are 3 numbers divisible by 7, i.e. 35, 42 and 49.

P(drawing a number that is divisible by 7) = $\frac{3}{21} = \frac{1}{7}$

4. **A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing the following:**

1) the ace of spades

Total number of possible outcomes = 52

There is only one ace of spades in the pack.

P(drawing the ace of spades) = $\frac{1}{52}$

2) a heart or a club

There are 13 hearts and 13 clubs in the pack.

P(drawing a heart or a club) = $\frac{26}{52} = \frac{1}{2}$

3) Jack, Queen or King of each suit

There are 3 × 4 = 12 Jack, Queen or King of each suit in the pack.

P(drawing Jack, Queen or King of each suit) = $\frac{12}{52} = \frac{3}{13}$

4) a card which is not Jack, Queen or King of each suit

Method 1:

There are 10 × 4 = 40 cards which are not Jack, Queen or King of each suit in the pack.

P(drawing a card which is not Jack, Queen or King of each suit) = $\frac{40}{52} = \frac{10}{13}$

Method 2:

P(getting a card which is not Jack, Queen or King of each suit)

= 1 - P(getting Jack, Queen or King of each suit)

= $1 - \frac{3}{13} = \frac{10}{13}$

5. **Each of the letters of the word ‘THAILAND’ is written on a card. All the cards are well-shuffled and placed face down on a table. A card is turned over. Find the probability that the card shows the following:**

1) the letter ‘I’

Total number of possible outcomes = 8

There is 1 ‘I’ in the cards. P(card shows the letter ‘I’) = $\frac{1}{8}$

2) the letter ‘A’

There are 2 ‘A’s in the cards.

P(card shows the letter ‘A’) = $\frac{2}{8} = \frac{1}{4}$

3) a vowel

There are 3 vowels in the cards, i.e. 2 ‘A’s and 1 ‘I’.

P(card shows a vowel) = $\frac{3}{8}$

4) a consonant

P(card shows a consonant) = 1 - P(card shows a vowel)

= $1 - \frac{3}{8} = \frac{5}{8}$

6. **A group of 30 people consisting of 9 men, 6 women, 12 boys and 3 girls are waiting to get their passport photographs taken. A person is selected at random from the group. Find the probability that the person is the following:**

1) a male

Total number of possible outcomes = 30

There are 9 + 12 = 21 males in the group.

P(person is a male) = $\frac{21}{30} = \frac{7}{10}$

2) either a women, a boy or a girl

Method 1:

There are 6 + 12 + 3 = 21 people in the group that is either a woman, a boy or a girl.

P(person is either a woman, a boy or a girl)

$= \frac{21}{30} = \frac{7}{10}$

Method 2:

There are 9 men in the group.

P(person is a man) $= \frac{9}{30} = \frac{3}{10}$

P(person is either a woman, a boy or a girl)

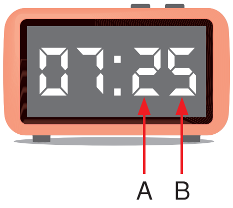
= P(person is not a man)

= 1 - P(person is a man)

$= 1 - \frac{3}{13} = \frac{10}{13}$

Intermediate Level

7. **Nyra wakes up in the morning and notices that her digital clock reads 07:25, as shown in the picture. After noon, she looks at the clock again. Find the probability that:**



1) the number in column A is a 4

The sample space in column A consists of the integers 0, 1, 2, 3, 4 and 5.

Total number of possible outcomes = 6

P(number in column A is a 4) $= \frac{1}{6}$

2) the number in column B is an 8

The sample space in column B consists of the integers 0, 1, 2, ..., 9.

Total number of possible outcomes = 10

P(number in column B is an 8) $= \frac{1}{10}$

3) the number in column A is less than 6

There are 6 numbers that are less than 6 in column A, i.e. 0, 1, 2, 3, 4 and 5

P(number in column A is less than 6) $= \frac{6}{6} = 1$

4) the number in column B is greater than 5

There are 4 numbers that are greater than 5 in column B, i.e. 6, 7, 8 and 9.

P(number in column B is greater than 5) $= \frac{4}{10} = \frac{2}{5}$

8. The table shows the number of each type of school personnel at a school.

Section	Management	Teaching	Laboratory	Administrative	Maintenance
Number of personnel	26	62	8	9	12

Two teachers and an administrative staff resign from the school. A school personnel is selected at random from the remaining staff. Find the probability that the school personnel is the following:

1) an administrative staff

Total number of possible outcomes = total number of staff - number of resigned staff

= 117 - 2 - 1 = 114

There are 9 - 1 = 8 administrative staff in the school.

P(school personnel is an administrative staff) = $\frac{8}{114} = \frac{4}{57}$

2) not a laboratory staff

There are 8 laboratory staff in the school.

P(school personnel is a laboratory staff) = $\frac{8}{114} = \frac{4}{57}$

P(school personnel is not a laboratory staff)

= 1 - P(school personnel is a laboratory staff)

= $1 - \frac{4}{57} = \frac{53}{57}$

9. There are a total of 117 pairs of socks in a cloths bin. Each pair of socks is placed in a bag. The probabilities of selecting a yellow pair of socks and a gray pair of socks at random from the bin are $\frac{2}{9}$ and $\frac{3}{13}$, respectively. Find the number of pairs of socks in the bin that is the following:

1) yellow

Total number of possible outcomes = 117

P(pair of socks is yellow) = $\frac{2}{9}$

$$\frac{\text{Number of yellow pairs of socks}}{\text{Total number of pairs of socks}} = \frac{2}{9}$$

Therefore, the number of yellow pairs of socks = $\frac{2}{9} \times 117 = 26.$

2) neither yellow nor gray

P(pair of socks is neither yellow nor gray)

$= 1 - P(\text{pair of socks in yellow}) - P(\text{pair of socks in gray})$

$= 1 - \frac{2}{9} - \frac{3}{13} = \frac{64}{117}$

$$\frac{\text{Number of pairs of socks ncither yellow nor gray}}{\text{Total number of pairs of socks}} = \frac{64}{117}$$

Therefore, the number of pairs of socks neither yellow nor gray $= \frac{64}{117} \times 117 = 64$.

Advanced Level

10. An IQ test consists of 80 multiple-choice questions. A question is selected at random. Find the probability that:

1) the question number contains only a single digit

The sample space consists of the questions numbers 1, 2, 3, ..., 80.

Total number of possible outcomes = 80

There are 9 question numbers containing only a single digit, i.e. 1, 2, 3, ..., 9.

P(question number contains only a single digit) $= \frac{9}{80}$

2) the question number is greater than 67

There are 13 question numbers greater than 67, i.e. 68, 69, 70, ..., 80.

P(question number is greater than 67) $= \frac{13}{80}$

3) the question number contains exactly one ‘7’

There are 8 + 9 = 17 question numbers containing exactly one ‘7’, i.e. 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78 and 79.

P(question number contains exactly one ‘7’) $= \frac{17}{80}$

4) the question number is divisible by both 2 and 5

There are 8 question numbers divisible by both 2 and 5, i.e. 10, 20, 30, 40, 50, 60, 70 and 80.

P(question number is divisible by both 2 and 5) $= \frac{8}{80} = \frac{1}{10}$

6.4 Real-life Applications of Probability

In this section, we will take a look at more examples that involve probability.

Worked Example 6

In a class of 28 students, there are 12 girls and 5 of them play a musical instrument. 13 of the boys do not play any musical instruments. If a student is chosen at random, find the probability that:

1) the student is a boy

2) the student plays a musical instrument

Solution:

1) Number of boys = 28 - 12 = 16

$$P(\text{student chosen is a boy}) = \frac{16}{28} = \frac{4}{7}$$

2) Number of girls who play a musical instrument = 5

Number of boys who play a musical instrument = 16 - 13 = 3

Number of students who play a musical instrument = 5 + 3 = 8

$$P(\text{student chosen plays a musical instrument}) = \frac{8}{28} = \frac{2}{7}$$

Similar Questions

Exercise 6B Questions 1, 3

Practice Now

In a class of 32 students, there are 18 boys and 3 of them are in the school's track and field team. 9 of the girls are not in the school's track and field team. If a student is chosen at random, find the probability that:

1) the student is a girl

Number of girls = 32 - 18 = 14

$P(\text{student chosen is a girl}) = \frac{14}{32} = \frac{7}{16}$

2) the student is in the school's track and field team

Number of boys who are in the school's track and field team = 3

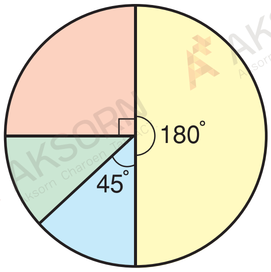
Number of girls who are in the school's track and field team = 14 - 9 = 5

Number of students who are in the school's track and field team = 3 + 5 = 8

$P(\text{student is in the school's track and field team}) = \frac{8}{32} = \frac{1}{4}$

Worked Example 7

A circle is divided into four sectors of different colors.
A point is selected at random in the circle.
Find the probability that:



1) the point lies in the yellow sector

2) the point lies in the green sector

3) the point lies in the black sector

Solution:

1) P(point selected lies in the yellow sector)

$$= \frac{\text{Area of the yellow sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the yellow sector}}{360^\circ}$$

$$= \frac{180^\circ}{360^\circ} = \frac{1}{2}$$

ATTENTION

Since a point is selected at random, any point in the circle will have the same chance of being selected. We assume that the point will not fall on any of the lines separating the four sectors.

PROBLEM SOLVING TIP

$$\frac{\text{Number of points in the selected sector}}{\text{Total number of points in the circle}}$$

$$= \frac{\text{Area of the selected sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the selected sector}}{\text{Angle of the circle}}$$

2) Angle of the green sector

$= 360^\circ - 180^\circ - 90^\circ - 45^\circ = 45^\circ$

P(point selected lies in the green sector)

$$= \frac{\text{Area of the green sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the green sector}}{360^\circ}$$

$$= \frac{45^\circ}{360^\circ} = \frac{1}{8}$$

3) P(point selected lies in the black sector)

$$= \frac{\text{Area of the black sector}}{\text{Area of the circle}}$$

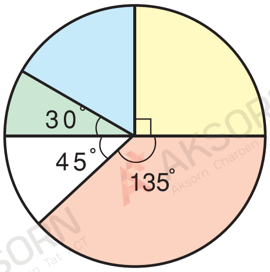
$$= \frac{\text{Angle of the black sector}}{360^\circ}$$

$= 0$

Practice Now

Similar Questions
Exercise 6B Question 2

A circle is divided into five sectors of different colors. A point is selected at random in the circle. Find the probability that:



1) the point lies in the blue sector

Angle of the blue sector = $360^\circ - 90^\circ - 135^\circ - 45^\circ - 30^\circ = 60^\circ$

P(point selected lies in the blue sector) $= \frac{\text{Area of the blue sector}}{\text{Area of the circle}}$

$= \frac{\text{Angle of the blue sector}}{360^\circ}$

$= \frac{60^\circ}{360^\circ} = \frac{1}{6}$

2) the point lies in the purple sector

P(point selected lies in the purple sector) $= \frac{\text{Area of the purple sector}}{\text{Area of the circle}}$

$= \frac{0}{\text{Area of the circle}}$

$= 0$

3) the point lies in the green or white sector

P(point selected lies in the green or white sector)

$= \frac{\text{Area of the green sector + area of the white sector}}{\text{Area of the circle}}$

$= \frac{\text{Angle of the green sector + angle of the white sector}}{360^\circ}$

$= \frac{30^\circ + 45^\circ}{360^\circ} = \frac{75^\circ}{360^\circ} = \frac{5}{24}$

Worked Example 8

A box contains x red marbles, $(x + 3)$ yellow marbles and $(4x - 15)$ blue marbles.

1) Find an expression, in terms of x, for the total number of marbles in the box.
2) A marble is drawn at random from the box. Write down an expression, in terms of x, for the probability that the marble is blue.
3) Given that the probability in question 2 is $\frac{1}{2}$, find the value of x.

Solution:

1) Total number of marbles $= x + (x + 3) + (4x - 15) = 6x - 12$

2) P(drawing a blue marble) $= \frac{4x - 15}{6x - 12}$

3) Given that

$$\begin{aligned} \frac{4x - 15}{6x - 12} &= \frac{1}{2} \\ 2(4x - 15) &= 6x - 12 \\ 8x - 30 &= 6x - 12 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

KEY

Similar Questions

Exercise 6B Questions 4, 5

Practice Now

There are 12 green balls and $(x + 2)$ yellow balls in a box.

1) Find an expression, in terms of x, for the total number of balls in the box.

Total number of balls $= 12 + (x + 2) = x + 14$

2) A ball is drawn at random from the box. Write down an expression, in terms of x, for the probability that the ball is yellow.

P(drawing a yellow ball) $= \frac{x + 2}{x + 14}$

3) Given that the probability in question 2 is $\frac{2}{5}$, find the value of x.

Given that

$$\begin{aligned} \frac{x + 2}{x + 14} &= \frac{2}{5} \\ 5(x + 2) &= 2(x + 14) \\ 5x + 10 &= 2x + 28 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

Exercise 6B

Basic Level

1. There are 6 servers and 3 cooks in a restaurant. 3 of the staff are female, 2 of whom are servers. The rest of the staff are male. If a staff is selected at random for a lucky draw, find the probability of picking the following:

1) a female staff

Total number of staff $= 6 + 3 = 9$

P(staff picked is female) $= \frac{3}{9} = \frac{1}{3}$

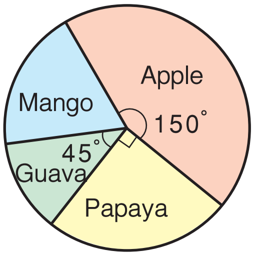
2) a male cook

Number of female cooks $= 3 - 2 = 1$

Number of male cooks $= 3 - 1 = 2$

P(staff picked is a male cook) $= \frac{2}{9}$

2. A survey is conducted to find out which of the four fruits, apple, papaya, guava and mango, the students in a class prefer. The pie chart shows the results of the survey. A student is selected at random. Find the probability that:



1) the student prefers apples

P(student prefers apple) $= \frac{\text{Area of the sector denoting apple}}{\text{Area of the circle}}$

$= \frac{\text{Angle of the sector denoting apple}}{360^\circ}$

$= \frac{150^\circ}{360^\circ} = \frac{5}{12}$

2) the student prefers papayas or guavas

Angle of sector denoting papaya or guava $= 90^\circ + 45^\circ = 135^\circ$

P(student prefers papaya or guava) $= \frac{\text{Area of the sector denoting papaya or guava}}{\text{Area of the circle}}$

$= \frac{\text{Angle of the sector denoting papaya or guava}}{360^\circ}$

$= \frac{135^\circ}{360^\circ} = \frac{3}{8}$

Intermediate Level

3. The table shows the number of students for each level in the school's bowling team.

Level	Number of boys	Number of girls
Secondary 1	22	38
Secondary 2	19	25
Secondary 3	25	35
Secondary 4	24	22

A month later, an additional two Secondary 1 boy and three Secondary 3 girls join the bowling team. A student is now selected at random to attend another bowling workshop. Find the probability that:

1) the student is a Secondary 3 student who is a girl

Number of students in the school's bowling team = 22 + 38 + 19 + 25 + 25 + 35 + 24 + 22 + 2 + 3 = 215

P(student chosen is a Secondary 3 student who is a girl) = $\frac{35 + 3}{215} = \frac{38}{215}$

2) the student is either a Secondary 2 student or a Secondary 4 student

P(student chosen is a Secondary 2 or Secondary 4 student) $= \frac{19 + 25 + 24 + 22}{215}$

$= \frac{90}{215} = \frac{18}{43}$

4. There are 23 boys and 35 girls on the school's track and field team. After q boys and $(q + 4)$ girls graduate at the end of this year, the probability of selecting a boy at random to represent the school for an event becomes $\frac{2}{5}$. Find the value of q.

Total number of boys and girls remaining after graduation = $23 + 35 - q - (q + 4) = 54 - 2q$

Total number of boys remaining after graduation = $23 - q$

P(boy is selected for event) = $\frac{23 - q}{54 - 2q}$

We get $\frac{23 - q}{54 - 2q} = \frac{2}{5}$

$5(23 - q) = 2(54 - 2q)$

$115 - 5q = 108 - 4q$

$q = 7.$

Advanced Level

5. There are 50 students in an auditorium, of which $2x$ are boys and y are girls. After $(y - 6)$ boys leave the auditorium and $(2x - 5)$ girls enter the auditorium, the probability of selecting a girl at random becomes $\frac{9}{13}$. Find the value of x and of y.

Total number of students $= 50$

$2x + y = 50$ .....①

After some boys left and some girls enter the auditorium,

total number of students $= 50 - (y - 6) + (2x - 5)$

$= 50 - y + 6 + 2x - 5$

$= 2x - y + 51$

Total number of girls $= y + 2x - 5$

$= 2x + y - 5$

P(girl is selected at random) $= \frac{2x + y - 5}{2x - y + 51}$

We get $\frac{2x + y - 5}{2x - y + 51} = \frac{9}{13}$

$13(2x + y - 5) = 9(2x - y + 51)$

$26x + 13y - 65 = 18x - 9y + 459$

$8x + 22y = 524$

$4x + 11y = 262.$ .....②

From $2x + y = 50$

$y = 50 - 2x,$ .....③

substitute ③ into ②; $4x + 11(50 - 2x) = 262$

$4x + 550 - 22x = 262$

$18x = 288$

$x = 16.$

Substitute $x = 16$ into ③; $y = 50 - 2(16) = 18$

Therefore, the value of x and y are 16 and 18, respectively.

6.5 Probability of Simple Combined Events

In this section, we will learn how to list the sample space of **simple combined events**, e.g. rolling 2 dice at the same time, tossing a coin 3 times, and how to calculate probabilities for simple combined events.

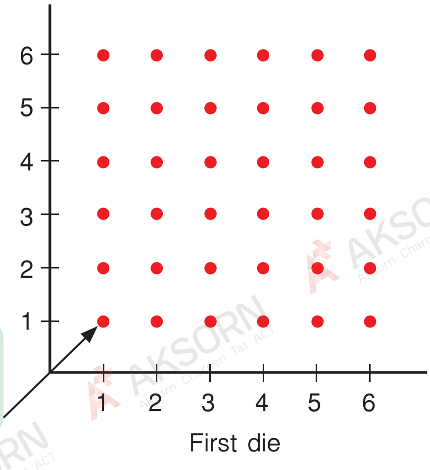
1. Possibility Diagram

The possible outcomes for rolling a fair die are 1, 2, 3, 4, 5 and 6. How do we write the possible outcomes for rolling two fair dice?

We can represent a possible outcome by using an ordered pair, e.g. (2, 3) means that the first die shows a '2' and the second die shows a '3'; which is different from (3, 2). So what does (3, 2) mean?

The possible outcomes for rolling two fair dice are (1, 1), (1, 2), (1, 3), …, (6, 6). However, listing out all the outcomes would be very tedious, and we may miss some outcomes.

Therefore, there is a need to use a different method to represent the sample space. Fig. 6.1 shows one way of drawing a possibility diagram to represent the sample space for rolling two fair dice.



The red spot means that the outcome of the first die is 1 and that of the second die is 1.

Fig. 6.1

A possibility diagram is used when each outcome of the sample space has two components. For example, in the above case, an outcome (represented by a red dot ●) is determined by the values displayed by the first and second die. From the above possibility diagram, we observe that the total number of possible outcomes is 6 × 6 = 36. We can also calculate the probability of certain events using a possibility diagram as shown in **Worked Example 9**.

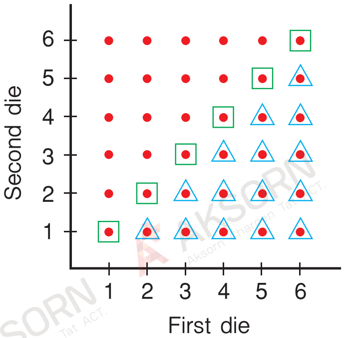
Worked Example 9

Two fair dice are rolled. Find the possibility that:

1) both dice show the same number

2) the number shown on the first die is greater than the number shown on the second die

Solution:



PROBLEM SOLVING TIP

Mark out the favorable outcomes on the possibility diagram.

- Count the number of □ for Question 1.
- Count the number of △ for Question 2.

1) P(both dice show the same number) $= \frac{6}{36} = \frac{1}{6}$

2) P(number shown on first die is greater than the number shown on second die)

$= \frac{15}{36} = \frac{5}{12}$

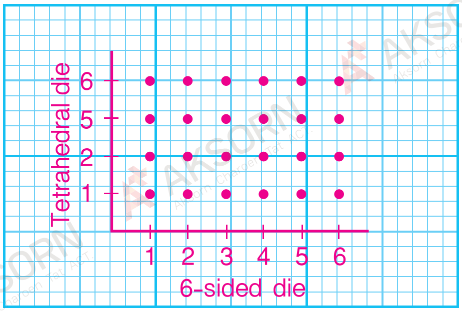
Practice Now

Similar Questions

Exercise 6C Questions 1, 4

A fair tetrahedral die (4-sided die) and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral die are 1, 2, 5 and 6, while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6.

1) Display all the outcomes of the experiment using a possibility diagram.



2) Using the possibility diagram or otherwise, find the probability that:

(1) both dice show the same number

P(both dice show the same number) $= \frac{4}{24} = \frac{1}{6}$

(2) the numbers shown on both dice are prime numbers

P(the numbers shown on both dice are prime numbers) $= \frac{6}{24} = \frac{1}{4}$

There is another way to draw a possibility diagram to represent the sample space for rolling two fair dice, as shown in **Worked Example 10**.

Worked Example 10

Two fair dice are rolled. Find the probability that the sum of the numbers shown on the dice is the following:

1) equal to 5

2) even

Solution:

First die

Second die

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

1) P(sum is equal to 5) $= \frac{4}{36} = \frac{1}{9}$

2) P(sum is even) $= \frac{18}{36} = \frac{1}{2}$

Similar Questions

Exercise 6C Questions 2, 5, 7

Practice Now

The numbers on a fair tetrahedral die are 1, 2, 5 and 6, while the numbers on a fair 6-sided die are 1, 2, 3, 4, 5 and 6. The two dice are rolled at the same time, and the scores on both dice are recorded. The possibility diagrams below display separately some of the values of the sum and product of the two scores.

Tetrahedral die

6-sided die

+	1	2	5	6
1		3		
2				
3			8	
4				
5				
6	7			

Tetrahedral die

6-sided die

×	1	2	5	6
1				
2		4		
3				18
4				
5				
6			30	

1) Copy and complete the possibility diagrams.

Tetrahedral die

6-sided die

+	1	2	5	6
1	2	3	6	7
2	3	4	7	8
3	4	5	8	9
4	5	6	9	10
5	6	7	10	11
6	7	8	11	12

Tetrahedral die

6-sided die

×	1	2	5	6
1	1	2	5	6
2	2	4	10	12
3	3	6	15	18
4	4	8	20	24
5	5	10	25	30
6	6	12	30	36

2) Using the possibility diagrams, find the probability that the sum of the scores is the following:

(1) even

P(sum of the scores is even)

$= \frac{12}{24} = \frac{1}{2}$

(2) divisible by 3

P(sum of the scores is divisible by 3)

$= \frac{8}{24} = \frac{1}{3}$

3) Using the possibility diagrams, find the probability that the product of the scores is the following:

(1) a prime number

P(product of the scores is a prime number)

$= \frac{5}{24}$

(2) less than 37

P(product of the scores is less than 37)

$= \frac{24}{24} = 1$

2. Tree Diagrams

The sample space for tossing a fair coin is {H, T}.

The sample space for tossing two fair coins can be represented by a possibility diagram as shown in Fig. 6.2. However, for tossing three fair coins, we use a different type of diagram called a tree diagram to represent the sample space as shown in Fig. 6.3. The following steps show how the tree diagram is constructed.

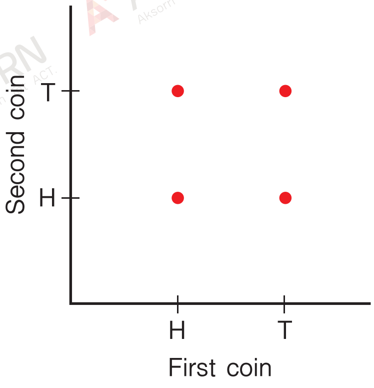
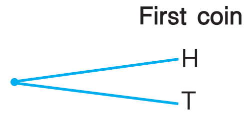
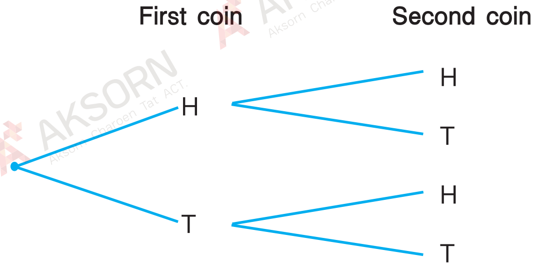


Fig. 6.2

1) When the first coin is tossed, there are two possible outcomes, head (H) or tail (T), so we start with a point and draw two branches H and T.



2) The second coin is then tossed. Regardless of the outcome of the first toss, the second coin would also yield either a H or a T, thus we draw two branches after the H and the T from the first toss as shown below. There are a total of 2 × 2 = 4 branches, i.e. there are 4 possible outcomes at this stage.



3) The third coin could also yield two outcomes when the first two outcomes are HH, HT, TH or TT. Thus, we obtain the tree diagram as shown in Fig. 6.3.

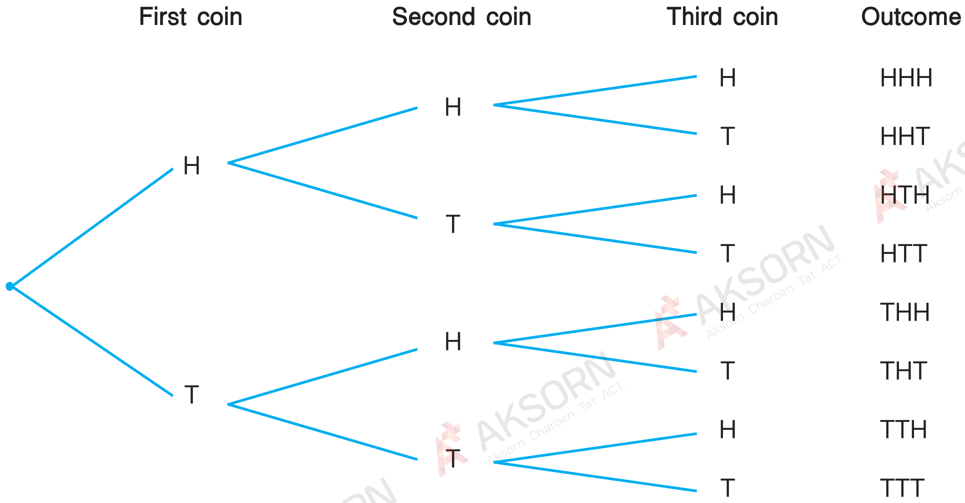


Fig. 6.3

From the tree diagram, we observe that there are a total of 2 × 2 × 2 = 8 branches, i.e. the total number of possible outcomes is 8.

From the different representations of the sample space, we can conclude that:

Example of experiment	Component of each outcome	Representation of sample space
Tossing 1 coin	1	List of outcomes in a set
Tossing 2 coins	2	Possibility diagram or tree diagram
Tossing 3 coins	3	Tree diagram

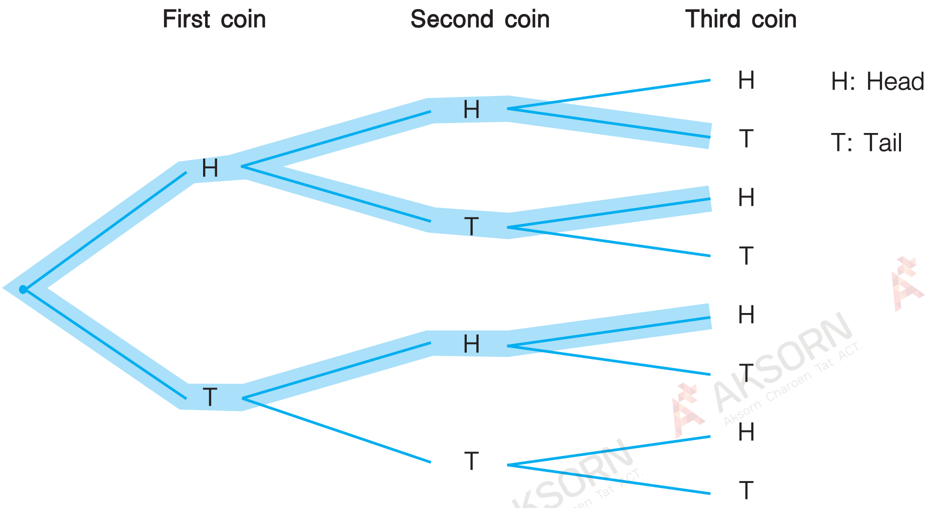
Worked Example 11

Three fair coins are tossed. Find the probability that:

1) there are two heads and one tail

2) there is at least one tail

Solution:



1) P(two heads and one tail) = $\frac{3}{8}$ *(see shaded regions)*

2) P(at least one tail) $= 1 - \text{P(no tail)}$

$= 1 - \text{P(three heads)}$

$= 1 - \frac{1}{8} = \frac{7}{8}$

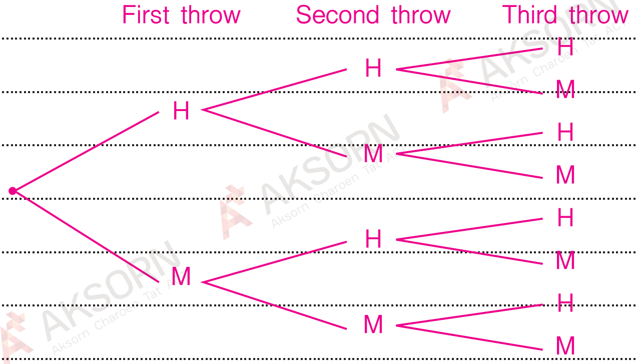
Practice Now

Similar Questions

Exercise 6C Questions 3, 6

Max is a darts player. There is an equal probability that he will hit or miss the bull's-eye. He aims for the bull's-eye and attempts 3 throws. Using a tree diagram, find the following:

1) sample space displayed by a tree diagram



H: hits the bull's-eye

M: misses the bull's-eye

2) the probability that he misses the bull's-eye once

P(misses the bull's-eye once) = $\frac{3}{8}$

3) the probability that he hits the bull's-eye at least once

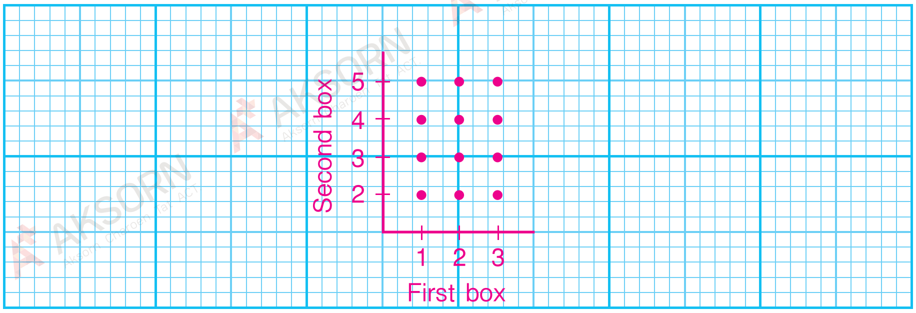
P(hits the bull's-eye at least once) = $\frac{7}{8}$

Exercise 6C

Basic Level

1. **A box contains three cards bearing the numbers 1, 2 and 3. A second box contains four cards bearing the numbers 2, 3, 4 and 5. A card is chosen at random from each box.**

1) Display all the possible outcomes of the experiment using a possibility diagram.



2) With the help of the possibility diagram, calculate the following:

(1) the probability that the cards bear the same number

P(cards bear the same number)

$= \frac{2}{12} = \frac{1}{6}$

(2) the probability that the numbers on the cards are different

Method 1:

P(numbers on the cards are different)

$= \frac{10}{12}$

$= \frac{5}{6}$

Method 2:

P(numbers on the cards are different)

= 1 - P(cards bear the same number)

$= 1 - \frac{1}{6} = \frac{5}{6}$

(3) the probability that the larger of the two numbers on the cards is 3

P(larger of the two numbers on the cards is 3)

$= \frac{3}{12}$

$= \frac{1}{4}$

2. **Six cards numbered 0, 1, 2, 3, 4 and 5 are placed in a box and well-mixed. A card is drawn at random from the box, and the number on the card is noted before it is replaced in the box. The cards in the box are thoroughly mixed again, and a second card is drawn at random form the box. The sum of the two numbers is then obtained.**

1) Copy and complete the possibility diagram below, giving all the possible sums of the two numbers. Some of the possible outcomes are as shown:

First number

Second number

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

2) How many possible outcomes are there in the sample space of this experiment?

Total number of possible outcomes = 36

3) Find the probability of the sum of two numbers from the following:

(1) the sum will be 7

P(sum of the two numbers is 7) $= \frac{4}{36}$

$= \frac{1}{9}$

(2) the sum will be a prime a number

P(sum of the two numbers is a prime number) $= \frac{17}{36}$

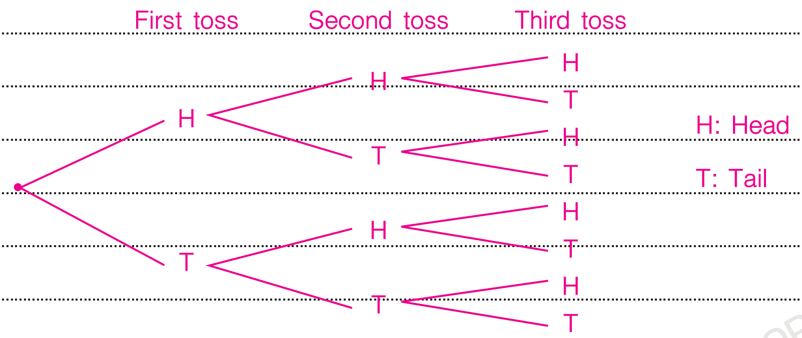
(3) the sum will not be a prime a number

P(sum of the two numbers is not a prime number) $= 1 - \frac{17}{36}$

$= \frac{19}{36}$

3. A fair coin is tossed three times. Find the following:

1) sample space displayed by a tree diagram



2) the probability of obtaining three heads

P(three heads) $= \frac{1}{8}$

3) the probability of obtaining exactly two heads

P(exactly two heads) $= \frac{3}{8}$

4) the probability of obtaining at least two heads

Method 1:

P(at least two heads)

$= \frac{4}{8} = \frac{1}{2}$

Method 2:

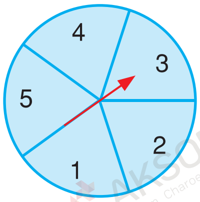
P(at least two heads)

= P(exactly two heads) + P(three heads)

$= \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

Intermediate Level

4. In an experiment, two spinners are constructed with spinning pointers as shown in diagrams below. Both printers are spun. Each time the pointer is spun, it is equally likely to stop at any sector. Find the probability that the pointers will point at the following:

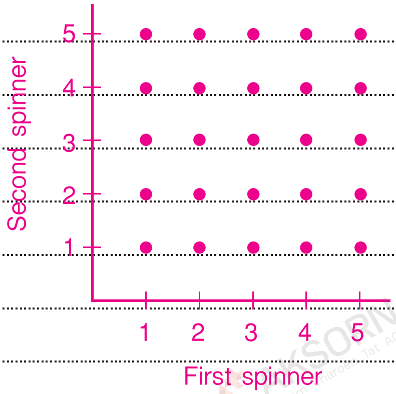


First spinner

Second spinner

1) numbers on the spinners whose sum is 6
2) the same numbers on both spinners
3) two different prime numbers on both spinners
4) numbers on the first spinner which are less than those on the second spinner

We can display the sample space of this event by using a possibility diagram as follows:



1) P(numbers on the spinners whose sum is 6) = $\frac{5}{25}$ = $\frac{1}{5}$

2) P(the same numbers on both spinners) = $\frac{5}{25}$ = $\frac{1}{5}$

3) P(two different prime numbers on both spinners) = $\frac{6}{25}$

4) P(numbers on the first spinner which are less than those on the second spinner)
= $\frac{10}{25}$ = $\frac{2}{5}$

5. **A bag contains 5 identical balls which are numbered 1, 2, 4, 5 and 7. Two balls are drawn at random, one after another and without replacement. Find the probability of the following:**

1) the numbers obtained on both balls are prime
2) the sum of the numbers obtained is odd
3) the difference in the numbers obtained is less than 7
4) the product of the numbers obtained is divisible by 9

We can display the sample space of this event by using a possibility diagram as follows:

Second ball

7	1, 7	2, 7	4, 7	5, 7	
5	1, 5	2, 5	4, 5		7, 5
4	1, 4	2, 4		5, 4	7, 4
2	1, 2		4, 2	5, 2	7, 2
1		2, 1	4, 1	5, 1	7, 1
	1	**2**	**4**	**5**	**7**

First ball

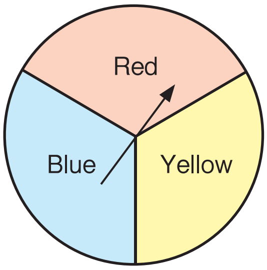
1) P(numbers obtained on both balls are prime) $= \frac{6}{20} = \frac{3}{10}$

2) P(sum of the numbers obtained is odd) $= \frac{12}{20} = \frac{3}{5}$

3) P(difference in the numbers obtained is less than 7) $= \frac{20}{20} = 1$

4) P(product of the numbers obtained is divisible by 9) $= \frac{0}{20} = 0$

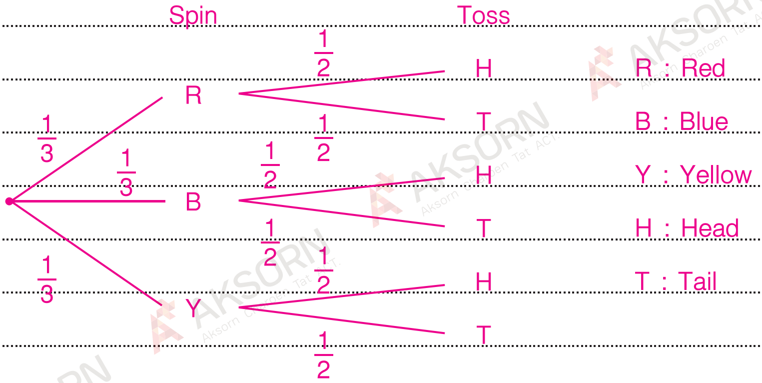
6. **A spinner with three equal sectors and a fair coin are used in a game. The spinner is spun once, and the coin is tossed once. Each time the spinner is spun, it is equally likely to stop at any sector. Calculate the probability of getting the following:**



1) red on the spinner and tail on the coin

2) blue or yellow on the spinner and head on the coin

We can display the sample space of this event by using a tree diagram as follows:



1) P(red on the spinner and tail on the coin) $= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

2) P(blue or yellow on the spinner and head on the coin) $= \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \frac{1}{3}$

Advanced Level

7. Two fair tetrahedral dice and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral dice are 1, 2, 3 and 4, while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6. What is the probability that the score on the 6-sided die is greater than the sum of scores of the two tetrahedral dice?

Total number of possible outcomes $= 4 \times 4 \times 6 = 96$

Let the event of the score on the 6-sided die greater than the sum of the scores of the two tetrahedral dice be A.

20 outcomes for event A: 611, 612, 613, 614, 621, 622, 623, 631, 632, 641, 511, 512, 513, 521, 522, 531, 411, 412, 421 and 311

Therefore, $P(A) = \frac{20}{96} = \frac{5}{24}$.

Summary

1. Probability is a measure of chance, i.e. from 0 to 1.
2. A sample space is the collection of all the possible outcomes of a probability experiment.
3. In a probability experiment with m equally likely outcomes, if k of these outcomes favor the occurrence of an event E, then the probability, P(E), of the event happening is given by:

$$P(E) = \frac{\text{Number of favorable outcomes for event E}}{\text{Total number of possible outcomes}} = \frac{k}{m}$$

4. For any event E, $0 \leq P(E) \leq 1$.
    - P(E) = 0 if and only if E is an impossible event, i.e. it will never occur.
    - P(E) = 1 if and only if E is a certain event, i.e. it will definitely occur.
5. For any event E, P(not E) = 1 - P(E).
6. From the different representations of the sample space, we can conclude that:

Example of experiment	Component of each outcome	Representation of sample space
Tossing 1 coin	1	List of outcomes in a set
Tossing 2 coin	2	Possibility diagram or tree diagram
Tossing 3 coin	3	Tree diagram

Review Exercise 6

1. **Each of the numbers 5, 6 and 8 is written on a card. One or more of these cards are drawn at random to form a one-, two- or three-digit number.**

1) For this experiment, write down the sample space and state the total number of possible outcomes.

The sample space consists of the numbers 5, 6, 8, 56, 58, 65, 68, 85, 86, 568, 586, 658, 685, 856 and 865.

Therefore, the total number of possible outcomes = 15

2) Find the probability of the following:

(1) the number formed consists of two digits

There are 6 two-digit numbers, i.e. 56, 58, 65, 68, 85 and 86.

P(number consists of two digits) = $\frac{6}{15} = \frac{2}{5}$

(2) the number formed is a multiple of 5

There are 5 numbers that are a multiple of 5, i.e. 5, 65, 85, 685 and 865.

P(number is a multiple of 5) = $\frac{5}{15} = \frac{1}{3}$

2. **All the 26 red cards from a standard pack of playing cards are mixed thoroughly. A card is then drawn at random. Find the probability of drawing the following:**

1) the Queen of hearts

Total number of possible outcomes = 26

There is only one Queen of hearts in the pack.

P(drawing the Queen of hearts) = $\frac{1}{26}$

2) the Jack of clubs

Since the given cards are all red, there is no Jack of clubs in the pack.

P(drawing the Jack of clubs) = $\frac{0}{26} = 0$

3) either the six of hearts or the seven of diamonds

There is 1 six of hearts and 1 seven of diamonds in the pack.

P(drawing either the six of hearts or the seven of diamonds) = $\frac{2}{26} = \frac{1}{13}$

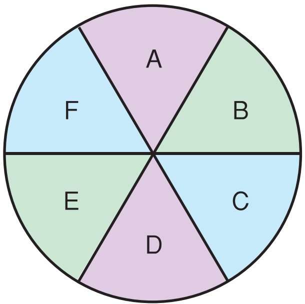
4) a card that is not a nine

There are 2 cards that are a nine in the pack, namely, the nine of hearts and the nine of diamonds.

P(drawing a card that is a nine) = $\frac{2}{26} = \frac{1}{13}$

P(drawing a card that is not a nine) = $1 - \frac{1}{13} = \frac{12}{13}$

3. **In a shopping mall, if a customer spends a minimum of 1,000 baht in a single receipt, he has the chance to spin a wheel to win a prize. The wheel is divided into 6 equal sectors. The prizes correspond to the letters on the wheel.**



A: 50 baht shopping voucher

B: 500 baht supermarket voucher

C: a smartphone

D: an umbrella

E: 100 baht dining voucher

F: 1 kg cheesecake

Find the probability that a customer who spins the wheel wins the following prizes:

1) an umbrella

Total number of possible outcomes = 6

There is only one sector that has an umbrella as a prize.

P(wins an umbrella) = $\frac{1}{6}$

2) a voucher

There are 3 sectors that have a voucher as a prize.

P(wins a voucher) = $\frac{3}{6} = \frac{1}{2}$

3) 1,000 cash

There are no sectors with a prize of 1,000 baht cash.

P(wins 1,000 baht cash) = $\frac{0}{6} = 0$

4. A bag contains 20 sweets, of which 7 are toffee wrapped in green paper, 6 are mints wrapped in green paper, 3 are toffee wrapped in red paper, and 4 are mints wrapped in red paper. If a sweet is drawn at random from the bag, find the probability that the sweet is the following:

1) a mint wrapped in red paper

There are 4 mints wrapped in red paper.

P(sweet is a mint wrapped in red paper) $= \frac{4}{20} = \frac{1}{5}$

2) a toffee

There are 7 + 3 = 10 toffees in the bag.

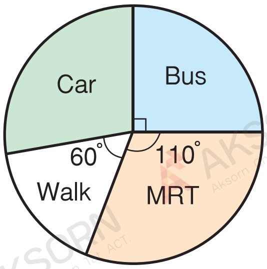
P(sweet is a toffee) $= \frac{10}{20} = \frac{1}{2}$

3) a sweet wrapped in green paper

There are 7 + 6 = 13 sweets wrapped in green paper.

P(sweet is wrapped in green paper) $= \frac{13}{20}$

5. A survey is conducted to find out how the students in a class travel to school. The students either take the bus, MRT, the car or walk to school. The pie chart shows the results of the survey. A student is selected at random. Find the probability that the student travels to school by the following:



1) by car

Angle of the sector denoting travel by car

$= 360^\circ - 90^\circ - 110^\circ - 60^\circ = 100^\circ$

P(student travels to school by car) $= \frac{\text{Area of the sector denoting travel by car}}{\text{Area of the circle}}$

$= \frac{\text{Angle of the sector denoting travel by car}}{360^\circ}$

$= \frac{100^\circ}{360^\circ} = \frac{5}{18}$

2) by MRT or on foot

Angle of the sector denoting travel by MRT or on foot

$= 110^\circ + 60^\circ = 170^\circ$

P(student travels to school by MRT or on foot)

$= \dfrac{\text{Area of the sector denoting MRT or on foot}}{\text{Area of the circle}}$

$= \dfrac{\text{Angle of the sector denoting MRT or on foot}}{360^\circ} = \dfrac{170^\circ}{360^\circ} = \dfrac{17}{36}$

3) by bicycle

Angle of the sector denoting travel by bicycle $= 0^\circ$

P(student travels to school by bicycle) $= \dfrac{\text{Area of the sector denoting travel by bicycle}}{\text{Area of the circle}}$

$= \dfrac{\text{Angle of the sector denoting travel by bicycle}}{360^\circ}$

$= \dfrac{0^\circ}{360^\circ} = 0$

6. A bed of flowers consists of 100 stalks of flowers, of which 20 are lilies, *h* are roses, and the rest are tulips.

1) Given that the probability of picking a stalk of tulip at random is $\frac{1}{4}$, find the value of h.

Number of tulips $= 100 - 20 - h = 80 - h$

P(picking a stalk of tulip) $= \dfrac{80 - h}{100}$

We get $\dfrac{80 - h}{100} = \dfrac{1}{4}$

$4(80 - h) = 100$

$320 - 4h = 100$

$4h = 220$

$h = 55.$

2) 10 stalks of lilies are removed from the bed. A stalk of flower is picked at random from the remaining stalks of flowers. Find the probability that a stalk of rose is picked.

Number of flowers remaining $= 100 - 10 = 90$

Number of roses $= 55$

P(picking a stalk of rose) $= \dfrac{55}{90}$

$= \dfrac{11}{18}$

7. **In a car park, there are 125 cars, $3p$ motorcycles, $2q$ trucks and 20 buses. One of the vehicles leaves the car park at random.**

1) Given that the probability that the vehicle is a motorcycle is $\frac{3}{40}$, form an equation in p and q.

Total number of vehicles $= 125 + 3p + 2q + 20 = 3p + 2q + 145$

P(vehicle is a motorcycle) $= \dfrac{3p}{3p + 2q + 145}$

We get $\dfrac{3p}{3p + 2q + 145} = \dfrac{3}{40}$

$40(3p) = 3(3p + 2q + 145)$

$120p = 9p + 6q + 435$

$37p - 2q - 145 = 0.$

2) Given that the probability that the vehicle is a bus is $\frac{1}{10}$, form an equation in p and q.

P(vehicle is a bus) $= \dfrac{20}{3p + 2q + 145}$

We get $\dfrac{20}{3p + 2q + 145} = \dfrac{1}{10}$

$10(20) = 3p + 2q + 145$

$200 = 3p + 2q + 145$

$3p + 2q - 55 = 0.$

3) Find the value of p and of q in questions 1 and 2.

$37p - 2q - 145 = 0$ .....①

$3p + 2q - 55 = 0$ .....②

① + ②; $(37p - 2q - 145) + (3p + 2q - 55) = 0 + 0$

$40p - 200 = 0$

$40p = 200$

$p = 5$

Substitute $p = 5$ into ②:

$3(5) + 2q - 55 = 0$

$15 + 2q - 55 = 0$

$2q = 40$

$q = 20$

Therefore, the value of p and of q are 5 and 20, respectively.

8. **50 lottery tickets, numbered from 1 to 50, are placed in a bowl. One lottery ticket is picked at random. Find the probability of the following:**

1) the number on the lottery ticket is greater than 28

P(number is greater than 28) = $\frac{22}{50} = \frac{11}{25}$

2) the number on the lottery ticket includes the digit '3'

P(number includes the digit '3') = $\frac{14}{50} = \frac{7}{25}$

3) the number on the lottery ticket is prime

P(number is prime) = $\frac{15}{50} = \frac{3}{10}$

4) the number on the lottery ticket is divisible by 4

P(number is divisible by 4) = $\frac{12}{50} = \frac{6}{25}$

9. **Six discs, with the numbers 1 to 6 written on each of them, are placed in a bag. Two discs are drawn at random from the bag and placed side by side to form a two-digit number. By drawing a possibility diagram, find the probability that the number formed is the following:**

1) divisible by 2

2) divisible by 5

3) a prime number

We can display the sample space of this event by using a possibility diagram as follows:

Second disc	**6**	16	26	36	46	56	
	5	15	25	35	45		65
	4	14	24	34		54	64
	3	13	23		43	53	63
	2	12		32	42	52	62
	1		21	31	41	51	61
		1	**2**	**3**	**4**	**5**	**6**

First disc

P(number formed is divisible by 2) = $\frac{15}{30} = \frac{1}{2}$

P(number formed is divisible by 5) = $\frac{5}{30} = \frac{1}{6}$

P(number formed is a prime number) = $\frac{7}{30}$

Challenge Yourself

Fiona writes 3 letters to 3 of her friends, Fred, Vince, and Mason. She types each of their addresses on each of the 3 envelopes and puts the letters into the envelopes randomly before she sends them out. Find the probability of the following:

1) Exactly one of her friends receives the correct letter.
2) Exactly two of her friends receive the correct letters.
3) All three of her friends receive the correct letters.

Let the 3 friends Fred, Vince, and Mason be A, B and C, and the letters be 1, 2 and 3, i.e. Fred should receive letter 1, Vince should receive letter 2, and Mason should receive letter 3. A ↔ 1, B ↔ 2, C ↔ 3.

The sample space consists of (A_1, B_2, C_3), (A_1, B_3, C_2), (A_2, B_1, C_3), (A_2, B_3, C_1), (A_3, B_1, C_2) and (A_3, B_2, C_1)

Total number of possible outcomes = 6

1) There are 3 ways where exactly one of her friends receives the correct letter, i.e. (A_1, B_3, C_2), (A_2, B_1, C_3) and (A_3, B_2, C_1)

$$P(\text{exactly one of her friends receives the correct letter}) = \frac{3}{6}$$

$$= \frac{1}{2}$$

2) There are no ways where exactly two of her friends receive the correct letters.

$$P(\text{exactly two of her friends receive the correct letters}) = \frac{0}{6}$$

$$= 0$$

3) There is only one way where all three of her friends receive the correct letters, i.e. (A_1, B_2, C_3).

$$P(\text{all three of her friends receive the correct letters}) = \frac{1}{6}$$

Problems in Real-World Contexts

Problem 1 Singapore Sports Hub

Singapore Sports Hub is a new sports complex in Singapore, in which the highest point above the ground is 82.5 m. Its shape is a dome with a diameter of 310 m as in the figures:

Fig. a

Fig. b

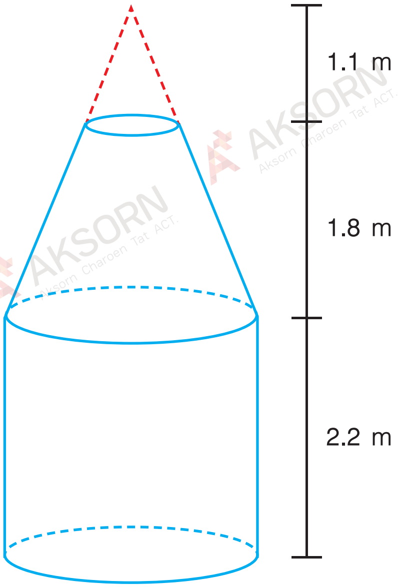
According to the dome shape on a two-dimensional surface, we see that the dome is similar to a parabola. So, it can be expressed by the quadratic equation $y = ax^2 + bx + c$ when x is the distance on the X-axis (in meters) and y is the distance in the Y-axis (in meters), measured from point O (see the picture) where a, b and c are real numbers.

1. According to the data, give 2 more coordinates other than point O.
2. Draw the graph of the dome with the obtained coordinates in Question 1.
3. Determine the general form of the quadratic function $y = ax^2 + bx + c$ and the obtained coordinates on the dome in Question 1, and answer the following questions:
    1) Write down the system of equations that has 3 equations where a, b and c are variables.
    2) Find the values of a, b and c.
    3) Write down the quadratic equation of the dome.
4. From figure b, change the position of the X-axis or Y-axis and find the quadratic equation of the dome one more time by using the above method.

Problem 2 Bottle Tree

Bottle trees are plants in the family of Brachychiton, in which the name is derived from the outer part that is similar to a bottle. The trunk of a bottle tree is thick and can be 18-20 m high when fully developed. If we want to transport a bottle tree to plant somewhere else, the difficulty for traveling depends on how big the tree is, i.e. the bigger, the more difficult.

Take a look at the trunk of a bottle tree compared to three-dimensional geometric figures, which are similar to a conical frustum with the common base to a cylinder. They have the sizes as follows:



If the circumference of the conical trunk is approximately 2.5 m long; the cylinder is approximately 2.2 m high; the conical frustum is approximately 108 m high, and the circumference of the top-frustum surface area is approximately 0.95 m, in which the conical frustum is obtained by cutting the top as a small cone with a circumference of 0.95 m long out of the big cone with a circumference of 2.5 m long, answer the following questions:

1. Find the radius of the trunk of this bottle tree.
2. Find the volume of the bottle tree in cubic meters.
3. We want to transport the bottle tree via a truck, and there are 4 trucks that can carry objects weighing up to half a ton, 1 ton, 3 tons and 5 tons, respectively. If the density of the bottle tree is up to 600 kg per cubic meter, then which truck is the most suitable for this task?

Key Journal Writing

(Page 10)

Suggested answer

Other real-life applications of inequalities: body mass index (BMI), grades, travel times, speed limits, height limits for vehicles, etc.

Performance Task

(Page 28)

Suggested answer

For postage shipping to Singapore and Malaysia, the two countries are in the same zone, so the EMS prices for international parcels are as follows:

Mass (x g)	Postage fee (baht)
$0 < x \leq 250$	720
$250 < x \leq 500$	760
$500 < x \leq 1{,}000$	860
$1{,}000 < x \leq 1{,}500$	970
$1{,}500 < x \leq 2{,}000$	1,070

For postage shipping to New Zealand and Finland, the two countries are in the same zone, so the EMS prices for international parcels are as follows:

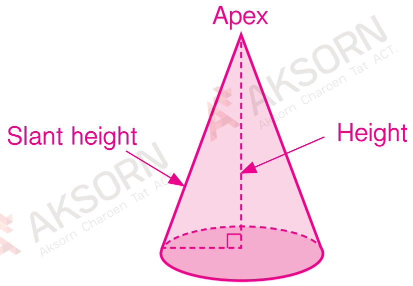
Mass (x g)	Postage fee (baht)
$0 < x \leq 250$	1,760
$250 < x \leq 500$	1,840
$500 < x \leq 1{,}000$	2,020
$1{,}000 < x \leq 1{,}500$	2,210
$1{,}500 < x \leq 2{,}000$	2,390

(Source: Thailand Post)

Key Journal Writing

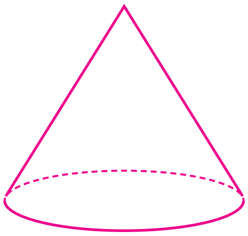
(Page 168)

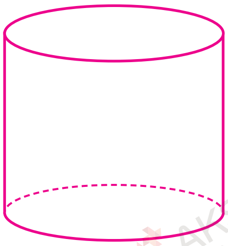
Suggested answer



A cone is a solid in which the base is bounded by a simple closed curve, and the curved surface tapers into a point called the apex, which is opposite the base. The perpendicular height (or height) of a cone is the perpendicular distance from the apex to the base of the cone. The slant height of a right circular cone is the distance from the apex to the circumference of the base.

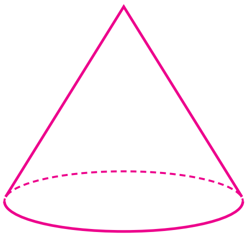
The differences between cones and cylinders are shown below:


Cone

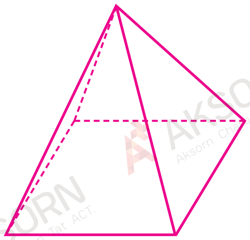

Cylinder

Cone	Cylinder
A cone has one circular base.	A cylinder has two circular bases.
A cone has an apex opposite its base.	A cylinder does not have an apex.

The differences between cones and pyramids are shown below:



Cone



Pyramid

Cone	**Pyramid**
The base of a cone is a circle.	The base of a pyramid is a polygon.
The side of a cone is a curved surface.	The sides of a pyramid are made up of triangles.

Key Problems in Real-World Contexts

Problem 1 (Page 290)

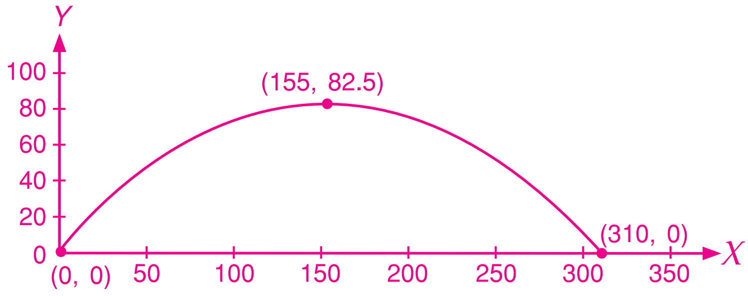
Suggested answer

1. According to figure b, we can find the 2 coordinates of the dome other than point O as follows:

   We know that the highest point above the ground is 82.5 m, and the diameter is 310 m.

   Therefore, the highest point of the dome is the highest point of the parabola, i.e. (155, 82.5), and the point where the dome touches the ground is the X-intercept of the parabola, i.e. (310, 0)

2.



3. 1) The general form of a quadratic function is $y = ax^2 + bx + c$.

   From (0, 0),

   we get $0 = a(0)^2 + b(0) + c$

   $c = 0.$ .....①

   From (155, 82.5),

   we get $82.5 = a(155)^2 + b(155) + 0$

   $82.5 = 155^2a + 155b.$ .....②

   From (310, 0),

   we get $0 = a(310)^2 + b(310) + 0$

   $0 = 310^2a + 310b.$ .....③

2) From equation ② in Question 1,

we get $82.5 = 155^2a + 155b$

$$155b = 82.5 - 155^2a$$

$$b = \frac{82.5 - 155^2a}{155}$$

$$= \frac{33}{62} - 155a. \qquad .....④$$

Substitute ④ into ③

We get $310^2a + 310\left(\frac{33}{62} - 155a\right) = 0$

$$310\left[310a + \left(\frac{33}{62} - 155a\right)\right] = 0$$

$$310a + \frac{33}{62} - 155a = 0$$

$$155a + \frac{33}{62} = 0$$

$$a = -\frac{33}{9{,}610}.$$

Substitute $a = -\frac{33}{9{,}610}$ into ④

We get $b = \frac{33}{62} - 155\left(-\frac{33}{9{,}610}\right)$

$$= \frac{33}{31}.$$

Therefore, $a = -\frac{33}{9{,}610}$, $b = \frac{33}{31}$ and $c = 0$.

3) Method 1

Since the shape of the dome on a plane can be explained by quadratic function $y = ax^2 + bx + c$ where x is distance (in meters) on the X-axis and y is distance (in meters) on the Y-axis, we substitute a, b and c into $y = ax^2 + bx + c$.

Therefore, the shape of the dome is $y = -\frac{33}{9{,}610}x^2 + \frac{33}{31}x$.

Method 2

We know that the graph of the quadratic equation passes through the X-axis at 2 points. The equation can be expressed as $y = a(x - h)(x - k)$.

Since the graph passes through the X-axis at points (310, 0),

we get $y = a(x - 310)(x - 0)$

$$y = ax(x - 310).$$

Since the graph passes through (155, 82.5),

$$\begin{aligned} \text{we get} \quad 82.5 &= 155a(155 - 310) \\ &= -155a \\ a &= -\frac{33}{9,610}. \end{aligned}$$

Therefore, the quadratic equation of the dome is $y = -\frac{33}{9,610}x(x - 310)$.

Method 3

From the fact that we know the highest point on the graph of the quadratic equation, we can make it quadratic by expressing the equation in the form of $y = a(x - 155)^2 + 82.5$.

Since the graph passes through (0, 0),

$$\begin{aligned} \text{we get} \quad 0 &= a(0 - 155)^2 + 82.5 \\ a &= -\frac{33}{9,610}. \end{aligned}$$

Therefore, the quadratic equation of the dome is $y = -\frac{33}{9,610}(x - 155)^2 + 82.5$.

4. We can make the Y-axis the line of symmetry where the X-axis stays the same so that the 3 sets of coordinates are known, i.e. (-155, 0), (0, 82.5) and (155, 0).

Problem 2

(Page 291)

Suggested answer

1. Let r be the radius of the bottle tree trunk.

$$\begin{aligned} \text{We get} \quad 2\pi r &= 2.5 \\ r &= \frac{2.5}{2\pi} \\ &\approx 0.398. \end{aligned}$$

Therefore, the radius of the trunk is approximately 0.398 m.

2. Let r_1 be the radius of the topmost surface area.

$$\begin{aligned} \text{We get} \quad 2\pi r &= 0.95 \\ r &= \frac{0.95}{2\pi} \\ &\approx 0.151. \end{aligned}$$

Therefore, the radius of the topmost surface area is approximately 0.151 m.

Let h_1 be the height of the small cone and h be the height of the big cone.

By using the knowledge of similar triangles, the volume of the bottle tree:

$$= \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_1^2 h_1 + \pi r^2(2.2)$$

$$= \frac{1}{3}\pi\left(\frac{2.5}{2\pi}\right)^2(2.9) - \frac{1}{3}\pi\left(\frac{0.95}{2\pi}\right)^2(1.1) + \pi\left(\frac{2.5}{2\pi}\right)^2(2.2)$$

$$= 1.55 \text{ m}^3$$

Therefore, the volume of the tree is approximately 1.55 m^3.

3 $$\frac{\text{Mass of the bottle tree}}{\text{Volume of the bottle tree}} \leq 600$$

$$\text{Mass of bottle tree} \leq 1.55 \times 600$$

$$\text{Mass of bottle tree} \leq 930 \text{ kg}$$

Therefore, we should select the truck that can carry objects weighing up to 1 ton since the mass of the bottle tree is more than 0.5 ton, but not exceeding 0.93 ton.