

SECOND EDITION

PROBLEMS AND SOLUTIONS IN ENGINEERING MATHEMATICS

PART - I



Dr. T.C. Gupta

**PROBLEMS AND SOLUTIONS
IN
ENGINEERING MATHEMATICS**

PROBLEMS AND SOLUTIONS IN ENGINEERING MATHEMATICS

For
B.E./B.Tech. 1st Year (I & II Semesters)
(Volume-I)

By
Dr. T.C. GUPTA
M.A., Ph. D
Deputy Director-General (Retired),
Indian Council of Medical Research
Ansari Nagar, New Delhi
Formerly Scientist,
Defence Research and Development Organisation,
New Delhi

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PREFACE

I have no words to express my gratitude towards my worthy students on account of whose keen interest and continuous suggestions this book is appearing in its present form as the new Revised Second Edition (Part-I) Keeping in view the changes done by Some Universities in the Syllabus of First and Second Semesters, I have revised it thoroughly to make a Comprehensive Book by rearranging some topics/chapters, adding new and important problems and all the questions asked/set in the previous university examinations.

The response to the First Edition of this book (All the three Volumes for respective Semesters), has been overwhelming and very encouraging which amply indicates that this book has proved extremely useful and helpful to all the B.E./B.Tech. students of Engineering colleges and Institutes throughout the country. Obviously it has helped them to be better equipped and more confident in solving the problems asked in several university examinations.

All the problems have been solved systematically and logically so that even an average student can become familiar with the techniques to solve the mathematical problems independently. Mathematics has always been a problematic subject for the students, hence they have been depending and relying upon private tuitions and coaching academies. It is hoped that this book in its new form will provide utmost utility to its readers.

—Author

SYMBOLS

Greek Alphabets

A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
D	δ	Delta	M	μ	Mu	Y	υ	Upsilon
E	ϵ	Epsilon	N	ν	Nu	Φ	ϕ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	$\omicron$	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega
	$\exists$	There exists		$\forall$	For all			

Metric Weights and Measures

LENGTH

10 millimetres	= 1 centimetre
10 centimetres	= 1 decimetre
10 decimetres	= 1 metre
10 metres	= 1 decametre
10 decametres	= 1 hectometre
10 hectometres	= 1 kilometre

CAPACITY

10 millilitres	= 1 centilitre
10 centilitres	= 1 decilitre
10 decilitres	= 1 litre
10 litres	= 1 dekalitre
10 decalitres	= 1 hectolitre
10 hectolitres	= 1 kilolitre

VOLUME

1000 cubic centimetres	= 1 centigram
1000 cubic decimetres	= 1 cubic metre

AREA

100 square metres	= 1 are
100 ares	= 1 hectare
100 hectares	= 1 square Kilometre

WEIGHT

10 milligrams	= 1 centigram
10 centigrams	= 1 decigram
10 decigrams	= 1 gram
10 grams	= 1 decagram
10 dekagrams	= 1 hectogram
10 hectograms	= 1 kilogram
100 kilograms	= 1 quintal
10 quintals	= 1 metric ton (tonne)

ABBREVIATIONS

kilometre	km	tonne	t
metre	m	quintal	q
centimetre	cm	kilogram	kg
millimetre	mm	gram	g
kilolitre	kl	are	a
litre	l	hectare	ha
millilitre	ml	centiare	ca

IMPORTANT DEFINITIONS AND FORMULAE

1. Convergent, Divergent and Oscillating Sequences:

A Sequence $\{a_n\}$ is said to be **convergent or divergent** if $\lim_{n \rightarrow \infty} a_n$ is **finite or not finite** respectively.

For example, consider the sequence $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots$

Here $a_n = \frac{1}{2^n}$, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ which is finite.

$\Rightarrow$ The sequence $\{a_n\}$ is **convergent**.

Consider the sequences $\{n^2\}$ or $\{-2^n\}$.

Here $a_n = n^2$ or -2^n

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad -\infty.$$

$\Rightarrow$ Both these sequences are **divergent**.

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an **Oscillatory sequence**.

Oscillatory sequences are of 2 types:

(i) A bounded sequence which does not converge, is said to **oscillate finitely**.

For example, consider the sequence $\{(-1)^n\}$.

Here $a_n = (-1)^n$. It is a bounded sequence because there exist two real numbers k and K ($k \leq K$) such that $k \leq a_n \leq K \quad \forall n \in \mathbb{N}$.

$$\{a_n\} = \{-1, 1, -1, 1, -1, \dots\} \Rightarrow -1 \leq a_n \leq 1$$

Now
$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} (-1)^{2n} = 1$$

$$\lim_{n \rightarrow \infty} a_{2n+1} = \lim_{n \rightarrow \infty} (-1)^{2n+1} = -1$$

Thus $\lim_{n \rightarrow \infty} a_n$ does not exist $\Rightarrow$ The sequence does not converge. Hence this sequence **oscillates finitely**.

(ii) An unbounded sequence which does not diverge, is said to **oscillate infinitely**.

Note: When we say $\lim_{n \rightarrow \infty} a_n = l$, it means

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{2n+1} = l.$$

2. Infinite Series: If $\{u_n\}$ is a sequence of real numbers, then the **expression** $u_1 + u_2 + \dots + u_n + \dots$ [i.e., the sum of the terms of the sequence, which are infinite in number] is called **an infinite**

series, usually denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by Σu_n .

3. Partial Sums: If Σu_n is an infinite series where the terms may be +ve or -ve, then $S_n = u_1 + u_2 + \dots + u_n$ is called the n^{th} partial sum of Σu_n . Thus, the n^{th} partial sum of an infinite series is the sum of its first n terms. $S_1, S_2, S_3, \dots$ are the first, second, third, ... partial sums of the series. Since $n \in \mathbb{N}$ (set of natural numbers), $\{S_n\}$ is a sequence called the sequence of partial sums of the infinite series Σu_n . Therefore, *to every infinite series Σu_n , there corresponds a sequence $\{S_n\}$ of its partial sums.*

4. Behaviour of an Infinite Series: An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

5. Geometric Series: The geometric series $1 + x + x^2 + x^3 + \dots$ to ∞

(i) converges if $-1 < x < 1$ i.e., $|x| < 1$

(ii) diverges if $x \geq 1$

(iii) oscillates finitely if $x = -1$

(iv) oscillates infinitely if $x < -1$

6. Theorem: If a series Σu_n is convergent, then

$$\lim_{n \rightarrow \infty} u_n = 0.$$

However, converse of the above theorem is not always true i.e., the n^{th} term may tend to zero as $n \rightarrow \infty$ even if the series is not convergent.

Thus $\lim_{n \rightarrow \infty} u_n = 0 \Rightarrow \Sigma u_n$ may or may not be convergent.

Also $\lim_{n \rightarrow \infty} u_n \neq 0 \Rightarrow \Sigma u_n$ is not convergent.

7. A positive term series either converges or diverges to $+\infty$.

8. There are six different comparison tests which can be used to examine the nature of infinite series. These are described in detail in question number 18 of this chapter.

9. General procedure for testing a series for convergence is given under question 127, depending upon the type of series whether it is alternating, positive term series or a power series.

1. Give an example of a monotonic increasing sequence which is (i) convergent (ii) divergent.

SOLVED PROBLEMS

Sol. (i) Consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$

Since $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \dots$, the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1, \text{ which is finite.}$$

$\therefore$ The sequence is convergent.

(ii) Consider the sequence $1, 2, 3, \dots, n, \dots$

Since $1 < 2 < 3 < \dots < n < \dots$, the sequence is monotonic increasing,

$$a_n = n, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n = \infty$$

$\therefore$ The sequence diverges to $+\infty$.

2. Give an example of a monotonic decreasing sequence which is (i) convergent (ii) divergent.

Sol. (i) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$

Since $1 > \frac{1}{2} > \frac{1}{3} > \dots$, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$\therefore$ The sequence converges to 0.

(ii) Consider the sequence $-1, -2, -3, \dots, -n, \dots$

Since $-1 > -2 > -3 > \dots$, the sequence is monotonic decreasing.

$$a_n = -n, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-n) = -\infty$$

$\therefore$ The sequence diverges to $-\infty$.

3. Discuss the convergence of the sequence $\{a_n\}$, where

$$(i) a_n = \frac{n}{n^2 + 1}$$

$$(ii) a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$$

$$(iii) a_n = \frac{n+1}{n}.$$

Sol. (i) Here,

$$a_n = \frac{n}{n^2 + 1}$$

$$\begin{aligned} \therefore a_{n+1} - a_n &= \frac{n+1}{(n+1)^2 + 1} - \frac{n}{n^2 + 1} = \frac{(n+1)(n^2 + 1) - n(n^2 + 2n + 2)}{(n^2 + 2n + 2)(n^2 + 1)} \\ &= \frac{-n^2 - n + 1}{(n^2 + 2n + 2)(n^2 + 1)} < 0 \quad \forall n \Rightarrow a_{n+1} < a_n \end{aligned}$$

$\Rightarrow \{a_n\}$ is a decreasing sequence

Also, $a_n = \frac{n}{n^2 + 1} > 0 \quad \forall n \Rightarrow \{a_n\}$ is bounded below by 0.

$\therefore \{a_n\}$ is decreasing and bounded below, it is convergent.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = 0$$

$\therefore$ The sequence $\{a_n\}$ converges to zero.

(ii) Here,

$$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$$

= sum of $(n+1)$ terms of a G.P. whose first term is 1 and common ratio is $\frac{1}{3}$

$$= \frac{1 \left(1 - \frac{1}{3^{n+1}} \right)}{\left(1 - \frac{1}{3} \right)}$$

$$\therefore S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$= \frac{3}{2} \left(1 - \frac{1}{3^{n+1}} \right)$$

Now,
$$a_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$$\therefore a_{n+1} - a_n = \frac{1}{3^{n+1}} > 0 \quad \forall n \Rightarrow a_{n+1} > a_n \quad \forall n$$

$\Rightarrow \{a_n\}$ is an increasing sequence.

Also,
$$a_n = \frac{3}{2} \left(1 - \frac{1}{3^{n+1}} \right) < \frac{3}{2} \quad \forall n \Rightarrow \{a_n\} \text{ is bounded above by } \frac{3}{2}.$$

$\therefore \{a_n\}$ is increasing and bounded above, it is convergent.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3}{2} \left(1 - \frac{1}{3^{n+1}} \right) = \frac{3}{2} (1 - 0) = \frac{3}{2}$$

$\therefore$ The sequence $\{a_n\}$ converges to $\frac{3}{2}$.

(iii)
$$a_n = \frac{n+1}{n}$$

$$\therefore a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0 \quad \forall n$$

$$\Rightarrow a_{n+1} < a_n \quad \forall n$$

$\Rightarrow \{a_n\}$ is a decreasing sequence.

Also,
$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded below by 1.

$\therefore \{a_n\}$ is decreasing and bounded below, it is convergent.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1$$

$\therefore$ The sequence $\{a_n\}$ converges to 1.

4. What is an infinite series ? When does it converge, diverge or oscillates (finitely or infinitely) ?

Sol. If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \dots + u_n + \dots$ (i.e., the sum of the terms of the sequence, which are infinite in number) is called an infinite series. The

infinite series $u_1 + u_2 + \dots + u_n + \dots$ is denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by Σu_n .

To every infinite series Σu_n , there corresponds a sequence $\{S_n\}$, where $S_n = u_1 + u_2 + u_3 + \dots + u_n$ is called the partial sum of its first n terms.

The infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely)

(i) Series Σu_n is convergent if $\lim_{n \rightarrow \infty} S_n = \text{finite}$.

(ii) Series Σu_n is divergent if $\lim_{n \rightarrow \infty} S_n = +\infty$ or $-\infty$

(iii) Series Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(iv) Series Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

5. Discuss whether the following series converges or otherwise, $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} + \dots \infty$

Sol. Here,
$$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Putting $n = 1, 2, 3, \dots, n$, we have

$$u_1 = 1 - \frac{1}{2}, u_2 = \frac{1}{2} - \frac{1}{3}, u_3 = \frac{1}{3} - \frac{1}{4}$$

$$u_4 = \frac{1}{4} - \frac{1}{5}, \dots, u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding,
$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = 1 - 0 = 1$$

$\Rightarrow \{S_n\}$ converges to 1 $\Rightarrow \sum u_n$ converges to 1.

6. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

Sol. Let,
$$u_n = \frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \frac{1}{2} \left[\left(\frac{1}{n} - \frac{1}{n+1} \right) + \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right]$$

Putting $n = 1, 2, 3, \dots, n$, we obtain

$$u_1 = \frac{1}{2} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) \right], \quad u_2 = \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) \right]$$

$$u_3 = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) \right], \quad \dots\dots\dots$$

$$u_n = \frac{1}{2} \left[\left(\frac{1}{n} - \frac{1}{n+1} \right) + \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right]$$

Adding,
$$S_n = \frac{1}{2} \left[\left(1 - \frac{1}{n+1} \right) + \left(\frac{1}{2} - \frac{1}{n+2} \right) \right]$$

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{2} \left(1 - 0 + \frac{1}{2} - 0 \right)$$

$$= \frac{3}{4}, \text{ a finite quantity.}$$

$\Rightarrow$ the given sequence $\langle S_n \rangle$ converges to $\frac{3}{4}$. Hence the given infinite series $\sum_{n=1}^{\infty} u_n$ converges

to $\frac{3}{4}$.

7. Show that the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1}$ converges to 4.

Sol. Let $u_n = \left(\frac{3}{4}\right)^{n-1}$
 then $S_n = u_1 + u_2 + u_3 + \dots + u_n$

$$= 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots + \left(\frac{3}{4}\right)^{n-1} = \frac{1 \left[1 - \left(\frac{3}{4}\right)^n \right]}{1 - \frac{3}{4}} = 4 \left(1 - \left(\frac{3}{4}\right)^n \right)$$

$$\lim_{n \rightarrow \infty} S_n = 4[1 - 0] \quad [\because \text{if } |x| < 1, \text{ then } x^n \rightarrow 0 \text{ as } n \rightarrow \infty]$$

$$= 4, \text{ a finite quantity.}$$

$\Rightarrow$ the sequence $\langle S_n \rangle$ converges to 4. Hence the given series $\sum_{n=1}^{\infty} u_n$ converges to 4.

8. Examine convergence or otherwise of the series, $1^2 + 2^2 + 3^2 + \dots + n^2 + \dots$

Sol. $S_n = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\lim_{n \rightarrow \infty} S_n = +\infty$$

$\Rightarrow \langle S_n \rangle$ diverges to $+\infty$

$\Rightarrow$ the given series diverges to $+\infty$.

9. Show that the series $-1 - 2 - 3 - \dots - n - \dots$ diverges to $-\infty$.

Sol. $S_n = -1 - 2 - 3 - \dots - n = -(1 + 2 + 3 + \dots + n)$

$$= -\frac{n(n+1)}{2}$$

$$\lim_{n \rightarrow \infty} S_n = -\infty \Rightarrow \langle S_n \rangle \text{ diverges to } -\infty.$$

$\Rightarrow$ the given series diverges to $-\infty$.

10. Examine the convergence or otherwise of the series $\sum_{n=1}^{\infty} (-1)^{n-1}$

Sol. $S_n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ to n terms
 $= 1$ or 0 according as n is odd or even.

The subsequence $\langle S_{2n-1} \rangle$ converges to 1 while the subsequence $\langle S_{2n} \rangle$ converges to 0.

$\Rightarrow \langle S_n \rangle$ is not convergent.

Since $\langle S_n \rangle$ is bounded, $\therefore \langle S_n \rangle$ oscillates finitely

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \text{ oscillates finitely.}$$

11. Test the convergence of the series $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \dots$ to ∞ .

Sol. Here $S_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \dots$ to n terms
 $= 0, 5$ or 1 according as the number of terms is $3m, 3m+1, 3m+2$.

Clearly, S_n does not tend to a unique limit. Since $\langle S_n \rangle$ is bounded, it oscillates finitely.

$\therefore$ the given series oscillates finitely.

12. Show that the series $\sum_{n=1}^{\infty} n(-1)^n$ oscillates infinitely.

Sol. Here,

$$S_n = -1 + 2 - 3 + 4 - 5 + 6 + \dots + \text{to } n \text{ terms}$$

$$= \begin{cases} -\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

The subsequence $\langle S_{2n-1} \rangle$ diverges to $-\infty$, while the subsequence $\langle S_{2n} \rangle$ diverges to $+\infty$
 $\therefore \langle S_n \rangle$ oscillates infinitely.

$$\Rightarrow \sum_{n=1}^{\infty} n(-1)^n \text{ oscillates infinitely.}$$

13. Test the nature of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \infty$.

Sol. Here,

$$u_n = \frac{1}{2^{n-1}}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} = \frac{1 \left[1 - \left(\frac{1}{2} \right)^n \right]}{1 - \frac{1}{2}} = 2 \left[1 - \frac{1}{2^n} \right]$$

$$\lim_{n \rightarrow \infty} S_n = 2(1 - 0) = 2, \text{ a finite quantity.}$$

$\Rightarrow$ The sequence $\langle S_n \rangle$ converges to 2.

$\Rightarrow$ The infinite series $\sum_{n=1}^{\infty} u_n$ converges to 2.

14. Test the nature of the series $-1 - 8 - 27 - 64 - \dots - \infty$

Sol.

$$S_n = -1 - 8 - 27 - 64 - \dots - n^3 = -(1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3)$$

$$= - \left[\frac{n(n+1)}{2} \right]^2 \quad (\text{Sum of cubes of } n \text{ natural numbers})$$

$$= - \frac{n^2}{4} (n^2 + 2n + 1) = - \frac{n^4}{4} \left[1 + \frac{2}{n} + \frac{1}{n^2} \right]$$

$$\lim_{n \rightarrow \infty} S_n = -\infty \Rightarrow \langle S_n \rangle \text{ diverges to } -\infty$$

$\therefore$ The given series diverges to $-\infty$.

15. Examine the series $1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \dots \infty$ for its nature.

Sol. The given series is $1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \dots \infty$

It is a geometric series with common ratio $r = -\frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, the given series is convergent.

- 16.** Test the nature of the series $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 \dots \infty$

Sol. Here,

$$\begin{aligned} S_n &= 7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 + \dots \text{ to } n \text{ terms} \\ &= 0, 7 \text{ or } 1 \text{ according as the number of terms is } 3m, 3m + 1, \\ &\quad 3m + 2. \end{aligned}$$

Clearly, S_n does not tend to a unique limit. Since $\langle S_n \rangle$ is bounded, it oscillates finitely.

$\Rightarrow$ the given series oscillates finitely.

- 17.** Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \text{ does not converge using Cauchy's general principle of convergence.}$$

Sol. Cauchy's General Principle of Convergence is that necessary and sufficient condition for the infinite series $\sum_{n=1}^{\infty} u_n$ to converge is that given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$|u_{m+1} + u_{m+2} + \dots + u_n| < \varepsilon \quad \forall n > m$$

Let us suppose that the given series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

Let us take $\varepsilon = \frac{1}{2}$

By Cauchy's general principle of convergence, there exists a positive integer m such that

$$\left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| < \frac{1}{2} \quad \forall n > m$$

$$\text{or} \quad \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} < \frac{1}{2} \quad \forall n > m \quad \dots(1)$$

By taking $n = 2m$ in (1), we observe that

$$= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \dots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$$

$$\text{i.e.,} \quad \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} > \frac{1}{2} \quad \text{where } n = 2m > m.$$

This contradicts (1).

$\therefore$ Our supposition is wrong.

Thus, the given series does not converge.

- 18.** What are the various comparison tests which can be used to examine the nature of infinite series ?

Sol. Following tests may be used to find the nature of infinite series :

Test I. If Σu_n and Σv_n are two series of positive terms and Σv_n is convergent and there is a positive constant k such that $u_n \leq kv_n, \forall n > m$, then Σu_n is also convergent.

Test II. If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall n > m$, then Σu_n is also divergent.

Test III. If Σu_n and Σv_n are two positive term series and there exist two positive constants H and

K (independent of n) and a positive integer m such that $H < \frac{u_n}{v_n} < K \forall n > m$, then the two series

Σu_n and Σv_n converge or diverge together.

Test IV. Let Σu_n and Σv_n be two positive term series :

(i) If $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.

(ii) If $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.

(iii) If $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \infty$ and Σv_n diverges, then Σu_n also diverges.

Test V. Let Σu_n and Σv_n be two positive term series.

(i) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \forall n > m$ and Σv_n is convergent, then Σu_n is also convergent.

(ii) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \forall n > m$ and Σv_n is divergent then Σu_n is also divergent.

Test VI. The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \infty$ converges if $p > 1$ and diverges if $p \leq 1$

This series is also called Hyper Harmonic series or P-series. This test is an important test for comparison.

19. Test the convergence of the series: $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots \infty$

Sol. Given series is

$$\begin{aligned} \frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots \infty \\ = \left(\frac{3}{5} + \frac{3}{5^3} + \dots \infty \right) + \left(\frac{4}{5^2} + \frac{4}{5^4} + \dots \infty \right) = \Sigma u_n + \Sigma v_n \text{ (say)} \end{aligned}$$

Now Σu_n is a G.P. with common ratio $\frac{1}{5^2}$ which is numerically less than 1.

$\therefore \Sigma u_n$ is convergent.

Further Σv_n is also a G.P. with common ratio $= \frac{1}{5^2}$ which is numerically less than 1.

$\therefore \Sigma v_n$ is convergent.

Hence the given series viz. $\Sigma(u_n + v_n)$ is also convergent.

20. Examine the convergence of the series : $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots$

Sol. Here,
$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let us compare this series with $\frac{1}{n^2}$ i.e., Σu_n with Σv_n , where $\Sigma v_n = \frac{1}{n^2}$

Now,
$$\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{(1)(1)} = 2.$$

which is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore \Sigma v_n$ is convergent. Hence, this implies Σu_n is convergent.

21. Test the convergence of the series : $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \dots \infty$

Sol. Given series can be written as

$$\begin{aligned} & 1 + \frac{1}{(2^2)^{2/3}} + \frac{1}{(3^2)^{2/3}} + \frac{1}{(4^2)^{2/3}} + \dots \infty \\ & = \frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} + \dots \infty = \Sigma \frac{1}{n^{4/3}} = \Sigma \frac{1}{n^p} \text{ with } p = \frac{4}{3} > 1 \end{aligned}$$

$\therefore$ By p -series test (Q. 18, Test VI), the given series is convergent.

22. Examine the convergence of the series : $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots$

Sol. Here
$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n , where $v_n = \frac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \text{ which is finite and non-zero.}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{1}{2}$.

$\therefore \Sigma v_n$ is divergent $\Rightarrow \Sigma u_n$ is divergent.

23. Test the convergence of the series : $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \dots + \sqrt{\frac{n}{2(n+1)}} + \dots$

Sol.

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow \Sigma u_n$ does not converge. Since the given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence, the given series is divergent.

24. Examine the convergence of the series : $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots$ (M.D.U., Dec., 2008)

Sol. As addition or deletion of a finite number of terms does not alter the nature of the series, ignoring 1st term of the series, we have

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n} = \frac{n^n}{n\left(1+\frac{1}{n}\right)n^n \cdot \left(1+\frac{1}{n}\right)^n} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Let us take $v_n = \frac{1}{n}$,

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \quad \left[\because \lim_{n \rightarrow \infty} \left(1+\frac{1}{n}\right)^n = e \right]$$

This is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 1$. $\therefore \Sigma v_n$ is divergent

Thus, Σu_n is divergent.

25. Examine the convergence of the series $\sum \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$.

Sol. Here,

$$u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$$

Rationalising numerator and denominator

$$u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{n^p \cdot \sqrt{n} \left(\sqrt{\frac{n+1}{n}} + 1 \right)} = \frac{1}{n^{p+\frac{1}{2}} \left(\sqrt{1+\frac{1}{n}} + 1 \right)}$$

Let us take

$$v_n = \frac{1}{n^{p+\frac{1}{2}}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}, \text{ which is finite and non-zero.}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^{p+\frac{1}{2}}}$ is convergent if $p + \frac{1}{2} > 1$ and divergent if $p + \frac{1}{2} \leq 1$ (Test VI, Q. 18)

i.e., convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

$\therefore \Sigma u_n$ is convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

26. Test the convergence of the series : $(n^3 + 1)^{\frac{1}{3}} - n$.

(M.D.U. May 2007)

Sol. Here

$$\begin{aligned} u_n &= (n^3 + 1)^{\frac{1}{3}} - n \\ &= \left[n^3 \left(1 + \frac{1}{n^3} \right)^{\frac{1}{3}} \right] - n = n \left(1 + \frac{1}{n^3} \right)^{\frac{1}{3}} - n = n \left[\left(1 + \frac{1}{n^3} \right)^{\frac{1}{3}} - 1 \right] \\ &= n \left[1 + \frac{1}{3} \cdot \frac{1}{n^3} + \frac{\frac{1}{3} \left(\frac{1}{3} - 1 \right)}{2!} \left(\frac{1}{n^3} \right)^2 - 1 \right] \end{aligned}$$

(Using binomial expansion)

$$= \frac{n}{n^3} \left[\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \dots \right] = \frac{1}{n^2} \left[\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \dots \right]$$

Take,
$$v_n = \frac{1}{n^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \dots \right) = \frac{1}{3}, \text{ which is finite and non-zero.}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$. $\therefore \Sigma v_n$ is convergent

Thus, Σu_n is convergent.

27. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$, $x > 0$.

Sol. Here, $u_n = \frac{1}{x^n + x^{-n}} = \frac{x^n}{x^{2n} + 1}$

Case I : When $0 < x < 1$, take $v_n = x^n$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{x^{2n} + 1} = \frac{1}{0 + 1} = 1 \quad [\because x^{2n} \rightarrow 0 \text{ as } n \rightarrow \infty]$$

which is non-zero and finite. $\therefore \Sigma u_n$ and Σv_n converge or diverge together

But $\Sigma v_n = \Sigma x^n = x + x^2 + x^3$ is an infinite geometric series with common ratio $x < 1$.

$\therefore \Sigma v_n$ is convergent. Thus, Σu_n is convergent.

Case II : When $x > 1$ so that $0 < \frac{1}{x} < 1$

$$u_n = \frac{x^n}{x^{2n} + 1} = \frac{\frac{1}{x^n}}{1 + \frac{1}{x^{2n}}}$$

Take, $v_n = \frac{1}{x^n}$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{x^{2n}}} = \frac{1}{1 + 0} = 1$$

which is non-zero and finite. $\therefore \Sigma u_n$ and Σv_n converge or diverge together

But $\Sigma v_n = \Sigma \frac{1}{x^n}$ is an infinite geometric series with common ratio $\frac{1}{x} < 1$.

$\therefore \Sigma v_n$ is convergent. Thus, Σu_n is convergent.

Case III : When $x = 1$, $u_n = \frac{1}{2}$

$\lim_{n \rightarrow \infty} u_n = \frac{1}{2} \neq 0$. Therefore, Σu_n is divergent. Hence Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

28. Show that $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$ converges.

Sol. Here, $u_n = \frac{1}{n} \sin \frac{1}{n}$

Since $\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$, take $v_n = \frac{1}{n^2}$

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sin \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

which is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore \Sigma v_n$ is convergent. Thus, Σu_n is convergent.

29. Examine the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$.

Sol. We know that $\log n \leq n \forall n \geq 2$

$$\therefore \frac{1}{\log n} \geq \frac{1}{n} \forall n \geq 2$$

Since $\Sigma \frac{1}{n}$ is a divergent series

$\therefore$ By comparison test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is divergent.

[Reference Test II, Q. 18]

30. Test the convergence of the series $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots$

Sol. Here,

$$\begin{aligned} u_n &= \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n} \left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{n^3 \left[\left(1 + \frac{2}{n} \right)^3 - \frac{1}{n^3} \right]} \\ &= \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{5/2} \left[\left(1 + \frac{2}{n} \right)^3 - \frac{1}{n^3} \right]} \end{aligned}$$

Take,

$$v_n = \frac{1}{n^{5/2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1 + \frac{2}{n} \right)^3 - \frac{1}{n^3}} = \frac{\sqrt{1+0} - 0}{(1-0)^3 - 0} = 1$$

which is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^{5/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{5}{2} > 1$.

[Ref. Test VI, Q. 18]

$\therefore \Sigma v_n$ is convergent. Thus, Σu_n is convergent.

31. Test the convergence or divergence of the series, $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \dots \infty$. (M.D.U., May 2008)

Sol. Here,

$$u_n = \frac{n+1}{n^p}$$

∴ Now, let $v_n = \frac{n}{n^p}$ or $\frac{1}{n^{p-1}}$ be series for comparison (auxiliary series)

$$\therefore \frac{u_n}{v_n} = \frac{n+1}{n} \quad \text{or} \quad 1 + \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$$

which is finite and non-zero.

Hence, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^{p-1}}$ is

(i) convergent if $p-1 > 1$ i.e., $p > 2$. (ii) divergent if $p-1 \leq 1$ i.e., $p \leq 2$.

∴ Σu_n converges if $p > 2$ and diverges if $p \leq 0$.

32. Test for convergence the series $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \dots + \frac{1}{n^2+1} + \dots$

Sol. Here, $u_n = \frac{1}{n^2+1}$ and let us take $\Sigma v_n = \Sigma \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{n^2}{n^2+1} = \frac{1}{\left(1 + \frac{1}{n^2}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n^2}\right)} = 1 \quad (\text{finite and non-zero})$$

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent ($p = 2 > 1$), therefore Σu_n is also convergent.

33. Discuss the nature of the series, $\Sigma u_n = \Sigma \left(\frac{2}{5n+1}\right)$.

Sol. Here, $u_n = \frac{2}{5n+1}$ and let, $\Sigma v_n = \Sigma \frac{1}{n}$

$$\therefore v_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{2n}{5n+1} = \frac{2}{\left(5 + \frac{1}{n}\right)}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2}{\left(5 + \frac{1}{n}\right)} = \frac{2}{5}$$

which is a finite non-zero quantity. ∴ Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is known to be divergent. $\Rightarrow \Sigma u_n$ is also divergent series.

34. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^p}$. (M.D.U., Dec., 2007)

Sol. Here,

$$u_n = \frac{1}{n^p (n+1)^p} \quad \text{or} \quad u_n = \frac{1}{n^{2p} \left(1 + \frac{1}{n}\right)^p}$$

Take,

$$v_n = \frac{1}{n^{2p}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^p} = \frac{1}{(1+0)^p} = 1$$

which is finite and non-zero. $\therefore \Sigma u_n$ and Σv_n converge or diverge together.

Now, $\Sigma v_n = \Sigma \frac{1}{n^{2p}}$ will be convergent for $2p > 1$ and divergent for $2p \leq 1$.

(Please See Q. 18, Test VI ; Hyper Harmonic series or P series test).

Hence Σu_n i.e., the given series will converge for $2p > 1$ or $p > \frac{1}{2}$ and will diverge for $p \leq \frac{1}{2}$.

35. Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$.

Sol. Here,

$$\begin{aligned} u_n &= \frac{\sqrt{n}}{n^2 + 1} \\ &= \frac{\sqrt{n}}{n^2 + 1} \times \frac{\sqrt{n}}{\sqrt{n}} = \frac{n}{\sqrt{n} (n^2 + 1)} = \frac{1}{n^{3/2} \left(1 + \frac{1}{n^2}\right)} \end{aligned}$$

Let us take $v_n = \frac{1}{n^{3/2}}$ for the comparison series Σv_n .

$$\Rightarrow \frac{u_n}{v_n} = \frac{1}{1 + \frac{1}{n^2}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = \frac{1}{1+0} = 1 \quad (\text{finite and non-zero})$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

$\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ is of the form $\Sigma \frac{1}{n^p}$, where $p = \frac{3}{2} > 1$ is convergent by P -series test.

(Test VI, Q. 18).

Hence Σu_n i.e., the given series is also convergent.

Q. 36. Examine the series $\sum \frac{2n^3 + 5}{4n^5 + 1}$ for convergence or otherwise.

Sol. Here,

$$u_n = \frac{2n^3 + 5}{4n^5 + 1} = \frac{\left(1 + \frac{5}{2n^3}\right)}{2\left(1 + \frac{1}{4n^5}\right)n^2}$$

Let

$$v_n = \frac{1}{n^2}$$

$\therefore$

$$\frac{u_n}{v_n} = \frac{1 + \frac{5}{2n^3}}{2\left(1 + \frac{1}{4n^5}\right)}$$

$\Rightarrow$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2(1+0)} = \frac{1}{2} \text{ (finite and non-zero)}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent (p series, $p = 2 > 1$)

Hence, the given series Σu_n is also convergent.

37. Test the convergence or divergence of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots \infty$

Sol. Here,

$$u_n = \frac{1}{n(n+1)(n+2)} \quad \text{or} \quad u_n = \frac{1}{n^3 \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}$$

Let us take

$$v_n = \frac{1}{n^3}$$

$\therefore$

$$\frac{u_n}{v_n} = \frac{1}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{(1+0)(1+0)} = 1 \text{ (finite and non-zero)}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

But the series $\Sigma v_n = \Sigma \frac{1}{n^3}$ is of the form $\Sigma \frac{1}{n^p}$, where $p = 3 > 1$ and converges.

$\therefore \Sigma u_n$ i.e., the given series is also convergent.

38. Examine the series for convergence or otherwise $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \dots \infty$

Sol. Here,

$$u_n = \frac{\sqrt{n}}{2n+3} = \frac{\sqrt{n}}{2n\left(1 + \frac{3}{2n}\right)} \quad \text{or} \quad \frac{1}{2n^{\frac{1}{2}}\left(1 + \frac{3}{2n}\right)}$$

Let us take

$$v_n = \frac{1}{\sqrt{n}}$$

$\therefore$

$$\frac{u_n}{v_n} = \frac{1}{2\left(1 + \frac{3}{2n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2(1+0)} = \frac{1}{2} \text{ (finite and non-zero)}$$

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is of the form $\Sigma \frac{1}{n^p}$ where $p = \frac{1}{2} < 1$ and is divergent. (See Test VI, Q. 18)

$\therefore$ The given series i.e., Σu_n is also divergent.

39. Test the series for convergence or divergence. $\sum (\sqrt{n^4 + 1} - \sqrt{n^4 - 1})$

(M.D.U., Dec. 2005 ; May 2009)

Sol. Here,

$$\begin{aligned} u_n &= (n^4 + 1)^{\frac{1}{2}} - (n^4 - 1)^{\frac{1}{2}} = n^2 \left(1 + \frac{1}{n^4}\right)^{\frac{1}{2}} - n^2 \left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \\ &= n^2 \left[\left(1 + \frac{1}{n^4}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \right] \\ &= n^2 \left[\left(1 + \frac{1}{2n^4} + \frac{\frac{1}{2} \left(\frac{1}{2} - 1\right)}{2!} \cdot \frac{1}{n^8} + \dots \right) - \left(1 + \frac{1}{2n^4} + \frac{\frac{1}{2} \left(\frac{1}{2} - 1\right)}{2!} \cdot \left(\frac{-1}{n^4}\right)^2 + \dots \right) \right] \\ &\quad \text{(Using Binomial Expansion)} \\ &= n^2 \left[\left(1 + \frac{1}{2n^4} - \frac{1}{8n^8} + \dots\right) - \left(1 - \frac{1}{2n^4} - \frac{1}{8n^8} - \dots\right) \right] \\ &= n^2 \left[2 \cdot \frac{1}{2n^4} \right] = \frac{1}{n^2} \quad \therefore \Sigma u_n = \Sigma \frac{1}{n^2} \end{aligned}$$

which is of $\Sigma \frac{1}{n^p}$, $p = 2 > 1$ and is convergent.

Alternatively :

We may rationalise the u_n and apply the comparison test to examine the given series for convergence as follows :

$$u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1} > 0 \quad \forall n$$

$\therefore$ The given series is a series of positive terms.

Also,

$$\begin{aligned} u_n &= \frac{\sqrt{n^4 + 1} - \sqrt{n^4 - 1}}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} \times \sqrt{n^4 + 1} + \sqrt{n^4 - 1} \\ &= \frac{n^4 + 1 - n^4 + 1}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} = \frac{2}{n^2 \left[\sqrt{1 + \frac{1}{n^4}} + \sqrt{1 - \frac{1}{n^4}} \right]} \end{aligned}$$

For sufficiently large n , u_n behaves like $\frac{1}{n^2}$

Let us take $v_n = \frac{1}{n^2}$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left\{ \frac{2}{\sqrt{1 + \frac{1}{n^4}} + \sqrt{1 - \frac{1}{n^4}}} \right\} = \frac{2}{1+1} = 1$$

which is finite and non-zero.

Therefore, by comparison test, the series Σu_n and Σv_n converge or diverge together.

Since $\Sigma \frac{1}{n^2}$ is convergent, the given series is convergent.

40. Test the convergence or divergence of the series $\sum (\sqrt{n^2 + 1} - n)$. (M.D.U. Dec., 2006)

Sol. Here,

$$\begin{aligned} u_n &= \sqrt{n^2 + 1} - n \\ &= (n^2 + 1)^{\frac{1}{2}} - n = n \left(1 + \frac{1}{n^2} \right)^{\frac{1}{2}} - n \\ &= n \left[1 + \frac{1}{2n^2} + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \cdot \frac{1}{n^4} + \dots \right] = \frac{1}{2n} - \frac{1}{8n^3} + \dots \\ &= \frac{1}{2n} \left[1 - \frac{1}{4n^2} + \dots \right] \end{aligned}$$

Let us take $v_n = \frac{1}{n}$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left[1 - \frac{1}{4n^2} + \dots \right] = \frac{1}{2} (1 - 0) = \frac{1}{2}$$

which is finite and non-zero.

$\therefore \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \frac{1}{n}$ is divergent ($p = 1$). Hence the given series i.e., Σu_n is divergent.

41. Test the convergence or divergence of the series : $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots$

$$\text{Sol. Here, } u_n = \frac{1+2+3+\dots+n}{1^2+2^2+3^2+\dots+n^2} \quad \text{or} \quad u_n = \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}$$

(Using formulae for sum and sum of squares of n natural numbers)

$$\Rightarrow u_n = \frac{3}{2n+1}$$

$$\text{Take } v_n = \frac{1}{n}$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \frac{3n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left(\frac{6n+3-3}{2n+1} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left(3 - \frac{3}{2n+1} \right) = \frac{1}{2} (3-0) = \frac{3}{2} \end{aligned}$$

which is finite and non-zero. $\therefore \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent.

$\therefore \Sigma u_n$ or the given series is *divergent*.

42. State D'Alembert's Ratio Test. Use it to test the convergence of the series

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \dots$$

Sol. D'Alembert's Ratio Test is that

If Σu_n is a positive term series and $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$, then

(i) Σu_n is convergent if $l < 1$

(ii) Σu_n is divergent if $l > 1$.

The test fails if $l = 1$; i.e., no conclusion can be drawn about the convergence or divergence of the series. The series may converge or it may diverge.

In practice, D'Alembert's Ratio Test is used in the following form :

When Σu_n is a positive term series and $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = l$, then Σu_n is convergent if $l > 1$ and is divergent if $l < 1$.

Let us use this test to examine the convergence or otherwise of the given series :

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots$$

$$u_n = \frac{n^2 (n+1)^2}{n!}$$

$$\therefore u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 \cdot (n+1)^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2 (n+2)^2} = \frac{n^2 \cdot (n+1) \cdot n!}{n! (n+2)^2}$$

$$= \frac{n^3 \cdot \left(1 + \frac{1}{n}\right)}{n^2 \cdot \left(1 + \frac{2}{n}\right)^2} = n \cdot \frac{1 + \frac{1}{n}}{\left(1 + \frac{2}{n}\right)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} n \cdot \frac{\left(1 + \frac{1}{n}\right)}{\left(1 + \frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

43. Examine the convergence of the series : $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2^{n-1} + 1} + \dots$

Sol. Here,
$$u_n = \frac{1}{2^{n-1} + 1}, \quad \therefore u_{n+1} = \frac{1}{2^n + 1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{2^n + 1}{2^{n-1} + 1} = \frac{2^n \left(1 + \frac{1}{2^n}\right)}{2^{n-1} \left(1 + \frac{1}{2^{n-1}}\right)} = 2 \cdot \frac{1 + \frac{1}{2^n}}{1 + \frac{1}{2^{n-1}}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{2 \left(1 + \frac{1}{2^n}\right)}{\left(1 + \frac{1}{2^{n-1}}\right)} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, the given series Σu_n is convergent.

44. Test the series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots (p > 0)$ for convergence or otherwise.

Sol. Here,
$$u_n = \frac{n^p}{n!}$$

$$\therefore u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{n^p \cdot (n+1)!}{n! (n+1)^p} = \frac{n^p \cdot (n+1) \cdot n!}{n! (n+1) \cdot (n+1)^{p-1}} = \frac{n^p}{(n+1)^{p-1}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{n^p}{n^{p-1} \left(1 + \frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1 + \frac{1}{n}\right)^{p-1}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{n}{\left(1 + \frac{1}{n}\right)^{p-1}} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, the series Σu_n is convergent.

45. Examine the series $\sum_{n=1}^{\infty} \frac{3.6.9. \dots 3n}{4.7.10. \dots (3n+1)} \cdot \frac{5^n}{3n+2}$ for convergence by applying D'Alembert's Ratio Test.

Sol. Here,
$$u_n = \frac{3.6.9. \dots 3n}{4.7.10. \dots u_{n+1}} \cdot \frac{5^n}{3n+2}$$

$$u_{n+1} = \frac{3.6.9. 3n \cdot (3n+3)}{4.7.10. u_{n+1} \cdot (3n+1)(3n+4)} \cdot \frac{5^{n+1}}{3n+5}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(3n+4)(3n+5) \cdot 1}{(3n+3)(3n+2) \cdot 5} = \frac{\left(1 + \frac{4}{3n}\right) \left(1 + \frac{5}{3n}\right)}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{3n}\right)} \cdot \frac{1}{5}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1 \cdot 1}{1 \cdot 1} \cdot \frac{1}{5} \quad \text{or} \quad \frac{1}{5} < 1$$

$\therefore$ By D'Alembert's Ratio Test, the series Σu_n is divergent.

46. Test the convergence of the series $1 + \frac{1^2 \cdot 2^2}{1 \cdot 3 \cdot 5} + \frac{1^2 \cdot 2^2 \cdot 3^2}{1 \cdot 3 \cdot 5 \cdot 7} + \dots \infty$

Sol. Here,
$$u_n = \frac{1^2 \cdot 2^2 \cdot 3^2 \dots n^2}{1 \cdot 3 \cdot 5 \cdot 7 \dots (4n-5)(4n-3)}$$

It is seen that in the denominator, 2 new factors are added with every term and u_n has $(2n-1)$ factors in the denominator.

$$u_{n+1} = \frac{1^2 \cdot 2^2 \cdot 3^2 \dots n^2 (n+1)^2}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \dots (4n-5)(4n-3)(4n-1)(4n+1)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(4n-1)(4n+1)}{(n+1)^2} = \frac{16n^2-1}{(n+1)^2} = \frac{16 - \frac{1}{n^2}}{\left(1 + \frac{1}{n}\right)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{16 - \frac{1}{n^2}}{\left(1 + \frac{1}{n}\right)^2} = \frac{16-0}{(1+0)^2} = 16 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

47. Examine the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n \cdot (x > 0)$

Sol. Here,
$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$$

$$\therefore u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot x^{n+1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1 + \frac{1}{n}} \cdot \frac{1 + \frac{2}{n} + \frac{2}{n^2}}{1 + \frac{1}{n^2}}} \cdot \frac{1}{x}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\frac{1}{x} > 1$, i.e., $x < 1$ and diverges if $\frac{1}{x} < 1$ or $x > 1$.

When $x = 1$, the test fails.

When $x = 1$,

$$u_n = \sqrt{\frac{n}{n^2 + 1}} = \sqrt{\frac{n}{n^2 \left(1 + \frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1 + \frac{1}{n^2}}}$$

Let us take $v_n = \frac{1}{\sqrt{n}}$. Then $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^2}}} = 1$, which is finite and non-zero.

$\therefore$ By Comparison test, Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is divergent as

it is of the form $\Sigma \frac{1}{n^p}$ ($p = \frac{1}{2} < 1$). $\therefore \Sigma u_n$ is divergent.

Hence, the given series Σu_n

(i) converges if $x < 1$

(ii) diverges if $x \geq 1$.

Note : Two tests have been applied here to examine the nature of the given series.

48. Discuss the convergence or otherwise of the series for $x > 0$: $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$

(M.D.U., Dec., 2008)

Sol. Here,

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$\therefore u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2} = \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \cdot \sqrt{1 + \frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D'Alembert's Ratio Test Σu_n converges if $\frac{1}{x^2} > 1$ i.e., $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$ i.e., $x^2 > 1$

When $x^2 = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1 + \frac{1}{n}\right)}$

Take $v_n = \frac{1}{n^{3/2}}$, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$, which is finite and non-zero. By comparison test, Σu_n is

convergent since Σv_n is of the form $\Sigma \frac{1}{n^p}$ ($p = 3/2 > 1$).

Hence the given series is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

49. Examine the convergence or divergence of the series

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots + \frac{2^{n+1}-2}{2^{n+1}+1} \cdot x^n + \dots \quad (x > 0).$$

Sol. Leaving a side the first term (because addition or deletion of a finite number of terms does not alter the nature of the series), here we have

$$u_n = \frac{2^{n+1}-2}{2^{n+1}+1} \cdot x^n$$

$$\therefore u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1} \cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{2^{n+2}+1}{2^{n+2}-2} \cdot \frac{1}{x} = \frac{2^{n+1}\left(1 - \frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1 + \frac{1}{2^{n+1}}\right)} \cdot \frac{2^{n+2}\left(1 + \frac{1}{2^{n+2}}\right)}{2^{n+2}\left(1 - \frac{2}{2^{n+2}}\right)} \cdot \frac{1}{x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^{n+1}}} \cdot \frac{1 + \frac{1}{2^{n+1}}}{1 - \frac{1}{2^{n+1}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\frac{1}{x} > 1$ i.e., $x < 1$ and diverges if $\frac{1}{x} < 1$ i.e., $x > 1$.

When $x = 1$, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1} = \frac{2^{n+1}\left(1 - \frac{1}{2^n}\right)}{2^{n+1}\left(1 + \frac{1}{2^{n+1}}\right)}$

$\lim_{n \rightarrow \infty} u_n = 1 \neq 0 \Rightarrow \Sigma u_n$ does not converge. Being a series of positive terms, it must diverge.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Note : Here $\lim_{n \rightarrow \infty} u_n = 1$. Now it may be noted that necessary condition for convergence of a series

Σu_n is that $\lim_{n \rightarrow \infty} u_n = 0$. However, the converse is not true always i.e., the n^{th} term may tend to

zero as $n \rightarrow \infty$ even if the series is not convergent. Thus $\lim_{n \rightarrow \infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of $\sum u_n$. As a corollary, if limit of u_n (when $n \rightarrow \infty$) $\neq 0$, $\sum u_n$ is not convergent.

Hence in this question, $\lim_{n \rightarrow \infty} u_n = 1 \neq 0$ indicates that the given series $\sum u_n$ is not convergent.

If $\sum u_n$ is a series of positive terms, it must be divergent when limit of u_n is non-zero as $n \rightarrow \infty$. The student is strongly advised to remember this result while examining nature of any series for its convergence or divergence, especially when the ratio test fails.

50. Test for convergence the series : $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \dots$

Sol. Here,

$$u_n = \frac{n}{1+2^n}$$

$$\therefore u_{n+1} = \frac{n+1}{1+2^{n+1}}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n}{1+2^n} \cdot \frac{1+2^{n+1}}{n+1} = \frac{2^{n+1} \left(1 + \frac{1}{2^{n+1}}\right)}{2^n \left(1 + \frac{1}{2^n}\right)} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{2 \cdot 1 \cdot 1}{1 \cdot 1} = 2 > 1$$

Hence, the given series is convergent.

51. Test for convergence the series $1 + 3x + 5x^2 + 7x^3 + \dots$ ($x > 0$)

Sol. Here,

$$u_n = (2n-1)x^{n-1}$$

$$\therefore u_{n+1} = (2n+1)x^n$$

$$\frac{u_n}{u_{n+1}} = \frac{2n-1}{2n+1} \cdot \frac{1}{x} = \frac{2 - \frac{1}{n}}{2 + \frac{1}{n}} \cdot \frac{1}{x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(2 - \frac{1}{n}\right)}{\left(2 + \frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

The given series converges if $\frac{1}{x} > 1$, i.e., $x < 1$ and diverges if $\frac{1}{x} < 1$, i.e., $x > 1$ by using D'Alembert's

Ratio Test.

When $x = 1$, Ratio Test fails.

$$u_n = 2n - 1$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (2n - 1) = \infty \neq 0$$

$\therefore$ The given series is divergent for $x = 1$.

Hence $\sum u_n$ is convergent for $x < 1$ and is divergent when $x \geq 1$.

52. Test for convergence the positive term series :

$$1 + \frac{\alpha + 1}{\beta + 1} + \frac{(\alpha + 1)(2\alpha + 1)}{(\beta + 1)(2\beta + 1)} + \frac{(\alpha + 1)(2\alpha + 1)(3\alpha + 1)}{(\beta + 1)(2\beta + 1)(3\beta + 1)} + \dots$$

Sol. Leaving the first term, we have

$$u_n = \frac{(\alpha + 1)(2\alpha + 1) \dots (n\alpha + 1)}{(\beta + 1)(2\beta + 1) \dots (n\beta + 1)}$$

$$\therefore u_{n+1} = \frac{(\alpha + 1)(2\alpha + 1) \dots (n\alpha + 1)[(n+1)\alpha + 1]}{(\beta + 1)(2\beta + 1) \dots (n\beta + 1)[(n+1)\beta + 1]}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta + 1}{(n+1)\alpha + 1} = \frac{\left(1 + \frac{1}{n}\right)\beta + \frac{1}{n}}{\left(1 + \frac{1}{n}\right)\alpha + \frac{1}{n}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)\beta + \frac{1}{n}}{\left(1 + \frac{1}{n}\right)\alpha + \frac{1}{n}} = \frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{\beta}{\alpha} > 1$ i.e., $\beta > \alpha > 0$ and diverges if $\frac{\beta}{\alpha} < 1$ i.e., $\beta < \alpha$ or $\alpha > \beta > 0$. When $\alpha = \beta$, the Ratio Test Fails. When $\alpha = \beta$, $u_n = 1$ or $\lim_{n \rightarrow \infty} u_n = 1 \neq 0$.

$\Rightarrow \sum u_n$ does not converge. Being a series of positive terms, $\sum u_n$ must be divergent. Hence the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

53. Discuss the convergence and divergence of the series

$$(i) \sum \left[(n+1)^{1/3} - n^{1/3} \right] \quad (ii) \sum \sin \frac{1}{n}$$

Sol. (i) Here,

$$u_n = (n+1)^{1/3} - n^{1/3} = n^{1/3} \left(1 + \frac{1}{n} \right)^{1/3} - n^{1/3} = n^{1/3} \left[\left(1 + \frac{1}{n} \right)^{1/3} - 1 \right]$$

$$= n^{1/3} \left[\left\{ 1 + \frac{1}{3} \cdot \frac{1}{n} + \frac{\frac{1}{3} \left(\frac{1}{3} - 1 \right)}{2!} \cdot \frac{1}{n^2} + \dots \right\} - 1 \right]$$

$$= n^{1/3} \left(\frac{1}{3n} - \frac{1}{9n^2} + \dots \right) = \frac{1}{n^3} \left(\frac{1}{3} - \frac{1}{9n} + \dots \right)$$

Let us take $v_n = \frac{1}{n^{2/3}}$. Applying the comparison test, we have $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{9n} + \dots \right) = \frac{1}{3}$ (finite and non-zero quantity). Therefore, both the series $\sum u_n$ and $\sum v_n$ either converge or diverge together.

But $\sum v_n = \sum \frac{1}{n^{2/3}}$ is of the form $\sum \frac{1}{n^p}$, where $p = \frac{2}{3} < 1$, which is divergent.
 $\Rightarrow \sum u_n$ is also divergent.

(ii) Here, $u_n = \sin \frac{1}{n}$

Let $v_n = \frac{1}{n}$

Now, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$

where, $\frac{1}{n} = x$

But $\sum v_n$ is a divergent series (form $\sum \frac{1}{n^p}$, $p = 1$)

Therefore $\sum u_n = \sum \sin \frac{1}{n}$ is also divergent series.

54. Discuss the convergence or divergence of the series $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots \infty$

Sol. Here, $u_n = \frac{n!}{n^n}$

$\therefore u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{n!(n+1)^{n+1}}{n^n \cdot (n+1)!} = \frac{n!(n+1)(n+1)^n}{n^n \cdot (n+1) \cdot n!} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n$

$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e > 1$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ is convergent.

55. Test for convergence the series : $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 + \dots \infty$

(M.D.U., May 2009)

Sol. Here, $u_n = \left[\frac{1.2.3. \dots n}{3.5.7. \dots (2n+1)} \right]^2$

$$u_{n+1} = \left[\frac{1.2.3 \dots n.(n+1)}{3.5.7 \dots (2n+1)(2n+3)} \right]^2$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(n+1)^2} = \frac{4n^2 \left(1 + \frac{3}{2n}\right)^2}{n^2 \left(1 + \frac{1}{n}\right)^2}$$

$$\text{or } \frac{u_n}{u_{n+1}} = \frac{4 \cdot \left(1 + \frac{3}{2n}\right)^2}{\left(1 + \frac{1}{n}\right)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{x \rightarrow \infty} 4 \cdot \frac{\left(1 + \frac{3}{2n}\right)^2}{\left(1 + \frac{1}{n}\right)^2} = 4 > 1$$

$\therefore \Sigma u_n$ is convergent by D'Alembert's Ratio Test.

56. Test the convergence of the series $\sum \frac{n^2 (n+1)^2}{n!}$.

Sol. Here, $u_n = \frac{n^2 (n+1)^2}{n!}$

$$\therefore u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 (n+1)!}{n! (n+2)^2} = \frac{n^2 (n+1)}{(n+2)^2} = \frac{n^3 \left(1 + \frac{1}{n}\right)}{n^2 \left(1 + \frac{2}{n}\right)^2} = \frac{n \left(1 + \frac{1}{n}\right)}{\left(1 + \frac{2}{n}\right)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{n \left(1 + \frac{1}{n}\right)}{\left(1 + \frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Hence, the given series is convergent.

57. Examine the convergence of the series $\sum \frac{n}{n^2+1} \cdot x^n$, $x > 0$.

Sol. $u_n = \frac{n}{n^2+1} \cdot x^n$

$$u_{n+1} = \frac{n+1}{(n+1)^2+1} \cdot x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n \cdot [(n+1)^2+1]}{(n^2+1)(n+1)} \cdot \frac{1}{x} = \frac{n(n^2+2n+2)}{(n+1)(n^2+1)} \cdot \frac{1}{x}$$

$$= \frac{n^3 \left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}{n^3 \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n^2}\right)} \cdot \frac{1}{x}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n^2}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent if $\frac{1}{x} > 1$ i.e., $x < 1$ and diverges when $x > 1$.

When $x = 1$, Ratio Test fails. For $x = 1$, we have

$$u_n = \frac{n}{n^2 + 1} = \frac{1}{n \left(1 + \frac{1}{n^2}\right)}, \text{ let us take } v_n = \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n^2}} \right) = 1$$

which is finite and non-zero.

$\therefore$ By Comparison test, both Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ ($p = 1$), it is divergent.

$\therefore \Sigma u_n$ is divergent when $x = 1$.

Hence Σu_n is convergent when $x < 1$ and divergent when $x \geq 1$.

58. Test the convergence of the series : $\sum \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right)$.

Sol. Let us assume that $u_n = \frac{n^2}{2^n}, \quad u'_n = \frac{1}{n^2}$

$$\text{Now,} \quad \frac{u_n}{u_{n+1}} = \frac{n^2}{2^n} \cdot \frac{2^{n+1}}{(n+1)^2} = \frac{2}{\left(1 + \frac{1}{n}\right)^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{2}{(1+0)^2} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent ...(1)

Now let us consider the series $\Sigma u'_n, u'_n = \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$, where $p = 2 > 1$ and is convergent by applying Comparison Test. $\therefore \Sigma u'_n$ is convergent ...(2)

$\Rightarrow$ By (1) and (2), $\Sigma(u_n + u'_n)$ or $\Sigma \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right)$ is convergent i.e., the given series is also convergent.

59. Examine the convergence or otherwise of the series : $x + 2x^2 + 3x^3 + 4x^4 + \dots \infty$

Sol. In the given series Σu_n ,

$$\begin{aligned} u_n &= nx^n \\ u_{n+1} &= (n+1)x^{n+1} \\ \Rightarrow \frac{u_n}{u_{n+1}} &= \frac{n}{(n+1)x} = \frac{1}{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test Σu_n is convergent when $x < 1$ and is divergent when $x > 1$.
But the Ratio Test fails if $x = 1$.

$$\text{When } x = 1, u_n = n, \Sigma u_n = \Sigma n = \frac{n(n+1)}{2}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} n = \infty \neq 0$$

$\Rightarrow \Sigma u_n$ does not converge. Being a series of positive terms, it must diverge.

$\therefore \Sigma u_n$ is divergent for $x \geq 1$ and is convergent when $x < 1$.

60. Test the series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots + \frac{x^n}{n^2 + 1} + \dots \infty$

Sol.

$$\begin{aligned} u_n &= \frac{x^n}{n^2 + 1} \\ u_{n+1} &= \frac{x^{n+1}}{(n+1)^2 + 1} \\ \therefore \frac{u_n}{u_{n+1}} &= \frac{x^n \cdot (n^2 + 2n + 2)}{x^{n+1} \cdot (n^2 + 1)} = \frac{\left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}{\left(1 + \frac{1}{n^2}\right)} \cdot \frac{1}{x} \\ \therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}{\left(1 + \frac{1}{n^2}\right)} \cdot \frac{1}{x} = \frac{1}{x} \end{aligned}$$

By D'Alembert's Ratio Test, the series Σu_n converges when $\frac{1}{x} > 1$ or $x < 1$ and is divergent for

$\frac{1}{x} < 1$ or $x > 1$. The Ratio Test fails when $x = 1$.

$$\text{When } x = 1, u_n = \frac{1}{n^2 + 1} \quad \text{or} \quad \frac{1}{n^2 \left(1 + \frac{1}{n^2}\right)}$$

$$\text{Let us take } v_n = \frac{1}{n^2} \quad \therefore \frac{u_n}{v_n} = \frac{1}{1 + \frac{1}{n^2}}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n^2}} \right) = 1$, which is finite and non-zero. By Comparison test Σu_n is convergent

since $\Sigma v_n = \Sigma \frac{1}{n^2}$ ($p = 2 > 1$) is convergent.

Hence Σu_n converges for $x \leq 1$ and Σu_n diverges for $x > 1$.

61. Test the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ for its convergence or otherwise.

Sol.

$$u_n = \frac{x^n}{(2n)!}$$

$$u_{n+1} = \frac{x^{n+1}}{(2n+2)!} = \frac{x^{n+1}}{(2n+2)(2n+1)(2n)!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+1)}{x} = \frac{4n^2 \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{2n}\right)}{x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{4n^2 \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{2n}\right)}{x}$$

$$= \infty > 1$$

Hence by D'Alembert's Ratio Test, Σu_n is convergent.

62. Examine the series $\frac{4}{18} + \frac{4.12}{18.27} + \frac{4.12.20}{18.27.36} + \dots \infty$.

Sol. In the given series, it is observed that in the numerator with every term, one factor (8) is added. In an A.P. where $a = 4$, $d = 8$, n^{th} term is given by $a + (n-1)d$, i.e., $4 + 8(n-1)$ or $8n - 4$. Hence in the numerator of the n^{th} term, factor $4(2n-1)$ will appear.

$\Rightarrow$ Numerator will be $4.12.20. \dots 4(2n-1)$

Similarly, the denominator will be $18.27.36. \dots 9n + 9 \dots$

Thus,

$$u_n = \frac{4.12.20. \dots 4(2n-1)}{18.27.36. \dots 9(n+1)}$$

$$u_{n+1} = \frac{4.12.20. \dots 4(2n-1) \cdot 4(2n+1)}{18.27.36. \dots 9(n+1) \cdot 9(n+2)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{9(n+2)}{4(2n+1)} \quad \text{or} \quad \frac{9 \left(1 + \frac{2}{n}\right)}{8 \left(1 + \frac{1}{2n}\right)}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{9 \left(1 + \frac{2}{n}\right)}{8 \left(1 + \frac{1}{2n}\right)} = \frac{9}{8} > 1$$

which implies that by D'Alembert's Ratio Test, the given series Σu_n is convergent.

63. Test the convergence of the series $\sum \frac{n^3 - n + 1}{n!}$.

Sol. Here,

$$u_n = \frac{n^3 - n + 1}{n!}$$

$$u_{n+1} = \frac{(n+1)^3 - (n+1) + 1}{(n+1)!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n^3 - n + 1)(n+1)}{n^3 + 3n^2 + 2n + 1} = \frac{n^3 \left(1 - \frac{1}{n^2} + \frac{1}{n^3}\right)(n+1)}{n^3 \left(1 + \frac{3}{n} + \frac{2}{n^2} + \frac{1}{n^3}\right)}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1 - \frac{1}{n^2} + \frac{1}{n^3}\right)(n+1)}{\left(1 + \frac{3}{n} + \frac{2}{n^2} + \frac{1}{n^3}\right)} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, the given series $\sum u_n$ is convergent.

64. What is Cauchy's Root Test? Use this test to examine the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

Sol. Cauchy's Root Test states that if $\sum u_n$ is a positive term series and $\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = l$, then

(i) $\sum u_n$ is convergent if $l < 1$

(ii) $\sum u_n$ is divergent if $l > 1$.

When $l = 1$, the test fails i.e., no conclusion can be drawn about the convergence or divergence of the series.

In the given series,

$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$\therefore (u_n)^{\frac{1}{n}} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left[\left(1 + \frac{1}{n}\right)^n\right]^{-1}$$

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \quad (\because e = 2.7)$$

$\therefore$ By Cauchy's Root Test, the given series $\sum u_n$ is convergent.

65. Examine the series $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} \cdot x^n$ for convergence.

Sol. Here,

$$u_n = \frac{(n+1)^n \cdot x^n}{n^{n+1}} = \left[\frac{(n+1)x}{n}\right]^n \cdot \frac{1}{n}$$

$$\begin{aligned}
\Rightarrow (u_n)^{\frac{1}{n}} &= \frac{(n+1) \cdot x}{n} \cdot \frac{1}{n^{1/n}} = \left(1 + \frac{1}{n}\right) x \cdot \frac{1}{n^{1/n}} \\
\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} &= \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) x \right] \left[\lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} \right] \\
&= (1+0) x \cdot \frac{1}{1} = x \qquad \left\{ \because \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1 \right\}
\end{aligned}$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent if $x < 1$ and divergent when $x > 1$. When $x = 1$, this test fails.

Now when $x = 1$,

$$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{n} \cdot \left(1 + \frac{1}{n}\right)^n$$

Let us take $v_n = \frac{1}{n}$, $\frac{u_n}{v_n} = \left(1 + \frac{1}{n}\right)^n$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

which is finite and non-zero.

$\therefore$ By Comparison Test, Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 1$, Σv_n is divergent.

$\Rightarrow \Sigma u_n$ is divergent.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

66. Test the series $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \dots (x > 0)$.

Sol. Here,

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\Rightarrow (u_n)^{\frac{1}{n}} = \frac{n+1}{n+2} \cdot x = \left(\frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}\right) \cdot x$$

$$\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}\right) \cdot x = x$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent if $x < 1$ and divergent if $x > 1$. The test fails when $x = 1$.

For $x = 1$,

$$u_n = \frac{\left(\frac{n+1}{n+2}\right)^n}{\left(1 + \frac{2}{n}\right)^n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n}{\left[\left(1 + \frac{2}{n}\right)^{n/2}\right]^2} = \frac{e}{e^2} \quad \text{or} \quad \frac{1}{e} \neq 0$$

Since $\lim_{n \rightarrow \infty} u_n \neq 0$, Σu_n can't converge. Being a series of positive terms, it must diverge.

Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

67. Test the series $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$ (M.D.U., May 2007)

Sol. Here,
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n} \right]^{-n}$$

$$\Rightarrow (u_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n} \right]^{-1}$$

$$\Rightarrow (u_n)^{\frac{1}{n}} = \left[\left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n}\right) \right]^{-1} = \left[\left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right) - \left(1 + \frac{1}{n}\right) \right]^{-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) - \left(1 + \frac{1}{n}\right) \right]^{-1} \\ &= (e \cdot 1 - 1)^{-1} = \frac{1}{e-1} < 1 \quad (\because e = 2.7) \end{aligned}$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

68. Discuss the convergence of the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots + \frac{1}{n^n} + \dots$

Sol. Here,
$$u_n = \frac{1}{n^n} \quad \text{or} \quad (u_n)^{\frac{1}{n}} = \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0 < 1$$

Hence the given series is convergent.

69. Discuss the convergence and divergence of the series $\sum \left(1 + \frac{1}{n}\right)^{n^2}$.

Sol. Here,
$$u_n = \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\therefore (u_n)^{\frac{1}{n}} = \left(1 + \frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

Therefore, the given series by Cauchy's Root Test, is divergent.

70. Test the series for convergence and divergence,

$$\frac{1}{2} \cdot x + \left(\frac{2}{3}\right)^4 \cdot x^2 + \left(\frac{3}{4}\right)^9 \cdot x^3 + \dots + \left(\frac{n}{n+1}\right)^{n^2} \cdot x^n + \dots$$

Sol. Here,
$$u_n = \left(\frac{n}{n+1}\right)^{n^2} \cdot x^n$$

$$\therefore (u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n \cdot x = \left(\frac{1}{1+\frac{1}{n}}\right)^n \cdot x = \frac{x}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{x}{\left(1+\frac{1}{n}\right)^n} = \frac{x}{e}$$

The given series converges if $\frac{x}{e} < 1$ i.e., $x < e$ and diverges if $\frac{x}{e} > 1$ i.e., $x > e$.

Cauchy's Root Test fails when $x = e$.

For $x = e$,
$$u_n = \left(\frac{n}{n+1}\right)^{n^2} \cdot e^n$$

Now,
$$\lim_{n \rightarrow \infty} u_n \neq 0$$

Therefore, the given series diverges at $x = e$.

Hence the given series converges for $x < e$ and diverges for $x \geq e$.

71. Discuss the convergence of the series $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Sol. Here,
$$u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \left(\frac{\sqrt{n}+1}{\sqrt{n}}\right)^{-n^{3/2}} = \left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)^{n^{3/2}}$$

$$\therefore (u_n)^{\frac{1}{n}} = \left[\left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)^{n^{3/2}}\right]^{\frac{1}{n}} = \left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)^{\sqrt{n}}$$

or
$$(u_n)^{\frac{1}{n}} = \left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)^{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{-1}$$

$$= e^{-1} \quad \text{or} \quad \frac{1}{e} < 1 \quad (\because e \approx 2.7)$$

$\therefore$ By Cauchy's Root Test, $\sum u_n$ is convergent.

72. Test the convergence or divergence of the series $1 + \frac{x^1}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \dots (x > 0)$.

Sol. Here, (Leaving the first term), we have

$$u_n = \frac{x^n}{(n+1)^n}$$

$$\therefore (u_n)^{\frac{1}{n}} = \frac{x}{n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) \cdot x = 0 \quad (\because x > 0)$$

$$\therefore \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = 0 < 1. \text{ Hence by Cauchy's Root Test, } \Sigma u_n \text{ is convergent.}$$

73. What are the other Tests of Convergence which may be used when the Ratio Test Fails.

Sol. (A) *Raabe's Test* :

If Σu_n is a series of positive terms and $\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = l$, then the series is convergent if $l > 1$ and divergent if $l < 1$.

This test fails when $l = 1$.

Note : Raabe's test is used when D'Alembert's Ratio test fails and when in the ratio test, $\frac{u_n}{u_{n+1}}$ does not involve the number e .

When $\frac{u_n}{u_{n+1}}$ involves e , we apply logarithmic test after the ratio test and not Raabe's Test.

(B) *Logarithmic Test* :

A positive term series Σu_n converges or diverges according as $\lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} > 1$ or < 1 .

The test fails if $\lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} = 1$.

This test is applied after the failure of Ratio Test and generally when in Ratio test,

$\frac{u_n}{u_{n+1}}$ involves e .

(C) *Gauss Test* :

If for the series Σu_n of positive terms, $\frac{u_n}{u_{n+1}}$ can be expanded in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n converges if $\lambda > 1$ and diverges if $\lambda \leq 1$.

Note : The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of Ratio Test and when it is possible to expand $\frac{u_n}{u_{n+1}}$ in

powers of $\frac{1}{n}$ by Binomial Theorem or by any other method.

(D) *Cauchy's Integral Test* :

If for $x \geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that

$f(n) = u_n$ for all positive integral values of n , then the series $\sum u_n$ and the integral $\int_1^{\infty} f(x)$ converge or diverge together.

If $x \geq k$, then $\sum u_n$ and $\int_k^{\infty} f(x) dx$ converge or diverge together.

74. Discuss the convergence of the series $\frac{1^2}{2^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} + \dots$ (M.D.U., Dec., 2007)

Sol. Here,

$$u_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots \cdot (2n)^2}$$

$$u_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2 \cdot (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots \cdot (2n)^2 \cdot (2n+2)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2 \left(1 + \frac{1}{n}\right)^2}{4n^2 \left(1 + \frac{1}{2n}\right)^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{1}{2n}\right)^2} = 1$$

Hence the Ratio Test fails.

$$\begin{aligned} n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= n \left[\frac{(2n+2)^2}{(2n+1)^2} - 1 \right] = n \left[\frac{(2n+2)^2 - (2n+1)^2}{(2n+1)^2} \right] \\ &= n \frac{(4n+3)}{(2n+1)^2} = \frac{4n^2 + 3n}{(2n+1)^2} = \frac{4n^2 \left(1 + \frac{3}{4n}\right)}{4n^2 \left(1 + \frac{1}{2n}\right)^2} \rightarrow 1 \text{ as } n \rightarrow \infty \end{aligned}$$

$\therefore$ Raabe's test also fails.

When D'Alembert's Ratio test fails, we can directly use Gauss Test.

$$\begin{aligned} \text{Now, } \frac{u_n}{u_{n+1}} &= \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{1}{2n}\right)^2} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2} \\ &= \left(1 + \frac{2}{n} + \frac{1}{n^2} + \dots\right) \left(1 - \frac{2}{2n} + \frac{3}{4n^2} - \dots\right) \end{aligned}$$

or

$$\begin{aligned}\frac{u_n}{u_{n+1}} &= 1 + \frac{1}{n} + \frac{1}{n^2} \left(1 - 2 + \frac{3}{4}\right) + \dots \\ &= 1 + \frac{1}{n} - \frac{1}{4n^2} + \dots = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

Comparing it with $\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$ we have $\lambda = 1$. Thus by Gauss Test, the series $\sum u_n$ diverges.

75. Discuss the convergence of the series : $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \dots$

Sol. Here,

$$u_n = \frac{1.3.5. \dots (2n-1)}{2.4.6. \dots 2n}$$

$$\therefore u_{n+1} = \frac{1.3.5. \dots (2n-1)(2n+1)}{2.4.6. \dots 2n \cdot (2n+2)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} = \frac{1 + \frac{1}{n}}{1 + \frac{1}{2n}} \rightarrow 1 \text{ as } n \rightarrow \infty$$

$\therefore$ Ratio Test fails.

Now,

$$n \left[\frac{u_n}{u_{n+1}} - 1 \right] = n \left[\frac{2n+2}{2n+1} - 1 \right] = \frac{n}{2n+1} = \frac{1}{2 + \frac{1}{n}}$$

$$\therefore \lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] = \frac{1}{2} < 1$$

Hence by Raabe's Test, $\sum u_n$ diverges.

76. Examine the convergence of the series $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots \infty$, ($x > 0$)

(M.D.U., May, 2008)

Sol. Neglecting the first term, we have

$$u_n = \frac{1.3.5. \dots (2n-1)}{2.4.6. \dots 2n} \cdot \frac{x^{2n+1}}{2n+1}$$

and

$$u_{n+1} = \frac{1.3.5. \dots (2n-1)(2n+1)}{2.4.6. \dots 2n \cdot (2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \cdot \frac{1}{x^2}$$

$$= \frac{2n \left(1 + \frac{1}{n}\right) \cdot 2n \left(1 + \frac{3}{2n}\right)}{2n \left(1 + \frac{1}{2n}\right) \cdot 2n \left(1 + \frac{1}{2n}\right)} \cdot \frac{1}{x^2} = \frac{\left(1 + \frac{1}{n}\right) \left(1 + \frac{3}{2n}\right)}{\left(1 + \frac{1}{2n}\right) \cdot \left(1 + \frac{1}{2n}\right)} \cdot \frac{1}{x^2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1 \cdot \frac{1}{x^2} \quad \text{or} \quad \frac{1}{x^2} \quad \text{as} \quad n \rightarrow \infty.$$

$\therefore$ By Ratio Test, $\sum u_n$ is convergent if $\frac{1}{x^2} > 1$ i.e., $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$ i.e., $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.

$$\text{When } x^2 = 1, \text{ we have} \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2 + 10n + 6}{4n^2 + 4n + 1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= \lim_{n \rightarrow \infty} n \left(\frac{4n^2 + 10n + 6}{4n^2 + 4n + 1} - 1 \right) = \lim_{n \rightarrow \infty} \frac{6n^2 + 5n}{4n^2 + 4n + 1} \\ &= \lim_{n \rightarrow \infty} \frac{6 + \frac{5}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1 \end{aligned}$$

$\therefore$ By Raabe's Test, the series converges.

Hence $\sum u_n$ is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

77. Discuss the convergence of the series : $1 + \frac{x}{2} + \frac{2!}{3^2} \cdot x^2 + \frac{3!}{4^3} \cdot x^3 + \frac{4!}{5^4} \cdot x^4 + \dots$

(M.D.U., May, 2006, 2007)

Sol. Neglecting the first term, we have

$$\begin{aligned} u_n &= \frac{n!}{(n+1)^n} \cdot x^n, \quad u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}} \cdot x^{n+1} \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} &= \lim_{n \rightarrow \infty} \frac{n! (n+2)^{n+1}}{(n+1)^n (n+1)!} \cdot \frac{1}{x} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^n \cdot \left(1 + \frac{1}{n}\right)^n} \cdot \frac{n^{n+1} \left(1 + \frac{2}{n}\right)^{n+1}}{(n+1)} \cdot \frac{1}{x} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{2}{n}\right)^n \left(1 + \frac{2}{n}\right)}{\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{e^2}{e} \cdot \frac{1}{x} = \frac{e}{x} \\ &\left\{ \because \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{a}{n}\right)^{\frac{n}{a}} \right]^a = e^a \right\} \end{aligned}$$

$\therefore$ By D'Alembert's Ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if

$x > e$. If $x = e$, the ratio test fails since $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1$.

When $x = e$,

$$\frac{u_n}{u_{n+1}} = \frac{\left(1 + \frac{2}{n}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n+1}} \cdot \frac{1}{e}$$

Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e , we do not apply Raabe's Test but apply logarithmic test.

$$\begin{aligned} \therefore \log \frac{u_n}{u_{n+1}} &= (n+1) \log \left(1 + \frac{2}{n}\right) - (n+1) \log \left(1 + \frac{1}{n}\right) - \log e \\ &= (n+1) \left[\log \left(1 + \frac{2}{n}\right) - \log \left(1 + \frac{1}{n}\right) \right] - 1 \\ &= (n+1) \left[\left(\frac{2}{n} - \frac{1}{2} \cdot \frac{4}{n^2} + \frac{1}{3} \cdot \frac{8}{n^3} - \dots \right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \dots \right) \right] \\ &= (n+1) \left[\frac{1}{n} - \frac{3}{2n^2} + \dots \right] - 1 = 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + \dots - 1 \\ &= -\frac{1}{2n} - \frac{3}{2n^2} + \dots \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} n \left[-\frac{1}{2n} - \frac{3}{2n^2} + \dots \right] = -\frac{1}{2} < 1$$

$\therefore$ By log test, the series diverges.

Hence the given series $\sum u_n$ converges for $x < e$ and diverges when $x \geq e$.

78. Test the following series for convergence : $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \dots \infty$

Sol. Here,

$$u_n = \frac{(a+nx)^n}{n!}$$

$$\therefore u_{n+1} = \frac{[a+(n+1)x]^{n+1}}{(n+1)!}$$

$$\begin{aligned} \Rightarrow \frac{u_n}{u_{n+1}} &= \frac{(a+nx)^n}{[a+(n+1)x]^{n+1}} \cdot \frac{(n+1)!}{n!} \\ &= \frac{n^n \cdot \left(1 + \frac{a}{nx}\right)^n}{(n+1)^n \left[1 + \frac{a}{(n+1)x}\right]^{n+1}} \cdot \frac{1}{x} = \frac{\left(1 + \frac{a}{nx}\right)^n}{\left(1 + \frac{1}{n}\right)^n \cdot \left[1 + \frac{a}{(n+1)x}\right]^{n+1}} \cdot \frac{1}{x} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left[\left(1 + \frac{a}{nx} \right)^{\frac{nx}{a}} \right]^{\frac{a}{x}}}{\left(1 + \frac{1}{n} \right)^n \cdot \left[\left\{ 1 + \frac{a}{(n+1)x} \right\}^{\frac{(n+1)x}{a}} \right]^{\frac{a}{x}}} \cdot \frac{1}{x} = \frac{e^{a/x}}{e \cdot e^{a/x} \cdot x} = \frac{1}{ex}$$

$\therefore$ By D'Alembert's Ratio Test, the series converges if $\frac{1}{ex} > 1$ i.e., if $x < \frac{1}{e}$ and diverges when $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$. If $x = \frac{1}{e}$, the test fails.

When $x = \frac{1}{e}$, we have

$$\frac{u_n}{u_{n+1}} = \frac{e \cdot \left(1 + \frac{ae}{n} \right)^n}{\left(1 + \frac{1}{n} \right)^n \left(1 + \frac{ae}{n+1} \right)^{n+1}}.$$

Since $\frac{u_n}{u_{n+1}}$ involves e , we will use logarithmic test after the Ratio Test has failed.

$$\begin{aligned} \therefore \log \frac{u_n}{u_{n+1}} &= \log \left[\frac{e \left(1 + \frac{ae}{n} \right)^n}{\left(1 + \frac{1}{n} \right)^n \left(1 + \frac{ae}{n+1} \right)^{n+1}} \right] \\ &= \log e + n \log \left(1 + \frac{ae}{n} \right) - n \log \left(1 + \frac{1}{n} \right) - (n+1) \log \left(1 + \frac{ae}{n+1} \right) \\ &= 1 + n \cdot \left(\frac{ae}{n} - \frac{a^2 e^2}{2n^2} + \frac{a^3 e^3}{3n^3} - \dots \right) - n \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \right) \\ &\quad - (n+1) \left[\frac{ae}{n+1} - \frac{a^2 e^2}{2(n+1)^2} + \frac{a^3 e^3}{3(n+1)^3} \right] \\ &= 1 + \left(ae - \frac{a^2 e^2}{2n} + \frac{a^3 e^3}{3n^2} - \dots \right) - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots \right) - \left[ae - \frac{a^2 e^2}{2(n+1)} + \frac{a^3 e^3}{3(n+1)^2} \right] \\ \Rightarrow \log \frac{u_n}{u_{n+1}} &= \left(-\frac{a^2 e^2}{2n} + \frac{a^3 e^3}{3n^2} - \dots \right) + \left(\frac{1}{2n} - \frac{1}{3n^2} + \dots \right) + \left[\frac{a^2 e^2}{2(n+1)} - \frac{a^3 e^3}{3(n+1)^2} + \dots \right] \\ \Rightarrow n \cdot \log \frac{u_n}{u_{n+1}} &= \left(-\frac{a^2 e^2}{2} + \frac{a^3 e^3}{3n} - \dots \right) + \left(\frac{1}{2} - \frac{1}{3n} + \dots \right) + \left[\frac{a^2 e^2}{2} \cdot \frac{n}{n+1} - \frac{a^3 e^3 \cdot n}{3(n+1)^2} + \dots \right] \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{a^2 e^2}{2} + \frac{a^3 e^3}{3n} - \dots \right) + \left(\frac{1}{2} - \frac{1}{3n} + \dots \right) \\
&\quad + \left[\frac{a^2 e^2}{2} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)} - \frac{a^3 e^3}{3(n+1)} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)} + \dots \right] \\
\Rightarrow \lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} &= \left(-\frac{a^2 e^2}{2} \right) + \frac{1}{2} + \left(\frac{a^2 e^2}{2} \right) \quad (\because \text{the last term vanishes when } n \rightarrow \infty) \\
&= \frac{1}{2} < 1
\end{aligned}$$

$\therefore$ By Logarithmic test, the series is divergent. [Please See Q. 73 (B)]

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

79. Examine the series for convergence : $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \dots \infty$.

Sol. Here,

$$u_n = x^{n+1} [\log(n+1)]^q$$

$\therefore$

$$u_{n+1} = x^{n+2} [\log(n+2)]^q$$

$$\begin{aligned}
\Rightarrow \frac{u_n}{u_{n+1}} &= \frac{1}{x} \cdot \left[\frac{\log(n+1)}{\log(n+2)} \right]^q = \frac{1}{x} \cdot \left[\frac{\log n \left(1 + \frac{1}{n}\right)}{\log n \left(1 + \frac{2}{n}\right)} \right]^q \\
&= \frac{1}{x} \cdot \left[\frac{\log n + \log \left(1 + \frac{1}{n}\right)}{\log n + \log \left(1 + \frac{2}{n}\right)} \right]^q = \frac{1}{x} \cdot \left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \dots}{\log n + \frac{2}{n} - \frac{1}{2} \left(\frac{2}{n}\right)^2 + \dots} \right]^q
\end{aligned}$$

Dividing the numerator and denominator by $\log n$ and taking limit as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{x} \cdot \left[\frac{1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots}{1 + \frac{2}{n \log n} - \frac{2}{n^2 \log n} + \dots} \right]^q = \frac{1}{x}$$

$\therefore$ By Ratio test, the given series $\sum u_n$ converges if $\frac{1}{x} > 1$ i.e., if $x < 1$ and diverges if $\frac{1}{x} < 1$ or $x > 1$.

However, if $x = 1$, the Ratio Test fails. When $x = 1$, we have

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= \left[\frac{1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots}{1 + \frac{2}{n \log n} - \frac{2}{n^2 \log n}} \right]^q \\
&= \left[1 + \left(\frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots \right) \right]^q \left[1 + \left(\frac{2}{n \log n} - \frac{2}{n^2 \log n} + \dots \right) \right]^{-q}
\end{aligned}$$

Expanding by Binomial Theorem, we have

$$\frac{u_n}{u_{n+1}} = \left[1 + q \left(\frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots \right) + \dots \right] \left[1 - q \left(\frac{2}{n \log n} - \frac{2}{n^2 \log n} + \dots \right) + \dots \right]$$

$$\frac{u_n}{u_{n+1}} = 1 + q \left(\frac{1}{n \log n} - \frac{2}{n \log n} \right) + \dots$$

$$\frac{u_n}{u_{n+1}} = 1 - \frac{q}{n \log n} + \dots$$

or $\frac{u_n}{u_{n+1}} - 1 = -\frac{q}{n \log n} + (\text{terms containing squares and higher powers of } n \text{ and } \log n \text{ in the denominator.})$

$$\Rightarrow n \left(\frac{u_n}{u_{n+1}} - 1 \right) = -\frac{q}{\log n} + (\text{terms containing } n, n^2; (\log n)^2, (\log n)^3, \dots \text{ in the denominator.})$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = 0 < 1.$$

$\therefore$ By Raabe's Test, the series $\sum u_n$ is divergent.

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Note : Raabe's test has been used here when for $x = 1$, D'Alembert's Ratio Test failed and $\frac{u_n}{u_{n+1}}$ did not involve the number e .

80. Discuss the convergence of the series :

$$x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2 \cdot \gamma(\gamma+1)} \cdot x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3 \cdot \gamma(\gamma+1)(\gamma+2)} \quad (\text{M.D.U., Dec., 2005})$$

Sol. Ignoring the first term, we have

$$u_n = \frac{\alpha(\alpha+1) \dots (\alpha+n-1) \beta(\beta+1) \dots (\beta+n-1)}{1.2.3 \dots n \cdot \gamma(\gamma+1) \dots (\gamma+n-1)} \cdot x^n$$

$$u_{n+1} = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} \frac{\alpha(\alpha+1) \dots (\alpha+n-1)(\alpha+n) \beta(\beta+1) \dots (\beta+n-1)(\beta+n)}{1.2.3 \dots n(n+1) \cdot \gamma(\gamma+1)(\gamma+2) \dots (\gamma+n-1)(\gamma+n)} \cdot x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \frac{\left(1 + \frac{1}{n}\right) \left(1 + \frac{\gamma}{n}\right)}{\left(1 + \frac{\alpha}{n}\right) \left(1 + \frac{\beta}{n}\right)} \cdot \frac{1}{x}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, the series $\sum u_n$ converges if $\frac{1}{x} > 1$ i.e., if $x < 1$ and diverges if $\frac{1}{x} < 1$

or if $x > 1$. If $x = 1$, $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1$, therefore the Ratio Test fails.

Putting $x = 1$ in $\frac{u_n}{u_{n+1}}$, we have

$$\frac{u_n}{u_{n+1}} = \frac{\left(1 + \frac{1}{n}\right)\left(1 + \frac{\gamma}{n}\right)}{\left(1 + \frac{\alpha}{n}\right)\left(1 + \frac{\beta}{n}\right)}$$

$$\begin{aligned} \Rightarrow \frac{u_n}{u_{n+1}} &= \left(1 + \frac{1}{n}\right)\left(1 + \frac{\gamma}{n}\right)\left(1 + \frac{\alpha}{n}\right)^{-1}\left(1 + \frac{\beta}{n}\right)^{-1} \quad (\text{Expand by Binomial Theorem}) \\ &= \left(1 + \frac{1}{n}\right)\left(1 + \frac{\gamma}{n}\right)\left(1 - \frac{\alpha}{n} + \frac{\alpha^2}{n^2} + \dots\right)\left(1 - \frac{\beta}{n} + \frac{\beta^2}{n^2} + \dots\right) \\ &= \left(1 + \frac{1}{n} + \frac{\gamma}{n} + \frac{\gamma}{n^2}\right)\left(1 - \frac{\alpha}{n} - \frac{\beta}{n} + \frac{\alpha\beta}{n^2} + \frac{\alpha^2}{n^2} + \frac{\beta^2}{n^2} + \dots\right) \\ &= 1 + \frac{1}{n}(1 + \gamma - \alpha - \beta) + 0\left(\frac{1}{n^2}\right) \end{aligned}$$

$\therefore$ **By Gauss Test**, the series $\sum u_n$ converges if $1 + \gamma - \alpha - \beta > 1$ i.e., if $\gamma > \alpha + \beta$ and diverges if $1 + \gamma - \alpha - \beta \leq 1$ i.e., if $\gamma \leq \alpha + \beta$.

Thus the given series converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if $\gamma > \alpha + \beta$ and diverges when $\gamma \leq \alpha + \beta$.

81. Discuss the convergence of the series $x + \frac{2^2 \cdot x^2}{2!} + \frac{3^3 \cdot x^3}{3!} + \frac{4^4 \cdot x^4}{4!} + \frac{5^5 \cdot x^5}{5!} + \dots \infty$

(M.D.U., May 2009)

Sol. Here,

$$u_n = \frac{n^n \cdot x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n \cdot x^n \cdot (n+1)!}{n! (n+1)^{n+1} \cdot x^{n+1}} = \frac{n^n \cdot (n+1) \cdot n!}{n! (n+1) (n+1)^n \cdot x} \cdot \frac{1}{x}$$

or

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{(n+1)^n \cdot x} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

Thus by Ratio test, the given series converges for $x < \frac{1}{e}$ and diverges for $x > \frac{1}{e}$. But it fails for $x = \frac{1}{e}$. Let us now try logarithmic test.

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1 + \frac{1}{n}\right)^n}$$

$$\begin{aligned}\therefore \log \frac{u_n}{u_{n+1}} &= \log_e e - n \log \left(1 + \frac{1}{n}\right) \\ &= 1 - n \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right) = \frac{1}{2n} - \frac{1}{3n^2} + \dots\end{aligned}$$

$\therefore \lim_{n \rightarrow \infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = \frac{1}{2}$, which is less than one. Therefore, by logarithmic test the series diverges.

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

82. Test for convergence the series

$$\frac{2^2}{3 \cdot 4} + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \dots \infty$$

Sol. Here,

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3 \cdot 4 \cdot 5 \dots (2n+2)}$$

$$u_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 \cdot (2n+2)^2}{3 \cdot 4 \cdot 5 \dots (2n+2) \cdot (2n+3) \cdot (2n+4)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3 \cdot 4 \cdot 5 \dots (2n+2)} \cdot \frac{3 \cdot 4 \cdot 5 \dots (2n+2) \cdot (2n+3) \cdot (2n+4)}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 \cdot (2n+2)^2}$$

$$= \frac{(2n+3)(2n+4)}{(2n+2)^2} = \frac{\left(2 + \frac{3}{n}\right)\left(2 + \frac{4}{2n}\right)}{\left(2 + \frac{2}{n}\right)^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left[\frac{\left(2 + \frac{3}{n}\right)\left(2 + \frac{4}{2n}\right)}{\left(2 + \frac{2}{n}\right)^2} \right] = 1$$

Hence the D'Alembert's Ratio Test fails and we apply Raabe's Test.

Now,

$$\begin{aligned}n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= n \left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1 \right] = \frac{n(6n+8)}{(2n+2)^2} \\ &= \frac{n \cdot 2n \left(3 + \frac{4}{n}\right)}{4n^2 \left(1 + \frac{1}{n}\right)^2} = \frac{\left(3 + \frac{4}{n}\right)}{2 \left(1 + \frac{1}{n}\right)^2}\end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} \left\{ \frac{\left(3 + \frac{4}{n}\right)}{2 \left(1 + \frac{1}{n}\right)^2} \right\} = \frac{3}{2} > 1$$

$\therefore$ The given series converges by Raabe's Test.

83. Discuss the convergence of the series $\frac{1}{3}x + \frac{1.2}{3.5} \cdot x^2 + \frac{1.2.3}{3.5.7} \cdot x^3 + \dots \infty$, ($x > 0$)

Sol. Here,

$$u_n = \frac{1.2.3 \dots n}{3.5.7 \dots (2n+1)} \cdot x^n$$

$$u_{n+1} = \frac{1.2.3 \dots n \cdot (n+1)}{3.5.7 \dots (2n+1)(2n+3)} \cdot x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(n+1)} \cdot \frac{1}{x} = \frac{\left(2 + \frac{3}{n}\right)}{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(2 + \frac{3}{n}\right)}{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{2}{x}$$

The given series converges if $\frac{2}{x} > 1$ i.e., $x < 2$ and diverges if $\frac{2}{x} < 1$ i.e., $x > 2$.

The test fails at $x = 2$.

When $x = 2$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{2(n+1)}$$

$$\therefore n \left(\frac{u_n}{u_{n+1}} - 1 \right) = n \left[\frac{2n+3}{2n+2} - 1 \right] = \frac{n}{2n+2}$$

$$\therefore \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{n}{2n+2} \right) = \frac{1}{2} (< 1)$$

Thus the given series diverges when $x = 2$. Hence the given series converges when $x < 2$ and diverges when $x \geq 2$.

84. Test the convergence of the series $\sum \frac{(n-1)^{n-1}}{n^n}$.

Sol. Here,

$$u_n = \frac{(n-1)^{n-1}}{n^n} \quad \text{and} \quad u_{n+1} = \frac{n^n}{(n+1)^{n+1}}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{n^{n-1} \left(1 - \frac{1}{n}\right)^{n-1} \cdot n^{n+1} \cdot \left(1 + \frac{1}{n}\right)^{n+1}}{n^{2n}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \left(1 - \frac{1}{n}\right)^{n-1} \cdot \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 - \frac{1}{n}\right)^{n-1} \cdot \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right)$$

$$\therefore n \log \frac{u_n}{u_{n+1}} = n \left[(n-1) \log \left(1 - \frac{1}{n}\right) + \log \left(1 + \frac{1}{n}\right)^n + \log \left(1 + \frac{1}{n}\right) \right]$$

$$\begin{aligned}
&= n \left[(n-1) \left(-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + \dots \right) + \log \left(1 + \frac{1}{n} \right)^n + \left(\frac{1}{n} - \frac{1}{2n^2} + \dots \right) \right] \\
&= n \left[\left(-1 + \frac{1}{n} - \frac{1}{2n} + \dots \right) + \log \left(1 + \frac{1}{n} \right)^n + \left(\frac{1}{n} - \frac{1}{2n^2} + \dots \right) \right] \\
\therefore \lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} &= \lim_{n \rightarrow \infty} n \left[\left(-1 + \frac{1}{n} - \frac{1}{2n} + \dots \right) + \log \left(1 + \frac{1}{n} \right)^n + \left(\frac{1}{n} - \frac{1}{2n^2} + \dots \right) \right] \\
&= \lim_{n \rightarrow \infty} n \left[-1 + \frac{3}{2n} + 1 + \dots \right] = \frac{3}{2} > 1 \quad [\because \log e = 1]
\end{aligned}$$

Hence the given series converges.

85. Test the convergence of the series $1 + \frac{2 \cdot x}{2!} + \frac{3^2 \cdot x^2}{3!} + \frac{4^3 \cdot x^3}{4!} + \frac{5^4 \cdot x^4}{5!} + \dots$

Sol. Here,

$$u_n = \frac{n^{n-1} \cdot x^{n-1}}{n!} \quad \text{and} \quad u_{n+1} = \frac{(n+1)^n \cdot x^n}{(n+1)!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^{n-1} (n+1)!}{n! (n+1)^n} \cdot \frac{1}{x} = \frac{n^{n-1}}{(n+1)^{n-1}} \cdot \frac{1}{x} = \frac{1}{\left(1 + \frac{1}{n}\right)^{n-1}} \cdot \frac{1}{x}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{x \left(1 + \frac{1}{n}\right)^n} \right] = \frac{1}{ex}$$

$\therefore$ The series is convergent when $\frac{1}{ex} > 1$ i.e., $ex < 1$ and divergent when $ex > 1$.

When $ex = 1$ or $x = \frac{1}{e}$, D'Alembert's Ratio Test fails. We apply logarithmic test now since the expression u_n/u_{n+1} involves the number e .

$$\begin{aligned}
\lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} &= \lim_{n \rightarrow \infty} n \log \left\{ \frac{n^{n-1}}{(n+1)^{n-1}} e \right\} = \lim_{n \rightarrow \infty} n \log \left\{ \frac{e}{\left(1 + \frac{1}{n}\right)^{n-1}} \right\} \\
&= \lim_{n \rightarrow \infty} n \left\{ \log e - (n-1) \log \left(1 + \frac{1}{n} \right) \right\} \\
&= \lim_{n \rightarrow \infty} n \left\{ 1 - (n-1) \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \right) \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ n - (n-1) + \frac{(n-1)}{2n} - \frac{(n-1)}{3n^2} + \dots \right\} \\
&= \lim_{n \rightarrow \infty} \left\{ \frac{3}{2} - \frac{1}{2n} - \frac{n-1}{3n^2} + \dots \right\} = \frac{3}{2} > 1
\end{aligned}$$

As such the given series is convergent when $x = 1/e$.

Hence the series is convergent when $ex \leq 1$ and divergent when $ex > 1$.

86. State, with reason, the value of x for which the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$ converges.

(M.D.U., Dec., 2005, May 2009)

Sol. Here, $u_n = (-1)^{n-1} \cdot \frac{x^n}{n}$ and $u_{n+1} = (-1)^n \cdot \frac{x^{n+1}}{n+1}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{-n}{n+1} \cdot x$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| |x| = |x|$$

$\therefore$ By Ratio Test, the given series converges for $|x| < 1$ and diverges for $|x| > 1$.
Let us examine when $x = \pm 1$.

Putting $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

which is an alternating series and is convergent. Putting $x = -1$, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\right)$, which is a divergent series as can be seen by comparison with series

$\sum \frac{1}{n^p}$, $p = 1$. Hence the given series converges for $-1 < x \leq 1$.

87. Discuss the convergence of the series $\sum \frac{x^n}{n}$, $x > 0$.

(M.D.U., May, 2005)

Sol. Here, $u_n = \frac{x^n}{n}$ and $u_{n+1} = \frac{x^{n+1}}{n+1}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{x^n}{n} \cdot \frac{n+1}{x^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{x}$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ i.e., for $x < 1$ and diverges if $x > 1$.

When $x = 1$, $u_n = \frac{1}{n}$. Take $v_n = \frac{1}{n}$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$, which is finite and $\neq 0$.

$\therefore$ By Comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n$ is divergent. Hence $\sum u_n$ is divergent when $x = 1$.

$\therefore$ The given series is convergent when $x < 1$ and divergent when $x \geq 1$.

88. Discuss the convergence of the series $\sum \left(\frac{n+1}{3n}\right)^n$.

(M.D.U., May, 2005)

Sol.
$$u_n = \left(\frac{n+1}{3n} \right)^n = \left(\frac{1}{3} + \frac{1}{3n} \right)^n$$

$$\therefore (u_n)^{\frac{1}{n}} = \frac{1}{3} + \frac{1}{3n}$$

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3n} \right] = \frac{1}{3} < 1$$

$\therefore$ By Cauchy's Root Test, the given series $\sum u_n$ is convergent.

89. Test the convergence of the series $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \dots \infty$. (M.D.U., May, 2005)

Sol.
$$u_n = \frac{1}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n^2 \left(1 + \frac{1}{n} \right)}} = \frac{1}{n \sqrt{1 + \frac{1}{n}}}$$

Let us compare it with $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{1 + \frac{1}{n}}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1 + \frac{1}{n}}} \right) = 1 \text{ (finite and } \neq 0)$$

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ (where $p = 1$), it is divergent.

$\Rightarrow \sum u_n$ is divergent.

90. Discuss the convergence of the series $1 + \frac{1}{2} \cdot \frac{x^2}{4} + \frac{1.3.5}{2.4.6} \cdot \frac{x^4}{8} + \frac{1.3.5.7.9}{2.4.6.8.10} \cdot \frac{x^6}{12} + \dots \infty$.

(M.D.U., Dec., 2006)

Sol. Here,
$$u_n = \frac{1.3.5 \dots (4n-3) \cdot x^{2n}}{2.4.6 \dots (4n-2) \cdot 4n}$$

$$u_{n+1} = \frac{1.3.5 \dots (4n-3)(4n-1)(4n+1)}{2.4.6 \dots (4n-2)4n(4n+2)} \cdot \frac{x^{2n+2}}{4n+4}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{(4n+2)(4n+4)}{(4n-1)(4n+1)} \cdot \frac{1}{x^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{2n} \right) \left(1 + \frac{1}{n} \right)}{\left(1 - \frac{1}{4n} \right) \left(1 + \frac{1}{4n} \right)} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore \sum u_n$ is convergent if $\frac{1}{x^2} > 1$ i.e., if $x^2 < 1$ and it is divergent if $x^2 > 1$. However when $x^2 = 1$, the Ratio Test fails.

When $x^2 = 1$, we have

$$\begin{aligned}\frac{u_n}{u_{n+1}} &= \frac{(4n+2)(4n+4)}{(4n-1)(4n+1)} \\ n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= n \left[\frac{(4n+2)(4n+4)}{(4n-1)(4n+1)} - 1 \right] = \frac{n(24n+9)}{16n^2-1} = \frac{24 + \frac{9}{n}}{16 - \frac{1}{n^2}} \\ \lim_{n \rightarrow \infty} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] &= \lim_{n \rightarrow \infty} \left[\frac{24 + \frac{9}{n}}{16 - \frac{1}{n^2}} \right] = \frac{24}{16} \quad \text{or} \quad \frac{3}{2} > 1\end{aligned}$$

$\therefore$ By Raabe's Test, the series converges.

Hence $\sum u_n$ is convergent if $x^2 \leq 1$ and is divergent when $x^2 > 1$.

91. Test for convergence the series $\sum \frac{1}{n^2 + 1}$.

Sol. Here, $u_n = \frac{1}{n^2 + 1} = f(n)$

$$\therefore f(x) = \frac{1}{x^2 + 1}$$

For $x \geq 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable [Reference Question 73 (D)].

$$\text{Now,} \quad \int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{dx}{x^2 + 1} = \left[\tan^{-1} x \right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \text{finite}$$

$$\Rightarrow \int_1^{\infty} f(x) dx \text{ converges and hence by Integral Test, } \sum u_n \text{ also converges.}$$

92. Show that the series $\sum_1^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Sol. Here, $u_n = \frac{1}{n^p} = f(n)$

$$\therefore f(x) = \frac{1}{x^p}$$

For $x \geq 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Case I : When $p \neq 1$

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]$$

Sub-case 1. When $p > 1$, $p - 1$ is +ve, so that

$$\int_1^{\infty} f(x) dx = -\frac{1}{p-1} \left[\frac{1}{x^{p-1}} \right]_1^{\infty} = -\frac{1}{p-1} (0 - 1) = \frac{1}{p-1} = \text{finite}$$

$$\Rightarrow \int_1^{\infty} f(x) dx \text{ converges} \Rightarrow \Sigma u_n \text{ is convergent.}$$

Sub-case 2. When $0 < p < 1$, $1 - p$ is +ve, so that

$$\int_1^{\infty} f(x) dx = \frac{1}{1-p} \left[x^{1-p} \right]_1^{\infty} = \frac{1}{1-p} (\infty - 1) = \infty$$

$$\Rightarrow \int_1^{\infty} f(x) dx \text{ diverges} \Rightarrow \Sigma u_n \text{ is divergent.}$$

Case II : When $p = 1$, $f(x) = \frac{1}{x}$

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1}{x} dx = [\log x]_1^{\infty} = \infty - \log 1 = \infty - 0 = \infty$$

$$\Rightarrow \int_1^{\infty} f(x) dx \text{ diverges} \Rightarrow \Sigma u_n \text{ is divergent.}$$

Hence Σu_n converges if $p > 1$ and diverges if $p \leq 1$.

93. Examine convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, ($p > 0$). (M.D.U., Dec., 2007)

Sol. Here, $u_n = \frac{1}{n(\log n)^p} = f(n)$

$$\therefore f(x) = \frac{1}{x(\log x)^p}$$

For $x \geq 2$, $p > 0$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test, $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x) dx$ converge or diverge together.

Case I. When $p \neq 1$

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} (\log x)^{-p} \cdot \frac{1}{x} dx = \left[\frac{(\log x)^{-p+1}}{-p+1} \right]_2^{\infty}$$

Sub-case 1. When $p > 1$, $p - 1$ is +ve, so that

$$\begin{aligned} \int_2^{\infty} f(x) dx &= \frac{-1}{p-1} \left[\frac{1}{(\log x)^{p-1}} \right]_2^{\infty} = \frac{-1}{p-1} \left[0 - \frac{1}{(\log 2)^{p-1}} \right] \\ &= \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite} \end{aligned}$$

$$\Rightarrow \int_2^{\infty} f(x) dx \text{ converges} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ converges.}$$

Sub-case 2. When $p < 1$, $1 - p$ is +ve, so that

$$\int_2^{\infty} f(x) dx = \frac{1}{1-p} \left[(\log x)^{1-p} \right]_2^{\infty} = \frac{1}{1-p} [\infty - (\log 2)^{1-p}] = \infty$$

$$\Rightarrow \int_2^{\infty} f(x) dx \text{ diverges} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges.}$$

Case II. When $p = 1$, $f(x) = \frac{1}{x \log x}$

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{dx}{x \log x} = \int_2^{\infty} \frac{1/x}{\log x} dx = [\log \log x] = \infty - \log \log 2 = \infty$$

$$\Rightarrow \int_2^{\infty} f(x) dx \text{ diverges} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges.}$$

Hence, $\sum u_n$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

94. Using the integral test, discuss the convergence of the series : $\sum \frac{1}{(n+1)^2}$. [M.D.U., May, 2008]

Sol. Here, $u_n = \frac{1}{(n+1)^2}$

$\therefore f(x) = \frac{1}{(x+1)^2}$

For $x \geq 1$, $f(x)$ is +ve and monotonically decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Now,
$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1}{(x+1)^2} dx = \left[\frac{(x+1)^{-1}}{-1} \right]_1^{\infty} = \left[\frac{-1}{(x+1)} \right]_1^{\infty} = -\frac{1}{2} \text{ (finite)}$$

$\Rightarrow \int_1^{\infty} f(x) dx$ converges and hence by integral test $\sum u_n$ also converges.

95. State Leibnitz's Test on Alternating series.

Sol. The alternating series $\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$ ($u_n > 0 \forall n$) converges if

(i) $u_n > u_{n+1} \forall n$ and (ii) $\lim_{n \rightarrow \infty} u_n = 0$.

Notes : 1. The alternating series will not be convergent **if any one of the two conditions** is not satisfied.

2. When $\lim_{n \rightarrow \infty} u_n \neq 0$, the alternating series is **oscillatory**.

96. Examine the convergence of the series : $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Sol. It is an alternating series $\sum (-1)^{n-1} u_n$

(i) $u_n = \frac{1}{n}, u_{n+1} = \frac{1}{n+1}$

$\therefore \frac{1}{n} > \frac{1}{n+1} \forall n \quad \therefore u_n > u_{n+1} \forall n$

$$(ii) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$\therefore$ Both the conditions of Leibnitz's Test are satisfied. Hence, the given series is convergent.

97. Test the convergence of the series $\sum \frac{(-1)^{n-1} \cdot n}{2n-1}$.

Sol. The given series is $\sum \frac{(-1)^{n-1} \cdot n}{2n-1} = \frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \dots$.

It is an alternating series $\sum (-1)^{n-1} u_n$.

$$(i) u_n = \frac{n}{2n-1}, u_{n+1} = \frac{n+1}{2n+1}$$

$$u_n - u_{n+1} = \frac{1}{4n^2-1} > 0 \forall n \Rightarrow u_n > u_{n+1} \forall n$$

$$(ii) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2} \neq 0$$

Here the second condition of Leibnitz's Test is not satisfied. Hence the given series is not convergent. It is oscillatory.

98. Test the convergence of the series $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \dots$

Sol. It is an alternating series $\sum (-1)^{n-1} u_n$

$$(i) u_n = \frac{1}{(n+1)^3} \cdot [1+2+3+\dots+n] = \frac{1}{(n+1)^3} \cdot \frac{n(n+1)}{2} = \frac{1}{2} \cdot \frac{n}{(n+1)^2}$$

$$u_{n+1} = \frac{1}{2} \cdot \frac{n+1}{(n+2)^2}$$

$$u_n - u_{n+1} = \frac{1}{2} \left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2} \right]$$

$$= \frac{1}{2} \cdot \frac{n(n+2)^2 - (n+1)^3}{(n+1)^2(n+2)^2} = \frac{1}{2} \cdot \frac{n^2+n-1}{(n+1)^2(n+2)^2} > 0 \forall n$$

$$\Rightarrow u_n > u_{n+1} \forall n$$

$$(ii) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n}{2(n+1)^2}$$

Dividing numerator and denominator by n^2 on right hand side, we have

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{2 \left(1 + \frac{1}{n} \right)^2} = 0$$

Since both the conditions of Leibnitz Test are satisfied, the given series is convergent.

99. Examine the convergence of the series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots$.

Sol. It is an alternating series $\sum (-1)^{n-1} u_n$

$$(i) \quad u_n = \frac{n+1}{n} \quad \text{and} \quad u_{n+1} = \frac{n+2}{n+1}$$

$$u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{(n+1)^2 - n(n+2)}{n(n+1)} = \frac{1}{n(n+1)} > 0 \quad \forall n$$

$$\Rightarrow \quad u_n > u_{n+1} \quad \forall n$$

$$(ii) \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$$

Since the second condition of Leibnitz's Test is not satisfied, the series is not convergent.

100. Show that the series $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} + \dots$ converges.

Sol. The given series is an alternating series $\sum (-1)^{n-1} u_n$ with $u_n = \frac{\log(n+1)}{(n+1)^2} \quad \forall n \geq 1$

Now consider the function $f(x) = \frac{\log x}{x^2}, x > 0$

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - 2x \log x}{x^4} = \frac{1 - 2 \log x}{x^3} < 0 \quad \text{if } 1 - 2 \log x < 0 \text{ or } \log x > \frac{1}{2}$$

$$\Rightarrow \text{if } x > e^{1/2}, f'(x) < 0$$

$$\Rightarrow f(x) \text{ is monotonically decreasing for } x > \sqrt{e}$$

$$\text{In particular } f(n+1) > f(n+2) \quad \forall n \geq 1$$

$$\text{i.e.,} \quad u_n > u_{n+1} \quad \forall n$$

$$\text{Also,} \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\log(n+1)}{(n+1)^2} \quad \left[\text{Form } \frac{\infty}{\infty} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{2(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{2(n+1)^2}$$

Thus both the conditions of Leibnitz's Test are satisfied. Hence the given series is convergent.

101. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot x^n}{n(n-1)}, 0 < x < 1$.

Sol. The given series is an alternating series $\sum_{n=2}^{\infty} (-1)^{n-1} u_n$ with

$$u_n = \frac{x^n}{n(n-1)}, \quad 0 < x < 1, n \geq 2$$

$$\Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)n}$$

$$\therefore u_n - u_{n+1} = \frac{x^n}{n} \left(\frac{1}{n-1} - \frac{x}{n+1} \right)$$

$$= \frac{x^n}{n} \left[\frac{(n+1) - (n-1)x}{(n-1)(n+1)} \right] = \frac{x^n}{n} \left[\frac{n(1-x) + (1+x)}{n^2 - 1} \right]$$

Since $0 < x < 1$ $\therefore x^n$, $(1-x)$ and $(1+x)$ are all positive.

Since $n \geq 2$ $\therefore n^2 - 1 > 0$

Thus $u_n - u_{n+1} > 0 \Rightarrow u_n > u_{n+1} \forall n \geq 2$

Also,
$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} \cdot x^n = 0 \times 0 = 0 \quad [\because 0 < x < 1]$$

Both the conditions of Leibnitz's Test are satisfied.

Hence, the given series is convergent.

102. Test the convergence of the series $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$

Sol. The terms of the series are alternatively positive and negative and also each term is numerically less than the preceding term, i.e.,

$$u_n > u_{n+1} \text{ for all } n \quad \dots(1)$$

Also,
$$u_n = \frac{1}{(2n-1)^2}$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{(2n-1)^2} = 0 \quad \dots(2)$$

From the conditions (1) and (2), the given series converges.

103. Test the convergence of the series : $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots \infty$.

Sol. Here, we shall first apply D' Alembert's Ratio Test.

$$u_n = (-1)^{n-1} \cdot \frac{x^n}{n^2} \quad \text{and} \quad u_{n+1} = (-1)^n \cdot \frac{x^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = - \frac{1}{\left(1 + \frac{1}{n}\right)^2} \cdot \frac{1}{x}$$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{-1 \cdot x}{\left(1 + \frac{1}{n}\right)^2} \right| = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \cdot |x|$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{\left(1 + \frac{1}{n}\right)^2} \right| |x| = |x|$$

Therefore, the given series converges when $|x| < 1$ and diverges for $|x| > 1$.

When $|x| = 1$, D'Alembert's Ratio Test fails.

Now when $|x| = 1$, the series can be shown to be convergent by Leibnitz's Test.

- 104.** When is a series absolutely convergent and when is it called conditionally convergent? Test whether the following series is absolutely convergent or conditionally convergent : $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Sol. If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an absolutely convergent series, i.e., the series Σu_n is said to be absolutely convergent if $\Sigma |u_n|$ is a convergent series. A series is said to be conditionally convergent if it is convergent but does not converge absolutely.

In the given series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ which is an alternating series, each term is less than the preceding term numerically.

Moreover $u_n = \frac{1}{n^2}$, which $\rightarrow 0$ as $n \rightarrow \infty$. Hence the given series satisfies both the conditions of the test on alternating series and so the given series converges.

Again when all the terms of the series are made positive, the series becomes

$$\Sigma |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \Sigma \frac{1}{n^2}$$

which we know is a convergent series. Thus the given series converges absolutely.

- 105.** Examine for absolute convergence or conditionally convergence the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Sol.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \Sigma (-1)^{n-1} \cdot u_n \text{ (say)}$$

Putting $n = 1, 2, 3, \dots$ the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

The series is clearly an alternating series.

(i) The terms go on decreasing numerically and (ii) $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$

$\therefore$ By Leibnitz's Test, the series converges.

But when all terms are made positive, the series becomes,

$$\Sigma |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

Here, $|u_n| = \frac{1}{2n-1}$. Take $v_n = \frac{1}{n}$

$$\therefore \lim_{n \rightarrow \infty} \frac{|u_n|}{v_n} = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \left[\frac{1}{2 - \frac{1}{n}} \right] = \frac{1}{2} = \text{finite} \neq 0$$

Hence by comparison test, series $\Sigma |u_n|$ and Σv_n behave alike. But $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series.

($\because p = 1$ here)

$\therefore \Sigma |u_n|$ also diverges.

Since the given series converges and the series of absolute terms diverges, therefore the given series converges conditionally.

106. Test the following series for convergence and absolute convergence : $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

It is an alternating series.

Here,
$$a_n = \frac{1}{n^{3/2}} > 0 \quad \forall n$$

$$a_{n+1} = \frac{1}{(n+1)^{3/2}}$$

Since
$$\frac{1}{n^{3/2}} > \frac{1}{(n+1)^{3/2}} \quad \forall n, \quad \therefore a_n > a_{n+1} \quad \forall n$$

Also,
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} = 0$$

$\therefore$ By Leibnitz's test, the series Σu_n is convergent

Now,
$$|u_n| = \frac{1}{n^{3/2}}, \quad \therefore \Sigma |u_n| = \Sigma \frac{1}{n^{3/2}} \text{ is convergent.} \quad \left(\because p = \frac{3}{2} > 1 \right)$$

Hence the series Σu_n is absolutely convergent.

107. Test for convergence and absolute convergence the series $\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \frac{1}{4.6} + \dots$

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$$

It is an alternating series.

Here,
$$a_n = \frac{1}{n(n+2)} > 0 \quad \forall n \quad \text{and} \quad a_{n+1} = \frac{1}{(n+1)(n+3)}$$

$$\therefore a_n - a_{n+1} = \frac{1}{n(n+2)} - \frac{1}{(n+1)(n+3)} = \frac{2n+3}{n(n+1)(n+2)(n+3)} > 0 \quad \forall n$$

$$\Rightarrow a_n > a_{n+1} \quad \forall n$$

Also,
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n(n+2)} = 0$$

$\therefore$ By Leibnitz's test, the series Σu_n is convergent

Now,
$$|u_n| = \frac{1}{n(n+2)} = \frac{1}{n^2 \left(1 + \frac{2}{n}\right)}$$

Take,
$$v_n = \frac{1}{n^2}, \lim_{n \rightarrow \infty} \frac{|u_n|}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = 1$$

which is non-zero and finite.

$\therefore$ By comparison test, $\Sigma|u_n|$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ converges ($\because p = 2 > 1$), $\Sigma|u_n|$ also converges.

Hence the given series Σu_n is absolutely convergent.

108. Test for convergence and absolute convergence the series :

$$1 - \frac{1}{2^3}(1+2) + \frac{1}{3^3}(1+2+3) - \frac{1}{4^3}(1+2+3+4) + \dots$$

Sol. The given series is

$$\begin{aligned} \sum_{n=1}^{\infty} u_n &= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1+2+3+\dots+n}{n^3} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n(n+1)}{n^3} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n+1}{2n^2} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n \end{aligned}$$

It is an alternating series.

Here,

$$a_n = \frac{n+1}{2n^2}, a_{n+1} = \frac{n+2}{2(n+1)^2}$$

$$a_n - a_{n+1} = \frac{n+1}{2n^2} - \frac{n+2}{2(n+1)^2} = \frac{(n+1)^3 - n^2(n+2)}{2n^2(n+1)^2}$$

or
$$a_n - a_{n+1} = \frac{n^2 + 3n + 1}{2n^2(n+1)^2} > 0 \quad \forall n.$$

$\Rightarrow a_n > a_{n+1} \quad \forall n$

Also,
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{2n^2} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0$$

$\therefore$ By Leibnitz's test, the series Σu_n is convergent.

Now,
$$|u_n| = \frac{n+1}{2n^2} = \frac{n \left(1 + \frac{1}{n} \right)}{2n^2} = \frac{1 + \frac{1}{n}}{2n}$$

Take,
$$v_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{n} \right) = \frac{1}{2}$$

which is non-zero and finite.

$\therefore$ By comparison test, $\Sigma|u_n|$ and Σv_n converge or diverge together. Since $\Sigma v_n = \Sigma \frac{1}{n}$ diverges ($\because p = 1$).

$\Sigma|u_n|$ also diverges. Hence given series Σu_n is conditionally convergent.

109. Test $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ for convergence and absolute convergence.

Sol. The given series is

$$\sum u_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2} = \sum (-1)^n \cdot a_n$$

It is an alternating series.

Here,
$$a_n = \frac{1}{n(\log n)^2}, \quad a_{n+1} = \frac{1}{(n+1)[\log(n+1)]^2}$$

Clearly,
$$a_n > a_{n+1} \quad \forall \quad n \geq 2$$

Also,
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n(\log n)^2} = 0$$

$\therefore$ By Leibnitz's Test, the series $\sum_{n=2}^{\infty} u_n$ is convergent.

Now,
$$|u_n| = \frac{1}{n(\log n)^2} = f(n) \quad \therefore \quad f(x) = \frac{1}{x(\log x)^2}$$

For $x \geq 2$, $f(x)$ is positive and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test, $\sum_{n=2}^{\infty} |u_n|$ and $\int_2^{\infty} f(x) dx$ converge or diverge together.

Now,
$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{dx}{x(\log x)^2} = \int_2^{\infty} (\log x)^{-2} \cdot \frac{1}{x} dx = \left[\frac{(\log x)^{-1}}{-1} \right]_2^{\infty} = \left[\frac{-1}{\log x} \right]_2^{\infty} = \frac{1}{\log 2}$$

which is a finite quantity and non-zero.

$\Rightarrow$ The integral $\int_2^{\infty} f(x) dx$ is convergent.

$\Rightarrow$ The series $\sum_{n=2}^{\infty} |u_n|$ is convergent.

Hence the given series $\sum_{n=2}^{\infty} u_n$ is absolutely convergent.

110. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$.

Sol. The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$. It is an alternating series.

Here,
$$a_n = \frac{n}{5^n}, \quad a_{n+1} = \frac{n+1}{5^{n+1}}$$

$$a_n - a_{n+1} = \frac{n}{5^n} - \frac{n+1}{5^{n+1}} = \frac{5n - (n+1)}{5^{n+1}} = \frac{4n-1}{5^{n+1}} > 0 \quad \forall n.$$

$\Rightarrow$
$$a_n > a_{n+1} \quad \forall n$$

Also,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{5^n} \quad \left[\text{Form } \frac{\infty}{\infty} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{n}{5^n \log 5} = 0 \quad [\text{L-Hospital's Rule}]$$

$\therefore$ By Leibnitz's Test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Now,

$$|u_n| = \frac{n}{5^n}, |u_{n+1}| = \frac{n+1}{5^{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \rightarrow \infty} \frac{5n}{n+1} = \lim_{n \rightarrow \infty} \frac{5}{1 + \frac{1}{n}} = 5 > 1$$

$\therefore$ By ratio test, the series $\sum |u_n|$ is convergent.

Hence the given series $\sum u_n$ is absolutely convergent.

111. Test $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right]$ for convergence and absolute convergence.

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right] = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

It is an alternating series.

Here,

$$a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} > 0 \quad \forall n.$$

$$a_{n+1} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2}$$

$$a_n - a_{n+1} = \frac{1}{n^2} - \frac{1}{(n+2)^2} = \frac{4n+4}{n^2(n+2)^2} > 0 \quad \forall n$$

$$\Rightarrow a_n > a_{n+1} \quad \forall n$$

Also,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} \right] = 0$$

$\therefore$ By Leibnitz's Test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Now,

$$|u_n| = \frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2} \quad \forall n$$

Since the series $\sum \frac{2}{n^2} = 2 \sum \frac{1}{n^2}$ is convergent ($\because p = 2 > 1$), by comparison test $\sum |u_n|$ is also convergent.

Hence the given series is absolutely convergent.

112. Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$ converges absolutely. (M.D.U., Dec., 2006)

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n^3}$$

Since $|u_n| = \frac{|\sin nx|}{n^3} \leq \frac{1}{n^3} \forall n$ and $\sum \frac{1}{n^3}$ converges.

$\therefore$ By comparison test, the series $\sum |u_n|$ converges

$\Rightarrow$ The given series converges absolutely.

113. Discuss the convergence of the series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ for all values of x .

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Here, $u_n = \frac{x^{n-1}}{(n-1)!}$ and $u_{n+1} = \frac{x^n}{n!}$

$$\therefore \frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{n-1}}{(n-1)!} \times \frac{n!}{|x|^n} = \frac{n}{|x|} \text{ for } x \neq 0$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \rightarrow \infty} \frac{n}{|x|} = \infty \text{ for } x \neq 0$$

$\therefore$ By Ratio test, the series $\sum_{n=1}^{\infty} |u_n|$ is convergent for $x \neq 0$.

When $x = 0$, the series becomes $1 + 0 + 0 + \dots$ and is convergent.

Thus $\sum |u_n|$ is convergent for all x .

$\Rightarrow$ The given series is absolutely convergent for all x .

Since every absolutely convergent series is convergent therefore, the given series is convergent for all x .

Note : Here $\sum u_n$ is convergent for all x .

$$\Rightarrow \lim_{n \rightarrow \infty} u_{n+1} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \text{ for all } x.$$

114. Discuss the convergence and absolute convergence of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$, x being real.

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1}$$

$$\text{Here, } u_n = \frac{(-1)^{n-1} \cdot x^{2n-1}}{2n-1}, \quad u_{n+1} = (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

$$\therefore \frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{2n-1}}{2n-1} \times \frac{2n+1}{|x|^{2n+1}} = \frac{2n+1}{2n-1} \cdot \frac{1}{|x|^2} = \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} \cdot \frac{1}{x^2} \quad (\because |x|^2 = x^2)$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By ratio test, the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

If $\frac{1}{x^2} > 1$ i.e., if $x^2 < 1$, i.e., if $-1 < x < 1$ and divergent if $\frac{1}{x^2} < 1$ i.e., if $x^2 > 1$ i.e., if $x > 1$ or $x < -1$.

Ratio test fails when $x^2 = 1$.

When $x^2 = 1$, we have

$$\begin{aligned} \frac{|u_n|}{|u_{n+1}|} &= \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} = \left(1+\frac{1}{2n}\right) \left(1-\frac{1}{2n}\right)^{-1} \\ &= \left(1+\frac{1}{2n}\right) \left[1+\frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right] \\ &= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

$\therefore$ By Gauss test, the series $\sum |u_n|$ is divergent.

Hence the given series is **not** absolutely convergent when $x^2 = 1$.

When $x = 1$, the series becomes

$$\sum u_n = \sum (-1)^{n-1} \cdot \frac{1}{2n-1} = \sum (-1)^{n-1} \cdot v_n$$

It is an alternating series.

Here
$$v_n = \frac{1}{2n-1}, v_{n+1} = \frac{1}{2n+1}$$

Since
$$2n-1 < 2n+1 \quad \forall n$$

$$\therefore \frac{1}{2n-1} > \frac{1}{2n+1} \quad \forall n$$

or
$$v_n > v_{n+1} \quad \forall n$$

Also
$$\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

$\therefore$ By Leibnitz's test, the series is convergent.

When $x = -1$, the series becomes

$$\begin{aligned} \sum u_n &= \sum (-1)^{n-1} \cdot \frac{(-1)^{2n-1}}{2n-1} = \sum (-1)^{n-1} \cdot \frac{-1}{2n-1} \\ &= -\sum (-1)^{n-1} \cdot \frac{1}{2n-1} \end{aligned}$$

Since $\sum (-1)^{n-1} \cdot \frac{1}{2n-1}$ is convergent as shown above $\sum u_n$ is also convergent.

Hence $\sum u_n$ is convergent if $-1 \leq x \leq 1$ and absolutely convergent if $-1 < x < 1$.

115. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$$

Sol. Here, $u_n = \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$ and $u_{n+1} = \frac{(-1)^{n+1} \cdot (x+1)^{n+1}}{2^{n+1} \cdot (n+1)^2}$

$$\begin{aligned} \frac{|u_n|}{|u_{n+1}|} &= \frac{|x+1|^n}{2^n \cdot n^2} \times \frac{2^{n+1} \cdot (n+1)^2}{|x+1|^{n+1}} \\ &= 2 \cdot \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{|x+1|} = 2 \cdot \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \rightarrow \infty} 2 \cdot \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|} = \frac{2}{|x+1|}$$

$\therefore$ By Ratio test, the series $\sum |u_n|$ is convergent if $\frac{2}{|x+1|} > 1$, i.e., if $|x+1| < 2$.

i.e., if $-2 < x+1 < 2$, i.e., if $-3 < x < 1$.

Also $\sum |u_n|$ is divergent if $\frac{2}{|x+1|} < 1$

i.e., if $|x+1| > 2$ i.e., if $x+1 > 2$ or $x+1 < -2$

i.e., if $x > 1$ or $x < -3$.

Ratio test fails when $x = 1$ or $x = -3$.

When $x = 1$,

$$\sum u_n = \sum \frac{(-1)^n \cdot 2^n}{2^n \cdot n^2} = \sum \frac{(-1)^n}{n^2} = \sum (-1)^n \cdot v_n$$

It is an alternating series.

Here,
$$v_n = \frac{1}{n^2}, \quad v_{n+1} = \frac{1}{(n+1)^2}$$

Clearly,
$$v_n > v_{n+1} \quad \forall n$$

Also,
$$\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

$\therefore$ By Leibnitz's test, $\sum u_n$ is convergent.

When $x = -3$,

$$\sum u_n = \sum \frac{(-1)^n \cdot (-2)^n}{2^n \cdot n^2} = \sum \frac{(-1)^{2n}}{n^2} = \sum \frac{1}{n^2}$$

which is convergent.

Hence the given series is convergent if $-3 \leq x \leq 1$.

116. For what values of x , is the following series convergent :

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

(M.D.U., Dec., 2005)

Sol. Here,

$$u_n = (-1)^{n-1} \cdot \frac{x^n}{n} \quad \text{and} \quad u_{n+1} = (-1)^n \cdot \frac{x^{n+1}}{n+1}$$

$$\frac{u_{n+1}}{u_n} = -\frac{x \cdot n}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{1}{\left(1 + \frac{1}{n}\right)} \right] |x|$$

By Ratio Test the given series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$, which is an alternating series and is convergent.

When $x = -1$, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\right)$, which is divergent series as can be seen after comparison with $\frac{1}{n^p}$ ($p = 1$).

Hence the given series converges for $-1 < x \leq 1$.

117. For what values of x is the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \dots$ convergent. (M.D.U., Dec., 2008)

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^n}{\sqrt{n}}$$

$$|u_n| = \frac{|x|^n}{\sqrt{n}} \quad \text{or} \quad \frac{|x|^n}{\sqrt{n}} \quad \text{and} \quad |u_{n+1}| = \frac{|x|^{n+1}}{\sqrt{n+1}}$$

$$\therefore \frac{|u_n|}{|u_{n+1}|} = \sqrt{\frac{n+1}{n}} \cdot \frac{1}{|x|} = \sqrt{1 + \frac{1}{n}} \cdot \frac{1}{|x|}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_n}{u_{n+1}} \right| = \frac{1}{|x|}$$

$\therefore$ By Ratio Test, the series $\sum |u_n|$ is convergent.

If $\frac{1}{|x|} > 1$, i.e., if $|x| < 1$, i.e., if $-1 < x < 1$ and divergent if $\frac{1}{|x|} < 1$, i.e., if $|x| > 1$, i.e., if $x > 1$ or $x < -1$.

Ratio test fails when $|x| = 1$.

$\therefore$ i.e., when $x = 1$ or -1 .

When $x = 1$, the series becomes $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

which is an alternating series and is convergent.

When $x = -1$, the series becomes

$$-\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots\right) = -\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which is divergent by p -series test ($p < 1$).

Hence the given series converges for $-1 < x \leq 1$.

118. Discuss the convergence of the series $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \dots$

Sol. The terms of the given series are alternatively positive and negative.

$$u_n = \frac{2n+3}{2n}, \quad u_{n-1} = \frac{2n+1}{2n-2}$$

$$\therefore u_n - u_{n-1} = \frac{2n+3}{2n} - \frac{2n+1}{2n-2} = -\frac{6}{4n(n-1)} < 0 \quad (\text{for } n > 1)$$

$$\text{i.e.,} \quad u_n < u_{n-1} \text{ for } n > 1.$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{2n+3}{2n} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{2n}\right) = 1 \neq 0$$

Hence by Leibnitz's Test, the given series is oscillatory.

119. Test whether the series $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \dots$ is convergent or not.

Sol. The series of absolute terms is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots$ which is obviously convergent.

$\therefore$ The given series is absolutely convergent and hence it is convergent.

120. Test whether the series

$$\frac{1}{2^3} - \frac{1}{3^3} (1+2) + \frac{1}{4^3} (1+2+3) - \frac{1}{5^3} (1+2+3+4) + \dots \infty \text{ is conditionally convergent.}$$

Sol. Here,

$$\begin{aligned} u_n &= (-1)^{n-1} \cdot \frac{(1+2+3+4+\dots+n)}{(n+1)^3} \\ &= (-1)^{n-1} \cdot \frac{n(n+1)}{2(n+1)^3} = (-1)^{n-1} \cdot \frac{n}{2(n+1)^2} = (-1)^{n-1} \cdot a_n \text{ (say)} \end{aligned}$$

$$a_n = \frac{n}{2(n+1)^2}, \quad a_{n+1} = \frac{n+1}{2(n+2)^2}$$

$$\Rightarrow a_n - a_{n+1} = \frac{1}{2} \left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2} \right] = \frac{1}{2} \cdot \frac{(n^2+n-1)}{(n+1)^2(n+2)^2} > 0$$

$$\text{i.e.,} \quad a_n > a_{n+1} \text{ or } a_{n+1} < a_n.$$

$$\text{Also,} \quad \lim_{n \rightarrow \infty} a_n = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{(n+1)^2} = 0$$

Therefore, by Leibnitz's Test, series Σa_n and so Σu_n is convergent.

$$\text{Also,} \quad |u_n| = \frac{1}{2} \cdot \frac{n}{(n+1)^2}.$$

$$\text{Taking} \quad v_n = \frac{1}{n}.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|u_n|}{v_n} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(\frac{n^2}{n^2+1} \right) = \frac{1}{2} \neq 0$$

Since Σv_n is divergent $\left(\Sigma \frac{1}{n^p} \text{ series test; } p = 1 \right) \Sigma |u_n|$ is also divergent.

i.e., Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Thus the given series Σu_n is conditionally convergent.

121. For which values of x , the following series converges $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \dots \infty$

Sol. Here, $u_n = \frac{1}{n(1-x)^n}, u_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n \cdot (1-x)^n}{(n+1) \cdot (1-x)^{n+1}} \right| = \left| \frac{1}{1-x} \right| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = \left| \frac{1}{1-x} \right|$$

By ratio test, Σu_n converges for $\left| \frac{1}{1-x} \right| < 1$, i.e., $|1-x| > 1$, i.e., for $-1 > 1-x > 1$ or $x < 0$ and $x > 2$.

Let us examine the series for $x = 0$ and $x = 2$.

For $x = 0$, the given series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \dots + \dots \infty$

which is a divergent harmonic series.

For $x = 2$, the given series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + \frac{(-1)^n}{n} + \dots \infty$

It is an alternating series which is convergent by Leibnitz's Test

Since $u_n < u_{n-1}$ for all n

and $\lim_{n \rightarrow \infty} u_n = 0$.

Hence the given series converges for $x < 0$ and $x \geq 2$.

122. Examine the binomial series $1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} \cdot x^{r-1}$ for convergence for $|x| < 1$.

Sol. Here, $u_r = \frac{n(n-1)\dots(n-r)}{(r-1)!} \cdot x^{r-1}$

$$u_{r+1} = \frac{n(n-1)\dots(n-r+1)}{r!} \cdot x^n$$

$$\therefore \lim_{r \rightarrow \infty} \frac{u_{r+1}}{u_r} = \lim_{r \rightarrow \infty} \frac{n-r+1}{r} \cdot x = \lim_{r \rightarrow \infty} \left(\frac{n+1}{r} - 1 \right) \cdot x = -x \text{ for } r > n+1$$

Hence the series converges for $|x| < 1$.

123. State Weierstrass's M-Test. Use it to show that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ for all real x and $p > 1$, is uniformly convergent.

Sol. Weierstrass's M-Test is that a series $\sum_{n=1}^{\infty} u_n(x)$ of functions converges uniformly and absolutely on an interval I if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that

$$|u_n(x)| \leq M_n \quad \forall n \in N \text{ and } \forall x \in I.$$

Here

$$u_n(x) = \frac{\cos nx}{n^p}$$

$$\therefore |u_n(x)| = \left| \frac{\cos nx}{n^p} \right| = \frac{|\cos nx|}{n^p} \leq \frac{1}{n^p} \quad (= M_n) \quad \forall x \in R$$

Since
$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for $p > 1$, the given series by M-Test, is *uniformly convergent* for all real x and $p > 1$.

124. Show that the series $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$ is uniformly convergent for all real x .

Sol. Here,

$$u_n(x) = \frac{\sin(x^2 + nx)}{n(n+2)}$$

$$\begin{aligned} \therefore |u_n(x)| &= \left| \frac{\sin(x^2 + nx)}{n(n+2)} \right| = \frac{|\sin(x^2 + nx)|}{n(n+2)} \\ &\leq \frac{1}{n(n+2)} < \frac{1}{n^2} \quad (= M_n) \quad \forall x \in R \end{aligned}$$

Since
$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent, therefore, by M-test, the given series is uniformly convergent for all real x .

125. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q \cdot x^2}$ is uniformly convergent for all real x and $p > 1$.

Sol. Here,
$$u_n(x) = \frac{1}{n^p + n^q \cdot x^2}$$

Since $x^2 \geq 0$ for all real x , $n^q \cdot x^2 \geq 0$.

$$\Rightarrow n^p + n^q \cdot x^2 \geq n^p \Rightarrow \frac{1}{n^p + n^q \cdot x^2} \leq \frac{1}{n^p}$$

$$\therefore |u_n(x)| = \frac{1}{n^p + n^q \cdot x^2} \leq \frac{1}{n^p} \quad (= M_n) \quad \forall x \in R$$

Since,
$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for $p > 1$ therefore by M-test the given series is uniformly convergent for all real x and $p > 1$.

126. Show that the following series converge uniformly in any interval $\sum \frac{1}{n^3 + n^4 \cdot x^2}$ for all real x .

Sol. For all values of x ,

$$n^3 + n^4 \cdot x^2 > n^3 \quad (\because x^2 \geq 0)$$

$$\therefore \left| \frac{1}{n^3 + n^4 \cdot x^2} \right| < \frac{1}{n^3} (= M_n).$$

But $\sum M_n = \sum \frac{1}{n^3}$

is the p -series with $p = 3 > 1$ and is convergent.

$\therefore$ By M-test, the given series $\sum \frac{1}{n^3 + n^4 \cdot x^2}$ converges uniformly in any interval.

127. Describe the general procedure for testing a series for convergence.

Sol. First see whether the given series is

(i) a series with terms alternatively positive and negative;

(ii) a series of positive terms excluding Power Series of the form $\sum_{n=0}^{\infty} a_n x^n$ or

(iii) a Power Series $\sum_{n=0}^{\infty} a_n x^n$.

For alternating series (i), apply the Leibnitz's Test (Reference Q. 95).

For a series (ii), first find u_n and if possible evaluate limit u_n . If limit $u_n \neq 0$, the series is divergent. If limit $u_n = 0$, compare $\sum u_n$ with $\sum \frac{1}{n^p}$ and apply the comparison tests.

If the comparison tests are not applicable, apply the Ratio Test. If limit $\frac{u_n}{u_{n+1}} = 1$, i.e., the Ratio

Test fails, apply Raabe's Test. If Raabe's Test fails for a similar reason, apply Logarithmic test. If this test also fails, apply Cauchy's root test.

For the power series (iii), apply the Ratio Test limit $\left(\frac{a_{n+1}}{a_n} \right)$ which has an interval of convergence

$-\frac{1}{l} < x < \frac{1}{l}$. The series diverge for values of x outside this interval.

If the Ratio Test fails, examine the series as in case (ii) above.

The above is a general procedure which the student is advised to follow for testing an infinite series for its convergence.

128. Comment whether there is any relation between absolute and uniform convergence of a series.

Sol. There is no relation between absolute and uniform convergence. A series may converge absolutely but not uniformly while another series may converge uniformly but not absolutely.

For example, the series $\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \dots$

may be seen to converge uniformly but not absolutely, while the series

$$x^2 + \frac{x^2}{x^2+1} + \frac{x^2}{(x^2+1)^2} + \frac{x^2}{(x^2+1)^3} + \dots$$

can be shown to converge absolutely but not uniformly.

- 129.** Examine the geometric series $1 + x + x^2 + \dots + x^{n-1} + \dots \infty$ for uniform convergence in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Sol. Sum of the first n terms i.e., $S_n(x)$ is given by

$$S_n(x) = 1 + x + x^2 + \dots + x^{n-1} = \frac{1-x^n}{1-x}$$

and

$$S(x) = \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} = \frac{1}{1-x} \quad \text{for } |x| < 1$$

$$\therefore |S(x) - S_n(x)| = \left| \frac{x^n}{1-x} \right| = \frac{|x^n|}{1-x} = \frac{|x|^n}{1-x}$$

which will be $< \varepsilon$, if $|x|^n < \varepsilon(1-x)$.

Choose N such that $|x|^N = \varepsilon(1-x)$

or
$$N = \log \frac{[\varepsilon(1-x)]}{\log |x|}$$

Obviously, N increases with the increase of $|x|$ and in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$, it assumes a maximum value

$$N' = \frac{\log(\varepsilon/2)}{\log\left(\frac{1}{2}\right)} \quad \text{at } x = \frac{1}{2}$$

for a given ε .

Thus $|s(x) - s_n(x)| < \varepsilon$ for all $n \geq N$ for every value of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Hence the geometric series converges uniformly in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ because by definition the series $\sum u_n(x)$ is said to be uniformly convergent in the interval (a, b) , if for a given $\varepsilon > 0$, a number N can be found independent of x , such that for every x in the interval (a, b) ,

$$|S(x) - S_n(x)| < \varepsilon \quad \text{for all } n > N.$$

Note: The geometric series though convergent in the interval $(-1, 1)$, is not uniformly convergent in this interval, since we cannot find a fixed number N for every x in this interval

[because $N (= \log [\varepsilon(1-x)]/\log |x|) \rightarrow \infty$ as $|x| \rightarrow 1$].

□ □ □

Matrices and Its Applications

IMPORTANT DEFINITIONS AND FORMULAE

1. **Matrix:** A matrix is defined as a rectangular array (or arrangement in rows or columns) of scalars, the numbers being enclosed by brackets [] or () subject to certain rules of operations. If mn numbers (real or complex) are arranged in the form of a rectangular array A, having m rows (horizontal lines) and n columns (vertical lines), then A is called an $m \times n$ matrix. Each of the mn numbers is called an element of the matrix. An $m \times n$ matrix is also called a matrix of order $m \times n$ and is written as $A = [a_{ij}]_{m \times n}$ or $A = \{a_{ij}\}$, where $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$.

Each element has 2 suffixes indicating the row and the column in which the element lies.

2. **Real Matrix:** A matrix is said to be real if all the elements are real numbers.
3. **Square Matrix:** A matrix in which the number of rows is equal to the number of columns, is called a square matrix otherwise it is said to be a rectangular matrix. The elements of a square matrix are called its **diagonal elements** and the diagonal along which these elements lie, is called the principal diagonal. The sum of the diagonal elements of a square matrix, is called its **trace**. Thus, trace of a square matrix A of order n is given by:

$$a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}, \quad \text{where } A = [a_{ij}]$$

4. **Diagonal Matrix:** A square matrix in which all non-diagonal elements are zero, is called a diagonal matrix.
5. **Identity Matrix:** A scalar matrix in which each diagonal element is unity, is called a unit matrix or identity matrix.

i.e., $A = [a_{ij}]_{n \times n}$ is a unit matrix

$$\text{if } a_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$$

A unit matrix of order n is denoted by I_n or simply I.

6. **Product of Two Matrices:** Two matrices A and B are said to be conformable for the product AB if the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor).

Thus, if the orders of A and B are $m \times n$ and $p \times q$ respectively, then

(i) AB is defined if $n = p$

(ii) BA is defined if $q = m$.

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$ be two matrices conformable for the product AB, then AB is defined as the matrix $C = [c_{ij}]_{m \times p}$,

where

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} \\ &= \sum_{k=1}^n a_{ik}b_{kj}. \end{aligned}$$

i.e., $(i, j)^{\text{th}}$ element of AB = sum of the products of the elements of i^{th} row of A with the corresponding elements of j^{th} column of B.

7. Properties of Matrix Multiplication:

- (i) Matrix multiplication is not commutative in general *i.e.*, $AB \neq BA$.
- (ii) Matrix multiplication is associative *i.e.*, $(AB)C = A(BC)$
- (iii) It is distributive with respect to matrix addition *i.e.*, $A(B + C) = AB + AC$.
- (iv) If A and I are square matrices of the same order, then $AI = IA = A$.

8. Transpose of a Matrix: Given a matrix A , then the matrix obtained from A by changing its rows into columns and columns into rows is called the **transpose** of A and is denoted by A' or A^T .

- (i) $(A')' = A$
- (ii) $(AB)' = B'A'$
- (iii) $(A + B)' = A' + B'$

9. Symmetric Matrix: A square matrix $A = \{a_{ij}\}$ is said to be symmetric if $A' = A$ *i.e.*, if the transpose of the matrix is equal to the matrix itself.

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

are symmetric matrices.

10. Orthogonal Matrix: A square matrix A is called an orthogonal matrix if $AA' = A'A = I$.

11. Adjoint of a Square Matrix: It is the transpose of the matrix obtained by replacing each element of A by its co-factor in $|A|$, briefly written as $\text{adj. } A$.

12. Inverse of a Square Matrix: If $AB = BA = I$, then B is called the inverse (or reciprocal) of A , or A^{-1} .

The **inverse of a square matrix** if it exists, is **unique**. The necessary and sufficient condition for a square matrix A to have its inverse, is that **A is non-singular** *i.e.*, $|A| \neq 0$. Inverse of A is denoted by A^{-1} .

Thus, $B = A^{-1}$ and $AA^{-1} = A^{-1}A = I$.

Also, $A^{-1} = \frac{\text{adj. } A}{|A|}$; $|A| \neq 0$.

Note: (i) If A and B are two non-singular square matrices of the same order then $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

(ii) If A is a non-singular square matrix then so is A' and $(A')^{-1} = (A^{-1})'$.

13. Gauss-Jordan Method: This is a method to obtain inverse of a matrix by using elementary row transformations to reduce it to the unit matrix.

14. Rank of a Matrix: Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A . If all minors of order $(r + 1)$ are zero, but there is at least one non-zero minor of order r , then r is called the rank of A , written as $\rho(A) = r$.

- (i) If A is a null matrix, then $\rho(A) = 0$
- (ii) If A is not a null matrix, then $\rho(A) \geq 1$.
- (iii) If A is a non-singular $n \times n$ matrix, then $\rho(A) = n$.
- (iv) If A is an $m \times n$ matrix, then $\rho(A) \leq \text{minimum of } m, n$.

To Determine Rank of a Given Matrix:

Method I: Start with the highest order minor (or minors) of A . Let their order be r . If any one of them is non-zero, then $\rho(A) = r$. If all of them are zero, start with minors of next lower order $(r - 1)$ and so on till you get a non-zero minor. The order of that minor is the rank of A .

However, this method usually involves a lot of computational work since we have to evaluate several determinants.

Method II: We generally find the rank of a matrix **by reducing it to echelon form** in which:

- (i) All entries below the leading element are zero.
- (ii) The number of zeros before the first non-zero element in a row is more than the number of such zeros in the previous row.

For example, $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is in **echelon** form.

whereas $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix}$ is **non-echelon**.

Method III: Normal Form Method:

If A is an $m \times n$ matrix and by a series of elementary (row or column or both) operations, it can be put into one of the following forms (called normal or canonical forms):

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right], \left[\begin{array}{c|c} I_r & \\ \hline 0 & 0 \end{array} \right], [I_r \mid 0], [I_r]$$

where I_r is the unit matrix of order r .

Since the rank of a matrix is not changed as a result of elementary transformations, it follows that $\rho(A) = r$ because r^{th} order minor $|I_r| = 1 \neq 0$.

This method is also called sweep out method or **pivotal method**.

- 15. For an $m \times n$ matrix A , method to find square matrices P and Q of orders m and n respectively such that PAQ is in the normal form:**

Write $A = IAI$

Reduce the matrix on L.H.S. to normal form by affecting elementary row and/or column transformations. Every elementary row (column), transformation on A must be accompanied by the same transformation on the pre-factor (post-factor) on R.H.S. Please see Q's 28 and 29 as examples.

- 16. Solution of a System of Linear Equations**

- (i) Consider 3 equations in 3 unknowns in matrix notation as $AX = B$

where A is called the co-efficient matrix,

X is called the column matrix of unknowns and

B is the column matrix of constants.

If $B = O$, the matrix equation $AX = B$ reduces to $AX = O$.

Such a system of equations is called a system of **homogeneous linear equations**.

If $B \neq O$, it is a system of **non-homogeneous linear equations**.

- (ii) For a homogeneous linear equations system $AX = O$, $X = O$ is always a solution called Trivial solution. Therefore, such a system is always **consistent**.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solutions. If $\rho(A) < \text{number of unknowns}$, system has an infinite number of non-trivial solutions. If $\rho(A) = \text{no. of unknowns}$, the system has only the trivial solution.

- (iii) For $AX = B$, the system is

(a) inconsistent if $\rho[A : B] \neq \rho(A)$

(b) has a unique solution if $\rho[A : B] = \rho(A) = \text{number of unknowns}$.

(c) has an infinite number of solutions if $\rho[A : B] = \rho(A) < \text{no. of unknowns}$.

The matrix $[A : B]$ in which the elements of A and B are written side by side, is called the **augmented matrix**.

- (iv) If A is a non-singular matrix, then the matrix equation $AX = B$ has a unique solution.

- 17. Characteristic Equation and Cayley-Hamilton Theorem:**

- (i) If A is a square matrix of order n , we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n .

$|A - \lambda I| = 0$ is called the characteristic equation of A .

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form

$$(-1)^n \lambda^n + k_1 \cdot \lambda^{n-1} + k_2 \cdot \lambda^{n-2} + \dots + k_n = 0$$

The roots of this equation are called the **characteristic roots** or latent roots or **eigen values of A**.

Note 1. Sum of the eigen values of a matrix A is equal to trace of A which in turn is the **sum of its diagonal elements**.

2. Corresponding to each eigen value, the homogeneous system $(A - \lambda I)X = 0$ has a non-zero

solution $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which is called an eigen vector.

3. If X is a solution of $(A - \lambda I)X = 0$ then so is kX where k is an arbitrary constant. Therefore, the **eigen vector corresponding to an eigen value is not unique**.

(ii) **Cayley-Hamilton Theorem:** Every square matrix satisfies its own characteristic equation. This theorem gives **another method for computing the inverse** of a square matrix.

Note: Since this method expresses the inverse of a matrix of order n in terms of $(n - 1)$ powers of A , it is **most suitable for computing inverses of large matrices**.

SOLVED PROBLEMS

1. Find the inverse of the matrix A , $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by Gauss-Jordan Method. (M.D.U. Dec., 2007)

Sol. Writing the given matrix side by side with unit matrix I_3 , we get

$$\begin{aligned} [A : I_3] &= \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \text{Operating } R_{12} \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \text{Operating } R_3 - 3R_1 \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array} \right] \text{Operating } R_1 - 2R_2, R_3 + 5R_2 \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array} \right] \text{Operating } \frac{1}{2} R_3 \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array} \right] \text{Operating } R_1 + R_3, R_2 - 2R_3 \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{4}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array} \right] \equiv [I_3 : A^{-1}] \end{aligned}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{4}{5} & \frac{3}{2} & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

2. Find x, y, z and w if $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}.$

Sol. The given equation is

$$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

Equating the corresponding elements on both sides,

$$\begin{aligned} 3x &= x+4, & 3y &= 6+x+y, & 3z &= -1+z+w, & 3w &= 2w+3 \\ \Rightarrow & x=2, & 2y &= 6+x, & 2z &= -1+w, & w &= 3 \\ \Rightarrow & x=2, & y &= 4, & z &= 1, & w &= 3 \end{aligned}$$

Hence $x = 2, y = 4, z = 1$ and $w = 3$.

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$. Find the products AB and BA . Show that $AB \neq BA$.

Sol.

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

Orders of AB and BA are the same (3×3) but their corresponding elements are not equal.

Hence $AB \neq BA$.

4. Prove that the product of the matrices

$$\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$$

is a null matrix, where θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Sol.

$$\begin{aligned} & \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cos \phi \sin \phi & \cos^2 \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin^2 \phi \\ \cos \theta \sin \theta \cos^2 \phi + \sin^2 \theta \cos \phi \sin \phi & \cos \theta \sin \theta \cos \phi \sin \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) & \cos \theta \sin \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) \\ \sin \theta \cos \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) & \sin \theta \sin \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \phi \cos (\theta - \phi) & \cos \theta \sin \phi \cos (\theta - \phi) \\ \sin \theta \cos \phi \cos (\theta - \phi) & \sin \theta \sin \phi \cos (\theta - \phi) \end{bmatrix} \end{aligned}$$

$\therefore \theta - \phi = \text{an odd multiple of } \frac{\pi}{2}$

$\therefore \cos (\theta - \phi) = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2 \times 2}.$

5. Evaluate $A^2 - 3A + 9I$ if I is the unit matrix of order 3 and $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$.

Sol.

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$\begin{aligned} \therefore A^2 - 3A + 9I_3 &= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - 3 \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -12-3+9 & -5-(-6)+0 & 11-9+0 \\ 11-6+0 & 4-9+9 & 1-(-3)+0 \\ -7-(-9)+0 & 11-3+0 & -6-6+9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}. \end{aligned}$$

6. By Mathematical induction, prove that if $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

Sol.

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

When $n = 1$, $A^1 = \begin{bmatrix} 1+2 \cdot 1 & -4 \cdot 1 \\ 1 & 1-2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$

$\Rightarrow$ The result is true when $n = 1$.

Let us assume that the result is true for any positive integer k i.e.,

Let $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$... (1)

Now $A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 3(1+2k) - 4k & -4(1+2k) + 4k \\ 3k + 1 - 2k & -4k - (1-2k) \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

$\Rightarrow$ The result is true for $n = k + 1$. Hence by mathematical induction, the result is true for all positive integers n .

7. Prove that $A^3 - 4A^2 - 3A + 11I = 0$, where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Sol.

$$A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$$\begin{aligned}
\therefore A^3 - 4A^2 - 3A + 11I &= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4 \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 28 - 36 - 3 + 11 & 37 - 28 - 9 + 0 & 26 - 20 - 6 + 0 \\ 10 - 4 - 6 + 0 & 5 - 16 - 0 + 11 & 1 - 4 + 3 + 0 \\ 35 - 32 - 3 + 0 & 42 - 36 - 6 + 0 & 34 - 36 - 9 + 11 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.
\end{aligned}$$

8. By Mathematical Induction, prove that if $A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$.

Sol. $A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$

When $n = 1$, $A^1 = \begin{bmatrix} 1+10 \cdot 1 & -25 \cdot 1 \\ 4 \cdot 1 & 1-10 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = A$

$\Rightarrow$ The result is true when $n = 1$

Let us assume that the result is true for any positive integer k i.e.,

Let $A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$

Now, $A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$

$$\begin{aligned}
&= \begin{bmatrix} 11(1+10k) - 100k & -25(1+10k) + 9(25k) \\ 44k + 4(1-10k) & -100k - 9(1-10k) \end{bmatrix} \\
&= \begin{bmatrix} 11+10k & -25-25k \\ 4k+4 & -10k-9 \end{bmatrix} = \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix} \\
A^{k+1} &= \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}
\end{aligned}$$

$\Rightarrow$ The result is true for $n = k+1$. Hence by mathematical induction, the result is true for all positive integers n .

9. If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer.

(M.D.U., 2006)

Sol. $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

When $n = 1$, $A^1 = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = A$

$\Rightarrow$ The result is true when $n = 1$.

Let us assume that the result is true for any positive integer k i.e.,

Let $A^k = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}$

Now, $A^{k+1} = A^k \cdot A$

$$= \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} \cos k\alpha \cdot \cos \alpha - \sin k\alpha \sin \alpha & \cos k\alpha \sin \alpha + \sin k\alpha \cdot \cos \alpha \\ -\sin k\alpha \cos \alpha - \cos k\alpha \sin \alpha & -\sin k\alpha \cdot \sin \alpha + \cos k\alpha \cdot \cos \alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos (k\alpha + \alpha) & \sin (k\alpha + \alpha) \\ -\sin (k\alpha + \alpha) & \cos (k\alpha + \alpha) \end{bmatrix} = \begin{bmatrix} \cos (k+1)\alpha & \sin (k+1)\alpha \\ -\sin (k+1)\alpha & \cos (k+1)\alpha \end{bmatrix}
\end{aligned}$$

$\Rightarrow$ The result is true for $n = k + 1$.

Hence by mathematical induction, the result is true for all positive integers n .

Hence $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ is true.

10. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

Sol. Equating the corresponding elements on both sides, we get

$$\begin{aligned}
&x + 3 = 0, \quad 2y + x = -7, \quad z - 1 = 3, \quad 4a - 6 = 2a \\
\Rightarrow &x = -3, \quad y = -2, \quad z = 4, \quad a = 3
\end{aligned}$$

11. For what values of x , the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular? (M.D.U., May, 2005, 2007)

Sol. If the matrix is singular, the determinantal value is zero.

$$\begin{aligned}
i.e., \quad \Rightarrow \quad &\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0 \\
&\text{L.H.S.} = (3-x)[(4-x)(-1-x)+4] - 2[2(-1-x)+2] + 2[-8+2(4-x)] \\
&= (3-x)(x^2-3x)+4x-4x = x(3-x)(x-3) = -x(x-3)^2 \\
\Rightarrow &x = 0, 3.
\end{aligned}$$

12. Matrix A has x rows and $x+5$ columns. Matrix B has y rows and $11-y$ columns. Both AB and BA exist. Find x and y .

Sol. As AB exists, $x+5 = y$... (1)

If BA exists, number of columns in B should be equal to number of rows in A .

$$i.e., \quad 11-y = x \quad \dots (2)$$

Solving (1) and (2) for x and y ;

$$11 - (x+5) = x \quad \text{or} \quad 6 - x = x$$

$$2x = 6 \Rightarrow x = 3$$

$$\therefore y = x + 5 = 8$$

Hence $x = 3, y = 8$.

13. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ calculate the product AB .

Sol. On adding the matrices $A - B$ from $A + B$, we get

$$2A = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \quad i.e., \quad A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

On subtracting matrix $A - B$ from $A + B$, we get

$$2B = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \quad i.e., \quad B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\text{Hence} \quad AB = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix} = -2 \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

14. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ find AB or BA whichever exists.

Sol. It is observed that A is 3×4 matrix and B is 3×3 matrix. Hence AB does not exist.
 $\Rightarrow$ BA exists.

$$\therefore BA = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \quad \text{or} \quad BA = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}.$$

15. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ verify that $AB(C) = A(BC)$ and $A(B + C) = AB + AC$.

$$\text{Sol.} \quad A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} \quad \dots(1)$$

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \quad \dots(2)$$

$$AB(C) = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} = A(BC) \quad \dots(3)$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad \dots(4)$$

$$\begin{aligned} \text{Now} \quad A(B + C) &= \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2-3 & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} = AB + AC. \quad \text{Hence Proved.} \quad \dots(5) \end{aligned}$$

16. If $A = \begin{bmatrix} 0 & -\tan \alpha/2 \\ \tan \alpha/2 & 0 \end{bmatrix}$, show that $I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$. (M.D.U., Dec., 2008)

$$\text{Sol.} \quad I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \alpha/2 \\ \tan \alpha/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$(I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha + \sin \alpha \cdot \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \cdot \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} \cdot \cos \alpha + \sin \alpha & \sin \alpha \cdot \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix}$$

Since we know that

$$\sin \alpha = \frac{2 \tan \alpha/2}{1 + \tan^2 \alpha/2} \quad \text{and} \quad \cos \alpha = \frac{1 - \tan^2 \alpha/2}{1 + \tan^2 \alpha/2}$$

$$\therefore \cos \alpha + \sin \alpha \cdot \tan \alpha/2 = \frac{1 - \tan^2 \alpha/2}{1 + \tan^2 \alpha/2} + \frac{2 \tan \alpha/2}{1 + \tan^2 \alpha/2} \cdot \tan \alpha/2 = \frac{1 + \tan^2 \alpha/2}{1 + \tan^2 \alpha/2} = 1.$$

$$\begin{aligned} \text{and } -\sin \alpha + \cos \alpha \cdot \tan \alpha/2 &= \frac{-2 \tan \alpha/2}{1 + \tan^2 \alpha/2} + \frac{\tan \alpha/2 (1 - \tan^2 \alpha/2)}{1 + \tan^2 \alpha/2} \\ &= \frac{-2 \tan \alpha/2 + \tan \alpha/2 - \tan^3 \alpha/2}{1 + \tan^2 \alpha/2} \\ &= \frac{-\tan \frac{\alpha}{2} \left[1 + \tan^2 \frac{\alpha}{2} \right]}{\left[1 + \tan^2 \frac{\alpha}{2} \right]} = -\tan \alpha/2. \end{aligned}$$

Hence,

$$(I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & -\tan \alpha/2 \\ \tan \alpha/2 & 1 \end{bmatrix} = I + A. \text{ Hence shown.}$$

17. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$, where I is a matrix of order 2.

$$\begin{aligned} \text{Sol. } A^2 &= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} \\ \therefore A^2 - 5A + 7I &= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \text{ Hence shown.} \end{aligned}$$

18. Express $\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix}$ as the sum of a lower triangular matrix and an upper triangular matrix with zero leading diagonal.

$$\begin{aligned} \text{Sol. Let } L &= \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \text{ be the lower triangular matrix and} \\ U &= \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

be the upper triangular matrix with zero leading diagonal such that

$$\begin{aligned} \begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix} &= \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} + \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix} &= \begin{bmatrix} a & p & q \\ b & c & r \\ d & e & f \end{bmatrix} \end{aligned}$$

Equating the corresponding elements on the two sides, we get

$$\begin{aligned} 2 &= a, & 5 &= p, & -7 &= q, & -9 &= b, & 12 &= c, & 4 &= r, \\ 15 &= d, & -13 &= e, & 6 &= f \end{aligned}$$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ -9 & 12 & 0 \\ 15 & -13 & 6 \end{bmatrix} \text{ and } U = \begin{bmatrix} 0 & 5 & -7 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

19. Factorise the matrix A , $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU , where L is lower triangular with each diagonal element one and U is upper triangular matrix.

Sol. Let
$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, \quad U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix}$$

$$\text{R.H.S.} = \begin{bmatrix} d & e & f \\ ad & ae + g & af + h \\ bd & be + cg & bf + ch + p \end{bmatrix}$$

Equating the corresponding elements on the two sides, we get

$$\begin{aligned} 5 &= d, & -2 &= e, & 1 &= f, & 7 &= ad, & 1 &= ae + g, \\ -5 &= af + h, & 3 &= bd, & 7 &= be + cg, & 4 &= bf + ch + p \end{aligned}$$

$$\Rightarrow \quad d = 5, \quad a = 7/5, \quad 1 = \frac{7}{5}(-2) + g \Rightarrow g = \frac{19}{5}$$

$$e = -2, \quad f = 1,$$

$$\frac{7}{5}(1) + h = -5 \quad \text{or} \quad h = -5 - \frac{7}{5} \quad \text{or} \quad h = -\frac{32}{5}$$

$$b = \frac{3}{d} \quad \text{or} \quad \frac{3}{5}$$

$$7 = \frac{3}{5}(-2) + c\left(\frac{19}{5}\right) \Rightarrow \frac{41}{5} = \frac{19c}{5} \Rightarrow c = \frac{41}{19}$$

$$4 = \frac{3}{5}(1) + \frac{41}{19}\left(-\frac{32}{5}\right) + p$$

$$\frac{17}{5} = -\frac{41 \times 32}{19 \times 5} + p$$

or

$$p = \frac{41 \times 32}{19 \times 5} + \frac{17}{5} = \frac{41 \times 32 + 19 \times 17}{19 \times 5} = \frac{1635}{95} = \frac{327}{19}$$

Hence

$$d = 5, \quad e = -2, \quad f = 1$$

$$a = \frac{7}{5}, \quad g = \frac{19}{5}, \quad h = -\frac{32}{5}$$

$$b = \frac{3}{5}, \quad c = \frac{41}{19}, \quad p = \frac{327}{19}$$

$$\Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$$

20. Find the rank of the matrix

$$(i) \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \quad (ii) \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}. \quad (\text{M.D.U., May 2008, 2009})$$

Sol. (i) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is a 4×4 matrix.

$$\therefore \rho(A) \leq 4.$$

Operating $R_4 - (R_3 + R_2 + R_1)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Operating } R_2 - 2R_1, R_3 - 3R_1,$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Operating } R_{23}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The only 4th order minor is zero.

The third order minor

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & -3 \end{vmatrix} = 1(-4)(-3) = 12 \neq 0, \quad \therefore \rho(A) = 3.$$

Hence rank of the matrix A is 3.

(ii) $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ is a 3×4 matrix.

$$\therefore \rho(A) \leq 3.$$

Operating C_{14}

$$A \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -4 \end{bmatrix} \quad \text{Operating } R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix} \quad \text{Operating } R_1 + 3R_2, R_3 + 2R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the third order minors are zero.

Second order minor

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \neq 0, \quad \therefore \rho(A) = 2.$$

21. Reduce the following matrices to normal form and hence find their ranks :

$$(i) \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \text{ (M.D.U., Dec., 2009)} \quad (ii) \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}.$$

(U.P.T.U., 2007, M.D.U., May 2008, 2009)

Sol. (i) Let $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ Operating $\frac{1}{8}C_1$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix} \text{ Operating } R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{ Operating } C_2 - C_1, C_3 - 3C_1, C_4 - 6C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{ Operating } \frac{1}{3}C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{ Operating } C_3 - 2C_2, C_4 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \text{ Operating } \frac{1}{10}C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ Operating } C_{34}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} = [I_3 : O]$$

which is the required normal form

$$\rho(A) = 3.$$

(ii) Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ Operating R_{12}

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operating $C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1,$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix}$$

Operating $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1,$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operating $R_2 - R_3, R_4 - 2R_3,$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

Operating $C_3 + 6C_2, C_4 + 3C_2,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Operating $R_3 - 4R_2, R_4 - R_2,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $\frac{1}{33} C_3, \frac{1}{22} C_4,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_4 - C_3,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} I_3 & \vdots & O \\ O & \vdots & O \end{bmatrix}$$

which is the required normal form.

$$\therefore \rho(A) = 3.$$

22. Determine the rank of: $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}.$

Sol. Operating $R_2 - R_1$, $R_3 - 2R_1$, $R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix} \text{ Operating } R_4 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix} \text{ Operate } R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore$ **Rank (A) = 4.**

23. Find the rank of the following matrices:

$$(i) \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix} \qquad (ii) \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Sol. (i) Since it is a 3×4 matrix, the rank is ≤ 3 .

Operating $R_2 - R_1$, $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & 5 \end{bmatrix} \text{ Operate } R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$\text{Now } \begin{vmatrix} 1 & 3 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 9 \end{vmatrix} = -9 \neq 0; \quad \therefore \text{ **Rank} = 3.**$$

$$(ii) \quad A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Operate $R_4 - R_1 - R_3$, $R_3 - R_2 - R_1$

$$\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore \text{ **Rank} \leq 2**$$

$$\text{Since } \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0, \quad \text{ **Rank} = 2.**$$

24. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$,

(i) Find A^{-1}

(ii) Show that $A^3 = A^{-1}$.

(M.D.U., 2006, 2007, 2009, and U.P.T.U., 2005, 2008)

Sol. (i)

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} \text{ Operating } C_2 + C_3$$

$$= \begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} \text{ Expanding by 3rd row}$$

$$= 1.$$

$$\therefore |A| = 1 \neq 0, \therefore A^{-1} \text{ exists.}$$

$$A_1 = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1, A_2 = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1,$$

$$A_3 = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0, B_1 = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$$

$$B_2 = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3, B_3 = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$C_1 = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2, C_2 = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3, C_3 = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$$

$$\therefore \text{adj } A = \text{transpose of } \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \quad (\because |A| = 1)$$

$$(ii) \quad A^2 = A \times A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = I$$

$$\text{Now, } A^4 = I \Rightarrow A \cdot A^3 = I = A \cdot A^3$$

$$\Rightarrow A^3 \text{ is inverse of } A \text{ i.e., } A^3 = A^{-1}. \text{ Hence shown.}$$

25. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$. (M.D.U., May, 2009)

Sol.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, A' = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

Hence $(AB)' = B'A'$ is verified.

26. Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$. (U.P.T.U., 2009 and M.D.U., May, 2008)

Sol. The determinant of the given matrix is

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (say)}$$

If $A_1, A_2, \dots$ be the co-factors of $a_1, a_2, \dots$ in Δ , then $A_1 = -24, A_2 = -8, A_3 = -12, B_1 = 10, B_2 = 2, B_3 = 6, C_1 = 2, C_2 = 2, C_3 = 2$.

Thus,

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8$$

and

$$\text{adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence the inverse of the given matrix A

$$= \frac{\text{adj } A}{\Delta} = -\frac{1}{8} \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

27. Determine the rank of the following matrices :

$$(i) \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol. (i) Operate $R_2 - R_1$ and $R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$.

$\therefore \rho(A) = 2$. Hence the rank of the given matrix is 2.

(ii) Given matrix $A \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$ Operating $C_3 - C_1, C_4 - C_1$

Now operating $R_3 - R_1$ and $R_4 - R_1$, we get

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_3 - 3R_2, R_4 - R_2$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_3 + 3C_2, C_4 + C_2$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ (say)}$$

4th order minor of A is zero. Also every 3rd order minor of A is zero. But of all the 2nd order minors, only

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0. \quad \therefore \rho(A) = 2.$$

Hence rank of the given matrix is 2.

28. For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

(M.D.U., 2005, 2006, 2008)

Sol. We write $A = IAI$

i.e., $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (post-factor) of A to the same.

Operate $C_2 - C_1, C_3 - 2C_1$ (subjecting post-factor on R.H.S. to same operations)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - R_1$ (subjecting the pre-factor on R.H.S. to same operations)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_3 - C_2$ (subjecting the post-factor on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_3 + R_2$ (subjecting the pre-factor on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is of the normal form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

Hence
$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho(A) = 2.$$

- 29.** For the matrix $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of matrix A .

(M.D.U., May 2008, Dec., 2009 and U.P.T.U., 2007)

Sol. We write $A_{3 \times 4} = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 2R_1$,

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1, C_4 \rightarrow C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \rightarrow R_3 - 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & 0 & 28 & 56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \rightarrow -R_2$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 28 & 56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_4 \rightarrow C_4 - 2C_3$, $C_3 \rightarrow C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 28 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_4 \rightarrow \frac{1}{28}R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -3 & 1 & 9 \\ \frac{1}{28} & \frac{1}{28} & \frac{1}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow [I_3 : 0] = PAQ$$

where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -3 & 1 & 9 \\ \frac{1}{28} & \frac{1}{28} & \frac{1}{28} \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and Rank of matrix A is 3.

30. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{bmatrix}$, find $\text{adj } A$.

Sol. Here

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ say}$$

$\therefore$

$$A_1 = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -2, B_1 = -\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} = -2$$

$$C_2 = -\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = -9, C_1 = \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = 1, C_3 = 5$$

$$A_2 = -\begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} = -12, B_2 = \begin{vmatrix} 1 & 0 \\ -2 & 6 \end{vmatrix} = 6, A_3 = 8, B_3 = -4$$

Thus the matrix of co-factors is $\begin{bmatrix} -2 & -2 & 1 \\ -12 & 6 & -9 \\ 8 & -4 & 5 \end{bmatrix}$

Hence $\text{adj } A = \begin{bmatrix} -2 & -12 & 8 \\ -2 & 6 & -4 \\ 1 & -9 & 5 \end{bmatrix}$.

31. Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by elementary row operations.

Sol. Writing the given matrix side by side with unit matrix of order 3, we get

$$\begin{bmatrix} 2 & 3 & 4 & : & 1 & 0 & 0 \\ 4 & 3 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

Operate R_{13} ,

$$\sim \begin{bmatrix} 1 & 2 & 4 & : & 0 & 0 & 1 \\ 4 & 3 & 1 & : & 0 & 1 & 0 \\ 2 & 3 & 4 & : & 1 & 0 & 0 \end{bmatrix}$$

Operating $R_2 - 4R_1$, $R_3 - 2R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 4 & : & 0 & 0 & 1 \\ 0 & -5 & -15 & : & 0 & 1 & -4 \\ 0 & -1 & -4 & : & 1 & 0 & -2 \end{bmatrix}$$

Operating $\frac{-R_2}{5}$, $-R_3$, we get

$$\sim \begin{bmatrix} 1 & 2 & 4 & : & 0 & 0 & 1 \\ 0 & 1 & 3 & : & 0 & -\frac{1}{5} & \frac{4}{5} \\ 0 & 1 & 4 & : & -1 & 0 & 2 \end{bmatrix}$$

Operating $R_1 - 2R_2$, $R_3 - R_2$,

$$\sim \begin{bmatrix} 1 & 0 & -2 & : & 0 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 3 & : & 0 & -\frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & : & -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$$

Operating $R_1 + 2R_3$, $R_2 - 3R_3$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -2 & \frac{4}{5} & \frac{9}{5} \\ 0 & 1 & 0 & : & 3 & -\frac{4}{5} & -\frac{14}{5} \\ 0 & 0 & 1 & : & -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$$

Thus the inverse of the given matrix is

$$\begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} \text{ i.e., } \frac{1}{5} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}.$$

32. Solve the system of equations

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$

$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$

$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$

$$2x_1 + 2x_2 - x_3 + x_4 = 10.$$

(M.D.U., May 2005, 2006, 2008, 2009)

Sol. In matrix notation, the given system of equations can be written as $AX = B$ where

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 6 & -6 & 6 & 12 & : & 36 \\ 4 & 3 & 3 & -3 & : & -1 \\ 2 & 2 & -1 & 1 & : & 10 \end{bmatrix}$$

Operating $R_2 - 3R_1, R_3 - 2R_1, R_4 - R_1,$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 0 & -9 & 0 & 9 & : & 18 \\ 0 & 1 & -1 & -5 & : & -13 \\ 0 & 1 & -3 & 0 & : & 4 \end{bmatrix}$$

Operating $-\frac{1}{9} R_2,$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 1 & -1 & -5 & : & -13 \\ 0 & 1 & -3 & 0 & : & 4 \end{bmatrix}$$

Operating $R_1 - R_2, R_3 - R_2, R_4 - R_2,$

$$\sim \begin{bmatrix} 2 & 0 & 2 & 2 & : & 8 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & -3 & 1 & : & 6 \end{bmatrix}$$

Operating $R_4 - 3R_3, \frac{1}{2} R_1,$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & : & 4 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & 0 & +13 & : & 39 \end{bmatrix}$$

Operating $R_1 + R_3, \frac{1}{13} R_4,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & : & 7 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Operating $R_1 + 3R_4, R_2 + R_4, R_3 + 4R_4,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Operating $(-1) R_3,$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Hence $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3.$

33. Using matrix method, show that the equations

$$3x + 3y + 2z = 1$$

$$x + 2y + 0z = 4$$

$$0x + 10y + 3z = -2$$

$$2x - 3y - z = 5$$

are consistent and hence obtain the solutions for x, y and $z.$

Sol. In matrix notation, the given system of equations can be written as $AX = B,$ where

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ Operating } R_{12}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ Operating } R_2 - 3R_1, R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \text{ Operating } R_3 + 3R_2, R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 1 & 9 & : & -35 \\ 0 & -1 & -5 & : & 19 \end{bmatrix} \text{ Operating } R_1 - 2R_3, R_2 + 3R_3, R_4 + R_3$$

$$\begin{aligned}
& \sim \begin{bmatrix} 1 & 0 & -18 & 74 \\ 0 & 0 & 29 & -116 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 4 & -16 \end{bmatrix} \text{ Operating } R_{23}, \frac{1}{4} R_4 \\
& \sim \begin{bmatrix} 1 & 0 & -18 & 74 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 1 & -4 \end{bmatrix} \text{ Operating } R_1 + 18R_4, R_2 - 9R_4, R_3 - 29R_4 \\
& \sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & : & -4 \end{bmatrix} \text{ Operating } R_{34} \\
& \sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}
\end{aligned}$$

$$\rho(A) = \rho(A : B) = 3 = \text{number of unknowns.}$$

$\Rightarrow$ The given system of equations is consistent and the unique solution is $x = 2, y = 1, z = -4$.

34. For what values of λ and μ do the system of equations

$$\begin{aligned}
x + y + z &= 6 \\
x + 2y + 3z &= 10 \\
x + 2y + \lambda z &= \mu
\end{aligned}$$

have (i) no solution

(ii) unique solution

(iii) more than one solution ?

(U.P.T.U., 2007, M.D.U., May 2008, 2009, Dec., 2009)

Sol. In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix

$$\begin{aligned}
[A : B] &= \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix} \text{ Operating } R_2 - R_1, R_3 - R_1, \\
&\sim \begin{bmatrix} 0 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix} \text{ Operating } R_1 - R_2, R_3 - R_2 \\
&\sim \begin{bmatrix} 0 & 0 & -1 & : & 2 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}
\end{aligned}$$

Case I. If

$$\lambda = 3, \mu \neq 10$$

$$\rho(A) = 2, \rho(A : B) = 3$$

$\therefore$

$$\rho(A) \neq \rho(A : B)$$

$\therefore$ The system has no solution.

Case II. If $\lambda \neq 3$, μ may have any value.

$$\rho(A) = \rho(A : B) = 3 = \text{a number of unknowns.}$$

$\therefore$ The system has unique solution.

Case III. If $\lambda = 3$, $\mu = 10$

$$\rho(A) = \rho(A : B) = 2 < \text{number of unknowns.}$$

$\therefore$ The system has an infinite number of solutions.

35. Solve the equations :

$$x_1 + 3x_2 + 2x_3 = 0$$

$$2x_1 - x_2 + 3x_3 = 0$$

$$3x_1 - 5x_2 + 4x_3 = 0$$

$$x_1 + 17x_2 + 4x_3 = 0.$$

Sol. In matrix notation, the given system of equations can be written as

$$AX = 0$$

where
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Operating $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - R_1$,

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \text{ Operating } R_3 - 2R_2, R_4 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Operating } R_1 + 2R_2$$

$$\sim \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 < \text{number of unknowns.}$

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$$x_1 - 11x_2 = 0, \quad -7x_2 - x_3 = 0$$

$$\text{i.e.,} \quad x_1 = 11k, \quad x_2 = k, \quad x_3 = -7k,$$

where k is arbitrary. Different values of k will give different solutions.

36. Show that the system of equations :

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$

$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$

$$-x_1 + 2x_2 = \lambda x_3$$

can possess a non-trivial solution only if $\lambda = 1$ or -3 . Obtain the general solution in each case.

Sol. The given system of equations is

$$(2 - \lambda)x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - (3 + \lambda)x_2 + 2x_3 = 0$$

$$-x_1 + 2x_2 - \lambda x_3 = 0$$

In matrix notation, it can be written as

$$AX = O$$

where
$$A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For non-trivial solution, $|A| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[\lambda(3+\lambda)-4] + 2(-2\lambda+2) + [4-(3+\lambda)] = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$$

$$(\lambda-1)^2(\lambda+3) = 0$$

$$\therefore \lambda = 1 \quad \text{or} \quad -3.$$

When $\lambda = 1$, the equations become

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

which are identical.

The given system is equivalent to a single equation

$$x_1 - 2x_2 + x_3 = 0$$

Taking $x_2 = t$, $x_3 = s$, we get $x_1 = 2t - s$

$$\therefore x_1 = 2t - s, x_2 = t, x_3 = s$$

which give an infinite number of non-trivial solutions, t and s being the parameters.

When $\lambda = -3$, the equations become

$$5x_1 - 2x_2 + x_3 = 0$$

$$2x_1 + 2x_3 = 0$$

$$-x_1 + 2x_2 + 3x_3 = 0$$

Solving the first two, we have

$$\frac{x_1}{-4} = \frac{x_2}{2-10} = \frac{x_3}{4} \quad \text{or} \quad x_1 = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\therefore x_1 = t, x_2 = 2t, x_3 = -t$$

which give an infinite number of non-trivial solutions, t being the parameter.

37. Solve with the help of matrices, the simultaneous equations :

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$x + 4y + 9z = 6.$$

Sol. In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 1 & 4 & 9 & : & 6 \end{bmatrix} \quad \text{Operating } R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 8 & : & 3 \end{bmatrix} \text{ Operating } R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 2 & : & 0 \end{bmatrix} \text{ Operating } \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} \text{ Operating } R_1 - R_3, R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & 3 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} \text{ Operating } R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}$$

$$\therefore x = 2, y = 1, z = 0.$$

38. Solve the system of equations

$$x + 2y + 3z = 1$$

$$2x + 3y + 2z = 2$$

$$3x + 3y + 4z = 1 \text{ with the help of matrix inversion.}$$

Sol. The given system can be written in the form $AX = D$

...(1)

where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{-1}{7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

From equation (1), we have $X = A^{-1}D$

i.e.,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{-1}{7} \begin{bmatrix} 6 + 2 - 5 \\ -2 - 10 + 4 \\ -3 + 6 - 1 \end{bmatrix} = \frac{-1}{7} \begin{bmatrix} 3 \\ -8 \\ 2 \end{bmatrix}$$

$$\therefore x = \frac{-3}{7}, y = \frac{8}{7}, z = \frac{-2}{7}$$

which is the required result.

39. Test for consistency and solve by matrix method

$$x - 2y + 3z = 2$$

$$2x - 3z = 3$$

$$x + y + z = 0.$$

Sol. The augmented matrix is

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array} \right] \text{Operate } R_2 - 2R_1, R_3 - R_1 \\ & \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 4 & -9 & -1 \\ 0 & 3 & -2 & -2 \end{array} \right] \text{Operate } \frac{R_2}{4} \\ & \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -9/4 & -1/4 \\ 0 & 3 & -2 & -2 \end{array} \right] \text{Operate } R_3 - 3R_2 \\ & \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -9/4 & -1/4 \\ 0 & 0 & 19/4 & -5/4 \end{array} \right] \dots(1) \end{aligned}$$

$$\text{Now, } \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -9/4 & -1/4 \\ 0 & 0 & 19/4 & -5/4 \end{array} \right] = \frac{19}{4} \neq 0$$

$\therefore$ Rank of augmented matrix is 3.

$$\text{Since co-efficient matrix } \sim \left[\begin{array}{ccc} 1 & -2 & 3 \\ 0 & 1 & -9/4 \\ 0 & 0 & 19/4 \end{array} \right]$$

$\therefore$ Rank of co-efficient matrix is also 3.

As the rank of augmented matrix is equal to the rank of co-efficient matrix, the equations are consistent.

It follows from equation (1) that by elementary row transformations the given equations have been reduced to

$$x - 2y + 3z = 2$$

$$y - \frac{9z}{4} = -\frac{1}{4}$$

$$\frac{19z}{4} = -\frac{5}{4}$$

Solving these, we get

$$x = \frac{21}{19}, y = \frac{-16}{19}, z = \frac{-5}{19}, \text{ which is the required solution.}$$

Second Method

The given system can be written in the form $AX = D$...(1)

where,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(3) + 2(2+3) + 3(2) = 19 \neq 0$$

$\therefore$ Matrix A is non-singular and hence the equations are consistent.

We have

$$A^{-1} = \frac{1}{19} \begin{bmatrix} 3 & 5 & 6 \\ -5 & -2 & 9 \\ 2 & -3 & 4 \end{bmatrix}$$

From (1), $X = A^{-1}D$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 3 & 5 & 6 \\ -5 & -2 & 9 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 21 \\ -16 \\ -5 \end{bmatrix}$$

$\therefore$ $x = \frac{21}{19}, y = \frac{-16}{19}, z = \frac{-5}{19}$ which is the required solution.

40. (a) Discuss consistency of the system of equations:

$$2x - 3y + 6z - 5w = 3$$

$$y - 4z + w = 1$$

$$4x - 5y + 8z - 9w = \lambda$$

for various values of λ . If consistent, find the solution.

(M.D.U., May, 2008, 2009)

Sol. Here

$$A = \begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \\ \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The augmented matrix $[A : B]$

$$= \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 4 & -5 & 8 & -9 & : & \lambda \end{bmatrix}$$

Operating $R_3 \rightarrow R_3 - 2R_1$

$$= \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 1 & -4 & 1 & : & \lambda - 6 \end{bmatrix}$$

Operating $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 0 & 0 & 0 & : & \lambda - 7 \end{bmatrix}$$

(i) There is no solution if

$$\rho(A) \neq \rho(A : B)$$

i.e., if $\lambda - 7 \neq 0$ or $\lambda \neq 7$

$$\rho(A) = 2, \rho(A : B) = 3$$

(ii) There are infinite number of solutions.

If $\rho(A) = \rho(A : B) = 2$

i.e., $\lambda - 7 = 0$ or $\lambda = 7$

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$2x - 3y + 6z - 5w = 3 \quad \dots(1)$$

$$y - 4z + w = 1 \quad \dots(2)$$

Let

$$w = k_1, z = k_2$$

From (2), $y - 4k_2 + k_1 = 1, y = 1 + 4k_2 - k_1$

From (1), $2x - 3 - 12k_2 + 3k_1 + 6k_2 - 5k_1 = 3$

or $2x = 6 + 6k_2 + 2k_1$

$$x = 3 + 3k_2 + k_1$$

$$y = 1 + 4k_2 - k_1$$

$$z = k_2,$$

$$w = k_1$$

41. Solve the system of equations :

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 11z = 5$$

with the help of matrix inversion.

(M.D.U., May 2007, U.P.T.U., 2008)

Sol. In matrix notation, the given system of equations can be written as

$$AX = B \quad \dots(1)$$

where

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{vmatrix} = 5 \begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} - 3 \begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} + 7 \begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix} \\ &= 5(286 - 4) - 3(33 - 14) + 7(6 - 182) \\ &= 1410 - 57 - 1232 = 121 \neq 0 \end{aligned}$$

$\Rightarrow$ A is non-singular. $\therefore A^{-1}$ exists and the unique solution of (1) is

$$X = A^{-1} B \quad \dots(2)$$

Co-factors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ in the determinant A ;

$$A_1 = \begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} = 282, B_1 = - \begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} = -19, C_1 = \begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix} = -176$$

$$A_2 = - \begin{vmatrix} 3 & 7 \\ 2 & 11 \end{vmatrix} = -19, B_2 = \begin{vmatrix} 5 & 7 \\ 7 & 11 \end{vmatrix} = 6, C_2 = - \begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix} = 11$$

$$A_3 = \begin{vmatrix} 3 & 7 \\ 26 & 2 \end{vmatrix} = -176, B_3 = - \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} = 11, C_3 = \begin{vmatrix} 5 & 3 \\ 3 & 26 \end{vmatrix} = 121$$

$$\therefore \text{adj } A = \text{transpose of } \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{121} \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

From (2),

$$X = A^{-1}.B$$

$$= \frac{1}{121} \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \frac{1}{121} \begin{bmatrix} 282(4) - 19(9) - 176(5) \\ -19(4) + 6(9) + 11(5) \\ -176(4) + 11(9) + 121(5) \end{bmatrix}$$

$$= \frac{1}{121} \begin{bmatrix} 77 \\ 33 \\ 0 \end{bmatrix} = \begin{bmatrix} 77/121 \\ 33/121 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/11 \\ 3/11 \\ 0 \end{bmatrix}$$

Hence

$$x = \frac{7}{11}, y = \frac{3}{11}, z = 0.$$

42. Solve the simultaneous equations with the help of matrices :

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$x + 4y + 9z = 6.$$

Sol. In this question there is no restriction that the solution must be obtained by finding A^{-1} .

Augmented matrix,

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 1 & 4 & 9 & : & 6 \end{bmatrix}$$

Operating $R_2 - R_1, R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 8 & : & 3 \end{bmatrix} \text{ Operating } R_3 - 3R_2$$

$$\begin{aligned}
& \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix} \text{ Operating } \frac{1}{2} R_3 \\
& \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \text{ Operating } R_1 - R_3, R_2 - 2R_3 \\
& \sim \begin{bmatrix} 1 & 1 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \text{ Operating } R_1 - R_2 \\
& \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}
\end{aligned}$$

$$\therefore x = 2, y = 1, z = 0.$$

43. Test for consistency and solve

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5.$$

(M.D.U., Dec., 2008)

Sol. We have

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \text{ Operate } 3R_1, 5R_2$$

$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix} \text{ Operate } R_2 - R_1$$

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix} \text{ Operate } \frac{7}{3} R_1, 5R_3, \frac{1}{11} R_2$$

$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix} \text{ Operate } R_3 - R_1 + R_2, \frac{1}{7} R_1$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

The ranks of co-efficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4, 11y - z = 3, \therefore y = \frac{3}{11} + \frac{z}{11} \text{ and } x = \frac{7}{11} - \frac{16}{11}z$$

where z is a parameter.

$$\text{Hence } z = 0, x = \frac{7}{11}, y = \frac{3}{11}, \text{ is a particular solution.}$$

Note. This set of equation has been solved earlier but with the help of matrix inversion. Here we have performed elementary row transformations to find the rank of the co-efficient matrix and the augmented matrix (by reducing co-efficient matrix to the triangular form). It is observed that given

system of equations are consistent and there are infinite number of solutions. Putting an arbitrary value to one of the 3 unknowns *i.e.*, $z = 0$, we may express the other 2 unknowns *viz.* x and y in terms of these. This provides a particular solution.

44. Test for consistency the system of linear equations:

$$-2x + y + z = a$$

$$x - 2y + z = b$$

$$x + y - 2z = c \text{ where } a, b, c \text{ are constants.}$$

(M.D.U., Dec., 2009)

Sol. We have

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

operate $R_2 \rightarrow R_2 + 2R_1$ and $R_3 \rightarrow R_3 + 2R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The ranks of co-efficient matrix and augmented matrix for the last set of equations, are both 2. Hence, the equations are consistent. Also the given system is equivalent to

$$-2x + y + z = a \quad \dots(1)$$

$$3y = b \quad \dots(2)$$

$$3z = c \quad \dots(3)$$

$$\Rightarrow y = \frac{b}{3}, z = \frac{c}{3}$$

Put the values of y and z equation (1), we get

$$-2x + \frac{b}{3} + \frac{c}{3} = a$$

$$\text{or } x = \frac{b + c - 3a}{6}$$

45. Find the values of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$, have

(i) no solution

(ii) a unique solution and

(iii) an infinite number of solutions.

(M.D.U., Dec., 2008, 2009)

Sol. We have

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The system admits of unique solution if, and only if, the co-efficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have no solution for those values of μ for which the matrices

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But co-efficient matrix is of rank 2 and augmented matrix is not of rank 2 unless $\mu = 9$.

Thus if $\lambda = 5$ and $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

46. Solve the equations : $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$.

Sol. Let us find the rank of the co-efficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \text{ Operating } R_2 - 3R_1, \text{ we get}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \text{ Operating } R_3 - 7R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

We see rank of the co-efficient matrix is 3 which is equal to the number of variables.

Hence the equations have only a trivial zero solution :

$$x = y = z = 0.$$

47. Solve the equations :

$$4x + 2y + z + 3w = 0$$

$$6x + 3y + 4z + 7w = 0$$

$$2x + y + w = 0.$$

Sol. The co-efficient matrix is

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \text{ Operating } R_2 - \frac{3}{2} R_1, R_3 - \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \text{ Operating } R_3 + \frac{1}{5} R_2$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of the co-efficient matrix is 2 which is less than the number of variables (4).

$\therefore$ Number of independent solutions = $4 - 2 = 2$.

Given system is equivalent to

$$4x + 2y + z + 3w = 0, z + w = 0$$

$\therefore$ We have $z = -w$, $4x + 2y = -2w$ or $y = -2x - w$

which give an infinite number of non-trivial solutions, x and w being the parameters.

48. Find the values of λ for which the equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greater of these values. [M.D.U., Dec., 2007, U.P.T.U., 2008, Delhi (D.T.U.), 2006]

Sol. The given equations will be consistent

$$\text{if } \begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0$$

Operating $R_2 - R_1$

$$\text{or, if } \begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & 3 - \lambda \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0$$

Operating $C_3 + C_2$

$$\text{or, if } \begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ 0 & \lambda - 3 & 0 \\ 2 & 3\lambda + 1 & 6\lambda - 2 \end{vmatrix} = 0$$

Expanding by R_2

$$\text{or, if } (\lambda - 3) \begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 2 & 2(3\lambda - 1) \end{vmatrix} = 0$$

$$\text{or, if } 2(\lambda - 3)[(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$$

$$\text{or, if } 6\lambda(\lambda - 3)^2 = 0$$

$$\text{or, if } \lambda = 0 \text{ or } 3.$$

(a) When $\lambda = 0$, the equations become

$$-x + y = 0 \quad \dots(1)$$

$$-x - 2y + 3z = 0 \quad \dots(2)$$

$$2x + y - 3z = 0 \quad \dots(3)$$

Solving (2) and (3), we get

$$\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4}.$$

Hence

$$x = y = z$$

(b) When $\lambda = 3$, the equations become identical.

49. Investigate for consistency of the following equations and if possible, find the solutions :

$$4x - 2y + 6z = 8,$$

$$x + y - 3z = -1,$$

$$15x - 3y + 9z = 21.$$

(M.D.U., 2007, 2009)

Sol. The augmented matrix is

$$\sim \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array} \right]$$

$$\begin{aligned}
& \sim \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 4 & -2 & 6 & : & 8 \\ 15 & -3 & 9 & : & 21 \end{bmatrix} \text{ Operating } R_2 - 4R_1, R_3 - 15R_1 \\
& \sim \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -6 & +18 & : & 12 \\ 0 & -18 & 54 & : & 36 \end{bmatrix} \text{ Operating } \frac{R_2}{6}, \frac{R_3}{18} \\
& \sim \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -1 & 3 & : & 2 \\ 0 & -1 & 3 & : & 2 \end{bmatrix} \text{ Operating } R_3 - R_2 \\
& \sim \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -1 & 3 & : & 2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}
\end{aligned}$$

Clearly, the rank of augmented matrix is equal to the rank of co-efficient matrix = 2. Hence equations are **consistent**.

From the last matrix we find that the given system reduces to the form

$$x + y - 3z = -1 \quad \text{and} \quad -y + 3z = 2$$

$\therefore$

$$x = 1, y = 3z - 2 \text{ are the required solutions.}$$

50. For what values of k the equations

$$x + y + z = 1$$

$$2x + y + 4z = k$$

$$4x + y + 10z = k^2$$

have a solution and solve them completely in each case.

(M.D.U., 2006, 2008; U.P.T.U., 2008; V.T.U., 2006, Delhi, 2007)

Sol. The augmented matrix is

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix} \text{ Operating } R_2 - 2R_1, R_3 - 4R_1 \\
& \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2-4 \end{bmatrix} \text{ Operating } R_3/3 \\
& \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -1 & 2 & : & \frac{k^2-4}{3} \end{bmatrix} \text{ Operating } R_3 - R_2 \\
& \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & \frac{k^2-4}{3} - k + 2 \end{bmatrix}
\end{aligned}$$

The system of given equations will be consistent only if rank of augmented matrix is equal to the rank of co-efficient matrix. This means last element of row 3 = 0.

This can be possible only when

$$\frac{k^2 - 4}{3} - k + 2 = 0$$

or

$$k^2 - 3k + 2 = 0$$

$$(k - 2)(k - 1) = 0$$

$$\Rightarrow k = 1, 2.$$

When $k = 1$, We have $x + y + z = 1$ and $-y + 2z = -1$

or

$$x = -3\lambda, y = 1 + 2\lambda, z = \lambda \text{ is the solution.}$$

When $k = 2$, $x = 1 - 3\lambda, y = 2\lambda, z = \lambda$ is the solution, where λ is arbitrary.

51. Test for consistency and solve :

$$2x - 3y + 7z = 5$$

$$3x + y - 3z = 13$$

$$2x + 19y - 47z = 32.$$

Sol. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array} \right] \text{ Operating } R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array} \right] \text{ Operating } R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 0 & 22 & -54 & 27 \end{array} \right] \text{ Operating } 3R_2 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 0 & -11 & 27 & -11 \\ 0 & 22 & -54 & 27 \end{array} \right] \text{ Operating } R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array} \right] \text{ Operating } R_1/3$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array} \right]$$

It is observed that rank of co-efficient matrix is 2 but the rank of augmented matrix is 3. Hence the equations are inconsistent.

52. Test for consistency and solve:

$$x + 2y + z = 3$$

$$2x + 3y + 2z = 5$$

$$3x - 5y + 5z = 2$$

$$3x + 9y - z = 4.$$

Sol. The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{array} \right]$ Operating $R_2 - 2R_1, R_4 - R_3$

$$\begin{aligned}
& \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 3 & -5 & 5 & : & 2 \\ 0 & 14 & -6 & : & 2 \end{bmatrix} \text{ Operating } R_3 - 3R_1 \\
& \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & -11 & 2 & : & -7 \\ 0 & 14 & -6 & : & 2 \end{bmatrix} \text{ Operating } R_3 - 11R_2, R_4 + 14R_2 \\
& \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & -6 & : & -12 \end{bmatrix} \text{ Operating } R_4 + 3R_3 \\
& \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ Operating } \frac{1}{2} R_3 \\
& \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}
\end{aligned}$$

Rank of augmented matrix and the co-efficient matrix is equal to 3. Hence system is consistent and has a solution.

The last matrix indicates that

$$\begin{aligned}
x + 2y + z &= 3 \\
-y &= -1 \\
z &= 2
\end{aligned}$$

$\Rightarrow x = -1, y = 1$ and $z = 2$ is the solution.

53. Find the values of a and b for which the equations :

$$\begin{aligned}
x + ay + z &= 3 \\
x + 2y + 2z &= b \\
x + 5y + 3z &= 9
\end{aligned}$$

are consistent. When will these equations have a unique solution ?

(M.D.U., Dec., 2009, U.P.T.U., 2008 ; V.T.U. 2006, Delhi, 2007)

Sol. The augmented matrix is

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 1 & 2 & 2 & : & b \\ 1 & 5 & 3 & : & 9 \end{bmatrix}$$

Operating $R_2 - R_1, R_3 - R_1,$

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 0 & 2-a & 1 & : & b-3 \\ 0 & 5-a & 2 & : & 6 \end{bmatrix}$$

Operating $R_3 - 2R_2,$

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 0 & 2-a & 1 & : & b-3 \\ 0 & 1+a & 0 & : & 12-2b \end{bmatrix}$$

Here the co-efficient matrix is

$$A \sim \begin{bmatrix} 1 & a & 1 \\ 0 & 2-a & 1 \\ 0 & 1+a & 0 \end{bmatrix}$$

The given system of equations will be inconsistent if rank of A and rank of augmented matrix (A : B) are not equal.

$\Rightarrow$ When $1 + a = 0$ but $12 - 2b \neq 0$

or if $a = -1, b \neq 6$

Equations will be consistent if

$a = -1$ and $b = 6$

In this case, $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

which gives $x - y + z = 3$

and $3y + z = 3$ or $y = 1 - \frac{z}{3}$

$$x = 4y \text{ or } x = 4 - \frac{4z}{3}$$

i.e., $x = 4 - \frac{4\lambda}{3}, y = 1 - \frac{\lambda}{3}, z = \lambda$

where λ is a parameter. Thus, it has an infinite number of solutions.

When $a \neq -1$ and b has any value, the augmented matrix and co-efficient matrix will have same rank showing the system of equations is consistent but it has a unique solution since rank = 3 = number of unknowns (3).

- 54.** Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

Sol. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array} \right] \text{ Operating } R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array} \right] \text{ Operating } R_2 - 3R_1, R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda + 18 & 9 \end{array} \right] \text{ Operating } R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda + 5 & 0 \end{array} \right]$$

Case I. If $\lambda + 5 = 0$ or $\lambda = -5$

Rank (A) = 2

Rank (A : B) = 2

Thus rank (A) = Rank (A : B) = 2 < number of unknowns.

$\therefore$ The system is consistent and has an infinite number of solutions.

The equations become :

$$\begin{aligned}x + 2y - 3z &= -2 \\ -7y + 13z &= 9\end{aligned}$$

$$\Rightarrow y = \frac{13z - 9}{7}, x = \frac{4 - 5z}{7}$$

or $x = \frac{4 - 5k}{7}, y = \frac{13k - 9}{7}, z = k$, where k is arbitrary.

$$\lambda = -5.$$

Case II. When

$$\lambda \neq -5$$

Rank (A) = 3

Rank (B) = 3 = number of unknowns.

$\therefore$ The system has a unique solution.

This gives $\begin{aligned}x + 2y - 3z &= -2 \\ -7y + 13z &= 9\end{aligned}$

Put $\begin{aligned}z &= 0 \\ -7y &= 9 \\ y &= -\frac{9}{7}, x = \frac{4}{7}.\end{aligned}$

Hence $\lambda \neq -5$, $x = \frac{4}{7}, y = -\frac{9}{7}, z = 0$ is the solution of given system of equations.

55. Show that the equations

$$\begin{aligned}3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c\end{aligned}$$

do not have a solution unless $a + c = 2b$.

(U.P.T.U., 2009 ; M.D.U., Dec., 2007)

Sol. The augmented matrix is

$$\begin{aligned}&\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{array} \right] \text{ Operating } R_2 - R_1, R_3 - R_2 \\ &\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 1 & 1 & 1 & b-a \\ 1 & 1 & 1 & c-b \end{array} \right] \text{ Operating } R_3 - R_2 \\ &\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 1 & 1 & 1 & b-a \\ 0 & 0 & 0 & a+c-2b \end{array} \right]\end{aligned}$$

Here rank of A : 2

Rank of (A : B) i.e., augmented matrix = 3

i.e., Rank of A and (A : B) are not equal.

The system has no solution in such situation.

However if $a + c - 2b = 0$ or $a + c = 2b$

$$\rho(A : B) = \rho(A) = 2 < \text{number of unknowns.}$$

The system will have an infinite number of solutions.

56. Solve the system of equations by matrix method :

$$\begin{aligned}x + y + z &= 8 \\ x - y + 2z &= 6 \\ 3x + 5y - 7z &= 14.\end{aligned}$$

Sol. In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 2 & : & 6 \\ 3 & 5 & -7 & : & 14 \end{bmatrix} \text{ Operating } R_2 - R_1, R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 2 & -10 & : & -10 \end{bmatrix} \text{ Operating } R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 0 & -9 & : & -12 \end{bmatrix}$$

$\therefore \rho(A) = \rho(A : B) = 3 = \text{number of unknowns.}$

$\Rightarrow$ The given system of equations is consistent and has a unique solution.

From the last matrix, we have

$$\begin{aligned} x + y + z &= 8 \\ -2y + z &= -2 \\ -9z &= -12 \end{aligned}$$

On solving these, we get

$$z = \frac{4}{3}, y = \frac{5}{3} \text{ and } x = 5.$$

57. Examine if the following equations are consistent and solve them if they are consistent :

$$\begin{aligned} 2x + 6y + 11 &= 0 \\ 6x + 20y - 6z + 3 &= 0 \\ 6y - 18z + 1 &= 0. \quad (\text{M.D.U., 2006, 2008, 2009 ; U.P.T.U., 2005, 2007}) \end{aligned}$$

Sol. Augmented matrix is

$$\begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \text{ Operating } R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & -36 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \text{ Operating } R_3 - 3R_2$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & -36 \\ 0 & 0 & 0 & : & 107 \end{bmatrix}$$

$$\rho(A) = 2, \rho(A : B) = 3 \text{ i.e., } \rho(A : B) \neq \rho(A)$$

$\therefore$ The system is inconsistent and has no solution.

58. Find the values of λ for which the equations $(2 - \lambda)x + 2y + 3 = 0$, $2x + (4 - \lambda)y + 7 = 0$, $2x + 5y + (6 - \lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ .

Sol. Given equations are

$$\begin{aligned} (2 - \lambda)x + 2y &= -3 \\ 2x + (4 - \lambda)y &= -7 \\ 2x + 5y &= \lambda - 6 \end{aligned}$$

i.e., 3 equations in 2 variables or unknowns.

The given equations will be consistent, if

$$\begin{vmatrix} 2-\lambda & 2 & -3 \\ 2 & 4-\lambda & -7 \\ 2 & 5 & \lambda-6 \end{vmatrix} = 0$$

Operating $R_3 - R_2$

$$\text{or} \quad \begin{vmatrix} 2-\lambda & 2 & -3 \\ 2 & 4-\lambda & -7 \\ 0 & 1+\lambda & \lambda+1 \end{vmatrix} = 0$$

$$\text{or} \quad (2-\lambda) \{(4-\lambda)(\lambda+1) + 7(\lambda+1)\} - 2\{2(\lambda+1)\} - 3\{2(\lambda+1)\} = 0$$

$$\text{or} \quad (\lambda+1) \{(2-\lambda)(11-\lambda) - 10\} = 0$$

$$(\lambda+1)(\lambda^2 - 13\lambda + 12) = 0$$

$$\text{or} \quad (\lambda+1)(\lambda-1)(\lambda-12) = 0$$

$$\Rightarrow \lambda = 1, -1, 12$$

When $\lambda = 1$

Given equations are :

$$x + 2y + 3 = 0$$

$$2x + 3y + 7 = 0$$

$$2x + 5y + 5 = 0$$

On solving these, we get $x = -5, y = 1$

When $\lambda = -1$

Equations are :

$$3x + 2y + 3 = 0$$

$$2x + 5y + 7 = 0$$

$$2x + 5y + 7 = 0$$

$$\text{or} \quad 3x + 2y = -3$$

$$2x + 5y = -7$$

$$\text{On solving these, we get} \quad x = \frac{-1}{11}, \quad y = \frac{-15}{11}$$

When $\lambda = 12$

Equations become

$$-10x + 2y = -3$$

$$2x - 8y = -7$$

$$2x + 5y = 6$$

$$\text{On solving these, we get} \quad x = \frac{1}{2}, y = 1.$$

Hence the given system of equations, is consistent when $\lambda = 1, -1, 12$.

Values of x and y corresponding to these values of λ are

$$x = -5, y = 1$$

$$x = \frac{-1}{11}, y = \frac{-15}{11}$$

$$x = \frac{1}{2}, y = 1.$$

59. Solve the equations:

$$x + y + z + w = 0$$

$$x + 3y + 2z + w = 0$$

$$2x + 0 \cdot y + z - w = 0.$$

(M.D.U., May, 2009)

Sol. Given system is 3 equations in 4 variables, the solution is always other than $x = y = z = w = 0$. The number of solutions is infinite. Let us first, find the rank of the co-efficient matrix,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

Operating $R_2 - R_1, R_3 - 2R_1$,

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & -3 \end{bmatrix}$$

Operating $R_3 + R_2$,

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Rank of the co-efficient matrix is 2.

Rank is less than number of variables (4).

Therefore the given system of equations have $4 - 2 = 2$ (two) linearly independent solutions. Thus if arbitrary values are assigned to 2 variables, the remaining variables can be uniquely found.

Let $z = \lambda, w = \mu$.

Given system is equivalent to

$$x + y + z + w = 0$$

$$2y + z = 0$$

$$-3w = 0 \quad \text{or} \quad 2y + z - 3w = 0$$

As

$$w = \mu, \quad z = \lambda$$

$$2y + \lambda - 3\mu = 0 \quad \text{or} \quad y = -\frac{1}{2}(\lambda + 3\mu)$$

$$x = -y - \lambda - \mu = +\frac{1}{2}(\lambda + 3\mu) - \lambda - \mu$$

$$= -\frac{1}{2}\lambda + \frac{\mu}{2} \quad \text{or} \quad \frac{1}{2}(\mu - \lambda)$$

Hence, the solution of the given system is given by

$$x = \frac{1}{2}(\mu - \lambda)$$

$$y = \frac{-1}{2}(\lambda + 3\mu)$$

$$z = \lambda, w = \mu$$

Here, λ and μ are the parameters.

60. Find the values of λ for which the equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

have non-trivial solution and find the ratios $x : y : z$ when λ has the smallest of these values. What happens when λ has the greatest of these values ?

Sol. The given equations will be consistent, if

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \quad [\text{Operate } R_2 - R_1]$$

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & 3 - \lambda \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \quad [\text{Operate } C_3 + C_2]$$

or, if $\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ 0 & \lambda - 3 & 0 \\ 2 & 3\lambda + 1 & 6\lambda - 2 \end{vmatrix} = 0 \quad [\text{Expand by } R_2]$

or, if $(\lambda - 3) \begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 2 & 2(3\lambda - 1) \end{vmatrix} = 0$

or, if $2(\lambda - 3) [(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$

or, if $6\lambda(\lambda - 3) = 0$ or if $\lambda = 0$ or 3

(a) When $\lambda = 0$, the equations become

$$-x + y = 0 \quad \dots(1)$$

$$-x - 2y + 3z = 0 \quad \dots(2)$$

$$2x + y - 3z = 0 \quad \dots(3)$$

Solving (2) and (3), we get

$$\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4}.$$

Hence

$$x = y = z$$

(b) When $\lambda = 3$, equations become identical.

61. Using the loop current method on a circuit, the following equations were obtained :

$$7i_1 - 4i_2 = 12$$

$$-4i_1 + 12i_2 - 6i_3 = 0$$

$$-6i_2 + 14i_3 = 0$$

By matrix method, solve for i_1 , i_2 and i_3 .

Sol. Given system can be written as $AX = B$

where, $A = \begin{bmatrix} 7 & -4 & 0 \\ -4 & 12 & -6 \\ 0 & -6 & 14 \end{bmatrix}$, $X = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$

The required solution is given by

$$X = A^{-1}B$$

Now, $|A| = 7(132) - 4(56) = 700$

$$\text{adjoint } A = \begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adjoint } A = \frac{1}{700} \begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix}$$

$$X = \frac{1}{700} \begin{bmatrix} 132 & 56 & 24 \\ 56 & 98 & 42 \\ 24 & 42 & 68 \end{bmatrix} \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{700} \begin{bmatrix} 132 \times 12 \\ 56 \times 12 \\ 24 \times 12 \end{bmatrix}$$

$$\Rightarrow i_1 = \frac{132 \times 12}{700}, \quad i_2 = \frac{56 \times 12}{700}, \quad i_3 = \frac{24 \times 12}{700}$$

$$\text{or} \quad i_1 = \frac{396}{175}, \quad i_2 = \frac{24}{25}, \quad i_3 = \frac{72}{175}$$

Note. If $A = (a_{ij})$ is a square matrix and A_{ij} is the co-factor of a_{ij} in the determinant of A , then the transpose of the matrix (A_{ij}) is called the adjoint of the matrix A and is denoted by $\text{adj. } A$. Inverse of matrix A is not defined when it is singular i.e., $|A| = 0$.

62. Solve the equations :

$$2x + 5y + 3z = 1$$

$$-x + 2y + z = 2$$

$$x + y + z = 0.$$

(M.D.U. Dec., 2009)

Sol. In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A : B] &= \begin{bmatrix} 2 & 5 & 3 & : & 1 \\ -1 & 2 & 1 & : & 2 \\ 1 & 1 & 1 & : & 0 \end{bmatrix} \text{ Operating } R_{13} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ -1 & 2 & 1 & : & 2 \\ 2 & 5 & 3 & : & 1 \end{bmatrix} \text{ Operating } R_3 - 2R_1, R_2 + R_1 \\ &\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 0 & 3 & 2 & : & 2 \\ 0 & 3 & 1 & : & 1 \end{bmatrix} \text{ Operating } R_3 - R_2 \\ &\sim \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 0 & 3 & 2 & : & 2 \\ 0 & 0 & -1 & : & -1 \end{bmatrix} \end{aligned}$$

As rank of A and $[A : B]$ are equal and same as the number of unknowns (3), the system has a unique solution.

From the last matrix, we have equations

$$x + y + z = 0$$

$$3y + 2z = 2$$

$$-z = -1$$

This means $z = 1$, $y = 0$ and $x = -1$.

Thus, solution of the given system of equations is given by

$$x = -1, y = 0 \text{ and } z = 1.$$

63. For what values of a and b do the equations

$$x + 2y + 3z = 6$$

$$x + 3y + 5z = 9$$

$$2x + 5y + az = b$$

(M.D.U., Dec., 2008)

have (i) no solution (ii) a unique solution (iii) more than one solution ?

Sol. In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$$

$$\text{Augmented matrix } [A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & a & : & b \end{bmatrix}$$

Operating $R_2 - R_1, R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & a-6 & : & b-12 \end{bmatrix} \quad \text{Operating } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & a-8 & : & b-15 \end{bmatrix}$$

Case I. If $a = 8, b \neq 15$

$$\rho(A) = 2, \rho(A : B) = 3$$

i.e.,

$$\rho(A) \neq \rho(A : B)$$

$\therefore$ The system has no solution.

Case II. If $a \neq 8, b$ may have any value

$$\rho(A) = \rho(A : B) = 3 = \text{number of unknowns.}$$

The system has a unique solution.

Case III. If $a = 8, b = 15$

$$\rho(A) = 2 = \rho(A : B) < \text{number of unknowns.}$$

The system has an infinite number of solutions.

64. Determine the values of λ for which the following system of equations may possess non-trivial solution :

$$3x_1 + x_2 - \lambda x_3 = 0$$

$$4x_1 - 2x_2 - 3x_3 = 0$$

$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

For each permissible value of λ , determine the general solution.

Sol. In matrix notation, the given system can be written as $AX = 0$, where

$$A = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For non-trivial solution, $|A| = 0$

$$\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

or,

$$3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

or,

$$-6\lambda + 36 - 10\lambda - 16\lambda - 4\lambda^2 = 0$$

or,

$$4\lambda^2 + 32\lambda - 36 = 0$$

or,

$$\lambda^2 + 8\lambda - 9 = 0$$

$$(\lambda + 9)(\lambda - 1) = 0$$

$\Rightarrow$

$$\lambda = 1, -9.$$

When $\lambda = 1$

The equations become

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ 4x_1 - 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 + x_3 &= 0 \end{aligned}$$

Solving first 2 equations, we have

$$\frac{x_1}{-3-2} = \frac{x_2}{-4+9} = \frac{x_3}{-6-4}$$

or $x_1 = k, x_2 = -k, x_3 = 2k$, where k is arbitrary.

When $\lambda = -9$

The equations become ;

$$\begin{aligned} 3x_1 + x_2 + 9x_3 &= 0 \\ 4x_1 - 2x_2 - 3x_3 &= 0 \\ -18x_1 + 4x_2 - 9x_3 &= 0 \end{aligned}$$

Solving the first 2 equations viz.

$$\begin{aligned} 3x_1 + x_2 + 9x_3 &= 0 \\ 4x_1 - 2x_2 - 3x_3 &= 0 \end{aligned}$$

$$\frac{x_1}{-3+18} = \frac{x_2}{36+9} = \frac{x_3}{-6-4}$$

or
$$\frac{x_1}{15} = \frac{x_2}{45} = \frac{x_3}{-10}$$

$$\Rightarrow x_1 = 3k, x_2 = 9k, x_3 = -2k.$$

where k is any arbitrary.

65. Prove that the following matrix is orthogonal : $\frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.

Sol. A square matrix A is said to be orthogonal if $AA' = A'A = I$

We have

$$\begin{aligned} AA' &= \frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

Hence the matrix is orthogonal.

66. Show that the transformation ;

$$\begin{aligned} y_1 &= 2x_1 + x_2 + x_3 \\ y_2 &= x_1 + x_2 + 2x_3 \\ y_3 &= x_1 - 2x_3 \end{aligned}$$

(M.D.U., May, 2009)

is regular. Write down the inverse transformation.

Sol. The given transformation may be written as

$$Y = AX$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

The transformation will be regular if the matrix A is non-singular i.e.; $|A| \neq 0$.

Now $|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$

$$= 2(-2) - 1(-2 - 2) + 1(-1)$$

$$= -4 + 4 - 1$$

$$= -1 \neq 0$$

$\therefore$ The given transformation is regular.

The inverse transformation is given by

$$X = A^{-1}Y$$

Now, $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

$\therefore X = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow$

$$x_1 = 2y_1 - 2y_2 - y_3$$

$$x_2 = -4y_1 + 5y_2 + 3y_3$$

$$x_3 = y_1 - y_2 - y_3$$

is the inverse transformation.

67. Represent each of the transformations

$$x_1 = 3y_1 + 2y_2, y_1 = z_1 + 2z_2$$

and

$$x_2 = -y_1 + 4y_2, y_2 = 3z_1$$

by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2 .

Sol.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{or} \quad X = AY$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{or} \quad Y = BZ$$

$\Rightarrow X = [AB] Z$

$$[AB] = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{or} \quad X = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

68. If $\xi = x \cos \alpha - y \sin \alpha$

$$\eta = x \sin \alpha + y \cos \alpha,$$

write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.

Sol. In matrix notation we have

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix of transformation A is

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix},$$

Now

$$A' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

i.e., $A' = A^{-1}$. Inverse transformation is given by A^{-1} above.

- 69.** A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$ and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3 .

Sol. Given $Y = AX$... (1)

$Z = BY$... (2)

$\Rightarrow Z = (BA)X$... (3)

Now,

$$BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$$

Hence the required transformation from x_1, x_2, x_3 to z_1, z_2, z_3 is given by equation (3),

where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$.

- 70.** Find the inverse transformation of

$$y_1 = x_1 + 2x_2 + 5x_3$$

$$y_2 = 2x_1 + 4x_2 + 11x_3$$

$$y_3 = -x_2 + 2x_3$$

(M.D.U., Dec., 2009 and U.P.T.U., 2008)

Sol. Given equations can be written in the matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e., $Y = AX$ where $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix}$

$$Y = AX$$

$$A^{-1}Y = A^{-1}AX$$

or $X = A^{-1}Y$ gives the inverse transformation.

Now, $|A| = 1(19) - 2(4) + 5(-2) = 1 \neq 0$.

Thus matrix A is non-singular.

Let us now find A^{-1} .

Now we can see that

$$A^{-1} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Hence inverse transformation is given by

$$X = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} Y$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned} x_1 &= 19y_1 - 9y_2 + 2y_3 \\ x_2 &= -4y_1 + 2y_2 - y_3 \\ x_3 &= -2y_1 + y_2 \end{aligned}$$

71. Verify that the following matrix is orthogonal : $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$.

(M.D.U., Dec., 2007, 2008 ; U.P.T.U., 2009)

Sol. Let

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

If A is an orthogonal matrix, we should have $AA' = I$.

$$\begin{aligned} \text{Now } AA' &= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 1/9 + 4/9 + 4/9 & 2/9 + 2/9 - 4/9 & 2/9 - 4/9 + 2/9 \\ 2/9 + 2/9 - 4/9 & 4/9 + 1/9 + 4/9 & 4/9 - 2/9 - 2/9 \\ 2/9 - 4/9 + 2/9 & 4/9 - 2/9 - 2/9 & 4/9 + 4/9 + 1/9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

Hence given matrix A is orthogonal.

72. Determine a, b, c so that the matrix is orthogonal, where

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$

(M.D.U., May 2009, U.P.T.U., 2008)

Sol. A matrix is orthogonal if $AA' = I$.

$$\text{Now } A' = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

If given matrix A is orthogonal then we should have

$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$$

$$\begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 4b^2 + c^2 &= 1 && \dots(i) \\ 2b^2 - c^2 &= 0 && \dots(ii) \\ a^2 + b^2 + c^2 &= 1 && \dots(iii) \\ a^2 - b^2 - c^2 &= 0 && \dots(iv) \end{aligned}$$

Using (i) and (ii), we have

$$6b^2 = 1 \quad \text{or} \quad b^2 = \frac{1}{6}, \quad c^2 = \frac{1}{3}$$

Using (iii) and (iv), we have

$$2a^2 = 1 \Rightarrow a^2 = \frac{1}{2}$$

Hence,
$$a = \pm \frac{1}{\sqrt{2}}, b = \pm \frac{1}{\sqrt{6}}, c = \pm \frac{1}{\sqrt{3}}.$$

73. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol. The characteristic equation of the given matrix is

$$\begin{aligned} |A - \lambda I| &= 0 && \text{or} && \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0 \\ \text{or} &&& (1-\lambda)(4-\lambda) - 10 = 0 && \text{or} && \lambda^2 - 5\lambda - 6 = 0 \\ \text{or} &&& (\lambda-6)(\lambda+1) = 0 && \therefore && \lambda = 6, -1 \end{aligned}$$

Thus, the eigen values of A are 6, -1.

Corresponding to $\lambda = 6$, the eigen vectors are given by $(A - 6I)X = 0$

$$\text{or} \quad \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get only one independent equation $-5x_1 - 2x_2 = 0$, since rank of co-efficient matrix is 1.

$$\therefore \frac{x_1}{2} = \frac{x_2}{-5} \text{ giving the eigen vector } (2, -5).$$

Corresponding to $\lambda = -1$, the eigen vectors are given by

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get only one independent equation $2x_1 - 2x_2 = 0$, since rank of co-efficient matrix is 1.

$$\therefore x_1 = x_2 \text{ giving the eigen vector } (1, 1).$$

Hence, eigen values are 6, -1 and eigen vectors are (2, -5) and (1, 1).

74. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

(M.D.U., 2006, 2007 ; U.P.T.U., 2008 Dec., 2009)

Sol. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\text{or,} \quad \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\text{or,} \quad (-2 - \lambda) [-\lambda(1 - \lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + 1(1 - \lambda)] = 0$$

$$\text{or,} \quad \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trial, $\lambda = -3$ satisfies it.

$$\therefore \quad (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\text{or,} \quad (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$$

$$\Rightarrow \quad \lambda = -3, -3, 5.$$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$$(A + 3I)X = 0$$

$$\text{or,} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We get only one independent equation

$$x_1 + 2x_2 - 3x_3 = 0, \text{ since rank of co-efficient matrix is 1.}$$

Choosing $x_2 = 0$, we have $x_1 - 3x_3 = 0$

$$\therefore \quad \frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{1} \text{ giving the eigen vector } (3, 0, 1).$$

Choosing $x_3 = 0$, we have $x_1 + 2x_2 = 0$

$$\therefore \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0} \text{ giving the eigen vector } (2, -1, 0).$$

Any other eigen vector corresponding to $\lambda = -3$ will be a linear combination of these two.

$$\text{Corresponding to } \lambda = 5, \text{ the eigen vectors are given by } \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

Solving first 2 equations, we have

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\text{or } \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} \text{ giving the eigen vector } (1, 2, -1).$$

75. Find the sum and product of the eigen values of the matrices:

$$(i) \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$$

$$(ii) \quad \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

(M.D.U., May 2009)

(M.D.U., May, 2008, Dec. 2009 and U.P.T.U., 2007)

Sol. (i) The sum of eigen values is obtained by adding the elements of the principal diagonal i.e., $1 + 1 + 3 + 5 = 10$.

Product of eigen values is given by the determinant value of the matrix, i.e.,

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{vmatrix}$$

Apply $C_2 \rightarrow C_2 - 2C_1$, $C_3 \rightarrow C_3 - 3C_1$, $C_4 \rightarrow C_4 - 4C_1$

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & -1 & -2 \\ 7 & -10 & -18 & -26 \\ 4 & -5 & -12 & -11 \end{vmatrix}$$

Expanding through R_1 ,

$$\begin{aligned} |A| &= \begin{vmatrix} -3 & -1 & -2 \\ -10 & -18 & -26 \\ -5 & -12 & -11 \end{vmatrix} \\ &= (-3)(198 - 312) + (110 - 130) - 2(120 - 90) \\ &= 342 - 20 - 60 = 262. \end{aligned}$$

$$(ii) \quad |A| = \begin{vmatrix} 2 & 3 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \text{Product of eigen values.}$$

$$\begin{aligned} \text{i.e., Product is} &= 2(2 - 0) - 3(-4 - 1) - 1(-1) \\ &= 4 + 15 + 1 = 20. \end{aligned}$$

$$\begin{aligned} \text{Sum} &= 2 + 1 + 2 \text{ (elements of principal diagonal)} \\ &= 5. \end{aligned}$$

76. If λ is an eigen value of an orthogonal matrix, then prove that $\frac{1}{\lambda}$ is also its eigen value.

(M.D.U., Dec., 2008, and U.P.T.U., 2009)

Sol. Let λ be an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is an eigen value of A^{-1} .

$$\begin{aligned} \text{But} \quad AA' &= I \\ \therefore A^{-1} &= A' \quad (\text{Transpose}) \end{aligned}$$

However, the matrices A and A' have the same eigen values.

$$\therefore \frac{1}{\lambda} \text{ is also an eigen value of } A.$$

Hence proved.

77. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence compute A^{-1} .

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

(On simplification).

To verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = O \quad \dots(1)$$

Now,

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$- 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

This verifies Cayley-Hamilton theorem.

Now multiplying both sides of (1) by A^{-1} , we have

$$A^2 - 6A + 9I - 4A^{-1} = 0 \Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

78. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

(M.D.U., 2007, 2009 ; U.P.T.U., 2007)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = 0$... (1)

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= A^2 + A + I$$

[Using (1)]

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

79. Using Cayley-Hamilton theorem, find A^8 if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}. \quad (\text{U.P.T.U., 2008 ; M.D.U., Dec., 2007})$$

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

or

$$-(1 - \lambda^2) - 4 = 0 \quad \text{or} \quad \lambda^2 = 5 \quad \dots (1)$$

By Cayley-Hamilton theorem, A satisfies its characteristic equation (1).

$$\begin{aligned} \therefore & \quad A^2 = 5I \\ \Rightarrow & \quad (A^2)^4 = (5I)^4 \\ \Rightarrow & \quad A^8 = 625I \quad (\because I^4 = I) \end{aligned}$$

80. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

(M.D.U., May, 2009 ; U.P.T.U., 2007)

Sol. The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \quad \dots(1)$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation (1), so that

$$A^2 - 4A - 5I = 0 \quad \dots(2)$$

Now,

$$\begin{aligned} A^2 - 4A - 5I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \end{aligned}$$

This verifies the theorem.

Multiplying (2) by A^{-1} , we get

$$A - 4I - 5A^{-1} = 0$$

or

$$\begin{aligned} A^{-1} &= \frac{1}{5}(A - 4I) \\ &= \frac{1}{5} \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$$\begin{aligned} & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I \\ &= (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5 \\ &= \lambda + 5 \quad \text{[by (1)]} \end{aligned}$$

Hence $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5$

which is a linear polynomial in A.

81. Find the eigen values and eigen vectors of the matrix : $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Sol. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\text{or} \quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] &= 0 \\ (8-\lambda)(5-10\lambda+\lambda^2) + 12(3\lambda-5) + 4(5+\lambda) &= 0 \\ \lambda(\lambda^2-18\lambda+45) &= 0 \\ \lambda(\lambda-3)(\lambda-15) &= 0 \end{aligned}$$

$$\Rightarrow \lambda = 0, 3, 15$$

For $\lambda = 0$, we have

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The equations are

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving the first 2 equations, we have

$$\frac{x_1}{24 - 14} = \frac{x_2}{-12 + 32} = \frac{x_3}{56 - 36}$$

or $\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$ giving the eigen vector (1, 2, 2)

For $\lambda = 3$, the equations are

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 = 0$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{24 - 8} \text{ giving eigen vector } (2, 1, -2)$$

For $\lambda = 15$, the equations are

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

Solving the last 2 equations, we get

$$\frac{x_1}{96 - 16} = \frac{x_2}{-8 - 72} = \frac{x_3}{24 + 16} \quad \text{or} \quad \frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40} \text{ giving eigen vector } (2, -2, 1).$$

Hence eigen values of the given matrix are (0, 3, 15) and eigen vectors are given by (1, 2, 2); (2, 1, -2) and (2, -2, 1).

82. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied

by A and hence obtain the inverse of the given matrix.

(M.D.U., May 2008, Dec., 2009, Delhi, 2008, U.P.T.U., 2007)

Sol. Characteristic equation is

$$|A - \lambda I| = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1 - \lambda) [(2 - \lambda)(1 - \lambda) - 6] - 3[4(1 - \lambda) - 3] + 7[8 - 2 + \lambda] = 0$$

$$\lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda - 3 + 12\lambda + 42 + 7\lambda = 0$$

$$-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$$

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

If A satisfies it,

$$A^3 - 4A^2 - 20A - 35I = 0$$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now, $A^3 - 4A^2 - 20A - 35I$

$$\begin{aligned} &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

Hence A satisfies the characteristic equation.

To obtain inverse of given matrix A we proceed as follows :

Since $A^3 - 4A^2 - 20A - 35I = 0$

Multiplying by A^{-1} , we get

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

or

$$35A^{-1} = A^2 - 4A - 20I$$

or

$$A^{-1} = \frac{1}{35} [A^2 - 4A - 20I]$$

Now,

$$A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} A^2 - 4A - 20I &= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} \end{aligned}$$

$\therefore$

$$A^{-1} = \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

83. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

fied by A.

Sol. The characteristic equation is $|A - \lambda I| = 0$

or

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
\text{or } (1-\lambda) [(1+\lambda)^2 - 4] - 2[-2(1+\lambda) - 12] + 3[2 + 3(1+\lambda)] &= 0 \\
(1-\lambda) (\lambda^2 + 2\lambda - 3) + 4 + 4\lambda + 39 + 9\lambda &= 0 \\
\text{or } -\lambda^3 - \lambda^2 + 18\lambda + 40 &= 0 \\
\text{or } \lambda^3 + \lambda^2 - 18\lambda - 40 &= 0
\end{aligned}$$

is the characteristic equation.

If A satisfies it, then $A^3 + A^2 - 18A - 40I = 0$

$$\text{Now, } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$\text{Now, } \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Hence Proved.

$$84. \text{ Using Cayley-Hamilton theorem, find the inverse of } A = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

(M.D.U., 2006, 2007, 2009 ; U.P.T.U., 2008)

Sol. The characteristic equation of A is $|A - \lambda I| = 0$.

$$\text{i.e., } \begin{vmatrix} 7-\lambda & -1 & 3 \\ 6 & 1-\lambda & 4 \\ 2 & 4 & 8-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
(7-\lambda) [(1-\lambda)(8-\lambda) - 16] + 1[6(8-\lambda) - 8] + 3[24 - 2(1-\lambda)] &= 0 \\
(7-\lambda) (-8 - 9\lambda + \lambda^2) + (40 - 6\lambda) + (66 + 6\lambda) &= 0 \\
\lambda^3 - 16\lambda^2 + 55\lambda - 50 &= 0
\end{aligned}$$

(On simplification)

To verify Cayley-Hamilton theorem, we can show that

$$A^3 - 16A^2 + 55A - 50I = 0$$

$$\text{Now, } A^2 = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 449 & 119 & 491 \\ 566 & 171 & 644 \\ 754 & 324 & 986 \end{bmatrix}$$

$$\text{Now, } A^3 - 16A^2 + 55A - 50I$$

$$= \begin{bmatrix} 449 & 119 & 491 \\ 566 & 171 & 644 \\ 754 & 324 & 986 \end{bmatrix} - 16 \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} + 55 \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} - 50 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 449 - 784 + 385 - 50 & 119 - 64 - 55 & 491 - 656 + 165 \\ 566 - 896 + 330 & 171 - 176 + 55 - 50 & 644 - 864 + 220 \\ 754 - 864 + 110 & 324 - 544 + 220 & 986 - 1376 + 440 - 50 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence $A^3 - 16A^2 + 55A - 50I = 0$ is satisfied.

To find A^{-1} , we multiply

$$A^3 - 16A^2 + 55A - 50I = 0 \text{ by } A^{-1}.$$

$$A^2 - 16A + 55I - 50A^{-1} = 0$$

or

$$50A^{-1} = A^2 - 16A + 55I$$

or

$$A^{-1} = \frac{1}{50} [A^2 - 16A + 55I]$$

$$\text{Now, } A^2 - 16A + 55I = \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} - 16 \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} + 55 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{1}{50} \begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}.$$

85. Find the eigen values and eigen vectors of the matrix A ,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}. \quad (\text{M.D.U., Dec., 2008, 2009})$$

Sol. The characteristic equation of the given matrix A is $|A - \lambda I| = 0$

$$\text{or } \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\text{or } (6-\lambda) [(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$$

$$(6-\lambda) [9 + \lambda^2 - 6\lambda - 1] - 12 + 4\lambda + 4 + 4 - 12 + 4\lambda = 0$$

$$(6-\lambda) (\lambda^2 - 6\lambda + 8) + 8\lambda - 16 = 0$$

$$6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 8\lambda - 16 = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

or

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

By trial, $\lambda = 2$ satisfies it.

$$\therefore (\lambda - 2) (\lambda^2 - 10\lambda + 16) = 0$$

or

$$(\lambda - 2) (\lambda - 2) (\lambda - 8) = 0$$

$$\therefore \lambda = 2, 2, 8 \text{ are the eigen values.}$$

Corresponding to $\lambda = 2$, the eigen vectors are given by

$$(A - 2I)X = 0 \quad \text{or} \quad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The equations are :

$$\begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

We get only one independent equation :

$$2x_1 - x_2 + x_3 = 0$$

Choosing $x_2 = 0$, we have $2x_1 + x_3 = 0$

or
$$\frac{x_1}{1} = \frac{x_3}{-2} = \frac{x_2}{0}$$

$\therefore$ eigen vector is $(1, 0, -2)$.

Choosing $x_3 = 0$, we have $2x_1 = x_2$

or
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0} \text{ giving eigen vector } (1, 2, 0).$$

For $\lambda = 8$, the eigen vectors are given by

$$(A - 8I)X = 0$$

or
$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We get equations :

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \dots(1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \dots(2)$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \dots(3)$$

$$(1) \Rightarrow x_1 + x_2 = x_3$$

Adding (2) with (3) gives

$$-6x_2 - 6x_3 = 0 \text{ or } x_2 + x_3 = 0 \quad \dots(4)$$

From (1) and (4), we have $x_1 + 2x_2 = 0$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence eigen vector is $(2, -1, 1)$.

Thus for the given matrix A, eigen values are $(2, 2, 8)$. Eigen vectors are $(1, 0, -2)$, $(1, 2, 0)$ and $(2, -1, 1)$.

86. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

(U.P.T.U., 2007 ; M.D.U., May 2008, Dec., 2009)

Sol. The characteristic equation of the given matrix A is $|A - \lambda I| = 0$.

or
$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
 \text{or} \quad & (2 - \lambda)(2 - \lambda)^2 + 1(\lambda - 2) = 0 \\
 & (2 - \lambda)[(2 - \lambda)^2 - 1] = 0 \\
 & (2 - \lambda)[4 + \lambda^2 - 4\lambda - 1] = 0 \\
 & (2 - \lambda)[\lambda^2 - 4\lambda + 3] = 0 \\
 & (2 - \lambda)(\lambda - 3)(\lambda - 1) = 0
 \end{aligned}$$

$\Rightarrow \lambda = 1, 2, 3$ are the eigen values.

The matrix equation corresponding to $\lambda = 1$ is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{aligned}
 \text{or} \quad & x_1 + x_3 = 0 \\
 & x_2 = 0
 \end{aligned}$$

$$\text{giving} \quad x_1 = -x_3 \quad \text{or} \quad \frac{x_1}{1} = \frac{x_3}{-1} = \frac{x_2}{0}$$

The eigen vector is $(1, 0, -1)$

For $\lambda = 2$ the equation is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_3 = 0, \quad x_1 = 0$$

This gives $(0, 1, 0)$ as the eigen vector.

For $\lambda = 3$ the equation is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{aligned}
 -x_1 + x_3 &= 0 \\
 -x_2 &= 0 \\
 x_1 - x_3 &= 0
 \end{aligned}$$

giving eigen vector $(1, 0, 1)$.

87. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and use it to find the matrix represented

$$\text{by } A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I.$$

(M.D.U., May 2006, 2009 ; U.P.T.U., 2007)

Sol. Characteristic equation is

$$\begin{aligned}
 & |A - \lambda I| = 0 \\
 \text{or} \quad & \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} = 0 \\
 & (1 - \lambda)(3 - \lambda) - 8 = 0 \\
 & \lambda^2 - 4\lambda - 5 = 0 \\
 & (\lambda - 5)(\lambda + 1) = 0 \\
 \Rightarrow \quad & \lambda = -1, 5
 \end{aligned}$$

$$\text{For } \lambda = -1, \text{ we have } \begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} = 0$$

i.e., we get only one independent equation $2x_1 + 4x_2 = 0$ since rank of co-efficient matrix is 1.

$$\frac{x_1}{2} = \frac{x_2}{-1} \text{ giving the eigen vector } (2, -1).$$

Corresponding to $\lambda = 5$, the eigen vectors are given by

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get only one independent equation.

$$2x_1 - 2x_2 = 0 \text{ or } x_1 = x_2 \text{ giving the eigen vector } (1, 1).$$

To verify Cayley-Hamilton theorem, we can show that

$$A^2 - 4A - 5I = 0 \quad \dots(1)$$

Now dividing the polynomial $\lambda^5 + 5\lambda^4 - 6\lambda^3 + 2\lambda^2 - 4\lambda + 7$ by $\lambda^2 - 4\lambda - 5$, we get

$$\begin{aligned} \lambda^5 + 5\lambda^4 - 6\lambda^3 + 2\lambda^2 - 4\lambda + 7 &= (\lambda^2 - 4\lambda - 5)(\lambda^3 + 9\lambda^2 + 35\lambda + 187) + 919\lambda + 942 \\ &= 919\lambda + 942 \quad (\because \lambda^2 - 4\lambda - 5 = 0) \end{aligned}$$

$$\text{Hence } A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7 = 919A + 942I$$

$$\begin{aligned} &= 919 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 942 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1861 & 3676 \\ 1838 & 3699 \end{bmatrix}. \end{aligned}$$

88. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ and find A^{-1} .

(Delhi, 2008, U.P.T.U., 2006 ; M.D.U., Dec., 2007, 2008)

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\text{i.e., } \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (7-\lambda)[(1+\lambda)^2 - 4] - 2[6(1+\lambda) - 12] - 2[-12 + 6(1+\lambda)] &= 0 \\ \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 &= 0 \end{aligned}$$

To verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \dots(1)$$

$$\text{Now, } A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$\therefore A^3 - 5A^2 + 7A - 3I$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 5 \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 7 \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 79 - 125 + 49 - 3 & 26 - 40 + 14 & -26 + 40 - 14 \\ -78 + 120 - 42 & -25 + 35 - 7 - 3 & 26 - 40 + 14 \\ 78 - 120 + 42 & 26 - 40 + 14 & -25 + 35 - 7 - 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.
\end{aligned}$$

Hence (1) is satisfied.

To find A^{-1} : $A^3 - 5A^2 + 7A - 3I = 0$

Multiplying by A^{-1} we get,

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 5A + 7I$$

or

$$A^{-1} = \frac{1}{3} [A^2 - 5A + 7I]$$

Now,

$$\begin{aligned}
A^2 - 5A + 7I &= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5 \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}
\end{aligned}$$

$$\therefore A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}.$$

89. Using Cayley-Hamilton theorem, find A^6 if $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$.

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 2 - \lambda & 1 \\ 5 & -2 - \lambda \end{vmatrix} = 0$$

or

$$-(2 - \lambda)(2 + \lambda) - 5 = 0$$

$$\lambda^2 - 4 - 5 = 0$$

$$\text{or} \quad \lambda^2 = 9$$

...(1)

By Cayley-Hamilton theorem, A satisfies its characteristic equation (1).

$$\therefore A^2 = 9I$$

$$\Rightarrow A^6 = (A^2)^3 = (9I)^3 = 729I$$

$$(\because I^3 = I)$$

or

$$A^6 = 729I.$$

90. Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

(U.P.T.U., 2007, 2009 ; M.D.U., May 2006, 2008, 2009)

Sol. Characteristic equation of A is

$$|A - \lambda I| = 0$$

or

$$\begin{vmatrix} 1 - \lambda & 0 & 3 \\ 2 & 1 - \lambda & -1 \\ 1 & -1 & 1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda) [(1 - \lambda)^2 - 1] + 3[-2 - 1 + \lambda] = 0$$

$$(1 - \lambda)(\lambda^2 - 2\lambda) + 3\lambda - 9 = 0$$

$$\lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 3\lambda - 9 = 0$$

$$3\lambda^2 + \lambda - \lambda^3 - 9 = 0 \quad \text{or} \quad \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

Since A satisfies its characteristic equation by Cayley-Hamilton theorem,

$$A^3 - 3A^2 - A + 9I = 0$$

Multiplying by A^{-1} we have

$$A^2 - 3A - I + 9 \cdot A^{-1} = 0$$

$$\begin{aligned} \Rightarrow \quad A^{-1} &= \frac{1}{9} [-A^2 + 3A + I] \\ &= -\frac{1}{9} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{3}{9} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= -\frac{1}{9} \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \frac{3}{9} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

or

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}.$$

91. Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal matrix and obtain the modal matrix.

(M.D.U., Dec., 2007, 2008)

Sol. The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

(On simplification)

Solving it, we get $\lambda = 1, \pm\sqrt{5}$

When $\lambda = 1$, the corresponding eigen vector is given by

$$-2x_1 + 2y_1 - 2z_1 = 0$$

$$x_1 + y_1 + z_1 = 0$$

$$-x_1 - y_1 - z_1 = 0$$

Solving the first 2 equations,

$$\frac{x_1}{4} = \frac{y_1}{0} = \frac{z_1}{-4} \text{ giving the eigen vector } (1, 0, -1).$$

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by

$$(-1 - \sqrt{5})x_2 + 2y_2 - 2z_2 = 0$$

$$x_2 + (2 - \sqrt{5})y_2 + z_2 = 0$$

$$-x_2 - y_2 - \sqrt{5}z_2 = 0$$

Solving the last 2 equations,

$$\frac{x_2}{6-2\sqrt{5}} = \frac{y_2}{-1+\sqrt{5}} = \frac{z_2}{1-\sqrt{5}}$$

or

$$\frac{x_2}{(\sqrt{5}-1)^2} = \frac{y_2}{\sqrt{5}-1} = \frac{z_2}{1-\sqrt{5}}$$

or

$$\frac{x_2}{\sqrt{5}-1} = \frac{y_2}{1} = \frac{z_2}{-1} \quad \text{giving the eigen vector } (\sqrt{5}-1, 1, -1)$$

Similarly, the eigen vector corresponding to $\lambda = -\sqrt{5}$ is $(\sqrt{5}+1, -1, 1)$.

$\therefore$ The modal matrix is given by

$$B = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{or} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

is the diagonal matrix, obtained by diagonalising A.

Note. The matrix B which diagonalises A is called the modal matrix of A and is obtained by grouping the eigen vectors of A into a square matrix.

92. Diagonalise the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^4 .

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

or

$$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

or

$$(\lambda + 1)(\lambda - 3)(\lambda - 4) = 0$$

$$\therefore \lambda = -1, 3, 4$$

When $\lambda = -1$, the corresponding eigen vector is given by

$$2x_1 + 6y_1 + z_1 = 0$$

$$x_1 + 3y_1 = 0$$

$$4z_1 = 0$$

Solving the last 2, $\frac{x_1}{3} = \frac{y_1}{-1}$, $z_1 = 0$ giving the eigen vector $(3, -1, 0)$

When $\lambda = 3$, the corresponding eigen vector is given by

$$-2x_1 + 6y_1 + z_1 = 0$$

$$x_1 - y_1 = 0$$

Solving $\frac{x_1}{1} = \frac{y_1}{1} = \frac{z_1}{-4}$ giving the eigen vector $(1, 1, -4)$

When $\lambda = 4$, the corresponding eigen vector is given by

$$-3x_1 + 6y_1 + z_1 = 0$$

(On simplification)

$$\begin{aligned}x_1 - 2y_1 &= 0 \\z_1 &= 0\end{aligned}$$

Solving the last 2, $\frac{x_1}{2} = \frac{y_1}{1}$, $z_1 = 0$ giving the eigen vector (2, 1, 0).

$$\therefore \text{ The modal matrix } B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

and

$$B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is the required diagonal matrix.

To find B^{-1} to know A^4 .

$$D^4 = \begin{bmatrix} (-1)^4 & 0 & 0 \\ 0 & (3)^4 & 0 \\ 0 & 0 & (4)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{vmatrix} = 3(4) + 1(0) + 2(4) = 20$$

$$\text{adj } B = \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{20} \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$\therefore A^4 = BD^4B^{-1} = \frac{1}{20} \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix} \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$A^4 = \frac{1}{20} \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -405 \\ 1024 & 3072 & 1024 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 2060 & 6120 & 1640 \\ 1020 & 3080 & 620 \\ 0 & 0 & 1620 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} 103 & 306 & 82 \\ 51 & 154 & 31 \\ 0 & 0 & 81 \end{bmatrix}.$$

93. Diagonalise the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Sol. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\text{or } \lambda(\lambda^2 - 18\lambda + 45) = 0 \quad (\text{On simplification})$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\Rightarrow \lambda = 0, 3, 15$$

For $\lambda = 0$, eigen vector is given by $(1, 2, 2)$

For $\lambda = 3, 15$, eigen vectors are $(2, 1, -2)$ and $(2, -2, 1)$.

$$\therefore \text{ The modal matrix } B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\text{and } B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{or } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

is the required diagonal matrix.

94. Reduce the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ to the diagonal form.

Sol. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\text{or } (2-\lambda)(2-\lambda)^2 + (\lambda-2) = 0 \quad (\text{On simplification})$$

$$\text{or } (2-\lambda)(\lambda-3)(\lambda-1) = 0 \Rightarrow \lambda = 1, 2, 3$$

We can easily see that eigen vectors corresponding to

$$\lambda = 1 \quad \text{is} \quad (1, 0, -1),$$

$$\lambda = 2 \quad \text{is} \quad (0, 1, 0) \text{ and}$$

$$\lambda = 3 \quad \text{is} \quad (1, 0, 1) \text{ respectively.}$$

$$\therefore \text{ The modal matrix } B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B^{-1}AB = D$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{or } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

95. Show that the vectors

$$x_1 = (1, 2, 4)$$

$$x_2 = (2, -1, 3)$$

$$x_3 = (0, 1, 2)$$

$$x_4 = (-3, 7, 2)$$

are linearly dependent and find the relation between them.

Sol. Adding suitable multiples of x_1 to x_2 and x_4 so that the first component reduces to zero, we have

$$x_2 - 2x_1 = (2, -1, 3) - (2, 4, 8) = (0, -5, -5)$$

$$x_4 + 3x_1 = (-3, 7, 2) + (3, 6, 12) = (0, 13, 14)$$

$$\text{Also } x_3 = (0, 1, 2)$$

Adding suitable multiples of x_3 to the above vectors so that the second component reduces to zero, we have

$$(x_2 - 2x_1) + 5x_3 = (0, 0, 5)$$

and $(x_4 + 3x_1) - 13x_3 = (0, 13, 14) - (0, 13, 26) = (0, 0, -12)$

To reduce the third component to zero, multiplying the above vectors by 12 and 5 respectively and adding, we have

$$\begin{aligned} 12(x_2 - 2x_1 + 5x_3) + 5(x_4 + 3x_1 - 13x_3) \\ = (0, 0, 60) + (0, 0, -60) \\ \Rightarrow -9x_1 + 12x_2 - 5x_3 + 5x_4 = (0, 0, 0) \\ \Rightarrow 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0 \end{aligned} \quad \dots(1)$$

Thus, there exist numbers $k_1 = 9, k_2 = -12, k_3 = 5, k_4 = -5$ which are not all zero such that $k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$

Hence the vectors x_1, x_2, x_3 and x_4 are linearly dependent.

Also, (1) is the relation between them.

- 96.** Are the vectors $x_1 = (1, 3, 4, 2), x_2 = (3, -5, 2, 2)$ and $x_3 = (2, -1, 3, 2)$ linearly dependent? If so, express one of these as a linear combination of the others. (M.D.U., May, 2008)

Sol. The relation $k_1x_1 + k_2x_2 + k_3x_3 = 0$

i.e., $k_1(1, 3, 4, 2) + k_2(3, -5, 2, 2) + k_3(2, -1, 3, 2) = 0$ is equivalent to

$$\begin{aligned} k_1 + 3k_2 + 2k_3 &= 0 \\ 3k_1 - 5k_2 - k_3 &= 0 \\ 4k_1 + 2k_2 + 3k_3 &= 0 \\ 2k_1 + 2k_2 + 2k_3 &= 0 \end{aligned}$$

As these are satisfied by the values $k_1 = 1, k_2 = 1$ and $k_3 = -2$ which are not zero, the given vectors are linearly dependent.

Also we have the relation $x_1 + x_2 - 2x_3 = 0$

by means of which any of the given vectors can be expressed as a linear combination of the others.

- 97.** Are the vectors $x_1 = (1, 2, 4), x_2 = (2, -1, 3), x_3 = (0, 1, 2), x_4 = (-3, 7, 2)$ linearly dependent? If so, find the relation between them. (U.P.T.U., 2003 ; Nagpur, 2001)

Sol. The relation $k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$

i.e., $k_1(1, 2, 4) + k_2(2, -1, 3) + k_3(0, 1, 2) + k_4(-3, 7, 2) = 0$

is equivalent to $k_1 + 2k_2 - 3k_4 = 0 \quad \dots(1)$

$$2k_1 - k_2 + k_3 + 7k_4 = 0 \quad \dots(2)$$

$$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0 \quad \dots(3)$$

Thus we have a homogeneous system of 3 linear equations in 4 unknowns. The co-efficient matrix,

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \text{ is } 3 \times 4$$

Rank (A) $\leq \min(3, 4) = 3$. Since rank of A is less than the number of unknowns, the homogeneous system has infinitely many non-zero solution.

Thus there exist scalars k_1, k_2, k_3, k_4 not all zero, such that

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

$\therefore$ The vectors x_1, x_2, x_3, x_4 are linearly dependent.

Now multiplying equation (2) by 2 and subtracting from (3), we have

$$5k_2 - 12k_4 = 0 \Rightarrow k_2 = \frac{12}{5}k_4$$

From equation (1), $k_1 = 3k_4 - 2k_2 = 3k_4 - \frac{24}{5}k_4 = -\frac{9}{5}k_4$

From equation (2), $k_3 = -2k_1 + k_2 - 7k_4 = \frac{18}{5}k_4 + \frac{12}{5}k_4 - 7k_4 = -k_4$

Thus, $k_1 = -\frac{9}{5}k_4, k_2 = \frac{12}{5}k_4, k_3 = -k_4$ and k_4 is arbitrary.

Taking $k_4 = -5$, we have $k_1 = 9, k_2 = -12, k_3 = 5$

$\therefore 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

Since, k_1, k_2, k_3, k_4 are all non-zero, we can express each of the given vectors as a linear combination of the remaining three.

$$9x_1 = 12x_2 - 5x_3 + 5x_4$$

$\therefore x_1 = \frac{4}{3}x_2 - \frac{5}{9}x_3 + \frac{5}{9}x_4$

The elements in the 4 columns of matrix A are the components of the 4 given vectors, as we can see.

98. Are the following vectors $x_1 = (1, 2, 1), x_2 = (2, 1, 4), x_3 = (4, 5, 6)$ linearly dependent? If so, find a relation between them. (U.P.T.U., 2008)

Sol. The relation $k_1x_1 + k_2x_2 + k_3x_3 = 0$

i.e., $k_1(1, 2, 1) + k_2(2, 1, 4) + k_3(4, 5, 6) = 0$

is equivalent to $k_1 + 2k_2 + 4k_3 = 0$

$$2k_1 + k_2 + 5k_3 = 0$$

$$k_1 + 4k_2 + 6k_3 = 0$$

These are satisfied by $k_1 = 2, k_2 = 1, k_3 = -1$ which are not zero, the given vectors are linearly dependent. Also we have the relation

$$2x_1 + x_2 - x_3 = 0 \quad \text{or} \quad x_3 = 2x_1 + x_2.$$

99. Are the following vectors $x_1 = (2, -1, 4), x_2 = (0, 1, 2), x_3 = (6, -1, 16)$ linearly dependent? If so, find relation between them.

Sol. The relation $k_1x_1 + k_2x_2 + k_3x_3 = 0$

i.e., $k_1(2, -1, 4) + k_2(0, 1, 2) + k_3(6, -1, 16) = 0$ is equivalent to

$$2k_1 + 6k_3 = 0$$

$$-k_1 + k_2 - k_3 = 0$$

$$4k_1 + 2k_2 + 16k_3 = 0$$

These are satisfied by $k_1 = 3, k_2 = 2, k_3 = -1$ which are not zero, therefore the given vectors are linearly dependent.

Also we have the relation

$$3x_1 + 2x_2 - x_3 = 0$$

or

$$x_3 = 3x_1 + 2x_2,$$

by means of which any of the given vectors can be expressed as a linear combination of the others.



Applications of Differentiation

IMPORTANT DEFINITIONS AND FORMULAE

1. Branches of a Curve:

Consider the rectangular hyperbola

$$x^2 - y^2 = a^2$$

Solving for y , we get $y = \pm \sqrt{x^2 - a^2}$

If $x \rightarrow \pm \infty$, y also tends to $\pm \infty$.

In this case both the branches extend to infinity and are said to be the **infinite branches** of the rectangular hyperbola.

Now consider the curve $x^2 y^2 = x^2 - y^2$.

Solving for y , we get

$$y = \pm \frac{x}{\sqrt{x^2 + 1}}$$

Here, as $x \rightarrow \pm \infty$, $y \rightarrow \pm 1$ along the first branch.

Also in the case of second branch *i.e.*,

when $y = -\frac{x}{\sqrt{x^2 + 1}}$

as $x \rightarrow \infty$, $y \rightarrow -1$ and as $x \rightarrow -\infty$, $y \rightarrow 1$. Here both the branches are infinite and x is capable of taking arbitrarily large values whereas y remains finite.

Definition: A point $P(x, y)$ on an infinite branch of a curve **is said to tend to infinity along the curve** if either x or y or both tend to $+\infty$ or $-\infty$ as P travels along the branch of the curve.

2. Asymptote:

- (i) A **straight line**, at a finite distance from the origin is said to be an **Asymptote** to an infinite branch of a curve, if the perpendicular distance of a point P on that branch from the straight line tends to zero, as P tends to ∞ along the branch.

Thus, an asymptote to an infinite branch of a curve **is the limiting position of the tangent whose point of contact tends to infinity along the branch**, but which itself remains at a finite distance from the origin.

- (ii) If a straight line cuts a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity, it is called an asymptote to the curve.
- (iii) The asymptotes may be *parallel* to either x -axis or y -axis and accordingly they are called **horizontal and vertical asymptotes**. If an asymptote is **not** parallel to y -axis; it is called an **oblique asymptote**.

3. Working Rule to find Asymptotes Parallel to the axes (y-axis or x-axis)

Case I: Asymptotes Parallel to y-axis: Equate to zero the co-efficient of the highest power of y , present in the given equation of the curve.

Resolve it now into real linear factors. These are the asymptotes required. In case the co-efficient of the highest power of y is either a constant or not resolvable into real linear factors ; then there are No asymptotes parallel to y -axis in the given equation of the curve.

Case II: Asymptotes parallel to x-axis for the curve $f(x, y) = 0$: Equate to zero the co-efficient of the highest power of x , present in the given equation of the curve. Resolve into real linear factors. If the above co-efficient is either a *constant or not resolvable into real linear factors*, then there are *No asymptotes parallel to x-axis*.

4. Oblique Asymptotes: If $y = mx + c$ is an oblique asymptote to any curve $f(x, y) = 0$; then

$$m = \lim_{x \rightarrow \infty} \frac{y}{x} \text{ and } c = \lim_{x \rightarrow \infty} (y - mx)$$

This requires m and c to be finite.

In particular m may be zero and therefore the asymptotes parallel to x -axis can also be found by using this method.

5. Working Rule to find Asymptotes of Algebraic Curves: Put $mx + c$ for y in the given equation of the curve and arrange it in descending powers of x . Equate the coefficients of two highest degree terms to zero.

Solve these equations to find the values of m and c .

For every pair of values of m and c , there is one asymptote $y = mx + c$ of the given curve.

6. Rule to find Oblique Asymptotes of the General Rational Algebraic Curve of n^{th} Degree:

Step I: Put $y = m, x = 1$ in the highest degree terms of the equation of given curve to find $\phi_n(m)$, a polynomial.

Step II: Put $\phi_n(m) = 0$ and solve for m , let the roots be $m_1, m_2, m_3, \dots$

Step III: Put $x = 1, y = m$ in the next lower degree terms of the given equation to find the polynomial $\phi_{n-1}(m)$. Similarly we get $\phi_{n-2}(m)$ after putting $x = 1, y = m$ in the next lower degree terms of the equation, and so on.

Step IV: Find the values $c_1, c_2, c_3, \dots$ corresponding to values $m_1, m_2, m_3, \dots$ by using the relation

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} \quad [\text{provided } \phi_n'(m) \neq 0]$$

Step V: The required asymptotes are

$$y = m_1x + c_1, y = m_2x + c_2,$$

$$y = m_3x + c_3, \dots \dots$$

Step VI: If $\phi_n'(m) = 0$ for some value of m , but $\phi_{n-1}(m) \neq 0$, then corresponding to that value of m there is no asymptote.

Step VII: If $\phi_n'(m) = 0 = \phi_{n-1}(m)$, for some value of m (i.e., **two roots of $\phi_n(m) = 0$, are equal**), then the values of c are found from the equation

$$\frac{c^2}{2!} \cdot \phi_n''(m) + c \cdot \phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

This gives two values of c and therefore there are two parallel asymptotes corresponding to this value of m .

Step VIII: If the three roots of $\phi_n(m) = 0$ are equal;

[i.e., $\phi_n''(m) = 0, \phi_{n-1}'(m) = 0, \phi_{n-2}(m) = 0$], then values of c corresponding to that value of m are obtained from the equation;

$$\frac{c^3}{3!} \cdot \phi_n'''(m) + \frac{c^2}{2!} \phi_{n-1}''(m) + c \cdot \phi_{n-2}'(m) + \phi_{n-3}(m) = 0.$$

Note: When it is required to find the asymptote of a curve, it is advisable to first find the asymptotes parallel to axes, if there be any, and then search for oblique asymptotes.

7. **Asymptotes by Inspection:** If equation of a curve of degree n , is of the form $F_n + F_{n-2} = 0$ where F_n is of degree n [i.e., contains terms of degree n and may also contain lower degree terms], and F_{n-2} is of degree $(n - 2)$ at the most, **then every linear factor of F_n equated to zero, will be an asymptote, provided no two linear factors of F_n are either coincident or differ by a constant.**

Example: Find **asymptotes** of the curve whose equation is $xy(x^2 - y^2) + 2x^2 + 2y^2 + 1 = 0$

Sol. Equation is of the form $F_n + F_{n-2} = 0$ ($n = 4$). The linear factors of $F_4 = xy(x^2 - y^2)$ are $x, y, x - y$ and $x + y$. Since none of them is repeated, **the four asymptotes of the curve are $x = 0, y = 0, x - y = 0$, and $x + y = 0$.**

8. **Radius of Curvature at the Origin:**

(a) **Newton's Method:** If a curve passes through the origin and axis of x is the tangent at the origin, then

$$\rho \text{ (at the origin)} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{x^2}{2y} \right)$$

In case y -axis is the tangent at the origin, then

$$\rho \text{ (at the origin)} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{y^2}{2x} \right)$$

Radius of curvature at the origin when polar equation of the curve ($x = r \cos \theta, y = r \sin \theta$)

is given, may be calculated using formula: $\rho = \lim_{\substack{r \rightarrow 0 \\ \theta \rightarrow 0}} \left(\frac{1}{2} \cdot \frac{dr}{d\theta} \right)$

(b) **Method of Expansion:** Is used when x -axis or the y -axis is not the tangent at the origin.

Let $y = f(x)$ be equation of the curve.

Then, ρ (at the origin) is given by $\frac{(1 + p^2)^{3/2}}{q}$

To find the values of p and q ,

Let the equation of the curve be

$$y = px + q \frac{x^2}{2!} + \dots$$

Put this value of y in the given equation of the curve and equate coefficients of like powers of x on the two sides.

Note: Newton's method for finding the radius of curvature at the origin should be used when x -axis or y -axis is the tangent at the origin. Tangents at the origin to an algebraic curve are obtained by equating to zero the lowest degree terms.

9. **Evolute of the Curve:** Locus of the centres of curvature of the given curve, **is called the Evolute** of the curve.
10. **Rule to Find Evolute of a Curve:**

(i) Let (X, Y) be the coordinates of the centre of curvature, then

$$X = x - \frac{y_1(1 + y_1^2)}{y_2},$$

$$Y = y + \frac{1 + y_1^2}{y_2}$$

where (x, y) is any point on the curve $y = f(x)$

(ii) Eliminate the parameters x and y to find a relation between X and Y .

(iii) Generalise X and Y to get the required equation of the evolute.

For example, *Solution* to Questions 130–134 describe in detail and illustrate the method for finding *evolute* of the given curve.

11. Procedure for Tracing of Cartesian Curves:

(a) Symmetry:

(i) **Symmetry about the x -axis.** If the equation of a curve remains unaltered when y is changed to $-y$ i.e., if only even powers of y occur in the equation, then the curve is symmetrical about x -axis. Thus the parabola $y^2 = 4ax$ is symmetrical about the x -axis.

(ii) The curve $x^2 = 4y$ is symmetrical about the y -axis, since only even powers of x occur in the equation of the curve.

(iii) *Symmetry in opposite quadrants:*

If the equation of the curve remains unaltered when both x and y are changed to $-x$ and $-y$ respectively, the curve is symmetrical in opposite quadrants. Thus the curve $xy = c^2$ is symmetrical in opposite quadrants.

(iv) *Symmetry about the line $y = x$:* If the equation of the curve is **unchanged** when x and y are **interchanged**, the curve is symmetrical about the line $y = x$. Thus, the curve $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

(b) Origin:

(i) Find whether the curve passes through the origin or not. It will pass through the origin if the equation of the curve is free from the constant term.

(ii) If the curve passes through the origin, find the equations of the tangents at the origin by equating to zero the lowest degree terms in the equation of the curve.

(iii) If there is only one tangent, find whether the curve lies below or above the tangent in the neighbourhood of the origin. If there are two tangents, then origin is a double point. Find the **nature** of the double point.

If the two tangents are **real and distinct**, then origin is a **node**.

If the two tangents are **real and coincident**, then origin is a **cusp**.

If the two tangents are imaginary, then origin is a conjugate point (or an isolated point).

(c) Axes Intersection:

Find the points where the curve cuts the coordinate axes. Find also the tangents at these points if necessary and the position of the curve relative to these tangents. If $y = x$ or $y = -x$ is a line of symmetry, find the point of intersection of the curve and the line and also the tangents at that point.

(d) Asymptotes:

(i) Find all the asymptotes (parallel to the x -axis, y -axis and oblique) as was described in detail in **previous sections 5 and 6**.

(ii) Find the **position of the curve** w.r.t. oblique asymptotes (i.e., whether the **curve lies above or below the asymptote**).

If the equation of a curve is of the form $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots$, then $y = mx + c$ is an asymptote to the curve.

If y_c and y_a stand for the ordinates of a point on the curve and a point on the asymptote

(both having same abscissa), then $y_c - y_a = \frac{A}{x} + \frac{B}{x^2} + \dots$

If $y_c - y_a > 0$, the curve lies above the asymptote.

If $y_c - y_a < 0$, the curve lies below the asymptote.

(e) Special Points:

- (i) Find the points on the curve where $\frac{dy}{dx} = 0$ or ∞ i.e., the points where the tangent is $\parallel$ or $\perp$ to the x -axis.

At such points, the ordinates or abscissae generally change the character from increasing to decreasing or vice-versa.

- (ii) Find out the **points of inflexion**, if they exist and the multiple points, if any, and their nature, by finding 2nd and 3rd Derivatives.
- (iii) Find some special points on the curve if necessary.

(iv) Region

- (a) Find those regions of xy -plane where the curve does not exist. Such a region is obtained on solving the given equation for one variable in terms of the other variable and finding out the set of values of one variable which makes the other imaginary or undefined.

Consider the curve $y^2(2a - x) = x^3$

Here,
$$y = \pm x \sqrt{\frac{x}{2a - x}}$$

when x is +ve and $> 2a$, y is imaginary ($\because 2a - x$ is -ve) and so no part of the curve lies to the right of the line $x = 2a$. Again when $x < 0$, y is again imaginary, so the curve **does not lie to the left of the line $x = 0$ i.e., y -axis**

- (b) Solve the equation for y (say) in terms of x . Consider $x = 0$ and observe how y varies as x increases and finally tends to $+\infty$, paying special attention to those values of x for which $y = 0$ or $y \rightarrow \infty$.

If as $x \rightarrow a$, $y \rightarrow 0$, then the special point is $(a, 0)$. Shift the origin to the point $(a, 0)$ and study the curve at the new origin. Again if as $x \rightarrow b$, $y \rightarrow \infty$, then $x = b$ is an asymptote $\parallel$ to y -axis and we draw the shape of the curve near the line $x = b$.

- (c) Similarly observe the variations of y as x decreases from 0 to $-\infty$. But if the curve is symmetrical about y -axis, only +ve values of x need be considered and the curve for -ve values of x is traced by symmetry.
- (d) For limitations of the curve, we sometimes change from the cartesian to polar or parametric form.

12. Procedure For Tracing Polar Curves:

Let the equation of the curve be $f(r, \theta) = 0$

(a) Symmetry

- (i) *Symmetry about the initial line or x -axis*

If the equation of the curve remains unchanged when θ is changed to $-\theta$, the curve is symmetrical about the initial line. Thus the curve $r = a(1 + \cos \theta)$ is symmetrical about the initial line.

- (ii) *Symmetry about the line $\theta = \pi/2$ or y -axis*

If the equation of the curve remains unaltered when θ is changed to $\pi - \theta$, or when θ is changed to $-\theta$ and r to $-r$, the curve is symmetrical about the line $\theta = \pi/2$ or y -axis.

- (iii) *Symmetry about the line $\theta = \pi/4$ or the line $y = x$.*

When θ is changed to $\frac{\pi}{2} - \theta$ and equation of the curve remains unchanged, the curve is said to be symmetrical about the line $\theta = \pi/4$.

Thus the curve $r = a \sin 2\theta$ is symmetrical about the line $\theta = \frac{\pi}{4}$ or the line $y = x$.

(iv) *Symmetry about the line $\theta = \frac{3\pi}{4}$ or the line $y = -x$.*

When θ is changed to $\frac{3\pi}{2} - \theta$ and the equation of the curve remains unchanged, then the

curve is symmetrical about the line $\theta = \frac{3\pi}{4}$. For example, the curve $r = a \sin 2\theta$ is symmetrical

about the line $\theta = \frac{3\pi}{4}$.

(v) *Symmetry about the pole:* When r is changed to $-r$ and equation of the curve remains unchanged, then the curve is said to be symmetrical about the pole. Thus, the curve $r^2 = a^2 \cos 2\theta$ is symmetrical about the pole.

(b) **Origin or Pole**

(i) *Find whether the curve passes through the pole or not.* Put $r = 0$ in the equation and then find some real values of θ , if it is not possible to find a real value of θ for which $r = 0$, then the curve does not pass through the pole.

(ii) *Find tangents at the pole.* Putting $r = 0$, the real values of θ give the tangents at the pole. For

example, consider the curve $r = a(1 - \sin \theta)$. Putting $r = 0$, we get $\sin \theta = 1$ or $\theta = \frac{\pi}{2}$. Hence the

curve passes through the pole and $\theta = \frac{\pi}{2}$ is the tangent at the pole.

(iii) Find the points where the curve meets the initial line and the line $\theta = \frac{\pi}{2}$.

(c) **Value of ϕ :**

(i) Find $\tan \phi = r \frac{d\theta}{dr}$ and hence ϕ .

(ii) Find the points where $\theta = 0$ or $\theta = \frac{\pi}{2}$.

(d) **Asymptotes:**

If $r \rightarrow \infty$ and $\theta \rightarrow \theta_1$, then there is an asymptote. Find it by usual method. (θ_1 is a fixed number)

Working Rule for finding the Asymptotes of Polar Curves:

1. Write down the equation as $\frac{1}{r} = f(\theta)$
2. Equate $f(\theta)$ to zero and solve for θ . Let the roots be $\theta_1, \theta_2, \theta_3 \dots$
3. Find $f'(\theta)$ and calculate $f'(\theta)$ at $\theta = \theta_1, \theta_2, \theta_3 \dots$
4. Then write Asymptotes as

$$r \sin (\theta - \theta_1) = \frac{1}{f'(\theta_1)}, r \sin (\theta - \theta_2) = \frac{1}{f'(\theta_2)}, \dots$$

If $f'(\theta_1) = 0$ or ∞ , the curve has No asymptote.

- (e) **Special Points.** For convenient values of θ , find some points on the curve.
- (f) **Region:** Solve the given equation for r or θ . Find the regions in which the curve does not lie. This can be done in the following manner:
- (i) If r is imaginary when $\alpha < \theta < \beta$, then there is no part of the curve between the lines $\theta = \alpha$ and $\theta = \beta$. Consider the curve $r^2 = a^2 \sin 2\theta$. When $\frac{\pi}{2} < \theta < \pi$, $\sin 2\theta$ is -ve, $\therefore r^2$ is -ve and so r is imaginary. Hence the curve does not lie between the lines $\theta = \frac{\pi}{2}$ and $\theta = \pi$.
- (ii) If the greatest numerical value of r be a , the curve lies entirely within the circle $r = a$. Also, if the least numerical value of r be b , the curve lies outside the circle $r = b$.
- (iii) **Trace the variations of r** when θ varies in the intervals $(0, \infty)$ and $(-\infty, 0)$ marking values of θ for which $r = 0$ or attains a minimum or maximum value. Plot the points so obtained.
[Note: When r is a periodic function of θ , the -ve values of θ need not be taken into account. We may consider values from $\theta = 0$ to those values of θ , where the values begin to repeat.]
- (g) **Conversion into Cartesians:** Transform the equation to Cartesian co-ordinates whenever required.

13. Calculation of n^{th} Order Derivatives:

- (i) n^{th} differential co-efficient (or derivative) of x^m ($n < m$) is equal to $m(m-1)(m-2) \dots (m-n+1) x^{m-n}$
 If m be a positive integer, and if $n = m$ then $y_m = m(m-1) \dots 3.2.1$ or $m!$ (which is constant)
- (ii) **Table showing n^{th} Derivative of Some Standard Functions**

Table 'A'

Function	n^{th} Derivative
1. $(ax + b)^m, n < m$ - Do - when $n = m$	$m(m-1)(m-2) \dots (m-n+1) (ax + b)^{m-n} \cdot a^n$ $m! a^m$
2. $\frac{1}{ax + b} \left[x \neq -\frac{b}{a} \right]$ $\frac{1}{ax + b}, b = 0$	$\frac{(-1)^n \cdot n! a^n}{(ax + b)^{n+1}}$ $\frac{(-1)^n \cdot n!}{a \cdot x^{n+1}}$
3. $\log(ax + b)$ $\log x$	$\frac{(-1)^{n-1} \cdot (n-1)! a^n}{(ax + b)^n}$ $\frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$
4. a^{mx} e^{mx}	$m^n \cdot a^{mx} (\log a)^n$ $m^n \cdot e^{mx}$
5. $\sin(ax + b)$ If $b = 0$, $\sin ax$	$a^n \cdot \sin \left[ax + b + n \cdot \frac{\pi}{2} \right]$ $a^n \cdot \sin \left(ax + n \cdot \frac{\pi}{2} \right)$

6. $\cos(ax + b)$	$a^n \cdot \cos\left(ax + b + n \cdot \frac{\pi}{2}\right)$
$\cos ax$	$a^n \cdot \cos\left(ax + n \cdot \frac{\pi}{2}\right)$
$e^{ax} \cdot \sin(bx + c)$	$(a^2 + b^2)^{n/2} \cdot e^{ax} \cdot \sin\left[bx + c + n \cdot \tan^{-1} \frac{b}{a}\right]$
$e^{ax} \cdot \cos(bx + c)$	$(a^2 + b^2)^{n/2} \cdot e^{ax} \cdot \cos\left[bx + c + n \cdot \tan^{-1} \frac{b}{a}\right]$
$e^x \cdot \sin x$	$2^{n/2} \cdot e^x \cdot \sin\left[x + n \cdot \frac{\pi}{4}\right]$
$e^x \cdot \cos x$	$2^{n/2} \cdot e^x \cdot \cos\left[x + n \cdot \frac{\pi}{4}\right]$

(iii) **n^{th} Derivative of Any Rational Function by Use of Partial Fraction:**

Forming partial fractions is the converse process of taking L.C.M. **To resolve a fraction into partial fractions**, the **degree** of numerator must be **less than** the degree of denominator.

Partial fractions for :

$$\begin{aligned}
 (a) \quad \frac{f(x)}{(x-a)(x-b)(x-c)} & \text{ are } \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \\
 (b) \quad \frac{f(x)}{(x-a)^2(x-b)} & \text{ are } \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b} \\
 (c) \quad \frac{f(x)}{(x-a)^3(x-b)} & \text{ are } \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)} \\
 (d) \quad \frac{f(x)}{(x-a)(x-b)(px^2+qx+r)} & = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+D}{px^2+qx+r}
 \end{aligned}$$

To find A, B, C, D etc. we put each linear factor of L.C.M. equal to zero. Remaining constants are obtained by Comparing co-efficients of like powers on both sides.

(iv) **Some Important Formulae Used:**

1. If $\sin \theta = 0$, then $\theta = n\pi$
2. If $\cos \theta = 0$, then $\theta = (2n+1) \frac{\pi}{2}$
3. If $\sin \theta = \sin \alpha$, then $\theta = n\pi + (-1)^n \alpha$.
4. If $\cos \theta = \cos \alpha$, then $\theta = 2n\pi \pm \alpha$
5. If $\tan \theta = \tan \alpha$, then $\theta = n\pi + \alpha$
6. $\sin(n\pi + \theta) = (-1)^n \sin \theta$.
7. $\cos(n\pi + \theta) = (-1)^n \cos \theta$.
8. $\tan(n\pi + \theta) = \tan \theta$

where $n \in \mathbb{I}$

SOLVED PROBLEMS

1. State Rolle's Theorem, Lagrange's Mean Value Theorem, Taylor's Theorem with Lagrange's form of Remainder after n terms, Maclaurin's Theorem with Lagrange's Form of Remainder after n terms, Taylor's Infinite series and method of application of Taylor's Infinite series to expand $f(x + h)$. Hence, use Taylor's series to prove that :

$$\log \sin(x + h) = \log \sin x + h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x + \frac{h^3}{3} \cot x \cdot \operatorname{cosec}^2 x + \dots$$

Sol. A. Rolle's Theorem :

If a function $f(x)$ is such that

- (i) it is continuous in the closed interval $[a, b]$
- (ii) it is derivable in the open interval (a, b)
- (iii) $f(a) = f(b)$, then there exists at least one value 'c' of x in the open interval (a, b) such that $f'(c) = 0$.

Another Form of Rolle's Theorem :

If a function $f(x)$ is such that

- (i) it is continuous in the closed interval $[a, a + h]$
- (ii) it is derivable in the open interval $(a, a + h)$
- (iii) $f(a) = f(a + h)$

then there exists at least one number θ such that $f'(a + \theta h) = 0$, $0 < \theta < 1$.

B. Lagrange's Mean Value Theorem :

If a function $f(x)$ is such that

- (i) it is continuous in the closed interval $[a, b]$
- (ii) it is derivable in the open interval (a, b) , then there exists at least one value 'c' of x in the

open interval (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

Another Form of Lagrange's Mean Value Theorem :

If a function $f(x)$ is such that

- (i) it is continuous in the closed interval $[a, a + h]$
- (ii) it is derivable in the open interval $(a, a + h)$ then there exists at least one number θ such that

$$f(a + h) = f(a) + hf'(a + \theta h), \text{ where } 0 < \theta < 1.$$

C. Taylor's Theorem with Lagrange's Form of Remainder after n terms :

If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \dots, f^{n-1}(x)$ are continuous in the closed interval $[a, a + h]$
2. $f^n(x)$ exists in the open interval $(a, a + h)$, then there exists at least one number θ between 0 and 1 such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h).$$

D. Maclaurin's Theorem with Lagrange's Form of Remainder after n terms :

If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \dots, f^{n-1}(x)$ are continuous in the closed interval $[0, x]$
2. $f^n(x)$ exists in the open interval $(0, x)$ then there exists at least one number θ between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(0)$$

Thus we can get by putting $a = 0$ and $h = x$ in Taylor's Theorem.

E. Taylor's Infinite Series :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \dots \infty$$

F. Method of Application of Taylor's Infinite Series to Expand $f(x+h)$:

1. Let $f(x+h)$ = given function
2. Put $h = 0$ and find $f(x)$
3. Differentiate $f(x)$ a number of times and obtain $f'(x), f''(x), f'''(x), \dots$
4. Substitute the values of $f(x), f'(x), f''(x), \dots$ in $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$

To expand $\log \sin(x+h)$

Let $f(x+h) = \log \sin(x+h)$

Putting $h = 0, f(x) = \log \sin x$

$$\begin{aligned} \therefore f'(x) &= \frac{\cos x}{\sin x} = \cot x, \\ f''(x) &= -\operatorname{cosec}^2 x \\ f'''(x) &= -2 \operatorname{cosec} x (-\operatorname{cosec} x \cot x) = 2 \operatorname{cosec}^2 x \cdot \cot x. \end{aligned}$$

$$\begin{aligned} \therefore \log \sin(x+h) &= f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \\ &= \log \sin x + h \cot x + \frac{h^2}{2!}(-\operatorname{cosec}^2 x) + \frac{h^3}{3} \operatorname{cosec}^2 x \cdot \cot x + \dots \end{aligned}$$

Hence,

$$\log \sin(x+h) = \log \sin x + h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x + \frac{h^3}{3} \operatorname{cosec}^2 x \cot x + \dots$$

2. Use Taylor's series to prove that :

$$\sec^{-1}(x+h) = \sec^{-1}x + \frac{h}{x\sqrt{x^2-1}} - \frac{h^2}{2!} \cdot \frac{2x^2-1}{x^2(x^2-1)^{3/2}} + \dots$$

Sol. Let $f(x+h) = \sec^{-1}(x+h)$

Putting $h = 0, f(x) = \sec^{-1}x$

$$\begin{aligned} f'(x) &= \frac{1}{x\sqrt{x^2-1}} \\ f''(x) &= \frac{d}{dx} [x^{-1} \cdot (x^2-1)^{-1/2}] = -x^{-2} \cdot (x^2-1)^{-1/2} + x^{-1} \cdot \left(-\frac{1}{2}\right)(x^2-1)^{-3/2} (2x) \\ &= -\frac{1}{x^2(x^2-1)^{1/2}} - \frac{1}{(x^2-1)^{3/2}} = \frac{x^2-1+x^2}{x^2(x^2-1)^{3/2}} = \frac{2x^2-1}{x^2(x^2-1)^{3/2}} \end{aligned}$$

$$\begin{aligned} \therefore \sec^{-1}(x+h) &= f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots \\ &= \sec^{-1}x + \frac{h}{x\sqrt{x^2-1}} - \frac{h^2}{2!} \cdot \frac{2x^2-1}{x^2(x^2-1)^{3/2}} + \dots \end{aligned}$$

Hence proved.

3. Apply Taylor's series to prove that : $\cos\left(\frac{\pi}{4} + h\right) = \frac{1}{\sqrt{2}}\left(1 - h - \frac{h^2}{2!} + \frac{h^3}{3!} \dots\right)$

Sol. Let $f(x + h) = \cos(x + h)$

Putting $h = 0$, $f(x) = \cos x$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

.....

$$\begin{aligned} \therefore \cos(x + h) &= f(x + h) \\ &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \\ &= \cos x - h \sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \dots \end{aligned}$$

Now putting $x = \frac{\pi}{4}$, we have

$$\cos\left(\frac{\pi}{4} + h\right) = \cos \frac{\pi}{4} - h \sin \frac{\pi}{4} - \frac{h^2}{2!}\cos \frac{\pi}{4} + \frac{h^3}{3!}\sin \frac{\pi}{4} + \dots$$

Since $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$$\Rightarrow \cos\left(\frac{\pi}{4} + h\right) = \frac{1}{\sqrt{2}}\left[1 - h - \frac{h^2}{2!} + \frac{h^3}{3!} + \dots\right] \quad \text{Hence proved.}$$

4. Expand $\log \sin x$ in powers of $x - 3$.

Sol. Let $f(x) = \log \sin x$

$$\log \sin x = f(x) = f[3 + (x - 3)] = f(3 + h) \quad \text{where } h = x - 3$$

$$= f(3) + hf'(3) + \frac{h^2}{2!}f''(3) + \frac{h^3}{3!}f'''(3) + \dots \quad \dots(1)$$

Now

$$f(x) = \log \sin x, \quad f(3) = \log \sin 3$$

$$f'(x) = \cot x, \quad f'(3) = \cot 3$$

$$f''(x) = -\operatorname{cosec}^2 x, \quad f''(3) = -\operatorname{cosec}^2 3.$$

$$f'''(x) = -2 \operatorname{cosec} x (-\operatorname{cosec} x \cot x)$$

or

$$f'''(x) = 2 \operatorname{cosec}^2 x \cdot \cot x$$

$\therefore$

$$f'''(3) = 2 \operatorname{cosec}^2 3 \cdot \cot 3$$

$\therefore$ From (1),

$$\begin{aligned} \log \sin x &= \log \sin 3 + (x - 3) \cot 3 - \frac{(x - 3)^2}{2} \operatorname{cosec}^2 3 \\ &\quad + \frac{(x - 3)^3}{3} \operatorname{cosec}^2 3 \cdot \cot 3 + \dots \end{aligned} \quad \text{Hence proved.}$$

5. Use Taylor's Theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$.

Sol. Let $f(x) = 2x^3 + 7x^2 + x - 6$

$$\therefore 2x^3 + 7x^2 + x - 6 = f(x)$$

$$= f[2 + (x - 2)] = f(2 + h), \quad \text{where } h = x - 2.$$

$$= f(2) + hf'(2) + \frac{h^2}{2!}f''(2) + \frac{h^3}{3!}f'''(2) \quad \dots(1)$$

as higher derivatives of $f(x)$ vanish.

$$\begin{array}{ll} \text{Now} & f(x) = 2x^3 + 7x^2 + x - 6, & f(2) = 16 + 28 + 2 - 6 = 40 \\ & f'(x) = 6x^2 + 14x + 1, & f'(2) = 53 \\ & f''(x) = 12x + 14, & f''(2) = 38 \\ & f'''(x) = 12, & f'''(2) = 12. \end{array}$$

$$\therefore \text{ From (1), } 2x^3 + 7x^2 + x - 6 = 40 + 53h + 38\frac{h^2}{2!} + \frac{12h^3}{3!}$$

$$\text{Thus } 2x^3 + 7x^2 + x - 6 = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3. \quad \text{Hence proved.}$$

6. If $f(x) = x^3 + 2x^2 - 5x + 11$, calculate the value of $f\left(\frac{9}{10}\right)$ by the application of Taylor's series for $f(x+h)$.

Sol. By Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$$

$$\text{Put } x = 1 \text{ and } h = -\frac{1}{10}$$

$$f\left(\frac{9}{10}\right) = f(1) - \frac{1}{10}f'(1) + \frac{1}{2!}\left(\frac{1}{10}\right)^2 f''(1) - \frac{1}{3!}\left(\frac{1}{10}\right)^3 f'''(1) + \dots \quad (A)$$

Here

$$\begin{array}{ll} f(x) = x^3 + 2x^2 - 5x + 11 & \therefore f(1) = 9 \\ f'(x) = 3x^2 + 4x - 5 & \therefore f'(1) = 2 \\ f''(x) = 6x + 4 & \therefore f''(1) = 10 \\ f'''(x) = 6 & \therefore f'''(1) = 6 \end{array}$$

All higher derivatives vanish

$\therefore$ From (A), we get

$$\begin{aligned} f\left(\frac{9}{10}\right) &= 9 - \frac{1}{10}(2) + \frac{1}{2} \cdot \frac{1}{100} \cdot (10) - \frac{1}{6} \left(\frac{1}{1000}\right)(6) \\ &= 9 - .2 + .05 - .001 = 8.849 \end{aligned}$$

7. Calculate the value of $\cos 32^\circ$ upto four decimal places, using Taylor's series.

(M.D.U., May 2009)

Sol. Let

$$f(x+h) = \cos(x+h)$$

Put $h = 0$, then

$$f(x) = \cos x$$

$$f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, \dots$$

$\Rightarrow$

$$\cos(x+h) = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$= \cos x - h\sin x + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}\sin x + \dots$$

Putting $x = 30^\circ$ and $h = 2^\circ = \frac{\pi}{90} = .03490$, we have

$$\begin{aligned}\cos 32^\circ &= \cos 30^\circ - .03490 \sin 30^\circ - \frac{(.03490)^2 \cos 30^\circ}{2} + \frac{(.03490)^3 \sin 30^\circ}{6} + \dots \\ &= \frac{\sqrt{3}}{2} - .03490 \times \frac{1}{2} - .00060 \times \frac{\sqrt{3}}{2} + .0000 + \dots \\ &= .86602 - .01745 - .00060 \times .86602 \\ &= .84857 - .00051 \\ &= .84806 \quad \text{or} \quad 0.8481 \quad \text{after rounding off to four decimal places.}\end{aligned}$$

8. Compute $\sqrt{10}$ to 4 decimal places using Taylor's series.

Sol. Let

$$f(x+h) = \sqrt{x+h}$$

Put $h = 0$,

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}}$$

$\therefore$

$$\begin{aligned}\sqrt{x+h} &= f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \\ &= \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \frac{h^3}{16x^2\sqrt{x}} + \dots\end{aligned}$$

Put $x = 9$ and $h = 1$

$\Rightarrow$

$$\begin{aligned}\sqrt{10} &= 3 + \frac{1}{2\sqrt{9}} - \frac{1}{8 \cdot 9 \cdot 3} + \frac{1}{16(81)(3)} + \dots \\ &= 3 + .16666 - .00463 + .00027 + \dots \\ &= 3.1623\end{aligned}$$

9. Compute approximately $\log_{10} 404$, it is given that $\log_{10} 4 = 0.6021$.

(M.D.U., Dec., 2006)

Sol. Let

$$f(x+h) = \log(x+h)$$

Put $h = 0$, then

$$f(x) = \log_e x$$

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{2}{x^3} \text{ and so on.}$$

$\therefore$

$$\begin{aligned}\log_e(x+h) &= f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \\ &= \log_e x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \dots\end{aligned}$$

Multiplying by $\log_{10} e$, we have

$$\log_{10}(x+h) = \log_{10}x + \log_{10}e \cdot \left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \dots \right)$$

Putting $x = 400$ and $h = 4$, we have

$$\begin{aligned} \log_{10}404 &= \log_{10}400 + .4343 \left[\frac{4}{400} - \frac{1}{2} \left(\frac{4}{400} \right)^2 + \frac{1}{3} \left(\frac{4}{400} \right)^3 \dots \right] \\ &= \log_{10}(4 \times 10^2) + .4343 \left[.01 - \frac{1}{2}(.0001) + \frac{1}{3}(.000001) \dots \right] \\ &= \log_{10}4 + 2\log_{10}10 + .4343[.01 - .00005 + .00000] \\ &= .6021 + 2 + .4343 \times .00995 \\ &= 2.6021 + .00432 \\ &= 2.6064 \end{aligned}$$

10. Prove, by using Taylor's series, the following :

$$\begin{aligned} \tan^{-1}(x+h) &= \tan^{-1}x + (h \sin z) \cdot \frac{\sin z}{1} - (h \sin z)^2 \cdot \frac{\sin 2z}{2} \\ &\quad + (h \sin z)^3 \cdot \frac{\sin 3z}{3} - \dots, \end{aligned}$$

where $z = \cot^{-1}x$.

Sol. Given

$$z = \cot^{-1}x \Rightarrow \cot z = x$$

$$\therefore \frac{dz}{dx} = -\frac{1}{1+x^2} = -\frac{1}{1+\cot^2 z} = -\frac{1}{\operatorname{cosec}^2 z} = -\sin^2 z$$

Now let

$$f(x+h) = \tan^{-1}(x+h)$$

Putting $h = 0$,

$$f(x) = \tan^{-1}x$$

$\Rightarrow$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \frac{1}{\operatorname{cosec}^2 z} = \sin^2 z.$$

$$\begin{aligned} f''(x) &= 2 \sin z \cos z \cdot \frac{dz}{dx} = \sin 2z \cdot (-\sin^2 z) \\ &= -\sin^2 z \sin 2z \end{aligned}$$

$$f'''(x) = \frac{d}{dz} \left\{ -\sin^2 z \cdot \sin 2z \right\} \cdot \frac{dz}{dx}$$

$$\begin{aligned} \Rightarrow f'''(x) &= -[\sin^2 z \cdot (2 \cos 2z) + \sin 2z \cdot 2 \sin z \cos z] \cdot \frac{dz}{dx} \\ &= -2 \sin z (\sin z \cos 2z + \cos z \sin 2z)(-\sin^2 z) \\ &= 2 \sin^3 z \cdot \sin (z + 2z) \quad [\text{Using } \sin(A+B) \text{ formula}] \\ &= 2 \sin^3 z \cdot \sin 3z \end{aligned}$$

and similarly for the other derivatives.

Now applying Taylor's series,

$$\begin{aligned} \therefore \tan^{-1}(x+h) &= f(x+h) \\ &= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \end{aligned}$$

$$\begin{aligned}
&= \tan^{-1} x + h \sin^2 z + \frac{h^2}{2} \cdot (-\sin^2 z \cdot \sin 2z) + \frac{h^3}{6} (2 \sin^3 z \cdot \sin 3z) + \dots \\
&= \tan^{-1} x + (h \sin z) \cdot \frac{\sin z}{1} - (h \sin z)^2 \cdot \frac{\sin 2z}{2} + (h \sin z)^3 \cdot \frac{\sin 3z}{3} - \dots
\end{aligned}$$

Hence proved.

11. Expand $\sin^{-1}(x+h)$ in powers of h .**Sol.** Let

$$f(x+h) = \sin^{-1}(x+h)$$

Put $h=0$,

$$f(x) = \sin^{-1} x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$f''(x) = -\frac{1}{2} (1-x^2)^{-3/2} (-2x)$$

or

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$\begin{aligned}
f'''(x) &= \frac{(1-x^2)^{3/2} \cdot -x \cdot \frac{3}{2} (1-x^2)^{1/2} \cdot (-2x)}{(1-x^2)^3} \\
&= \frac{(1-x^2)^{1/2} \cdot [1-x^2+3x^2]}{(1-x^2)^3} = \frac{1+2x^2}{(1-x^2)^{5/2}}
\end{aligned}$$

Now,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

 $\therefore$

$$\sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{\sqrt{1-x^2}} + \frac{h^2}{2!} \cdot \frac{x}{(1-x^2)^{3/2}} + \frac{h^3}{3!} \cdot \frac{(1+2x^2)}{(1-x^2)^{5/2}} + \dots$$

12. If $f(x) = x^3 + 8x^2 + 15x - 24$, calculate the value of $f\left(\frac{11}{10}\right)$ by application of Taylor's series.**Sol.** By Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

To find $f\left(\frac{11}{10}\right)$ or $f\left(1+\frac{1}{10}\right)$, let us put $x=1$ and $h=1/10$ in the above series. $\therefore$

$$f\left(\frac{11}{10}\right) = f(1) + \frac{1}{10} f'(1) + \frac{1}{10^2} \cdot \frac{1}{2!} f''(1) + \frac{1}{10^3} \cdot \frac{1}{3!} f'''(1) + \dots \quad \dots (1)$$

Now,

$$\begin{aligned}
f(x) &= x^3 + 8x^2 + 15x - 24, & \therefore f(1) &= 1 + 8 + 15 - 24 = 0 \\
f'(x) &= 3x^2 + 16x + 15, & \therefore f'(1) &= 3 + 16 + 15 = 34 \\
f''(x) &= 6x + 16, & \therefore f''(1) &= 22 \\
f'''(x) &= 6, & \therefore f'''(1) &= 6
\end{aligned}$$

Substituting the values of $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ in equation (1), we get

$$f\left(\frac{11}{10}\right) = 0 + 3.4 + .11 + .001 \\ = 3.511$$

13. Find the value of $f(2.001)$ with the help of Taylor's series where $f(x) = x^3 - 2x + 5$. Find the approximate change in the value of $f(x)$ when x changes from 2 to 2.001.

Sol. In the Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Let us put $x = 2$ and $h = .001$.

$$f(2.001) = f(2) + .001.f'(2) + \frac{(.001)^2}{2!}f''(2) + \frac{(.001)^3}{3!}f'''(2) + \dots \quad \dots(1)$$

Now,

$$\begin{aligned} f(x) &= x^3 - 2x + 5, & f(2) &= 9 \\ f'(x) &= 3x^2 - 2, & \therefore f'(2) &= 10 \\ f''(x) &= 6x, & \therefore f''(2) &= 12 \\ f'''(x) &= 6, & \therefore f'''(2) &= 6 \\ f^{(4)}(x) &= 0 & \therefore f^{(4)}(2) &= 0 \end{aligned}$$

Substituting these values in equation (1), we get

$$f(2.001) = 9 + (.001).10 + \frac{1}{2!}(.001)^2(12) + \frac{(.001)^3}{3!}(6)$$

$$\begin{aligned} f(2.001) &= 9 + .01 + .000006 + .000000001 \\ &= 9.010006001 \text{ or } 9.01 \text{ approximately} \end{aligned}$$

Approximate change in the value of

$$f(2.001) - f(2) = 9.01 - 9 = .01$$

14. Expand $\sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$. (M.D.U., May 2007, Dec., 2008)

Sol.

$$f(x) = \sin x$$

$$f(x) = f\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) = f(a+h)$$

where $a = \frac{\pi}{2}$ and $h = x - \frac{\pi}{2}$

$$f(x) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f^{(4)}(a) + \dots$$

Putting these values of a and h , we get

$$\sin x = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \dots \quad \dots(1)$$

Now, $f(x) = \sin x$, $\therefore$ $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$$f'\left(\frac{\pi}{2}\right) = \cos x, \quad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x, \quad f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x, \quad f''' \left(\frac{\pi}{2} \right) = 0$$

$$f^{iv}(x) = \sin x, \quad f^{iv} \left(\frac{\pi}{2} \right) = 1$$

Putting these values in the above equation (1), we get

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \dots$$

15. Expand $\log x$ in powers of $(x-1)$ by Taylor's Theorem and hence find the value of $\log_e(1.1)$.

Sol.

$$\begin{aligned} f(x) &= \log x = \log \{1 + (x-1)\} \\ &= \log(a+h), \text{ where } a=1, h=x-1 \end{aligned}$$

$$\text{Now, } f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}, \quad f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4}, \quad f^{iv}(1) = -6$$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots$$

$$\begin{aligned} \Rightarrow \log x &= 0 + (x-1) \cdot 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \dots \\ &= x-1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{(x-1)^4}{4} + \dots \end{aligned}$$

Putting $x = 1.1$ in the above equation, we get

$$\log 1.1 = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \dots$$

$$\Rightarrow \log 1.1 = 0.095305$$

16. Using Taylor's series, prove that $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots$

Sol. Here

$$f(x+h) = \log(x+h)$$

Putting $h=0$, then $f(x) = \log x$

$$f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3} \text{ and so on.}$$

$$\therefore \log(x+h) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$= \log(x) + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots$$

Hence proved.

17. Using Taylor's series, prove that $\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \dots$

Sol. $f(x+h) = \tan^{-1}(x+h)$

Putting $h = 0$, we get $f(x) = \tan^{-1}x$.

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -(1+x^2)^{-2}(2x)$$

$$\Rightarrow f''(x) = -\frac{2x}{(1+x^2)^2} \text{ and so on.}$$

$$\therefore \tan^{-1}(x+h) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$= \tan^{-1}x + \frac{h}{1+x^2} - \frac{h^2x}{(1+x^2)^2} + \dots$$

Hence proved.

18. Calculate the approximate value of $\sqrt{17}$ to four decimal places using Taylor's series.

Sol. Let $f(x+h) = \sqrt{x+h}$

Put $h = 0$, then $f(x) = \sqrt{x}$

$$f'(x) = -\frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}} \text{ and so on}$$

$$\therefore \sqrt{x+h} = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\Rightarrow \sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \frac{h^3}{16x^2\sqrt{x}} + \dots$$

Putting $x = 16$ and $h = 1$, we have

$$\begin{aligned} \sqrt{17} &= 4 + \frac{1}{8} - \frac{1}{8 \cdot 16 \cdot 4} + \frac{1}{16 \cdot 256 \cdot 4} + \dots \\ &= 4 + 0.125 - .00195 + .0000 \\ &= 4.12305 \quad \text{or} \quad 4.1230 \end{aligned}$$

19. What is Maclaurin's Infinite series? Using Maclaurin's series, expand $\tan x$ upto the term containing x^5 .

Sol. Putting $a = 0$ and $h = x$ in the Taylor's infinite series, we get Maclaurin's infinite series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

where $f(x)$ possesses derivatives of all orders in the interval $(0, x)$ and Maclaurin's remainder

$$R_n = \frac{x^n}{n!} f^n(0x) \text{ tends to } 0 \text{ as } n \rightarrow \infty.$$

Using Maclaurin's infinite series, let us now expand $\tan x$.

$$\begin{aligned} f(x) &= \tan x, & \therefore & f(0) = 0 \\ f'(x) &= \sec^2 x, & & \\ &= 1 + \tan^2 x, & & f'(0) = 1 \end{aligned}$$

$$\begin{aligned}
f''(x) &= 2 \tan x \cdot \sec^2 x \\
&= 2 \tan x (1 + \tan^2 x) \\
&= 2 \tan x + 2 \tan^3 x, \quad f''(0) = 0 \\
f'''(x) &= 2 \sec^2 x + 6 \tan^2 x \cdot \sec^2 x \\
&= 2 (1 + \tan^2 x) + 6 \tan^2 x \cdot (1 + \tan^2 x) \\
&= 2 + 2 \tan^2 x + 6 \tan^2 x + 6 \tan^4 x \\
&= 2 + 8 \tan^2 x + 6 \tan^4 x, \quad \therefore f'''(0) = 2. \\
f^{iv}(x) &= 16 \tan x \cdot \sec^2 x + 24 \tan^3 x \cdot \sec^2 x \\
&= 16 \tan x (1 + \tan^2 x) + 24 \tan^3 x \cdot (1 + \tan^2 x) \\
&= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x, \quad \therefore f^{iv}(0) = 0 \\
f^v(x) &= 16 \sec^2 x + 120 \tan^2 x \cdot \sec^2 x + 120 \tan^4 x \cdot \sec^2 x.
\end{aligned}$$

$\therefore f^v(0) = 16$ and so on.

Putting these values in the Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^v(0) + \dots$$

we get

$$\begin{aligned}
\tan x &= 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 2 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 16 + \dots \\
&= x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots
\end{aligned}$$

20. Using Maclaurin's series, expand $\log(1+x)$. Hence deduce that

$$\log \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

Sol. Let $f(x) = \log(1+x)$, $\therefore f(0) = \log 1 = 0$

$$f'(x) = \frac{1}{1+x}, \quad \therefore f'(0) = 1.$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad \therefore f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3}, \quad \therefore f'''(0) = 2$$

$$f^{iv}(x) = -\frac{6}{(1+x)^4}, \quad \therefore f^{iv}(0) = -6 \text{ and so on}$$

$$\begin{aligned}
\therefore \log(1+x) &= f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \\
&= x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots
\end{aligned}$$

Changing x to $-x$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\therefore \log \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} [\log(1+x) - \log(1-x)]$$

$$\begin{aligned}
&= \frac{1}{2} \left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \right) \right] \\
&= \frac{1}{2} \left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots \right] \\
&= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots
\end{aligned}$$

Hence proved.

21. Prove that:

$$\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots$$

Sol. Let

$$y = \sin^{-1} x, \quad \therefore y(0) = 0$$

$$y_1 = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$= 1 + \frac{1}{2} \cdot x^2 + \frac{\frac{1}{2} \left(\frac{1}{2} + 1 \right)}{2!} \cdot x^4 + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{3!} \cdot x^6 + \dots$$

(By Binomial Theorem)

 $\Rightarrow$

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots$$

 $\therefore$

$$y_1(0) = 1$$

$$y_2 = x + \frac{3}{2}x^3 + \frac{15}{8}x^5 + \dots, \quad y_2(0) = 0$$

$$y_3 = 1 + \frac{9}{2}x^2 + \frac{75}{8}x^4 + \dots, \quad y_3(0) = 1$$

$$y_4 = 9x + \frac{75}{2}x^3 + \dots, \quad y_4(0) = 0$$

$$y_5 = 9 + \frac{225}{2}x^2 + \dots, \quad y_5(0) = 9$$

$$y_6 = 225x + \dots, \quad y_7 = 225 + \dots, \quad y_6(0) = 0, y_7(0) = 0$$

 $\therefore$

$$\sin^{-1} x = y(0) + x \cdot y_1(0) + \frac{x^2}{2!} y_2(0) + \dots$$

$$= x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \frac{225 \cdot x^7}{7!} + \dots$$

$$= x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots \quad \text{Hence proved.}$$

Note : Putting $x = \frac{1}{2}$ on both sides, we get

$$\sin^{-1} \left(\frac{1}{2} \right) = \frac{1}{2} + \frac{1}{6} \left(\frac{1}{2} \right)^3 + \frac{3}{40} \left(\frac{1}{2} \right)^5 + \frac{5}{112} \left(\frac{1}{2} \right)^7 + \dots$$

$$\Rightarrow \frac{\pi}{6} = 0.5 + .0208 + .0023 + .0003 + \dots$$

$$\Rightarrow \frac{\pi}{6} = 0.5234$$

$$\Rightarrow \pi = 3.1404 \text{ or } 3.14 \text{ approximately.}$$

22. Prove that: $\log (\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \dots$

Sol. Let

$$y = \log (\sec x + \tan x), \quad y(0) = 0$$

$$y_1 = \sec x, \quad y_1(0) = 1$$

$$y_2 = \sec x \cdot \tan x, \quad y_2(0) = 0$$

$$= y_1 \tan x.$$

$$y_3 = y_2 \tan x + y_1 \sec^2 x = y_2 \tan x + y_1 \cdot y_1^2$$

$$= y_2 \tan x + y_1^3, \quad \therefore y_3(0) = 1$$

$$y_4 = y_3 \tan x + y_2 \sec^2 x + 3y_1^2 \cdot y_2$$

$$= y_3 \tan x + y_2 y_1^2 + 3y_1^2 \cdot y_2$$

$$y_4 = y_3 \tan x + 4y_1^2 \cdot y_2, \quad \therefore y_4(0) = 0 + 4 \cdot 1 \cdot 0 = 0$$

$$y_5 = y_4 \tan x + y_3 \sec^2 x + 8y_1 \cdot y_2^2 + 4y_1^2 \cdot y_3$$

$$= y_4 \tan x + y_3 \cdot y_1^2 + 8y_1 \cdot y_2^2 + 4y_1^2 \cdot y_3$$

$$\Rightarrow y_5 = y_4 \tan x + 5y_1^2 \cdot y_3 + 8y_1 \cdot y_2^2$$

$$\therefore y_5(0) = 5 \cdot 1^2 \cdot 1 = 5 \text{ and so on.}$$

Hence, $\log (\sec x + \tan x) = y$

$$= y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \dots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} \cdot 5 \dots$$

$$= x + \frac{x^3}{6} + \frac{x^5}{24} + \dots$$

Hence proved.

23. Discuss the use of some well-known series to find the successive derivatives of a function.

Prove that $e^x \cdot \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \dots$

Sol. When it is very difficult to find the successive derivatives of a function, the use of the following well-known series is useful and often convenient :

$$(i) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$(ii) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$(iii) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$(iv) \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$(v) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$(vi) \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$(vii) \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$(viii) \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Now, the given series $e^x \log(1+x)$ may be written as :

$$e^x \cdot \log(1+x) = \left(1+x+\frac{x^2}{2!}+\dots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)$$

or

$$\begin{aligned} e^x \log(1+x) &= x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}-\frac{x^2}{2}-\frac{x^3}{2}-\frac{x^4}{4}-\frac{x^5}{12}-\dots \\ &\quad +\frac{x^3}{3}+\frac{x^4}{3}+\frac{x^5}{6}+\dots-\frac{x^4}{4}-\frac{x^5}{4}-\dots+\frac{x^5}{5}+\dots \\ &= x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{3x^5}{40}+\dots \end{aligned}$$

Hence proved.

24. Expand $\log(1+\sin^2 x)$ in powers of x as far as the term in x^6 .

Sol.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\begin{aligned} \sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \dots\right)^2 = \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)\right]^2 \\ &= x^2 - 2x\left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right) + \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)^2 = x^2 - \frac{x^4}{3} + \frac{x^6}{60} + \frac{x^6}{36} + \dots \\ &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \dots = z \text{ (say)} \end{aligned}$$

$$\therefore \log(1+\sin^2 x) = \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

$$= \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots\right)^2 + \frac{1}{3}\left(x^2 - \dots\right)^3 - \dots$$

$$\begin{aligned} \log(1+\sin^2 x) &= \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots\right) \\ &\quad - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots\right) - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} - \dots\right) + \frac{1}{3}\left(x^2 - \dots\right)^3 \\ &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^6}{3} = x^2 - x^4\left(\frac{1}{3} + \frac{1}{2}\right) + x^6\left(\frac{2}{45} + \frac{1}{3} + \frac{1}{3}\right) \\ &= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 - \dots \end{aligned}$$

25. Find the first four terms in the expansion of $\log(1+\tan x)$.

Sol. Since

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

$\therefore$

$$\log(1+\tan x) = \tan x - \frac{1}{2}\tan^2 x + \frac{1}{3}\tan^3 x - \frac{1}{4}\tan^4 x + \dots$$

Now,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$\begin{aligned}
\therefore \log(1 + \tan x) &= \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right) - \frac{1}{2} \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right)^2 \\
&\quad + \frac{1}{3} \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right)^3 - \frac{1}{4} \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right)^4 \\
&= \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right) - \frac{1}{2} \left[x^2 + 2x \left(\frac{x^3}{3} + \dots\right)\right] + \frac{1}{3} [x^3 + \dots] - \frac{1}{4} [x^4 + \dots] + \dots \\
&= x + \frac{x^3}{3} - \frac{1}{2}x^2 - \frac{x^4}{3} + \frac{x^3}{3} - \frac{1}{4}x^4 + \dots \\
\therefore \log(1 + \tan x) &= x - \frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{7}{12}x^4 + \dots
\end{aligned}$$

26. Show that : $\tan^{-1} \frac{2x}{1-x^2} = 2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right)$.

Sol. Let $y = \tan^{-1} \frac{2x}{1-x^2}$

Put $x = \tan \theta$, so that $\theta = \tan^{-1} x$

$$\begin{aligned}
\Rightarrow y &= \tan^{-1} \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right) = \tan^{-1} (\tan 2\theta) \\
&= 2\theta = 2 \tan^{-1} x, \quad y(0) = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y_1 &= \frac{2}{1+x^2} = 2(1+x^2)^{-1} = 2(1-x^2+x^4-x^6+\dots), \quad \therefore y_1(0) = 2 \\
y_2 &= 2(-2x+4x^3-6x^5+\dots), & y_2(0) &= 0 \\
y_3 &= 2(-2+12x^2-30x^4+\dots), & y_3(0) &= -4 \\
y_4 &= 2(24x-120x^3+\dots), & y_4(0) &= 0 \\
y_5 &= 2(24-360x^2+\dots), & y_5(0) &= 48 \text{ and so on}
\end{aligned}$$

$$\begin{aligned}
\therefore \tan^{-1} \left(\frac{2x}{1-x^2} \right) &= y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots \\
&= 2x - 4 \frac{x^3}{3!} + 48 \frac{x^5}{5!} + \dots \\
&= 2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \right).
\end{aligned}$$

Hence proved.

27. Show that : $\tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^6}{12} + \dots$

Sol. Let $y = \tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$

Put $x^2 = \cos \theta$, so that $\theta = \cos^{-1} x^2$

Then,

$$y = \tan^{-1} \frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}} = \tan^{-1} \left[\frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \right]$$

$$= \tan^{-1} \left[\frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \right] = \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

or

$$y = \frac{\pi}{4} - \frac{\theta}{2}$$

$$y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x^2.$$

$\therefore$

$$y_1 = 0 - \frac{1}{2} \cdot \frac{-1}{\sqrt{1-x^4}} \cdot 2x$$

$$= x(1-x^4)^{-1/2} = x \left[1 + \frac{1}{2}x^4 + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!} \cdot x^8 + \dots \right]$$

or

$$y_1 = x + \frac{x^5}{2} + \frac{3}{8}x^9 + \dots$$

$$y_2 = 1 + \frac{5}{2}x^4 + \frac{27}{8}x^8 + \dots$$

$$y_3 = 10x^3 + 27x^7 + \dots$$

$$y_4 = 30x^2 + 189x^6 + \dots$$

$$y_5 = 60x + 1134x^5 + \dots$$

$$y_6 = 60 + 5670x^4 + \dots$$

$$\vdots$$

$\therefore y(0) = 0, y_1(0) = 0, y_2(0) = 1, y_3(0) = y_4(0) = y_5(0) = 0, y_6(0) = 60, \dots$

$\therefore$

$$\tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0)$$

$$+ \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \frac{x^6}{6!}y_6(0) + \dots$$

or

$$\tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^6}{12} + \dots$$

Hence proved.

28. Expand a^x and e^x in powers of x , by Maclaurin's theorem.

Sol. Let

$$f(x) = a^x$$

$$\therefore f(0) = a^0 = 1$$

$$f'(x) = a^x \log a \quad \Rightarrow \quad f'(0) = \log(a)$$

$$f''(x) = a^x (\log a)^2 \quad \Rightarrow \quad f''(0) = (\log a)^2$$

$$f'''(x) = a^x (\log a)^3 \quad \Rightarrow \quad f'''(0) = (\log a)^3$$

and so on.

.....

Now,
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$\therefore$
$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \dots$$

Putting $a = e$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Both these series are called “exponential series”.

- 29.** Expand $\log(1+x)$ in powers of x .

Sol. Let $f(x) = \log(1+x)$ $\therefore f(0) = \log 1 = 0$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \quad \therefore f'(0) = 1$$

$$f''(x) = (-1)(1+x)^{-2} \quad \therefore f''(0) = -1$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} \quad \therefore f'''(0) = 2$$

$$f^{iv}(x) = (-1)(-2)(-3)(1+x)^{-4} \quad \therefore f^{iv}(0) = -6$$

and so on.

Substituting these values in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots$$

we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

- 30.** Expand $\tan^{-1}x$ by Maclaurin's theorem.

Sol. Let $f(x) = \tan^{-1}x$, $f(0) = 0$.

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}, \quad \therefore f'(0) = 1$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + \dots, \quad (\text{By Binomial Theorem})$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \dots, \quad \therefore f''(0) = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 + \dots, \quad \therefore f'''(0) = -2$$

$$f^{iv}(x) = 24x - 120x^3 + \dots, \quad \therefore f^{iv}(0) = 0$$

$$f^v(x) = 24 - 360x^2 + \dots, \quad \therefore f^v(0) = 24$$

and so on.

Putting these values in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$\therefore$
$$\tan^{-1}x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \dots$$

or
$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Note. This expansion is valid only if $-1 < x < 1$.

31. Expand $\frac{e^x}{e^x + 1}$ by Maclaurin's Theorem, as far as x^3 . (M.D.U., Dec., 2005)

Sol. Let
$$f(x) = \frac{e^x}{e^x + 1}, \quad \therefore f(0) = \frac{e^0}{e^0 + 1} = \frac{1}{2}$$

$$f'(x) = \frac{(e^x + 1)e^x - e^x \cdot e^x}{(e^x + 1)^2}$$

or
$$f'(x) = \frac{e^x}{(e^x + 1)^2}, \quad \therefore f'(0) = \frac{1}{2^2}$$

$$f''(x) = \frac{(e^x + 1)^2 \cdot e^x - e^x \cdot 2(e^x + 1)e^x}{(e^x + 1)^4}$$

$$= \frac{e^x - e^{2x}}{(e^x + 1)^3}, \quad \therefore f''(0) = 0$$

$$f'''(x) = \frac{(e^x + 1)^3 (e^x - 2e^{2x}) - (e^x - e^{2x})(e^x + 1)^2 \cdot 3 \cdot e^x}{(e^x + 1)^6}$$

$$= \frac{e^x - 4e^{2x} + e^{3x}}{(e^x + 1)^4}, \quad \therefore f'''(0) = -\frac{1}{8}$$

Hence,
$$\frac{e^x}{e^x + 1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$$

32. Apply Maclaurin's Theorem to obtain the expansion of the function $e^{ax} \sin bx$.

Sol. Let
$$f(x) = e^{ax} \sin bx$$

$$f'(x) = ae^{ax} \cdot \sin bx + e^{ax} \cdot b \cos bx = e^{ax} \{a \sin bx + b \cos bx\}$$

Let
$$a = r \cos \phi, b = r \sin \phi$$

Then

$$f'(x) = e^{ax} \cdot r \{\sin bx \cdot \cos \phi + \cos bx \cdot \sin \phi\} = re^{ax} \cdot \sin (bx + \phi)$$

Similarly, again differentiating and simplifying, we have

$$f''(x) = r^2 \cdot e^{ax} \sin (bx + 2\phi)$$

$$f'''(x) = r^3 \cdot e^{ax} \sin (bx + 3\phi)$$

$$f^{iv}(x) = r^4 \cdot e^{ax} \sin (bx + 4\phi)$$

and so on.

Here, $r = \sqrt{a^2 + b^2}, \phi = \tan^{-1} \frac{b}{a}.$

$\therefore \sin \phi = \frac{b}{\sqrt{a^2 + b^2}}, \cos \phi = \frac{a}{\sqrt{a^2 + b^2}}$

$f(0) = 0$

$f'(0) = r \sin \phi = \sqrt{a^2 + b^2} \cdot \frac{b}{\sqrt{a^2 + b^2}} = b$

$f''(0) = r^2 \sin 2\phi = (a^2 + b^2) \cdot 2 \sin \phi \cos \phi$

$= 2(a^2 + b^2) \cdot \frac{ab}{(\sqrt{a^2 + b^2})^2} = 2ab$

$f'''(0) = r^3 \sin 3\phi = (a^2 + b^2)^{3/2} \cdot [3 \sin \phi - 4 \sin^3 \phi]$

$= (a^2 + b^2)^{3/2} \cdot \left[\frac{3b}{\sqrt{a^2 + b^2}} - \frac{4b^3}{(a^2 + b^2)^{3/2}} \right] = 3b(a^2 + b^2) - 4b^3$

or $= b(3a^2 - b^2)$

$\therefore$ By Maclaurin's Theorem, we have

$$e^{ax} \sin bx = bx + abx^2 + \frac{b(3a^2 - b^2)}{3!} x^3 + \dots$$

33. Expand $e^{a \sin^{-1} x}$ by Maclaurin's Theorem. Hence show that $e^\theta = 1 + \sin \theta + \frac{1}{2!} \sin^2 \theta + \frac{2}{3!} \sin^3 \theta + \dots$

Sol. Let $y = e^{a \sin^{-1} x}, \therefore y(0) = 1$

$y_1 = e^{a \sin^{-1} x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}} \therefore y_1(0) = a$

$y_1^2 = \frac{a^2 y^2}{1-x^2}$

$y_1^2(1-x^2) = a^2 y^2$

Differentiating, we get

$$(1-x^2) 2y_1 y_2 - 2xy_1^2 = 2a^2 y y_1$$

or $(1-x^2) y_2 - xy_1 = a^2 y, \therefore y_2(0) = a^2$

Differentiating n times by **Leibnitz's Theorem**, we get

$$y_{n+2} (1-x^2) + ny_{n+1} (-2x) + \frac{n(n-1)}{2!} y_n (-2) - [y_{n+1} \cdot x + ny_n] = a^2 y_n$$

or $(1-x^2) y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2) y_n = 0$

Putting $x = 0,$ $y_{n+2}(0) = (n^2 + a^2) y_n(0) \dots(1)$

Putting $n = 1, 2, 3, \dots$ in (1)

$$y_3(0) = (1^2 + a^2) y_1(0) = a(1^2 + a^2)$$

$$y_4(0) = (2^2 + a^2) y_2(0) = a^2(2^2 + a^2)$$

$$y_5(0) = (3^2 + a^2) y_3(0) = a(1^2 + a^2)(3^2 + a^2)$$

.....

$$\therefore e^{a \sin^{-1} x} = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \dots$$

$$\text{or } e^{a \sin^{-1} x} = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a(1^2 + a^2)}{3!} x^3 + \frac{a^2(2^2 + a^2)}{4!} x^4 + \dots$$

Putting $a = 1$ and $\sin^{-1} x = \theta$, we get

$$e^\theta = 1 + \sin \theta + \frac{1}{2!} \sin^2 \theta + \frac{2}{3!} \sin^3 \theta + \frac{5}{4!} \sin^4 \theta + \dots$$

Hence shown.

34. Expand $\sin(m \sin^{-1} x)$ by Maclaurin's Theorem, as far as x^5 .

Sol.

$$\text{Let } y = \sin(m \sin^{-1} x),$$

$$\therefore y(0) = 0$$

Differentiating

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \therefore y_1(0) = m$$

$$y_1^2 = \cos^2(m \sin^{-1} x) \cdot \frac{m^2}{1-x^2}$$

or

$$(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)$$

$$(1-x^2)y_1^2 = m^2 [1 - \sin^2(m \sin^{-1} x)]$$

Differentiating

$$-2x \cdot y_1^2 + (1-x^2) \cdot 2y_1 y_2 = m^2 (-2y_1 x)$$

or

$$(1-x^2)y_2 - xy_1 + m^2 y = 0, \quad \therefore y_2(0) = 0$$

Differentiating n times by **Leibnitz's Theorem**,

we get

$$y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2) - [y_{n+1}x + ny_n \cdot 1] + m^2 y_n = 0$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$$

$$\text{Putting } x = 0, y_{n+2}(0) = (n^2 - m^2)y_n(0)$$

Putting $n = 1, 2, 3, \dots$

$$y_3(0) = (1^2 - m^2)y_1(0) = m(1^2 - m^2)$$

$$y_4(0) = (2^2 - m^2)y_2(0) = 0$$

$$y_5(0) = (3^2 - m^2)y_3(0) = m(1^2 - m^2)(3^2 - m^2)$$

.....

$$\begin{aligned}
 \therefore \sin(m \sin^{-1} x) &= y(0) + x \cdot y_1(0) + \frac{x^2}{2!} y_2(0) + \dots \\
 &= mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \dots
 \end{aligned}$$

35. Prove that : $e^x \cos x = 1 + x - \frac{2}{3!} x^3 - \frac{2^2}{4!} x^4 + \dots$

Sol. Let $f(x) = e^x \cos x$

$$f'(x) = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$$

$$f''(x) = e^x (\cos x - \sin x) + (-\sin x - \cos x) e^x$$

$$= e^x (-2 \sin x) = -2 e^x \sin x$$

$$f'''(x) = -2[e^x \sin x + e^x \cos x] = -2e^x (\sin x + \cos x)$$

$$f^{iv}(x) = -2[e^x (\sin x + \cos x) + e^x (\cos x - \sin x)] = -2e^x [2 \cos x] = -4 \cos x \cdot e^x$$

$\therefore f(0) = 1, f'(0) = 1, f''(0) = 0, f'''(0) = -2, f^{iv}(0) = -4$

Hence, $e^x \cos x = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$

$$= 1 + x - \frac{2}{3!} x^3 - \frac{2^2}{4!} x^4 + \dots$$

Hence proved.

36. Prove that : $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$

(M.D.U., Dec., 2007)

Sol. Let $f(x) = \log(1 + e^x), \therefore f(0) = \log 2$

$$f'(x) = \frac{e^x}{1 + e^x}, \therefore f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{e^x}{(1 + e^x)^2}, \therefore f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(e^x + 1)^2 e^x - e^x \cdot 2(e^x + 1)(e^x)}{(1 + e^x)^4}$$

$$= \frac{e^x - e^{2x}}{(e^x + 1)^3}, \therefore f'''(0) = 0$$

$$f^{iv}(x) = \frac{(e^x + 1)^3 (e^x - e^{2x}) - 3e^x (e^x - e^{2x})(e^x + 1)^2}{(e^x + 1)^6}$$

$$= \frac{e^x - 4e^{2x} + e^{3x}}{(e^x + 1)^4}, \therefore f^{iv}(0) = -\frac{1}{8}$$

$\therefore \log(1 + e^x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$

$\Rightarrow \log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$

Hence proved.

37. Prove that:
$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$$

Sol. We know

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin x = x \left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots \right]$$

$$\therefore \frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots$$

$$\Rightarrow \frac{x}{\sin x} = \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots \right)^{-1} = \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} \right) + \dots \right]^{-1}$$

$$= 1 + \frac{x^2}{6} - \frac{x^4}{120} + \frac{1.2}{2!} \left(\frac{x^2}{6} - \frac{x^4}{120} \right)^2 + \dots$$

$$\Rightarrow \frac{x}{\sin x} = 1 + \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^4}{36} + \dots = 1 + \frac{x^2}{6} + x^4 \left(\frac{1}{36} - \frac{1}{120} \right)$$

$$= 1 + \frac{x^2}{6} + x^4 \cdot \frac{7}{360} + \dots$$

or
$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$$

Hence proved.

38. Prove that: $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

Sol. Since $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$

$$\therefore \log(1 + \sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \dots$$

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\begin{aligned} \log(1 + \sin x) &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) - \frac{1}{2} \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right)^2 \\ &\quad + \frac{1}{3} \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right)^3 - \frac{1}{4} \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right)^4 + \dots \end{aligned}$$

or
$$\begin{aligned} \log(1 + \sin x) &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) - \frac{1}{2} \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \right)^2 \\ &\quad + \frac{1}{3} \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} \right) \right]^3 - \frac{1}{4} \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} \right) \right]^4 \end{aligned}$$

$$\begin{aligned}
&= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{1}{2} \left[x^2 - \frac{2x^4}{6} + \dots \right] + \frac{1}{3} [x^3 + \dots] - \frac{1}{4} [x^4 + \dots] \\
&\hspace{15em} \text{Neglecting higher power terms} \\
&= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\
&= x - \frac{x^2}{2} + \frac{x^3}{6} - x^4 \left(\frac{1}{6} - \frac{1}{4} \right) + \dots \\
&= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots
\end{aligned}$$

Hence proved.

39. Prove that: $\sin^{-1} \left(\frac{2x}{1+x^2} \right) = 2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right)$.

Sol. Let

$$y = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

Put $x = \tan \theta$,

$$y = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta)$$

$\Rightarrow$

$$y = 2\theta = 2 \tan^{-1} x$$

$\therefore$

$$y(0) = 0$$

$$y_1 = \frac{2}{1+x^2} = 2(1+x^2)^{-1} = 2[1 - x^2 + x^4 - x^6 + \dots]$$

$$y_1(0) = 2$$

$$y_2 = 2[-2x + 4x^3 - 6x^5 + \dots]$$

$$y_2(0) = 0$$

$$y_3 = 2[-2 + 12x^2 - 30x^4 + \dots]$$

$$y_3(0) = -4$$

$$y_4 = 2[24x - 120x^3 + \dots]$$

$$y_4(0) = 0$$

$$y_5 = [48 - 240 \cdot 3x^2]$$

$$y_5(0) = 48$$

Hence

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \dots$$

$\Rightarrow$

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = 2x - \frac{4}{3!} x^3 + \frac{x^5}{5!} \cdot (48) + \dots$$

or

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = 2x - \frac{2}{3} x^3 + \frac{2x^5}{5} - \dots$$

Hence

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = 2 \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right]$$

Hence proved.

40. Use the method of differential equations to expand $(\sin^{-1} x)^2$ and show that

$$\frac{(\sin^{-1} x)^2}{2} = \frac{x^2}{2!} + \frac{2^2}{4!} x^4 + \frac{2^2 \cdot 4^2}{6!} x^6 + \dots$$

Sol. Let $y = (\sin^{-1} x)^2$... (1)

$\therefore y_1 = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$... (2)

or $(1-x^2)y_1^2 = 4y$

Differentiating again

$$(1-x^2) \cdot 2y_1 y_2 + y_1^2 (-2x) = 4y_1$$

or Dividing by $2y_1$,

$$(1-x^2)y_2 - xy_1 = 2 \quad \dots (3)$$

Differentiating (3), n times by Leibnitz's Theorem,

$$(1-x^2)y_{n+2} + {}^nC_1 \cdot y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1} - {}^nC_1 \cdot y_n = 0$$

or $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$

or $(1-x^2)y_{n+2} - (2n-1)xy_{n+1} = n^2 y_n$... (4)

Putting $x = 0$ in (1), (2), (3) and (4), we get

$$y = 0, y_1 = 0, y_2 = 2, (y_{n+2})_0 = n^2 (y_n)_0$$

Putting $n = 1, 2, 3, 4$ in the last equation, we get

$$y_3 = 1^2 \cdot y_1 = 0$$

$$y_4 = 2^2 \cdot y_2 = 2^2 \cdot 2$$

$$y_5 = 3^2 \cdot y_3 = 0$$

$$y_6 = 4^2 \cdot y_4 = 2 \cdot 2^2 \cdot 4^2 \quad \text{and so on.}$$

By Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

or $y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \frac{x^6}{6!} y_6(0) + \dots$

or $y = 2 \cdot \frac{x^2}{2!} + (2 \cdot 2^2) \frac{x^4}{4!} + (2 \cdot 2^2 \cdot 4^2) \frac{x^6}{6!} + \dots$

$$\Rightarrow (\sin^{-1} x)^2 = 2 \left[\frac{x^2}{2!} + \frac{2^2 \cdot x^4}{4!} + 2^2 \cdot 4^2 \cdot \frac{x^6}{6!} + \dots \right]$$

and hence

$$\frac{1}{2} (\sin^{-1} x)^2 = \frac{x^2}{2!} + \frac{2^2}{4!} x^4 + \frac{2^2 \cdot 4^2}{6!} x^6 + \dots \quad \text{Hence shown.}$$

41. Expand $\tan\left(\frac{\pi}{4} + x\right)$ in ascending powers of x . Hence find the value of $\tan 45^\circ 30'$ to four places of decimals.

Sol. Let $f(x) = \tan\left(\frac{\pi}{4} + x\right), \therefore f(0) = \tan \frac{\pi}{4} = 1$

$$f'(x) = \sec^2\left(\frac{\pi}{4} + x\right), \quad \therefore f'(0) = \sec^2 \frac{\pi}{4} = 2$$

$$f''(x) = 2 \sec^2\left(\frac{\pi}{4} + x\right) \tan\left(\frac{\pi}{4} + x\right) = 2 \left[1 + \tan^2\left(\frac{\pi}{4} + x\right)\right] \cdot \tan\left(\frac{\pi}{4} + x\right)$$

or
$$f''(x) = 2 \left[\tan\left(\frac{\pi}{4} + x\right) + \tan^3\left(\frac{\pi}{4} + x\right) \right]$$

$$\therefore f''(0) = 2[1 + 1] = 4$$

$$f'''(x) = 2 \left[\sec^2\left(\frac{\pi}{4} + x\right) + 3 \tan^2\left(\frac{\pi}{4} + x\right) \cdot \sec^2\left(\frac{\pi}{4} + x\right) \right]$$

$$\therefore f'''(0) = 2[2 + 3 \cdot 2] = 16$$

$$f^{iv}(x) = 2 \left\{ 8 \tan\left(\frac{\pi}{4} + x\right) \sec^2\left(\frac{\pi}{4} + x\right) + 12 \tan^3\left(\frac{\pi}{4} + x\right) \sec^2\left(\frac{\pi}{4} + x\right) \right\}$$

$$\therefore f^{iv}(0) = 2(8 \cdot 1 \cdot 2 + 12 \cdot 1 \cdot 2) = 80$$

By Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$$

$$\begin{aligned} \therefore \tan\left(\frac{\pi}{4} + x\right) &= 1 + 2x + \frac{x^2}{2!} \cdot (4) + \frac{x^3}{3!} (16) + \frac{x^4}{4!} (80) + \dots \\ &= 1 + 2x + 2x^2 + \frac{8}{3} x^3 + \frac{10}{3} x^4 + \dots \end{aligned}$$

Putting $x = 30' = \frac{\pi}{360}$, we get

$$\begin{aligned} \tan 45^\circ 30' &= 1 + 2 \cdot \frac{\pi}{360} + 2 \left(\frac{\pi}{360}\right)^2 + \dots \\ &= 1 + 0.01745 + .00015 + \dots \\ &= 1.0176 \text{ (approximately).} \end{aligned}$$

42. Find the value of $\sin 91^\circ$ correct to 4 decimal places.

(M.D.U., May 2007, Dec., 2008)

Sol.

$$\begin{aligned} \sin x &= \sin\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) \\ &= \sin \frac{\pi}{2} + \left(x - \frac{\pi}{2}\right) \cos \pi/2 + \frac{(x - \pi/2)^2}{2!} f''\left(\frac{\pi}{2}\right) \\ &= 1 - \frac{(x - \pi/2)^2}{2} + \frac{(x - \pi/2)^4}{24} + \dots \end{aligned}$$

Put $x = 91^\circ$.

$$\begin{aligned} \sin 91^\circ &= 1 - \frac{1^2}{2} + \frac{1^4}{24} + \dots = 1 - \frac{1}{2} + \frac{1}{24} + \dots = \frac{1}{2} + \frac{1}{24} \\ &= \frac{13}{24} = 0.5417 \end{aligned}$$

43. Expand $e^{\sin x}$ by Maclaurin's series up to the term containing x^4 . [M.D.U., May 2008]

Sol. We have $e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \dots$

$$\begin{aligned} e^{\sin x} &= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \frac{1}{2!} \left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{3!} \left(x - \frac{x^3}{3!} + \dots\right)^3 \\ &\quad + \frac{1}{4!} \left(x - \frac{x^3}{3!} + \dots\right)^4 + \dots \\ &= 1 + \left(x - \frac{x^3}{6} + \dots\right) + \frac{1}{2} \left(x^2 - \frac{x^4}{3} + \dots\right) + \frac{1}{6} (x^3 - \dots) + \frac{1}{24} (x^4 + \dots) + \dots \\ &= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots \end{aligned}$$

44. Find the asymptotes, parallel to the axes for the curve. $x^2y^2 = a^2(x^2 + y^2)$

Sol. The equation of the curve is

$$x^2y^2 - a^2(x^2 + y^2) = 0$$

Equating to zero the co-efficient of x^2 , the highest power of x , we get

$$y^2 - a^2 = 0 \quad \text{or} \quad y = \pm a$$

which are the asymptotes parallel to x -axis.

Again equating to zero the co-efficient of y^2 , the highest power of y , we get $x^2 - a^2 = 0$ or $x = \pm a$.

which are the asymptotes parallel to y -axis.

45. Find the asymptotes parallel to the axes for the curve : $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$.

Sol. Equation of the curve is

$$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$$

$$\text{or} \quad a^2y^2 - b^2x^2 = x^2y^2$$

$$\text{or} \quad x^2y^2 + b^2x^2 - a^2y^2 = 0$$

Equating to zero the co-efficient of x^2 , the highest power of x , we get

$$y^2 + b^2 = 0 \quad \text{or} \quad y = \pm ib$$

which gives imaginary values of y , and therefore, there is no asymptote parallel to x -axis. Again equating to zero, the co-efficient of y^2 , the highest power of y , we get

$$x^2 - a^2 = 0 \quad \text{or} \quad x = \pm a,$$

which are the asymptotes parallel to y -axis.

46. Find the asymptotes parallel to the axes, for the curve : $xy^3 - x^3 = a(x^2 + y^2)$

Sol. Here the highest power of x is x^3 , having a constant co-efficient. Thus there is no asymptote parallel to x -axis.

Again equating to zero, the co-efficient of y^3 , the highest power of y , we get $x = 0$, which is the asymptote parallel to y -axis, i.e., y -axis itself is the asymptote.

47. Find the asymptotes parallel to the x -axis of the curve: $x^2y - 3x^2 - 5xy + 6y + 2 = 0$

Sol. To find asymptotes parallel to the x -axis, equating to zero the co-efficient of highest power of x , we get

$$y - 3 = 0$$

$$\Rightarrow y = 3, \quad \text{which is the required asymptote.}$$

48. Find the asymptotes parallel to y -axis of the curve $\frac{a^2}{x} + \frac{b^2}{y} = 1$.

Sol. The given curve is

$$a^2y + b^2x = xy$$

Equating the co-efficient of highest power of y to zero, we get

$$a^2 - x = 0$$

$$\Rightarrow x = a^2$$

which is the required asymptote parallel to y -axis of the given curve.

49. Find the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Sol. Given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{1}{a^2} - \frac{1}{b^2} \cdot \frac{y^2}{x^2} = \frac{1}{x^2}$$

Take limit as $x \rightarrow \infty$

$$\frac{1}{a^2} - \frac{1}{b^2} \cdot \lim_{x \rightarrow \infty} \left(\frac{y}{x} \right)^2 = 0$$

or

$$\lim_{x \rightarrow \infty} \left(\frac{y}{x} \right)^2 = \frac{b^2}{a^2}$$

i.e.,

$$\lim_{x \rightarrow \infty} \left(\frac{y}{x} \right) = \pm \frac{b}{a}$$

If $y = mx + c$ is an oblique asymptote to any curve $f(x, y) = 0$, then

$$m = \lim_{x \rightarrow \infty} \frac{y}{x} \quad \text{and} \quad c = \lim_{x \rightarrow \infty} (y - mx)$$

$\therefore$ Values of m are $\pm \frac{b}{a}$

$$\begin{aligned} \text{Now,} \quad c &= \lim_{x \rightarrow \infty} \left(y - x \frac{dy}{dx} \right) = \lim_{x \rightarrow \infty} \left(y - x \cdot \frac{b^2x}{a^2 \cdot y} \right) = \lim_{x \rightarrow \infty} \left(\frac{y^2}{b^2} - \frac{x^2}{a^2} \right) \cdot \frac{b^2}{y} \\ &= \lim_{x \rightarrow \infty} \left(-\frac{b^2}{y} \right) = 0, \quad \left[\because \lim_{x \rightarrow \infty} y = \pm \frac{b}{a} x \right] \end{aligned}$$

Hence the asymptotes are

$$y = \pm \frac{b}{a} \cdot x.$$

50. Find the asymptotes of $y^3 - x^2y + 2y^2 + 4y + 1 = 0$

Sol. Putting $y = mx + c$ in the equation of the curve, we get

$$(mx + c)^3 - x^2(mx + c) + 2(mx + c)^2 + 4(mx + c) + 1 = 0$$

or

$$(m^3 - m)x^3 + (3m^2c - c + 2m^2)x^2 + \dots \text{etc.} = 0$$

Equating to zero the co-efficients of x^3 and x^2 , we get

$$m^3 - m = 0 \quad \dots(1)$$

$$\text{and} \quad 3m^2c - c + 2m^2 = 0 \quad \dots(2)$$

From equation (1), $m = 0, 1, -1$

From equation (2),

$$c = \frac{2m^2}{1-3m^2}$$

When $m = 0$, we get $c = 0$

When $m = 1$, we get $c = -1$

When $m = -1$, we get $c = -1$.

$\therefore$ Three asymptotes are $y = 0$, $y = x - 1$ and $y = -x - 1$.

- 51.** Find the asymptotes of the curve: $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$

Sol. Put $y = mx + c$ in the given equation, we get

$$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + 4(mx + c)^2 + 2x(mx + c) + (mx + c) - 1 = 0$$

$$\Rightarrow (1 + 2m - m^2 - 2m^3)x^3 + 2(c - mc - 3m^2c - 2m^2 + m)x^2 + \dots = 0$$

Equating the co-efficients of x^2 and x^3 to zero, we get

$$1 + 2m - m^2 - 2m^3 = 0 \quad \dots(1)$$

$$\text{and } c - mc - 3m^2c + 2m^2 + m = 0 \quad \dots(2)$$

From equation (1), $(1 + 2m)(1 - m^2) = 0$

$$\Rightarrow m = \pm 1, -\frac{1}{2}$$

$$\text{From equation (2), } c = -\left(\frac{2m^2 + m}{1 - m - 3m^2}\right)$$

When $m = -1$, we have $c = 1$

When $m = 1$, we have $c = 1$

When $m = -\frac{1}{2}$, we have $c = 0$

Hence the asymptotes are $y = -x + 1$, $y = x + 1$ and $y = -\frac{1}{2}x$.

- 52.** Find all the asymptotes of the curve: $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0$.

Sol. Since the co-efficients of x^3 and y^3 , the highest degree terms in x and y , are constant, $\therefore$ There are no asymptotes parallel to x -axis or y -axis.

Now to find oblique asymptotes; putting $x = 1$, $y = m$ in the 3rd and 2nd terms in the given equation of the curve, we get

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3$$

$$\phi_2(m) = 3m + 3m^2$$

The slopes of asymptotes are the roots of $\phi_3(m) = 0$.

$$\text{i.e., } 1 + 2m - m^2 - 2m^3 = 0$$

Solving for m , we get

$$m = -1, 1, -\frac{1}{2}$$

Also,

$$\phi_3'(m) = 2 - 2m - 6m^2 = -(6m^2 + 2m - 2)$$

Now, c is given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = +\frac{3m + 3m^2}{6m^2 + 2m - 2}$$

When $m = -1, c = 0$

When $m = 1, c = 1$

$$\text{When } m = \frac{-1}{2}, c = \frac{-\frac{3}{2} + \frac{3}{4}}{\frac{3}{2} - 1 - 2} = \frac{1}{2}$$

Putting these values of m and c in $y = mx + c$, the corresponding asymptotes are

$$y = -x, y = x + 1, y = -\frac{1}{2}x + \frac{1}{2}$$

i.e., $x + y = 0, x - y + 1 = 0$ and $x + 2y - 1 = 0$.

53. Find all the asymptotes of the curve

$$x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0.$$

Sol. Since the co-efficient of x^3 , the highest degree term in x , is constant, $\therefore$ There is no asymptote parallel to x -axis.

Again equating to zero the co-efficient of y^2 , the highest degree term in y , the asymptote parallel to y -axis is given by the equation

$$4x + 10 = 0 \quad \text{or} \quad 2x + 5 = 0$$

Now to find oblique asymptotes, putting $x = 1$ and $y = m$ in the 3rd and 2nd degree terms in the equation of the curve, we get

$$\phi_3(m) = 1 + 4m + 4m^2$$

and

$$\phi_2(m) = 5 + 15m + 10m^2$$

Slopes of asymptotes are given by the roots of the equation $\phi_3(m) = 0$ i.e.,

$$\text{by roots of} \quad 1 + 4m + 4m^2 = 0$$

$$\text{or} \quad (1 + 2m)^2 = 0$$

$$\therefore m = -\frac{1}{2}, -\frac{1}{2} \quad (\text{Equal})$$

$\therefore c$ is now given by the equation

$$\frac{c^2}{2!} \phi_3''(m) + c \cdot \phi_2'(m) + \phi_1(m) = 0$$

Now,

$$\phi_3'(m) = 4 + 8m, \quad \phi_3''(m) = 8$$

$$\phi_2'(m) = 15 + 20m, \quad \phi_1(m) = -2m$$

$$\text{Thus,} \quad \frac{c^2}{2!} \cdot (8) + c \cdot (15 + 20m) - 2m = 0$$

$$\text{For } m = -\frac{1}{2}, \text{ it becomes} \quad 4c^2 + 5c + 1 = 0$$

$$\text{or} \quad (4c + 1)(c + 1) = 0$$

$$\Rightarrow c = -1, -1/4$$

Hence the corresponding asymptotes are

$$y = -\frac{1}{2}x - 1 \quad \text{and} \quad y = -\frac{1}{2}x - \frac{1}{4}$$

∴ The required asymptotes are

$$2x + 5 = 0, x + 2y + 2 = 0 \text{ and } 2x + 4y + 1 = 0.$$

54. Find asymptotes of the curve: $(x - y)^2 (x + 2y - 1) = 3x + y - 7$.

Sol. The equation of the curve is

$$(x - y)^2 (x + 2y - 1) - 3x - y + 7 = 0$$

$$\text{or } (x - y)^2 (x + 2y) - (x - y)^2 - 3x - y + 7 = 0 \quad \dots(1)$$

Since the co-efficients of x^3 and y^3 , the highest degree terms in x and y are constants, therefore there are no asymptotes parallel to x -axis or y -axis.

Now to find oblique asymptotes :

Putting $x = 1, y = m$ in the 3rd, 2nd and first degree terms in equation (1), we get

$$\phi_3(m) = (1 - m)^2 (1 + 2m)$$

$$\text{or } \phi_3(m) = 2m^3 - 3m^2 + 1$$

$$\phi_2(m) = -(1 - m)^2$$

$$\phi_1(m) = -3 - m$$

Slopes of the asymptotes are given by

$$\phi_3(m) = 0$$

$$\text{i.e., by } (1 - m)^2 (1 + 2m) = 0$$

$$\text{or } m = 1, 1, -\frac{1}{2}$$

c is given by the equation

$$c \cdot \phi_3'(m) + \phi_2(m) = 0$$

$$\text{or } c \cdot (6m^2 - 6m) - (1 - m)^2 = 0$$

For $m = -\frac{1}{2}$, we have

$$c \left(\frac{6}{4} + \frac{6}{2} \right) - \frac{9}{4} = 0 \Rightarrow 18 \frac{c}{4} = \frac{9}{4}$$

$$\text{or } c = \frac{1}{2}$$

∴ Equation of the asymptote is

$$y = mx + c$$

$$\text{or } y = -\frac{1}{2}x + \frac{1}{2}$$

$$\text{or } 2y + x - 1 = 0$$

For equal values of $m = 1$, values of c are given by

$$\frac{c^2}{2!} \phi_3''(m) + c \phi_2'(m) + \phi_1(m) = 0$$

$$\text{Now, } \phi_3''(m) = 12m - 6, \phi_2'(m) = 2(1 - m)$$

$$\therefore \frac{c^2}{2} \cdot (12m - 6) + 2c(1 - m) - 3 - m = 0$$

$$\text{For } m = 1, \text{ we have } \frac{c^2}{2} (6) - 4 = 0 \Rightarrow c^2 = \frac{4}{3}$$

or
$$c = \pm \frac{2}{\sqrt{3}}$$

$\therefore$ Equation of asymptote is
$$y = x \pm \frac{2}{\sqrt{3}}$$

Hence asymptotes are : $2y + x - 1 = 0$ and $y = x \pm \frac{2}{\sqrt{3}}$.

- 55.** Find the asymptotes of: $(x - y)^2 (x^2 + y^2) - 10(x - y)x^2 + 12y^2 + 2x + y = 0$.

Sol. The equation of the curve is

$$(x - y)^2 (x^2 + y^2) - 10(x - y)x^2 + 12y^2 + 2x + y = 0 \quad \dots(1)$$

Since co-efficients of x^4 and y^4 , the highest degree terms in x and y are constants, therefore there are no asymptotes parallel to x -axis or y -axis.

Now to find oblique asymptotes putting $x = 1, y = m$ in the fourth, third and second degree terms in (1), we get

$$\begin{aligned}\phi_4(m) &= (1 - m)^2 (1 + m^2) = m^4 - 2m^3 + 2m^2 - 2m + 1 \\ \phi_3(m) &= -10(1 - m) \\ \phi_2(m) &= 12m^2\end{aligned}$$

Slopes of the asymptotes are given by $\phi_4(m) = 0$

i.e., by $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$ or $(1 - m)^2 (1 + m^2) = 0 \Rightarrow m = 1, 1$

because $m^2 + 1 = 0$ gives only imaginary values of m .

For equal values of $m = 1$, values of c are given by

$$\frac{c^2}{2!} \phi_4''(m) + c \cdot \phi_3'(m) + \phi_2(m) = 0$$

or
$$c^2(6m^2 - 6m + 2) + 10c + 12m^2 = 0$$

For $m = 1$,
$$c^2(2) + 10c + 12 = 0$$

or
$$c^2 + 5c + 6 = 0$$

$\Rightarrow$
$$c = -3, -2$$

$\therefore$ Asymptotes are
$$y = mx + c$$

i.e.,
$$y = x - 3 \text{ and } y = x - 2.$$

- 56.** Find all the asymptotes of the curve: $(x + y)^2 (x + y + 2) = x + 9y - 2$.

Sol. Equation of the curve is $(x + y)^3 + 2(x + y)^2 - x - 9y + 2 = 0 \quad \dots(1)$

Since the co-efficients of x^3 and y^3 , the highest degree terms in x and y , are constants, therefore there are no asymptotes parallel to x -axis or y -axis.

Now to find oblique asymptotes :

Putting $x = 1, y = m$ in the third, second, first degree terms and constant terms in (1), we have

$$\begin{aligned}\phi_3(m) &= (1 + m)^3 & \phi_3'(m) &= 3(1 + m)^2 \\ \phi_2(m) &= 2(1 + m)^2 & \phi_3''(m) &= 6(1 + m) \\ \phi_1(m) &= -1 - 9m & \phi_3'''(m) &= 6 \\ \phi_0(m) &= 2 & \phi_2'(m) &= 4(1 + m) \\ & & \phi_2''(m) &= 4 \\ & & \phi_1'(m) &= -9\end{aligned}$$

Slopes of the oblique asymptotes are given by

$$\phi_3(m) = 0 \quad \text{or} \quad (1+m)^3 = 0 \quad \therefore m = -1, -1, -1.$$

For these three equal values of $m = -1$, values of c are given by

$$\frac{c^3}{3!} \phi_3'''(m) + \frac{c^2}{2!} \phi_2''(m) + c \cdot \phi_1'(m) + \phi_0(m) = 0$$

$$\text{or} \quad c^3 + 2c^2 - 9c + 2 = 0$$

Solving for c , we get

$$c = -2 \pm \sqrt{5} \quad \text{and} \quad 2.$$

$\therefore$ For $m = -1$, the three different values of c are 2 and $-2 + \sqrt{5}, -2 - \sqrt{5}$.

$\therefore$ Equations of the three asymptotes are $y = mx + c$

$$\text{i.e.,} \quad y = -x + 2, y = -x - 2 + \sqrt{5}, y = -x - 2 - \sqrt{5}.$$

57. Find all the asymptotes of the curve $x^3 + y^3 - 3axy = 0$.

Sol. The equation of the curve is

$$x^3 + y^3 - 3axy = 0 \quad \dots(1)$$

Since the co-efficients of x^3 and y^3 , the highest degree terms in x and y are constants, the curve has no asymptotes parallel to the x -axis or y -axis.

To find the oblique asymptotes, putting $x = 1, y = m$ in the two highest degree terms of (1),

$$\text{We have} \quad \phi_3(m) = 1 + m^3 \quad \text{and} \quad \phi_2(m) = -3am$$

$$\text{Now, } \phi_3(m) = 0 \text{ gives } 1 + m^3 = 0 \quad \text{or} \quad (1+m)(1-m+m^2) = 0$$

The only real value in $m = -1$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3am}{3m^2} = \frac{a}{m}$$

$$\text{When } m = -1, c = \frac{a}{-1} \quad \text{or} \quad -a.$$

$\therefore$ The required asymptote is

$$y = mx + c$$

$$\text{or} \quad y = -x - a$$

$$\text{or} \quad y + x + a = 0.$$

58. Find all the asymptotes of the curve: $(y-x)(y-2x)^2 + (y+3x)(y-2x) + 2x + 2y - 1 = 0$.

Sol. Since the co-efficient of x^3 and y^3 , the highest degree terms in x and y , are constants, there are no asymptotes parallel to the x -axis or y -axis.

To find oblique asymptotes, putting $x = 1, y = m$ in the third, second and first degree terms in the given equation of the curve

$$\phi_3(m) = (m-1)(m-2)^2 = (m-1)(m^2 - 4m + 4)$$

$$\text{or} \quad \phi_3(m) = m^3 - 5m^2 + 8m - 4$$

$$\phi_2(m) = (m+3)(m-2) = m^2 + m - 6$$

$$\phi_1(m) = 2 + 2m$$

The slopes of the asymptotes are the roots of $\phi_3(m) = 0$.

$$\phi_3(m) = 0 \quad \Rightarrow \quad (m-1)(m-2)^2 = 0 \quad \therefore m = 1, 2, 2$$

Also

$$\phi_3'(m) = 3m^2 - 10m + 8, \quad \phi_3''(m) = 6m - 10$$

$$\phi_2'(m) = 2m + 1$$

For the non-repeated value $m = 1$, c is given by

$$c = -\frac{\phi_2(m)}{\phi_3(m)} = -\frac{m^2 + m - 6}{3m^2 - 10m + 8} = 4$$

The corresponding asymptote is $y = mx + c$ i.e.,

$$y = x + 4 \quad \text{or} \quad x - y + 4 = 0$$

For the twice repeated value $m = 2$, c is given by

$$\frac{c^2}{2!} \phi_3''(m) + c \cdot \phi_2'(m) + \phi_1(m) = 0$$

$$\text{or} \quad \frac{c^2}{2} (6m - 10) + c \cdot (2m + 1) + 2 + 2m = 0$$

When $m = 2$, we have

$$\begin{aligned} c^2 + 5c + 6 &= 0 \\ (c + 2)(c + 3) &= 0 \end{aligned}$$

$$\therefore c = -2, -3$$

The corresponding asymptotes are $y = mx + c$

$$\text{i.e.,} \quad y = 2x - 2 \quad \text{and} \quad y = 2x - 3$$

$$\text{i.e.,} \quad 2x - y - 2 = 0 \quad \text{and} \quad 2x - y - 3 = 0$$

Hence all the asymptotes of the curve are

$$\begin{aligned} x - y + 4 &= 0, \\ 2x - y - 2 &= 0, \\ 2x - y - 3 &= 0. \end{aligned}$$

59. Find all the asymptotes of the curve: $y^2(x - 2a) = x^3 - a^3$. (M.D.U., 2005)

Sol. The highest power of y is y^2 and its co-efficient is $x - 2a$. Therefore, asymptote parallel to the y -axis is given by

$$\begin{aligned} x - 2a &= 0 \\ x &= 2a \end{aligned}$$

To find oblique asymptotes, the equation can be written as

$$x^3 - a^3 - xy^2 + 2ay^2 = 0$$

Putting $x = 1$, $y = m$ in the 3rd degree terms

$$\phi_3(m) = 1 - m^2 = 0 \Rightarrow m = \pm 1$$

Now,

$$\phi_2(m) = 2am^2$$

$$\therefore c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2am^2}{(-2m)} = am$$

When $m = 1$; $c = a$

When $m = -1$, $c = -a$

$\therefore$ The asymptotes are given by the equation

$$y = mx + c$$

$$\therefore y = x + a \quad \text{or} \quad x - y + a = 0$$

$$\text{and} \quad y = -x - a \quad \text{or} \quad x + y + a = 0$$

60. Find all the asymptotes of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ (M.D.U., May 2008)

Sol. Putting $y = m$ and $x = 1$ in the 3rd and second degree terms separately

$$\phi_3(m) = 1 + 3m - 4m^3, \quad \phi_2(m) = 0$$

$$\phi_3'(m) = 3 - 12m^2$$

To find slopes of asymptotes, we put $\phi_3(m) = 0$

$$\Rightarrow 1 + 3m - 4m^3 = 0$$

$$\text{or } (1-m)(2m+1)^2 = 0$$

$$\Rightarrow m = 1, -\frac{1}{2}, -\frac{1}{2}$$

$$c \text{ is given by } c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0$$

When $m = 1$; $c = 0$. Hence asymptote is $y = x$

When $m = -\frac{1}{2}$, c is given by

$$\frac{c^2}{2!} \phi_3''(m) + \frac{c}{1!} \phi_2'(m) + \phi_1(m) = 0$$

$$\text{Now, } \phi_3''(m) = -24m$$

$$\phi_2'(m) = 0$$

$$\phi_1(m) = m - 1$$

$$\therefore \frac{c^2}{2!} (-24m) + m - 1 = 0$$

$$\Rightarrow (-12) \left(-\frac{1}{2} \right) c^2 - \frac{1}{2} - 1 = 0$$

$$\Rightarrow 6c^2 = \frac{3}{2} \quad \text{or} \quad c = \pm \frac{1}{2}$$

Hence $y = -\frac{1}{2}x + \frac{1}{2}$ and $y = -\frac{1}{2}x - \frac{1}{2}$ are two parallel asymptotes.

i.e., $x + 2y = \pm 1$ and $x - y = 0$ are the asymptotes of the given curve.

61. Find all the asymptotes of: $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$. (M.D.U., May, 2005)

Sol. The highest power of x is x^2 and its co-efficient is $y^2 - y$.

$\therefore$ The asymptote parallel to the x -axis is given by

$$y(y-1) = 0 \quad \text{i.e., } y = 0 \quad \text{and} \quad y = 1.$$

The highest power of y is y^2 and its co-efficient is $x^2 - x$.

$\therefore$ The asymptotes parallel to the y -axis is given by

$$x(x-1) = 0 \quad \text{i.e., } x = 0 \quad \text{and} \quad x = 1.$$

Hence the asymptotes are

$$x = 0, x = 1, y = 0 \text{ and } y = 1.$$

62. Find all the asymptotes of the curve: $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 0$

(M.D.U., May, 2006, May 2007, Dec., 2008)

Sol. There is no asymptote parallel to x -axis and y -axis.

To find oblique asymptotes

Put $x = 1, y = m$ in the highest degree terms

$$\phi_3(m) = m^3 - m^2 - m + 1, \quad \phi_3'(m) = 3m^2 - 2m - 1$$

$$\phi_3(m) = 0 \Rightarrow m^3 - m^2 - m + 1 = 0$$

$$\Rightarrow m^2(m-1) - 1(m-1) = 0$$

$$\Rightarrow (m^2 - 1)(m - 1) = 0$$

$$\text{or } m = 1, 1, -1$$

$$\phi_3''(m) = 6m - 2$$

$$\phi_2(m) = 1 - m^2$$

$$\phi_2'(m) = -2m$$

For the twice repeated value of m , c is given by

$$\frac{c^2}{2!} \phi_3''(m) + c \cdot \phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2} (4) + c \cdot (-2) = 0$$

$$\text{or} \quad 2c^2 - 2c = 0$$

$$\Rightarrow \quad c = 0, 1$$

For the non-repeated value of $m = -1$, c is given by

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{m^2 - 1}{3m^2 - 2m - 1} = 0$$

Hence asymptotes are

$$y = mx + c$$

i.e.,

$$y = \pm x \quad \text{and} \quad y = x + 1$$

- 63.** Find all the asymptotes of the curve: $6x^2 + xy - 2y^2 + x + 2y + 1 = 0$.

Sol. Put $x = 1$, $y = m$ in the highest degree terms

$$\phi_2(m) = 6 + m - 2m^2$$

$$\phi_1(m) = 1 + 2m$$

The slopes of asymptotes are the roots of $\phi_2(m) = 0$

$$\therefore \quad 6 + m - 2m^2 = 0$$

$$\Rightarrow \quad m = 2, -\frac{3}{2}$$

c is given by

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = -\left(\frac{1 + 2m}{1 - 4m}\right)$$

When,

$$m = 2, \quad c = \frac{5}{7}$$

$$m = -\frac{3}{2}, \quad c = \frac{2}{7}$$

Hence asymptotes are given by

$$y = 2x + \frac{5}{7} \quad \text{and} \quad y = -\frac{3}{2}x + \frac{2}{7}$$

i.e.,

$$14x - 7y + 5 = 0 \quad \text{and} \quad 21x + 14y = 4.$$

- 64.** Find all the asymptotes of the curve:

(M.D.U., Dec., 2007)

$$y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2y + 2x + 1 = 0.$$

Sol. Putting $x = 1$, $y = m$ in the 3rd and 2nd degree terms, we get

$$\phi_3(m) = m^3 - 2m^2 - m + 2$$

$$\phi_2(m) = 3m^2 - 7m + 2$$

The slopes of asymptotes are given by

$$\begin{aligned}\phi_3(m) &= 0 \quad \text{or} \quad m^3 - 2m^2 - m + 2 = 0 \\ &\quad \text{or} \quad (m+1)(m-1)(m-2) = 0 \\ &\Rightarrow m = -1, 1, 2\end{aligned}$$

$$\phi'_3(m) = 3m^2 - 4m - 1$$

$$c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\left(\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}\right)$$

Thus,

When $m = -1$, $c = -2$

When $m = 1$, $c = -1$ and

When $m = 2$, $c = 0$

Hence the asymptotes are

$$\begin{aligned}y &= -x - 2 \quad \text{or} \quad x + y + 2 = 0 \\ y &= x - 1 \quad \text{or} \quad x - y - 1 = 0 \\ y &= 2x \quad \text{or} \quad 2x - y = 0\end{aligned}$$

65. Find the asymptotes of the curve: $x^3 + 4x^2y + 5xy^2 + 2y^3 + 2x + 4xy + 2y^2 - x - 9y + 2 = 0$.

Sol. Putting $x = 1$ and $y = m$ in the 3rd degree terms, we get

$$\begin{aligned}\phi_3(m) &= 1 + 4m + 5m^2 + 2m^3 \\ \phi_2(m) &= 2 + 4m + 2m^2 \\ \phi_1(m) &= -1 - 9m\end{aligned}$$

The slopes of asymptotes are given by $\phi_3(m) = 0$

$$\Rightarrow 1 + 4m + 5m^2 + 2m^3 = 0$$

$$\text{or} \quad (2m+1)(m+1)^2 = 0$$

$$\therefore m = -\frac{1}{2}, -1, -1.$$

Now, when

$$m = -\frac{1}{2},$$

$$c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{2 + 4m + 2m^2}{4 + 10m + 6m^2} = -1$$

Hence the asymptote corresponding to $m = -\frac{1}{2}$ is

$$y = -\frac{1}{2}x - 1 \quad \text{or} \quad 2y + x + 2 = 0$$

When $m = -1$ (repeated equal roots), c is given by

$$\frac{c^2}{2!} \phi''_3(m) + c \cdot \phi'_2(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2} (10 + 12m) + c(4 + 4m) - 9m - 1 = 0$$

$$c^2(5 - 6) + 9 - 1 = 0 \Rightarrow c^2 = 8$$

$$\therefore c = \pm 2\sqrt{2}$$

Thus the asymptotes are

$$y = -x \pm 2\sqrt{2} \quad \text{or} \quad y + x = 2\sqrt{2} \quad \text{and} \quad y + x + 2\sqrt{2} = 0.$$

66. Find the asymptotes of: $y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$.

Sol. Putting $x = 1, y = m$ in the highest degree terms, we get

$$\phi_4(m) = m^4 - 2m^3 + 2m - 1$$

$$\phi_3(m) = -3 + 3m + 3m^2 - 3m^3$$

$$\phi_2(m) = -2 + 2m^2$$

$$\phi_1(m) = 0$$

Slopes of the asymptotes are given by

$$\phi_4(m) = 0 \quad \text{or} \quad m^4 - 2m^3 + 2m - 1 = 0$$

$$m^4 - 1 - 2m(m^2 - 1) = 0$$

$$(m^2 - 1)(m^2 + 1) - 2m(m^2 - 1) = 0$$

$$(m^2 - 1)(m^2 + 1 - 2m) = 0$$

$$\text{or} \quad (m^2 - 1)(m - 1)^2 = 0$$

$$\text{or} \quad (m + 1)(m - 1)(m - 1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1, 1, -1$$

$$\text{When } m = -1, \quad c = -\frac{\phi_3(m)}{\phi_4'(m)} = 0 \quad [\because \phi_3(m) = 0 \text{ for } m = -1]$$

The corresponding asymptote is

$$y = -x \quad \text{or} \quad y + x = 0$$

When $m = 1$ (thrice repeated root) c is given by

$$\frac{c^3}{3!} \phi_4'''(m) + \frac{c^2}{2!} \phi_3''(m) + c \cdot \phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^3}{6} (24m - 12) + \frac{c^2}{2} (6 - 18m) + c(4m) = 0$$

$$\Rightarrow \quad 2c^3 - 6c^2 + 4c = 0$$

$$2c(c^2 - 3c + 2) = 0$$

$$2c(c - 1)(c - 2) = 0$$

$$\therefore \quad c = 0, 1, 2$$

Hence all the asymptotes are given by

$$y = x, y = x + 1, y = x + 2 \quad \text{and} \quad y + x = 0.$$

67. Find all the asymptotes of the following curves :

$$(a) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$(b) \quad xy(x^2 - y^2) + 3x^2 - 4y^2 + 1 = 0.$$

Sol. (a) The equation of the given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

The equation is of the form $F_n + F_{n-2} = 0$

where F_n is of degree $n = 2$ and F_{n-2} can be resolved into real linear non-repeated factors.

Thus the asymptotes are given by $F_n = 0$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y^2 = \frac{b^2}{a^2} x^2$$

$$\Rightarrow \quad y = \pm \frac{b}{a} x.$$

(b) The equation of the given curve is

$$[xy(x^2 - y^2)] + (3x^2 - 4y^2 + 1) = 0$$

The equation is of the form $F_n + F_{n-2} = 0$

[Here $n = 4$]

Where F_n can be split up into linear non-repeated factor.

Thus the asymptotes are given by

$$F_n = 0$$

$$\text{or} \quad xy(x^2 - y^2) = 0 \quad \text{or} \quad xy(x + y)(x - y) = 0$$

$\therefore x=0, y=0, x-y=0$ and $x+y=$ are the required asymptotes.

68. Discuss intersection of a curve and its asymptotes.

Sol. Let $y = mx + c$ be an asymptote of the curve

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1} \cdot \phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2} \phi_{n-2}\left(\frac{y}{x}\right) + \dots = 0$$

Eliminating y , we get abscissae of the points of intersection given by

$$x^n \phi_n\left(m + \frac{c}{x}\right) + x^{n-1} \cdot \phi_{n-1}\left(m + \frac{c}{x}\right) + x^{n-2} \phi_{n-2}\left(m + \frac{c}{x}\right) + \dots = 0$$

Expanding by Taylor's Theorem, we have

$$x^{n-2} \left[\frac{c^2}{2!} \phi_n''(m) + c \phi_n'(m) + \phi_{n-2}(m) \right] + x^{n-3} [\dots] = 0$$

which is of degree $(n-2)$ and this gives $n-2$ values of x . Hence if a curve of n^{th} degree has n asymptotes, then they all intersect the curve in $n(n-2)$ points.

For the cubic $n = 3$ and therefore the asymptotes cut the curve in $3(3-2) = 3$ points which lie on a curve of degree $(3-2) = 1$, i.e., on a straight line.

For a curve of 4^{th} degree, $n = 4$ and therefore the asymptotes cut the curve in $4(4-2) = 8$ points which lie on a curve of degree $4-2 = 2$, i.e., on a conic *Rule to find the equation of the curve on which lie the points of intersection of the curve and its asymptotes.*

1. Find all the asymptotes of the given curve and hence the joint equation of all the asymptotes.
2. Write this equation in such a way that the highest degree term in x is the same as in the equation of the curve. Denote this by $F_n = 0$.
3. Re-write the given equation of the curve in the form $F_n + F_{n-2} = 0$, where F_n is of step 2 above.
4. Then the equation $F_{n-2} = 0$ gives the curve on which lie the points of intersection of the curve and its asymptotes.

69. Show that the points of intersection of the curve.

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0.$$

and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.

Sol. The equation of the given curve is

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0 \quad \dots (1)$$

To find the asymptotes of Equation (1) :

The curve has no asymptotes parallel to x -axis or y -axis because the co-efficient of highest degree terms in x and y i.e., x^3 and y^3 , are constants.

Putting $x = 1, y = m$ in the 3rd and 2nd degree terms, we get $\phi_3(m) = 2m^3 - 2m - 4m^2 + 4$ and $\phi_{n-1}(m)$ or $\phi_2(m) = -14m + 6m^2 + 4$

The slopes of the asymptotes are given by

$$\begin{aligned} \phi_n(m) = 0 & \quad \text{or} \quad 2m^3 - 2m - 4m^2 + 4 = 0 \\ & \quad \text{or} \quad 2m(m^2 - 1) - 4(m^2 - 1) = 0 \end{aligned}$$

$$\text{or} \quad (2m - 4)(m^2 - 1) = 0$$

$$\therefore \quad m = 2, 1, -1$$

$$\text{Also,} \quad \phi'_n(m) = \phi'_3(m) = 6m^2 - 2 - 8m$$

$$c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)} = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{-14m + 6m^2 + 4}{6m^2 - 2 - 8m} = \frac{7m - 3m^2 - 2}{3m^2 - 4m - 1}$$

When $m = 2, c = 0$

When $m = 1, c = -1$

When $m = -1, c = -2$

Putting these values of m and c in $y = mx + c$, the asymptotes are

$$y = 2x, y = x - 1 \quad \text{and} \quad x + y + 2 = 0$$

Thus the 3 asymptotes will cut the curve in $3(3 - 2)$ or 3 points

The joint equation of the asymptotes is

$$(2x - y)(x - y - 1)(x + y + 2) = 0$$

$$\text{or } y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2y - 4x = 0$$

Multiplying by 2

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 4y - 8x = 0$$

Now the equation of the curve is

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$

which can be re-written as

$$[2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 4y - 8x] + [2y + 8x + 1] = 0$$

which is of the form $F_n + F_{n-2} = 0$

Hence the points of intersection lie on the curve $F_{n-2} = 0$ or $2y + 8x + 1 = 0$ or $8x + 2y + 1 = 0$ which is the required straight line.

70. Show that the asymptotes of the curve:

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in 8 points which lie on a circle of radius unity.

Sol. The given equation of the curve is

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0 \quad \dots(1)$$

Since co-efficients of x^4 and y^4 are constants, the curve has no asymptotes parallel to x -axis or y -axis.

Putting $x = 1, y = m$ in the two highest degree terms in (1), we have

$$\phi_n(m) = \phi_4(m) = (1 - m^2)(m^2 - 4)$$

and

$$\phi_{n-1}(m) = \phi_3(m) = 6 - 5m - 3m^2 + 2m^3$$

The slopes of the asymptotes are given by

$$\phi_n(m) = 0 \quad \text{or} \quad (1 - m^2)(m^2 - 4) = 0$$

$$\therefore m = \pm 1, \pm 2$$

Also, $\phi'_4(m) = (1 - m^2)2m + (m^2 - 4)(-2m) = -2m(2m^2 - 5)$

Now,
$$c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)} = -\frac{\phi_3(m)}{\phi'_4(m)} = \frac{6 - 5m - 3m^2 + 2m^3}{2m(2m^2 - 5)}$$

When $m = 1, c = 0$, When $m = 2, c = 0$

When $m = -1, c = 1$, When $m = -2, c = 1$

Putting these values of m and c in $y = mx + c$, the required asymptotes are

$$y = x, y = -x + 1, y = 2x \quad \text{and} \quad y = -2x + 1$$

or $x - y = 0, x + y - 1 = 0, y - 2x = 0 \quad \text{and} \quad y + 2x - 1 = 0$

The 4 asymptotes will cut the curve again in $4(4 - 2) = 8$ points.

The joint equation of all the asymptotes is

$$(x - y)(x + y - 1)(y - 2x)(y + 2x - 1) = 0$$

or $[(x^2 - y^2) - x + y][y^2 - 4x^2 - y + 2x] = 0 \quad \dots (2)$

or $(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 3xy^2 + 2y^3 - 2x^2 - y^2 + 3xy = 0$

The equation of the curve (1) can be written as

$$[(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 3xy^2 + 2y^3 - 2x^2 - y^2 + 3xy] + (x^2 + y^2 - 1) = 0 \quad [\text{Here } n = 4]$$

which is of the form $F_n + F_{n-2} = 0$

Hence the points of intersection lie on the curve $F_{n-2} = 0$ or $x^2 + y^2 - 1 = 0$, which is a circle of radius unity.

- 71.** Find the equation of the hyperbola having $x + y - 1 = 0$ and $x - y + 2 = 0$ as its asymptotes and passing through the origin.

Sol. The joint equation of the asymptotes is

$$F_2 = (x + y - 1)(x - y + 2) = 0 \quad \dots (1)$$

Since the equation of the curve which has asymptotes given by $F_n = 0$ is of the form

$$F_n + F_{n-2} = 0$$

$\therefore$ The equation of the curve which has asymptotes given by (1) is of the form

$$F_2 + F_0 = 0$$

or $(x + y - 1)(x - y + 2) + k = 0 \quad \dots (2)$

(Since F_0 of zero degree and so a constant)

This passes through the origin (0, 0) if

$$(-1)(2) + k = 0 \Rightarrow k = 2$$

Putting this value of k in (2), the required equation is

$$(x + y - 1)(x - y + 2) + 2 = 0$$

or $x^2 - y^2 + x + 3y = 0.$

- 72.** Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points (0, 0), (1, 0) and (0, 1).

Sol. The given equation of the curve is

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

or $(x - y)(x - 2y)(x - 3y) + (x + y + 1) = 0$

which is of the form $F_n + F_{n-2} = 0$ (where $n = 3$) and F_n consists of linear non-repeated factors.

Thus the asymptotes are given by $F_n = 0$

$$\text{or } (x-y)(x-2y)(x-3y) = 0$$

$$\text{i.e., } x-y = 0, x-2y = 0, x-3y = 0$$

The joint equation of the asymptotes is

$$F_3 = (x-y)(x-2y)(x-3y) = 0 \quad \dots(1)$$

$\therefore$ The equation of any cubic having the asymptotes given by (1) is of the form

$$F_3 + F_1 = 0$$

$$\text{or } F_3 + (ax + by + c) = 0$$

$$\text{or } (x-y)(x-2y)(x-3y) + (ax + by + c) = 0 \quad \dots(2)$$

where $ax + by + c$ is a general linear expression.

Since the curve (1) passes through (0, 0), (1, 0) and (0, 1)

$$\therefore c = 0, 1 + a + c = 0 \quad \text{and} \quad -6 + b + c = 0$$

$$\text{which gives } a = -1, b = 6, c = 0$$

Putting these values in (2), the required cubic is

$$(x-y)(x-2y)(x-3y) - x + 6y = 0$$

$$\text{or } x^3 - 6x^2y + 11xy^2 - 6y^3 - x + 6y = 0.$$

73. Find the asymptotes of the curve: $4x^3 + 2x^2 - 3xy^2 - y^3 - 1 - xy - y^2 = 0$.

Sol. The given equation of the curve may be written as

$$4x^3 - 4xy^2 + xy^2 - y^3 + 2x^2 - 2xy + xy - y^2 - 1 = 0$$

$$\text{or } (4x^2 + 4xy + y^2)(x-y) = 2xy + y^2 - 2x^2 - xy + 1$$

$$\Rightarrow (y+2x)^2(x-y) - (y+2x)(x-y) - 1 = 0$$

Asymptotes corresponding to $y+2x$ are given by

$$(y+2x)^2 + (y+2x) - \lim_{x \rightarrow \infty} \frac{1}{(y+2x)^2} = 0$$

$$\Rightarrow (y+2x)^2 + (y+2x) = 0$$

$$\Rightarrow (y+2x)(y+2x+1) = 0$$

$$\text{or } y+2x = 0, y+2x+1 = 0 \quad \dots(1)$$

Asymptote corresponding to $x-y$ is given by

$$(x-y) + \lim_{x \rightarrow \infty} \left(\frac{1}{y+2x} \right) - \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow 1}} \left\{ \frac{1}{(y+2x)^2} \right\} = 0$$

$$\Rightarrow x-y = 0 \quad \text{or} \quad y = x$$

Hence required asymptotes are $y = x, y+2x = 0, y+2x+1 = 0$.

Q. 74. Find the asymptotes of the curve

$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0.$$

Sol. Since the co-efficient of y^4 and x^4 , the highest degree terms in y and x , are constants, therefore there are no asymptotes parallel to y -axis or x -axis. The given equation may be written as

$$(y-x)^3(y+x) - 3(y-x)^2(y+x) + 2(y-x)(y+x) - 1 = 0$$

Asymptotes corresponding to $y+x$ is

$$(y+x) - 3 \cdot \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left(\frac{y+x}{y-x} \right) + 2 \cdot \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \frac{y+x}{(y-x)^2} = \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left\{ \frac{1}{(y-x)^2} \right\} = 0$$

$$\Rightarrow (y+x) - 3 \cdot \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left(\frac{\frac{y}{x} + 1}{\frac{y}{x} - 1} \right) + 2 \cdot \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \frac{\left(\frac{y}{x} + 1 \right)}{\left(\frac{y}{x} - 1 \right)^2} - \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \frac{1}{(y-x)^3} = 0$$

$$\Rightarrow y+x = 0 \quad \dots(1)$$

(Since all other terms vanish)

Again, asymptotes corresponding to $(y-x)$ is

$$(y-x)^3 - 3(y-x)^2 + 2(y-x) - \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow 1}} \left(\frac{1}{y+x} \right) = 0$$

$$\Rightarrow (y-x)^3 - 3(y-x)^2 + 2(y-x) = 0$$

$$\Rightarrow y-x = 0, y-x = 1, y-x = 2 \quad \dots(2)$$

Hence the required asymptotes are, $y+x=0$, $y-x=0$, $y-x=1$ and $y-x=2$.

75. Find the asymptotes of the curve: $(x^2 - y^2)^2 - 4y^2 + y = 0$.

Sol. There is no asymptote parallel to x -axis or y -axis because the co-efficients of x^4 and y^4 , the highest degree terms in x and y , are constants.

The given equation may be written as

$$(y+x)^2 (x-y)^2 = 4y^2 - y$$

Asymptotes corresponding to $(x+y)^2$ are

$$(x+y)^2 = \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left\{ \frac{4y^2 - y}{(x-y)^2} \right\} = \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left\{ \frac{4 \left(\frac{y}{x} \right)^2 - \frac{y}{x^2}}{\left(1 - \frac{y}{x} \right)^2} \right\}$$

$$= 1$$

$\therefore$ Asymptotes parallel to $y+x=0$ are $x+y = \pm 1$.

Again, Asymptotes corresponding to $(x-y)^2$ are

$$(x-y)^2 = \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow 1}} \left\{ \frac{4 \left(\frac{y}{x} \right)^2 - \frac{y}{x^2}}{\left(1 - \frac{y}{x} \right)^2} \right\} = 1, \quad \therefore x-y = \pm 1$$

$\therefore$ Required asymptotes are

$$x+y = \pm 1, x-y = \pm 1.$$

76. Find the asymptotes of the curve: $xy^2 + x^2y + xy + y^2 + 3x = 0$.

Sol. Asymptote parallel to y -axis is given by co-efficient of the equation

$$(x+1)y^2 = 0 \quad \text{or} \quad x+1 = 0$$

Asymptote parallel to x -axis is given by $y = 0$

Also, the curve is $xy(x+y) + y(x+y) + 3x = 0$

Asymptote corresponding to $x+y$ is

$$(x+y) + \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} \left(\frac{x+y}{x} \right) + \lim_{\substack{x \rightarrow \infty \\ \frac{y}{x} \rightarrow -1}} 3 \cdot \frac{1}{y} = 0$$

$$\Rightarrow x+y = 0$$

$\therefore$ Required asymptotes are $y = 0$, $x+1 = 0$ and $x+y = 0$.

77. Find the asymptotes of the curve: $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$.

Sol. The equation of curve is

$$x(x-y)^2 + x(x-y) + 2 = 0 \quad \dots(1)$$

Let $x = k$ be an asymptote of the curve.

From equation (1),

$$k(k-y)^2 + k(k-y) + 2 = 0$$

Equating the higher power co-efficient of y , to zero, we get

$$k = 0$$

$\therefore$ Asymptote parallel to $x = 0$ is $x = 0$ itself.

Again, put $x - y = k_1$ in equation (1), we get

$$xk_1^2 + xk_1 + 2 = 0 \Rightarrow x(k_1^2 + k_1) + 2 = 0$$

Equating the co-efficient of highest power of x to zero, we get

$$k_1^2 + k_1 = 0 \Rightarrow k_1 = 0, -1$$

Hence the asymptotes parallel to $x - y = 0$ are $x - y = 0$ itself and $x - y = -1$

$\therefore$ All asymptotes are : $x = 0$, $x = y$ and $x = y - 1$.

78. Find all the asymptotes of the curve $xy^2 + y^3 - x - y = 1$. [U.P.T.U., (AG) 2005]

Sol. The equation may be written as

$$x(y^2 - 1) + y^3 - y - 1 = 0$$

Asymptote parallel to x -axis is $y^2 - 1 = 0$ or $y = \pm 1$.

The curve has no asymptote parallel to y -axis because co-efficient of y^3 (the highest power) is a constant.

To find the oblique asymptote put $x = 1$, $y = m$ in the 3rd degree and 2nd degree terms.

$$\phi_3(m) = m^2 + m^3$$

$$\phi_2(m) = 0$$

$$\phi_1(m) = -1 - m$$

Slope of the asymptote is given by $\phi_3(m) = 0 \Rightarrow m^2(1 + m) = 0$

$$1 + m = 0 \Rightarrow m = -1$$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0$$

$\therefore$ Asymptote is $y = -x$ or $x + y = 0$.

Hence all the asymptotes are

$$y = \pm 1 \quad \text{and} \quad x + y = 0.$$

79. Show that asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$. (M.D.U., May 2009)

Sol. The given equation is

$$x^2y^2 - a^2x^2 - a^2y^2 = 0 \quad \dots(1)$$

Equating to zero the coefficient of the highest power of x i.e., x^2 in equation (1), we get

$$y^2 - a^2 = 0$$

or

$$y = \pm a$$

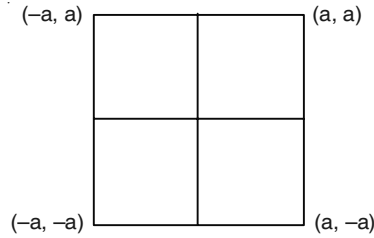
which are asymptotes parallel to x -axis.

Again equating to zero, the coefficient of the highest power of y i.e., y^2 in equation (1), we get

$$x^2 - a^2 = 0 \quad \text{or} \quad x = \pm a$$

which are asymptotes parallel to y -axis.

Hence all asymptotes are $x = \pm a, y = \pm a$ which form a square of side $2a$.



80. What is the working rule for finding polar equation of an asymptote to the curve $r = f(\theta)$?

Sol. Polar equation of an asymptote to the curve $r = f(\theta)$ is

$p = r \sin (\theta_1 - \theta)$, where $p = \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right)$, $u = \frac{1}{r}$ and θ_1 is a root of the equation obtained by putting $u = 0$.

Working Rule is

Step I : Put $r = \frac{1}{u}$ in the given equation. Also change all *T-ratios*, if any, into $\sin \theta$ and $\cos \theta$.

Step II : Find the limit of θ as $u \rightarrow 0$. Let θ_1 be this limit or one of the limits if more than one such limits exist.

Step III : Determine $p = \lim_{\substack{\theta \rightarrow \theta_1 \\ u \rightarrow 0}} \left(-\frac{d\theta}{du} \right)$ for value or values of θ obtained in Step II.

Step IV : Putting the values of p and θ_1 in the equation $p = r \sin (\theta_1 - \theta)$ we get the corresponding asymptote.

If $\lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right)$ does not tend to a finite limit; then there is no asymptote corresponding to the value $\theta = \theta_1$.

81. Find the asymptotes of the hyperbolic spiral $r\theta = a$.

Sol. The equation of the curve is

$$r\theta = a$$

$$\text{Put } r = \frac{1}{u} \quad \frac{\theta}{u} = a \Rightarrow u = \frac{\theta}{a}$$

$$\text{As } u \rightarrow 0, \frac{\theta}{a} \rightarrow 0 \text{ i.e., } \theta \rightarrow 0, \therefore \theta_1 = 0$$

$$\text{Since } u = \frac{\theta}{a}, \frac{du}{d\theta} = \frac{1}{a}$$

$$p = \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow 0} (-a) = -a$$

$\therefore$ Equation of asymptote is

$$p = r \sin (\theta_1 - \theta) = -r \sin \theta$$

$$\text{i.e., } -a = -r \sin \theta$$

$$\text{or } a = r \sin \theta.$$

82. Find the asymptotes of the curve: $r \cos \theta = a \sin^2 \theta$.

Sol. Equation of the curve is

$$r \cos \theta = a \sin^2 \theta$$

Put $r = \frac{1}{u} \quad \therefore \quad \frac{1}{u} \cos \theta = a \sin^2 \theta$

or $u = \frac{\cos \theta}{a \sin^2 \theta}$

As $u \rightarrow 0$, $\cos \theta \rightarrow 0 \quad \therefore \quad \theta \rightarrow (2n+1)\frac{\pi}{2}$ where n is a positive integer.

i.e., $\theta_1 = (2n+1)\frac{\pi}{2}$

Since $u = \frac{\cos \theta}{a \sin^2 \theta}$

$$\therefore \quad \frac{du}{d\theta} = \frac{1}{a} \left[\frac{\sin^2 \theta (-\sin \theta) - \cos \theta \cdot 2 \sin \theta \cos \theta}{\sin^4 \theta} \right] = \frac{1}{a} \left[\frac{-(\sin^2 \theta + 2 \cos^2 \theta)}{\sin^3 \theta} \right]$$

$$\therefore \quad p = \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow (2n+1)\frac{\pi}{2}} \left(\frac{a \sin^3 \theta}{\sin^2 \theta + 2 \cos^2 \theta} \right)$$

Since $\sin (2n+1)\frac{\pi}{2} = (-1)^n$

and $\cos (2n+1)\frac{\pi}{2} = 0$,

$$p = a \frac{[(-1)^n]^3}{[(-1)^n]^2 + 0} = a(-1)^n$$

$\therefore$ Equation of the asymptote is

$$p = r \sin (\theta_1 - \theta)$$

or $a(-1)^n = r \sin \left[(2n+1)\frac{\pi}{2} - \theta \right]$

$$\Rightarrow a(-1)^n = r \sin \left(n\pi + \frac{\pi}{2} - \theta \right)$$

$$\Rightarrow a(-1)^n = (-1)^n r \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\Rightarrow a(-1)^n = r \cos \theta (-1)^n$$

$$\Rightarrow a = r \cos \theta$$

83. Find the asymptotes of the curve: $r = a \sec \theta + b \tan \theta$.

Sol. Put $r = \frac{1}{u}$, $\frac{1}{u} = \frac{a}{\cos \theta} + \frac{b \sin \theta}{\cos \theta}$

$$\Rightarrow \frac{1}{u} = \frac{a + b \sin \theta}{\cos \theta}$$

or
$$u = \frac{\cos \theta}{a + b \sin \theta}$$

As $u \rightarrow 0$, $\cos \theta \rightarrow 0$ or $\theta \rightarrow (2n+1)\frac{\pi}{2}$

i.e.,
$$\theta_1 = (2n+1)\frac{\pi}{2}$$

Now,
$$\frac{du}{d\theta} = \frac{(a + b \sin \theta)(-\sin \theta) - \cos \theta(b \cos \theta)}{(a + b \sin \theta)^2} = -\frac{(a \sin \theta + b)}{(a + b \sin \theta)^2}$$

$$p = \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \frac{(a + b \sin \theta)^2}{a \sin \theta + b} = \frac{[a + b(-1)^n]^2}{[a(-1)^n + b]}$$

Equation of asymptotes is

$$p = r \sin(\theta_1 - \theta)$$

or

$$p = r \sin \left(n\pi + \frac{\pi}{2} - \theta \right)$$

$\Rightarrow$

$$p = (-1)^n \cdot r \cos \theta$$

i.e.,
$$\frac{[a + b(-1)^n]^2}{[a(-1)^n + b]} = (-1)^n r \cos \theta$$

If n is even, equation is $a + b = r \cos \theta$.

If n is odd, equation of asymptote is

$$\frac{(a-b)^2}{(-a+b)} = -r \cos \theta$$

i.e.,
$$r \cos \theta = a - b.$$

Hence equation of the asymptote is

$$r \cos \theta = a + b \quad \text{and} \quad r \cos \theta = a - b.$$

When n is even or odd respectively.

84. Prove that the curve: $r = \frac{a}{1 - \cos \theta}$ has no asymptotes.

Sol. The equation of the curve is

$$r = \frac{a}{1 - \cos \theta}$$

Put $r = \frac{1}{u}$, $\therefore \frac{1}{u} = \frac{a}{1 - \cos \theta} \Rightarrow u = \frac{1 - \cos \theta}{a}$

As $u \rightarrow 0$, $\frac{1 - \cos \theta}{a} \rightarrow 0$ i.e., $\cos \theta \rightarrow 1$ or $\cos \theta \rightarrow \cos 0$.

$\therefore \theta \rightarrow 2n\pi \pm 0 = 2n\pi$

$\therefore \theta_1 = 2n\pi$, where n is any integer.

Since

$$u = \frac{1 - \cos \theta}{a} \quad \therefore \quad \frac{du}{d\theta} = \frac{\sin \theta}{a}$$

$$\lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow 2n\pi} \left(-\frac{a}{\sin \theta} \right) = -\frac{a}{\sin 2n\pi} = \infty$$

$\therefore \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right)$ does not exist.

$\therefore$ The curve $r = \frac{a}{1 - \cos \theta}$ has no asymptote.

85. Find the asymptotes of the curve: $r(\theta^2 - \pi^2) = 2a\theta$.

Sol. Put $r = \frac{1}{u}$, $\frac{1}{u}(\theta^2 - \pi^2) = 2a\theta \Rightarrow u = \frac{\theta^2 - \pi^2}{2a\theta}$

As $u \rightarrow 0$, $\theta_1 = \pm \pi$

Since $u = \frac{\theta^2 - \pi^2}{2a\theta}$

$$\frac{du}{d\theta} = \frac{1}{2a} \cdot \frac{(\theta^2 + \pi^2)}{\theta^2}$$

$$\therefore \lim_{\theta \rightarrow \pi} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow \pi} \left(\frac{-2a\theta^2}{\theta^2 + \pi^2} \right) = -a$$

$$\text{and} \quad \lim_{\theta \rightarrow -\pi} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow -\pi} \left(\frac{-2a\theta^2}{\theta^2 + \pi^2} \right) = -a$$

$$\therefore p = -a$$

Equation of asymptotes is

$$\begin{aligned} p &= r \sin(\theta_1 - \theta) \\ -a &= r \sin(\pm \pi - \theta) \end{aligned}$$

Taking (+ve) sign,

$$-a = r \sin(\pi - \theta) = r \sin \theta$$

i.e.,

$$r \sin \theta + a = 0$$

Taking (-ve) sign,

$$-a = r \sin(-\pi - \theta)$$

$$-a = -r \sin(\pi + \theta)$$

$$-a = -r(-\sin \theta)$$

$$r \sin \theta = -a$$

$$r \sin \theta + a = 0, \text{ which is same as above.}$$

Hence there is only one asymptote

$$r \sin \theta + a = 0.$$

86. Find the asymptotes of the curve $r = \frac{2a}{1 - 2 \cos \theta}$.

Sol. Equation of the curve is $r = \frac{2a}{1 - 2 \cos \theta}$

$$\text{Put } r = \frac{1}{u}, \quad u = \frac{1 - 2 \cos \theta}{2a}$$

$$\text{As } u \rightarrow 0, 1 - 2 \cos \theta \rightarrow 0 \quad \text{or} \quad \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\begin{aligned}\therefore \quad \theta &= 2n\pi \pm \frac{\pi}{3} \quad \text{or} \quad \theta_1 = 2n\pi \pm \frac{\pi}{3} \\ \text{Since} \quad u &= \frac{1 - 2 \cos \theta}{2a} \\ \therefore \quad \frac{du}{d\theta} &= \frac{1}{2a} (2 \sin \theta) = \frac{\sin \theta}{a} \\ p &= \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow 2n\pi \pm \frac{\pi}{3}} \left(-\frac{a}{\sin \theta} \right) \\ p &= -\frac{a}{\sin \left(\pm \frac{\pi}{3} \right)},\end{aligned}$$

Equation of the asymptote is

$$\begin{aligned}p &= r \sin (\theta_1 - \theta) \\ p &= r \sin \left(2n\pi \pm \frac{\pi}{3} - \theta \right) \\ -\frac{a}{\sin (\pm \pi/3)} &= r \sin \left(2n\pi \pm \frac{\pi}{3} - \theta \right)\end{aligned}$$

Taking upper sign, we have

$$\begin{aligned}+\frac{a}{\sin (+\pi/3)} &= -r \sin \left(\frac{\pi}{3} - \theta \right) \\ \text{or} \quad r \sin \left(\frac{\pi}{3} - \theta \right) &= -\frac{2a}{\sqrt{3}}\end{aligned}$$

Taking lower sign, we have

$$r \sin \left(\frac{\pi}{3} + \theta \right) = -\frac{a}{\sin \frac{\pi}{3}} = -\frac{2a}{\sqrt{3}}$$

Hence the required asymptotes are

$$\begin{aligned}r \sin \left(\frac{\pi}{3} \pm \theta \right) &= -\frac{2a}{\sqrt{3}} \\ \Rightarrow \quad r \sin \left(\frac{\pi}{3} \pm \theta \right) + \frac{2a}{\sqrt{3}} &= 0.\end{aligned}$$

87. Find the asymptotes of the curve: $r = a \operatorname{cosec} \theta + b$.

(M.D.U., 2009; Amity Uni. 2008; U.P.T.U., 2007, 2009)

Sol. The equation of the given curve is

$$r = a \operatorname{cosec} \theta + b = \frac{a}{\sin \theta} + b = \frac{a + b \sin \theta}{\sin \theta}$$

$$\text{Put } r = \frac{1}{u}, \quad \therefore \quad u = \frac{\sin \theta}{a + b \sin \theta}$$

$$\text{As } u \rightarrow 0, \sin \theta = 0 \quad \therefore \quad \theta_1 = n\pi$$

$$\frac{du}{d\theta} = \frac{(a + b \sin \theta) \cos \theta - \sin \theta (b \cos \theta)}{(a + b \sin \theta)^2} = \frac{a \cos \theta}{(a + b \sin \theta)^2}$$

$$\begin{aligned}
 p &= \lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow n\pi} \left[-\frac{(a + b \sin \theta)^2}{a \cos \theta} \right] \\
 &= -\frac{a^2}{a \cdot (-1)^n} = \frac{-a}{(-1)^n}
 \end{aligned}$$

Equation of the asymptote is

$$p = r \sin (\theta_1 - \theta)$$

$$p = r \sin (n\pi - \theta)$$

$$\frac{-a}{(-1)^n} = r \sin (n\pi - \theta)$$

$$-(-1)^n r \sin (-\theta) = \frac{a}{(-1)^n} \quad \left| \sin (n\pi + \alpha) = (-1)^n \sin \alpha \right.$$

or $(-1)^{2n} r \sin \theta = a$

i.e., $r \sin \theta = a$

Hence there is only one asymptote

$$r \sin \theta = a$$

88. Find the asymptotes of the curve: $r = a \tan \theta$.

(M.D.U., Dec., 2006; M.D.U. May 2008, U.P.T.U., 2007 ; A.U.U.P., 2005, 2007)

Sol. Equation of the curve is

$$r = a \tan \theta$$

$$\therefore \frac{1}{r} = \frac{1}{a} \cot \theta$$

Put $r = \frac{1}{u}$, $\therefore u = \frac{\cos \theta}{a \sin \theta}$

As $u \rightarrow 0$, $\cos \theta = 0$

$$\therefore \theta_1 = (2n+1) \frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

Since

$$u = \frac{\cot \theta}{a},$$

$$\frac{du}{d\theta} = -\frac{1}{a} \operatorname{cosec}^2 \theta$$

$$-\frac{d\theta}{du} = \frac{a}{\operatorname{cosec}^2 \theta}$$

$$\lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = a \quad \left[\because \operatorname{cosec}^2 \left(n\pi + \frac{n\pi}{2} \right) = 1 \right]$$

Equation of asymptotes is

$$p = r \sin (\theta_1 - \theta)$$

or

$$a = r \sin \left[(2n+1) \frac{\pi}{2} - \theta \right]$$

When $n = 0$,

$$a = r \cos \theta$$

When $n = 1$,

$$a = r \sin \left(3 \frac{\pi}{2} - \theta \right)$$

or

$$a = -r \cos \theta \quad \text{i.e., } r \cos \theta = -a$$

When $n = 2$,

$$a = r \sin \left(5 \frac{\pi}{2} - \theta \right) = r \cos \theta$$

Which is same as corresponding to when $n = 0$ or $\theta_1 = \frac{\pi}{2}$.

Hence there are two asymptotes only

$$r \cos \theta = a$$

and

$$r \cos \theta = -a$$

i.e.,

$$r \cos \theta = \pm a.$$

89. Find the asymptotes of the curve: $r = a(\sec \theta + \tan \theta)$.

(M.D.U., 2008)

Sol. Equation of the curve is

$$r = a(\sec \theta + \tan \theta) = a \cdot \frac{(1 + \sin \theta)}{\cos \theta}$$

$$\text{Put } r = \frac{1}{u},$$

$$u = \frac{\cos \theta}{a(1 + \sin \theta)}$$

$$\text{As } u \rightarrow 0, \cos \theta = 0 \Rightarrow \theta = (2n+1) \frac{\pi}{2}$$

$$\begin{aligned} \frac{du}{d\theta} &= \frac{(1 + \sin \theta)(-\sin \theta) - \cos \theta \cdot \cos \theta}{a(1 + \sin \theta)^2} = \frac{-\sin \theta - \sin^2 \theta - \cos^2 \theta}{a(1 + \sin \theta)^2} \\ &= \frac{-1 + \sin \theta}{a(1 + \sin \theta)^2} = -\frac{1}{a(1 + \sin \theta)} \end{aligned}$$

$$\therefore -\frac{d\theta}{du} = a(1 + \sin \theta)$$

$$\lim_{\theta \rightarrow \theta_1} \left(-\frac{d\theta}{du} \right) = a \cdot \lim_{\theta \rightarrow (2n+1) \frac{\pi}{2}} (1 + \sin \theta) = a[1 + (-1)^n]$$

$\therefore$ Equation of asymptotes is

$$p = r \sin(\theta_1 - \theta) = r \sin \left[(2n+1) \frac{\pi}{2} - \theta \right]$$

$$a[1 + (-1)^n] = r \cos \theta (-1)^n \quad \text{i.e., when } n \text{ is even, we get } r \cos \theta = 2a$$

When n is odd,

$$r \cos \theta = 0$$

Hence the asymptotes of the curve $r = a(\sec \theta + \tan \theta)$ are $r \cos \theta = 0$ and $r \cos \theta = 2a$.

90. Find the circular asymptote of the curve: $r(\theta + \sin \theta) = 2\theta + \cos \theta$.

Sol. Equation of the given curve is $r = \frac{2\theta + \cos \theta}{\theta + \sin \theta} = \frac{2 + \frac{\cos \theta}{\theta}}{1 + \frac{\sin \theta}{\theta}}$

$$\lim_{\theta \rightarrow \infty} \frac{2 + \frac{\cos \theta}{\theta}}{1 + \frac{\sin \theta}{\theta}} = \frac{2 + 0 \times \text{a finite value between } -1 \text{ and } 1}{1 + 0 \times \text{a finite value between } -1 \text{ and } 1} = \frac{2 + 0}{1 + 0} = 2.$$

Hence $r = 2$ is the required circular asymptote.

- 91.** Prove that the circular asymptote of the curve $r(e^\theta - 1) = a(e^\theta + 1)$ is $r = a$.

Sol. The curve is
$$r = \frac{a(e^\theta + 1)}{e^\theta - 1} = \frac{a(1 + e^{-\theta})}{(1 - e^{-\theta})}$$

Taking limit as $\theta \rightarrow \infty$, we get

$$\lim_{\theta \rightarrow \infty} \frac{a(1 + e^{-\theta})}{(1 - e^{-\theta})} = a$$

$\therefore$ Circular asymptote is $r = a$.

- 92.** Find the circular asymptote of the curve: $r = \frac{\theta^2 - 1}{\theta^2 + 1}$.

Sol. The curve is
$$r = \frac{\theta^2 - 1}{\theta^2 + 1}$$

$$\lim_{\theta \rightarrow \infty} \left(\frac{\theta^2 - 1}{\theta^2 + 1} \right) = \lim_{\theta \rightarrow \infty} \left(\frac{1 - \frac{1}{\theta^2}}{1 + \frac{1}{\theta^2}} \right) = 1$$

$\therefore$ Circular asymptote is $r = 1$.

- 93.** Find the circular asymptote of the curve: $r(2\theta^2 - 3\theta + 7) = 6\theta^2 + 5\theta - 1$.

Sol. Curve is
$$r = \frac{6\theta^2 + 5\theta - 1}{2\theta^2 - 3\theta + 7} = \frac{6 + \frac{5}{\theta} - \frac{1}{\theta^2}}{2 - \frac{3}{\theta} + \frac{7}{\theta^2}}$$

$$\lim_{\theta \rightarrow \infty} \left[\frac{6 + \frac{5}{\theta} - \frac{1}{\theta^2}}{2 - \frac{3}{\theta} + \frac{7}{\theta^2}} \right] = 3 \quad \left(\because \frac{1}{\theta} \text{ and } \frac{1}{\theta^2} \text{ tend to zero when } \theta \rightarrow \infty \right)$$

Hence circular asymptote is $r = 3$.

- 94.** How do you find the radius of curvature of the curve of the types:

- (i) $s = f(\psi)$ at the point (s, ψ)
- (ii) Explicit Equation $y = f(x)$
- (iii) Implicit Equation $f(x, y) = 0$
- (iv) Parametric Equation $x = f(t), y = \phi(t)$.

Sol. (i) Radius of curvature at the point (s, ψ) denoted by ρ is given by $\rho = \frac{ds}{d\psi}$

(ii)
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

However, this formula does not hold good when the tangent at (x, y) is parallel to y -axis or perpendicular to x -axis for then $\frac{dy}{dx}$ = slope of the tangent is infinite.

(iii) For implicit equation $f(x, y) = 0$,

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}$$

(iv) For parametric equations $x = f(t)$, $y = \phi(t)$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

where dashes denote differentiation w.r.t., parameter t .

95. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Sol. Equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating w.r.t. x , we have

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Differentiating again w.r.t. x , we have

$$\frac{d^2y}{dx^2} = -\frac{x^{1/2} \cdot \frac{1}{2} \cdot y^{-1/2} \frac{dy}{dx} - y^{1/2} \cdot \frac{1}{2} \cdot x^{-1/2}}{x} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} = \frac{\sqrt{a}}{2x\sqrt{x}}$$

At $\left(\frac{a}{4}, \frac{a}{4}\right)$,
$$\frac{dy}{dx} = -1$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{a}}{2x\sqrt{x}} = \frac{\sqrt{a}}{2 \cdot \frac{a}{4} \sqrt{\frac{a}{4}}} = \frac{4}{a}$$

$$\therefore \rho \text{ at } \left(\frac{a}{4}, \frac{a}{4}\right) = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{\frac{4}{a}} = \frac{a}{\sqrt{2}}.$$

96. Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

(M.D.U., Dec., 2007)

Sol. Equation of curve is $x^3 + y^3 = 3axy$

Differentiating w.r.t. x , we have

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left(x \frac{dy}{dx} + y \right)$$

$$\text{or} \quad (y^2 - ax) \frac{dy}{dx} = ay - x^2 \quad \dots(1)$$

$$\text{At } \left(\frac{3a}{2}, \frac{3a}{2} \right),$$

$$\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right) \frac{dy}{dx} = \frac{3a^2}{2} - \frac{9a^2}{4}$$

$$\Rightarrow \quad \frac{dy}{dx} = -1$$

Differentiating (1) w.r.t. x , we have

$$(y^2 - ax) \frac{d^2y}{dx^2} + \left(2y \frac{dy}{dx} - a \right) \frac{dy}{dx} = a \frac{dy}{dx} - 2x$$

$$\text{At } \left(\frac{3a}{2}, \frac{3a}{2} \right), \text{ we have}$$

$$\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right) \frac{d^2y}{dx^2} + [3a(-1) - a](-1) = a(-1) - 3a$$

$$\text{or} \quad \frac{3a^2}{4} \cdot \frac{d^2y}{dx^2} = -8a, \quad \therefore \quad \frac{d^2y}{dx^2} = -\frac{32}{3a}$$

$$\text{Hence } \rho \text{ at } \left(\frac{3a}{2}, \frac{3a}{2} \right) = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\sqrt{8}}{-\frac{32}{3a}} = \frac{3a}{8\sqrt{2}}.$$

- 97.** Prove that for the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2b^2}{p^3}$. Where p is the perpendicular from the centre upon the tangent at (x, y) . (M.D.U., Dec., 2005 May 2009.)

Sol. Any point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(a \cos \theta, b \sin \theta)$ or simply ' θ '.

Parametric equations of the ellipse are $x = a \cos \theta$ and $y = b \sin \theta$.

$$\Rightarrow \quad \begin{aligned} x' &= -a \sin \theta, & y' &= b \cos \theta \\ x'' &= -a \cos \theta, & y'' &= -b \sin \theta \end{aligned}$$

$$\begin{aligned} \therefore \quad \rho \text{ at } ' \theta ' &= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab \sin^2 \theta + ab \cos^2 \theta} \\ &= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \end{aligned}$$

Equation of tangent at ' θ ' is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad \dots(1)$$

p = length of $\perp$ from centre $(0, 0)$ on tangent

$$= \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

$$\Rightarrow p^3 = \frac{a^3 b^3}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}} \quad \dots(2)$$

Multiplying (1) and (2),

$$\rho p^3 = a^2 b^2$$

$$\text{or} \quad \rho = \frac{a^2 b^2}{p^3}.$$

- 98.** Prove that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.

(U.P.T.U., 2008, M.D.U., 2007 ; Amity Univ. Noida 2009)

Sol. The equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$
Its parametric equations are

$$x = \cos^3 t, \quad y = a \sin^3 t$$

$$\Rightarrow x' = -3a \cos^2 t \sin t, \quad y' = 3a \sin^2 t \cos t$$

$$x'' = -3a (\cos^3 t - 2 \cos t \sin^2 t)$$

$$\text{or} \quad x'' = 3a \cos t (2 \sin^2 t - \cos^2 t)$$

$$y'' = 3a (2 \sin t \cos^2 t - \sin^3 t) = 3a \sin t (2 \cos^2 t - \sin^2 t)$$

$$\therefore x'^2 + y'^2 = 9a^2 (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)$$

$$\text{or} \quad x'^2 + y'^2 = 9a^2 \sin^2 t \cos^2 t$$

$$\begin{aligned} x'y'' - y'x'' &= -9a^2 \sin^2 t \cos^2 t (2 \cos^2 t - \sin^2 t) - 9a^2 \sin^2 t \cos^2 t (2 \sin^2 t - \cos^2 t) \\ &= -9a^2 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) = -9a^2 \sin^2 t \cos^2 t \end{aligned}$$

$$\therefore \rho \text{ at any point } t \text{ i.e., } (a \cos^3 t, a \sin^3 t)$$

$$\begin{aligned} &= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(9a^2 \cos^2 t \sin^2 t)^{3/2}}{-9a^2 \sin^2 t \cos^2 t} \\ &= \frac{27a^3 \cos^3 t \sin^3 t}{-9a^2 \sin^2 t \cos^2 t} = -3a \sin t \cos t \end{aligned}$$

$$\text{or} \quad 3a \sin t \cos t \text{ (in magnitude)}$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{y'}{x'} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\tan t.$$

$\therefore$ Equation of tangent at $(a \cos^3 t, a \sin^3 t)$ is

$$y - a \sin^3 t = -\tan t (x - a \cos^3 t)$$

$$\text{or} \quad y - a \sin^3 t = -\frac{\sin t}{\cos t} (x - a \cos^3 t)$$

$$\begin{aligned}
 y \cos t - a \sin^3 t \cos t &= -x \sin t + a \sin t \cos^3 t \\
 \text{or } x \sin t + y \cot t - a \sin t \cos t (\sin^2 t + \cos^2 t) &= 0 \\
 \Rightarrow x \sin t + y \cos t - a \sin t \cos t &= 0 \\
 p &= \text{length of } \perp \text{ from the origin } (0, 0) \text{ on the tangent at } t \\
 &= \frac{|0 + 0 - a \sin t \cos t|}{\sqrt{\sin^2 t + \cos^2 t}} = a \sin t \cos t
 \end{aligned}$$

Obviously,

$$\rho = 3p.$$

Hence proved.

99. The tangents at two points P, Q on the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ are at right angles.

Show that if ρ_1, ρ_2 be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

Sol. Here,

$$\begin{aligned}
 x &= a(\theta - \sin \theta), & y &= a(1 - \cos \theta) \\
 x' &= a(1 - \cos \theta), & y' &= a \sin \theta \\
 x'' &= a \sin \theta, & y'' &= a \cos \theta
 \end{aligned}$$

$$\therefore x'^2 + y'^2 = a^2(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)$$

$$\text{or } x'^2 + y'^2 = 2a^2(1 - \cos \theta) = 4a^2 \sin^2 \frac{\theta}{2}$$

$$\begin{aligned}
 x'y'' - y'x'' &= a^2 \cos \theta (1 - \cos \theta) - a^2 \sin^2 \theta \\
 &= a^2(\cos \theta - 1) = -2a^2 \sin^2 \frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \rho \text{ at } \theta &= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{8a^3 \sin^3 \theta/2}{-2a^2 \sin^2 \theta/2} \\
 &= 4a \sin \theta/2 \text{ (in magnitude)}
 \end{aligned}$$

Let the points P and Q correspond to $\theta = \theta_1$ and $\theta = \theta_2$, then

$$\rho_1 = 4a \sin \frac{\theta_1}{2}, \quad \rho_2 = 4a \sin \frac{\theta_2}{2}$$

$$\text{Now, } \frac{dy}{dx} = \frac{y'}{x'} = \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

$$\Rightarrow \text{Slope of tangent at } \theta = \cot \frac{\theta}{2}$$

$$\therefore \text{Slopes of tangents at } P \text{ and } Q \text{ are } \cot \frac{\theta_1}{2} \text{ and } \cot \frac{\theta_2}{2}.$$

Since the tangents at P and Q are perpendicular,

$$\cot \frac{\theta_1}{2} \cdot \cot \frac{\theta_2}{2} = -1$$

$$\Rightarrow \cot \frac{\theta_2}{2} = -\tan \frac{\theta_1}{2} = \cot \left(\frac{\pi}{2} + \frac{\theta_1}{2} \right)$$

$$\Rightarrow \theta_2 = \pi + \theta_1$$

$$\therefore \rho_2 = 4a \sin \left(\frac{\pi}{2} + \frac{\theta_1}{2} \right) = 4a \cos \frac{\theta_1}{2}$$

Hence,

$$\rho_1^2 + \rho_2^2 = (4a)^2 \left(\sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_1}{2} \right)$$

or

$$\rho_1^2 + \rho_2^2 = 16a^2.$$

- 100.** Show that for the parabola $y^2 = 4ax$, ρ^2 varies as $(SP)^3$, where ρ is the radius of curvature at any point P of the parabola and S is the focus of the parabola. (M.D.U., May 2009)

Sol. Equation of parabola is $y^2 = 4ax$

Parametric equations are

$$x = at^2, \quad y = 2at$$

Any point P on it is $(at^2, 2at)$

$$x' = 2at, \quad y' = 2a$$

$$x'' = 2a, \quad y'' = 0$$

$$\begin{aligned} \therefore \rho \text{ at } P \text{ 't'} &= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(4a^2t^2 + 4a^2)^{3/2}}{-4a^2} \\ &= \frac{8a^3(1+t^2)^{3/2}}{4a^2} = 2a(1+t^2)^{3/2} \end{aligned}$$

Also, SP = distance between $S(a, 0)$ and $P(at^2, 2at)$

$$= \sqrt{(at^2 - a)^2 + (2at - 0)^2} = \sqrt{a^2(t^2 - 1)^2 + 4a^2t^2} = a\sqrt{(t^2 + 1)^2} = a(t^2 + 1)$$

$$\begin{aligned} \therefore \rho^2 &= 4a^2(1+t^2)^3 = \frac{4}{a} \cdot a^3(1+t^2)^3 \\ &= \frac{4}{a} \cdot (SP)^3 \quad \text{or} \quad \propto (SP)^3 \end{aligned}$$

Hence, ρ^2 varies as $(SP)^3$.

- 101.** Show that the radius of curvature at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, is $3a \sin \theta \cos \theta$.

Sol. Equation of curve is

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta (-\sin \theta)} = -\tan \theta$$

Also,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\sec^2 \theta \cdot \frac{d\theta}{dx} \\ &= -\sec^2 \theta \cdot \frac{1}{(-3a \cos^2 \theta \sin \theta)} = \frac{1}{3a \cos^4 \theta \sin \theta} \end{aligned}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\sec^3 \theta}{\frac{1}{3a \cos^4 \theta \sin \theta}}$$

$\Rightarrow$

$$\rho = 3a \cos^4 \theta \sin \theta \sec^3 \theta \quad \text{or} \quad 3a \sin \theta \cos \theta.$$

102. Show that for rectangular hyperbola: $xy = c^2$, $\rho = \frac{(x^2 + y^2)^{3/2}}{2c^2}$.

Sol. Equation of the hyperbola is

$$xy = c^2 \quad \text{or} \quad y = \frac{c^2}{x}$$

Differentiating the above equation w.r.t. x , we get

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2c^2}{x^3}$$

$$\begin{aligned} \therefore \rho &= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{c^4}{x^4}\right)^{3/2}}{2c^2/x^3} \\ &= \frac{(x^4 + c^4)^{3/2}}{2c^2 \cdot (x^4)^{3/2}} \cdot x^3 = \frac{x^3(x^2 + y^2)^{3/2} \cdot x^3}{2c^2 \cdot x^6} \end{aligned}$$

$$\Rightarrow \rho = \frac{(x^2 + y^2)^{3/2}}{2c^2}$$

Hence shown.

103. Find the radius of curvature of the curve: $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$.

Sol. Given equation is

$$y^2 = a^2 \left[\frac{a}{x} - 1 \right]$$

$$\text{or} \quad \frac{y^2}{a^2} = \frac{a}{x} - 1$$

$$\text{or} \quad \frac{y^2}{a^2} + 1 = \frac{a}{x}$$

$$\Rightarrow x = \frac{a^3}{y^2 + a^2}$$

$$\therefore \frac{dx}{dy} = a^3 \left[\frac{-2y}{(y^2 + a^2)^2} \right] = \frac{-2a^3y}{(y^2 + a^2)^2}$$

At $(a, 0)$, we have $\frac{dx}{dy} = 0$

$$\frac{d^2x}{dy^2} = -2a^3 \cdot \left[\frac{(y^2 + a^2)^2 - y \cdot 2(y^2 + a^2) \cdot 2y}{(y^2 + a^2)^4} \right]$$

$$= \frac{-2a^3(y^2 + a^2)}{(y^2 + a^2)^4} [-4y^2 + y^2 + a^2] = -\frac{2a^3(a^2 - 3y^2)}{(y^2 + a^2)^3}$$

$$\therefore \frac{d^2x}{dy^2} \text{ (at } a, 0) = -\frac{2a^5}{a^6} \quad \text{or} \quad -\frac{2}{a}$$

$$\text{Hence,} \quad \rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{1}{-2/a} = -\frac{a}{2}$$

$$\text{or} \quad \rho = \frac{a}{2} \text{ (in magnitude).}$$

104. Find the radius of curvature at any point of the curve:

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta). \quad (\text{M.D.U., May 2008})$$

Sol. We have

$$x' = a(1 + \cos \theta), \quad y' = a \sin \theta$$

$$x'' = -a \sin \theta, \quad y'' = a \cos \theta$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{[a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}}{a^2 \cos \theta (1 + \cos \theta) + a^2 \sin^2 \theta}$$

$$= \frac{a[2(1 + \cos \theta)]^{3/2}}{(1 + \cos \theta)} = a \cdot 2^{3/2} \cdot (1 + \cos \theta)^{1/2}$$

$$= 2^{3/2}a \cdot (2 \cos^2 \theta/2)^{1/2} = 4a \cos \theta/2$$

$$\text{Hence,} \quad \rho = 4a \cos \theta/2.$$

105. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then prove that: $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. (M.D.U., Dec., 2006, Delhi, 2009)

Sol. For the parabola $y^2 = 4ax$, the extremities of a focal chord will be the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

$$\text{Hence,} \quad t_1 t_2 = -1 \quad \text{or} \quad t_2 = -\frac{1}{t_1}.$$

$x = at^2, y = 2at$ are the parametric equations of parabola $y^2 = 4ax$

$$x' = 2at, \quad y' = 2a$$

$$x'' = 2a, \quad y'' = 0$$

$$\therefore \rho \text{ (at point } t) = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - y'x'')}$$

$$\rho = \frac{(4a^2t^2 + 4a^2)^{3/2}}{(-4a^2)} = \frac{8a^3(1+t^2)^{3/2}}{-4a^2} = -2a(1+t^2)^{3/2}$$

(Neglecting -ve sign)

$$\therefore \rho_1 = 2a(1+t_1^2)^{3/2} \text{ and } \rho_2 = 2a(1+t_2^2)^{3/2}$$

$$\begin{aligned}
 \text{Hence, } \rho_1^{-2/3} + \rho_2^{-2/3} &= (2a)^{-2/3} \left[\frac{1}{1+t_1^2} + \frac{1}{1+t_2^2} \right] = (2a)^{-2/3} \left[\frac{1+t_1^2+1+t_2^2}{(1+t_1^2)(1+t_2^2)} \right] \\
 &= (2a)^{-2/3} \left[\frac{2+t_1^2+t_2^2}{1+t_1^2+t_2^2+t_1^2 t_2^2} \right] = (2a)^{-2/3} \cdot 1. \quad (\because t_1^2 + t_2^2 = 1) \\
 &= (2a)^{-2/3}.
 \end{aligned}$$

- 106.** Show that the radius of curvature ρ at any point (x, y) of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ satisfies $\rho^3 = 27axy$.

Sol. We know that ρ at any point (x, y) of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a \sin \theta \cos \theta$.

Any point (x, y) is given by $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ [Pl. see solution of Q. 101]

$$\begin{aligned}
 \text{Now, } \rho^3 &= (3a \sin \theta \cos \theta)^3 \\
 &= 27a^3 \sin^3 \theta \cos^3 \theta = 27a(a^2 \sin^3 \theta \cos^3 \theta)
 \end{aligned}$$

$$\text{or } \rho^3 = 27a(a \sin^3 \theta)(a \cos^3 \theta)$$

$$\Rightarrow \rho^3 = 27axy.$$

Hence shown.

- 107.** Find the radius of curvature at any point $(0, c)$ of the catenary $y = c \cosh \frac{x}{c}$

Sol. Given equation is $y = c \cosh \frac{x}{c}$

$$\frac{dy}{dx} = c \cdot \sinh \left(\frac{x}{c} \right) \cdot \frac{1}{c} = \sinh \frac{x}{c} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{c} \cdot \cosh \frac{x}{c}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \sinh^2 \frac{x}{c} \right)^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}}$$

$$\text{At } (0, c); \quad \rho = \frac{(1+0)^{3/2}}{\frac{1}{c} \cdot 1} = c.$$

- 108.** Prove that the radius of curvature for the catenary $y = c \cosh \frac{x}{c}$ is equal to the portion of the normal intercepted between the curve and the x -axis and that it varies as the square of the ordinate.

Sol. As shown in the previous question, radius of curvature is given by

$$\rho = \frac{\left(1 + \sinh^2 \frac{x}{c} \right)^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}} = c \cosh^2 \frac{x}{c}$$

Now portion of the normal intercepted between the curve and the x -axis is called the length of the normal and is given by $y\sqrt{1+y_1^2}$

$$= c \cosh \frac{x}{c} \cdot \sqrt{1 + \left(\sinh \frac{x}{c} \right)^2}$$

$$= c \cosh \frac{x}{c} \cdot \cosh \frac{x}{c} = c \cosh^2 \frac{x}{c} = \rho.$$

Also $\rho \propto y^2$, which is obvious.

- 109.** Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7 \frac{13}{16}$.

Sol. Equation of the parabola is $y^2 = 8x$.

Any point on the parabola is $(2t^2, 4t)$

$$\begin{aligned} \text{i.e.,} \quad x &= 2t^2, & y &= 4t \\ x' &= 4t, & y' &= 4 \\ x'' &= 4, & y'' &= 0 \end{aligned}$$

$$\therefore \rho \text{ at } t = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{[(4t)^2 + 16]^{3/2}}{-16}$$

$$\Rightarrow \rho = \frac{64(t^2 + 1)^{3/2}}{16} \quad \text{or} \quad 4(1 + t^2)^{3/2} \quad (\text{neglecting -ve sign})$$

It is given that radius of curvature is $7 \frac{13}{16}$ or $\frac{125}{16}$.

$$\therefore \frac{125}{16} = 4(1 + t^2)^{3/2}$$

$$\text{or} \quad (1 + t^2)^{3/2} = \frac{125}{64}$$

$$\Rightarrow 1 + t^2 = \left(\frac{125}{64}\right)^{2/3} = \frac{25}{16}$$

$$\text{i.e.,} \quad t^2 = \frac{9}{16} \quad \text{or} \quad t = \pm \frac{3}{4}$$

Points on the parabola are given by $(2t^2, 4t)$

For $t = \frac{3}{4}$, the point is $\left(\frac{9}{8}, 3\right)$.

For $t = -\frac{3}{4}$, the point is $\left(\frac{9}{8}, -3\right)$.

- 110.** Show that for the curve $x = a(1 + \sin \theta) \cos \theta$, $y = a(1 + \cos \theta) \sin \theta$, the radius of curvature at

$\theta = -\frac{\pi}{4}$ is a .

$$\begin{aligned} \text{Sol.} \quad x' &= a[-\sin \theta (1 + \sin \theta) + \cos^2 \theta] = a[-\sin \theta - \sin^2 \theta + \cos^2 \theta] \\ x'' &= a[-\cos \theta - 2 \sin \theta \cos \theta + 2 \cos \theta (-\sin \theta)] = -a \cos \theta [1 + 4 \sin \theta] \\ y &= a \sin \theta (1 + \cos \theta) \\ y' &= a[\cos \theta (1 + \cos \theta) + \sin \theta (-\sin \theta)] = a[\cos \theta + \cos^2 \theta - \sin^2 \theta] \\ y'' &= a[-\sin \theta + 2 \cos \theta (-\sin \theta) - 2 \sin \theta \cos \theta] = -a \sin \theta [1 + 4 \cos \theta] \end{aligned}$$

ρ at any point ' θ ' is

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$\begin{aligned}
&= \frac{[a^2(-\sin \theta - \sin^2 \theta + \cos^2 \theta)^2 + a^2(\cos \theta - \sin^2 \theta + \cos^2 \theta)^2]^{3/2}}{a(-\sin \theta - \sin^2 \theta + \cos^2 \theta)(-a \sin \theta)(1 + 4 \cos \theta) + a^2 \cos \theta (\cos \theta + \cos^2 \theta - \sin^2 \theta)(1 + 4 \sin \theta)} \\
&= \frac{a^3}{a^2} \text{ or } a \text{ (at } \theta = -\pi/4 \text{) after simplification.}
\end{aligned}$$

- 111.** Show that the radius of curvature ρ at P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is given by $\rho = \frac{CD^3}{ab}$, where CD is the semi-diameter conjugate to CP .

Sol. Parametric equations of the ellipse are

$$x = a \cos t, \quad y = b \sin t$$

$$\therefore \rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - y'x'')} = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab(\sin^2 t + \cos^2 t)}$$

$$\text{or} \quad \rho = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab} \quad \dots(1)$$

Now, P and D are the ends of semi-conjugate diameters.

$$P = (a \cos t, b \sin t)$$

Changing t to $\frac{\pi}{2} + t$, the co-ordinates of D are $\left[a \cos \left(\frac{\pi}{2} + t \right), b \sin \left(\frac{\pi}{2} + t \right) \right]$ or $(-a \sin t, b \cos t)$

Now, $C(0, 0)$ is the centre of the ellipse.

$$\therefore CD = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$$

$$\text{or} \quad CD^2 = a^2 \sin^2 t + b^2 \cos^2 t \quad \dots(2)$$

From (1) and (2), we have $\rho = \frac{CD^3}{ab}$.

Hence proved.

- 112.** Show that radius of curvature at the end of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is equal to the semi-latus rectum. [M.D.U., Dec., 2008]

Sol. At the ends of major axis, $\theta = 0, \pi$

Any point on the ellipse is $(a \cos \theta, b \sin \theta)$

$$x = a \cos \theta, \quad y = b \sin \theta$$

$$x' = -a \sin \theta, \quad y' = b \cos \theta$$

$$x'' = -a \cos \theta, \quad y'' = -b \sin \theta$$

$$\text{Radius of curvature} = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab \sin^2 \theta + ab \cos^2 \theta}$$

$$\text{Radius of curvature} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab}$$

Since, $\theta = 0$ and π at the ends of major axis,

$$\text{Radius of curvature (at } \theta = 0) = \frac{(b^2)^{3/2}}{ab} = \frac{b^2}{a} \text{ and}$$

$$\text{Radius of curvature (at } \theta = \pi) = \frac{(b^2)^{3/2}}{ab} = \frac{b^2}{a}$$

Hence, radius of curvature at the end of the major axis is $\frac{b^2}{a}$ i.e., semi-latus rectum of the ellipse.

$$\text{Latus rectum of ellipse} = \frac{2b^2}{a}.$$

- 113.** Find the radius of curvature at the origin of the two branches of the curve given by $x = 1 - t^2$, $y = t - t^3$.

Sol. At the origin

$$\begin{aligned} x = y = 0 & \Rightarrow t = \pm 1 \\ x = 1 - t^2, & y = t - t^3 \\ x' = -2t, & y' = 1 - 3t^2 \\ x'' = -2, & y'' = -6t \end{aligned}$$

Radius of curvature is given by

$$\begin{aligned} \rho &= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{[4t^2 + (1 - 3t^2)^2]^{3/2}}{(-2t)(-6t) + 2(1 - 3t^2)} = \frac{(4t^2 + 1 + 9t^4 - 6t^2)^{3/2}}{12t^2 + 2 - 6t^2} \\ \rho &= \frac{(1 - 2t^2 + 9t^4)^{3/2}}{2 + 6t^2} \end{aligned}$$

At the origin we have, $t = \pm 1$

Radius of curvature at the origin :

$$\text{When } t = 1, \quad \rho = \frac{(8)^{3/2}}{8} = 2\sqrt{2}$$

$$\text{When } t = -1, \quad \rho = 2\sqrt{2} \text{ (same)}$$

Hence, radius of curvature at the origin of the two branches of the given curve is $2\sqrt{2}$.

- 114.** Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

Sol. The curve $y = e^x$ crosses the y-axis when $x = 0$

$$\Rightarrow y = e^0 = 1$$

$$\frac{dy}{dx} = e^x, \quad \frac{d^2y}{dx^2} = e^x$$

$$\rho = \frac{(1 + e^{2x})^{3/2}}{e^x}$$

$$\text{At } x = 0, \quad \rho = \frac{(2)^{3/2}}{1} = (8)^{1/2} = 2\sqrt{2}.$$

- 115.** Show that the radius of curvature at any point of the cardioid $r = a(1 + \cos \theta)$ is $\frac{2}{3}\sqrt{2ar}$ and

prove that $\frac{\rho^2}{r}$ is constant.

Sol. Radius of curvature for the curve

$$r = f(\theta) \text{ or } f(r, \theta) = 0, \text{ is given by}$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2} \quad \text{where, } r_1 = \frac{dr}{d\theta},$$

and $r_2 = \frac{d^2r}{d\theta^2}$

Here, $r = a(1 + \cos \theta)$

$\therefore r_1 = -a \sin \theta$

and $r_2 = -a \cos \theta$

$$\rho = \frac{[a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}}{a^2(1 + \cos \theta)^2 - a(1 + \cos \theta)(-a \cos \theta) + 2a^2 \sin^2 \theta}$$

$$= \frac{a^3[(1 + \cos \theta)^2 + (1 - \cos \theta)(1 + \cos \theta)]^{3/2}}{a^2[(1 + \cos \theta)(1 + \cos \theta + \cos \theta + 2 - 2 \cos \theta)]}$$

$$= \frac{a[(1 + \cos \theta)(1 + \cos \theta + 1 - \cos \theta)]^{3/2}}{3(1 + \cos \theta)}$$

$$\Rightarrow \rho = \frac{a(1 + \cos \theta)^{3/2} \cdot (2)^{3/2}}{3(1 + \cos \theta)} = \frac{a(1 + \cos \theta)^{1/2} \cdot 2\sqrt{2}}{3}$$

$$= \frac{a}{3} \cdot \sqrt{\frac{r}{a}} \cdot 2\sqrt{2} \quad \left| \because 1 + \cos \theta = \frac{r}{a} \right.$$

or $\rho = \frac{2}{3} \sqrt{2ar}$

Now, $\rho^2 = \frac{8}{9} ar$

or $\frac{\rho^2}{r} = \frac{8a}{9}$, which is constant.

Hence proved.

Note : Here, $\rho^2 = \frac{8}{9} ar$ or $\rho = \frac{2\sqrt{2a}}{3} \sqrt{r} \therefore \rho$ varies as $\sqrt{r}$.

116. Find the radius of curvature at any point of the curve $r^n = a^n \cos n\theta$.

Sol. Equation of the curve is $r^n = a^n \cos n\theta$... (1)

Taking logarithms of both sides, we have

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating w.r.t. θ , we have

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} (-n \sin n\theta)$$

or $\frac{dr}{d\theta} = -r \tan n\theta$

$\therefore r_1 = -r \tan n\theta$

Differentiating again w.r.t. θ , we have

$$r_2 = -r(n \sec^2 n\theta) - r_1 \tan n\theta = -nr \sec^2 n\theta + r \tan^2 n\theta$$

$$\begin{aligned} \therefore \rho &= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r(r \tan^2 n\theta - nr \sec^2 n\theta)} \\ \text{or} \quad \rho &= \frac{r^3 \sec^2 n\theta}{r^2[1 + \tan^2 n\theta + n \sec^2 n\theta]} = \frac{r \sec^3 n\theta}{\sec^2 n\theta + n \sec^2 n\theta} \\ \text{or} \quad \rho &= \frac{r \sec n\theta}{n+1} = \frac{r}{n+1} \cdot \frac{1}{\cos n\theta} \\ \text{or} \quad \rho &= \frac{r}{n+1} \cdot \frac{a^n}{r^n} = \frac{a^n}{(n+1)r^{n-1}}. \end{aligned}$$

117. Find the radius of curvature at any point (r, θ) on the curve $\frac{l}{r} = 1 + e \cos \theta$.

Sol. Given curve is a polar equation, $\frac{l}{r} = 1 + e \cos \theta$

If the equation of a curve is given in the polar form $u = f(\theta)$, where $u = \frac{1}{r}$,

Then,
$$\rho = \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)}, \quad \text{where } u_1 = \frac{du}{d\theta}, u_2 = \frac{d^2u}{d\theta^2}.$$

Here,
$$u = \frac{1}{l}(1 + e \cos \theta).$$

$$\therefore \frac{du}{d\theta} = u_1 = -\frac{e \sin \theta}{l} \quad \text{and} \quad u_2 = \frac{d^2u}{d\theta^2} = -\frac{e \cos \theta}{l}$$

$$\begin{aligned} \therefore \rho &= \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)} \\ &= \frac{\left[\frac{(1 + e \cos \theta)^2}{l^2} + \frac{e^2 \sin^2 \theta}{l^2} \right]^{3/2}}{\frac{(1 + e \cos \theta)^3}{l^3} \left[\frac{1 + e \cos \theta}{l} - \frac{e \cos \theta}{l} \right]} \\ &= \frac{(1 + e^2 + 2e \cos \theta)^{3/2}}{l^3} \cdot \frac{l^4}{(1 + e \cos \theta)^3} = \frac{l(1 + e^2 + 2e \cos \theta)^{3/2}}{(1 + e \cos \theta)^3}. \end{aligned}$$

118. Find the radius of curvature at any point of the curve $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1} \frac{a}{r}$.

Sol. Equation of the curve is $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1} \frac{a}{r}$

$$\therefore \frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{2r}{2\sqrt{r^2 - a^2}} + \frac{1}{\sqrt{1 - \frac{a^2}{r^2}}} \left(-\frac{a}{r^2} \right) = \frac{r}{a\sqrt{r^2 - a^2}} - \frac{a}{r\sqrt{r^2 - a^2}}$$

$$= \frac{r^2 - a^2}{ar\sqrt{r^2 - a^2}} = \frac{\sqrt{r^2 - a^2}}{ar}$$

Since $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{a^2 r^2}{(r^2 - a^2)} = \frac{1}{r^2} + \frac{a^2}{r^2(r^2 - a^2)} = \frac{1}{r^2 - a^2}$$

$$\therefore p^2 = r^2 - a^2 \quad \text{(Pedal equation)}$$

$$\therefore \rho = r \frac{dr}{dp} \quad \text{(Formula for } \rho \text{ for a pedal equation)}$$

Now, $\frac{dr}{dp}$ can be found out from $p^2 = r^2 - a^2$ by differentiating w.r.t. p which gives

$$2p = 2r \frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{p}{r}$$

Hence $\rho = r \frac{dr}{dp} = r \cdot \frac{p}{r} = p$

$$\therefore \rho = p = \sqrt{r^2 - a^2}$$

119. If ϕ be the angle which the radius vector of the curve $r = f(\theta)$ makes with the tangent, prove that

$$\rho = \frac{r \operatorname{cosec} \phi}{\left(1 + \frac{d\phi}{d\theta}\right)}. \text{ Apply this result to show that } \rho = \frac{a}{2} \text{ for the circle } r = a \cos \theta.$$

Sol. We know that $\psi = \theta + \phi$

Differentiating w.r.t. s

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} \left[1 + \frac{d\phi}{d\theta} \right]$$

$$\frac{1}{\rho} = \frac{\sin \phi}{r} \left[1 + \frac{d\phi}{d\theta} \right] \quad \left| \because \sin \phi = r \frac{d\theta}{ds} \right.$$

Hence $\rho = \frac{r \operatorname{cosec} \phi}{\left(1 + \frac{d\phi}{d\theta}\right)} \quad \dots(1)$

Equation of the circle is $r = a \cos \theta$

To find ϕ : $\frac{dr}{d\theta} = -a \sin \theta, \quad \tan \phi = r \frac{d\theta}{dr}$

$$\therefore \tan \phi = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta = \tan \left(\frac{\pi}{2} + \theta \right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + \theta$$

$$\Rightarrow \frac{d\phi}{d\theta} = 1.$$

$$\begin{aligned} \therefore \text{From (1),} \quad \rho &= \frac{r \operatorname{cosec} \phi}{\left(1 + \frac{d\phi}{d\theta}\right)} = \frac{a \cos \theta \cdot \operatorname{cosec} \left(\frac{\pi}{2} + \theta\right)}{2} \\ &= \frac{a}{2} \cos \theta \cdot \sec \theta = \frac{a}{2} \end{aligned}$$

Hence $\rho = \frac{a}{2}$ for the circle.

120. Show that the radius of curvature at any point of the cardioid $r = a(1 - \cos \theta)$ varies as $\sqrt{r}$.

Sol.

$$r = a(1 - \cos \theta)$$

$\therefore$

$$r_1 = a \sin \theta$$

$$r_2 = a \cos \theta$$

$\therefore$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{[a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}}{a^2(1 - \cos \theta)^2 + 2a^2 \sin^2 \theta - a^2 \cos \theta(1 - \cos \theta)}$$

or

$$\begin{aligned} \rho &= \frac{a[(1 - \cos \theta)^2 + (1 - \cos^2 \theta)]^{3/2}}{(1 - \cos \theta)^2 + 2(1 - \cos^2 \theta) - \cos \theta(1 - \cos \theta)} \\ &= \frac{a[(1 - \cos \theta)(1 - \cos \theta + 1 + \cos \theta)]^{3/2}}{(1 - \cos \theta)[1 - \cos \theta + 2 + 2 \cos \theta - \cos \theta]} = \frac{a(1 - \cos \theta)^{3/2} \cdot (2)^{3/2}}{3(1 - \cos \theta)} \\ &= \frac{a}{3} (1 - \cos \theta)^{1/2} \cdot (2)^{3/2} = \frac{a}{3} \cdot \sqrt{\frac{r}{a}} \cdot 2\sqrt{2} \end{aligned}$$

$\Rightarrow$

$$\rho = \frac{2\sqrt{2} \cdot a}{3} \sqrt{\frac{r}{a}} = \frac{2\sqrt{2a}}{3} \cdot \sqrt{r}$$

or

$$\frac{\rho}{\sqrt{r}} = \frac{2\sqrt{2a}}{3} \quad (a \text{ constant})$$

$\Rightarrow \rho$ varies as $\sqrt{r}$.

121. Show that for the curve $r = a(1 + \cos \theta)$, the radius of curvature $\rho = \frac{4a}{3} \cos \frac{\theta}{2}$.

Sol. As shown in question 115 earlier, the radius of curvature

$$\rho = \frac{2\sqrt{2}}{3} \cdot a \cdot (1 + \cos \theta)^{\frac{1}{2}}$$

Since

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$\Rightarrow$

$$\rho = \frac{2\sqrt{2}}{3} \cdot a \cdot \sqrt{2} \cdot \cos \frac{\theta}{2}$$

or

$$\rho = \frac{4a}{3} \cos \frac{\theta}{2}.$$

122. Find the radius of curvature at any point (r, θ) on the curve $\sqrt{r} \cos \theta/2 = \sqrt{a}$.

Sol. Here, $\sqrt{r} \cos \frac{\theta}{2} = \sqrt{a}$

$$r \cos^2 \frac{\theta}{2} = a$$

or $r = a \sec^2 \frac{\theta}{2}$

$$r_1 = a \cdot 2 \sec \frac{\theta}{2} \cdot \sec \frac{\theta}{2} \tan \frac{\theta}{2} \cdot \frac{1}{2} = a \sec^2 \frac{\theta}{2} \tan \frac{\theta}{2}$$

$\Rightarrow r_1 = r \tan \frac{\theta}{2}$

Now, $r_2 = r \sec^2 \frac{\theta}{2} \cdot \frac{1}{2} + r_1 \tan \frac{\theta}{2} = \frac{1}{2} r \sec^2 \frac{\theta}{2} + r \tan^2 \frac{\theta}{2} \quad [\because r_1 = r \tan \frac{\theta}{2}]$

Now,
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2 \tan^2 \frac{\theta}{2})^{3/2}}{r^2 + 2r^2 \tan^2 \frac{\theta}{2} - r \left(r \tan^2 \frac{\theta}{2} + \frac{1}{2} r \sec^2 \frac{\theta}{2} \right)}$$

$$= \frac{r^3 \cdot \sec^3 \frac{\theta}{2}}{r^2 \left(1 + \tan^2 \frac{\theta}{2} \right) - \frac{1}{2} r^2 \sec^2 \frac{\theta}{2}} = \frac{r^3 \sec^3 \frac{\theta}{2}}{\frac{1}{2} r^2 \sec^2 \frac{\theta}{2}} = 2r \sec \frac{\theta}{2}$$

But $r = a \sec^2 \frac{\theta}{2}$

$\therefore \rho = 2r \cdot \sqrt{\frac{r}{a}}$

123. Find the radius of curvature at the point (p, r) on the lemniscate $r^3 = a^2 p$.

Sol. Here $r^3 = a^2 p$

$\therefore 3r^2 \frac{dr}{dp} = a^2$

or $r \frac{dr}{dp} = \frac{a^2}{3r}$

$\therefore \rho = \frac{a^2}{3r}$

124. Find the radius of curvature for the parabola $\frac{2a}{r} = 1 + \cos \theta$.

Sol. Let us first find the Pedal equation of the given parabola, $\frac{2a}{r} = 1 + \cos \theta \quad \dots(1)$

Taking logs on both sides, we get

$$\log 2a - \log r = \log (1 + \cos \theta)$$

Differentiating both sides w.r.t. θ , we get

$$-\frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin \theta}{1 + \cos \theta} = -\tan \frac{\theta}{2}$$

$$r \frac{d\theta}{dr} = \cot \frac{\theta}{2}$$

Since $r \frac{d\theta}{dr} = \tan \phi$

$\therefore \tan \phi = \cot \frac{\theta}{2} = \tan \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$

or $\phi = \frac{\pi}{2} - \frac{\theta}{2}$

$$p = r \sin \phi = r \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$$

$$p = r \cos \frac{\theta}{2}$$

$$p^2 = r^2 \cos^2 \frac{\theta}{2} = r^2 \left(\frac{1 + \cos \theta}{2} \right) = r^2 \left(\frac{2a}{2r} \right) = ar$$

$$p^2 = ar \text{ is the pedal equation}$$

$$2p \frac{dp}{dr} = a \quad \text{or} \quad \frac{dp}{dr} = \frac{a}{2p} = \frac{a}{2\sqrt{ar}} = \frac{1}{2} \sqrt{\frac{a}{r}}$$

$\therefore \rho = \frac{rdr}{dp} = r \cdot 2\sqrt{\frac{r}{a}} = \frac{2}{\sqrt{a}} r^{3/2}$

125. Find the radius of curvature at the origin for the curve

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

Sol. Equation of the curve is

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0 \quad \dots (1)$$

The curve passes through the origin.

Equating to zero the terms of the lowest degree, we have $y = 0$ i.e., x -axis is the tangent at the origin. Radius of curvature at the origin is found out by *Newton's Method* in this situation.

$$\rho = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{x^2}{2y} \right)$$

Dividing equation (1) by $2y$, we have

$$x \cdot \frac{x^2}{2y} - x^2 + \frac{3}{2}xy - 2y^2 + 5 \cdot \frac{x^2}{2y} - 3x + \frac{7}{2}y - 4 = 0$$

Taking limits as $x \rightarrow 0, y \rightarrow 0$, we have

$$5 \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{x^2}{2y} \right) - 4 = 0 \quad \text{or} \quad 5\rho - 4 = 0$$

$$\therefore \quad \rho = \frac{4}{5}$$

126. Find the radius of curvature at the origin for the curve

$$2x^4 + 4x^3y + xy^2 + 6y^3 - 3x^2 - 2xy + y^2 - 4x = 0.$$

Sol. Equation of curve is

$$2x^4 + 4x^3y + xy^2 + 6y^3 - 3x^2 - 2xy + y^2 - 4x = 0 \quad \dots(1)$$

The curve passes through the origin.

Equating to zero the terms of the lowest degree, we have $x = 0$ i.e., y -axis is the tangent at the origin.

$$\therefore \quad \rho \text{ (at the origin)} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{y^2}{2x} \right)$$

Dividing equation (1) by $2x$, we have

$$x^3 + 2x^2y + \frac{1}{2}y^2 + 6y \cdot \frac{y^2}{2x} - \frac{3}{2}x - y + \frac{y^2}{2x} - 2 = 0$$

Taking limits as $x \rightarrow 0, y \rightarrow 0$, we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{y^2}{2x} \right) - 2 = 0 \quad (\because \text{Other terms vanish})$$

$$\text{or} \quad \rho - 2 = 0$$

$$\Rightarrow \quad \rho = 2.$$

127. Apply Newton's method to find the radius of curvature at the origin for the cycloid

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta).$$

$$\textbf{Sol.} \quad x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta)$$

$$\frac{dx}{d\theta} = a(1 + \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta} = \tan \frac{\theta}{2}$$

When $\theta = 0$, we have $x = 0$ and $y = 0$

$$\text{Also,} \quad \frac{dy}{dx} = 0$$

$\therefore$ The curve passes through the origin and x -axis is the tangent at the origin.

$$\begin{aligned} \therefore \quad \rho \text{ (at the origin)} &= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{x^2}{2y} \right) = \lim_{\theta \rightarrow 0} \frac{a^2(\theta + \sin \theta)^2}{2a(1 - \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \cdot \frac{a}{2} \cdot \frac{2(\theta + \sin \theta)(1 + \cos \theta)}{\sin \theta} \quad \left(\text{Form } \frac{0}{0} \right) \end{aligned}$$

$$\text{or} \quad \rho = \lim_{\theta \rightarrow 0} a \cdot \frac{(\theta + \sin \theta)(-\sin \theta) + (1 + \cos \theta)^2}{\cos \theta} = a \cdot \frac{4}{1} \quad \text{or} \quad 4a$$

Hence, $\rho = 4a$.

- 128.** Find the radius of curvature of the curve $r = a \sin n\theta$ at the pole.

Sol. Equation of the curve is $r = a \sin n\theta$... (1)

$$\frac{dr}{d\theta} = na \cos n\theta$$

$$\therefore \tan \phi = r \frac{d\theta}{dr} = \frac{a \sin n\theta}{na \cos n\theta} = \frac{1}{n} \tan n\theta \quad \dots (2)$$

When $\theta = 0$, from (1) $r = 0$

$\therefore$ The curve passes through the pole.

Also from (2), $\tan \phi = 0 \quad \therefore \phi = 0$

$\therefore$ Initial line is the tangent to the curve at pole.

$$\text{By Newton's method, } \rho \text{ (at pole)} = \lim_{\substack{r \rightarrow 0 \\ \theta \rightarrow 0}} \left(\frac{1}{2} \frac{dr}{d\theta} \right)$$

$$\text{or} \quad \rho = \lim_{\substack{r \rightarrow 0 \\ \theta \rightarrow 0}} \left(\frac{1}{2} \cdot na \cos n\theta \right) = \frac{na}{2}$$

- 129.** Show that the radius of curvature of the curve $y^2 = x^2 \frac{(a+x)}{(a-x)}$ at the origin is $a\sqrt{2}$.

Sol. Equation of the curve is $y^2(a-x) = x^2(a+x)$... (1)

It passes through the origin.

Equating to zero the lowest degree terms, we have $a(y^2 - x^2) = 0$

$\therefore y = \pm x$ are the tangents at the origin. So, Newton's method is not applicable. Therefore, we use the method of expansion.

$$\text{Let} \quad y = px + q \frac{x^2}{2!} + \dots$$

Putting this value of y in equation (1), we get

$$\left(px + q \frac{x^2}{2!} + \dots \right)^2 (a-x) = x^2(a+x) \quad \dots (2)$$

Equating coefficients of x^2 on both sides,

$$ap^2 = a \quad \text{or} \quad p = \pm 1$$

Equating coefficients of x^3 on both sides,

$$-p^2 + apq = 1 \quad \dots (3)$$

When $p = 1$, $aq = 2$, or $q = \frac{2}{a}$

$$\therefore \rho \text{ (at the origin)} = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\frac{2}{a}} = 2\sqrt{2} \cdot \frac{a}{2} = \sqrt{2} \cdot a$$

When $p = -1$, from equation (3), $-1 - aq = 1$

$$\text{or} \quad q = -\frac{2}{a}$$

$$\therefore \quad \rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{-2/a} = \sqrt{2} \cdot a \quad (\text{neglecting } - \text{ve sign})$$

Hence ρ at the origin is $a\sqrt{2}$.

- 130.** Find the coordinates of the centre of curvature for any point (x, y) on the parabola $y^2 = 4ax$. Also find the equation of the evolute (locus of the centre of curvature) of the parabola.

(M.D.U., U.P.T.U., 2009., May, 2007)

Sol. The parametric equations of the parabola $y^2 = 4ax$, are

$$x = at^2, \quad y = 2at$$

$$\therefore \quad \frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t} \quad \text{and} \quad y_2 = -\frac{1}{t^2} \cdot \frac{dt}{dx} = -\frac{1}{t^2} \cdot \frac{1}{2at} = -\frac{1}{2at^3}$$

If (X, Y) is the centre of curvature at ' t ', then

$$X = x - \frac{y_1(1+y_1^2)}{y_2}, \quad Y = y + \frac{1+y_1^2}{y_2}$$

$$\therefore \quad X = at^2 - \frac{\frac{1}{t} \left(1 + \frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = at^2 + 2at^2 \left(1 + \frac{1}{t^2}\right)$$

$$\text{or} \quad X = 3at^2 + 2a \quad \dots(1)$$

$$Y = y + \frac{1+y_1^2}{y_2} = 2at + \frac{1 + \frac{1}{t^2}}{-\frac{1}{2at^3}} = 2at - 2at^3 \left(1 + \frac{1}{t^2}\right)$$

$$\Rightarrow \quad Y = 2at - 2at^3 - 2at$$

$$\text{or} \quad Y = -2at^3 \quad \dots(2)$$

$\therefore$ The co-ordinates of the centre of curvature at any point (x, y) i.e., at any point ' t ' are $(2a + 3at^2, -2at^3)$.

Now evolute is locus of the centre of curvature.

To eliminate the parameter ' t ' between (1) and (2).

$$\text{From (1),} \quad t^2 = \frac{X - 2a}{3a} \quad \dots(3)$$

$$\text{From (2),} \quad t^3 = -\frac{Y}{2a} \quad \dots(4)$$

Cubing (3), squaring (4) and equating the two values of t^6 , we get

$$\frac{(X - 2a)^3}{27a^3} = \frac{Y^2}{4a^2}$$

$$\text{or} \quad 27aY^2 = 4(X - 2a)^3$$

Changing X to x and Y to y , the locus of (X, Y) i.e., the evolute of the parabola is

$$27ay^2 = 4(x - 2a)^3.$$

- 131.** Find the co-ordinates of the centre of curvature of the ellipse $x = a \cos \theta$, $y = b \sin \theta$. Hence show that the equation of its evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

or

Find the co-ordinates of the centre of curvature at any point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the evolute of the ellipse.

Sol. Equations of ellipse are

$$x = a \cos \theta, \quad y = b \sin \theta$$

$$\therefore \frac{dx}{d\theta} = -a \sin \theta, \quad \frac{dy}{d\theta} = b \cos \theta$$

$$y_1 = \frac{dy}{dx} = -\frac{b \cos \theta}{a \sin \theta} = -\frac{b}{a} \cot \theta$$

$$y_2 = \frac{b}{a} \operatorname{cosec}^2 \theta \cdot \frac{d\theta}{dx} = -\frac{b}{a^2} \operatorname{cosec}^3 \theta$$

Let (X, Y) be the co-ordinates of the centre of curvature. Then

$$X = x - \frac{y_1(1 + y_1^2)}{y_2} = a \cos \theta - \frac{-\frac{b}{a} \cot \theta \left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)}{-\frac{b}{a^2} \operatorname{cosec}^3 \theta}$$

$$\Rightarrow X = a \cos \theta - a \cot \theta \sin^3 \theta \left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)$$

$$\Rightarrow X = a \cos \theta - a \cot \theta \sin^3 \theta - \frac{b^2}{a} \cos^3 \theta$$

$$\Rightarrow X = a \cos \theta - a \cos \theta \sin^2 \theta - \frac{b^2}{a} \cos^3 \theta$$

$$\Rightarrow X = a \cos \theta (1 - \sin^2 \theta) - \frac{b^2}{a} \cos^3 \theta$$

$$\therefore X = \frac{a^2 - b^2}{a} \cos^3 \theta \quad \dots(1)$$

and $Y = y + \frac{1 + y_1^2}{y_2} = b \sin \theta + \frac{1 + \frac{b^2}{a^2} \cot^2 \theta}{-\frac{b}{a^2} \operatorname{cosec}^3 \theta}$

$$= b \sin \theta - \frac{a^2}{b} \sin^3 \theta \left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)$$

$$= b \sin \theta - \frac{a^2}{b} \sin^3 \theta - b \sin \theta \cos^2 \theta$$

$$\begin{aligned}
&= b \sin \theta (1 - \cos^2 \theta) - \frac{a^2}{b} \sin^3 \theta \\
&= b \sin^3 \theta - \frac{a^2}{b} \sin^3 \theta = \frac{b^2 - a^2}{b} \sin^3 \theta \\
&= -\frac{a^2 - b^2}{b} \sin^3 \theta \quad \dots(2)
\end{aligned}$$

To find the equation of the evolute, we have to eliminate θ between (1) and (2)

$$\begin{aligned}
\text{From (1),} \quad aX &= (a^2 - b^2) \cos^3 \theta \\
\therefore (aX)^{2/3} &= (a^2 - b^2)^{2/3} \cos^2 \theta \quad \dots(3)
\end{aligned}$$

$$\text{From (2),} \quad (bY)^{2/3} = (a^2 - b^2)^{2/3} \sin^2 \theta \quad \dots(4)$$

Adding (3) and (4)

$$(aX)^{2/3} + (bY)^{2/3} = (a^2 - b^2)^{2/3}$$

$\therefore$ Locus of X, Y is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

which is the required evolute.

- 132.** Show that equation of the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$.

Sol. The parametric equations of the curve are

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

$$\frac{dy}{dx} = -\tan \theta$$

$$\frac{d^2y}{dx^2} = -\sec^2 \theta \cdot \frac{d\theta}{dx} = -\frac{1}{\cos^2 \theta} \cdot \frac{1}{-3a \cos^2 \theta \sin \theta} = \frac{1}{3a \sin \theta \cos^4 \theta}$$

Let (X, Y) be the co-ordinates of the centre of curvature, then

$$\begin{aligned}
X &= x - \frac{y_1(1 + y_1^2)}{y_2} = a \cos^3 \theta - \frac{-\tan \theta(1 + \tan^2 \theta)}{\frac{1}{3a \sin \theta \cos^4 \theta}} \\
&= a \cos^3 \theta + 3a \sin \theta \cos^4 \theta \cdot \tan \theta \cdot \sec^2 \theta = a \cos^3 \theta + 3a \sin^2 \theta \cos \theta \\
&= a \cos \theta (\cos^2 \theta + 3 \sin^2 \theta) \quad \dots(1)
\end{aligned}$$

$$\begin{aligned}
Y &= y + \frac{1 + y_1^2}{y_2} = a \sin^3 \theta + \frac{\sec^2 \theta}{\frac{1}{3a \sin \theta \cos^4 \theta}} \\
&= a \sin^3 \theta + 3a \sin \theta \cos^2 \theta \\
&= a \sin \theta (\sin^2 \theta + 3 \cos^2 \theta) \quad \dots(2)
\end{aligned}$$

(1) and (2) give the co-ordinates of the centre of curvature.

To find the equation of the Evolute, we have to eliminate θ between (1) and (2)

$$\begin{aligned}
X + Y &= a(\cos^3 \theta + \sin^3 \theta) + 3a \sin \theta \cos \theta (\sin \theta + \cos \theta) \\
&= a[(\cos \theta + \sin \theta)^3 - 3 \cos \theta \sin \theta (\cos \theta + \sin \theta) \\
&\quad + 3a \sin \theta \cos \theta (\sin \theta + \cos \theta)] \\
&= a(\cos \theta + \sin \theta)^3
\end{aligned}$$

$$\therefore (X + Y)^{2/3} = a^{2/3} (\cos \theta + \sin \theta)^2 \quad \dots(3)$$

Also

$$X - Y = a(\cos \theta - \sin \theta)^3$$

$$\Rightarrow (X - Y)^{2/3} = a^{2/3} (\cos \theta - \sin \theta)^2 \quad \dots(4)$$

Adding (3) and (4), we have

$$\begin{aligned} (X + Y)^{2/3} + (X - Y)^{2/3} &= a^{2/3} [(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2] \\ &= a^{2/3} [2(\cos^2 \theta + \sin^2 \theta)] \\ &= 2a^{2/3} \end{aligned}$$

$\therefore$ Locus of (X, Y) is $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$, which is the required evolute.

133. Show that the equation of the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is

$$\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \frac{1}{2}a^2. \quad (\text{U.P.T.U., 2008})$$

Sol. Equation of curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \dots(1)$$

Differentiating w.r.t. x ,

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot y_1 = 0 \quad \dots(2)$$

Differentiating again,

$$-\frac{1}{4}x^{-3/2} - \frac{1}{4}y^{-3/2} \cdot y_1 \cdot y_1 + \frac{1}{2}y^{-1/2} \cdot y_2 = 0 \quad \dots(3)$$

At the point $\left(\frac{a}{4}, \frac{a}{4}\right)$,

$$\text{From equation (2), } \frac{1}{2} \cdot \frac{2}{\sqrt{a}} + \frac{1}{2} \cdot \frac{2}{\sqrt{a}} y_1 = 0$$

$$\text{or } y_1 = -1$$

$$\text{From equation (3), } -\frac{1}{4} \cdot \frac{4}{a} \cdot \frac{2}{\sqrt{a}} - \frac{1}{4} \cdot \frac{4}{a} \cdot \frac{2}{\sqrt{a}} (-1)^2 + \frac{1}{2} \cdot \frac{2}{\sqrt{a}} \cdot y_2 = 0$$

$$\text{or } -\frac{4}{a\sqrt{a}} + \frac{1}{\sqrt{a}} y_2 = 0$$

$$\therefore y_2 = \frac{4}{a}$$

$$\rho \text{ (at the given point)} = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + 1)^{3/2}}{4/a} = 2\sqrt{2} \cdot \frac{a}{4} = \frac{a}{\sqrt{2}}$$

Let (α, β) be the centre of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

$$\text{Then, } \alpha = x - \frac{y_1(1 + y_1^2)}{y_2}$$

or

$$\alpha = \frac{a}{4} - \frac{(-1)(1+1)}{4/a} = \frac{a}{4} + \frac{a}{2} = \frac{3}{4}a$$

$$\beta = y + \frac{1+y_1^2}{y_2} = \frac{a}{4} + \frac{1+1}{4/a} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

Equation of circle of curvature is

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

134. Show that the evolute of the tractrix $x = c \cos t + c \log \tan \frac{t}{2}$, $y = c \sin t$ is the catenary $y = c \cosh \frac{x}{c}$.

Sol. Equation of the curve is $x = c \cos t + c \log \tan \frac{t}{2}$, $y = c \sin t$

$$\begin{aligned} \frac{dx}{dt} &= -c \sin t + \frac{c}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2} \\ &= -c \sin t + \frac{c}{2 \sin \frac{t}{2} \cos \frac{t}{2}} = -c \sin t + \frac{c}{\sin t} \\ &= \frac{c(1 - \sin^2 t)}{\sin t} = \frac{c \cos^2 t}{\sin t} \end{aligned}$$

$$\frac{dy}{dt} = c \cos t,$$

 $\therefore$

$$y_1 = \frac{dy}{dx} = \frac{c \cos t \cdot \sin t}{c \cos^2 t} = \tan t$$

$$y_2 = \sec^2 t \frac{dt}{dx} = \frac{1}{\cos^2 t} \cdot \frac{\sin t}{c \cos^2 t} = \frac{\sin t}{c \cos^4 t}$$

If (X, Y) is the centre of curvature at any point on the curve, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = c \cos t + c \log \tan \frac{t}{2} - \frac{\tan t (\sec^2 t)}{\frac{\sin t}{c \cos^4 t}}$$

$$= c \cos t + c \log \tan \frac{t}{2} - \frac{c \cos^4 t \cdot \tan t \cdot \sec^2 t}{\sin t}$$

$$= c \cos t + c \log \tan \frac{t}{2} - c \cos t$$

or

$$X = c \log \tan \frac{t}{2} \quad \dots(1)$$

and

$$Y = y + \frac{1+y_1^2}{y_2} = c \sin t + \frac{\sec^2 t}{\frac{\sin t}{c \cos^4 t}}$$

$$\begin{aligned}
 &= c \sin t + \frac{c \cos^4 t \cdot \sec^2 t}{\sin t} = c \sin t + \frac{c \cos^2 t}{\sin t} \\
 &= \frac{c \sin^2 t + c \cos^2 t}{\sin t}
 \end{aligned}$$

$$Y = \frac{c}{\sin t} \quad \dots(2)$$

Evolute of the given curve is the locus of (X, Y) . Let us eliminate ' t ' between (1) and (2).

From (1),
$$X = c \log \tan \frac{t}{2}$$

or
$$\tan \frac{t}{2} = e^{\frac{X}{c}}$$

From (2),
$$\frac{Y}{c} = \frac{1}{\sin t} = \frac{1 + \tan^2 \frac{t}{2}}{2 \tan \frac{t}{2}}$$

or
$$\frac{Y}{c} = \frac{1}{2} \cdot \frac{1 + e^{\frac{2X}{c}}}{e^{\frac{X}{c}}} = \frac{1}{2} \cdot \left(e^{-\frac{X}{c}} + e^{\frac{X}{c}} \right) = \cosh \frac{X}{c}$$

$\therefore Y = c \cosh \frac{X}{c}$

Changing X to x , Y to y , the locus of (X, Y) is

$y = c \cosh \frac{x}{c}$, which is the required equation of evolute.

135. Find the centre of curvature of the parabola $y^2 = 4ax$ at the point $(a, 2a)$.

Sol. Here,

$$y^2 = 4ax$$

$$y = 2\sqrt{a} \cdot \sqrt{x}$$

$$y_1 = 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}} \quad \text{and} \quad y_2 = -\frac{1}{2} \cdot \frac{a^{1/2}}{x^{3/2}}$$

$$y_1 \text{ at } (a, 2a) = 1, \quad \text{and} \quad y_2 \text{ at } (a, 2a) = -\frac{1}{2a}$$

If (X, Y) is the centre of curvature, then

$$\therefore X = x - \frac{y_1(1 + y_1^2)}{y_2} = a - \frac{1}{-1/2a} = a + 4a = 5a$$

$$Y = y + \frac{1 + y_1^2}{y_2} = 2a + \frac{2}{-1/2a} = 2a - 4a = -2a$$

Hence the co-ordinates of the centre of curvature are $(5a, -2a)$.

136. Show that the chord of curvature parallel to y -axis for the curve $y = c \log \sec \frac{x}{c}$ is of constant length.

Sol. Chord of curvature parallel to y -axis $= 2\rho \cos \psi$, where ψ is the angle made by the tangent with the x -axis.

Here,

$$y = c \log \sec \frac{x}{c}$$

$$\therefore y_1 = c \cdot \frac{1}{\sec \frac{x}{c}} \cdot \sec \frac{x}{c} \cdot \tan \frac{x}{c} \cdot \frac{1}{c} \text{ or } y_1 = \tan \frac{x}{c} \quad \dots(1)$$

$$y_2 = \frac{1}{c} \cdot \sec^2 \frac{x}{c}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\sec^3 \frac{x}{c}}{\frac{1}{c} \sec^2 \frac{x}{c}} = c \cdot \sec \frac{x}{c}$$

Now the chord of curvature parallel to y -axis $= 2\rho \cos \psi$

$$= 2c \sec \frac{x}{c} \cdot \cos \psi \quad \dots(2)$$

Also, $\frac{dy}{dx} = \tan \frac{x}{c}$

or $\tan \psi = \tan \frac{x}{c}$

$\therefore \cos \psi = \cos \frac{x}{c}$

From (2), we have

$$\begin{aligned} \text{Required chord} &= 2c \sec \frac{x}{c} \cos \frac{x}{c} \\ &= 2c \text{ (which is a constant)} \end{aligned}$$

Hence shown.

137. Find the chord of curvature through the pole of the cardioid $r = a(1 + \cos \theta)$.

Sol. Here,

$$r = a(1 + \cos \theta)$$

$$\therefore \rho = \frac{2\sqrt{2ar}}{3} \quad (\text{See question. 115})$$

Chord of curvature through the pole

$= 2\rho \sin \phi$, where ϕ is the angle made by the chord at any point on the circumference of the circle of curvature. Chord of curvature through the pole

$$\begin{aligned} &= 2 \cdot \frac{2\sqrt{2ar}}{3} \cdot \sin \phi = \frac{4\sqrt{2}}{3} \sqrt{ar} \cdot r \frac{d\theta}{ds} = \frac{4\sqrt{2ar}}{3} \cdot \frac{r}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} \\ &= \frac{4\sqrt{2ar}}{3} \cdot \frac{r}{\sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta}} = \frac{4}{3} \sqrt{2ar} \cdot \frac{r}{\sqrt{2ar}} = \frac{4}{3} r \end{aligned}$$

Hence chord of curvature $= \frac{4}{3} r$.

- 138.** If C_x and C_y be the chords of curvature parallel to x -axis and y -axis respectively at any point of the curve $y = e^{x/a}$, show that

$$\frac{1}{(C_x)^2} + \frac{1}{(C_y)^2} = \frac{1}{(2a C_x)^2}$$

Sol. Equation of the curve is

$$y = ae^{\frac{x}{a}} \quad \dots(1)$$

$$y_1 = ae^{\frac{x}{a}} \cdot \frac{1}{a} = e^{\frac{x}{a}} = \frac{y}{a} \text{ and } y_2 = \frac{1}{a} \cdot y_1 = \frac{y}{a^2}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{y^2}{a^2}\right)^{3/2}}{\frac{y}{a^2}} = \frac{(a^2 + y^2)^{3/2}}{a^3} \cdot \frac{a^2}{y} = \frac{(a^2 + y^2)^{3/2}}{ay}$$

$$\tan \psi = y_1 = \frac{y}{a}, \quad \sin \psi = \frac{y}{\sqrt{a^2 + y^2}}, \quad \cos \psi = \frac{a}{\sqrt{a^2 + y^2}}$$

$\therefore$

$$C_x = 2\rho \sin \psi$$

or

$$C_x = 2 \cdot \frac{(a^2 + y^2)^{3/2}}{ay} \cdot \frac{y}{\sqrt{a^2 + y^2}} = \frac{2(a^2 + y^2)}{a}$$

$$C_y = 2\rho \cos \psi$$

$$= 2 \cdot \frac{(a^2 + y^2)^{3/2}}{ay} \cdot \frac{a}{\sqrt{a^2 + y^2}} = \frac{2(a^2 + y^2)}{y}$$

$\therefore$

$$\begin{aligned} \frac{1}{(C_x)^2} + \frac{1}{(C_y)^2} &= \frac{a^2}{4(a^2 + y^2)^2} + \frac{y^2}{4(a^2 + y^2)^2} \\ &= \frac{a^2 + y^2}{4(a^2 + y^2)^2} = \frac{1}{4(a^2 + y^2)} = \frac{1 \cdot 2}{4 \cdot a C_x} = \frac{1}{2a \cdot C_x} \end{aligned}$$

Hence

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2a \cdot C_x}$$

- 139.** Find the length of the chord of curvature through the pole for a cardioid $r = a(1 + \cos \theta)$.

Sol. Note this is the same question as question number 137 but we will solve it by a different method using Pedal equation

$$\frac{dr}{d\theta} = -a \sin \theta$$

$$\tan \phi = r \frac{d\theta}{dr} = \frac{-a(1 + \cos \theta)}{a \sin \theta} = \tan \left(\frac{\pi}{2} + \frac{\theta}{2} \right)$$

$\therefore$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$p = r \sin \phi = r \sin \left(\frac{\pi}{2} + \frac{\theta}{2} \right) = r \cos \frac{\theta}{2}$$

$$\begin{aligned} \therefore p^2 &= r^2 \cos^2 \frac{\theta}{2} = r^2 \cdot \frac{1 + \cos \theta}{2} = \frac{r^3}{2a} \\ \text{i.e., } 2ap^2 &= r^3 \text{ (Pedal equation)} \\ p &= \sqrt{\frac{r^3}{2a}}, \quad \therefore \frac{dp}{dr} = \frac{3}{2} \cdot \frac{r^{\frac{1}{2}}}{\sqrt{2a}} \\ \therefore C_0 &= 2p \frac{dr}{dp} = 2 \cdot \sqrt{\frac{r^3}{2a}} \cdot \frac{2\sqrt{2a}}{3 \cdot r^{1/2}} = \frac{4}{3} r. \end{aligned}$$

140. Show that the evolute of the rectangular hyperbola $xy = c^2$ is the curve

$$(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}.$$

(Chennai, 2005)

Sol. Given curve is

$$xy = c^2$$

or

$$y = \frac{c^2}{x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\frac{c^2}{x^2} \\ \frac{d^2y}{dx^2} &= \frac{2c^2}{x^3} \end{aligned}$$

If (X, Y) is the centre of curvature at any point on the curve, then

$$\begin{aligned} X &= x - \frac{y_1(1 + y_1^2)}{y_2} = x - \frac{(-c^2/x^2)(1 + c^4/x^4)}{2c^2/x^3} \\ &= x + \frac{x}{2} \left(1 + \frac{c^4}{x^4} \right) = \frac{3x}{2} + \frac{c^4}{2x^3} \end{aligned} \quad \dots(1)$$

and

$$\begin{aligned} Y &= y + \frac{\left(1 + \frac{c^4}{x^4} \right)}{2c^2/x^3} = \frac{c^2}{x} + \frac{x^3}{2c^2} \left(1 + \frac{c^4}{x^4} \right) = \frac{c^2}{x} + \frac{x^3}{2c^2} + \frac{c^2}{2x} \\ &= \frac{3c^2}{2x} + \frac{x^3}{2c^2} \end{aligned} \quad \dots(2)$$

To find the evolute we have to eliminate x from equations (1) and (2).

$$\begin{aligned} X + Y &= \frac{3x}{2} + \frac{c^4}{2x^3} + \frac{3c^2}{2x} + \frac{x^3}{2c^2} \\ &= \frac{3c^2x^4 + c^6 + 3c^4x^2 + x^6}{2c^2x^3} = \frac{c^6 + x^6 + 3c^2x^2(x^2 + c^2)}{2c^2x^3} \\ X + Y &= \frac{(x^2 + c^2)^3}{2c^2x^3} \end{aligned} \quad \dots(3)$$

$$X - Y = \frac{3x}{2} + \frac{c^4}{2x^3} - \frac{3c^2}{2x} - \frac{x^3}{2c^2} = \frac{3c^2x^4 + c^6 - 3x^2c^4 - x^6}{2x^3c^2}$$

$$= \frac{c^6 - x^6 - 3x^2c^4 + 3c^2x^4}{2x^3c^2}$$

$$X - Y = \frac{(c^2 - x^2)^3}{2x^3c^2} \quad \dots(4)$$

Now, $(X + Y)^{2/3} = \frac{(x^2 + c^2)^2}{(2c^2)^{2/3} \cdot x^2} \quad \dots(5)$

$$(X - Y)^{2/3} = \frac{(c^2 - x^2)^2}{(2c^2)^{2/3} \cdot x^2} \quad \dots(6)$$

$$(X + Y)^{2/3} - (X - Y)^{2/3} = \frac{1}{(2c^2)^{2/3} \cdot x^2} [(c^2 + x^2)^2 - (c^2 - x^2)^2]$$

$$= \frac{4c^2x^2}{(2c^2)^{2/3} \cdot x^2} \quad \text{or} \quad \frac{4c^2}{(2c^2)^{2/3}}$$

$$\Rightarrow (X + Y)^{2/3} - (X - Y)^{2/3} = \frac{4c^2}{(2c^2)^{2/3}} = \frac{(2c)^2}{(2c^2)^{2/3}} = \frac{(8c^3)^{2/3}}{(2c^2)^{2/3}} = (4c)^{2/3}$$

$\therefore$ Locus of (X, Y) i.e., evolute of the given hyperbola is

$$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}$$

Hence proved.

141. Find the equation of the evolute of the parabola $x^2 = 4ay$.

Sol. Equation of parabola is

$$x^2 = 4ay$$

or

$$y = \frac{x^2}{4a}$$

$\therefore$

$$\frac{dy}{dx} = \frac{2x}{4a} \quad \text{or} \quad \frac{x}{2a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

If (X, Y) is the centre of curvature at (x, y) , then

$$X = x - \frac{\frac{x}{2a} \left(1 + \frac{x^2}{4a^2} \right)}{\frac{1}{2a}} = x - x - \frac{x^3}{4a^2} = -\frac{x^3}{4a^2} \quad \dots(1)$$

$$Y = y + \frac{1 + \frac{x^2}{4a^2}}{\frac{1}{2a}} = y + 2a + \frac{x^2}{2a} = 2a + \frac{x^2}{4a} + \frac{x^2}{2a} = 2a + \frac{3}{4a} x^2 \quad \dots(2)$$

$\therefore$ The centre of curvature at (x, y) is $\left(-\frac{x^3}{4a^2}, 2a + \frac{3x^2}{4a} \right)$

To find the evolute we have to eliminate x from (1) and (2)

From (1), $X = -\frac{x^3}{4a^2}$

From (2), $Y - 2a = \frac{3x^2}{4a}$

$$\begin{aligned}
 (Y - 2a)^3 &= \frac{27x^6}{64a^3} = \frac{27 \cdot (4a^2X)^2}{64a^3} = \frac{27aX^2}{4} \\
 \Rightarrow 4(Y - 2a)^3 &= 27aX^2 \\
 \therefore \text{Locus of } (X, Y) \text{ i.e., the evolute of parabola } x^2 = 4ay \text{ is} \\
 4(y - 2a)^3 &= 27ax^2.
 \end{aligned}$$

- 142.** Find the co-ordinates of the centre of curvature for any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence find its evolute.

Sol. Take the parametric equations of the hyperbola as

$$x = a \sec \theta, \quad y = b \tan \theta.$$

$$\therefore \frac{dx}{d\theta} = a \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\therefore \frac{dy}{dx} = \frac{b \sec^2 \theta}{a \sec \theta \cdot \tan \theta} = \frac{b \sec \theta}{a \tan \theta} = \frac{b}{a \sin \theta}$$

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{b}{a} (-\operatorname{cosec} \theta \cot \theta) \cdot \frac{d\theta}{dx} \\
 &= -\frac{b}{a} \cdot \frac{\operatorname{cosec} \theta \cdot \cot \theta}{a \sec \theta \cdot \tan \theta} = -\frac{b}{a^2} \cdot \frac{\cos^3 \theta}{\sin^3 \theta}
 \end{aligned}$$

Let (X, Y) be the co-ordinates of the centre of curvature, then

$$X = x - \frac{y_1(1 + y_1^2)}{y_2} = a \sec \theta - \frac{\frac{b}{a \sin \theta} \left(1 + \frac{b^2}{a^2 \sin^2 \theta}\right)}{-\frac{b}{a^2} \cdot \frac{\cos^3 \theta}{\sin^3 \theta}}$$

$$\begin{aligned}
 \text{or } X &= a \sec \theta + \frac{a \sin^2 \theta}{\cos^3 \theta} \left(1 + \frac{b^2}{a^2 \sin^2 \theta}\right) = a \sec \theta + \frac{a \sin^2 \theta}{\cos^3 \theta} + \frac{b^2}{a \cos^3 \theta} \\
 &= \frac{a^2 \cos^2 \theta + a^2 \sin^2 \theta + b^2}{a \cos^3 \theta}
 \end{aligned}$$

$$\Rightarrow X = \frac{a^2 + b^2}{a} \cdot \sec^3 \theta \quad \dots(1)$$

$$\begin{aligned}
 \text{and } Y &= y + \frac{1 + y_1^2}{y_2} = b \tan \theta + \frac{1 + \frac{b^2}{a^2 \sin^2 \theta}}{-\frac{b}{a^2} \cdot \frac{\cos^3 \theta}{\sin^3 \theta}} = b \tan \theta - \frac{a^2 \sin^3 \theta}{b \cos^3 \theta} \left(1 + \frac{b^2}{a^2 \sin^2 \theta}\right) \\
 &= b \tan \theta - \frac{a^2 \sin^3 \theta}{b \cos^3 \theta} - \frac{b \sin \theta}{\cos^3 \theta} = \frac{b^2 \sin \theta \cos^2 \theta - a^2 \sin^3 \theta - b^2 \sin \theta}{b \cos^3 \theta} \\
 &= \frac{-a^2 \sin^3 \theta - b^2 \sin \theta + b^2 \sin \theta \cos^2 \theta}{b \cos^3 \theta} \\
 &= \frac{-a^2 \sin^3 \theta - b^2 \sin \theta (1 - \cos^2 \theta)}{b \cos^3 \theta} = -\frac{\sin^3 \theta (a^2 + b^2)}{b \cos^3 \theta} \\
 &= -\left(\frac{a^2 + b^2}{b}\right) \tan^3 \theta \quad \dots(2)
 \end{aligned}$$

Hence centre of curvature is $\left(\frac{a^2 + b^2}{b} \sec^3 \theta, -\frac{a^2 + b^2}{b} \tan^3 \theta \right)$

To find the equation of the evolute, we have to eliminate θ between (1) and (2).

$$\begin{aligned} aX &= (a^2 + b^2) \sec^3 \theta \\ bY &= -(a^2 + b^2) \tan^3 \theta \\ \therefore (aX)^{2/3} - (bY)^{2/3} &= (a^2 + b^2)^{2/3} \sec^2 \theta - (a^2 + b^2)^{2/3} \tan^2 \theta \\ &= (a^2 + b^2)^{2/3} [\sec^2 \theta - \tan^2 \theta] = (a^2 + b^2)^{2/3} \end{aligned}$$

Hence the evolute is

$$(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}.$$

143. Find the circle of curvature for the curve $x^3 + y^3 = 3xy$ at the point $\left(\frac{3}{2}, \frac{3}{2} \right)$ on it.

Sol. Equation of the curve is

$$x^3 + y^3 = 3xy \quad \dots(1)$$

Differentiating w.r.t. x ,

$$3x^2 + 3y^2 \frac{dy}{dx} = 3 \left[y + x \frac{dy}{dx} \right]$$

$$(3y^2 - 3x) \frac{dy}{dx} = 3y - 3x^2$$

$$(y^2 - x) \frac{dy}{dx} = y - x^2 \quad \dots(2)$$

Differentiating again,

$$2y \left(\frac{dy}{dx} \right)^2 + y^2 \frac{d^2y}{dx^2} - \frac{dy}{dx} - x \frac{d^2y}{dx^2} = \frac{dy}{dx} - 2x$$

$$2y \left(\frac{dy}{dx} \right)^2 + (y^2 - x) \frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 2x \quad \dots(3)$$

At the point $\left(\frac{3}{2}, \frac{3}{2} \right)$, $\frac{dy}{dx} = -1$

From equation (3),

$$2 \cdot \frac{3}{2} (-1)^2 + \left(\frac{9}{4} - \frac{3}{2} \right) \frac{d^2y}{dx^2} = 2(-1) - 3$$

$$3 + \frac{3}{4} \frac{d^2y}{dx^2} = -5$$

$$\frac{3}{4} \frac{d^2y}{dx^2} = -8 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\frac{32}{3}$$

$$\rho \text{ (at the given point)} = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(2)^{3/2}}{-\frac{32}{3}} = \frac{3\sqrt{8}}{-32} = -\left(\frac{3\sqrt{8}}{32} \right)$$

Let (α, β) be the centre of curvature at $\left(\frac{3}{2}, \frac{3}{2}\right)$

Then
$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} = \frac{3}{2} - \frac{(-1)(2)}{-32/3} = \frac{3}{2} - \frac{6}{32} = \frac{21}{16}$$

$$\begin{aligned}\beta &= y + \frac{1+y_1^2}{y_2} \\ &= \frac{3}{2} + \frac{2}{-32/3} = \frac{3}{2} - \frac{3}{16} = \frac{21}{16}\end{aligned}$$

$\therefore$ Equation of circle of curvature is

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2$$

or
$$\left(x - \frac{21}{16}\right)^2 + \left(y - \frac{21}{16}\right)^2 = \frac{72}{(32)^2}$$

$$x^2 + \frac{441}{256} - \frac{21x}{8} + y^2 - \frac{21}{8}y + \frac{441}{256} = \frac{72}{1024}$$

$$x^2 - \frac{21}{8}x + y^2 - \frac{21}{8}y + \frac{441}{128} = \frac{72}{1024}$$

$$8(x^2 + y^2) - 21x - 21y + \frac{441}{16} = \frac{72}{128}$$

$$8(x^2 + y^2) - 21(x + y) + \frac{441}{16} - \frac{72}{128} = 0$$

or
$$8(x^2 + y^2) - 21(x + y) + \frac{432}{16} = 0$$

or
$$8(x^2 + y^2) - 21(x + y) + 27 = 0.$$

144. Show that the evolute of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is another equal cycloid.

Sol. $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta$$

$\therefore y_1 = \frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)} = \cot \frac{\theta}{2}$

$$y_2 = \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{d\theta}{dx} \cdot \frac{1}{2} = -\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{a(1 - \cos \theta)} = -\frac{1}{4a \sin^4 \frac{\theta}{2}}$$

If (X, Y) is the centre of curvature, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin \theta) - \frac{\cot \frac{\theta}{2} \left(1 + \cot^2 \frac{\theta}{2}\right)}{\frac{-1}{4a \sin^4 \frac{\theta}{2}}}$$

$$= a(\theta - \sin \theta) + \cot \frac{\theta}{2} \left(4a \sin^4 \frac{\theta}{2}\right) \left(1 + \cot^2 \frac{\theta}{2}\right)$$

$$= a(\theta - \sin \theta) + \frac{\cos \theta/2}{\sin \theta/2} \cdot 4a \sin^4 \frac{\theta}{2} \cdot \operatorname{cosec}^2 \frac{\theta}{2}$$

$$\begin{aligned}
&= a(\theta - \sin \theta) + 4a \sin \theta/2 \cos \frac{\theta}{2} = a(\theta - \sin \theta) + 2a \sin \theta \\
&= a(\theta + \sin \theta) \\
Y &= y + \frac{1 + y_1^2}{y_2} \\
&= a(1 - \cos \theta) - 4a \sin^4 \frac{\theta}{2} \left(1 + \cot^2 \frac{\theta}{2}\right) \\
&= a(1 - \cos \theta) - 4a \sin^4 \frac{\theta}{2} \cdot \operatorname{cosec}^2 \frac{\theta}{2} \\
&= a(1 - \cos \theta) - 4a \sin^2 \frac{\theta}{2} = a(1 - \cos \theta) - 2a(1 - \cos \theta) \\
&= -a(1 - \cos \theta)
\end{aligned}$$

Hence the locus of (X, Y) is given by

$x = a(\theta + \sin \theta)$, $y = -a(1 - \cos \theta)$, which is another equal cycloid.

145. Show that the evolute of the curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ is $x^2 + y^2 = a^2$.

Sol. Given curve is

$$x = a(\cos \theta + \theta \sin \theta), \quad y = a(\sin \theta - \theta \cos \theta)$$

$$\frac{dx}{d\theta} = a(-\sin \theta + \theta \cos \theta + \sin \theta) = a \theta \cos \theta$$

$$\frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta) = a \theta \sin \theta$$

$\therefore$

$$y_1 = \frac{dy}{dx} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta$$

$$y_2 = \frac{d^2y}{dx^2} = \sec^2 \theta \cdot \frac{d\theta}{dx} = \sec^2 \theta \cdot \frac{1}{a \theta \cos \theta} = \frac{\sec^3 \theta}{a \theta}$$

If (X, Y) is the centre of curvature, then

$$\begin{aligned}
X &= x - \frac{y_1(1 + y_1^2)}{y_2} = a(\cos \theta + \theta \sin \theta) - \frac{\tan \theta \cdot \sec^2 \theta}{\sec^3 \theta / a \theta} \\
&= a(\cos \theta + \theta \sin \theta) - a \theta \sin \theta \\
&= a \cos \theta \quad \dots(1)
\end{aligned}$$

and

$$\begin{aligned}
Y &= y + \frac{1 + y_1^2}{y_2} = a(\sin \theta - \theta \cos \theta) + \frac{\sec^2 \theta \cdot a \theta}{\sec^3 \theta} \\
&= a \sin \theta - a \theta \cos \theta + a \theta \cos \theta \\
&= a \sin \theta \quad \dots(2)
\end{aligned}$$

Locus of (X, Y) i.e., evolute of the given curve is obtained by eliminating θ between (1) and (2).

$$\begin{aligned}
\frac{x}{a} &= \cos \theta, \quad \frac{y}{a} = \sin \theta \\
\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 &= \cos^2 \theta + \sin^2 \theta = 1 \\
\text{or } x^2 + y^2 &= a^2.
\end{aligned}$$

146. Trace the curve $y = x^3 - 3ax^2$.

Sol. Equation of the curve is $y = x^3 - 3ax^2$

- Symmetry** : The curve is not symmetrical about any line.
- Origin** : The curve passes through the origin equating to zero the lowest degree terms, the tangent at the origin is given by $y = 0$, i.e., x -axis.
- Axes Intersection** : The curve meets x -axis, where putting $y = 0$, we get

$$x^3 - 3ax^2 = 0$$

$$x^2(x - 3a) = 0$$

$$\Rightarrow x = 0, 3a$$

$\therefore$ The curve meets x -axis in the points $(0, 0)$ and $(3a, 0)$. The curve meets y -axis only at the origin.

- Asymptotes** : The curve has no asymptote.

- Special Points** : $\frac{dy}{dx} = 3x^2 - 6ax = 3x(x - 2a)$

$$\therefore \frac{dy}{dx} = 0, \text{ when } x = 0 \text{ or } x = 2a.$$

When $x = 0$, $y = 0$ and when $x = 2a$, $y = 8a^3 - 12a^3$ or $y = -4a^3$. Thus the tangent at $(2a, -4a^3)$ is parallel to x -axis and the tangent at $(0, 0)$ coincides with the x -axis.

$\frac{dy}{dx}$ is not infinite for any real value of x .

- Region** :

The equation of the curve can be written as

$$y = x^2(x - 3a)$$

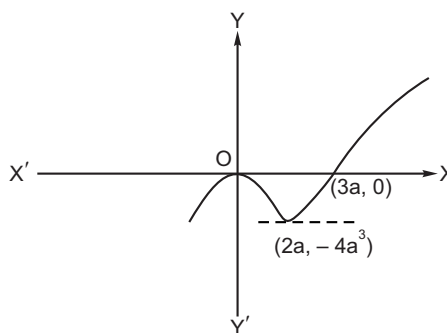
For values of $x < 3a$, y is -ve and for values of $x > 3a$, y is +ve.

When $x = 0$, $y = 0$

As x increases from 0 to $2a$, y decreases from 0 to $-4a^3$.

As x increases from $2a$ to $3a$, y increases from $-4a^3$ to 0 and as x further increases from $3a$ to ∞ , y also increases from 0 to ∞ .

As x decreases from 0 to $-\infty$, y also decreases from 0 to $-\infty$. Hence shape of the curve is as shown in the figure.



147. Trace the curve $y^2(a + x) = x^2(3a - x)$.

Sol. The equation of the curve is $y^2(a + x) = x^2(3a - x)$... (1)

- Symmetry** : Since the equation (1) contains only even powers of y , the curve is symmetrical about x -axis. But the curve is *NOT* symmetrical about y -axis.
- Origin** : The curve passes through the origin. Equating to zero, the lowest degree terms, the tangents at the origin are given by

$$ay^2 - 3ax^2 = 0 \text{ or } a(y^2 - 3x^2) = 0 \Rightarrow y = \pm \sqrt{3}x$$

Since the two tangents are real and different,

$\therefore$ Origin is a node.

$$\text{From (1), we have } y^2 = x^2 \left(\frac{3a - x}{a + x} \right)$$

Since $\frac{3a-x}{a+x} < \frac{3a}{a}$ i.e., 3 for small +ve x

$\therefore y$ is less than $\sqrt{3}x$. Hence the curve lies below the tangent $y = \sqrt{3}x$ for small +ve values of x .

3. **Axes Intersection** : The curve meets x -axis, where putting $y = 0$, we have $x^2(3a-x) = 0$
 $\therefore x = 0$ or $3a$. i.e., in the points $(0, 0)$ and $A(3a, 0)$. Similarly, it meets y -axis in the origin only.

Shifting the origin to $(3a, 0)$, the equation transforms to

$$Y^2(4a+X) = (3a+X)^2(-X)$$

Equating to zero the lowest degree terms, the tangent at the new origin is given by $-9a^2X = 0$ or $X = 0$. Thus, the tangent at $A(3a, 0)$ is parallel to y -axis.

4. **Asymptotes** : Equating to zero the co-efficient of y^2 , the asymptote parallel to ay -axis is $x+a=0$. There is no other asymptote of the curve.
5. **Region** : Solving for y and considering the +ve values ($\because$ the curve is symmetrical about the x -axis), we have

$$y = x\sqrt{\frac{3a-x}{a+x}} \quad \dots(2)$$

- (i) When $x = 0$, $y = 0$. When x is +ve and small, y is real. As x goes on increasing, y also increases upto the point for which $x = \sqrt{3}a$ and then decreases and finally becomes zero at $x = 3a$.
- (ii) When $x > 3a$, y is imaginary. Thus, no part of the curve lies beyond the line $x = 3a$.
- (iii) When x is -ve and numerically small, from (2), y is real. Also for values of x under consideration, $3a-x > 3(a+x)$.

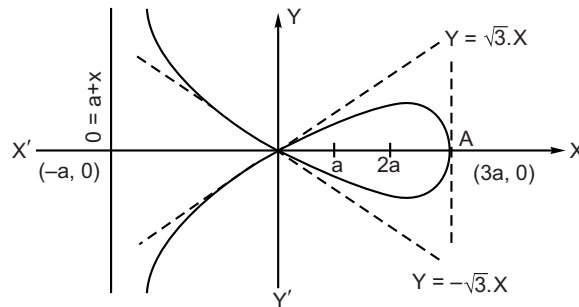
$\therefore y$ is numerically $> -\sqrt{3}x$

$\therefore$ The curve lies above the tangent in the second quadrant.

- (iv) When $x < -a$, from (2), y is imaginary.

$\therefore$ The curve does not lie beyond $x = -a$. Again when x is -ve and numerically increases from 0 to a , y is -ve and decreases from 0 to $-\infty$.

Hence the shape of the curve is as shown in the figure.



148. Trace the curve $y^2(2a-x) = x^3$ (Cissoid).

(U.P.T.U., 2005)

Sol. The equation of the curve is

$$y^2(2a-x) = x^3 \quad \dots(1)$$

1. *Symmetry* : Since (1) contains even powers of y , the curve is symmetrical about x -axis.
2. *Origin* :
 - (i) The curve passes through the origin.
 - (ii) The tangents at the origin are given by $y^2 = 0$. Since the two tangents are real and coincident, $\therefore$ origin is a cusp.
3. *Axes Intersection* : The curve meets x -axis and y -axis at the origin only.
4. *Asymptotes* : Equating to zero the co-efficient of y^2 , the highest degree term in y , the asymptote parallel to y -axis is $x - 2a = 0$. There is no other asymptote of the curve.
5. *Special Points* :

From equation (1),
$$y = \frac{x^{3/2}}{\sqrt{2a-x}} = x^{3/2} (2a-x)^{-1/2} \quad \dots(2)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{3}{2} x^{1/2} (2a-x)^{-1/2} - \frac{1}{2} (2a-x)^{-3/2} \cdot (-1) \cdot x^{3/2} \\ &= \frac{1}{2} (2a-x)^{-3/2} \cdot x^{1/2} [3(2a-x) + x] = \frac{\sqrt{x} \cdot (3a-x)}{(2a-x)^{3/2}} \end{aligned}$$

$$\therefore \frac{dy}{dx} = 0 \text{ when } \sqrt{x} (3a-x) = 0 \text{ or } x = 0, 3a$$

Rejecting $x = 3a$ since when $x = 3a$ from (2), y is imaginary.

$\therefore$ The tangent at $x = 0$ is parallel to x -axis.

$\frac{dy}{dx} \rightarrow \infty$ when $x \rightarrow 2a$. From (2), when $x \rightarrow 2a$, $y \rightarrow \infty$. Thus $x = 2a$ is an asymptote.

6. *Region* :

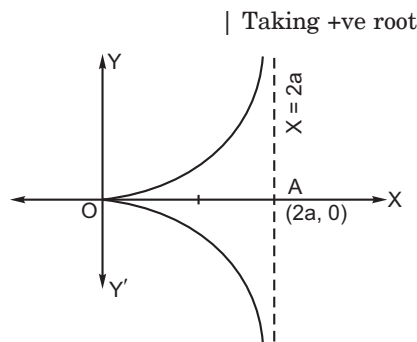
From (1),
$$y = x \sqrt{\frac{x}{2a-x}}$$

When $x < 0$ or when $x > 2a$, y is imaginary.

Thus the curve does not lie to the left of y -axis and to the right of the line $x = 2a$.

When $x = 0$, $y = 0$. As x increases from 0 to $2a$, y also increases from 0 to ∞ .

Hence the shape of the curve is as shown the adjoining figure.



- 149.** Trace the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ or $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.

Sol. The equation of the curve is

$$y^2(a^2 + x^2) = x^2(a^2 - x^2) \quad \dots(1)$$

1. *Symmetry* : Since in equation (1), powers of both x and y are even, the curve is symmetrical about both axes.
2. *Origin* : The curve passes through the origin. Tangents at the origin are $y^2 = x^2$ or $y = \pm x$ which are real and distinct showing that origin is a Node.
3. *Axes Intersection* : The curve meets x -axis at $A(a, 0)$ and $A'(-a, 0)$. Transferring the origin to $(a, 0)$ the equation (1) transforms to

$$Y^2[a^2 + (X+a)^2] = (X+a)^2 [a^2 - (X+a)^2]$$

$$\text{or } Y^2[2a^2 + X^2 + 2aX] = (X+a)^2 [-2aX - X^2]$$

Equating to zero the lowest degree terms, the tangent at the new origin is $X = 0$ i.e., new y -axis. Thus the tangent at $A(a, 0)$ is parallel to y -axis. Similarly, tangent at $A'(-a, 0)$ is also parallel to y -axis (by symmetry).

4. *Asymptotes* : The curve has no asymptotes.

5. *Region* :

(i) Solving (1) for y , we have

$$y = \pm x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

Now y is real if and only if $a^2 - x^2 \geq 0$ or if $x^2 \leq a^2$ or $|x| \leq a$.

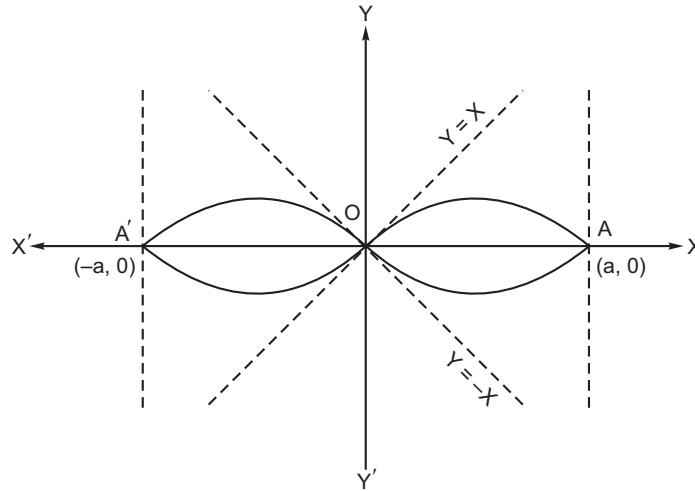
$\therefore$ The whole curve lies between the lines $x = \pm a$.

(ii) For the branch in the first quadrant $y = x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

When $x = 0$, $y = 0$. As x increases, y also increases and goes on increasing until $x = a\sqrt{(\sqrt{2} - 1)}$, where $\frac{dy}{dx} = 0$, i.e., the tangent is parallel to x -axis. As x increases from

$a\sqrt{(\sqrt{2} - 1)}$ to a , y decreases and ultimately becomes zero when $x = a$.

The shape of the curve is as shown in the figure.



150. Trace the curve $r = a(1 + \cos \theta)$ (Cardioid.)

[P.T.U., 2006., U.P.T.U., 2008., M.D.U., 2009]

Sol. The equation of the curve is

$$r = a(1 + \cos \theta) \quad \dots(1)$$

1. *Symmetry* : When θ is changed to $-\theta$, equation of the curve remains unchanged.

$\therefore$ The curve is symmetrical about initial line.

2. *Origin or Pole* :

(i) When $\theta = \pi$, $r = 0$, $\therefore$ Pole lies on the curve.

(ii) The tangent at the pole is $\theta = \pi$.

(iii) The curve cuts the initial line $\theta = 0$ at $(2a, 0)$ and the lines $\theta = \pm \frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right)$.

3. Value of ϕ :

$$\tan \phi = r \frac{d\theta}{dr} = -\frac{1 + \cos \theta}{\sin \theta} = -\cot \frac{\theta}{2} = \tan \left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2} \quad \therefore \phi = \frac{\pi}{2}, \text{ when } \theta = 0 \text{ and } r = 2a$$

i.e., at $(2a, 0)$ the tangent is perpendicular to initial line.

4. *Asymptotes* : There are no asymptotes, because for any finite value of θ , r does not tend to infinity.

5. *Special Points* : The corresponding values of θ and r are given below :

$\theta :$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π
$r :$	$2a$	$a \left(1 + \frac{1}{\sqrt{2}}\right)$	a	0

6. *Region* :

(i) r is never greater than $2a$ $\therefore$ No portion of the curve lies to the right of the tangent at $(2a, 0)$.

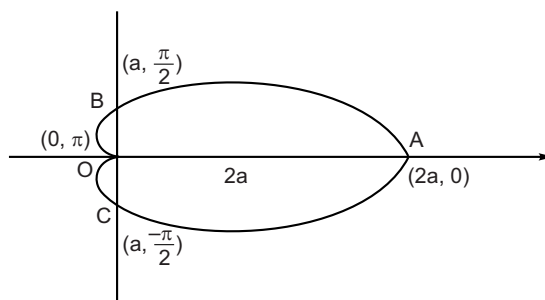
(ii) Since $|\cos \theta| \leq 1$, $\therefore r \leq 2a$

$\therefore$ curve lies entirely within the circle $r = 2a$.

(iii) When θ increases from 0 to π , r remains positive and decreases from $2a$ to 0.

When θ increases from π to 2π , r remains positive and increases from 0 to $2a$.

The shape of the curve is as shown in the figure.



151. Trace the curve $r = 2 + 3 \cos \theta$.

Sol. The equation of the curve

$$r = 2 + 3 \cos \theta \quad \dots(1)$$

1. *Symmetry* : Equation of curve remains unchanged when θ is changed to $-\theta$, $\therefore$ curve is symmetrical about the initial line.

2. *Origin or Pole* : When $r = 0$, $\cos \theta = -\frac{2}{3} < 1$ (numerically)

$\therefore$ The curve passes through the pole.

$$\text{Now,} \quad \cos \theta = -\frac{2}{3} = \cos(\pi - \alpha) \text{ (say)}$$

$$\therefore \theta = (\pi - \alpha) \text{ is tangent to the curve at the pole where } \alpha \text{ is given by } \cos(\pi - \alpha) = -\frac{2}{3}.$$

3. Value of ϕ :

From (1), $\frac{dr}{d\theta} = -3 \sin \theta$

$$\tan \phi = r \frac{d\theta}{dr} = \frac{2 + 3 \cos \theta}{-3 \sin \theta} = 0$$

When $2 + 3 \cos \theta = 0$ or $\cos \theta = -\frac{2}{3}$ or $\theta = (\pi - \alpha)$

$\therefore$ at $(0, \pi - \alpha)$ the tangent to curve is parallel to initial line.

$\tan \phi = \infty$, when $\sin \theta = 0$ i.e., when $\theta = 0$ or π

$\therefore$ At $(5, 0)$ and $(-1, \pi)$, the tangent is perpendicular to the initial line.

4. Asymptotes : Curve has no asymptotes because for any real value of θ , r is never infinite.

5. Special Points :

$\theta = 0$	$\frac{\pi}{2}$	π
$r = 5$	2	-1

6. Region :

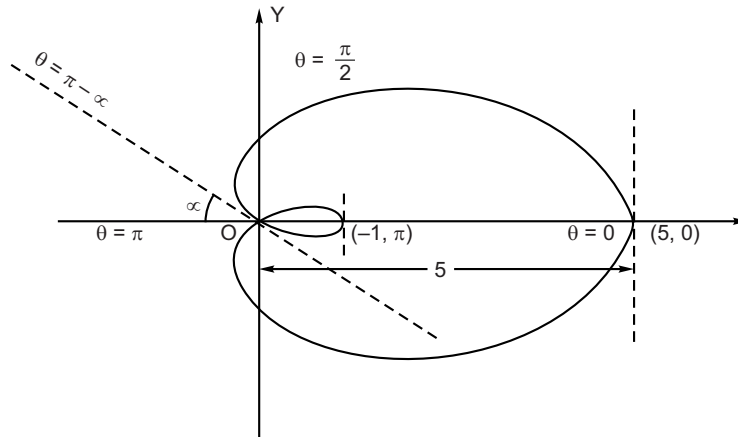
(i) Since $|\cos \theta| \leq 1$, $\therefore r \leq 5$.

$\therefore$ Entire curve lies within the circle $r = 5$.

(ii) As θ increases from 0 to $\frac{\pi}{2}$, r remains +ve but decreases from 5 to 2.

When θ increases from $\frac{\pi}{2}$ to π , r decreases from 2 to 0 and then from 0 to -1 .

Thus the shape of the curve is as shown in the figure.



152. Trace the curve $r^2 = a^2 \cos 2\theta$.

(U.P.T.U., 2008 ; A.U.U.P., 2007)

Sol. The equation of the curve is

$$r^2 = a^2 \cos 2\theta \quad \dots(1)$$

1. Symmetry :

(i) Since equation (1) remains unchanged when θ is changed to $-\theta$, the curve is symmetrical about the initial line.

(ii) Since the equation remains unchanged when ϕ is changed to $\pi - \theta$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(iii) Since equation (1) remains unaltered when r is changed to $-r$, the curve is symmetrical about the pole.

2. *Origin or Pole :*

(i) Putting $r = 0$, we get $\cos 2\theta = 0$ or $2\theta = \pm \frac{\pi}{2}$ or $\theta = \pm \frac{\pi}{4}$. Since it is possible to find real values of θ for which $r = 0$, therefore the curve passes through the pole and $\theta = \pm \frac{\pi}{4}$ are the tangents at the pole.

(ii) When $\theta = 0$, $r^2 = a^2$ or $r = \pm a$. Thus the curve meets the initial line ($\theta = 0$) in the points $(\pm a, 0)$.

3. *Asymptotes :* Since there is no finite value of θ for which $r \rightarrow \infty$, therefore the curve has no asymptote.

4. *Value of ϕ :*

$$\text{From (1),} \quad 2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\therefore \quad \tan \phi = r \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r^2}{-a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta$$

$$\text{or} \quad \tan \phi = \tan \left(\frac{\pi}{2} + 2\theta \right)$$

$$\therefore \quad \phi = \frac{\pi}{2} + 2\theta$$

When $\theta = 0$, $\phi = \frac{\pi}{2}$. Thus at the points $(\pm a, 0)$, the tangents are perpendicular to the initial line.

5. *Special Points and Region :*

$$\text{From (1),} \quad \rho = a \sqrt{\cos 2\theta} \quad \left[\text{Taking +ve root, } \therefore \text{ the curve is symmetrical about the pole.} \right]$$

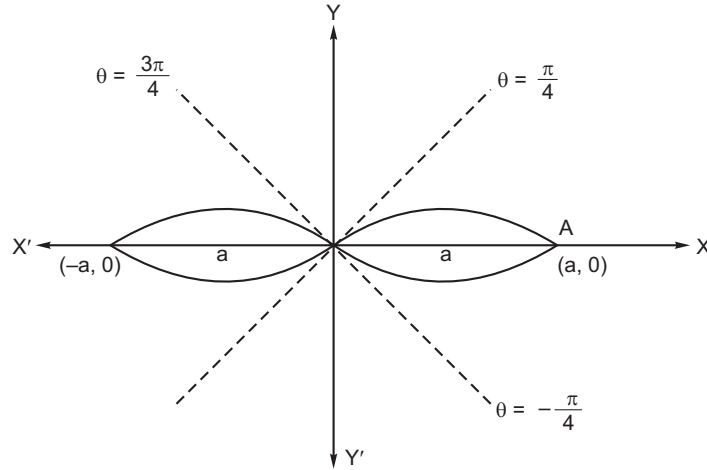
$$\therefore \quad \frac{dr}{d\theta} = -\frac{a \sin 2\theta}{\sqrt{\cos 2\theta}}$$

$\therefore \frac{dr}{d\theta}$ is -ve for $0 < \theta < \frac{\pi}{4}$, $\therefore r$ decreases in this range.

Again because $\frac{dr}{d\theta}$ is +ve for $\frac{3\pi}{4} < \theta < \pi$, $\therefore r$ increases in this range.

When $\theta = 0$, $r = a$. As θ increases from 0 to $\frac{\pi}{4}$, r decreases from a to 0. For $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, r is imaginary. Thus no portion of the curve lies between the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Again as θ increases from $\frac{3\pi}{4}$ to π , r is +ve and increases from 0 to a . Thus we trace the part of the curve above the initial line. The part of the curve below the initial line can be traced by symmetry. Hence the shape of the curve is as shown in the figure.



153. Trace the curve $x^3 + y^3 = 3axy$.

(M.D.U., 2008 U.P.T.U., 2007)

Sol. The equation of the given curve is

$$x^3 + y^3 = 3axy \quad \dots(1)$$

1. *Symmetry* : Since (1) remains unaltered when x and y are interchanged, $\therefore$ The curve is symmetrical about the line $y = x$.
2. *Origin* : The curve passes through the origin and equating to zero the lowest degree terms, the tangents at the origin are given by $xy = 0$ or $x = 0, y = 0$. These tangents being real and distinct, origin is a node.
3. *Axes Intersection* :

(i) The curve meets the axes only at the origin $(0, 0)$.

(ii) The line $y = x$ meets the curve in the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ besides the origin and the tangent

at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is $x + y = 3a$.

Equation of the tangent has been obtained as follows : $x^3 + y^3 = 3axy$

$$\therefore \frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2} = -1 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right).$$

Thus the equation of the tangent is

$$y - \frac{3a}{2} = -1 \left(x - \frac{3a}{2}\right)$$

$$\text{or} \quad y + x = \frac{3a}{2} + \frac{3a}{2} = 3a.$$

4. *Asymptotes* :

(i) The curve has no asymptotes $\parallel$ to axes.

(ii) There is an oblique asymptote which is determined as below :

Putting $x = 1, y = m$ in the highest (*i.e.*, third) degree terms and equating to zero, we get

$$\phi_3(m) = 1 + m^3 = 0 \quad \text{which gives } m = -1$$

$$\phi_2(m) = -3am \quad \text{Putting } x = 1, y = m \text{ in the second degree terms}$$

$$\therefore C = -\frac{\phi_2(m)}{\phi_3(m)} = -\frac{-3am}{3m^2} = \frac{a}{m} = -a \text{ for } m = -1.$$

$\therefore$ The asymptote is $y = mx + c$ or $y = -x - a$ or $x + y + a = 0$.

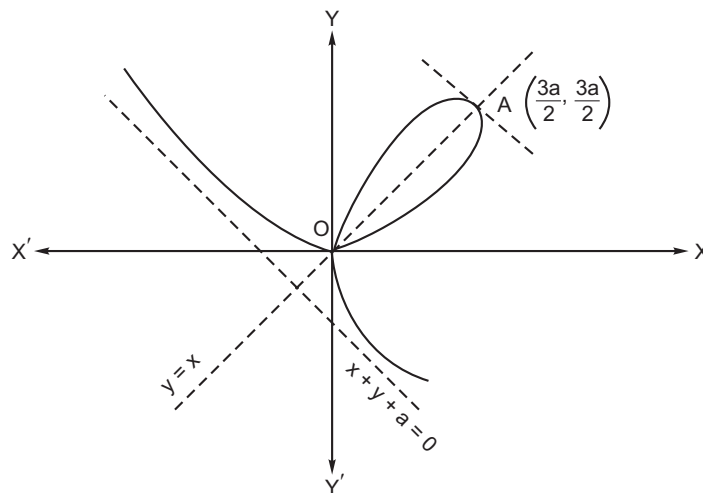
5. *Region :*

(i) x and y can't both be $-ve$ $\therefore$ if x and y are both $-ve$, then L.H.S. of equation (1) is $-ve$ and R.H.S. of (1) is $+ve$, which is impossible.

$\therefore$ No portion of the curve lies in the 3rd quadrant.

(ii) Transforming (1) to polars by putting $x = r \cos \theta$, $y = r \sin \theta$, we get

$$r = \frac{3a \sin \theta \cos \theta}{\cos^3 \theta + \sin^3 \theta} \quad \dots(2)$$



When $\theta = 0$, $r = 0$. When $\theta = \frac{\pi}{2}$, $r = 0$.

For $0 < \theta < \frac{\pi}{2}$, r is $+ve$. $\therefore r$ must first increase from 0 and then decrease to 0.

Hence there is a loop between $\theta = 0$ and $\theta = \frac{\pi}{2}$. Also when $\theta = \frac{\pi}{4}$, $r = \frac{3a}{\sqrt{2}}$.

Again as θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, $\sin \theta$ remains $+ve$ and decreases while $\cos \theta$ becomes $-ve$ and numerically increases. $\therefore r$ is $-ve$ in this range and numerically increases from 0 to ∞ . Thus we have the portion of the curve in the 4th quadrant.

As θ increases from $\frac{3\pi}{4}$ to π , r remains $+ve$ and decreases from ∞ to 0.

Thus we have the portion of the curve in the 2nd quadrant. As θ increases from π to 2π , we do not get any new point on the curve. Hence the shape of the curve is as shown in the figure.

154. Trace the curve $r = a \sin 3\theta$.

(M.D.U., 2009)

Sol. The equation of the curve is

$$r = a \sin 3\theta \quad \dots(1)$$

1. *Symmetry :* Since equation (1) remains unaltered when θ is changed to $\pi - \theta$, therefore the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

2. *Pole* : Putting $r = 0$, we get $\sin 3\theta = 0$, which gives

$$3\theta = 0, \quad \pi, \quad 2\pi, \quad 3\pi, \dots$$

or
$$\theta = 0, \quad \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \pi, \dots$$

$\therefore$ The curve passes through the pole and the tangents at the pole are $\theta = 0$, $\theta = \frac{\pi}{3}$ and

$\theta = \frac{2\pi}{3}$, the other values of θ give the same tangents.

3. *Asymptotes* : Since for no values of θ , $r \rightarrow \infty$, therefore the curve has no asymptote.

4. *Special Points and Region* :

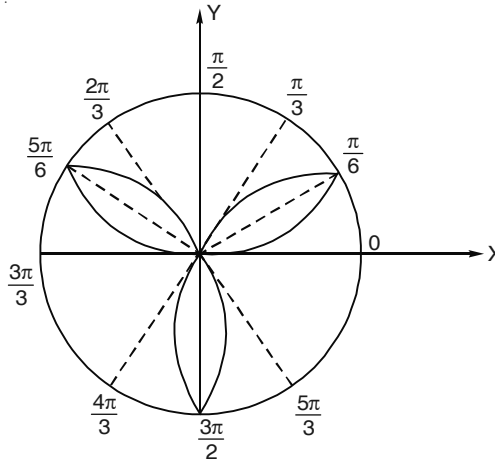
(i) From (1), $r = a \sin 3\theta$

$\therefore |\sin 3\theta| \leq 1$, $\therefore r$ is never greater than a . Thus the curve lies entirely within the circle $r = a$.

(ii) Some special points are

$\theta = 0$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$r = 0$	a	0	$-a$	0	a	0

When θ increases from 0 to $\frac{\pi}{6}$, r is +ve and increases from 0 to a . When θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is +ve and decreases from a to 0. Thus we get a loop between the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$. As θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{2}$, r is -ve and numerically increases from 0 to a .



As θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r is -ve and numerically decreases from a to 0. Thus we get a second loop between the lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ in the opposite direction. ($\because r$ is -ve)

As θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r remains +ve and increases from 0 to a . As θ increases from $\frac{5\pi}{6}$ to π , r remains +ve and decreases from a to 0. Thus we have the third loop between the lines $\theta = \frac{2\pi}{3}$ and $\theta = \pi$. When θ varies from π to 2π , the same loops are

repeated and we do not get any new loop. As r is periodic, therefore values of θ outside $[0, 2\pi]$ need not be considered. Thus the shape of the curve is as shown in the figure.

Note : The curve $r = a \sin n\theta$ or $r = a \cos n\theta$ consists of n or $2n$ loops according as n is odd or even.

155. Trace the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$. [U.P.T.U., 2007 ; A.U.U.P., 2009]

Sol. The equations of the curve are

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta) \quad \dots(1)$$

- Limits :** The greatest value of y is $2a$ and the least value is 0 . Hence the curve lies between the lines $y = 0$ and $y = 2a$.
- Symmetry :** Since $x = a(\theta + \sin \theta)$ is an odd function of θ [$\therefore f(-\theta) = -f(\theta)$] and $y = a(1 - \cos \theta)$ is an even function of θ [$\therefore f(-\theta) = f(\theta)$]
 $\therefore$ The curve is symmetrical about the y -axis.
- Origin and Axes Intersection :**
 - When $x = 0$, $\theta + \sin \theta = 0$ or $\theta = 0$ and then $y = 0$. $\therefore$ The curve passes through the origin.
 - The curve cuts the x -axis (putting $y = 0$) where $1 - \cos \theta = 0$ or $\cos \theta = 1$ or $\theta = 0$ and then $x = 0$ i.e., at $(0, 0)$.
 The curve cuts y -axis (putting $x = 0$) where $\theta + \sin \theta = 0$ or $\theta = 0$ and then $y = 0$ i.e., at $(0, 0)$.
- Asymptotes :** When $\theta \rightarrow \infty$, $x \rightarrow \infty$ then y does not tend to a limit. $\therefore$ The curve has no asymptote parallel to x -axis.
 Also y cannot tend to ∞ [$\therefore |\cos \theta| \leq 1$]. $\therefore$ The curve has no asymptote parallel to y -axis.
- Special Points :**

$$\frac{dx}{d\theta} = a(1 + \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta} = \tan \frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = 0 \quad \text{when} \quad \frac{\theta}{2} = 0 \quad \text{i.e., when} \quad \theta = 0.$$

But when $\theta = 0$, $x = 0$, $y = 0$. $\therefore$ tangent at $(0, 0)$ is parallel to x -axis i.e., x -axis itself.

Again $\frac{dy}{dx} = \infty$ when $\frac{\theta}{2} = \frac{\pi}{2}$ or $\theta = \pi$

$\theta = \pi$ gives $x = a\pi$, $y = 2a$. Thus at $(a\pi, 2a)$ the tangent is perpendicular to x -axis.

6. Region :

$$(i) \quad y = a(1 - \cos \theta) = 2a \sin^2 \frac{\theta}{2}$$

$$\therefore \sin \frac{\theta}{2} = \sqrt{\frac{y}{2a}}$$

When y is -ve, $\sin \frac{\theta}{2}$ is imaginary.

$\therefore$ No portion of the curve lies below the x -axis.

(ii) We have the following table :

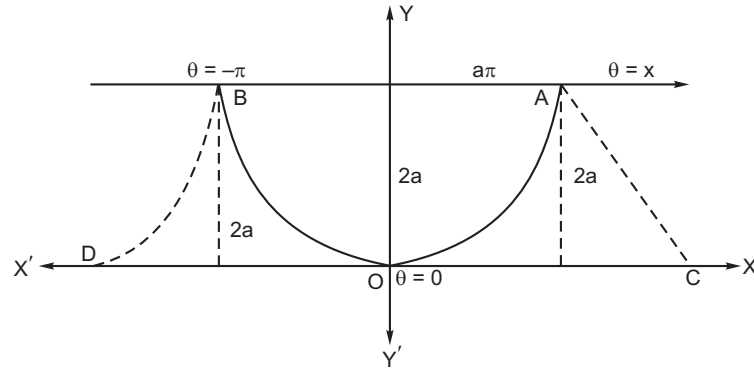
θ	$= 0$	$\frac{\pi}{2}$	π	2π	$-\frac{\pi}{2}$	$-\pi$
x	$= 0$	$a\left(\frac{\pi}{2} + 1\right)$	$a\pi$	$2a\pi$	$-a\left(\frac{\pi}{2} + 1\right)$	$-a\pi$

y	$= 0$	a	$2a$	0	0	$2a$
$\frac{dy}{dx}$	$= 0$	1	∞	0	-1	$-\infty$

(iii) As x increases from 0 to $a\pi$, y also increases from 0 to $2a$.

But as x further increases to $2a\pi$, y decreases to 0 again.

$\therefore$ For +ve values of θ from 0 to π , we can draw the portion OAC of the curve in the first quadrant.



In fact as x further increases, y again increases to $2a$ and then decreases to 0 again and so on.

The curve for -ve values of x can be traced by symmetry [$\because$ the curve is symmetrical about y -axis]. Shape of the curve is as shown in the figure.

156. Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.D.U., 2008 ; U.P.T.U., 2007)

Sol. The parametric equations of the curve are

$$x = a \cos^3 t, \quad y = a \sin^3 t$$

1. *Limits* :

$$|x| \leq a \quad \text{and} \quad |y| \leq a$$

$\Rightarrow$ The curve lies entirely within the square bounded by the lines $x = \pm a, y = \pm a$.

2. *Symmetry* : x is an even function of t and y is an odd function of t .

$\Rightarrow$ The curve is symmetrical about the x -axis.

3. *Origin and Axes Intersection* :

$$\text{When } x = 0, \cos^3 t = 0 \Rightarrow t = \frac{\pi}{2} \text{ and then } y = a \neq 0$$

$\therefore$ The curve does not pass through the origin.

The curve meets x -axis (putting $y = 0$) where $\sin^3 t = 0$ or $t = 0, \pi, 2\pi, \dots$

$\therefore$ The curve meets x -axis at $(a, 0)$ and $(-a, 0)$.

Similarly, the curve meets y -axis at $(0, a)$ and $(0, -a)$.

4. *Asymptotes* : The curve has no asymptotes.

5. *Special Points* :

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\Rightarrow \frac{dy}{dx} = -\tan t$$

$$\therefore \frac{dy}{dx} = 0 \quad \text{When } t = 0 \text{ or } \pi$$

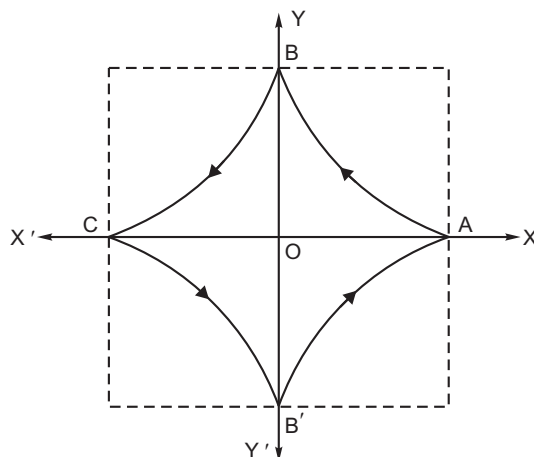
and $\frac{dy}{dx} = \infty$ When $t = \frac{\pi}{2}$

As t increases from 0 to $\frac{\pi}{2}$

x is +ve and decreases from a to 0,

y is +ve and increases from 0 to a .

Also $\frac{dy}{dx}$ varies from 0 to ∞ . Thus the portion AB is traced.



As t increases from $\frac{\pi}{2}$ to π

x is -ve and increases numerically from 0 to $-a$.

y is +ve and decreases from a to 0.

Also, $\frac{dy}{dx}$ varies from ∞ to 0.

Thus the portion BC is traced.

As t increases from π to 2π

We get the curve $CB'A$ which is the reflection of the curve CBA in the x -axis. Since x and y are periodic functions with period 2π , no new points are obtained.

Hence the shape of the curve is as shown in the figure.

157. Trace the curve: $9ay^2 = (x - 2a)(x - 5a)^2$.

Sol. The equation of the curve is

$$9ay^2 = (x - 2a)(x - 5a)^2 \quad \dots(1)$$

- Symmetry :** Since equation (1) contains only even powers of y , the curve is symmetrical about x -axis.
- Origin :** The curve does not pass through the origin because $(0, 0)$ does not satisfy the given equation.
- Asymptotes :** The curve has no asymptotes.
- Axes Intersection :**

Putting $y = 0$ in (1), we get $x = 2a, 5a$

Thus the curve meets x -axis in the points $A(2a, 0)$ and $B(5a, 0)$.

Putting $x = 0$ in (1), we get $9ay^2 = -50a^3$ which gives imaginary values of y . Therefore, the curve does not meet y -axis.

5. *Region :*

$$\text{From (1),} \quad y = \frac{1}{\sqrt{9a}} (x - 5a)\sqrt{x - 2a} \quad \dots(2)$$

When $x < 2a$, y is imaginary.

Thus the curve does not lie to the left of the line $x = 2a$.

When $x > 2a$, y is real.

6. *Special Points :*

$$\begin{aligned} \text{From (2),} \quad \frac{dy}{dx} &= \frac{1}{\sqrt{9a}} \cdot \left[\frac{(x - 5a) \cdot 1}{2\sqrt{x - 2a}} + \sqrt{x - 2a} \right] \\ &= \frac{x - 3a}{2\sqrt{a}\sqrt{x - 2a}} \end{aligned} \quad \dots(3)$$

$$\frac{dy}{dx} = 0 \text{ gives } x = 3a$$

But when $x = 3a$, then from (1), $9ay^2 = 4a^3$

$$\Rightarrow y = \pm \frac{2a}{3}$$

$\therefore$ Tangents at the points $\left(3a, \pm \frac{2a}{3}\right)$ are parallel to x -axis.

When $x > 3a$, $\frac{dy}{dx}$ is +ve

From (3)

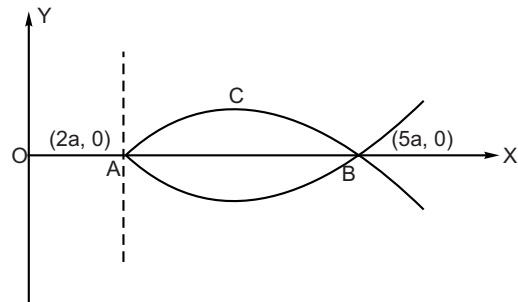
$\therefore$ The curve rises for all values of $x > 3a$ (if y is +ve).

When $x < 3a$, $\frac{dy}{dx}$ is -ve, $\therefore$ The curve is falling for values of $x < 3a$.

Again $\frac{dy}{dx} \rightarrow \infty$ when $x = 2a$.

$\therefore$ Tangent at $(2a, 0)$ is parallel to y -axis.

Thus the shape of the curve is as shown in the figure.



158. Trace the curve $x^2y^2 = x^2 - a^2$.

Sol. The equation of the curve is

$$x^2y^2 = x^2 - a^2 \quad \dots(1)$$

- Symmetry :* Since equation (1) contains only even powers of y and only even powers of x , therefore, the curve is symmetrical about both the axes. Also the curve is symmetrical in opposite quadrants.
- Origin :* The curve does not pass through the origin because $(0, 0)$ does not satisfy the given equation.
- Asymptotes :* Asymptotes parallel to x -axis are given by $y^2 - 1 = 0 \therefore y = 1$ and $y = -1$ are the asymptotes. Asymptotes parallel to y -axis are $x = 0$. Oblique asymptotes : $\phi_4(m) = m^2$, $\phi_4(m) = 0$ gives $m = 0$ and this value of m gives asymptotes parallel to x -axis (which we have already obtained).

4. *Axes Intersection :*

Putting $y = 0$ in (1), $x = \pm a$

$\therefore$ Curve meets the x -axis in the points $(a, 0)$ and $(-a, 0)$. Also $x = 0$ and $x = -a$ are the tangents at the points $(a, 0)$ and $(-a, 0)$ respectively. The curve does not meet the y -axis.

5. *Region :*

$$\text{From (1),} \quad y = \frac{\sqrt{x^2 - a^2}}{x} \quad \dots(2)$$

For y to be real, $x^2 > a^2$

$$\therefore \quad x > a \quad \text{or} \quad x < -a$$

$\therefore$ No portion of the curve lies between the lines $x = a$ and $x = -a$.

$$\text{Again from (1),} \quad x^2(1 - y^2) = a^2$$

$$\therefore \quad x = \frac{a}{\sqrt{1 - y^2}}$$

For x to be real, $1 - y^2 > 0$ i.e., $y^2 < 1$

$$\therefore \quad -1 < y < 1.$$

$\therefore$ The curve lies entirely between the lines $y = 1$ and $y = -1$.

6. *Special Points :*

$$\text{From (2),} \quad \frac{dy}{dx} = \frac{a^2}{x^2 \sqrt{x^2 - a^2}}$$

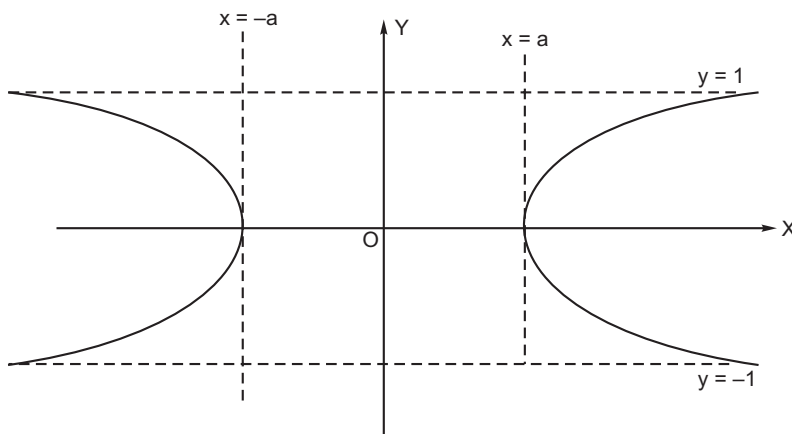
$\frac{dy}{dx}$ never becomes zero. $\frac{dy}{dx}$ is always +ve.

$$\frac{dy}{dx} \rightarrow \infty \quad \text{when} \quad x^2 - a^2 = 0 \quad \text{i.e.,} \quad x = \pm a$$

For $x = \pm a, y = 0$ from (1).

$\therefore$ Tangents at the points $(a, 0)$ and $(-a, 0)$ are parallel to y -axis.

Thus, the shape of the curve is as shown in the figure.



159. Trace the curve $xy^2 = a^2(a - x)$.

Sol. Equation of the curve is

$$xy^2 = a^2(a - x) \quad \dots(1)$$

1. *Symmetry* : Since (1) contains only even powers of y , so the curve is symmetrical about x -axis.
2. *Origin* : The curve does not pass through the origin.
3. *Axes Intersection* : The curve meets x -axis at the point $A(a, 0)$. It does not meet y -axis. Shifting the origin to $A(a, 0)$, Equation (1) becomes

$$(X + a)Y^2 = a^2[a - (X + a)]$$

$$\Rightarrow (X + a)Y^2 + a^2X = 0$$

Equating to zero, the lowest degree terms, the tangent at the new origin is $X = 0$ i.e., new y -axis.

Hence at the new origin i.e., at $A(a, 0)$, the tangent is perpendicular to x -axis.

4. *Asymptotes* : The only asymptote is $x = 0$.
5. *Region* :

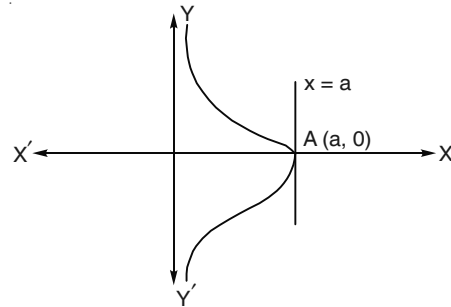
From (1),
$$y = a\sqrt{\frac{a - x}{x}}$$

When $x < 0$ or $x > a$, y is imaginary.

Thus, no portion of the curve lies to the left of y -axis and to the right of $x = a$.

When $x = a$, $y = 0$. As x decreases from a to 0, y increases from 0 to ∞ .

Hence the shape of the curve is as shown in the figure.



160. Trace the curve $y^2(a - x) = x^2(a + x)$.

Sol. The given curve is

$$y^2(a - x) = x^2(a + x) \quad \dots(1)$$

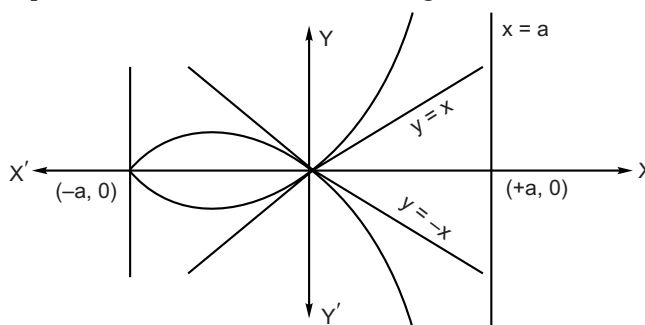
1. *Symmetry* : Curve is symmetrical about x -axis since equation (1) contains only even power of y .
2. *Origin* : The curve passes through the origin. Tangents at the origin are $y = \pm x$. Since the tangents are real and distinct, therefore origin is a node.
3. *Axes Intersection* : The curve intersects x -axis at $(0, 0)$ and $(-a, 0)$. The curve intersects y -axis at $(0, 0)$ only. The tangent at $(-a, 0)$ can be found to be $x + a = 0$.
4. *Asymptotes* : The only asymptote is $x = a$.
5. *Region* :

From (1),
$$y = x\sqrt{\frac{a + x}{a - x}}$$

For $x > a$ or $x < -a$, y is imaginary.

Hence, the curve does not exist for $x > a$ or $x < -a$.

Hence shape of the curve is as shown in the figure.



161. Trace the curve $x^3 + y^3 = ax^2$ ($a > 0$).

Sol. The equation of the curve is

$$x^3 + y^3 = ax^2 \quad \dots(1)$$

1. *Symmetry* : The curve is neither symmetrical about the co-ordinate axes nor about the line $y = x$.

2. *Origin* : It passes through the origin. Tangent at the origin are given by

$$x^2 = 0 \text{ i.e., } x = 0 \text{ and } x = 0$$

$\therefore$ The origin is a cusp.

From (1), on neglecting x^3 , we have

$$ax^2 = y^3$$

$$\text{or} \quad x = \pm \sqrt{\frac{y^3}{a}}$$

The roots are real only when y is +ve.

Thus the cusp is a single cusp. Again, the two values of x are of opposite sign, so that it is a cusp of the first species.

3. *Axes Intersection* : The curve meets the x -axis at $(a, 0)$.

4. *Asymptotes* :

$$(i) \text{ From (1), } y^3 = ax^2 - x^3$$

$$\text{or } y^3 = -x^3 \left(1 - \frac{a}{x}\right)$$

$$y = -x \left(1 - \frac{a}{x}\right)^{1/3}$$

$$= -x \left[1 - \frac{a}{3x} - \frac{a^2}{9x^2} + \dots\right] = -x + \frac{a}{3} + \frac{a^2}{9x} + \dots$$

$\therefore$ The only real asymptote is

$$y = -x + \frac{a}{3}$$

It meets the curve in the point $\left(\frac{a}{9}, \frac{2a}{9}\right)$.

(ii) If y_c and y_a denote the ordinates of the curve and that of the asymptote for the same value of x , then

$$y_c - y_a = \frac{a^2}{9x} - \dots$$

which for large values of x is > 0 for $x > 0$ and < 0 for $x < 0$.

Thus, the curve lies above the asymptote to the right of y -axis (i.e., for $x > 0$) and below the asymptote to the left of y -axis (i.e., for $x < 0$).

5. *Special Points :*

From (1),

$$y = x^{2/3} (a - x)^{1/3}$$

$$\therefore \frac{dy}{dx} = \frac{2}{3} x^{-1/3} (a - x)^{1/3} - \frac{1}{3} (a - x)^{-2/3} \cdot x^{2/3} = \frac{2a - 3x}{3x^{1/3} \cdot (a - x)^{2/3}}$$

$$\frac{dy}{dx} = 0 \quad \text{When} \quad x = \frac{2a}{3}$$

$\therefore$ The tangent is parallel to x -axis at the point $\left(\frac{2a}{3}, \frac{(4)^{1/3} a}{3}\right)$.

Also, $\frac{dy}{dx} = \infty$, when $x = a$ or $x = 0$.

$\therefore$ Tangent is parallel to y -axis at $(a, 0)$.

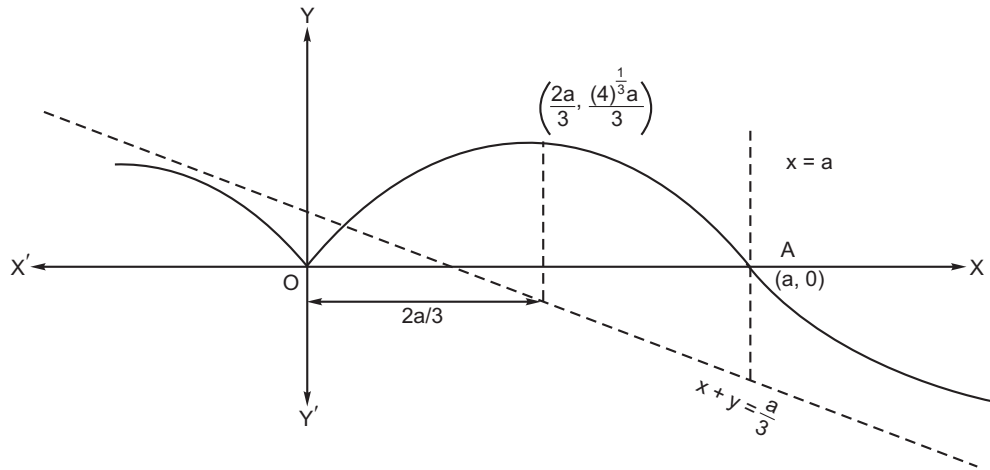
6. *Region :*

- (i) x and y cannot be both negative, for this would make the L.H.S. of (1) negative and the R.H.S. positive. Hence, no part of the curve exists in the third quadrant.
- (ii) The equation (1) of the curve can be written as $y^3 = x^2(a - x)$.

Now, when x is negative, y is positive and as x decreases from 0 to $-\infty$, y increases from 0 to ∞ .

When x is positive and lies between 0 and a , y is also positive. When $x > a$, y is $-ve$. This is possible only if the curve crosses the line $x = a$ which is tangent to the curve, at the point $(a, 0)$. $\therefore (a, 0)$ is the point of the inflexion. As $x \rightarrow \infty$, y also $\rightarrow \infty$.

The approximate shape of the curve is as shown in the figure.



162. Trace the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$. (U.P.T.U., 2007 ; M.D.U., 2008 ; A.U.U.P., 2008)

Sol. The equation of the curve is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$... (1)

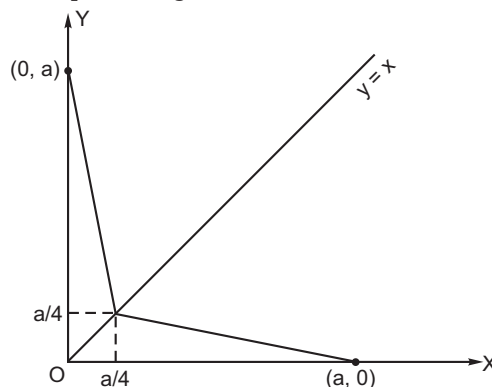
1. *Symmetry :* The curve is symmetrical about the line $y = x$.
2. *Origin :* The curve does not pass through the origin because $(0, 0)$ does not satisfy the equation (1).
3. *Asymptotes :* No asymptotes parallel to the co-ordinate axes.

4. *Axes Intersection* : When $x = 0$, $y = a$, when $y = 0$, $x = a$. Hence the curve meets the co-ordinate axes at $(0, a)$ and $(a, 0)$.

5. *Special Points* :

$x : a$	$\frac{a}{4}$	$\frac{a}{2}$	$\frac{3a}{4}$	0
$y : 0$	$\frac{a}{4}$	$\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 \cdot a$	$\left(\frac{2-\sqrt{3}}{2}\right)^2 \cdot a$	a

The approximate shape of the given curve is as shown in the figure.



163. Trace the curve $y^2(x+3a) = x(x-a)(x-2a)$.

Sol. 1. *Symmetry* : The curve is symmetrical about x -axis because equation has even power of y .

2. *Origin* : The curve passes through the origin where the tangent is $x = 0$.

3. *Asymptotes and Region* :

(a) (i) When $x = 0$, $y = 0$; when $0 < x < a$, y is real.

(ii) When $x = a$, $y = 0$ again.

When $a < x < 2a$, y is imaginary.

When $x = 2a$, $y = 0$ and when $x > 2a$, y is real.

Tangent at $(2a, 0)$ is parallel to y -axis obtained by shifting origin to it and then equating the lowest degree terms to zero.

When $x = 3a$, $y = a$ and when $x = 4a$, $y = 1.8a$

When $x \rightarrow \infty$, $y \rightarrow \infty$.

When x, y are very large

$$\begin{aligned} \frac{y^2}{x^2} &= \frac{(x-a)(x-2a)}{x(x+3a)} = \frac{\left(1-\frac{a}{x}\right)\left(1-\frac{2a}{x}\right)}{\left(1+\frac{3a}{x}\right)} = 1 - \frac{6a}{x} + \frac{20a^2}{x^2} + \dots \\ \Rightarrow \quad \frac{y}{x} &= \pm \left(1 - \frac{3a}{x} + \frac{11a^2}{2x^2} + \dots\right) \\ \text{or} \quad y &= x - 3a + \frac{11a^2}{2x} + \dots \quad \dots(1) \end{aligned}$$

leaving -ve values of y/x .

$y = x - 3a$ is an asymptote and the curve lies above this asymptote in the I quadrant.

(b) When x is -ve and numerically less than $3a$, y is imaginary.

Therefore, the curve does not exist between $x = -3a$ and $x = 0$.

When $x = -3a$, the expression for y becomes meaningless but $x \rightarrow -3a$ from left, $y^2 \rightarrow \infty$. In fact $x = -3a$ is an asymptote.

When $x \rightarrow \infty$, $y^2 \rightarrow \infty$.

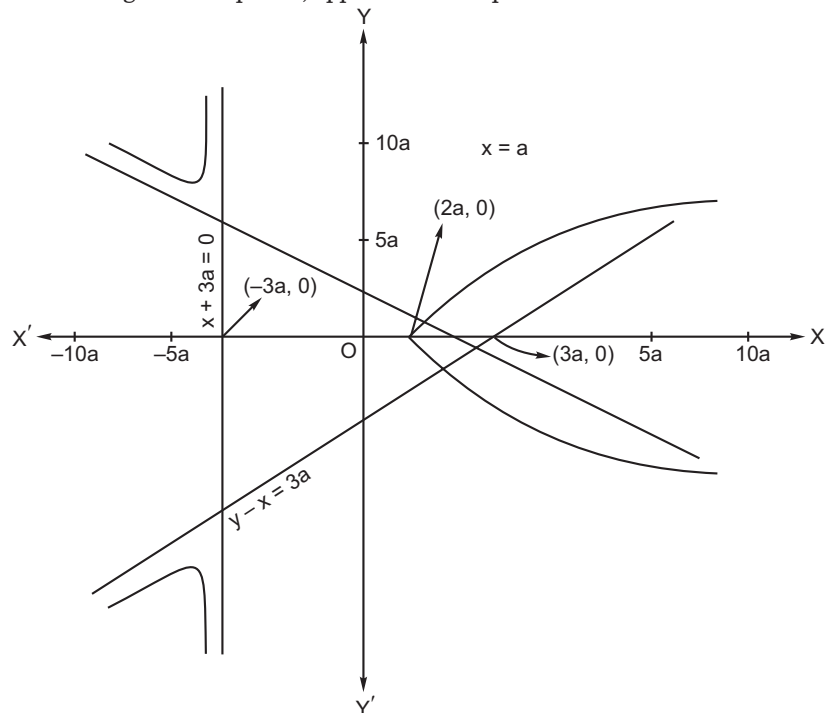
For large values of x and y , taking negative values of $\frac{y}{x}$ from (1), we get

$$y = -x + 3a - \frac{11a^2}{2x} + \dots$$

$\therefore y = 3a - x$ is an asymptote and the curve lies above the asymptote in the IIInd quadrant.

$\therefore y$ is very large in the neighbourhood of $x = -3a$ in IIInd quadrant and is again very large when x is -ve and numerically large.

Considering all these points, approximate shape of the curve is as shown in the figure.



164. Trace the curve $r = a(1 - \cos \theta)$. (P.T.U., B.Tech. Dec., 2005 ; A.U.U.P., 2007 ; M.D.U., 2008)

Sol. Equation of the curve is $r = a(1 - \cos \theta)$... (1)

1. *Symmetry* : The curve is symmetrical about the initial line because the equation of the curve remains unaltered when θ is changed to $-\theta$.

2. *Pole or Origin* :

(i) When $\theta = 0$, $r = 0$ hence the curve passes through the pole and the tangent at the pole is the line $\theta = 0$ i.e., the initial line.

(ii) The curve meets the initial line $\theta = 0$ at $(0, 0)$ and the lines $\theta = \frac{\pi}{2}$ and $\theta = \pi$ in the

points $\left(a, \frac{\pi}{2}\right)$ and $(2a, \pi)$ respectively.

3. *Asymptote* : Since for any finite value of θ , r does not tend to ∞ , $\therefore$ the curve has no asymptote.

4. *Special Points* : The corresponding values of θ and r are given below :

$\theta : 0$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$r : 0$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

5. *Region* :

(i) $\because |\cos \theta| \leq 1 \therefore$ From (1), $r \leq 2a$

$\therefore$ The curve lies entirely within the circle $r = 2a$.

(ii) When θ increases from 0 to π , r remains +ve and increases from 0 to $2a$.

When θ increases from π to 2π , r is +ve and decreases from $2a$ to 0.

6. *Value of ϕ* :

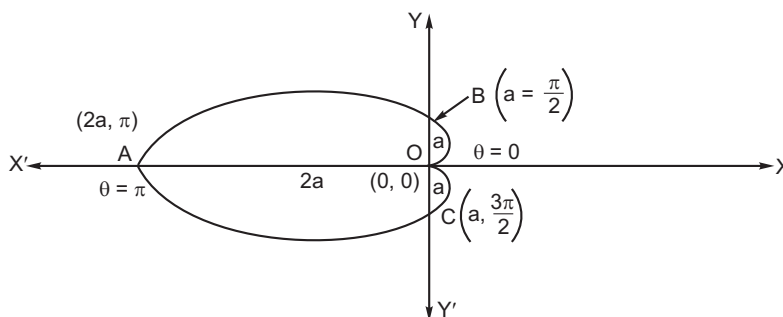
$$\text{From (1),} \quad \frac{dr}{d\theta} = a \sin \theta$$

$$\therefore \tan \phi = r \frac{d\theta}{dr} = \frac{a(1 - \cos \theta)}{a \sin \theta} = \tan \frac{\theta}{2}$$

$$\Rightarrow \phi = \frac{\theta}{2}$$

Now, $\phi = 0$ when $\theta = 0$ and $\phi = \frac{\theta}{2}$ when $\theta = \pi \therefore$ at $(2a, \pi)$ the tangent is perpendicular to the line $\theta = \pi$.

Shape of the curve is as shown in the figure.



165. Trace the curve $r = ae^{m\theta}$. (a, m are +ve)

Sol. The equation of the curve is $r = ae^{m\theta}$... (1)

1. *Symmetry* : Curve has no symmetry.

2. *Pole* : r is not zero for any finite real value of θ . $\therefore$ The curve does not pass through the pole.

3. *Value of ϕ* :

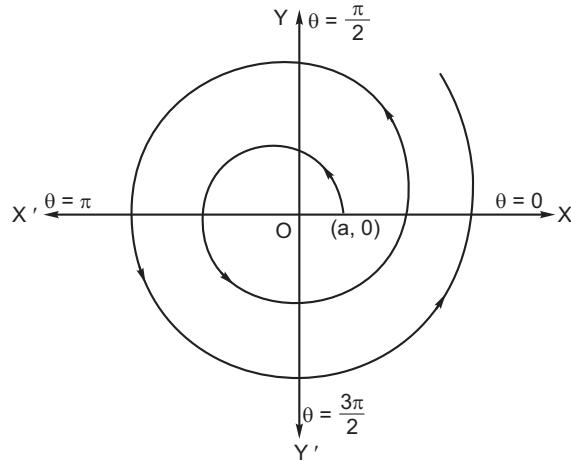
$$\tan \phi = r \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{ae^{m\theta}}{ame^{m\theta}} = \frac{1}{m}$$

$\Rightarrow \phi$ is constant.

4. Region :

- (i) r is always +ve but there is no limit to its values.
- (ii) When $\theta = 0, r = a$. $\therefore (a, 0)$ lies on the curve.
- (iii) As θ increases from 0 to ∞, r remains +ve and increases from a to ∞ .
- (iv) As θ decreases from 0 to $-\infty, r$ still remains positive and decreases from a to 0.

Hence the shape of the curve is as shown in the figure.



166. Trace the curve $r^2 \cos \theta = a^2 \sin 3\theta$.

Sol. The equation of the curve is $r^2 \cos \theta = a^2 \sin 3\theta$

...(1)

1. **Symmetry** : On changing r to $-r$, equation (1) remains unchanged.

$\therefore$ The curve is symmetrical about the pole i.e., the curve is symmetrical in opposite quadrants.

2. **Origin or Pole** : Putting $r = 0$ in equation (1), we get $\sin 3\theta = 0$

$\therefore 3\theta = n\pi$, where n is 0 or any other integer.

$$\therefore \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}$$

$\therefore$ The curve passes through the pole and the tangents at the pole are

$$\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \dots, \theta = \frac{6\pi}{3}.$$

3. **Asymptotes** :

Putting $r = \frac{1}{u}$, from (1)

$$u^2 = \frac{\cos \theta}{a^2 \sin 3\theta} \quad \dots(2)$$

Let $u \rightarrow 0$. Therefore $\cos \theta \rightarrow 0$ i.e., $\theta \rightarrow \frac{\pi}{2}, \frac{3\pi}{2}$

Differentiating (2) w.r.t., θ , we have

$$2u \frac{du}{d\theta} = \frac{-1}{a^2} \left[\frac{\sin 3\theta \cdot \sin \theta + 3 \cos \theta \cos 3\theta}{\sin^2 3\theta} \right]$$

$$\therefore \frac{du}{d\theta} = - \frac{\sin 3\theta \sin \theta + 3 \cos \theta \cos 3\theta}{2a^2 u \sin^2 3\theta}$$

For $\theta_1 = \frac{\pi}{2}$,

$$p = \lim_{\substack{\theta \rightarrow \theta_1 \\ u \rightarrow 0}} \left(- \frac{d\theta}{du} \right) = 0 \quad (\because u \rightarrow 0)$$

$\therefore$ The asymptote is

$$p = r \sin (\theta_1 - \theta)$$

$$\text{or} \quad 0 = r \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\text{i.e.,} \quad r \cos \theta = 0$$

$$\text{or} \quad x = 0$$

i.e., y -axis is an asymptote to the curve.

Again for $\theta_1 = \frac{3\pi}{2}$, y -axis is an asymptote.

4. *Axes Intersection :*

When $\theta = 0$, from (1), $r = 0$

$$\text{When } \theta = \frac{\pi}{6}, \quad r^2 = \frac{2a^2}{\sqrt{3}} \quad \therefore \quad r = \pm a \sqrt{\frac{2}{\sqrt{3}}}$$

$\therefore$ The two points are $\left(\pm a \sqrt{\frac{2}{\sqrt{3}}}, \frac{\pi}{6} \right)$

When $\theta = \frac{\pi}{3}$, $r = 0$ again.

5. *Region w.r.t., tangents at the pole :*

$$\text{From (1),} \quad r^2 = \frac{a^2 \sin 3\theta}{\cos \theta}$$

When $0 \leq \theta \leq \frac{\pi}{3}$, r^2 is +ve $\therefore r$ is real

When $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, r^2 is -ve $\therefore r$ is imaginary

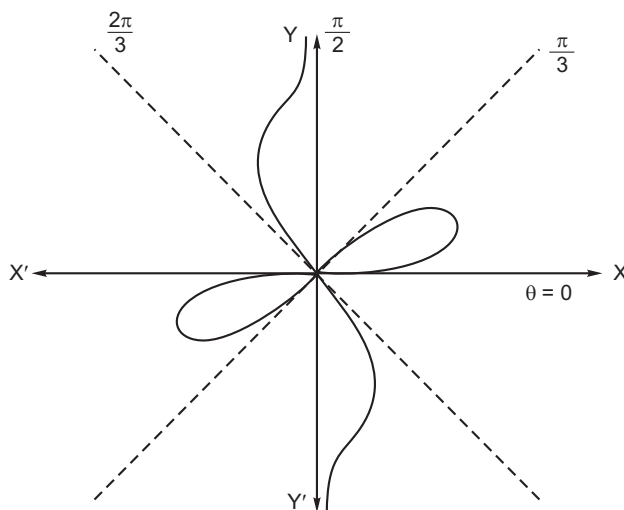
$\therefore$ No portion of the curve lies between the lines $\theta = \pi/3$ and $\theta = \frac{\pi}{2}$.

When $\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$, r^2 is +ve $\therefore r$ is real

When $\frac{2\pi}{3} \leq \theta \leq \pi$, r^2 is -ve $\therefore r$ is imaginary

$\therefore$ No portion of the curve lies between the lines $\theta = \frac{2\pi}{3}$ and $\theta = \pi$.

Hence the shape of the curve is as shown in the figure.



167. Trace the curve $r\theta = a$.

Sol. The equation of the curve is

$$r\theta = a \quad \dots(1)$$

Firstly, we consider positive values of θ , i.e., values of θ in $(0, \infty)$

1. *Symmetry* :

Equation (1) remains unchanged where θ is changed to $-\theta$ and r to $-r$, $\therefore$ the curve is

symmetrical about the line $\theta = \frac{\pi}{2}$ i.e., y -axis.

2. *Origin* :

Since r does not become zero for any real finite value of θ , therefore the curve does not pass through the pole.

3. *Asymptotes* :

Equation (1) is $r\theta = a$

Putting $r = \frac{1}{u}$, we have $\frac{\theta}{u} = a, u = \frac{\theta}{a}$

$u \rightarrow 0$ gives $\theta \rightarrow 0 \quad \therefore \theta_1 = 0$

$$\frac{du}{d\theta} = \frac{1}{a}$$

$$p = \lim_{\substack{\theta \rightarrow \theta_1 \\ u \rightarrow 0}} \left(-\frac{d\theta}{du} \right) = \lim_{\theta \rightarrow 0} (-a) = -a$$

Putting the values of p and θ_1 in

$p = r \sin(\theta_1 - \theta)$, the equation of the asymptote is

$$-a = r \sin(-\theta)$$

$$\text{or} \quad r \sin \theta = a$$

$$\text{or} \quad y = a \quad (\because r \sin \theta = y)$$

4. *Axes Intersection* :

$$\text{When} \quad \theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{3\pi}{2}, 2\pi \dots$$

$$\therefore \text{From (1);} \quad r = \infty, \frac{6a}{\pi}, \frac{3a}{\pi}, \frac{2a}{\pi}, \frac{3a}{2\pi}, \frac{a}{\pi}, \frac{2a}{3\pi}, \frac{a}{2\pi}, \dots$$

5. *Region* :

$$(i) \text{ From (1)} \quad r = \frac{a}{\theta}$$

$$r \sin \theta = \frac{a}{\theta} \sin \theta = a \cdot \frac{\sin \theta}{\theta} < a \quad \left[\because \frac{\sin \theta}{\theta} < 1 \right]$$

$$\text{or} \quad y < a \quad (\because y = r \sin \theta)$$

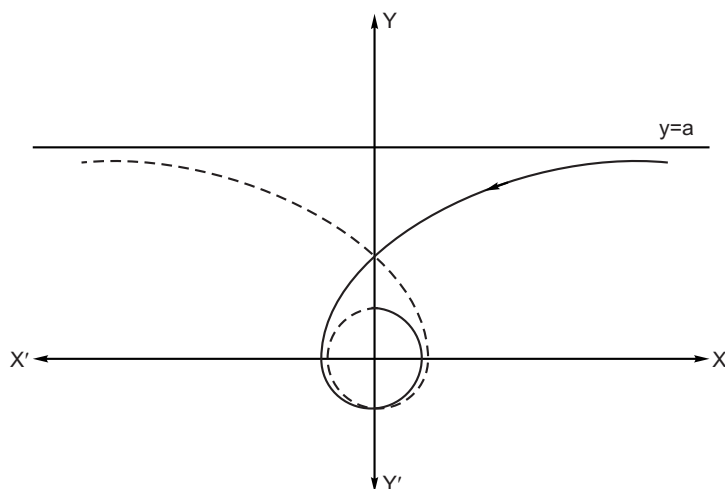
$\therefore$ The curve lies entirely below the asymptote $r \sin \theta = a$ or $y = a$.

(ii) For positive values of θ , r is positive.

6. Value of ϕ :

$$\text{From (1),} \quad r = \frac{a}{\theta} \quad \therefore \quad \frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

$\frac{dr}{d\theta}$ is -ve for all values of θ .



$\therefore r$ decreases as θ increases.

As θ increases from 0 to ∞ , r is +ve and decreases from ∞ to 0.

$$\tan \phi = \frac{r}{\frac{dr}{d\theta}} = \frac{a}{\theta} \left(-\frac{\theta^2}{a} \right) = -\theta$$

Hence for +ve values of θ , the shape of the curve is as shown by thick lines in the figure. The part of the curve for -ve values of θ (or here r also because from $r = \frac{a}{\theta}$, r is -ve when θ is negative) is obtained by turning the curve for positive value of θ through two right angles. This branch is shown in the figure by the dotted line.

168. Trace the curve $r = a(\sec \theta + \cos \theta)$.

Sol. The equation of the curve is $r = a(\sec \theta + \cos \theta)$

...(1)

Changing to cartesians, we get

$$r = a \left(\frac{r}{x} + \frac{x}{r} \right) \quad \text{or} \quad r^2 x = a(r^2 + x^2)$$

or

$$(x^2 + y^2)x = a(2x^2 + y^2)$$

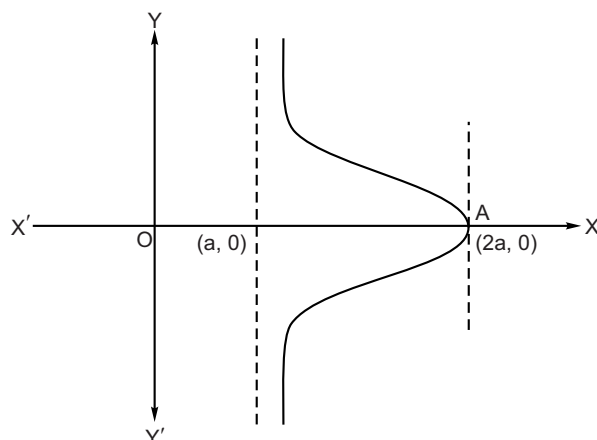
$$[\because r^2 = x^2 + y^2]$$

or

$$y^2(x - a) = x^2(2a - x)$$

Now the curve can be easily traced.

Shape of the curve is as shown in the figure.



169. Trace the curve $x^5 + y^5 = 5a^2x^2y$.

Sol. The equation of the curve is

$$x^5 + y^5 = 5a^2x^2y \quad \dots(1)$$

1. *Symmetry* :

The curve is symmetrical in opposite quadrants because equation (1) remains unchanged when both x and y are changed to $-x$ and $-y$ respectively.

2. *Origin* :

The curve passes through the origin. The tangents at the origin are $x^2y = 0$, i.e., $x = 0$, $x = 0$, $y = 0$. Hence y -axis is a cuspidal tangent. In fact, the origin is a node as well as a cusp.

3. *Asymptotes* : Since coefficients of x^5 and y^5 are constants, therefore, the curve has no asymptotes parallel to axes.

Let us now find oblique asymptotes.

$$\text{From (1),} \quad \phi_5(m) = 1 + m^5, \quad \phi_4(m) = 0$$

$$\text{Putting } \phi_5(m) = 0, \text{ we have } m^5 = -1 \quad \therefore m = -1$$

$$\text{To find } c, \quad c\phi'_5(m) + \phi_4(m) = 0$$

$$c(5m^4) = 0 \quad \text{or} \quad c = 0$$

$\therefore$ Equation of the asymptote is $y = mx + c$ or $y = -x$.

Position of the curve w.r.t., asymptote :

In the second quadrant, x is -ve and y is +ve. $\therefore$ L.H.S. of (1) namely $x^5 + y^5$ must be +ve because R.H.S. of (1) is +ve.

Hence, y should be numerically $> x$.

Thus, the curve lies above the asymptote $y = -x$. Because of symmetry in opposite quadrants, the curve approaches the other end (in the fourth quadrant) from below.

4. *Axes Intersection* :

(i) The curve meets the co-ordinate axes only at $(0, 0)$.

(ii) Intersection with line $y = x$;

Putting $y = x$ in equation (1), we have

$$2x^5 = 5a^2x^3$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = \pm \sqrt{\frac{5}{2}} \cdot a$$

$$\therefore \quad y = 0 \quad \text{or} \quad y = \pm \sqrt{\frac{5}{2}} \cdot a$$

$$\therefore \text{ Points of intersection are } (0, 0); \left(\pm \sqrt{\frac{5}{2}} \cdot a, \pm \sqrt{\frac{5}{2}} \cdot a \right)$$

(iii) Intersection with the asymptote $y = -x$.

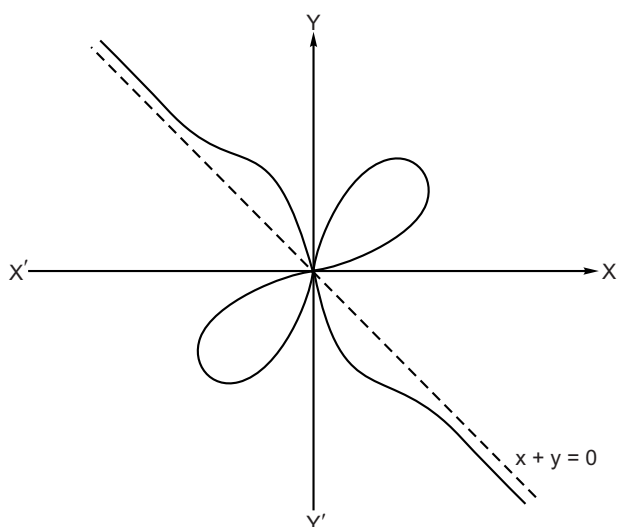
Solving (1) and $y = -x$, we again have $(0, 0)$ as their intersection.

5. *Region* :

Transforming to polars by putting $x = r \cos \theta$ and $y = r \sin \theta$ in (1), we have

$$r^2 = \frac{5a^2 \cos^2 \theta \sin \theta}{\cos^5 \theta + \sin^5 \theta}$$

For values of θ between $\frac{3\pi}{4}$ and π , r^2 is negative and so r is imaginary. Thus, no portion of the curve lies between the lines $\theta = \frac{3\pi}{4}$ and $\theta = \pi$. Thus, the shape of the curve is as shown in the figure.



170. Trace the four-leaved rose $r = a \sin 2\theta$.

Sol. The equation of the given curve is

$$r = a \sin 2\theta \quad \dots(1)$$

1. *Symmetry* : The curve is symmetrical about the line $\theta = \frac{\pi}{4}$ and the line $\theta = \frac{\pi}{2}$.

2. *Pole* : Put $r = 0$ in (1), we get

$$\sin 2\theta = 0$$

$$\Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2}, \quad \text{where, } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$\Rightarrow \theta = 0, \pm \frac{\pi}{2}, \pm \pi, \pm \frac{3\pi}{2}, \dots$$

Hence, the curve passes through the pole and the tangents at the pole are $\theta = 0, \theta = \frac{\pi}{2}$ because the other values of θ give the same tangents.

3. *Asymptotes* : The curve has no asymptotes.

4. *Special Points and Region* :

$$(i) \because |\sin 2\theta| \leq 1$$

$$\therefore \text{From (1), } |r| \leq a$$

Hence, the curve lies entirely within the circle $r = a$.

(ii) From (1), $\frac{dr}{d\theta} = 2a \cos 2\theta$

and $\frac{d^2r}{d\theta^2} = -4a \sin 2\theta$

For r to be maximum or minimum, $\frac{dr}{d\theta} = 0$

$\Rightarrow 2a \cos 2\theta = 0$ or $\cos 2\theta = 0$

$\Rightarrow 2\theta = (2n+1)\frac{\pi}{2}$ where $n \in I$

$\Rightarrow \theta = (2n+1)\frac{\pi}{4}$ where $n \in I$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$

For r to be maximum, $\frac{d^2r}{d\theta^2}$ should be -ve and we find that $\frac{d^2r}{d\theta^2}$ is -ve for

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$ and the maximum value of r is $r_{\max} = a \sin \left(2 \cdot \frac{\pi}{4} \right)$ or $r_{\max} = a \sin \frac{\pi}{2} = a$.

The corresponding values of r and θ are given below :

$\theta :$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$r :$	0	a	0	$-a$	0

As θ increases from 0 to $\frac{\pi}{4}$, r is +ve and increases from 0 to a . As θ increases from

$\frac{\pi}{4}$ to $\frac{\pi}{2}$, r is +ve and decreases from a to 0. Thus we get a loop between the lines

$\theta = 0$ and $\theta = \frac{\pi}{2}$.

Similarly, we get a loop between the line $\theta = \frac{\pi}{2}$

and $\theta = \pi$ but in opposite direction ($\because r$ is -ve).

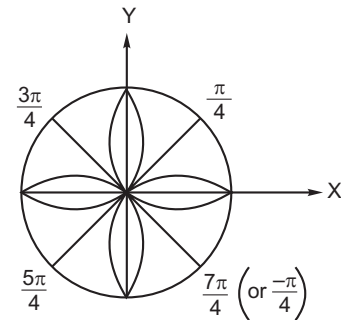
Thus, we trace the two loops I and II.

Since the curve is symmetrical about the line

$\theta = \frac{\pi}{2}$, the other two loops are traced by symmetry.

Thus, in all we get four equal loops.

Hence the slope of the curve is as shown in the figure.



171. Trace the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$.

Sol. The parametric equations of the cycloid are

$$x = a(\theta + \sin \theta), \quad y = a(1 + \cos \theta) \quad \dots(1)$$

Let us trace this curve firstly for values of θ in the interval $(-\pi, \pi)$.

1. *Symmetry* :

On changing θ to $-\theta$ in (1), changes to $-x$ and y remains unchanged. $\therefore$ The curve is symmetrical about y -axis.

2. *Origin* :

Putting $y = 0$, $a(1 + \cos \theta) = 0 \quad \therefore \quad \cos \theta = -1$

or $\theta = \pi$

For $\theta = \pi$, $x = a(\pi + \sin \pi) = a\pi \neq 0$.

$\therefore$ The curve does not pass through the origin.

3. *Asymptotes* :

The curve has no asymptotes.

4. *Points of Intersection* :

(i) Intersections with x -axis :

Putting $y = 0$ in (1), we get $\theta = \pi$ which gives $x = a\pi$.

$\therefore$ Intersection with x -axis is $(a\pi, 0)$.

(ii) Intersection with y -axis :

Putting $x = 0$ in (1), we have $\theta + \sin \theta = 0$ which is satisfied by only $\theta = 0$ and for $\theta = 0$, $y = a(1 + \cos \theta) = 2a$.

$\therefore$ Intersection with y -axis is $(0, 2a)$.

5. *Region* :

We know that $-1 \leq \cos \theta \leq 1$

$\therefore \quad 1 - 1 \leq 1 + \cos \theta \leq 1 + 1$

or $0 \leq 1 + \cos \theta \leq 2$

$\therefore \quad 0 \leq a(1 + \cos \theta) \leq 2a$

or $0 \leq y \leq 2a$

$\therefore$ Curve lies entirely between the lines $y = 0$ and $y = 2a$.

6. *Special Points* :

From (1), $\frac{dx}{d\theta} = a(1 + \cos \theta)$

$$\frac{dy}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a \sin \theta}{a(1 + \cos \theta)} = -\tan \frac{\theta}{2} \quad \dots(2)$$

Corresponding values of x , y and $\frac{dy}{dx}$ for different values of θ are given below :

	$\theta = -\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
From (1),	$x = -a\pi$	$-a\left(\frac{\pi}{2} + 1\right)$	0	$a\left(\frac{\pi}{2} + 1\right)$	$a\pi$
	$y = 0$	a	$2a$	a	0
From (2),	$\frac{dy}{dx} = \infty$	1	0	-1	$-\infty$

$\therefore$ Points on the curve are $(-a\pi, 0)$, $\left[-a\left(\frac{\pi}{2} + 1\right), a\right]$, $[0, 2a]$, $\left[a\left(\frac{\pi}{2} + 1\right), a\right]$, $[a\pi, 0]$.

Tangents at the points $[-a\pi, 0]$ and $[a\pi, 0]$ are parallel to y -axis. $\left(\because \frac{dy}{dx} = \infty \text{ or } -\infty\right)$

Tangent at the point $(0, 2a)$ is parallel to x -axis. $\left(\because \frac{dy}{dx} = 0\right)$

From (2), $\frac{dy}{dx} = -\tan \frac{\theta}{2}$

Again differentiating w.r.t. x ,

$$\frac{d^2y}{dx^2} = -\left(\sec^2 \frac{\theta}{2}\right) \frac{1}{2} \cdot \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2} \sec^2 \frac{\theta}{2} \cdot \frac{1}{2a} \sec^2 \frac{\theta}{2}$$

or $\frac{d^2y}{dx^2} = -\frac{1}{4a} \sec^4 \frac{\theta}{2}$

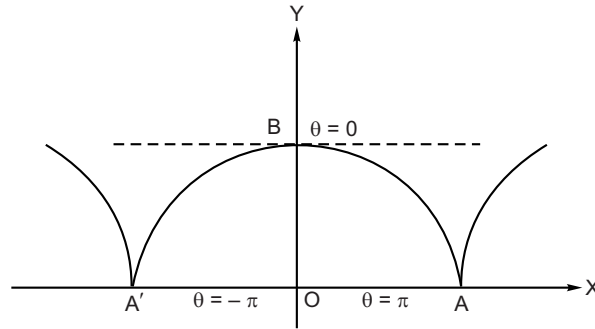
$\therefore \frac{d^2y}{dx^2}$ is -ve for all values of θ .

$\therefore$ The curve is concave downwards.

For values of $\theta > \pi$ or $< -\pi$, the same types of branches of the curve will be obtained.

The curve consists of congruent arches on both sides of y -axis which extend to infinity.

Hence the shape of the curve is as shown in the figure.



172. Trace the curve $x = a \left[\cos t + \frac{1}{2} \log \tan^2 \frac{t}{2} \right]$, $y = a \sin t$.

Sol. The equations of the curve are

$$x = a \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2}, \quad y = a \sin t \quad \dots(1)$$

1. *Symmetry* :

(i) On changing t to $-t$ in (1), x remains unchanged and y changes to $-y$. $\therefore$ The curve is symmetrical about x -axis.

(ii) On changing t to $\pi - t$ in (1), y is unchanged and x changes to $-x$.
 $\therefore$ The curve is symmetrical about y -axis.

2. *Origin* :

Putting $y = 0$, we get $\sin t = 0$ or $t = 0$ and then

$$x = a + \frac{a}{2} \log 0 = -\infty$$

$\therefore$ The curve does not pass through the origin.

3. *Asymptotes* :

When $t = 0$, $x \rightarrow -\infty$ and $y = 0$

Thus $y = 0$, i.e., x -axis is an asymptote to the curve.

4. *Points of Intersection* :

The curve does not meet x -axis.

The curve meets y -axis (putting $x = 0$)

where,
$$x = a \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2} = 0$$

which is satisfied by $t = \pm \frac{\pi}{2}$.

and then $y = a \sin \left(\pm \frac{\pi}{2} \right) = \pm a$. Thus, the curve meets y -axis in the points $(0, \pm a)$.

5. *Region :*

We know that
$$\begin{aligned} -1 &\leq \sin t \leq 1 \\ -a &\leq a \sin t \leq a \quad \text{or} \quad -a \leq y \leq a \end{aligned}$$

Thus, the curve lies entirely between the lines $y = \pm a$.

6. *Special Points :*

From (1),
$$\frac{dx}{dt} = -a \sin t + \frac{1}{2} a \cdot \frac{1}{\tan^2 \frac{t}{2}} \cdot 2 \tan \frac{t}{2} \sec^2 \frac{t}{2} \cdot \frac{1}{2}$$

$$\begin{aligned} \Rightarrow \quad \frac{dx}{dt} &= a \left(-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\ &= a \left(-\sin t + \frac{1}{\sin t} \right) = \frac{a \cos^2 t}{\sin t} \end{aligned}$$

and
$$\frac{dy}{dt} = a \cos t$$

$$\therefore \quad \frac{dy}{dx} = \tan t \quad \dots(2)$$

$\frac{dy}{dx} = \infty$ when $t = \pm \frac{\pi}{2}$ and then $x = 0, y = \pm a$

Thus, at the points $(0, \pm a)$, the tangent is parallel to y -axis.

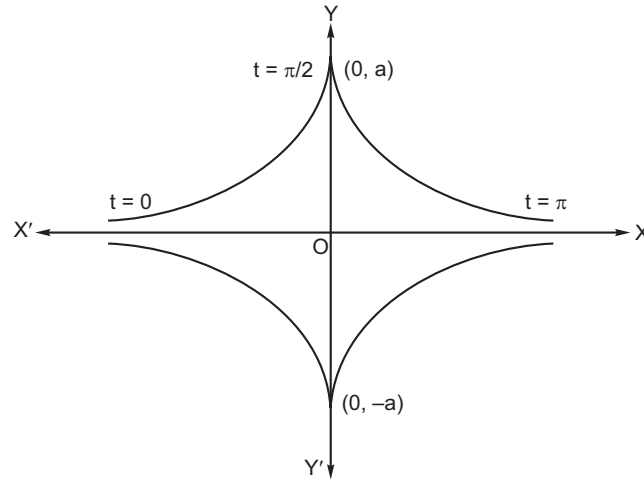
i.e., y -axis itself is a tangent at the points $(0, \pm a)$. $\frac{dy}{dx} = 0$ when $t = 0$ and then $x \rightarrow \infty$, $y = 0$, showing that $y = 0$ or x -axis is an asymptote as proved earlier.

From (2),
$$\frac{dy}{dx} = \tan t$$

Differentiating w.r.t. x ,
$$\begin{aligned} \frac{d^2 y}{dx^2} &= \sec^2 t \cdot \frac{dt}{dx} \\ &= \sec^2 t \cdot \frac{\sin t}{a \cos^2 t} \\ &= \frac{\sin t}{a \cos^4 t} \end{aligned}$$

$\therefore \frac{d^2 y}{dx^2}$ is +ve for values of t in $[0, \pi]$. Curve is concave upwards for values of t in $[0, \pi]$.

The shape of the curve is as shown in the figure.



173. Trace the curve $x = \frac{a(t+t^3)}{1+t^4}$, $y = \frac{a(t-t^3)}{1+t^4}$.

Sol. We have,

$$x = \frac{at(1+t^2)}{1+t^4} \quad \dots(1)$$

$$y = \frac{at(1-t^2)}{1+t^4} \quad \dots(2)$$

Let us eliminate t from equations (1) and (2) to form Cartesian equation.

Dividing equation (1) by equation (2), we have

$$\frac{x}{y} = \frac{1+t^2}{1-t^2} \quad \text{or} \quad t^2 = \frac{x-y}{x+y} \quad \dots(3)$$

From (1), we have

$$x^2(1+t^4)^2 = a^2t^2(1+t^2)^2$$

Putting the value of t^2 from (3),

$$x^2 \left[1 + \frac{(x-y)^2}{(x+y)^2} \right]^2 = \frac{a^2(x-y)}{x+y} \left(1 + \frac{x-y}{x+y} \right)^2$$

$$\text{or} \quad x^2 \frac{[(x+y)^2 + (x-y)^2]^2}{(x+y)^4} = \frac{a^2(x-y)}{x+y} \left[\frac{2x}{(x+y)} \right]^2$$

$$\text{or} \quad \frac{4x^2(x^2+y^2)^2}{(x+y)^4} = \frac{4a^2x^2(x-y)}{(x+y)^3}$$

$$\text{or} \quad (x^2+y^2)^2 = a^2(x^2-y^2) \quad \dots(4)$$

1. *Symmetry* : Curve (4) is symmetrical both about x-axis and y-axis.

2. *Origin* : Curve (4) passes through the origin. Tangents at the origin are :

$$a^2(x^2-y^2) = 0. \quad (\text{Equating lowest degree terms to zero})$$

But

$$a \neq 0$$

$$\therefore x^2 - y^2 = 0 \text{ or } y^2 = x^2 \therefore y = \pm x.$$

3. *Asymptotes* : The curve has no asymptotes.

4. *Points of Intersection* :

$$\text{Putting } y = 0 \text{ in (4), } x^4 = a^2 x^2 \text{ or } x^2(x^2 - a^2) = 0$$

$$\therefore \text{ Either } x = 0 \text{ or } x^2 = a^2 \text{ i.e., } x = \pm a$$

$\therefore$ Intersections of the curve (4) with x -axis are $(-a, 0)$, $(0, 0)$, $(a, 0)$.

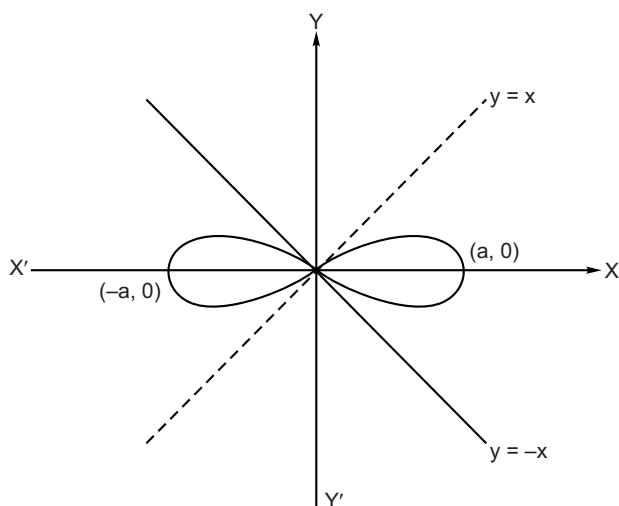
Similarly putting $x = 0$ in equation (4), we get

$$y^4 = -a^2 y^2 \text{ or } y^2(y^2 + a^2) = 0$$

$$\Rightarrow y = 0 \text{ or } y^2 + a^2 = 0 \text{ or } y = \pm ia$$

The only point of intersection of curve (4) with y -axis is $(0, 0)$.

Thus the shape of the curve is as shown in the figure.



174. If $y = \sin(a \sin^{-1} x)$, prove that $(1 - x^2) \frac{d^2 y}{dx^2} = x \frac{dy}{dx} - a^2 y$.

[A.U.U.P. 2007 ; M.D.U., 2008 ; U.P.T.U., 2009]

$$\text{Sol. } y = \sin(a \sin^{-1} x) \quad \dots(1)$$

$$\therefore \frac{dy}{dx} = \cos(a \sin^{-1} x) \cdot \frac{a}{\sqrt{1-x^2}} = \frac{a \cos(a \sin^{-1} x)}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} \cdot \frac{dy}{dx} = a \cos(a \sin^{-1} x)$$

Squaring both sides,

$$(1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 \cos^2(a \sin^{-1} x)$$

$$\text{or } (1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 [1 - \sin^2(a \sin^{-1} x)]$$

$$\text{or } (1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 (1-y^2) \quad [\text{By (1)}]$$

Again differentiating w.r.t. x , we have

$$(1-x^2) \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 + \left(\frac{dy}{dx} \right)^2 \frac{d}{dx} (1-x^2) = a^2 \left[0 - \frac{d}{dx} (y)^2 \right]$$

$$\Rightarrow (1-x^2) \cdot 2 \left(\frac{dy}{dx} \right) \cdot \frac{d^2 y}{dx^2} - 2x \left(\frac{dy}{dx} \right)^2 = -2a^2 y \frac{dy}{dx}$$

Dividing every term by $2 \frac{dy}{dx}$.

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = -a^2 y$$

or $(1-x^2) \frac{d^2 y}{dx^2} = x \frac{dy}{dx} - a^2 y$.

Hence proved.

175. If $y = \sin 2x \sin 3x$, find y_n .

Sol.

$$y = \sin 2x \sin 3x = \frac{1}{2} [2 \sin 3x \sin 2x]$$

$$= \frac{1}{2} [\cos x - \cos 5x]$$

[$\because 2 \sin A \sin B = \cos (A-B) - \cos (A+B)$]

$$\therefore y_n = \frac{1}{2} \left[\frac{d^n}{dx^n} \cos x - \frac{d^n}{dx^n} \cos 5x \right]$$

$$= \frac{1}{2} \left[\cos \left(x + \frac{n\pi}{2} \right) - 5^n \cos \left(5x + \frac{n\pi}{2} \right) \right].$$

176. If $y = \frac{1}{1-5x+6x^2}$, find the n th derivative of y (or y_n).

[U.P.T.U., (B.T.), 2005 and A.U.U.P., 2008 ; M.D.U., 2009]

Sol.

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)} = \frac{2}{2x-1} - \frac{3}{3x-1}$$

Now we know that n th differential co-efficient of $\frac{1}{ax+b} \left[x \neq -\frac{b}{a} \right]$ is given by $\frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}$

$$\therefore D^n \left(\frac{1}{2x-1} \right) = \frac{(-1)^n \cdot n! (2)^n}{(2x-1)^{n+1}}$$

and $D^n \left(\frac{1}{3x-1} \right) = \frac{(-1)^n \cdot n! (3)^n}{(3x-1)^{n+1}}$

Hence,

$$y_n = 2 \left[\frac{(-1)^n \cdot n! (2)^n}{(2x-1)^{n+1}} \right] - 3 \left[\frac{(-1)^n \cdot n! (3)^n}{(3x-1)^{n+1}} \right]$$

$$= (-1)^n \cdot n! \left[\frac{(2)^{n+1}}{(2x-1)^{n+1}} - \frac{(3)^{n+1}}{(3x-1)^{n+1}} \right].$$

177. If $y = \frac{x^2}{(x-1)^2 (x+2)}$, find n th derivative of y .

[U.P.T.U., 2006, 2008 ; M.D.U., 2007]

Sol. Firstly, we will split y into partial fractions.

Let $x-1=z$, then $y = \frac{1}{z^2} \cdot \frac{1+2z+z^2}{3+z} = \frac{1}{z^2} \left(\frac{1}{3} + \frac{5z}{9} + \frac{4 \cdot z^2}{9 \cdot (3+z)} \right)$

$$= \frac{1}{3z^2} + \frac{5}{9z} + \frac{4}{9(3+z)} = \frac{1}{3(x-1)^2} + \frac{5}{9(x-1)} + \frac{4}{9(x+2)}$$

Hence,

$$y_n = \frac{(-1)^n (n+1)!}{3(x-1)^{n+2}} + \frac{5(-1)^n \cdot n!}{9(x-1)^{n+1}} + \frac{4(-1)^n \cdot n!}{9(x+2)^{n+1}}.$$

178. Find the n th derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$. [U.P.T.U., 2007 ; M.D.U., 2009]

Sol. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2 \tan^{-1}x$ (A Standard Result)

$\therefore y = 2 \tan^{-1}x$. Now we have to find y_n .

$$y_1 \left(\text{or } \frac{dy}{dx} \right) = \frac{2}{1+x^2} = \frac{2}{2i} \left[\frac{1}{x-i} - \frac{1}{x+i} \right] = \frac{1}{i} \left[\frac{1}{x-i} - \frac{1}{x+i} \right]$$

$$\text{Hence, } y_n = \frac{1}{i} (-1)^{n-1} \cdot (n-1)! \left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n} \right]$$

[Differentiating $(n-1)$ times]

Suppose $x = r \cos \theta$, $1 = r \sin \theta$ so that

$$r^2 = x^2 + 1 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Thus,

$$y_n = \frac{(-1)^{n-1} (n-1)!}{i r^n} \{(\cos \theta - i \sin \theta)^{-n} - (\cos \theta + i \sin \theta)^{-n}\}$$

$$\Rightarrow y_n = \frac{(-1)^{n-1} (n-1)!}{i \cdot r^n} \cdot 2i \sin n\theta \quad \text{[Using De-Moivre's Theorem]}$$

$$\Rightarrow y_n = 2(-1)^{n-1} \cdot (n-1)! \sin n\theta \cdot \sin^n \theta$$

$$\text{where, } \theta = \tan^{-1}\left(\frac{1}{x}\right).$$

179. If $y = x \log \frac{x-1}{x+1}$, show that: $y_n = (-1)^{n-2} (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n} \right]$. [A.U.U.P., 2008]

Sol. $y = x \log \frac{x-1}{x+1}$

or $y = x[\log(x-1) - \log(x+1)]$... (1)

Differentiating y w.r.t. x , we get

$$\begin{aligned} y_1 &= x \left(\frac{1}{x-1} - \frac{1}{x+1} \right) + \log(x-1) - \log(x+1) \\ &= \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1) \end{aligned} \quad \dots (2)$$

Again differentiating (2), $n-1$ times w.r.t. x , we get

$$y_n = \frac{(-1)^{n-1} (n-1)!}{(x-1)^n} + \frac{(-1)^{n-1} (n-1)!}{(x+1)^n} + \frac{(-1)^{n-2} (n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2} (n-2)!}{(x+1)^{n-1}}$$

$$= (-1)^{n-2} (n-2)! \left[\frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n} + \frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} \right]$$

$$\Rightarrow y_n = (-1)^{n-2} (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n} \right].$$

180. State Leibnitz Theorem and hence find the n th derivative of $e^x \log x$.

[M.D.U., 2008; U.P.T.U., 2007; A.U.U.P., 2009]

Sol. Let

$$u = e^x$$

and

$$v = \log x$$

$$\therefore u_n = e^x,$$

$$v_1 = \frac{1}{x},$$

$$u_{n-1} = u_{n-2} = u_2 = u_1 = e^x$$

$$v_2 = -\frac{1}{x^2},$$

$$v_3 = \frac{2}{x^3},$$

$$\vdots$$

$$v_n = \frac{(-1)^{n-1} (n-1)!}{x^n}$$

Leibnitz Theorem helps us to find the n th derivative of the product of two functions in terms of the successive derivatives of the functions.

Statement : If u and v are 2 functions of x , having derivatives of the n th order, then $\frac{d^n}{dx^n} (uv)$

$= u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + u^n C_n v_n$ where the suffixes of u and v denote differentiations w.r.t. x .

$$\Rightarrow \frac{d^n}{dx^n} (e^x \log x) = e^x \log x + {}^nC_1 e^x \left(\frac{1}{x} \right) + {}^nC_2 e^x \left(-\frac{1}{x^2} \right) + \dots$$

$$+ {}^nC_n \cdot e^x \cdot \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$$

or R.H.S. is (i.e., the n^{th} derivative of $e^x \log x$)

$$= e^x \left[\log x + \frac{n}{x} - \frac{n(n-1)}{2} \cdot \frac{1}{x^2} + \dots + \frac{(-1)^{n-1} (n-1)!}{x^n} \right]$$

181. If $y = a \cos (\log x) + b \sin (\log x)$, show that

$$x^2 \cdot y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0.$$

Sol. Given that

$$y = a \cos (\log x) + b \sin (\log x) \quad \dots(1)$$

$$y_1 = -a \sin (\log x) \cdot \frac{1}{x} + b \cos (\log x) \cdot \frac{1}{x}$$

$\Rightarrow$

$$x y_1 = -a \sin (\log x) + b \cos (\log x)$$

Again

$$x y_2 + y_1 = -a \cos (\log x) \cdot \frac{1}{x} - b \sin (\log x) \cdot \frac{1}{x}$$

$\Rightarrow$

$$x^2 y_2 + x y_1 = -y$$

[Using (1)]

or

$$x^2 y_2 + x y_1 + y = 0 \quad \dots(2)$$

Differentiating equation (2) n times by Leibnitz Theorem, we get

$$\{y_{n+2} \cdot (x^2) + {}^nC_1 \cdot y_{n+1} (2x) + {}^nC_2 \cdot y_n (2)\} + y_{n+1} \cdot (x) + {}^nC_1 \cdot y_n (1) + y_n = 0$$

$\Rightarrow$

$$x^2 \cdot y_{n+2} + y_{n+1} (2nx + x) + y_n \{n(n-1) + (n+1)\} = 0$$

$$\Rightarrow x^2 \cdot y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Hence the result.

182. If $y = (x^2 - 1)^n$, prove that: $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

Hence, if P_n is the n th derivative of y , show that $\frac{d}{dx} \left\{ (1-x^2) \frac{dP_n}{dx} \right\} + n(n+1)P_n = 0$.

[A.U.U.P., 2009]

$$\text{Sol.} \quad y = (x^2 - 1)^n \quad \dots(1)$$

$$\therefore y_1 = n(x^2 - 1)^{n-1} \cdot 2x$$

$$\Rightarrow (x^2 - 1)y_1 = 2nxy \quad \dots(2)$$

Differentiating again, we get

$$(x^2 - 1)y_2 + 2xy_1 = 2n(xy_1 + y)$$

$$\Rightarrow (x^2 - 1)y_2 + 2(1-n)xy_1 - 2ny = 0$$

By Leibnitz Theorem, differentiating n times, we get

$$y_{n+2}(x^2 - 1) + {}^nC_1 \cdot y_{n+1}(2x) + {}^nC_2 y_n(2) + 2(1-n) \{y_{n+1}(x) + {}^nC_1 y_n(1)\} - 2n y_n = 0$$

$$\Rightarrow \{x^2 - 1\} y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0 \quad \dots(3)$$

$$\text{Again,} \quad P_n = \frac{d^n}{dx^n} (x^2 - 1)^n = y_n \quad \dots(4)$$

$$\begin{aligned} \therefore \frac{d}{dx} \left\{ (1-x^2) \frac{dP_n}{dx} \right\} &= \frac{d}{dx} \left\{ (1-x^2) y_{n+1} \right\} = (1-x^2)y_{n+2} - 2xy_{n+1} \\ &= - \left\{ (x^2 - 1)y_{n+2} + 2xy_{n+1} \right\} \\ &= -n(n+1)y_n \quad \text{[Using (3)]} \\ &= -n(n+1)P_n \quad \text{[Using (4)]} \end{aligned}$$

$$\text{Hence,} \quad \frac{d}{dx} \left\{ (1-x^2) \frac{dP_n}{dx} \right\} + n(n+1)P_n = 0.$$

183. Find the n^{th} derivative of $x^2 \log x$. [U.P.T.U., 2008]

Sol. Let $u = \log x$ and $v = x^2$

$$\therefore u_n = \frac{(-1)^{n-1}(n-1)!}{x^n}; \quad v_1 = 2x, v_2 = 2 \text{ and } v_3, v_4 \text{ etc.} = 0$$

By applying Leibnitz's Theorem, we have

$$\begin{aligned} \frac{d^n (x^2 \log x)}{dx^n} &= \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot x^2 + n \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot 2x \\ &\quad + \frac{n(n-1)}{2!} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} \cdot 2 \\ &= \frac{(-1)^{n-1} \cdot (n-3)!}{x^{n-2}} [(n-1)(n-2) - 2n(n-2) + n(n-1)] \\ &= \frac{(-1)^{n-1} \cdot 2(n-3)!}{x^{n-2}} \end{aligned}$$

184. If $y = \sin (m \sin^{-1} x)$, then prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n \quad \text{[A.U.U.P., 2009; M.D.U., 2008; P.T.U., 2007]}$$

$$\text{Sol.} \quad y = \sin (m \sin^{-1} x) \quad \dots(1)$$

$$\therefore y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} \cdot y_1 = m \cos(m \sin^{-1} x)$$

Squaring both sides,

$$(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x) = m^2 [1 - \sin^2(m \sin^{-1} x)] = m^2(1-y^2)$$

Again differentiating both sides w.r.t. x ,

$$(1-x^2)2y_1y_2 - 2xy_1^2 = -2m^2yy_1$$

Dividing every term by $2y_1$, we have

$$(1-x^2)y_2 - xy_1 = -m^2y$$

$$\text{or } (1-x^2)y_2 - xy_1 + m^2y = 0$$

Now differentiating every term n times by Leibnitz Theorem, we have

$$y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2) - y_{n+1} \cdot x - {}^nC_1y_n + m^2y_n = 0$$

$$\text{or } (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$$

$$\text{Hence, } (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$$

185. If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$. (U.P.T.U., 2007)

Sol. Given that

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x. \quad \text{Put } y^{\frac{1}{m}} = z.$$

$$\therefore z + \frac{1}{z} = 2x \quad \text{or } z^2 - 2xz + 1 = 0$$

$$\therefore z = x \pm \sqrt{x^2-1}$$

$$\text{i.e., } y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow y = \left(x \pm \sqrt{x^2-1}\right)^m$$

$$\text{If, } y = \left[x + \sqrt{x^2-1}\right]^m, \text{ then}$$

$$y_1 = m \left(x + \sqrt{x^2-1}\right)^{m-1} \left(1 + \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-1}} \cdot 2x\right)$$

$$= m \left(x + \sqrt{x^2-1}\right)^{m-1} \left(\frac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{m \left(x + \sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

Similarly, if

$$y = \left(x - \sqrt{x^2-1}\right)^m$$

$$y_1 = -\frac{my}{\sqrt{x^2-1}}$$

In either case, on squaring, we get

$$y_1^2 = \frac{m^2 y^2}{x^2-1}$$

$$\Rightarrow (x^2-1)y_1^2 = m^2 y^2$$

Differentiating again,

$$(x^2-1)2y_1 y_2 + y_1^2 \cdot 2x = m^2 \cdot 2y y_1$$

Dividing both sides by $2y_1$, we have

$$(x^2-1)y_2 + xy_1 - m^2 y = 0$$

Differentiating 'n' times by Leibnitz Theorem,

$$y_{n+2}(x^2-1) + n \cdot y_{n+1} \cdot 2x + \frac{n(n-1)}{2!} \cdot y_n \cdot 2 + [y_{n+1} \cdot x + n y_n \cdot 1] - m^2 y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$$

$$\text{or} \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Hence proved.

186. If $y = e^{\tan^{-1}x}$, prove that : $(1+x^2)y_{n+2} + [(2n+2)x-1]y_{n+1} + n(n+1)y_n = 0$. [A.U.U.P., 2009]

Sol.
$$y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = y$$

Differentiating 'n' times by Leibnitz theorem, we get

$$\{y_{n+1}(1+x^2) + {}^nC_1 y_n(2x) + {}^nC_2 \cdot y_{n-1}(2)\} = y_n$$

$$\Rightarrow (1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$$

Differentiating again w.r.t. x and on simplification, we obtain

$$(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + \{(2n+2)x-1\}y_{n+1} + n(n+1)y_n = 0$$

Hence proved.

187. If $y = (\sin^{-1}x)^2$, prove that

$$(i) \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 \cdot y_n = 0$$

(ii) Hence, find the value of y_n at $x = 0$. (M.D.U., 2008; A.U.U.P., 2007; U.P.T.U., 2006)

Sol. (i) $y = (\sin^{-1}x)^2$

Differentiating both sides w.r.t. x ,

$$y_1 = 2(\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}} \quad \dots (1)$$

Squaring both sides, we get

$$(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$$

$$\Rightarrow (1-x^2)y_1^2 = 4y$$

Differentiating again, we have

$$2(1-x^2)y_1 y_2 - 2xy_1^2 = 4y_1$$

On dividing both sides by $2y_1$,

$$(1-x^2)y_2 - xy_1 - 2 = 0 \quad \dots (2)$$

Differentiating 'n' times by Leibnitz Theorem, we get

$$\begin{aligned} (1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1} - {}^nC_1 y_n &= 0 \\ (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n &= 0. \end{aligned} \quad \dots (3)$$

Hence proved.

(ii) Put $x = 0$ in (1), (2) and (3), then

$$\begin{aligned} y_1(0) &= 0 \text{ and } y_2(0) = 2 \\ \text{and } y_{n+2}(0) &= n^2 y_n(0) \end{aligned} \quad \dots (4)$$

Putting $n = 1, 2, 3, 4, \dots$ in (4), we get

$$\begin{aligned} y_3(0) &= (1)^2 \cdot y_1(0) = 0 \\ y_4(0) &= (2)^2 \cdot y_2(0) = 2 \cdot 2^2 \\ y_5(0) &= (3)^2 \cdot y_3(0) = 0 \\ y_6(0) &= (4)^2 \cdot y_4(0) = 2 \cdot 2^2 \cdot 4^2 \\ y_7(0) &= (5)^2 \cdot y_5(0) = 0 \\ y_8(0) &= (6)^2 \cdot y_6(0) = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \\ &\dots \dots \dots \end{aligned}$$

In general, when n is odd, $y_n(0) = 0$

When n is even, $y_n(0) = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \dots (n-2)^2$.

188. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)$, prove that: $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + m^2)y_n = 0$.

Sol. Given $\cos^{-1}\left(\frac{y}{b}\right) = m \log\left(\frac{x}{m}\right)$

$$\Rightarrow \frac{y}{b} = \cos\left\{m \log\left(\frac{x}{m}\right)\right\}$$

$$\therefore y = b \cos\left\{m \log\left(\frac{x}{m}\right)\right\} \quad \dots (1)$$

$$y_1 = -b \sin\left\{m \log\left(\frac{x}{m}\right)\right\} \cdot m \cdot \frac{1}{(x/m)} \cdot \frac{1}{m}$$

$$\Rightarrow xy_1 = -bm \sin\left\{m \log\left(\frac{x}{m}\right)\right\}$$

Again differentiating, we get

$$xy_2 + y_1 = -b m \cos\left\{m \log\left(\frac{x}{m}\right)\right\} \cdot m \cdot \frac{1}{(x/m)} \cdot \frac{1}{m}$$

$$\Rightarrow x^2 y_2 + xy_1 + m^2 y = 0 \quad \dots (2)$$

Differentiating equation (2) n times by Leibnitz theorem,

$$\begin{aligned} &\left\{y_{n+2}(x^2) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2)\right\} \\ &+ \left\{y_{n+1}(x) + {}^nC_1 y_n(1)\right\} + m^2 y_n = 0 \\ \Rightarrow &x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + m^2)y_n = 0. \end{aligned}$$

Hence proved.

189. If $y = x^2 e^x$, show that $\frac{d^n y}{dx^n} = \frac{1}{2} n(n-1) \frac{d^2 y}{dx^2} - n(n-2) \frac{dy}{dx} + \frac{1}{2} (n-1)(n-2) y$.

Sol. Given $y = x^2 e^x$ or $e^x \cdot x^2 (= u \cdot v, \text{ say})$

Differentiating n times by Leibnitz Theorem, we get

$$y_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2$$

where

$$u = e^x, v = x^2$$

$$u_n = u_{n-1} = u_{n-2} = e^x,$$

$$v_1 = 2x, v_2 = 2,$$

$\Rightarrow$

$$y_n = e^x \cdot x^2 + {}^n C_1 \cdot e^x \cdot 2x + {}^n C_2 \cdot e^x \cdot 2$$

or

$$y_n = e^x [x^2 + 2nx + n(n-1)] \quad \dots (1)$$

Put $n = 1, 2$ in equation (1), successively.

we then have,

$$y_1 = e^x (x^2 + 2x) \quad \dots (2)$$

$$y_2 = e^x (x^2 + 4x + 2) \quad \dots (3)$$

Now, R.H.S. of the given identity (to be proved) is $\frac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \frac{1}{2} (n-1)(n-2) y$

which on using (2) and (3) becomes

$$\begin{aligned} & \frac{1}{2} n(n-1) \cdot e^x (x^2 + 4x + 2) - n(n-2) e^x (x^2 + 2x) + \frac{1}{2} (n-1)(n-2) x^2 e^x \\ &= \frac{1}{2} e^x [n(x^2 + 4x + 2)(n-1) - 2n(n-1)(x^2 + 2x) + (n-1)(n-2)x^2] \\ &= \frac{1}{2} e^x [x^2 \{n(n-1) - 2n(n-2) + (n-1)(n-2)\} + x \{4n(n-1) - 4n(n-2)\} + 2n(n-1)] \\ &= \frac{1}{2} e^x [2x^2 + 4nx + 2n(n-1)] = e^x [x^2 + 2nx + n(n-1)] = y_n. \end{aligned}$$

Hence, the given relation (identity) is proved.

190. If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

Sol. The given equations define u as a composite function of t .

$$\begin{aligned} \therefore \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 + \frac{1}{\sqrt{1-(x-y)^2}} \cdot (-1) \cdot 12t^2 = \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} \\ &= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} \\ \Rightarrow \frac{du}{dt} &= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}} \end{aligned}$$

Hence proved.

191. Show that: $D^{n-1}(x^{n-1} \log x) = \frac{(n-1)!}{x}$.

Sol. Let $y = x^{n-1} \log x$... (1)

$\therefore y_1 = x^{n-1} \left(\frac{1}{x} \right) + (n-1) x^{n-2} \log x$

$\Rightarrow xy_1 = x^{n-1} + (n-1) x^{n-1} \cdot \log x$

or $xy_1 = x^{n-1} + (n-1)y$... (2) [Using equation. (1)]

Differentiating equation (2), $(n-1)$ times, we get

$$y_n(x) + {}^{n-1}C_1 \cdot y_{n-1}(1) = (n-1)! + (n-1)y_{n-1}$$

$\Rightarrow xy_n + (n-1)y_{n-1} = (n-1)! + (n-1)y_{n-1}$

$\Rightarrow y_n = \frac{(n-1)!}{x} \quad \text{or} \quad D^{n-1}(x^{n-1} \log x) = \frac{(n-1)!}{x}$

Hence proved.

192. If $y = x^n \log x$, prove that: $y_n = ny_{n-1} + (n-1)!$ [M.D.U., 2007; U.P.T.U., 2008; A.U.U.P., 2009]

Sol. $y_n = \frac{d^n}{dx^n}(x^n \log x) = \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{d}{dx}(x^n \log x) \right\}$... (1)

$$= \frac{d^{n-1}}{dx^{n-1}} \left\{ x^n \cdot \frac{1}{x} + nx^{n-1} \cdot \log x \right\} = n \cdot \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$\Rightarrow y_n = n \cdot y_{n-1} + (n-1)!$

Hence proved.

193. If $y = \tan^{-1} x$, prove that [M.D.U., 2009; U.P.T.U., 2007; A.U.U.P., 2008]

(i) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

(ii) Hence, determine the values of all the derivatives of y w.r.t. x when $x = 0$.

Sol. (i) Here $y = \tan^{-1} x$, $y_1 = \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) = 1$

Differentiating ' n times' by Leibnitz Theorem, we have

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0 \quad \dots(1)$$

Hence proved.

(ii) Putting $x = 0$ in (1), we have

$$y(0) = 0, \quad y_1(0) = 1$$

and $y_{n+1}(0) = -n(n-1)y_{n-1}(0) \quad \dots(2)$

Putting $n = 1, 2, 3, 4$ in equation (2), we get

$$y_2(0) = -1(0)y_0(0) = 0$$

$$y_3(0) = -2(1)y_1(0) = -2 = (-1)^1 \cdot 2!$$

$$y_4(0) = -3 \cdot (2) \cdot y_2(0) = 0$$

$$y_5(0) = -4 \cdot (3) \cdot y_3(0) = (-4)(3)(-2)(1) = (-1)^2 \cdot 4!$$

.....
.....

In general,

When n is even, $y_n(0) = 0$.

When n is odd, $y_n(0) = (-1)^{\frac{n-1}{2}} (n-1)!$

194. If $y = e^{\tan^{-1}x}$, find y_3, y_4 and y_5 at $x = 0$.

Sol. It has been proved earlier (see the solution) in question number 186, that when

$$y = e^{\tan^{-1}x} \quad \dots(1)$$

$$(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0 \quad \dots(2)$$

Now to find y_3, y_4 and y_5 when $x = 0$, we should determine y_1 and y_2 .

$$\Rightarrow y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2} \quad \text{or} \quad (1+x^2)y_1 = y$$

$$\text{and} \quad (1+x^2)y_2 + 2xy_1 = y_1$$

$$\text{when } x = 0, \quad y_1 = y = 1, \quad y_2 = 1 \quad \dots(3)$$

$$\text{and} \quad (y_3)_0 = -1, \quad (y_4)_0 = -7, \quad (y_5)_0 = 5.$$

195. If $y = \sin(m \sin^{-1}x)$, find $y_n(0)$. [U.P.T.U./M.T.U., 2009; A.U.U.P., 2008; M.D.U. 2007]

Sol. Please see solution given in question number 184,

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$$

$$\text{or} \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0 \quad \dots(1)$$

Putting $x = 0$, we obtain

$$(y_{n+2})_0 = (n^2-m^2)(y_n)_0 \quad \dots(2)$$

$$y = \sin(m \sin^{-1}x)$$

$$y_0 = 0, \quad y_1 = \frac{m\sqrt{1-x^2}}{\sqrt{1-x^2}} \Rightarrow (y_1)_0 = m \quad \dots(3)$$

Putting $n = 2, 4, 6, \dots$ in equation (2), we get

$$(y_2)_0 = (y_4)_0 = 0 = \dots$$

Putting $n = 1, 3, 5$ in (2), we get

$$(y_3)_0 = (1-m^2)(y_1)_0 = (1-m^2)m$$

$$(y_5)_0 = (3^2-m^2)(y_3)_0 = m(1-m^2)(3^2-m^2)$$

$$(y_n)_0 = m(1-m^2)(3-m^2)\dots [(2k-1)^2-m^2]$$

In general,

when n is odd and $= 2k+1$.

□ □ □

Partial Differentiation

BRIEF DEFINITIONS AND IMPORTANT FORMULAE

1. Functions of Two Variables

If three variables x, y, z are so related that the value of z depends upon the values of x and y , then z is called a function of two variables x and y , and this is denoted by $z = f(x, y)$, z is called **dependent variable** and x, y are called **independent variables**.

2. Partial Derivatives of First Order

The derivative of $z = f(x, y)$ with respect to x , **treating y as constant**, is called *partial derivative of z w.r.t. x* and is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x .

Similarly, the derivative of z w.r.t. y , by treating x as a constant, is called *partial derivative of z w.r.t. y* and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y .

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z** .

In general, if z is a function of two or more independent variables, then the **partial derivative of z w.r.t. any one** of the independent variables is the **ordinary derivative of z w.r.t. that variable**, **treating all other variables as constant**.

3. Partial Derivatives of Higher Order

Since first order p.d. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y , they can be further differentiated partially w.r.t. x as well as y . These are called second order partial derivatives of z . Usual notations for these second order partial derivatives are:

$$\frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}, \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}, \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}, \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}, f_{xy} = f_{yx}.$$

(i) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y .

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

(ii) If $z = uv$ then

$$\frac{\partial z}{\partial x} = u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y}$$

(iii) If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left(\frac{u}{v} \right) = \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left(\frac{u}{v} \right) = \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2}$$

(iv) If $z = f(u)$, where $u = \phi(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \quad \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

4. Homogeneous Functions

A function $f(x, y)$ is said to be *homogeneous* of degree (or order) n in the variables x and y if it can be expressed in the form

$$x^n \phi\left(\frac{y}{x}\right) \quad \text{or} \quad y^n \cdot \phi\left(\frac{x}{y}\right)$$

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) is that $f(tx, ty) = t^n \cdot f(x, y)$.

For example, if $f(x, y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$, then

$$(i) \quad f(x, y) = \frac{x \left(1 + \frac{y}{x}\right)}{\sqrt{x} \left(1 + \sqrt{\frac{y}{x}}\right)} = x^{1/2} \cdot \phi\left(\frac{y}{x}\right)$$

$\Rightarrow f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y .

$$(ii) \quad f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = \sqrt{t} \cdot f(x, y)$$

$\Rightarrow f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y .

5. Euler's Theorem on Homogeneous Functions

If u is a homogeneous function of degree n in x and y , then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$

Also, if u is a homogeneous function of degree n in x, y and z , then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu.$$

Important Result: For u ,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

6. Derivatives of Composite Functions

If u is a composite function of t , defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$\frac{du}{dt}$ is called the **total derivative** of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

7. If We are Given an Implicit Function

$f(x, y) = c$, then $u = f(x, y)$ where $u = c$.

Derivative of $f(x, y)$ w.r.t. x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \quad [\text{if } f(x, y) = c]$$

Differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$$

8. Jacobians

If u and v are functions of two independent variables x and y , then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ is called } \mathbf{Jacobian} \text{ of } u, v \text{ w.r.t. } x, y \text{ and is denoted by } \mathbf{J} \left(\begin{matrix} u, v \\ x, y \end{matrix} \right) \text{ or } \frac{\partial(u, v)}{\partial(x, y)}.$$

9. Conditions for $f(x, y)$ to be Maximum or Minimum

Necessary conditions for $f(x, y)$ to have a maximum or minimum value at (a, b) are $f_x(a, b) = 0$, $f_y(a, b) = 0$

10. Working Rule to Find Maximum or Minimum (or Extreme Values) of a Function $f(x, y)$

Step I: Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

Step II: Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

Let (a, b) ; (c, d) ... be the solution of these equations.

Step III: At each points obtained in step-II, find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Step IV: (a) If $rt - s^2 > 0$ and $r > 0$ at point (a, b) (say), then f has a **minimum** value at (a, b) .

(b) If $rt - s^2 > 0$ and $r < 0$ at some point, say, (a, b) , then f has a **maximum** value at point (a, b) .

(c) If $rt - s^2 < 0$ at (a, b) , then f has **No extreme values** at (a, b) .

(d) If $rt - s^2 = 0$ at (a, b) , then f **may or may not have extreme values**. In this case we apply the following method:

Step I. If $f(a + h, b + k) - f(a, b) > 0 \forall h, k$ +ve or -ve (very small) **then f has minimum value.**

Step II. If $f(a + h, b + k) - f(a, b) < 0 \forall h, k$ +ve or -ve, then f has a **maximum** value.

Step III. If $f(a + h, b + k) - f(a, b)$ **does not keep the same sign**, $\forall h, k$, +ve, h -ve (very small), **then f does not have extreme values.**

11. Differentiation under Integral Sign

If a function $f(x, \alpha)$ of the two variables x and α , α being a parameter, be integrated w.r.t. x

between the limits a and b , $\int_a^b f(x, \alpha) dx$ is a function of α .

For example,
$$\int_0^{\pi/2} \sin \alpha x \cdot dx = - \left[\frac{\cos \alpha x}{\alpha} \right]_0^{\pi/2}$$

$$= - \frac{1}{\alpha} \left(\cos \frac{\pi}{2} \alpha - 1 \right) = \frac{1}{\alpha} \left(1 - \cos \frac{\pi}{2} \alpha \right)$$

Thus in general
$$\int_a^b f(x, \alpha) dx = F(\alpha).$$

12. Leibnitz Rule

If $f(x, \alpha)$ and $\frac{\partial}{\partial \alpha} [f(x, \alpha)]$ be continuous functions of x and α then

$$\frac{d}{d\alpha} \left[\int_a^b f(x, \alpha) dx \right] = \int_a^b \frac{\partial}{\partial \alpha} \{f(x, \alpha)\} dx \quad \text{where } a \text{ and } b \text{ are constants independent of } \alpha.$$

SOLVED PROBLEMS

1. Find the first order partial derivative of the following functions :

(i) $u = \tan^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$ (ii) $u = \cos^{-1} \left(\frac{x}{y} \right)$. (U.P.T.U., 2008)

Sol. (i)

$$u = \tan^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{x^2 + y^2}{x + y} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{x^2 + y^2}{x + y} \right)$$

$$\begin{aligned}
&= \frac{(x+y)^2}{(x+y)^2 + (x^2+y^2)^2} \cdot \frac{(x+y) \frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2) \frac{\partial}{\partial x}(x+y)}{(x+y)^2} \\
&= \frac{(x+y) \cdot 2x - (x^2+y^2) \cdot 1}{(x+y)^2 + (x^2+y^2)^2} \\
\Rightarrow \quad \frac{\partial u}{\partial x} &= \frac{x^2 + 2xy - y^2}{(x+y)^2 + (x^2+y^2)^2} \quad \dots(1)
\end{aligned}$$

Since u remains the same if we interchange x and y , u is symmetrical w.r.t. x and y . In equation (1), on interchanging x and y , we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{y^2 + 2xy - x^2}{(x+y)^2 + (x^2+y^2)^2} \\
(ii) \quad u &= \cos^{-1} \left(\frac{x}{y} \right) \\
\frac{\partial u}{\partial x} &= \frac{-1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial x} \left(\frac{x}{y} \right) = \frac{-y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y} = -\frac{1}{\sqrt{y^2 - x^2}} \\
\frac{\partial u}{\partial y} &= \frac{-1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial y} \left(\frac{x}{y} \right) = \frac{-y}{\sqrt{y^2 - x^2}} \left(-\frac{x}{y^2} \right) = \frac{x}{y\sqrt{y^2 - x^2}}
\end{aligned}$$

2. If $z(x+y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$.

$$\begin{aligned}
\text{Sol.} \quad z &= \frac{x^2 + y^2}{x+y} \\
\therefore \quad \frac{\partial z}{\partial x} &= \frac{(x+y) \frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2) \frac{\partial}{\partial x}(x+y)}{(x+y)^2} \\
\text{or} \quad \frac{\partial z}{\partial x} &= \frac{(x+y) \cdot 2x - (x^2+y^2) \cdot 1}{(x+y)^2} \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{x^2 + 2xy - y^2}{(x+y)^2} \\
\text{Similarly,} \quad \frac{\partial z}{\partial y} &= \frac{y^2 + 2xy - x^2}{(x+y)^2} \\
\text{Now,} \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 &= \left[\frac{2x^2 - 2y^2}{(x+y)^2} \right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2} \quad \dots(1) \\
4 \left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right) &= 4 \left(1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2} \right) \\
&= 4 \left[\frac{x^2 + y^2 + 2xy - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2} \right] \\
&= \frac{4(x^2 - 2xy + y^2)}{(x+y)^2} = \frac{4(x-y)^2}{(x+y)^2} \quad \dots(2)
\end{aligned}$$

From (1) and (2), $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Hence proved.

3. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.

$$\begin{aligned} u &= x^y \\ \frac{\partial u}{\partial y} &= x^y \cdot \log x \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} = x^{y-1} [y \log x + 1] \\ \frac{\partial^3 u}{\partial x^2 \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial x \partial y} \right) = \frac{\partial}{\partial x} [x^{y-1} (y \log x + 1)] \quad \dots(1) \\ \frac{\partial u}{\partial x} &= yx^{y-1} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = x^{y-1} + y \cdot x^{y-1} \log x = x^{y-1} (y \log x + 1) \\ \frac{\partial^3 u}{\partial x \partial y \partial x} &= \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial y \partial x} \right) = \frac{\partial}{\partial x} [x^{y-1} (y \log x + 1)] \quad \dots(2) \end{aligned}$$

From (1) and (2), $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Hence proved.

4. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) = \frac{\partial \theta}{\partial t}$.

Sol.

$$\begin{aligned} \theta &= t^n \cdot e^{-\frac{r^2}{4t}} \\ \frac{\partial \theta}{\partial r} &= t^n \cdot e^{-\frac{r^2}{4t}} \cdot \left(-\frac{2r}{4t} \right) = -\frac{1}{2} r t^{n-1} \cdot e^{-\frac{r^2}{4t}} \\ \therefore r^2 \frac{\partial \theta}{\partial r} &= -\frac{1}{2} r^3 \cdot t^{n-1} \cdot e^{-\frac{r^2}{4t}} \\ \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) &= -\frac{1}{2} t^{n-1} \left[3r^2 \cdot e^{-\frac{r^2}{4t}} + r^3 \cdot e^{-\frac{r^2}{4t}} \left(-\frac{2r}{4t} \right) \right] = -\frac{1}{2} t^{n-1} \cdot r^2 \cdot e^{-\frac{r^2}{4t}} \cdot \left(3 - \frac{r^2}{2t} \right) \\ \therefore \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) &= \frac{1}{2} t^{n-1} \cdot e^{-\frac{r^2}{4t}} \left(\frac{r^2}{2t} - 3 \right) \end{aligned}$$

Also, $\frac{\partial \theta}{\partial t} = n t^{n-1} \cdot e^{-\frac{r^2}{4t}} + t^n \cdot e^{-\frac{r^2}{4t}} \cdot \left(\frac{r^2}{4t^2} \right) = t^{n-1} \cdot e^{-\frac{r^2}{4t}} \left(n + \frac{r^2}{4t} \right)$

Since, $\frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) = \frac{\partial \theta}{\partial t}$ (given)

$$\therefore \frac{1}{2} t^{n-1} \cdot e^{-\frac{r^2}{4t}} \left(\frac{r^2}{2t} - 3 \right) = t^{n-1} \cdot e^{-\frac{r^2}{4t}} \left(n + \frac{r^2}{4t} \right)$$

$$\frac{1}{2} \left(\frac{r^2}{2t} - 3 \right) = n + \frac{r^2}{4t}$$

$$\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \Rightarrow n = -\frac{3}{2}.$$

5. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $\frac{\partial}{\partial x} \left\{ (1 - x^2) \frac{\partial u}{\partial x} \right\} + \frac{\partial}{\partial y} \left\{ y^2 \cdot \frac{\partial u}{\partial y} \right\} = 0$. (M.D.U., Dec., 2006)

Sol. Given $u = (1 - 2xy + y^2)^{-1/2}$
 $= V^{-1/2}$, where $V = 1 - 2xy + y^2$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} V^{-3/2} \cdot \frac{\partial V}{\partial x} = -\frac{1}{2} V^{-3/2} \cdot (-2y) = yV^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = y \cdot \frac{\partial}{\partial x} (V^{-3/2}) = y \cdot \left(-\frac{3}{2} V^{-5/2} \right) \cdot \frac{\partial V}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{3y}{2} V^{-5/2} (-2y) = 3y^2 V^{-5/2}$$

$$\therefore \frac{\partial}{\partial x} \left\{ (1 - x^2) \frac{\partial u}{\partial x} \right\} = (1 - x^2) \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x} (1 - x^2) = (1 - x^2) 3y^2 V^{-5/2} + yV^{-3/2} \cdot (-2x)$$

$$= yV^{-3/2} [3yV^{-1}(1 - x^2) - 2x] \quad \dots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2} V^{-3/2} \cdot \frac{\partial V}{\partial y} = -\frac{1}{2} V^{-3/2} \cdot (-2x + 2y) = V^{-3/2} \cdot (x - y)$$

$$\frac{\partial^2 u}{\partial y^2} = V^{-3/2} \cdot \frac{\partial}{\partial y} (x - y) + (x - y) \cdot \frac{\partial}{\partial y} (V^{-3/2}) = V^{-3/2} \cdot (-1) + (x - y) \left(-\frac{3}{2} V^{-5/2} \right) \cdot \frac{\partial V}{\partial y}$$

$$= -V^{-3/2} + (x - y) \left(-\frac{3}{2} V^{-5/2} \right) \cdot (-2x + 2y) = -V^{-3/2} + 3(x - y)^2 \cdot V^{-5/2}$$

$$\therefore \frac{\partial}{\partial y} \left\{ y^2 \frac{\partial u}{\partial y} \right\} = y^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial y} (y^2) = y^2 \frac{\partial^2 u}{\partial y^2} + 2y \frac{\partial u}{\partial y}$$

$$= y^2 \left[-V^{-3/2} + 3(x - y)^2 V^{-5/2} \right] + 2y \cdot V^{-3/2} \cdot (x - y)$$

$$= yV^{-3/2} \left[-y + 3y(x - y)^2 V^{-1} + 2(x - y) \right] = yV^{-3/2} \left[3y(x - y)^2 V^{-1} + (2x - 3y) \right] \quad \dots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x} \left\{ (1 - x^2) \frac{\partial u}{\partial x} \right\} + \frac{\partial}{\partial y} \left\{ y^2 \frac{\partial u}{\partial y} \right\} = yV^{-3/2} \left[3yV^{-1}(1 - x^2) - 2x + 3y(x - y)^2 V^{-1} + 2x - 3y \right]$$

$$\text{R.H.S.} = yV^{-3/2} \left[3yV^{-1}(1 - x^2 + x^2 + y^2 - 2xy) - 3y \right]$$

$$= yV^{-3/2} \left[3yV^{-1}(1 + y^2 - 2xy) - 3y \right]$$

$$= yV^{-3/2} \left[3yV^{-1} \cdot V - 3y \right] = 0 = \text{L.H.S.} \quad \text{Hence proved.}$$

6. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Sol. $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\text{Adding,} \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x+y+z}$$

$$\begin{aligned} \text{Now,} \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{3}{x+y+z}\right) \\ &= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}. \end{aligned}$$

7. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Sol. $x^x y^y z^z = c$ defines z as a function of x and y .

Taking logs, we have $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y , we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + 1 \cdot \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$\text{or} \quad 1 + \log y + (1 + \log z) \frac{\partial z}{\partial y} = 0 \quad \dots(1)$$

$$\text{or} \quad \frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z}$$

$$\text{Similarly,} \quad \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \quad \dots(2)$$

Differentiating (1) partially w.r.t. x , we have

$$\left(\frac{1}{z} \frac{\partial z}{\partial x}\right) \frac{\partial z}{\partial y} + (1 + \log z) \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1 + \log z)} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \quad \dots(3)$$

when $x = y = z$

$$\text{From (2),} \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1.$$

$$\text{From (3),} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1 + \log x)} (-1)(-1) = -\frac{1}{x(\log e + \log x)} = -\frac{1}{x \log ex}$$

$$\text{Hence,} \quad \frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

Hence shown.

8. If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

(M.D.U., Dec., 2007; U.P.T.U., 2007; A.U.U.P., 2009)

Sol. Given,

$$\begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta \\ \Rightarrow x^2 + y^2 &= r^2 \quad \dots(1) \end{aligned}$$

Differentiating partially w.r.t. x , we get

$$2x = 2r \frac{\partial r}{\partial x} \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Now,

$$u = f(r)$$

$$\therefore \frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

Differentiating again w.r.t. x , we get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{r} f'(r) + x \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) f'(r) + \frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x} \\ &= \frac{1}{r} f'(r) - \frac{x}{r^2} \cdot \frac{x}{r} \cdot f'(r) + \frac{x}{r} f''(r) \cdot \frac{x}{r} = \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} \cdot f''(r) \\ &= \frac{r^2 - x^2}{r^3} \cdot f'(r) + \frac{x^2}{r^2} f''(r) = \frac{y^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \end{aligned}$$

$$\text{Similarly,} \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^3} \cdot f'(r) + \frac{x^2 + y^2}{r^2} f''(r) = \frac{1}{r} f'(r) + f''(r) \quad (\because x^2 + y^2 = r^2)$$

9. If $x = r \cos \theta$, $y = r \sin \theta$, prove that

$$(i) \quad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \quad (ii) \quad \frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r \frac{\partial \theta}{\partial x} \quad (iii) \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial y^2} = 0.$$

Sol. (i) $\frac{\partial r}{\partial x}$ means partial derivative of r w.r.t. x , treating y as constant.

$\therefore$ We express r in terms of x and y .

$$\begin{aligned} \text{As} \quad x &= r \cos \theta, \quad y = r \sin \theta, \\ x^2 + y^2 &= r^2 \end{aligned}$$

Differentiating partially w.r.t. x , we get

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$x = r \cos \theta$$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos \theta = \frac{x}{r} \quad \therefore \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

(ii) Expressing x in terms of r and θ , we have

$$x = r \cos \theta$$

$$\Rightarrow \frac{\partial x}{\partial \theta} = -r \sin \theta = -y \quad \Rightarrow \quad \frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = -\frac{y}{r}$$

Expressing θ in terms of x and y , we have

$$\tan \theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}$$

$$\Rightarrow \quad r \frac{\partial \theta}{\partial x} = -\frac{y}{r} \quad \therefore \quad r \frac{\partial \theta}{\partial x} = \frac{1}{r} \cdot \frac{\partial x}{\partial \theta}$$

(iii) Expressing θ in terms of x and y , we have

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -y(x^2 + y^2)^{-1}$$

$$\frac{\partial^2 \theta}{\partial x^2} = y(x^2 + y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = x(x^2 + y^2)^{-1}$$

$$\frac{\partial^2 \theta}{\partial y^2} = -x(x^2 + y^2)^{-2} \cdot (2y) = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

10. If $x = e^{r \cos \theta} \cos (r \sin \theta)$, $y = e^{r \cos \theta} \sin (r \sin \theta)$, prove that $\frac{\partial x}{\partial r} = \frac{1}{r} \frac{\partial y}{\partial \theta}$, $\frac{\partial y}{\partial r} = -\frac{1}{r} \frac{\partial x}{\partial \theta}$. Hence

deduce that $\frac{\partial^2 x}{\partial r^2} + \frac{1}{r} \frac{\partial x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 x}{\partial \theta^2} = 0$.

Sol.

$$x = e^{r \cos \theta} \cdot \cos (r \sin \theta)$$

$$\begin{aligned} \therefore \quad \frac{\partial x}{\partial r} &= e^{r \cos \theta} \cdot \cos \theta \cdot \cos (r \sin \theta) - e^{r \cos \theta} \cdot \sin (r \sin \theta) \cdot \sin \theta \\ &= e^{r \cos \theta} [\cos \theta \cdot \cos (r \sin \theta) - \sin \theta \cdot \sin (r \sin \theta)] \\ &= e^{r \cos \theta} \cdot [\cos (\theta + r \sin \theta)] \end{aligned} \quad \dots(1)$$

$$\begin{aligned} \frac{\partial x}{\partial \theta} &= e^{r \cos \theta} (-r \sin \theta) \cdot \cos (r \sin \theta) - e^{r \cos \theta} \cdot \sin (r \sin \theta) \cdot r \cos \theta \\ &= -r e^{r \cos \theta} [\sin \theta \cdot \cos (r \sin \theta) + \cos \theta \cdot \sin (r \sin \theta)] \\ &= -r e^{r \cos \theta} \cdot \sin (\theta + r \sin \theta) \end{aligned} \quad \dots(2)$$

Also, $y = e^{r \cos \theta} \cdot \sin (r \sin \theta)$

$$\begin{aligned} \therefore \quad \frac{\partial y}{\partial r} &= e^{r \cos \theta} \cdot \cos \theta \cdot \sin (r \sin \theta) + e^{r \cos \theta} \cdot \cos (r \sin \theta) \cdot \sin \theta \\ &= e^{r \cos \theta} [\sin \theta \cos (r \sin \theta) + \cos \theta \cdot \sin (r \sin \theta)] \end{aligned}$$

or $\frac{\partial y}{\partial r} = e^{r \cos \theta} \cdot (\sin \theta + r \sin \theta) \quad \dots(3)$

$$\begin{aligned} \frac{\partial y}{\partial \theta} &= e^{r \cos \theta} (-r \sin \theta) \sin (r \sin \theta) + e^{r \cos \theta} \cdot \cos (r \sin \theta) \cdot r \cos \theta \\ &= r e^{r \cos \theta} \cdot \cos (\theta + r \sin \theta) \end{aligned} \quad \dots(4)$$

From (1) and (4), $\frac{\partial x}{\partial r} = \frac{1}{r} \cdot \frac{\partial y}{\partial \theta} \quad \dots(5)$

From (2) and (3),
$$\frac{\partial y}{\partial r} = -\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} \quad \dots(6)$$

From (5),
$$\frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} + \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \cdot \partial \theta}$$

From (6),
$$\frac{\partial x}{\partial \theta} = -r \frac{\partial y}{\partial r}$$

$$\therefore \frac{\partial^2 x}{\partial \theta^2} = -r \frac{\partial^2 y}{\partial \theta \partial r} = -r \frac{\partial^2 y}{\partial r \partial \theta}$$

$$\therefore \frac{\partial^2 x}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial x}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} + \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} + \frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

Hence proved.

11. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2 \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right).$

Sol. Given
$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad \dots(1)$$

or
$$x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$$

Differentiating partially w.r.t. x , we have

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2} \cdot \frac{\partial u}{\partial x} - y^2(b^2+u)^{-2} \cdot \frac{\partial u}{\partial x} - z^2(c^2+u)^{-2} \cdot \frac{\partial u}{\partial x} = 0$$

or
$$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} \right] \cdot \frac{\partial u}{\partial x}$$

$$\frac{2x}{a^2+u} = V \cdot \frac{\partial u}{\partial x}, \text{ where } V = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{V(a^2+u)}$$

Similarly,
$$\frac{\partial u}{\partial y} = \frac{2y}{V(b^2+u)} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{2z}{V(c^2+u)}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{V^2} \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} \right]$$

$$= \frac{4}{V^2} \cdot (V) = \frac{4}{V} \quad \dots(2)$$

Now,
$$2 \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right) = 2 \left[\frac{2x^2}{V(a^2+u)} + \frac{2y^2}{V(b^2+u)} + \frac{2z^2}{V(c^2+u)} \right]$$

$$= \frac{4}{V} \left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} \right] = \frac{4}{V} (1) = \frac{4}{V}$$

[Using equation (1)]

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \quad \text{[Using equation (2)]}$$

12. If $u = lx + my$, $v = mx - ly$, show that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$ and $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$.

Sol. Given $u = lx + my$... (1) and $v = mx - ly$... (2)

(i) $\left(\frac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x , keeping y as constant.

$\therefore$ We need a relation expressing u as a function of x and y .

From (1), $\left(\frac{\partial u}{\partial x}\right)_y = l$

$\left(\frac{\partial x}{\partial u}\right)_v$ = The partial derivative of x w.r.t. u , keeping v as constant.

From (1) and (2), $lu + mv = l^2x + lmy + m^2x - lmy = (l^2 + m^2)x$

$$x = \frac{lu + mv}{l^2 + m^2}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2}$$

$$\text{Hence, } \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = l \cdot \frac{l}{l^2 + m^2} = \frac{l^2}{l^2 + m^2}$$

(ii) $\left(\frac{\partial y}{\partial v}\right)_x$ = The partial derivative of y w.r.t. v keeping x constant

From (2), $y = \frac{mx - v}{l}$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{l}$$

Also, $\left(\frac{\partial v}{\partial y}\right)_u$ = Partial derivative of v w.r.t. y keeping u constant.

$\therefore$ We need a relation expressing v as a function of y and u .

Eliminating x between equations (1) and (2), we have

$$mu - lv = (l^2 + m^2)y$$

$$\text{or } v = \frac{mu - (l^2 + m^2)y}{l}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

$$\text{Hence, } \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right) \left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}$$

This proves the result.

13. Prove that if $f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}}$, then $f_{xy} = f_{yx}$.

$$\text{Sol. } f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}} = y^{-1/2} \cdot e^{-\frac{(x-a)^2}{4y}}$$

$$\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = y^{-1/2} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x} \left[-\frac{(x-a)^2}{4y} \right] \\
&= y^{-1/2} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \left[-\frac{2(x-a)}{4y} \right] = -\frac{1}{2} y^{-3/2} \cdot (x-a) e^{-\frac{(x-a)^2}{4y}} \\
f_y &= \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-3/2} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-1/2} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial y} \left[-\frac{(x-a)^2}{4y} \right] \\
&= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-3/2} + y^{-1/2} \cdot \frac{1}{4y^2} \cdot (x-a)^2 \right] = \frac{1}{4} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} [-2 + y^{-1} (x-a)^2] \\
f_{xy} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \\
&= \frac{1}{4} y^{-3/2} \left\{ e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x} \left[-\frac{(x-a)^2}{4y} \right] [-2 + y^{-1} (x-a)^2] + e^{-\frac{(x-a)^2}{4y}} \cdot 2y^{-1} \cdot (x-a) \right\} \\
&= \frac{1}{4} y^{-3/2} \cdot e^{-\frac{(x-a)^2}{4y}} \left\{ -\frac{2(x-a)}{4y} [-2 + y^{-1} (x-a)^2] + 2y^{-1} (x-a) \right\} \\
&= \frac{1}{4} y^{-3/2} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{x-a}{y} \left\{ -\frac{1}{2} [-2 + y^{-1} (x-a)^2] + 2 \right\} \\
&= \frac{1}{4} y^{-5/2} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right] \\
f_{yx} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\frac{1}{2} (x-a) \left[-\frac{3}{2} y^{-5/2} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-3/2} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2} \right] \\
&= -\frac{1}{4} (x-a) y^{-5/2} \cdot e^{-\frac{(x-a)^2}{4y}} \left[-3 + \frac{(x-a)^2}{2y} \right] \\
&= \frac{1}{4} y^{-5/2} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right] \\
\therefore f_{xy} &= f_{yx}
\end{aligned}$$

Hence proved.

14. If $z = e^{ax+by} f(ax-by)$, show that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.

Sol. We have $z = e^{ax+by} \cdot f(ax-by)$... (1)

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial z}{\partial x} = e^{ax+by} \cdot f'(ax-by) \cdot a + f(ax-by) \cdot e^{ax+by} \cdot a$$

$$\therefore b \frac{\partial z}{\partial x} = abe^{ax+by} [f'(ax-by) + f(ax-by)] \quad \dots (2)$$

Again differentiating (1) w.r.t. y partially, we get

$$\begin{aligned}\frac{\partial z}{\partial y} &= e^{ax+by} \cdot f'(ax-by) \cdot (-b) + f(ax-by) \cdot e^{ax+by} \cdot b \\ a \frac{\partial z}{\partial y} &= abe^{ax+by} \cdot [-f'(ax-by) + f(ax-by)]\end{aligned}\quad \dots(3)$$

Adding (2) and (3), we get

$$b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz.$$

15. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

Sol. Given $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$...(1)

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2} + \frac{1}{y} \quad \therefore \quad x \frac{\partial u}{\partial x} = -\frac{z}{x} + \frac{x}{y} \quad \dots(2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{z} - \frac{x}{y^2} \quad \therefore \quad y \frac{\partial u}{\partial y} = \frac{y}{z} - \frac{x}{y} \quad \dots(3)$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x} \quad \therefore \quad z \frac{\partial u}{\partial z} = -\frac{y}{z} + \frac{z}{x} \quad \dots(4)$$

Adding (2), (3) and (4), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{x}{y} + \frac{y}{z} - \frac{x}{y} - \frac{y}{z} + \frac{z}{x} = 0.$$

16. If $u = e^{xyz}$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2) e^{xyz}$.

(U.P.T.U., 2007; A.U.U.P., 2008; M.D.U., 2009)

Sol. Given

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = xye^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial y} [xye^{xyz}] = xe^{xyz} + e^{xyz} \cdot xy \cdot xz = xe^{xyz} [1 + xyz]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x} [x \cdot e^{xyz} + x^2 yz e^{xyz}] = e^{xyz} + x y z e^{xyz} + 2 x y z e^{xyz} + y z \cdot x^2 e^{xyz} \cdot yz$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + 3xyz \cdot e^{xyz} + x^2 y^2 z^2 \cdot e^{xyz} = e^{xyz} (1 + 3xyz + x^2 y^2 z^2).$$

Hence proved.

17. Find p and q , if $x = \sqrt{a} (\sin u + \cos v)$, $y = \sqrt{a} (\cos u - \sin v)$, $z = 1 + \sin (u - v)$

where p and q mean $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.

Sol.

$$x^2 = a(\sin u + \cos v)^2$$

$$y^2 = a(\cos u - \sin v)^2$$

$$\begin{aligned}\therefore x^2 + y^2 &= a[(\sin u + \cos v)^2 + (\cos u - \sin v)^2] \\ &= a[\sin^2 u + \cos^2 v + 2 \sin u \cos v + \cos^2 u + \sin^2 v - 2 \sin v \cos u] \\ &= a[2 + 2(\sin u \cos v - \cos u \sin v)] = 2a[1 + \sin (u - v)] = 2az\end{aligned}$$

$$x^2 + y^2 = 2az$$

$$\therefore z = \frac{x^2 + y^2}{2a}$$

$$\frac{\partial z}{\partial x} = \frac{2x}{2a} \quad \text{or} \quad \frac{x}{a}; \quad \frac{\partial z}{\partial y} = \frac{2y}{2a} \quad \text{or} \quad \frac{y}{a}$$

$$\text{Hence,} \quad p = \frac{x}{a}, \quad q = \frac{y}{a}.$$

$$18. \text{ If } V = (x^2 + y^2 + z^2)^{-1/2}, \text{ prove that } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (\text{U.P.T.U., 2009})$$

$$\text{Sol. Given } V = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial V}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 V}{\partial x^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - x \cdot \left(-\frac{3}{2}\right)(x^2 + y^2 + z^2)^{-\frac{5}{2}}(2x)$$

$$= 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3x^2(x^2 + y^2 + z^2)^{-1} - 1]$$

$$\text{Similarly,} \quad \frac{\partial^2 V}{\partial y^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3y^2(x^2 + y^2 + z^2)^{-1} - 1]$$

$$\frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3z^2(x^2 + y^2 + z^2)^{-1} - 1]$$

Adding these, we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3x^2(x^2 + y^2 + z^2)^{-1} - 1 + 3y^2(x^2 + y^2 + z^2)^{-1} - 1 + 3z^2(x^2 + y^2 + z^2)^{-1} - 1]$$

$$\text{RHS} = \frac{(x^2 + y^2 + z^2)^{-\frac{3}{2}}}{(x^2 + y^2 + z^2)} [3x^2 - (x^2 + y^2 + z^2) + 3y^2 - (x^2 + y^2 + z^2) + 3z^2 - (x^2 + y^2 + z^2)]$$

$$\text{or} \quad \text{RHS} = (x^2 + y^2 + z^2)^{-\frac{5}{2}} [2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2]$$

$$\Rightarrow \quad \text{RHS} = (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 0 = 0$$

$$\text{Hence L.H.S.} = 0 \text{ i.e., } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$$19. \text{ If } r^2 = x^2 + y^2 + z^2 \text{ and } V = r^m, \text{ show that } V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}.$$

(M.D.U., Dec., 2005, 2008 and A.U.U.P., 2007; U.P.T.U., 2008)

Sol. Given
and

$$r^2 = x^2 + y^2 + z^2$$

$$V = r^m$$

$$\therefore \quad \frac{\partial V}{\partial x} = mr^{m-1} \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial^2 V}{\partial x^2} = m(m-1)r^{m-2} \left(\frac{\partial r}{\partial x} \right)^2 + mr^{m-1} \cdot \frac{\partial^2 r}{\partial x^2}$$

$$\text{i.e., } V_{xx} = m(m-1)r^{m-2} \left(\frac{\partial r}{\partial x} \right)^2 + mr^{m-1} \cdot \frac{\partial^2 r}{\partial x^2}$$

$$\therefore V_{xx} + V_{yy} + V_{zz} = m(m-1)r^{m-2} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 + \left(\frac{\partial r}{\partial z} \right)^2 \right] + mr^{m-1} \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} \right]$$

$$\text{Now, } \frac{\partial r}{\partial x} = \frac{2x}{2r} \quad \text{or} \quad \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{2y}{2r} \quad \text{or} \quad \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{2z}{2r} \quad \text{or} \quad \frac{z}{r}$$

$$\text{Now, } \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \frac{\partial r}{\partial x}}{r^2} = \frac{r - \frac{x^2}{r}}{r^2} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3}$$

$$\text{Similarly, } \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} \quad \text{and} \quad \frac{\partial^2 r}{\partial z^2} = \frac{r^2 - z^2}{r^3}$$

$$\begin{aligned} \therefore \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} &= \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3} \\ &= \frac{3r^2 - (x^2 + y^2 + z^2)}{r^3} = \frac{3r^2 - r^2}{r^3} = \frac{2r^2}{r^3} \quad \text{or} \quad \frac{2}{r} \end{aligned}$$

$$\begin{aligned} \text{Hence, } V_{xx} + V_{yy} + V_{zz} &= m(m-1)r^{m-2} \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \right] + mr^{m-1} \cdot \frac{2}{r} \\ &= m(m-1)r^{m-2} \cdot 1 + 2mr^{m-2} \\ &= r^{m-2} \cdot [m(m-1) + 2m] = r^{m-2} \cdot [m^2 + m] \\ V_{xx} + V_{yy} + V_{zz} &= m(m+1)r^{m-2} \end{aligned}$$

Hence shown.

20. If $x^2 + y^2 + z^2 = \frac{1}{u^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

$$\text{Sol. Given } u^2 = (x^2 + y^2 + z^2)^{-1} \quad \text{or} \quad u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot (2x) = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3x^2(x^2 + y^2 + z^2)^{-1} - 1]$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3y^2(x^2 + y^2 + z^2)^{-1} - 1]$$

$$\frac{\partial^2 u}{\partial z^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} [3z^2(x^2 + y^2 + z^2)^{-1} - 1]$$

Adding these, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2 + y^2 + z^2)^{-\frac{5}{2}} [2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2]$$

$$= (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 0 = 0$$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

21. If $x = r \cos \theta$, $y = r \sin \theta$, prove that

$$(i) \left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = 1.$$

$$(ii) \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right].$$

(U.P.T.U., 2008)

Sol. (i)

$$\therefore \quad \begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta \\ x^2 + y^2 &= r^2 \end{aligned}$$

$$2r \cdot \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\therefore \quad \left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = 1.$$

$$(ii) \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \therefore \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3},$$

Similarly, $\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$

$$\begin{aligned} \therefore \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} &= \frac{r^2 - x^2 + r^2 - y^2}{r^3} = \frac{2r^2 - (x^2 + y^2)}{r^3} \\ &= \frac{r^2}{r^3} = \frac{1}{r} \cdot 1 = \frac{1}{r} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right]. \end{aligned}$$

22. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0. \quad [\text{M.D.U., May 2007}]$$

Sol.

$$V = r^n (3 \cos^2 \theta - 1)$$

$$\frac{\partial V}{\partial r} = (3 \cos^2 \theta - 1) \cdot n r^{n-1}$$

$$r^2 \frac{\partial V}{\partial r} = n(3 \cos^2 \theta - 1) r^{n+1}$$

$$\frac{\partial}{\partial r} \left[r^2 \frac{\partial V}{\partial r} \right] = n(n+1) (3 \cos^2 \theta - 1) r^n = n(n+1) V$$

$$\frac{\partial V}{\partial \theta} = r^n (-6 \cos \theta \sin \theta)$$

$$\sin \theta \cdot \frac{\partial V}{\partial \theta} = -6 r^n \cos \theta \sin^2 \theta$$

$$\frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial V}{\partial \theta} \right) = -6 r^n [\cos \theta \cdot 2 \sin \theta \cos \theta + \sin^2 \theta (-\sin \theta)]$$

$$= -6r^n \cdot \sin \theta (2 \cos^2 \theta - \sin^2 \theta)$$

$$\therefore \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = -6r^n (2 \cos^2 \theta - \sin^2 \theta)$$

Now, $2 \cos^2 \theta - \sin^2 \theta + \cos^2 \theta - \cos^2 \theta = 3 \cos^2 \theta - 1$

$\therefore$ RHS of above equation becomes $-6r^n (3 \cos^2 \theta - 1)$ or $-6V$.

By putting the values in the given equation, we have

$$n(n+1)V - 6V = 0 \quad \text{or} \quad n(n+1) - 6 = 0$$

$$n^2 + n - 6 = 0$$

$$(n+3)(n-2) = 0 \Rightarrow n = 2, -3.$$

Thus required values of n are 2, -3.

23. If $z = f(x+ay) + \phi(x-ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}$.

[U.P.T.U., (AG) 2005; M.D.U., 2008; A.U.U.P., 2009]

Sol.

$$z = f(x+ay) + \phi(x-ay)$$

$$\frac{\partial z}{\partial x} = f'(x+ay) \cdot 1 + \phi'(x-ay) \cdot 1 = f'(x+ay) + \phi'(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \dots(1)$$

$$\text{Now,} \quad \frac{\partial z}{\partial y} = f'(x+ay) \cdot a + \phi'(x-ay) \cdot (-a) = a [f'(x+ay) - \phi'(x-ay)]$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a [f''(x+ay) \cdot a - \phi''(x-ay) \cdot (-a)]$$

$$\text{or} \quad \frac{\partial^2 z}{\partial y^2} = a^2 [f''(x+ay) + \phi''(x-ay)] \quad \dots(2)$$

$$\text{From (1) and (2), we have} \quad \frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}.$$

Hence proved.

24. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, show that $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x+y)^{-2}$.

Sol. Given

$$u = \log(x^3 + y^3 - x^2y - xy^2) = \log[x^2(x-y) - y^2(x-y)]$$

$$= \log[(x^2 - y^2)(x-y)] = \log[(x+y)(x-y)^2]$$

or

$$u = \log(x+y) + 2 \log(x-y)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2 \cdot 1}{x-y} \quad \dots(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y} \quad \dots(2)$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - 2 \cdot (x-y)^2 \quad \dots(3)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2} \quad \dots(4)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{(x+y)^2} + 2 \cdot \frac{1}{(x-y)^2} \quad \dots(5)$$

From (3), (4) and (5), we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= -\frac{4}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{2}{(x-y)^2} + \frac{4}{(x-y)^2} \\ \therefore \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= -\frac{4}{(x+y)^2} \quad \text{or} \quad -4(x+y)^{-2}. \end{aligned} \quad \text{Hence shown.}$$

25. If $u = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$, 3, find the value of $\frac{\partial^2 u}{\partial x \partial y}$. (U.P.T.U., 2007)

Sol.

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \tan^{-1} \frac{y}{x} + x^2 \cdot \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(\frac{1}{y}\right) \\ &= 2x \tan^{-1} \frac{y}{x} - \frac{x^2 y}{x^2 + y^2} - \frac{y^3}{x^2 + y^2} \\ \frac{\partial^2 u}{\partial x \partial y} &= 2x \frac{1}{\frac{y^2}{x^2} + 1} \cdot \left(\frac{1}{x}\right) - x^2 \cdot \frac{[(x^2 + y^2) \cdot 1 - y \cdot 2y]}{(x^2 + y^2)^2} - \frac{[(x^2 + y^2) \cdot 3y^2 - y^3 (2y)]}{(x^2 + y^2)^2} \\ &= \frac{2x^2}{x^2 + y^2} - \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2} - \frac{3x^2 y^2 + y^4}{(x^2 + y^2)^2} \\ &= \frac{2x^2(x^2 + y^2) - x^2(x^2 - y^2) - y^2(3x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{2x^4 + 2x^2 y^2 - x^4 + x^2 y^2 - 3x^2 y^2 - y^4}{(x^2 + y^2)^2} = \frac{x^4 - y^4}{(x^2 + y^2)^2} = \frac{(x^2 + y^2)(x^2 - y^2)}{(x^2 + y^2)^2} \end{aligned}$$

Hence, $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

26. State Euler's Theorem on homogeneous functions and verify Euler's Theorem for the function $u = (x^{1/2} + y^{1/2})(x^n + y^n)$. (U.P.T.U., 2006; M.D.U., 2007)

Sol. A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if

it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $f(x, y) = \frac{x+y}{\sqrt{x} + \sqrt{y}}$, then

$$(i) \quad f(x, y) = \frac{x \left(1 + \frac{y}{x}\right)}{\sqrt{x} \left(1 + \sqrt{\frac{y}{x}}\right)} = \frac{\sqrt{x} \left(1 + \frac{y}{x}\right)}{\left(1 + \sqrt{\frac{y}{x}}\right)} = x^{1/2} \cdot \phi\left(\frac{y}{x}\right)$$

$\Rightarrow f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y .

$$(ii) \quad f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}(\sqrt{x} + \sqrt{y})} = \sqrt{t} \cdot f(x, y)$$

$\Rightarrow f(x, y)$ is a homogeneous function of degree $1/2$ in x and y .

Euler's Theorem :

If u is a homogeneous function of degree n in x and y , then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$.

Also,
$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

To verify Euler's Theorem for the function :

$$\begin{aligned} u &= \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n) \quad \dots(1) \\ &= x^{\frac{1}{2}} \left(1 + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right) x^n \left(1 + \frac{y^n}{x^n}\right) = x^{n+\frac{1}{2}} \left[1 + \left(\frac{y}{x}\right)^{\frac{1}{2}}\right] \left[1 + \left(\frac{y}{x}\right)^n\right] = x^{n+\frac{1}{2}} \cdot f\left(\frac{y}{x}\right) \end{aligned}$$

$\Rightarrow u$ is a homogeneous function of degree $n + \frac{1}{2}$ in x and y .

$\therefore$ By Euler's Theorem, we should have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u \quad \dots(2)$$

From (1),

$$\frac{\partial u}{\partial x} = \frac{1}{2} x^{-1/2} (x^n + y^n) + nx^{n-1} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)$$

$$x \frac{\partial u}{\partial x} = \frac{1}{2} x^{\frac{1}{2}} (x^n + y^n) + nx^n \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}} (x^n + y^n) + ny^{n-1} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)$$

$$y \frac{\partial u}{\partial y} = \frac{1}{2} y^{\frac{1}{2}} (x^n + y^n) + ny^n \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= \frac{1}{2} (x^{\frac{1}{2}} + y^{\frac{1}{2}}) (x^n + y^n) + n(x^n + y^n) \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \\ &= \frac{1}{2} u + nu = \left(n + \frac{1}{2}\right)u \end{aligned}$$

which is the same as (2). Hence the verification.

27. Verify Euler's Theorem for the function : $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$.

Sol.

$$u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x} \quad \dots(1)$$

$$= \operatorname{cosec}^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x} = x^0 \cdot f\left(\frac{y}{x}\right)$$

$\Rightarrow u$ is a homogeneous function of degree 0 in x and y .

$\therefore$ By Euler's Theorem, we should have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad \dots(2)$$

From (1),

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\therefore x \frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$\therefore y \frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0, \text{ which is the same as (2). Hence the verification.}$$

28. If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$.

Sol. Here u is not a homogeneous function but $\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3(1 + y^3/x^3)}{x(1 - y/x)} = x^2 \cdot f\left(\frac{y}{x}\right)$

is a homogeneous function of degree 2 in x and y .

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$$

$$x \cdot \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u.$

Hence proved.

29. If $u = \log \frac{x^4 + y^4}{x + y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$.

Sol. Here u is not a homogeneous function

$$u = \log \frac{x^4 + y^4}{x + y} \Rightarrow e^u = \frac{x^4 + y^4}{x + y}$$

which is a homogeneous function of degree 3 in x, y .

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = 3 \cdot e^u$$

or
$$x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3 e^u \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.$$

30. If $u = \sin^{-1} \frac{x+2y+3z}{\sqrt{(x^8+y^8+z^8)}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0$.

Sol. Here u is not a homogeneous function

$$\sin u = f(x, y, z) = \frac{x+2y+3z}{\sqrt{(x^8+y^8+z^8)}}$$

$$f(tx, ty, tz) = \frac{t(x+2y+3z)}{t^4 \sqrt{(x^8+y^8+z^8)}} = t^{-3} f(x, y, z)$$

⇒ $\sin u$ is a homogeneous function of degree -3 in x, y, z .

∴ By Euler's Theorem, we have

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) + z \frac{\partial}{\partial z} (\sin u) = -3 \sin u$$

or
$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} + 3 \sin u = 0$$

or
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$

31. If $u = \frac{x^2 y}{x+y}$, show that $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}$.

Sol. Here,

$$u = \frac{x^2 y}{x+y} = \frac{x^3 \left(\frac{y}{x}\right)}{x \left(1 + \frac{y}{x}\right)} = x^2 \cdot f\left(\frac{y}{x}\right)$$

is a homogeneous function of degree 2 in x and y .

∴ By Euler's theorem, we have $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$... (1)

Differentiating (1) partially w.r.t. x ,

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot 1 + y \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x} \Rightarrow x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}.$$

32. If $u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4 \cos^3 u}$.

Sol. $u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is not a homogeneous function but $\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous

function of degree $\frac{1}{2}$ in x and y .

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = \frac{1}{2} \sin u \quad \text{or} \quad x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$\text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \quad \dots(1)$$

Differentiating (1) partially w.r.t. x , we have

$$x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2} \sec^2 u \cdot \frac{\partial u}{\partial x}$$

$$\text{or} \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{2} \sec^2 u - 1 \right) \frac{\partial u}{\partial x} \quad \dots(2)$$

Differentiating (1) partially w.r.t. y , we have

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + 1 \cdot \frac{\partial u}{\partial y} = \frac{1}{2} \sec^2 u \cdot \frac{\partial u}{\partial y}$$

$$\text{or} \quad x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2} \sec^2 u - 1 \right) \frac{\partial u}{\partial y} \quad \dots(3)$$

Multiplying (2) by x , (3) by y and adding, we get

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= \left(\frac{1}{2} \sec^2 u - 1 \right) \cdot \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \left(\frac{1}{2} \sec^2 u - 1 \right) \cdot \frac{1}{2} \tan u \\ &= -\frac{2 \cos^2 u - 1}{2 \cos^2 u} \cdot \frac{1}{2} \frac{\sin u}{\cos u} = -\frac{\sin u \cos 2u}{4 \cos^3 u}. \quad \text{Hence proved.} \end{aligned}$$

33. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, show that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$. (U.P.T.U., 2006, 2008)

Sol. Let $u = xf\left(\frac{y}{x}\right)$ and $v = g\left(\frac{y}{x}\right)$

so that $z = u + v \quad \dots(1)$

Since u is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0 \quad \dots(2)$$

Since v is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0 \quad \dots(3)$$

Adding (2) and (3), we have

$$x^2 \frac{\partial^2}{\partial x^2} (u+v) + 2xy \frac{\partial^2}{\partial x \partial y} (u+v) + y^2 \frac{\partial^2}{\partial y^2} (u+v) = 0$$

$$\text{or} \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$

34. If $u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$. (A.U.U.P., 2009)

Sol.
$$u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right) \Rightarrow \sin u = \frac{x^2 + y^2}{x + y} = \frac{x^2 \left(1 + \frac{y^2}{x^2} \right)}{x \left(1 + \frac{y}{x} \right)} = x \cdot f \left(\frac{y}{x} \right)$$

is a homogeneous function of degree 1 in x, y .

$\therefore$ By Euler's theorem,

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \sin u \quad \text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

35. If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2 \cos 3u \sin u$.

(M.D.U., Dec., 2008)

Sol. As given in question 27, we have already shown that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Here,

$$\phi(u) = \sin 2u, \quad \therefore \phi'(u) = 2 \cos 2u.$$

$\therefore$ By deduction from Euler's Theorem, we know that

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= \phi(u) [\phi(u) - 1] = \sin 2u [2 \cos 2u - 1] \\ &= 2 \sin 2u \cos 2u - \sin 2u = \sin 4u - \sin 2u \\ &= 2 \cos 3u \sin u. \end{aligned}$$

Hence proved.

36. If $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$, then find the value of $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$.

(U.P.T.U., 2007)

Sol.

$$\begin{aligned} u &= \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x} \quad \dots(1) \\ &= \operatorname{cosec}^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x} = x^0 \cdot f \left(\frac{y}{x} \right) \end{aligned}$$

$\Rightarrow u$ is a homogeneous function of degree 0 in x and y .

$\therefore$ By Euler's theorem, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Here from (1), we get

$$\begin{aligned} x \frac{\partial u}{\partial x} &= \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \\ y \frac{\partial u}{\partial y} &= -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0. \end{aligned}$$

37. If $u = \cos^{-1} \frac{x + y}{\sqrt{x} + \sqrt{y}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$.

Sol. Given

$$\cos u = \frac{x + y}{\sqrt{x} + \sqrt{y}}$$

is a homogeneous function of degree $\frac{1}{2}$ in x, y .

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\cos u) + y \frac{\partial}{\partial y} (\cos u) = \frac{1}{2} \cos u$$

$$-x \sin u \frac{\partial u}{\partial x} - y \sin u \frac{\partial u}{\partial y} = \frac{1}{2} \cos u$$

or

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \cos u \left(-\frac{1}{\sin u} \right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cot u \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0. \quad \text{Hence shown.}$$

38. Given $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, prove that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n^2 z$.

(M.D.U., Dec., 2005, 2007)

Sol.

$$z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right) = x^n \left[f_1\left(\frac{y}{x}\right) + \frac{y^{-n}}{x^n} f_2\left(\frac{x}{y}\right) \right]$$

∴ z is a homogeneous function of degree n in x and y .

$$\therefore \text{By Euler's theorem } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \dots(1)$$

Differentiating w.r.t. x , we get

$$x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

Multiplying by x , we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial x} + xy \frac{\partial^2 z}{\partial x \partial y} = nx \frac{\partial z}{\partial x} \quad \dots(2)$$

Differentiating (1) w.r.t. y , we get

$$x \frac{\partial^2 z}{\partial x \partial y} + y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n \frac{\partial z}{\partial y}$$

Multiplying by y , we get

$$xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + y \frac{\partial z}{\partial y} = ny \frac{\partial z}{\partial y} \quad \dots(3)$$

Adding (2) and (3), we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nx \frac{\partial z}{\partial x} + ny \frac{\partial z}{\partial y}$$

$$= n \left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right] = n \cdot nz$$

[Using equation (1)]

$$= n^2 z.$$

Hence proved.

39. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

Sol.

$$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = f(x, y, z)$$

$$f(tx, ty, tz) = \frac{tx}{ty + tz} + \frac{ty}{tz + tx} + \frac{tz}{tx + ty} = t^0 \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

$\therefore u$ is a homogeneous function of degree or order 0 in the variables x, y, z .

$\therefore$ By Euler's Theorem, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0 \cdot u = 0$.

Hence shown.

40. If $u = \operatorname{cosec}^{-1} \left(\frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}} \right)^{\frac{1}{2}}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144} (13 + \tan^2 u)$.

(M.D.U., May 2006, Dec., 2006; U.P.T.U., 2009; A.U.U.P., 2007, 2008)

Sol.

$$u = \operatorname{cosec}^{-1} \left(\frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}} \right)^{\frac{1}{2}}$$

$$\operatorname{cosec} u = \sqrt{\frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}}} = \left[\frac{x^{\frac{1}{2}} \left\{ 1 + \left(\frac{y}{x} \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}}{x^{\frac{1}{3}} \left\{ 1 + \left(\frac{y}{x} \right)^{\frac{1}{3}} \right\}} \right]^{\frac{1}{2}}$$

$$\therefore \sin u = \left[\frac{x^{\frac{1}{3}} \left\{ 1 + \left(\frac{y}{x} \right)^{\frac{1}{3}} \right\}^{\frac{1}{2}}}{x^{\frac{1}{2}} \left\{ 1 + \left(\frac{y}{x} \right)^{\frac{1}{2}} \right\}} \right]^{\frac{1}{2}} = x^{-\frac{1}{12}} \left[\frac{1 + \left(\frac{y}{x} \right)^{\frac{1}{3}}}{1 + \left(\frac{y}{x} \right)^{\frac{1}{2}}} \right]^{\frac{1}{2}}$$

Let $z = \sin u$, then z is a homogeneous function of (x, y) of order $\left(-\frac{1}{12} \right)$.

$\therefore$ By Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -\frac{1}{12} z$$

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = -\frac{1}{12} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = -\frac{1}{12} \sin u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{12} \tan u \quad \dots(1)$$

Differentiating equation (1) w.r.t. x and then multiplying both sides by x , we get

$$x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{12} x \sec^2 u \cdot \frac{\partial u}{\partial x} \quad \dots(2)$$

Differentiating equation (1) w.r.t. y and then multiplying both sides by y , we get

$$xy \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{1}{12} y \sec^2 u \cdot \frac{\partial u}{\partial y} \quad \dots(3)$$

Adding (2) and (3), we get

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = -\frac{1}{12} \sec^2 u \cdot \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right]$$

Using equation (1), we get

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left[-\frac{1}{12} \tan u \right] = -\frac{1}{12} \sec^2 u \cdot \left[-\frac{1}{12} \tan u \right] = \frac{1}{144} \tan u \sec^2 u$$

$$\text{or} \quad x^2 \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u + \frac{1}{144} \tan u \sec^2 u = \frac{1}{144} \tan u (\sec^2 u + 12)$$

$$= \frac{1}{144} \tan u (13 + \tan^2 u)$$

Hence proved.

41. If $u = \tan^{-1} \left(\frac{y^2}{x} \right)$, show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

Sol. $\tan u = \frac{y^2}{x}$ is a homogeneous function of degree 1 in x and y .

$$\therefore x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \cdot \frac{\partial u}{\partial y} = \tan u$$

$$\text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \frac{1}{2} \sin 2u \quad \dots(1)$$

Differentiating (1) w.r.t. x , we get

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2} \cos 2u \cdot \frac{\partial u}{\partial x} \cdot 2$$

$$\text{or} \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (\cos 2u - 1) \frac{\partial u}{\partial x} \quad \dots(2)$$

Differentiating (1) w.r.t. y , we get

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \cos 2u \cdot 2 \cdot \frac{\partial u}{\partial y}$$

$$\text{or} \quad x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (\cos 2u - 1) \frac{\partial u}{\partial y} \quad \dots(3)$$

Multiplying (2) by x , we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x (\cos 2u - 1) \frac{\partial u}{\partial x} \quad \dots(4)$$

Multiplying (3) by y , we have

$$xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = y (\cos 2u - 1) \frac{\partial u}{\partial y} \quad \dots(5)$$

Adding (4) and (5), we get

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} &= (\cos 2u - 1) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] \\ &= (\cos 2u - 1) \left(\frac{1}{2} \sin 2u \right) = (-2 \sin^2 u) \left(\frac{1}{2} \sin 2u \right) \\ &= -\sin 2u \cdot \sin^2 u. \end{aligned}$$

Hence shown.

42. If $u = x^2 \tan^{-1} \left(\frac{y}{x} \right) - y^2 \tan^{-1} \left(\frac{x}{y} \right)$, then evaluate $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$.

(M.D.U., May 2007, May 2009; U.P.T.U., 2008, A.U.U.P., 2007)

Sol. Let

$$u = v - w$$

Consider first term $v = x^2 \tan^{-1} \left(\frac{y}{x} \right)$. v is a homogeneous function of degree 2 in x and y .

$$\therefore \text{By Euler's Theorem, } x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2v \quad \dots(1)$$

Partial differentiating (1) w.r.t. x , we have

$$x \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \frac{\partial^2 v}{\partial x \partial y} = 2 \frac{\partial v}{\partial x} \quad \dots(2)$$

Partial differentiating (1) w.r.t. y , we have

$$x \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial y} + y \frac{\partial^2 v}{\partial y^2} = 2 \frac{\partial v}{\partial y} \quad \dots(2A)$$

After multiplying (2) by x and (2A) by y and then adding, we obtain,

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = 2v \quad \dots(3)$$

Similarly, for the second term

$$w = y^2 \tan^{-1} \left(\frac{x}{y} \right)$$

w is homogeneous function of degree 2 in x and y

$\therefore$ By Euler's Theorem,

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 2w$$

Partial differentiating w.r.t. x , we have

$$x \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x} + y \frac{\partial^2 w}{\partial x \partial y} = 2 \frac{\partial w}{\partial x} \quad \dots(4)$$

Partial differentiating w.r.t. y , we have

$$x \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial w}{\partial y} + y \frac{\partial^2 w}{\partial y^2} = 2 \frac{\partial w}{\partial y} \quad \dots(5)$$

Multiplying equation (4) by x and equation (5) by y and on adding, we obtain

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial y \partial x} + y^2 \frac{\partial^2 w}{\partial y^2} = 2w \quad \dots(6)$$

Subtracting (6) from equation (3) gives :

$$x^2 \frac{\partial^2}{\partial x^2} (v - w) + 2xy \frac{\partial^2}{\partial x \partial y} (v - w) + y^2 \frac{\partial^2}{\partial y^2} (v - w) = 2(v - w)$$

$$\text{or } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$$

Hence the result.

43. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

Sol. The given equations define u as a composite function of t .

$$\begin{aligned} \therefore \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 + \frac{1}{\sqrt{1-(x-y)^2}} \cdot (-1) \cdot 12t^2 \\ &= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} \end{aligned}$$

Hence $\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$.

44. If z is a function of x and y , where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.

Sol. Here z is a composite function of u and v .

(M.D.U., 2007)

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u + \frac{\partial z}{\partial y} \cdot (-e^{-u})$$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot (-e^{-v}) + \frac{\partial z}{\partial y} \cdot (e^v)$

Subtracting, we get $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v}) \frac{\partial z}{\partial x} - (-e^{-u} - e^v) \frac{\partial z}{\partial y} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$. **Hence shown.**

45. If $u = f(y - z, z - x, x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

[U.P.T.U., (AG) 2005; M.D.U., 2008; A.U.U.P., 2009]

Sol. Here,

$$u = f(X, Y, Z)$$

where $X = y - z$, $Y = z - x$, $Z = x - y$

$\therefore u$ is a composite function of x, y and z .

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X} \cdot (0) + \frac{\partial u}{\partial Y} \cdot (-1) + \frac{\partial u}{\partial Z} \cdot (1) \quad \dots(1)$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} (1) + \frac{\partial u}{\partial Y} (0) + \frac{\partial u}{\partial Z} (-1) \quad \dots(2)$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} (-1) + \frac{\partial u}{\partial Y} (1) + \frac{\partial u}{\partial Z} (0) \quad \dots(3)$$

On adding (1), (2) and (3), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

46. If $w = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$, show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

Sol. The given equations define w as a composite function of r and θ .

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \cos \theta + \frac{\partial w}{\partial y} \cdot \sin \theta \quad \text{or} \quad \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta \quad \dots(1)$$

[$\because w = f(x, y)$]

Also,

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x} (-r \sin \theta) + \frac{\partial w}{\partial y} (r \cos \theta)$$

or

$$\frac{1}{r} \cdot \frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x} \sin \theta + \frac{\partial f}{\partial y} \cos \theta \quad \dots(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

47. If u is a homogeneous function of n^{th} degree in x, y, z , prove that (U.P.T.U., 2008)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu.$$

Sol. Since u is a homogeneous function of degree n in x, y, z

let $u = x^n \cdot f\left(\frac{y}{x}, \frac{z}{x}\right)$ or $u = x^n \cdot f(t, s)$, where $t = \frac{y}{x}$, $s = \frac{z}{x}$

Here, u is a composite function of x, y, z .

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} &= nx^{n-1} f(t, s) + x^n \left(\frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} \right) \\ &= nx^{n-1} f(t, s) + x^n \left[\frac{\partial f}{\partial t} \cdot \left(-\frac{y}{x^2}\right) + \frac{\partial f}{\partial s} \cdot \left(-\frac{z}{x^2}\right) \right] \\ \Rightarrow x \frac{\partial u}{\partial x} &= nx^n f(t, s) - yx^{n-1} \frac{\partial f}{\partial t} - zx^{n-1} \frac{\partial f}{\partial s} \quad \dots(1) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= x^n \left(\frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} \right) = x^n \left(\frac{\partial f}{\partial t} \cdot \frac{1}{x} \right) \\ y \frac{\partial u}{\partial y} &= yx^{n-1} \frac{\partial f}{\partial t} \quad \dots(2) \end{aligned}$$

$$\Rightarrow \frac{\partial u}{\partial z} = x^n \left[\frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial z} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial z} \right] = x^n \cdot \frac{\partial f}{\partial s} \cdot \frac{1}{x} \Rightarrow z \frac{\partial u}{\partial z} = zx^{n-1} \frac{\partial f}{\partial s} \quad \dots(3)$$

Adding (1), (2) and (3), we have $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nx^n f(t, s) = nu.$

48. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$ show that (U.P.T.U., 2007; M.D.U., 2008; A.U.U.P., 2009)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = u(x^2 + y^2) \left[\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} \right].$$

Sol. Here, $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$

$$\begin{aligned} \therefore \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial u} \cdot 2x + \frac{\partial \theta}{\partial v} \cdot 2y \\ &= 2 \left(x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \right) \theta \quad \text{or} \quad \frac{\partial}{\partial x} \equiv 2 \left(x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \right) \end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2 \left(x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \right) \left[2 \left(x \frac{\partial \theta}{\partial u} + y \frac{\partial \theta}{\partial v} \right) \right] \\ &= 4 \left[x^2 \frac{\partial^2 \theta}{\partial u^2} + 2xy \frac{\partial^2 \theta}{\partial u \partial v} + y^2 \frac{\partial^2 \theta}{\partial v^2} \right] \quad \dots(1)\end{aligned}$$

Also, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial u} (-2y) + \frac{\partial \theta}{\partial v} (2x)$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = -2 \left(y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v} \right) \theta \quad \text{or} \quad \frac{\partial}{\partial y} = -2 \left(y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v} \right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 4 \left[y^2 \frac{\partial^2 \theta}{\partial u^2} - 2xy \frac{\partial^2 \theta}{\partial u \partial v} + x^2 \frac{\partial^2 \theta}{\partial v^2} \right] \quad \dots(2)$$

Adding (1) and (2), we get

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 4 \left[(x^2 + y^2) \frac{\partial^2 \theta}{\partial u^2} + (x^2 + y^2) \frac{\partial^2 \theta}{\partial v^2} \right] \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 4 (x^2 + y^2) \left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} \right).\end{aligned}$$

49. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$, when $x = y = a$.

Sol. The given equations are of the form

$$z = f(x, y) \quad \text{and} \quad \phi(x, y) = c$$

$\therefore z$ is a composite function of x and y .

$$\Rightarrow \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \quad \dots(1)$$

Now, $\frac{\partial z}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}}$

Similarly, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$ w.r.t. x , we have

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0$$

or $(y^2 + ax) \frac{dy}{dx} = -(x^2 + ay)$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (1), $\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}} \left(-\frac{x^2 + ay}{y^2 + ax} \right)$

When $x = y = a$, $\frac{dz}{dx} = \frac{a}{\sqrt{2a^2}} + \frac{a}{\sqrt{2a^2}} \left(-\frac{a^2 + a^2}{a^2 + a^2} \right) = 0.$

50. If $x + y = 2e^\theta \cos \phi$ and $x - y = 2ie^\theta \sin \phi$, show that

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}.$$

Sol. Here,

$$x + y = 2e^\theta \cos \phi$$

$$x - y = 2ie^\theta \sin \phi$$

On adding, we get

$$2x = 2e^\theta (\cos \phi + i \sin \phi)$$

or

$$x = e^\theta \cdot e^{i\phi}$$

Subtracting

$$y = e^\theta \cdot e^{-i\phi}$$

Clearly, u is a composite function of θ and ϕ

$$\begin{aligned} \therefore \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (e^\theta \cdot e^{i\phi}) + \frac{\partial u}{\partial y} (e^\theta \cdot e^{-i\phi}) \\ &= e^\theta \left(e^{i\phi} \frac{\partial u}{\partial x} + e^{-i\phi} \frac{\partial u}{\partial y} \right) = e^\theta \left(e^{i\phi} \frac{\partial}{\partial x} + e^{-i\phi} \frac{\partial}{\partial y} \right) u \end{aligned}$$

$$\text{or} \quad \frac{\partial}{\partial \theta} = e^\theta \left(e^{i\phi} \frac{\partial}{\partial x} + e^{-i\phi} \frac{\partial}{\partial y} \right)$$

$$\begin{aligned} \Rightarrow \frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial \theta} \right) = e^\theta \left(e^{i\phi} \frac{\partial}{\partial x} + e^{-i\phi} \frac{\partial}{\partial y} \right) \left[e^\theta \left(e^{i\phi} \frac{\partial}{\partial x} + e^{-i\phi} \frac{\partial}{\partial y} \right) u \right] \\ &= e^{2\theta} \left[e^{2i\phi} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} + e^{-2i\phi} \frac{\partial^2 u}{\partial y^2} \right] \\ &= e^{2\theta} \left[e^{2i\phi} \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + e^{-2i\phi} \frac{\partial^2 u}{\partial y^2} \right] \quad \dots(1) \end{aligned}$$

Similarly, we will obtain $\frac{\partial^2 u}{\partial \phi^2}$.

$$\begin{aligned} \frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x} (e^\theta \cdot ie^{i\phi}) + \frac{\partial u}{\partial y} [e^\theta \cdot (-ie^{i\phi})] \\ &= ie^\theta \left(e^{i\phi} \frac{\partial u}{\partial x} - e^{-i\phi} \frac{\partial u}{\partial y} \right) = ie^\theta \left(e^{i\phi} \frac{\partial}{\partial x} - e^{-i\phi} \frac{\partial}{\partial y} \right) u \end{aligned}$$

$$\text{or} \quad \frac{\partial}{\partial \phi} = ie^\theta \left(e^{i\phi} \frac{\partial}{\partial x} - e^{-i\phi} \frac{\partial}{\partial y} \right)$$

$$\begin{aligned} \therefore \frac{\partial^2 u}{\partial \phi^2} &= \frac{\partial}{\partial \phi} \left(\frac{\partial u}{\partial \phi} \right) = i^2 e^{2\theta} \left[e^{2i\phi} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} + e^{-2i\phi} \frac{\partial^2 u}{\partial y^2} \right] \\ &= -e^{2\theta} \left[e^{2i\phi} \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + e^{-2i\phi} \frac{\partial^2 u}{\partial y^2} \right] \quad \dots(2) \end{aligned}$$

Adding (1) and (2), we have

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4e^{2\theta} \cdot \frac{\partial^2 u}{\partial x \partial y} = 4xy \frac{\partial^2 u}{\partial x \partial y} \quad (\because e^{2\theta} = xy)$$

51. If $u = xe^y \cdot z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, find $\frac{du}{dx}$.

Sol. Here, u is a function of x, y, z while y and z are functions of x .

$$\begin{aligned}
 \therefore \frac{du}{dx} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx} \\
 &= e^y \cdot z \cdot 1 + xe^y \cdot z \cdot \frac{1}{2} (a^2 - x^2)^{-1/2} (-2x) + xe^y \cdot 2 \sin x \cos x \\
 \text{or } \frac{du}{dx} &= e^y \cdot z - x^2 e^y z (a^2 - x^2)^{-1/2} + xe^y \cdot \sin 2x \\
 \Rightarrow \frac{du}{dx} &= e^y \left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x \sin 2x \right].
 \end{aligned}$$

- 52.** If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm per second when it passes through the value $x = 3$ cm, show that if y is passing through the value $y = 1$ cm, y must be decreasing at the rate of $2 \frac{2}{15}$ cm per second, in order that z shall remain constant.

Sol. Given : $z = 2xy^2 - 3x^2y$ and $\frac{dx}{dt} = 2$ cm/sec, when $x = 3$ cm, we have to find $\frac{dy}{dt}$ when $y = 1$ cm.

Now,
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (2y^2 - 6xy) \frac{dx}{dt} + (4xy - 3x^2) \frac{dy}{dt}$$

Since z remains constant, $\frac{dz}{dt} = 0$.

$$\therefore 0 = (2y^2 - 6xy) 2 + (4xy - 3x^2) \frac{dy}{dt} \quad \text{or} \quad 0 = 4y^2 - 36y + (12y - 27) \frac{dy}{dt}$$

When $y = 1$ cm, we have

$$0 = 4 - 36 + (12 - 27) \frac{dy}{dt} \quad \text{or} \quad \frac{dy}{dt} = -\frac{32}{15} \quad \text{which is negative.}$$

$\therefore$ When $y = 1$ cm, y is decreasing at the rate of $2 \frac{2}{15}$ cm/sec.

- 53.** Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.

Sol. The given equations are of the form $u = f(x, y)$ and $\phi(x, y) = k$.

$\therefore u$ is a composite function of x .

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \text{or} \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \dots(1)$$

Now,
$$\frac{\partial u}{\partial x} = 2x \cos(x^2 + y^2),$$

$$\frac{\partial u}{\partial y} = 2y \cos(x^2 + y^2)$$

Also, differentiating $a^2x^2 + b^2y^2 = c$ w.r.t. x , we have

$$2a^2x + 2b^2y \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$$\therefore \text{From (1),} \quad \frac{du}{dx} = 2x \cos(x^2 + y^2) + 2y \cos(x^2 + y^2) \cdot \left[-\frac{a^2x}{b^2y} \right]$$

$$\text{or} \quad \frac{du}{dx} = 2 \left[x - \frac{a^2x}{b^2} \right] \cos(x^2 + y^2) = \frac{2(b^2 - a^2)}{b^2} x \cdot \cos(x^2 + y^2).$$

54. Find $\frac{dy}{dx}$ when $x^y + y^x = c$.

Sol. Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$. If we are given an implicit function $f(x, y) = c$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{fx}{fy} \Rightarrow \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \cdot \log y}{x^y \log x + xy^{x-1}}.$$

55. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

Sol. Let $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\frac{fx}{fy} = -\frac{y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log(\sin y)}{(\cos x)^y \log \cos x - x(\sin y)^{x-1} \cos y} \\ &= \frac{y(\cos x)^{y-1} \sin x + (\sin y)^x \log(\sin y)}{(\cos x)^y \log \cos x - x(\cos x)^y (\sin y)^{-1} \cos y} \quad [\because (\sin y)^x = (\cos x)^y] \\ &= \frac{(\cos x)^y \left[y \cdot \frac{\sin x}{\cos x} + \log \sin y \right]}{(\cos x)^y [\log \cos x - x \cot y]} \end{aligned}$$

Hence,
$$\frac{dy}{dx} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}.$$

56. Given $x = u \cos \alpha - v \sin \alpha$ and $y = u \sin \alpha + v \cos \alpha$. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$.

Sol. Here z is a composite function of u and v .

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \cos \alpha \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \sin \alpha$$

or
$$\frac{\partial}{\partial u}(z) = \left(\cos \alpha \cdot \frac{\partial}{\partial x} + \sin \alpha \cdot \frac{\partial}{\partial y} \right) z$$

$\Rightarrow$
$$\frac{\partial}{\partial u} = \cos \alpha \cdot \frac{\partial}{\partial x} + \sin \alpha \cdot \frac{\partial}{\partial y} \quad \dots(1)$$

$\therefore$
$$\begin{aligned} \frac{\partial^2 z}{\partial u^2} &= \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) = \left(\cos \alpha \cdot \frac{\partial}{\partial x} + \sin \alpha \cdot \frac{\partial}{\partial y} \right) \left(\cos \alpha \cdot \frac{\partial z}{\partial x} + \sin \alpha \cdot \frac{\partial z}{\partial y} \right) \\ &= \cos^2 \alpha \cdot \frac{\partial^2 z}{\partial x^2} + 2 \sin \alpha \cos \alpha \cdot \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \cdot \frac{\partial^2 z}{\partial y^2} \quad \dots(2) \end{aligned}$$

Also,
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = (-\sin \alpha) \frac{\partial z}{\partial x} + (\cos \alpha) \frac{\partial z}{\partial y}$$

or
$$\frac{\partial}{\partial v}(z) = \left(-\sin \alpha \cdot \frac{\partial}{\partial x} + \cos \alpha \cdot \frac{\partial}{\partial y} \right) z$$

$\Rightarrow$
$$\frac{\partial}{\partial v} = -\sin \alpha \cdot \frac{\partial}{\partial x} + \cos \alpha \cdot \frac{\partial}{\partial y} \quad \dots(3)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) = \left(-\sin \alpha \cdot \frac{\partial}{\partial x} + \cos \alpha \cdot \frac{\partial}{\partial y} \right) \left(-\sin \alpha \cdot \frac{\partial z}{\partial x} + \cos \alpha \cdot \frac{\partial z}{\partial y} \right)$$

$$= \sin^2 \alpha \frac{\partial^2 z}{\partial x^2} - 2 \sin \alpha \cos \alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2 \alpha \frac{\partial^2 z}{\partial y^2} \quad \dots(4)$$

Adding (2) and (4), we get $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$.

57. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$. [U.P.T.U., (AG) 2005]

Sol. $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \quad \dots(1)$

$$f(x, y) = 0 \text{ gives } \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}; \quad \phi(y, z) = 0 \text{ gives } \frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

$\therefore$ From (1), $\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}} \Rightarrow \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$.

58. If $\phi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$. (U.P.T.U., 2009)

Sol. The given relation defines y as a function of x and z .

Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y . Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

Similarly, $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$

Multiplying all these, we get the desired result.

59. Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, into polar co-ordinates.

Sol. The relations connecting cartesian co-ordinates (x, y) with polar co-ordinates are

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Squaring and adding, $r^2 = x^2 + y^2$

Dividing, we have $\tan \theta = \frac{y}{x}$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\text{Thus, } r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r \cos \theta}{r} = \cos \theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right),$$

$$\text{or} \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin \theta}{r}$$

$$\frac{\partial \theta}{\partial y} = -\frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = -\frac{x}{x^2 + y^2} = -\frac{r \cos \theta}{r^2} = -\frac{\cos \theta}{r}$$

Here u is a composite function of x and y .

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \cos \theta \cdot \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{\partial}{\partial x}(u) = \left(\cos \theta \cdot \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial}{\partial \theta} \right) u$$

$$\Rightarrow \quad \frac{\partial}{\partial x} \equiv \cos \theta \cdot \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial}{\partial \theta} \quad \dots(1)$$

$$\text{Also,} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \sin \theta \cdot \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \cdot \frac{\partial u}{\partial \theta}$$

$$\text{or} \quad \frac{\partial}{\partial y}(u) = \left(\sin \theta \cdot \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \cdot \frac{\partial}{\partial \theta} \right) u$$

$$\Rightarrow \quad \frac{\partial}{\partial y} \equiv \sin \theta \cdot \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \cdot \frac{\partial}{\partial \theta} \quad \dots(2)$$

Now, we shall make use of relations (1) and (2).

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\cos \theta \cdot \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial}{\partial \theta} \right) \times \left(\cos \theta \cdot \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial u}{\partial \theta} \right) \\ &= \cos \theta \cdot \frac{\partial}{\partial r} \left(\cos \theta \cdot \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \cdot \frac{\partial}{\partial \theta} \left(\cos \theta \cdot \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \cdot \frac{\partial u}{\partial \theta} \right) \\ &= \cos \theta \left[\cos \theta \cdot \frac{\partial^2 u}{\partial r^2} - \sin \theta \cdot \frac{\partial}{\partial \theta} \left(-\frac{1}{r^2} \right) - \frac{\sin \theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} \right] \\ &\quad - \frac{\sin \theta}{r} \left[-\sin \theta \cdot \frac{\partial u}{\partial r} + \cos \theta \cdot \frac{\partial^2 u}{\partial \theta \partial r} - \frac{\cos \theta}{r} \cdot \frac{\partial u}{\partial \theta} - \frac{\sin \theta}{r} \cdot \frac{\partial^2 u}{\partial \theta^2} \right] \end{aligned}$$

$$= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{2 \cos \theta \sin \theta}{r^2} \cdot \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r} \cdot \frac{\partial u}{\partial r} - \frac{2 \cos \theta \sin \theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \quad \dots(3)$$

Similarly, we get :

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) &= \sin^2 \theta \cdot \frac{\partial^2 u}{\partial r^2} - \frac{2 \cos \theta \sin \theta}{r^2} \cdot \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r} \cdot \frac{\partial u}{\partial r} \\ &\quad + \frac{2 \cos \theta \sin \theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \quad \dots(4) \end{aligned}$$

Adding (3) and (4), we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2}$

$\therefore$ The transformed equation in polar co-ordinates is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$.

60. If $u = \tan^{-1} \left(\frac{y}{x} \right)$, where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$, find $\frac{du}{dt}$.

(M.D.U., 2009; U.P.T.U., 2007; A.U.U.P., 2008)

Sol. The given equations define u as a composite function of t .

$$\begin{aligned} \therefore \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2} \right) \cdot (e^t + e^{-t}) + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} \cdot (e^t - e^{-t}) \\ &= -\frac{y}{x^2 + y^2} \cdot (e^t + e^{-t}) + \frac{x(e^t - e^{-t})}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} \quad \dots(1) \end{aligned}$$

$$x^2 - y^2 = (x + y)(x - y) = (2e^t)(-2e^{-t}) = -4 \quad \dots(2)$$

$$x^2 + y^2 = (e^t - e^{-t})^2 + (e^t + e^{-t})^2 = 2(e^{2t} + e^{-2t}) \quad \dots(3)$$

Using (2) and (3), we get from equation (1), $\frac{du}{dt} = \frac{x^2 - y^2}{x^2 + y^2} = \frac{-4}{2(e^{2t} + e^{-2t})} = -\frac{2}{e^{2t} + e^{-2t}}$.

61. If $u = u \left(\frac{y-x}{xy}, \frac{z-x}{xz} \right)$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$. (U.P.T.U., 2005)

Sol. Let $u = u(v, w)$... (1)

where, $v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$... (2) and $w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$... (3)

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v} \left(-\frac{1}{x^2} \right) + \frac{\partial u}{\partial w} \left(-\frac{1}{x^2} \right)$

$\Rightarrow x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}$... (4)

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial u}{\partial v} \cdot \left(\frac{1}{y^2} \right) + \frac{\partial u}{\partial w} \cdot (0)$

$\Rightarrow y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}$... (5)

$$\begin{aligned} \text{and} \quad \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z} = \frac{\partial u}{\partial v} \cdot (0) + \frac{\partial u}{\partial w} \left(\frac{1}{z^2} \right) \\ \Rightarrow \quad z^2 \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial w} \end{aligned} \quad \dots(6)$$

$$\text{Adding (4), (5) and (6), we get} \quad x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Hence proved.

62. If $\phi(cx - az, cy - bz) = 0$, show that $a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = c$.

$$\begin{aligned} \text{Sol. Let} \quad cx - az &= v \\ cy - bz &= w \\ \text{then, we have} \quad \phi(v, w) &= 0 \end{aligned} \quad \dots(1)$$

$$\begin{aligned} \therefore \quad \frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w} \cdot \frac{\partial w}{\partial x} \\ \text{or} \quad 0 &= \frac{\partial \phi}{\partial v} \left(c - a \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial w} \left(-b \frac{\partial z}{\partial x} \right) = c \frac{\partial \phi}{\partial v} + \frac{\partial z}{\partial x} \left(-a \frac{\partial \phi}{\partial v} - b \frac{\partial \phi}{\partial w} \right) \end{aligned}$$

$$\Rightarrow \quad a \frac{\partial z}{\partial x} = \frac{ac \frac{\partial \phi}{\partial v}}{a \frac{\partial \phi}{\partial v} + b \frac{\partial \phi}{\partial w}} \quad \dots(2)$$

$$\begin{aligned} \text{Also,} \quad \frac{\partial \phi}{\partial y} &= \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w} \cdot \frac{\partial w}{\partial y} \\ 0 &= \frac{\partial \phi}{\partial v} \left(-a \frac{\partial z}{\partial y} \right) + \frac{\partial \phi}{\partial w} \left(c - b \frac{\partial z}{\partial y} \right) \\ 0 &= c \frac{\partial \phi}{\partial w} + \frac{\partial z}{\partial y} \left(-a \frac{\partial \phi}{\partial v} - b \frac{\partial \phi}{\partial w} \right) \\ \Rightarrow \quad b \frac{\partial z}{\partial y} &= \frac{bc \frac{\partial \phi}{\partial w}}{a \frac{\partial \phi}{\partial v} + b \frac{\partial \phi}{\partial w}} \quad \dots(3) \end{aligned}$$

Adding (2) and (3), we get

$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \frac{c \left(a \frac{\partial \phi}{\partial v} + b \frac{\partial \phi}{\partial w} \right)}{\left(a \frac{\partial \phi}{\partial v} + b \frac{\partial \phi}{\partial w} \right)} = c. \quad \text{Hence shown.}$$

63. If $z = e^{ax+by} f(ax - by)$, prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.

(V.T.U., 2005, 2007; U.P.T.U., 2006; M.D.U., 2008; A.U.U.P., 2009)

$$\begin{aligned} \text{Sol.} \quad z &= e^{ax+by} \cdot f(ax - by) \\ \frac{\partial z}{\partial x} &= e^{ax+by} \cdot a \cdot f(ax - by) + e^{ax+by} f'(ax - by) \cdot a \\ \frac{\partial z}{\partial x} &= ae^{ax+by} [f(ax - by) + f'(ax - by)] \\ \frac{\partial z}{\partial y} &= b \cdot e^{ax+by} f(ax - by) + e^{ax+by} \cdot f'(ax - by) \cdot (-b) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= be^{ax+by} [f(ax-by) - f'(ax-by)] \\ \therefore b \frac{\partial z}{\partial y} + a \frac{\partial z}{\partial y} &= abe^{ax+by} [f(ax-by) + f'(ax-by)] + abe^{ax+by} [f(ax-by) - f'(ax-by)] \\ \text{or } b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} &= 2abe^{ax+by} \cdot f(ax-by) = 2abz. \quad \text{Hence proved.} \end{aligned}$$

64. If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, prove that : $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$. (M.D.U., 2007)

Sol.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x} = \frac{\partial u}{\partial r} \cdot \left(\frac{1}{y}\right) + \frac{\partial u}{\partial s} \cdot (0) + \frac{\partial u}{\partial t} \cdot \left(-\frac{z}{x^2}\right) \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{1}{y} - \frac{z}{x^2} \cdot \frac{\partial u}{\partial t} \\ \Rightarrow x \frac{\partial u}{\partial x} &= \frac{x}{y} \cdot \frac{\partial u}{\partial r} - \frac{z}{x} \cdot \frac{\partial u}{\partial t} \quad \dots(1) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = \frac{\partial u}{\partial r} \cdot \left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s} \cdot \left(\frac{1}{z}\right) + \frac{\partial u}{\partial t} \cdot (0) \\ \frac{\partial u}{\partial y} &= -\frac{x}{y^2} \cdot \frac{\partial u}{\partial r} + \frac{1}{z} \cdot \frac{\partial u}{\partial s} \\ \Rightarrow y \frac{\partial u}{\partial y} &= -\frac{x}{y} \cdot \frac{\partial u}{\partial r} + \frac{y}{z} \cdot \frac{\partial u}{\partial s} \quad \dots(2) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z} \\ \text{or } \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} \cdot (0) + \frac{\partial u}{\partial s} \cdot \left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t} \cdot \left(\frac{1}{x}\right) = -\frac{y}{z^2} \cdot \frac{\partial u}{\partial s} + \frac{1}{x} \cdot \frac{\partial u}{\partial t} \\ \Rightarrow z \frac{\partial u}{\partial z} &= -\frac{y}{z} \cdot \frac{\partial u}{\partial s} + \frac{z}{x} \cdot \frac{\partial u}{\partial t} \quad \dots(3) \end{aligned}$$

Adding (1), (2) and (3), we get

$$\begin{aligned} x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= \frac{x}{y} \frac{\partial u}{\partial r} - \frac{z}{x} \frac{\partial u}{\partial t} - \frac{x}{y} \frac{\partial u}{\partial r} + \frac{y}{z} \frac{\partial u}{\partial s} - \frac{y}{z} \frac{\partial u}{\partial s} + \frac{z}{x} \frac{\partial u}{\partial t} \\ &= 0. \end{aligned}$$

Hence proved.

65. If $u = x \log(xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Sol.

$$u = x \log(xy) \quad \dots(1)$$

$$\therefore \frac{\partial u}{\partial x} = x \left(\frac{1}{xy} \cdot y \right) + \log xy = 1 + \log xy$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x = \frac{x}{y}$$

Also, $x^3 + y^3 + 3xy = 1 \quad \dots(2)$

Differentiating (2), we get

$$3x^2 + 3y^2 \frac{dy}{dx} + 3 \left(x \frac{dy}{dx} + y \right) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = - \left(\frac{x^2 + y}{x + y^2} \right)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 1 + \log xy + \frac{x}{y} \cdot \left[-\left(\frac{x^2 + y}{x + y^2} \right) \right]$$

or
$$\frac{du}{dx} = 1 + \log xy - \frac{x(x^2 + y)}{y(x + y^2)}.$$

66. If $u = f(r, s)$, $r = x + y$, $s = x - y$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial r}.$

Sol.
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \dots(1) \quad \left| \because \frac{\partial r}{\partial x} = \frac{\partial s}{\partial x} = 1 \right|$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad \dots(2)$$

Adding (1) and (2), we get
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial r}.$$

67. If $u = x^2 - y^2 + \sin yz$, where $y = e^x$ and $z = \log x$, find $\frac{du}{dx}.$ (U.P.T.U., (AG) 2005)

Sol.
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx} = 2x + (-2y + z \cos yz) e^x + (y \cos zy) \frac{1}{x}$$

$$= 2x + [-2e^x + \log x \cos (e^x \log x)]e^x + \left\{ e^x \cos (e^x \log x) \cdot \frac{1}{x} \right\}$$

$$\Rightarrow \frac{du}{dx} = 2(x - e^{2x}) + e^x \cos (e^x \log x) \left(\log x + \frac{1}{x} \right).$$

68. If $v = f(r)$, $r^2 = x^2 + y^2 + z^2$, then $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r}$

(M.D.U., May 2008; U.P.T.U., 2009, A.U.U.P., 2007)

Sol.
$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now,
$$v = f(r)$$

$$\therefore \frac{\partial v}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r) = \frac{x}{r} \cdot \frac{\partial v}{\partial r}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{1}{r} f'(r) + x \cdot \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) f'(r) + \frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{1}{r^2} x \frac{\partial r}{\partial x} \cdot f'(r) + \frac{x}{r} f''(r) \frac{\partial r}{\partial x} = \frac{1}{r} f'(r) - \frac{1}{r^2} \cdot \frac{x^2}{r} \cdot f'(r) + \frac{x^2}{r^2} f''(r)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \quad \dots(1)$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2} = \frac{r^2 - y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \quad \dots(2)$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{r^2 - z^2}{r^3} f'(r) + \frac{z^2}{r^2} f''(r) \quad \dots(3)$$

Adding (1), (2) and (3), we get

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} &= \frac{f'(r)}{r^3} \{3r^2 - x^2 - y^2 - z^2\} + \frac{f''(r)}{r^2} (x^2 + y^2 + z^2) \\ &= f'(r) \cdot \frac{2r^2}{r^3} + f''(r) = \frac{2}{r} f'(r) + f''(r) \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} &= \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r} \end{aligned}$$

69. If z is a function of x and y , and u and v be two other variables such that $u = lx + my$, $v = ly - mx$, show

$$\text{that } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right).$$

Sol.

$$u = lx + my$$

$$\frac{\partial u}{\partial x} = l, \quad \frac{\partial u}{\partial y} = m, \quad \frac{\partial v}{\partial x} = -m, \quad \frac{\partial v}{\partial y} = l$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \left(l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \right) z$$

$$\Rightarrow \frac{\partial}{\partial x} = l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v}$$

$$\begin{aligned} \therefore \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \left(l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \right) \left(l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v} \right) \\ &= l^2 \frac{\partial^2 z}{\partial u^2} + m^2 \frac{\partial^2 z}{\partial v^2} - 2lm \frac{\partial^2 z}{\partial u \partial v} \quad \dots(1) \end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot m + l \frac{\partial z}{\partial v} = \left(m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v} \right) z$$

$$\Rightarrow \frac{\partial}{\partial y} = m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v}$$

$$\begin{aligned} \therefore \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \left(m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v} \right) \left(m \frac{\partial z}{\partial u} + l \frac{\partial z}{\partial v} \right) \\ &= m^2 \frac{\partial^2 z}{\partial u^2} + l^2 \frac{\partial^2 z}{\partial v^2} + 2lm \frac{\partial^2 z}{\partial u \partial v} \quad \dots(2) \end{aligned}$$

Adding (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = l^2 \frac{\partial^2 z}{\partial u^2} + m^2 \frac{\partial^2 z}{\partial v^2} + m^2 \frac{\partial^2 z}{\partial u^2} + l^2 \frac{\partial^2 z}{\partial v^2} = (l^2 + m^2) \frac{\partial^2 z}{\partial u^2} + (l^2 + m^2) \frac{\partial^2 z}{\partial v^2}$$

or
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right).$$

70. If $x = e^r \cos \theta$, $y = e^r \sin \theta$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r} \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right).$

Sol.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \left(\frac{1}{e^r \cos \theta} \right) + \frac{\partial u}{\partial \theta} \left(\frac{1}{-e^r \sin \theta} \right)$$

$$= \frac{1}{e^r \cos \theta} \frac{\partial u}{\partial r} - \frac{1}{e^r \sin \theta} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = \frac{1}{e^r} \left(\frac{1}{\cos \theta} \cdot \frac{\partial}{\partial r} - \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \right) u$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{1}{e^r} \left(\frac{1}{\cos \theta} \cdot \frac{\partial}{\partial r} - \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \right)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{1}{e^r} \left(\frac{1}{\cos \theta} \cdot \frac{\partial}{\partial r} - \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \right) \left[\frac{1}{e^r} \left(\frac{1}{\cos \theta} \cdot \frac{\partial}{\partial r} - \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \right) u \right] \\ &= \frac{1}{e^{2r}} \cdot \frac{1}{\cos^2 \theta} \frac{\partial^2 u}{\partial r^2} + \frac{1}{e^{2r}} \cdot \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{\sin \theta \cos \theta} \cdot \frac{\partial^2 u}{\partial r \partial \theta} \end{aligned} \quad \dots(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

As $y = e^r \sin \theta, \quad \frac{\partial y}{\partial \theta} = e^r \cos \theta, \quad \frac{\partial y}{\partial r} = e^r \sin \theta$

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{e^r \sin \theta} \cdot \frac{\partial u}{\partial r} + \frac{1}{e^r \cos \theta} \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{1}{e^r} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial r} + \frac{1}{\cos \theta} \cdot \frac{\partial}{\partial \theta} \right) u$$

$$\Rightarrow \frac{\partial}{\partial y} = \frac{1}{e^r} \left(\frac{1}{\sin \theta} \cdot \frac{\partial}{\partial r} + \frac{1}{\cos \theta} \cdot \frac{\partial}{\partial \theta} \right)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{1}{e^r} \left(\frac{1}{\sin \theta} \cdot \frac{\partial}{\partial r} + \frac{1}{\cos \theta} \cdot \frac{\partial}{\partial \theta} \right) \left[\frac{1}{e^r} \left(\frac{1}{\sin \theta} \cdot \frac{\partial u}{\partial r} + \frac{1}{\cos \theta} \cdot \frac{\partial u}{\partial \theta} \right) \right] \\ &= \frac{1}{e^{2r}} \left(\frac{1}{\sin^2 \theta} \cdot \frac{\partial^2 u}{\partial r^2} + \frac{1}{\cos^2 \theta} \cdot \frac{\partial^2 u}{\partial \theta^2} + \frac{2}{\sin \theta \cos \theta} \frac{\partial^2 u}{\partial r \partial \theta} \right) \end{aligned} \quad \dots(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{e^{2r}} \left[\frac{\partial^2 u}{\partial r^2} \cdot \left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \right) + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \right) \right] = \frac{1}{e^{2r}} \left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right].$$

71. By changing the independent variables x and y to u and v by means of the relations $u = x - ay$,

$v = x + ay$, show that $a^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ transforms into $4a^2 \frac{\partial^2 z}{\partial u \partial v}$.

Sol.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

or

$$\frac{\partial(z)}{\partial x} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) z$$

$\therefore$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2 \frac{\partial^2 z}{\partial u \partial v} \quad \dots(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -a \frac{\partial z}{\partial u} + a \frac{\partial z}{\partial v}$$

or

$$\frac{\partial z}{\partial y} = -a \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) = -a \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) z$$

$\therefore$

$$\frac{\partial}{\partial y} = -a \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right)$$

$\Rightarrow$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = a^2 \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) = a^2 \left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} - 2 \frac{\partial^2 z}{\partial u \partial v} \right] \quad \dots(2)$$

Multiplying (1) by a^2 , we have

$$a^2 \frac{\partial^2 z}{\partial x^2} = a^2 \left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2 \frac{\partial^2 z}{\partial u \partial v} \right]$$

$$\therefore a^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 4a^2 \frac{\partial^2 z}{\partial u \partial v} \quad \text{Hence shown.}$$

72. If u and v are functions of 2 independent variables x and y , then define Jacobian of u, v with respect to x, y . If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$ evaluate $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Sol. The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v w.r.t. x and y and is denoted by the

symbol $J \left(\frac{u, v}{x, y} \right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

Given

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2},$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\begin{aligned} \therefore \frac{\partial(r, \theta)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} \\ &= \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{\sqrt{x^2 + y^2}}. \end{aligned}$$

73. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$.

$$\text{Sol.} \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and $r \sin \theta$ from third column)

$$= r^2 \sin \theta \begin{vmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{vmatrix}$$

Expanding by third row

$$\begin{aligned} &= r^2 \sin \theta \left\{ \cos \theta \begin{vmatrix} \cos \theta \cos \phi & -\sin \phi \\ \cos \theta \sin \phi & \cos \phi \end{vmatrix} + \sin \theta \begin{vmatrix} \sin \theta \cos \phi & -\cos \theta \cos \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi \end{vmatrix} \right\} \\ &= r^2 \sin \theta [(\cos \theta \cos^2 \phi + \cos \theta \sin^2 \phi) \cos \theta + \sin \theta (\sin \theta \cos^2 \phi + \sin \theta \sin^2 \phi)] \\ &= r^2 \sin \theta [\cos^2 \theta + \sin^2 \theta] = r^2 \sin \theta. \end{aligned}$$

74. If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.

$$\text{Sol. Here,} \quad u = \frac{x}{y-z}, \quad v = \frac{y}{z-x}, \quad w = \frac{z}{x-y}$$

$$\Rightarrow \log u = \log x - \log(y-z) \quad \dots(1)$$

$$\log v = \log y - \log(z-x) \quad \dots(2)$$

$$\log w = \log z - \log(x-y) \quad \dots(3)$$

Differentiating (1) partially w.r.t. x

$$\frac{1}{u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x} \Rightarrow \frac{\partial u}{\partial x} = \frac{u}{x}$$

$$\text{Similarly, we have} \quad \frac{\partial u}{\partial y} = -\frac{u}{y-z}, \quad \frac{\partial u}{\partial z} = \frac{u}{y-z}$$

Similarly from (2) and (3), we have

$$\frac{\partial v}{\partial x} = \frac{v}{z-x}, \quad \frac{\partial v}{\partial y} = \frac{v}{y}, \quad \frac{\partial v}{\partial z} = -\frac{v}{z-x}$$

$$\frac{\partial w}{\partial x} = -\frac{w}{x-y}, \quad \frac{\partial w}{\partial y} = \frac{w}{x-y}, \quad \frac{\partial w}{\partial z} = \frac{w}{z}$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{u}{x} & -\frac{u}{y-z} & \frac{u}{y-z} \\ \frac{v}{z-x} & \frac{v}{y} & -\frac{v}{z-x} \\ -\frac{w}{x-y} & \frac{w}{x-y} & \frac{w}{z} \end{vmatrix} = uvw \begin{vmatrix} \frac{1}{x} & -\frac{1}{y-z} & \frac{1}{y-z} \\ \frac{1}{z-x} & \frac{1}{y} & -\frac{1}{z-x} \\ -\frac{1}{x-y} & \frac{1}{x-y} & \frac{1}{z} \end{vmatrix}$$

Multiplying R_1, R_2, R_3 by $y-z, z-x, x-y$ respectively

$$= \frac{uvw}{(y-z)(z-x)(x-y)} \begin{vmatrix} \frac{y-z}{x} & -1 & 1 \\ 1 & \frac{z-x}{y} & -1 \\ -1 & 1 & \frac{x-y}{z} \end{vmatrix}$$

Multiplying C_1, C_2, C_3 by x, y, z respectively, we get

$$= \frac{uvw}{xyz(y-z)(z-x)(x-y)} \begin{vmatrix} y-z & -y & z \\ x & z-x & -z \\ -x & y & x-y \end{vmatrix}$$

Operating $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \frac{1}{(y-z)^2(z-x)^2(x-y)^2} \begin{vmatrix} 0 & -y & z \\ 0 & z-x & -z \\ 0 & y & x-y \end{vmatrix} = 0 \quad \left[\because \frac{u}{x} = \frac{1}{y-z}, \frac{v}{y} = \frac{1}{z-x}, \frac{w}{z} = \frac{1}{x-y} \right]$$

Hence $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$

75. If $x = r \cos \theta$, $y = r \sin \theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Sol. $\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$

$\therefore \frac{\partial(x, y)}{\partial(r, \theta)} = r, \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}.$

76. If $x = a \cosh \alpha \cos \beta$, $y = a \sinh \alpha \sin \beta$, then show that $\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \frac{a^2}{2} [\cosh 2\alpha - \cos 2\beta]$

Sol. $\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{vmatrix} \quad \dots(1)$

Differentiating partially w.r.t. α and β , the equations

$$x = a \cosh \alpha \cos \beta \quad \dots(2)$$

and $y = a \sinh \alpha \sin \beta \quad \dots(3)$

We get $\frac{\partial x}{\partial \alpha} = a \sinh \alpha \cos \beta, \quad \frac{\partial x}{\partial \beta} = -a \cosh \alpha \sin \beta$

$$\frac{\partial y}{\partial \alpha} = a \cosh \alpha \sin \beta, \quad \frac{\partial y}{\partial \beta} = a \sinh \alpha \cos \beta$$

Putting in (1), we have

$$\begin{aligned} \frac{\partial(x, y)}{\partial(\alpha, \beta)} &= \begin{vmatrix} a \sinh \alpha \cos \beta & -a \cosh \alpha \sin \beta \\ a \cosh \alpha \sin \beta & a \sinh \alpha \cos \beta \end{vmatrix} = a^2(\sinh^2 \alpha \cos^2 \beta + \cosh^2 \alpha \sin^2 \beta) \\ &= a^2[(\cosh^2 \alpha - 1) \cos^2 \beta + \cosh^2 \alpha (1 - \cos^2 \beta)] \\ &= a^2[\cosh^2 \alpha \cos^2 \beta - \cos^2 \beta + \cosh^2 \alpha - \cosh^2 \alpha \cos^2 \beta] \\ &= a^2[\cosh^2 \alpha - \cos^2 \beta] = \frac{a^2}{2} [\cosh 2\alpha + 1 - 1 - \cos 2\beta] = \frac{a^2}{2} [\cosh 2\alpha - \cos 2\beta]. \end{aligned}$$

77. If $x = \sin \theta \sqrt{1 - c^2 \sin^2 \phi}$ and $y = \cos \theta \cos \phi$, then $\frac{\partial(x, y)}{\partial(\theta, \phi)} = -\sin \phi \frac{(1 - c^2) \cos^2 \theta + c^2 \cos^2 \phi}{\sqrt{1 - c^2 \sin^2 \phi}}.$

Sol. Differentiating partially x and y w.r.t. θ and ϕ respectively, we get

$$\frac{\partial x}{\partial \theta} = \cos \theta \sqrt{1 - c^2 \sin^2 \phi}, \quad \frac{\partial x}{\partial \phi} = \frac{\sin \theta [-2c^2 \sin \theta \cos \phi]}{2\sqrt{1 - c^2 \sin^2 \phi}}$$

$$\frac{\partial y}{\partial \theta} = -\sin \theta \cos \phi, \quad \frac{\partial y}{\partial \phi} = -\cos \theta \sin \phi$$

Now,
$$\begin{aligned} \frac{\partial(x, y)}{\partial(\theta, \phi)} &= \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos \theta \sqrt{1 - c^2 \sin^2 \phi} & \frac{\sin \theta (-2c^2 \sin \theta \cos \phi)}{2\sqrt{1 - c^2 \sin^2 \phi}} \\ -\sin \theta \cos \phi & -\cos \theta \sin \phi \end{vmatrix} \\ &= -\cos^2 \theta \sin \phi \sqrt{1 - c^2 \sin^2 \phi} - \frac{c^2 \sin^2 \theta \cos^2 \phi \sin \phi}{\sqrt{1 - c^2 \sin^2 \phi}} \\ &= \frac{-\sin \phi [\cos^2 \theta (1 - c^2 \sin^2 \phi) + c^2 \sin^2 \theta \cos^2 \phi]}{\sqrt{1 - c^2 \sin^2 \phi}} \\ &= \frac{-\sin \phi [\cos^2 \theta - c^2 \cos^2 \theta \sin^2 \phi + c^2 (1 - \cos^2 \theta) \cos^2 \phi]}{\sqrt{1 - c^2 \sin^2 \phi}} \\ &= \frac{-\sin \phi [\cos^2 \theta - c^2 \cos^2 \theta (1 - \cos^2 \phi) + c^2 \cos^2 \phi - c^2 \cos^2 \theta \cos^2 \phi]}{\sqrt{1 - c^2 \sin^2 \phi}} \\ &= \frac{-\sin \phi [(1 - c^2) \cos^2 \theta + c^2 \cos^2 \phi]}{\sqrt{1 - c^2 \sin^2 \phi}} \end{aligned}$$

78. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, then show that Jacobian $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$.

(U.P.T.U., 2005; M.D.U., 2009; A.U.U.P. 2008; V.T.U., 2007)

Sol.

$$\begin{aligned} y_1 &= \frac{x_2 x_3}{x_1}, \quad \frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}, \quad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \quad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1} \\ y_2 &= \frac{x_1 x_3}{x_2}, \quad \text{then } \frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \quad \frac{\partial y_2}{\partial x_2} = -\frac{x_1 x_3}{x_2^2} \quad \text{and} \quad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2} \\ y_3 &= \frac{x_1 x_2}{x_3}, \quad \text{then } \frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}, \quad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3} \quad \text{and} \quad \frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} &= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix} \\ &= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4. \quad \text{Hence shown.} \end{aligned}$$

79. If $y_1 = 1 - x_1$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1 x_2(1 - x_3)$, ... $y_n = x_1 x_2 x_3 \dots x_{n-1}(1 - x_n)$, then show that

$$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \cdot x_1^{n-1} \cdot x_2^{n-2} \dots x_{n-1}.$$

Sol. We know that

$$\begin{aligned} \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} &= \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \dots \frac{\partial y_n}{\partial x_n} \\ &= (-1)(-x_1)(-x_1 x_2) \dots (-x_1 x_2 \dots x_{n-1}) \\ &= (-1)^n \cdot x_1^{n-1} \cdot x_2^{n-2} \dots x_{n-1}. \end{aligned}$$

80. If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$ then show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Sol. Since we know that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3}$

Now,

$$\begin{aligned} \frac{\partial y_1}{\partial x_1} &= -\sin x_1 \\ \frac{\partial y_2}{\partial x_2} &= \sin x_1 (-\sin x_2) = -\sin x_1 \sin x_2 \\ \frac{\partial y_3}{\partial x_3} &= \sin x_1 \sin x_2 (-\sin x_3) = -\sin x_1 \sin x_2 \sin x_3 \end{aligned}$$

Hence
$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \cdot \sin^2 x_2 \cdot \sin x_3.$$

81. (i) If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find the Jacobian $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

(U.P.T.U., 2008; M.D.U., 2007; A.U.U.P., 2009)

- (ii) If $x + y + z = u$, $y + z = uv$, $z = uvw$ then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$.

Sol. (i) Let

$$\begin{aligned} F_1 &= u - xyz \\ F_2 &= v - x^2 - y^2 - z^2 \\ F_3 &= w - x - y - z \end{aligned}$$

We know that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \cdot \left[\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \bigg/ \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \right] \quad \dots(1)$$

$$\begin{aligned} \text{Now, } \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} &= \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \\ \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -yz & -zx & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix} = 2(x-y)(y-z)(z-x) \end{aligned}$$

$$\therefore \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{-1}{2(x-y)(y-z)(z-x)} \quad [\text{From (1)}]$$

$$\begin{aligned} (ii) \quad x + y + z &= u \\ y + z &= uv \\ z &= uvw \\ \Rightarrow x &= u - uv = u(1-v) \\ y &= uv - uvw = uv(1-w) \\ z &= uvw \end{aligned}$$

$$\therefore \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + (R_2 + R_3)$, the R.H.S. is

$$\begin{aligned} &= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \\ &= u^2v(1-w) + u^2vw = u^2v \end{aligned}$$

Hence shown.

82. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If so, find the relationship. (M.D.U., 2007)

Sol.
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence u, v are not independent and are functionally related.

$$\tan^{-1}x + \tan^{-1}y = v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\therefore v = \tan^{-1}u \Rightarrow u = \tan v$$

which is the required relationship.

83. If $u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Are u, v functionally related? If so, find the relationship. (U.P.T.U., 2006, 2008; M.D.U., 2009; A.U.U.P. 2008)

Sol. $u = \sin^{-1}x + \sin^{-1}y$... (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} \quad \dots (2)$$

$$\frac{\partial v}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}, \quad \frac{\partial v}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{-xy}{\sqrt{(1-y^2)(1-x^2)}} + 1 - 1 + \frac{xy}{\sqrt{(1-y^2)(1-x^2)}} = 0.$$

Hence u and v are not independent. They are functionally related.

From (1), we have $u = \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$$u = \sin^{-1}v \Rightarrow v = \sin u.$$

which is the required relationship between u and v .

84. Determine functional dependence and find relation between $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$.

Sol.
$$\frac{\partial u}{\partial x} = \frac{-2y}{(x-y)^2}, \quad \frac{\partial u}{\partial y} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{-y(x+y)}{(x-y)^3}, \quad \frac{\partial v}{\partial y} = \frac{x(x+y)}{(x-y)^3}$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{-y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = \frac{2(x+y)}{(x-y)^5} \begin{vmatrix} -y & x \\ -y & x \end{vmatrix} = 0$$

Hence the functional relationship exists between u and v .

Now,
$$u = \frac{x+y}{x-y} \Rightarrow \frac{u+1}{u-1} = \frac{2x}{2y} = \frac{x}{y} \Rightarrow x = y \left(\frac{u+1}{u-1} \right) \quad \dots(1)$$

$$v = \frac{xy}{(x-y)^2} = \frac{y^2 \left(\frac{u+1}{u-1} \right)}{y^2 \left[\frac{u+1}{u-1} - 1 \right]^2}$$

or
$$v = \left(\frac{u+1}{u-1} \right) \cdot \frac{(u-1)^2}{4} = \frac{u^2-1}{4} \Rightarrow u^2 - 4v = 1$$

which is the required relationship between u and v .

85. If $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$. Are u and v functionally related? If so, find their relationship.

Sol.

$$\frac{\partial u}{\partial x} = \frac{2y}{(x+y)^2}, \quad \frac{\partial u}{\partial y} = \frac{-2x}{(x+y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2}{x^2(x+y)^2} \begin{vmatrix} y & -x \\ -y & x \end{vmatrix} = 0$$

Hence functional relationship exists between u and v .

Now,
$$u = \frac{x-y}{x+y}, \quad \frac{u+1}{u-1} = \frac{2x}{-2y} = -\frac{x}{y}$$

$$\Rightarrow x = y \left(\frac{1+u}{1-u} \right) \quad \dots(1)$$

We have,
$$v = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1-u}{1+u}$$

or
$$v = \frac{2}{1+u}$$

which is the required relationship between u and v .

86. Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$.

Sol. Let $u = ax^2 + 2hxy + by^2$, $v = Ax^2 + 2Hxy + By^2$. u and v will not be independent if there exists a relationship between them and in that case $\frac{\partial(u, v)}{\partial(x, y)}$ should vanish identically.

$$\text{i.e.,} \quad \begin{vmatrix} 2(ax + hy) & 2(hx + by) \\ 2(Ax + Hy) & 2(Hx + By) \end{vmatrix} = 0$$

$$\text{or} \quad (ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$$

$$\text{or} \quad (aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2 = 0$$

Now, the variables x and y are independent and as such the coefficients of x^2 and y^2 should vanish separately.

$$\therefore \quad aH - Ah = 0 \quad \text{or} \quad \frac{a}{A} = \frac{h}{B}$$

$$\text{and} \quad Bh - bH = 0 \quad \text{or} \quad \frac{h}{H} = \frac{b}{B}$$

Hence $\frac{a}{A} = \frac{h}{H} = \frac{b}{B}$ and these conditions also make the coefficient of xy zero.

Hence $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$ are the required conditions.

87. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}.$$

$$\begin{aligned} \text{Sol. Let} \quad F_1 &= u^3 + v^3 + w^3 - x - y - z \\ F_2 &= u^2 + v^2 + w^2 - x^3 - y^3 - z^3 \\ F_3 &= u + v + w - x^2 - y^2 - z^2 \end{aligned}$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \cdot \frac{\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}} \quad \dots(i)$$

Now, consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} \quad \dots(ii)$$

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$

$$\begin{aligned} &= \begin{vmatrix} 0 & 0 & -1 \\ 3(z^2 - x^2) & 3(z^2 - y^2) & 3z^2 \\ 2(z - x) & 2(z - y) & -2z \end{vmatrix} = 6(z - x)(z - y) \begin{vmatrix} 0 & 0 & -1 \\ z + x & z + y & z^2 \\ 1 & 1 & -z \end{vmatrix} \\ &= 6(z - x)(z - y)(-1)(x - y) = 6(x - y)(y - z)(z - x) \end{aligned}$$

Now, consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix} \quad \dots(iii)$$

Applying $C_1 \rightarrow C_1 - C_3$, $C_2 \rightarrow C_2 - C_3$

$$= 6 \begin{vmatrix} u^2 - w^2 & v^2 - w^2 & 3w^2 \\ u - w & v - w & 2w \\ 0 & 0 & 1 \end{vmatrix} = 6(u - w)(v - w) \begin{vmatrix} u + w & v + w & 3w^2 \\ 1 & 1 & 2w \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6(u - w)(v - w)(u - v)$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \cdot \frac{6(x - y)(y - z)(z - x)}{6(u - v)(v - w)(u - w)} = \frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}.$$

88. If $u = x(1 - r^2)^{-\frac{1}{2}}$, $v = y(1 - r^2)^{-\frac{1}{2}}$, $w = z(1 - r^2)^{-\frac{1}{2}}$, where $r^2 = x^2 + y^2 + z^2$, then show that
- $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (1 - r^2)^{-5/2}. \quad (\text{U.P.T.U., 2008, 2009; A.U.U.P., 2007})$$

Sol. $u = x(1 - r^2)^{-\frac{1}{2}} = x[1 - x^2 - y^2 - z^2]^{-1/2}$

$$x^2 = u^2(1 - x^2 - y^2 - z^2)$$

$$y^2 = v^2(1 - x^2 - y^2 - z^2)$$

$$z^2 = w^2(1 - x^2 - y^2 - z^2)$$

Let $f_1 = x^2 - u^2(1 - x^2 - y^2 - z^2) = 0 \quad \dots(1)$

$$f_2 = y^2 - v^2(1 - x^2 - y^2 - z^2) = 0 \quad \dots(2)$$

$$f_3 = z^2 - w^2(1 - x^2 - y^2 - z^2) = 0 \quad \dots(3)$$

We know that,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} \quad \dots(4)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x(1 + u^2) & 2yu^2 & 2zu^2 \\ 2xv^2 & 2y(1 + v^2) & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z(1 + w^2) \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1 + u^2 & u^2 & u^2 \\ v^2 & 1 + v^2 & v^2 \\ w^2 & w^2 & 1 + w^2 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + (R_2 + R_3)$

$$\begin{aligned}
 &= 8xyz \begin{vmatrix} 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix} \\
 &= 8xyz(1+u^2+v^2+w^2) \begin{vmatrix} 1 & 1 & 1 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}
 \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$

$$= 8xyz(1+u^2+v^2+w^2) \begin{vmatrix} 1 & 0 & 0 \\ v^2 & 1 & 0 \\ w^2 & 0 & 1 \end{vmatrix} = 8xyz(1+u^2+v^2+w^2) \quad \dots(5)$$

Now, consider

$$\begin{aligned}
 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} &= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \\
 &= \begin{vmatrix} -2u(1-x^2-y^2-z^2) & 0 & 0 \\ 0 & -2v(1-x^2-y^2-z^2) & 0 \\ 0 & 0 & -2w(1-x^2-y^2-z^2) \end{vmatrix} \\
 &= -8uvw(1-x^2-y^2-z^2)^3 = -8uvw(1-r^2)^3 \quad \dots(6)
 \end{aligned}$$

Using (5) and (6) in (4), we get

$$\begin{aligned}
 \frac{\partial(u, v, w)}{\partial(x, y, z)} &= (-1)^3 \frac{8(1+u^2+v^2+w^2)xyz}{-8uvw(1-r^2)^3} \\
 &= \frac{\left[1 + \frac{x^2}{1-r^2} + \frac{y^2}{1-r^2} + \frac{z^2}{1-r^2}\right]xyz}{xyz(1-r^2)^{-3/2} \cdot (1-r^2)^3} = \frac{1 + \frac{r^2}{1-r^2}}{(1-r^2)^{3/2}} = \frac{1}{(1-r^2)^{5/2}} = (1-r^2)^{-5/2}.
 \end{aligned}$$

89. If u, v, w are the roots of the equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$, then find $\frac{\partial(u, v, w)}{\partial(a, b, c)}$.

(U.P.T.U., 2009)

Sol. u, v, w are the roots of the equation

$$(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$$

$$\text{i.e.,} \quad 3x^3 - 3x^2(a+b+c) + 3x(a^2+b^2+c^2) - (a^3+b^3+c^3) = 0$$

then

$$S_1 = u+v+w = \frac{3(a+b+c)}{3} = a+b+c$$

$$S_2 = uv+vw+wu = a^2+b^2+c^2$$

$$S_3 = uvw = \frac{a^3+b^3+c^3}{3}$$

$$\begin{aligned}
 \text{Let } F_1 &= u + v + w - a - b - c = 0 \\
 F_2 &= uv + vw + wu - a^2 - b^2 - c^2 = 0 \\
 F_3 &= uvw - \frac{1}{3}(a^3 + b^3 + c^3) = 0 \\
 \therefore \frac{\partial(u, v, w)}{\partial(a, b, c)} &= (-1)^3 \left[\frac{\partial(F_1, F_2, F_3)}{\partial(a, b, c)} \middle/ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \right] \quad \dots(1)
 \end{aligned}$$

$$\text{Now, } \frac{\partial(F_1, F_2, F_3)}{\partial(a, b, c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = -2(a-b)(b-c)(c-a)$$

$$\text{Also, } \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix} = -(u-v)(v-w)(w-u)$$

Hence from (1),

$$\frac{\partial(u, v, w)}{\partial(a, b, c)} = (-1)^3 \cdot \frac{[-2(a-b)(b-c)(c-a)]}{[-(u-v)(v-w)(w-u)]} = -\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}.$$

90. If u, v, w are the roots of the cubic $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ , find $J(u, v, w)$.

Sol. If u, v, w are the roots of the equation

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

$$\text{i.e., } 3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$$

$$\text{Then } S_1 = u + v + w = x + y + z; \quad S_2 = uv + vw + wu = x^2 + y^2 + z^2$$

$$S_3 = uvw = \frac{x^3 + y^3 + z^3}{3}$$

$$\text{Let } F_1 = u + v + w - x - y - z; \quad F_2 = uv + vw + wu - x^2 - y^2 - z^2$$

$$F_3 = uvw - \frac{x^3 + y^3 + z^3}{3}$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \cdot \frac{\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}} \quad \dots(1)$$

$$\text{Now, } \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$$\begin{aligned}
 &= (-2) \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (-2)(y-x)(z-x)(z+x-y-x) \\
 &= -2(y-x)(z-x)(z-y) = -2(x-y)(-z+y)(z-x)
 \end{aligned}$$

$$\text{Also, } \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} = \begin{vmatrix} u-v & u-w \\ w(u-v) & v(u-w) \end{vmatrix}$$

$$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u)$$

Hence,
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{-2(x-y)(-z+y)(z-x)}{-(u-v)(v-w)(w-u)} \cdot (-1)^3 = -\left[\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)} \right].$$

91. If $f(0) = 0$, $f'(x) = \frac{1}{1+x^2}$, $f'(y) = \frac{1}{1+y^2}$, prove that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ without using the method of integration.

Sol. Let $u = f(x) + f(y)$... (1)

$$v = \frac{x+y}{1-xy} \quad \dots (2)$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \dots (3)$$

$$\frac{\partial u}{\partial x} = f'(x) = \frac{1}{1+x^2}, \quad \frac{\partial u}{\partial y} = f'(y) = \frac{1}{1+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(1-xy) + (x+y)y}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}, \quad \frac{\partial v}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

Putting these values in (3), we get

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix} = \frac{1}{(1-xy)^2} \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{1+y^2} & \frac{1+x^2}{1+x^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \left\{ \frac{1+x^2}{1+x^2} - \frac{1+y^2}{1+y^2} \right\} = 0$$

Thus there is a relation between u and v .

Let $u = \phi(v)$, then $f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right)$

Putting $y = 0$, we get $f(x) + 0 = \phi(x) \Rightarrow \phi(x) = f(x) \quad [\because f(0) = 0]$

Hence function $\phi = f$.

Therefore, $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$.

Hence proved.

92. If $u = x + 2y + z$, $v = x - 2y + 3z$, $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u , v and w .

Sol. u , v and w will not be independent if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

$$\text{Hence, } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & (-x+4y-4z) \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - 2C_1$, $C_3 \rightarrow C_3 - C_1$

$$\begin{aligned} &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x-4y+6z & -x+2y-3z \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ 2x-4y+6z & -x+2y-3z \end{vmatrix} \\ &= -4(-x+2y-3z) - 2(2x-4y+6z) \\ &= 4x-8y+12z-4x+8y-12z = 0. \end{aligned}$$

Hence u , v and w are not independent.

$$\begin{aligned} \text{Now, } u &= x+2y+z, \quad v = x-2y+3z, \quad w = 2xy-xz+4yz-2z^2 \\ u+v &= 2x+4z, \quad u-v = 4y-2z \end{aligned}$$

Multiplying these, we get

$$(u+v)(u-v) = (2x+4z)(4y-2z) = 4(x+2z)(2y-z) = 4[2xy-xz+4yz-2z^2]$$

$$\text{or } (u+v)(u-v) = 4w$$

i.e., $u^2 - v^2 = 4w$ is the relation between u , v and w .

93. If $u = \frac{x+y}{z}$, $v = \frac{y+z}{x}$, $w = \frac{y(x+y+z)}{xz}$, then show that u , v , w are not independent. Find the relation between them. (U.P.T.U., 2008; A.U.U.P., 2009)

Sol. u , v and w will not be independent if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

$$\begin{aligned} \text{Now, } \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{x+y}{z^2} \\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ \frac{-y^2-yz}{x^2z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix} \\ &= \frac{1}{x^4z^4} \begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -yz(y+z) & xz(x+2y+z) & -xy(x+y) \end{vmatrix} \end{aligned}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\begin{aligned} &= \frac{1}{x^4z^4} \begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & x+y+z & x+y+z \\ -yz(y+z) & z(x+y)(x+y+z) & y(z-x)(x+y+z) \end{vmatrix} \\ &= \frac{(x+y+z)^2}{x^4z^4} \begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & y(z-x) \end{vmatrix} \end{aligned}$$

Applying $C_1 \rightarrow C_1 + zC_3$

$$\begin{aligned}
 &= \frac{(x+y+z)^2}{x^4 z^4} \begin{vmatrix} 0 & 0 & -1 \\ -y & 1 & 1 \\ -yz(y+x) & z(x+y) & y(z-x) \end{vmatrix} \\
 &= -\frac{(x+y+z)^2}{x^4 z^4} \begin{vmatrix} -y & 1 \\ -yz(y+x) & z(x+y) \end{vmatrix} \\
 &= -\frac{(x+y+z)^2}{x^4 z^4} [-yz(x+y) + yz(y+x)] = 0
 \end{aligned}$$

Hence, u , v and w are not independent.

Now, $uv = \frac{x+y}{z} \cdot \frac{y+z}{x} = \frac{xy+xz+y^2+yz}{xz} = 1 + \frac{xy+y^2+yz}{xz} = 1 + \frac{y(x+y+z)}{xz}$

$\therefore uv = w + 1$ is the required relation between them.

94. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (M.D.U., 2007; U.P.T.U., 2008)

Sol.

$$\begin{aligned}
 \frac{\partial u}{\partial x} &= -\frac{y^2}{2x^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x} \\
 \frac{\partial v}{\partial x} &= \frac{-(x^2+y^2)2 + (2x)(2x)}{4x^2} = \frac{2(x^2-y^2)}{4x^2} = \frac{x^2-y^2}{2x^2} \\
 \frac{\partial v}{\partial y} &= \frac{1}{2x} \cdot 2y = \frac{y}{x}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} -\frac{y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix} = \frac{1}{4x^4} \begin{vmatrix} -y^2 & 2xy \\ x^2-y^2 & 2xy \end{vmatrix} = \frac{2xy}{4x^4} \begin{vmatrix} -y^2 & 1 \\ x^2-y^2 & 1 \end{vmatrix} \\
 &= \frac{y}{2x^3} (-y^2 - x^2 + y^2) = \frac{-x^2 y}{2x^3} = -\frac{y}{2x}
 \end{aligned}$$

95. In polar co-ordinates, $x = r \cos \theta$, $y = r \sin \theta$. Show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$.

Sol.

$$\begin{aligned}
 \frac{\partial x}{\partial r} &= \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta \\
 \frac{\partial y}{\partial r} &= \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta \\
 \therefore \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta) = r.
 \end{aligned}$$

96. If $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Sol.

$$\begin{aligned}
 \frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\
 &= r(\cos^2 \theta + \sin^2 \theta) = r.
 \end{aligned}$$

97. If $u = x(1 - y)$, $v = xy$, prove that $JJ' = 1$.

Sol. $J = \frac{\partial(u, v)}{\partial(x, y)}, \quad J' = \frac{\partial(x, y)}{\partial(u, v)}$

Now, $J = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(1-y) + xy = x$

$$J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Given $u = x(1 - y), \quad v = xy$

$\therefore u + v = x, \quad y = \frac{v}{x} = \frac{v}{u+v}$

$\therefore \frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = -\frac{v}{(u+v)^2}, \quad \frac{\partial y}{\partial v} = -\frac{u}{(u+v)^2}$

Substituting these values in J' , we get

$$J' = \begin{vmatrix} 1 & 1 \\ -\frac{v}{(u+v)^2} & -\frac{u}{(u+v)^2} \end{vmatrix} = \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x} \quad \therefore JJ' = x \cdot \frac{1}{x} = 1.$$

98. If $u = 2xy$, $v = x^2 - y^2$, $x = r \cos \theta$, $y = r \sin \theta$, evaluate $\frac{\partial(u, v)}{\partial(r, \theta)}$.

Sol. $\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$

Now, $u = 2xy, \quad \frac{\partial u}{\partial x} = 2y, \quad \frac{\partial u}{\partial y} = 2x$

$$v = x^2 - y^2, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2y$$

$$x = r \cos \theta, \quad \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$y = r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2 + y^2)$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

$\therefore \frac{\partial(u, v)}{\partial(r, \theta)} = -4(x^2 + y^2) \cdot r = -4r^3.$

99. State Taylor's Theorem for a function $f(x, y)$ of 2 independent variables x and y .

Sol. $f(x + h, y + k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \dots$

where, $f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and so on.

100. Expand $e^x \sin y$ in powers of x and y , as far as terms of the 3rd degree.

Sol. Here

$f(x, y) = e^x \sin y,$	$f(0, 0) = 0$
$f_x(x, y) = e^x \sin y,$	$f_x(0, 0) = 0$
$f_y(x, y) = e^x \cos y,$	$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \sin y,$	$f_{xx}(0, 0) = 0$
$f_{xy}(x, y) = e^x \cos y,$	$f_{xy}(0, 0) = 1$
$f_{yy}(x, y) = -e^x \sin y,$	$f_{yy}(0, 0) = 0$
$f_{xxx}(x, y) = e^x \sin y,$	$f_{xxx}(0, 0) = 0$
$f_{xxy}(x, y) = e^x \cos y,$	$f_{xxy}(0, 0) = 1$
$f_{xyy}(x, y) = -e^x \sin y,$	$f_{xyy}(0, 0) = 0$
$f_{yyy}(x, y) = -e^x \cos y,$	$f_{yyy}(0, 0) = -1$

$$\therefore e^x \sin y = f(x, y)$$

$$\begin{aligned}
 &= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\
 &\quad + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots \\
 &= 0 + [x \cdot 0 + y \cdot 1] + \frac{1}{2!} [x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0] + \frac{1}{3!} [x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot 0 + y^3 \cdot (-1)] + \dots \\
 &= y + xy + \frac{1}{2} x^2 y - \frac{1}{6} y^3 + \dots
 \end{aligned}$$

101. Expand $f(x, y) = \tan^{-1} \frac{y}{x}$ in the neighbourhood of $(1, 1)$ upto third degree terms. Hence compute $f(1.1, 0.9)$ approximately. (M.D.U., Dec., 2007)

Sol. Here, $f(x, y) = \tan^{-1} \frac{y}{x}$, $f(1, 1) = \tan^{-1}(1) = \frac{\pi}{4}$

$$f_x(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}, \quad f_x(1, 1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}, \quad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = -y(-1)(x^2 + y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2 + y^2)^2}, \quad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = x(-1)(x^2 + y^2)^{-2} \cdot (2y) = -\frac{2xy}{(x^2 + y^2)^2}, \quad f_{yy}(1, 1) = -\frac{1}{2}$$

$$f_{xxx}(x, y) = 2y \cdot \frac{(x^2 + y^2)^2 \cdot 1 - x(x^2 + y^2) \cdot 2 \cdot 2x}{(x^2 + y^2)^4}$$

$$f_{xxx}(x, y) = 2y \cdot \frac{(x^2 + y^2) - 4x^2}{(x^2 + y^2)^3} = \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$\begin{aligned}
\Rightarrow f_{xxx}(1, 1) &= -\frac{1}{2} \\
f_{xxy}(x, y) &= \frac{(x^2 + y^2)^2(-2x) - (y^2 - x^2) \cdot 2(x^2 + y^2) \cdot 2x}{(x^2 + y^2)^4} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}, \\
\therefore f_{xxy}(1, 1) &= -\frac{1}{2} \\
f_{xyy}(x, y) &= -2y \cdot \frac{(x^2 + y^2)^2 \cdot 1 - x \cdot 2(x^2 + y^2)(2x)}{(x^2 + y^2)^4} = \frac{-2y \cdot (x^2 + y^2 - 4x^2)}{(x^2 + y^2)^3} = \frac{-2y(y^2 - 3x^2)}{(x^2 + y^2)^3} \\
\therefore f_{xyy}(1, 1) &= \frac{-2 \cdot 1 \cdot (-2)}{8} = \frac{1}{2} \\
f_{yyy}(x, y) &= -2x \cdot \frac{(x^2 + y^2)^2 \cdot 1 - y(x^2 + y^2) \cdot 2 \cdot 2y}{(x^2 + y^2)^4} \\
\text{or } f_{yyy}(x, y) &= \frac{-2x(x^2 + y^2 - 4y^2)}{(x^2 + y^2)^3} = \frac{-2x(x^2 - 3y^2)}{(x^2 + y^2)^3} = \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3} \\
f_{yyy}(1, 1) &= \frac{1}{2} \\
\therefore \tan^{-1}\left(\frac{y}{x}\right) &= f(x, y) \\
&= f(1, 1) + [(x-1)f_x(1, 1) + (y-1)f_y(1, 1)] + \frac{1}{2!} [(x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1)f_{xy}(1, 1) \\
&\quad + (y-1)^2 f_{yy}(1, 1)] + \frac{1}{3!} [(x-1)^3 f_{xxx}(1, 1) + 3(x-1)^2(y-1)f_{xxy}(1, 1) \\
&\quad + 3(x-1)(y-1)^2 f_{xyy}(1, 1) + (y-1)^3 f_{yyy}(1, 1)] + \dots \\
&= \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2} \right] + \frac{1}{2} \left[(x-1)^2 \cdot \frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2 \left(-\frac{1}{2}\right) \right] \\
&\quad + \frac{1}{6} \left[(x-1)^3 \left(-\frac{1}{2}\right) + 3(x-1)^2(y-1) \left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2 \cdot \frac{1}{2} + (y-1)^3 \left(\frac{1}{2}\right) \right] + \dots \\
&= \frac{\pi}{4} - \frac{1}{2} [(x-1) - (y-1)] + \frac{1}{4} [(x-1)^2 - (y-1)^2] \\
&\quad - \frac{1}{12} [(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3] + \dots
\end{aligned}$$

Putting $x = 1.1$ and $y = 0.9$

$$\begin{aligned}
f(1.1, 0.9) &= \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}[0.1]^3 + 3(0.1)^2(-0.1) - 3(0.1)(-0.1)^2 - (0.1)^3 \\
&= 0.7854 - 0.1 + .0003 = 0.6857.
\end{aligned}$$

- 102.** Find the first six terms of the expansion of the function $e^x \log(1+y)$ in a Taylor's series in the neighbourhood of the point $(0, 0)$. (M.D.U., May 2008)

Sol. Here $f(x, y) = e^x \log(1+y), \quad f(0, 0) = 0$

$f_x(x, y) = e^x \log(1+y), \quad f_x(0, 0) = 0$

$f_y(x, y) = \frac{e^x}{1+y}, \quad f_y(0, 0) = 1$

$$f_{xx}(x, y) = e^x \log(1 + y), \quad f_{xx}(0, 0) = 0$$

$$f_{xy}(x, y) = \frac{e^x}{1 + y}, \quad f_{xy}(0, 0) = 1$$

$$f_{yy}(x, y) = -\frac{e^x}{(1 + y)^2}, \quad f_{yy}(0, 0) = -1$$

$$f_{xxx}(x, y) = e^x \log(1 + y), \quad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = \frac{e^x}{1 + y}, \quad f_{xxy}(0, 0) = 1$$

$$f_{xyy}(x, y) = -\frac{e^x}{(1 + y)^2}, \quad f_{xyy}(0, 0) = -1$$

$$f_{yyy}(x, y) = \frac{2e^x}{(1 + y)^3}, \quad f_{yyy}(0, 0) = 2$$

$$\therefore e^x \log(1 + y) = f(x, y)$$

$$= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots$$

$$= 0 + [x \cdot 0 + y \cdot 1] + \frac{1}{2} [x^2 \cdot 0 + 2xy \cdot 1 + y^2(-1)] + \frac{1}{6} [x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2(-1) + 2y^3] + \dots$$

$$= y + xy - \frac{1}{2} y^2 + \frac{1}{2} x^2 y - \frac{1}{2} xy^2 + \frac{1}{3} y^3 \dots$$

103. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $(y + 2)$ using Taylor's Theorem. (M.D.U., Dec., 2006)

Sol. Expansion of $f(x, y)$ in powers of $x - a$ and $y - b$ is given by

$$\begin{aligned} f(x, y) = & f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] + \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) \\ & + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!} [(x - a)^3 f_{xxx}(a, b) \\ & + 3(x - a)^2(y - b)f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)] \\ & + \dots \end{aligned} \quad \dots(1)$$

Here, $f(x, y) = x^2y + 3y - 2$, $a = 1$, $b = -2$

$$f(1, -2) = -2 - 6 - 2 = -10$$

$$f_x = 2xy, \quad f_x(1, -2) = -4; \quad f_y = x^2 + 3, \quad f_y(1, -2) = 4,$$

$$f_{xx} = 2y, \quad f_{xx}(1, -2) = -4; \quad f_{xy} = 2x, \quad f_{xy}(1, -2) = 2,$$

$$f_{yy} = 0, \quad f_{yy}(1, -2) = 0; \quad f_{xxx} = 0, \quad f_{xxx}(1, -2) = 0$$

$$f_{xxy} = 2, \quad f_{xxy}(1, -2) = 2, \quad f_{xyy} = 0, \quad f_{xyy}(1, -2) = 0$$

$$f_{yyy} = 0, \quad f_{yyy}(1, -2) = 0$$

$\therefore$ From (1), we have

$$x^2y + 3y - 2 = f(x, y)$$

$$= -10 + [(x - 1)(-4) + (y + 2)4] + \frac{1}{2} [(x - 1)^2(-4) + 2(x - 1)(y + 2)(2)$$

$$+ (y + 2)^2(0)] + \frac{1}{6} [(x - 1)^3 \cdot 0 + 3(x - 1)^2(y + 2)(2) + 3(x - 1)(y + 2)^2(0) + (y + 2)^3 \cdot 0]$$

$$= -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2).$$

104. Expand $e^{ax} \sin by$ in powers of x and y as far as the terms of third degree.

$$\begin{array}{ll} \text{Sol.} & f(x, y) = e^{ax} \sin by, & f(0, 0) = 0 \\ & f_x(x, y) = ae^{ax} \sin by, & f_x(0, 0) = 0 \\ & f_y(x, y) = be^{ax} \cos by, & f_y(0, 0) = b \\ & f_{xx}(x, y) = a^2 e^{ax} \sin by, & f_{xx}(0, 0) = 0 \\ & f_{xy}(x, y) = abe^{ax} \cos by, & f_{xy}(0, 0) = ab \\ & f_{yy}(x, y) = -b^2 e^{ax} \sin by, & f_{yy}(0, 0) = 0 \\ & f_{xxx}(x, y) = a^3 e^{ax} \sin by, & f_{xxx}(0, 0) = 0 \\ & f_{xxy}(x, y) = a^2 be^{ax} \cos by, & f_{xxy}(0, 0) = a^2 b \\ & f_{xyy}(x, y) = ae^{ax} \cdot (-b^2 \sin by), & f_{xyy}(0, 0) = 0 \\ & f_{yyy}(x, y) = -b^3 e^{ax} \cos by, & f_{yyy}(0, 0) = -b^3 \end{array}$$

Now,
 $e^{ax} \sin by = f(x, y)$

$$\begin{aligned} &= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) \\ &\quad + y^2 f_{yy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0) + \dots] \\ &= 0 + [x \cdot 0 + yb] + \frac{1}{2!} [x^2 \cdot 0 + 2xy \cdot ab + y^2 \cdot 0] + \frac{1}{3!} [x^3 \cdot 0 + 3x^2 y \cdot a^2 b + 3xy^2 \cdot 0 + y^3 \cdot (-b^3)] + \dots \\ &= by + abxy + \frac{1}{3!} (3a^2 bx^2 y - b^3 y^3) + \dots \end{aligned}$$

105. Expand e^{xy} at $(1, 1)$ in powers of $(x - 1)$ and $(y - 1)$.

Sol. Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ is given by

$$\begin{aligned} f(x, y) &= f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] \\ &\quad + \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + (y - b)^2 f_{yy}(a, b) + 2(x - a)(y - b) f_{xy}(a, b)] + \dots \end{aligned}$$

Now, $f(x, y) = e^{xy}$, $a = 1$, $b = 1$

$$\begin{array}{ll} f(1, 1) = e^1 & \text{or } e \\ f_x(x, y) = ye^{xy}, & f_x(1, 1) = e \\ f_y(x, y) = xe^{xy}, & f_y(1, 1) = e \\ f_{xx}(x, y) = y^2 e^{xy}, & f_{xx}(1, 1) = e \\ f_{yy}(x, y) = x^2 e^{xy}, & f_{yy}(1, 1) = e \\ f_{xy}(x, y) = xye^{xy}, & f_{xy}(1, 1) = e \end{array}$$

$$\begin{aligned} \therefore e^{xy} &= e + [(x - 1)e + (y - 1)e] + \frac{1}{2!} [(x - 1)^2 e + (y - 1)^2 e + 2(x - 1)(y - 1)e] + \dots \\ &= e \left[1 + (x - 1) + (y - 1) + \frac{1}{2!} [(x - 1)^2 + (y - 1)^2 + 2(x - 1)(y - 1)] \right] + \dots \end{aligned}$$

106. Expand $\sin xy$ in powers of $(x - 1)$ and $\left(y - \frac{\pi}{2}\right)$ upto the second degree terms.

(M.D.U., Dec., 2005)

Sol. Here, $f(x, y) = \sin xy$, $a = 1$, $b = \frac{\pi}{2}$

$$f\left(1, \frac{\pi}{2}\right) = 1$$

$$\begin{aligned}
f_x(x, y) &= y \cos xy, & f_x\left(1, \frac{\pi}{2}\right) &= \frac{\pi}{2} \cos \frac{\pi}{2} = 0 \\
f_y(x, y) &= x \cos xy, & f_y\left(1, \frac{\pi}{2}\right) &= \cos \frac{\pi}{2} = 0 \\
f_{xx}(x, y) &= -y^2 \sin xy, & f_{xx}\left(1, \frac{\pi}{2}\right) &= -\frac{\pi^2}{4} \\
f_{xy}(x, y) &= -xy \sin xy, & f_{xy}\left(1, \frac{\pi}{2}\right) &= -\frac{\pi}{2} \\
f_{yy}(x, y) &= -x^2 \sin xy, & f_{yy}\left(1, \frac{\pi}{2}\right) &= -1
\end{aligned}$$

Taylor's expansion of $f(x, y)$ in powers of $(x - 1)$ and $\left(y - \frac{\pi}{2}\right)$ is given by

$$\begin{aligned}
f(x, y) &= f\left(1, \frac{\pi}{2}\right) + (x - 1)f_x\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)f_y\left(1, \frac{\pi}{2}\right) \\
&\quad + \frac{1}{2!}\left[(x - 1)^2 f_{xx}\left(1, \frac{\pi}{2}\right) + 2(x - 1)\left(y - \frac{\pi}{2}\right)f_{xy}\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\left(1, \frac{\pi}{2}\right)\right] + \dots \\
&= 1 + (x - 1) \cdot 0 + \left(y - \frac{\pi}{2}\right) \cdot 0 + \frac{1}{2}\left[(x - 1)^2 \left(-\frac{\pi^2}{4}\right) + 2(x - 1)\left(y - \frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 (-1)\right] + \dots \\
&= 1 - \frac{1}{8}\pi^2(x - 1)^2 - \frac{\pi}{2}(x - 1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.
\end{aligned}$$

107. Expand $e^x \cos y$ at $\left(1, \frac{\pi}{4}\right)$.

(U.P.T.U., (AG) 2005)

Sol. $f(x, y) = e^x \cos y, \quad a = 1, \quad b = \frac{\pi}{4}$

$$f\left(1, \frac{\pi}{4}\right) = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$$

$$f_x(x, y) = e^x \cdot \cos y, \quad f_x\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_y(x, y) = -e^x \sin y, \quad f_y\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{xx}(x, y) = e^x \cos y, \quad f_{xx}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_{xy}(x, y) = -e^x \sin y, \quad f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{yy}(x, y) = -e^x \cos y, \quad f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$\begin{aligned}
f(x, y) &= f\left(1, \frac{\pi}{4}\right) + (x - 1)f_x\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)f_y\left(1, \frac{\pi}{4}\right) \\
&\quad + \frac{1}{2!}\left[(x - 1)^2 f_{xx}\left(1, \frac{\pi}{4}\right) + 2(x - 1)\left(y - \frac{\pi}{4}\right)f_{xy}\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2 f_{yy}\left(1, \frac{\pi}{4}\right)\right] + \dots
\end{aligned}$$

$$e^x \cos y = \frac{e}{\sqrt{2}} \left[1 + (x-1) - \left(y - \frac{\pi}{4} \right) + \frac{1}{2} (x-1)^2 - (x-1) \left(y - \frac{\pi}{4} \right) - \frac{1}{2} \left(y - \frac{\pi}{4} \right)^2 + \dots \right]$$

108. If $f(x, y) = \tan^{-1}(xy)$, compute $f(0.9, -1.2)$ approximately.

[M.D.U., May 2009]

Sol. $f(x, y) = \tan^{-1}(xy)$

$$f(0.9, -1.2) = f(1 - 0.1, -1 - 0.2)$$

So let us expand $f(x, y)$ near the point $(1, -1)$

$$f(x, y) = \tan^{-1}(xy), \quad f(1, -1) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$f_x = \frac{y}{1+x^2y^2}, \quad f_x(1, -1) = -\frac{1}{2}$$

$$f_y = \frac{x}{1+x^2y^2}, \quad f_y(1, -1) = \frac{1}{2}$$

$$f_{xx} = \frac{-2xy^3}{(1+x^2y^2)^2}, \quad f_{xx}(1, -1) = \frac{1}{2}$$

$$f_{yy} = \frac{-2x^3y}{(1+x^2y^2)^2}, \quad f_{yy}(1, -1) = +\frac{1}{2}$$

$$f_{xy} = \frac{1+x^2y^2-x(2xy^2)}{(1+x^2y^2)^2}, \quad f_{xy}(1, -1) = 0$$

Taylor's Theorem for a function of two independent variables x and y is

$$f(x+h, y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!} \{ h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} \} + \dots$$

Put, $x = 1, y = -1, h = -0.1, k = -0.2$.

$$\begin{aligned} \Rightarrow f(0.9, -1.2) &= f(1, -1) + \left[(-0.1) \left(-\frac{1}{2} \right) - (0.2) \left(\frac{1}{2} \right) \right] + \frac{1}{2!} \left[(0.01) \left(\frac{1}{2} \right) + 2 \cdot (0.02) (0) + (0.04) \left(+\frac{1}{2} \right) \right] \\ &= -\frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2} (0.005 + 0.02) = -\frac{\pi}{4} + 0.05 - 0.1 + 0.0125 \\ &= -0.786 + 0.05 - 0.1 + 0.0125 \\ &= -0.786 - 0.0375 = -0.8235 \end{aligned}$$

109. Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.

(U.P.T.U., 2005)

Sol. Here, $f(x, y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay, \quad f_y = 3y^2 - 3ax$$

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3a$$

$$\text{Now, } f_x = 0 \text{ and } f_y = 0 \quad \Rightarrow \quad 3x^2 - 3ay = 0 \quad \dots(1)$$

$$\Rightarrow \quad x^2 = ay \quad \text{or} \quad y = \frac{x^2}{a} \quad \text{and} \quad 3y^2 - 3ax = 0 \quad \dots(2)$$

$$\Rightarrow \quad y^2 = ax$$

$$\frac{x^4}{a^2} = ax \quad \text{or} \quad x^4 - a^3x = 0 \quad \text{or} \quad x(x^3 - a^3) = 0 \quad \Rightarrow \quad x = 0, a.$$

When $x = 0, y = 0$; when $x = a, y = a$

$\therefore$ There are 2 stationary points $(0, 0)$ and (a, a) .

Now, $f_{xx} = r = 6x$, $f_{xy} = s = -3a$, $f_{yy} = t = 6y$

$\therefore rt - s^2 = 36xy - 9a^2$

At $(0, 0)$, $rt - s^2 = -9a^2 < 0$

$\Rightarrow$ there is no extreme value at $(0, 0)$

At (a, a) , $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$.

$\Rightarrow f(x, y)$ has extreme value at (a, a) .

Now, $r = 6a$

If $a > 0$, $r > 0$ so that $f(x, y)$ has a minimum value at (a, a) .

Minimum value $= f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

If $a < 0$, $r < 0$ so that $f(x, y)$ has a maximum value at (a, a) .

Maximum value $= f(a, a) = a^3 + a^3 - 3a^3 = -a^3$.

110. Discuss the maxima and minima of $x^3y^2(1-x-y)$.

Sol. Here, $f(x, y) = x^3y^2(1-x-y)$

or $f(x, y) = x^3y^2 - x^4y^2 - x^3y^3$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2yx^4 - 3y^2x^3$$

$$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$

$$t = f_{yy} = 2x^3 - 2x^4 - 6yx^3$$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0 \quad \dots(1) \quad \text{and} \quad x^3y(2-2x-3y) = 0 \quad \dots(2)$

Solving (1) and (2), the stationary points are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Now, $rt - s^2 = (6xy^2 - 12x^2y^2 - 6xy^3)(2x^3 - 2x^4 - 6x^3y) - [x^2y(6 - 8x - 9y)]^2$
 $= 12x^4y^2(1-2x-y)(1-x-3y) - x^4y^2(6-8x-9y)^2$

At $(0, 0)$, $rt - s^2 = 0$

$\therefore$ at $(0, 0)$ further investigation is needed.

Consider $f(a+h, b+k) - f(a, b) = f(h, k) - f(0, 0)$

$$= h^3k^2(1-h-k)$$

$$= h^3k^2 \quad (\text{Neglecting } h^4k^2 + h^3k^3 \text{ which is very small as compared to } h^3k^2)$$

which is > 0 if $h > 0$ and < 0 if $h < 0$.

$\therefore f(a+h, b+k) - f(a, b)$ does not keep same sign for all small values of h and k (positive or negative).

$\therefore$ There is no extreme value at $(0, 0)$.

At $\left(\frac{1}{2}, \frac{1}{3}\right)$, $rt - s^2 = 12 \cdot \frac{1}{16} \cdot \frac{1}{9} \left(-\frac{1}{3}\right) \left(-\frac{1}{2}\right) - \frac{1}{16} \cdot \frac{1}{9} (1) = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$

Also, $r = 6xy^2(1-2x-y) = 6 \cdot \frac{1}{2} \cdot \frac{1}{9} \left(1-1-\frac{1}{3}\right) = -\frac{1}{9} < 0$

$\therefore f(x, y)$ has a maximum value at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Maximum value $= \frac{1}{8} \cdot \frac{1}{9} \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

111. Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature.

Sol. Here, $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$
 $f_x = 4x^3 - 4x + 4y, \quad f_y = 4y^3 + 4x - 4y, \quad r = f_{xx} = 12x^2 - 4$
 $s = f_{xy} = 4, \quad t = f_{yy} = 12y^2 - 4$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow x^3 - x + y = 0 \quad \dots(1) \quad y^3 + x - y = 0 \quad \dots(2)$

Adding (1) and (2), we get $x^3 + y^3 = 0 \Rightarrow y = -x$

Putting $y = -x$ in (1), we get $x^3 - 2x = 0$ or $x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$

Since $y = -x$, the stationary points are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

Now, $rt - s^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(0, 0), \quad rt - s^2 = 0$

$\therefore$ At $(0, 0)$, further investigation is needed.

Now, $f(0, 0) = 0$

$f(x, y) = x^4 + y^4 - 2(x^2 - 2xy + y^2) = x^4 + y^4 - 2(x - y)^2$

When h and k are small,

$f(h, k) = 2h^4 > 0$ for $h = k$

and $f(h, k) = -2(h - k)^2 < 0$ for $h \neq k$ (Neglecting $h^4 + k^4$ which is very small)

$\Rightarrow f(0, 0) < f(h, k)$ for $h = k$

$f(0, 0) > f(h, k)$ for $h \neq k$

$\Rightarrow$ There is no extreme value at $(0, 0)$.

At $(\sqrt{2}, -\sqrt{2}), \quad rt - s^2 = (12 \times 2 - 4)^2 - 16 = 384 > 0$
 $r = 20 > 0$

$\therefore f(x, y)$ has a minimum value at $(\sqrt{2}, -\sqrt{2})$

Minimum value $= 4 + 4 - 4 - 8 - 4 = -8$

Similarly, $f(x, y)$ has a minimum value -8 at $(-\sqrt{2}, \sqrt{2})$.

112. Examine $\sin x + \sin y + \sin(x + y)$ for minimum and maximum values. (U.P.T.U., 2008)

Sol. Here, $f(x, y) = \sin x + \sin y + \sin(x + y)$

$f_x = \cos x + \cos(x + y), \quad f_y = \cos y + \cos(x + y)$

$r = f_{xx} = -\sin x - \sin(x + y), \quad s = f_{xy} = -\sin(x + y)$

$t = f_{yy} = -\sin y - \sin(x + y)$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow \cos x + \cos(x + y) = 0 \quad \dots(1) \quad \cos y + \cos(x + y) = 0 \quad \dots(2)$

Subtracting (2) from (1), $\cos x - \cos y = 0$ or $\cos x = \cos y \therefore x = y$

From (1), $\cos x + \cos 2x = 0$ or $\cos 2x = -\cos x$

$\cos 2x = \cos(\pi - x) \Rightarrow 2x = \pi - x$

$\therefore x = \frac{\pi}{3}$

$\therefore x = y = \frac{\pi}{3}$ is a stationary point.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), \quad r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0, \quad s = \frac{\sqrt{3}}{2}, \quad t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

Also $r < 0$

$\therefore f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\text{Maximum value} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

- 113.** A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction. (U.P.T.U., 2006)

Sol. Let x, y and z be the length, breadth and height respectively. Let V be the given capacity and S , the surface. V is given $\Rightarrow V$ is constant.

$$V = xyz \quad \text{or} \quad z = \frac{V}{xy} \quad \dots(1)$$

$$S = xy + 2xz + 2yz = xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y)$$

$$f_x = y - \frac{2V}{x^2}, \quad f_y = x - \frac{2V}{y^2}$$

$$r = f_{xx} = \frac{4V}{x^3}, \quad s = f_{xy} = 1, \quad t = f_{yy} = \frac{4V}{y^3}$$

Now, $f_x = 0$ and $f_y = 0$

$$\Rightarrow y - \frac{2V}{x^2} = 0 \quad \dots(1) \quad x - \frac{2V}{y^2} = 0 \quad \dots(2)$$

From (1), $y = \frac{2V}{x^2}$,

$$\therefore \text{From (2), } x - \frac{2V}{4V^2} \cdot x^4 \quad \text{or} \quad x \left(1 - \frac{x^3}{2V}\right) = 0 \quad \text{or} \quad x = (2V)^{1/3}$$

and $y = \frac{2V}{x^2} = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3} \quad (\because x \neq 0)$

$\therefore x = y = (2V)^{1/3}$ is a stationary point.

At this point, $r = \frac{4V}{x^3} = \frac{4V}{2V} = 2 > 0$, $s = 1$, $t = \frac{4V}{2V} = 2$ so that $rt - s^2 = 4 - 1 = 3 > 0$ and $r > 0$.

$\Rightarrow S$ is minimum when $x = y = (2V)^{1/3}$.

Also,
$$z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}} = \frac{V^{1/3} \cdot 2^{1/3}}{2^{2/3} \cdot 2^{1/3}} = \frac{(2V)^{1/3}}{2} = \frac{y}{2}$$

Hence S is minimum when $x = y = 2z = (2V)^{1/3}$.

- 114.** Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral.

Sol. Let a, b, c be the sides of a triangle whose perimeter $2s$ is constant.

Then $2s = a + b + c$ or $c = 2s - a - b \quad \dots(1)$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(a+b-s)}$$

Let $z = \Delta^2 = s(s-a)(s-b)(a+b-s) = f(a, b)$

$$f_a = s(s-b) \frac{\partial}{\partial a} [(s-a)(a+b-s)]$$

$$= s(s-b) [-(a+b-s) + (s-a)] = s(s-b)(2s-2a-b)$$

$$f_b = s(s-a) \frac{\partial}{\partial b} [(s-b)(a+b-s)] = s(s-a) [-(a+b-s) + (s-b)]$$

or $f_b = s(s-a)(2s-2b-a)$

$$r = f_{aa} = -2s(s-b)$$

$$s = f_{ab} = s[-(2s-2b-a) - (s-a)] = s[-(3s-2b-2a)] = s(2b+2a-3s)$$

$$t = f_{bb} = -2s(s-a)$$

Now, $f_a = 0$ and $f_b = 0$

$$\Rightarrow s(s-b)(2s-2a-b) = 0 \quad \dots(1)$$

$$s(s-a)(2s-2b-a) = 0 \quad \dots(2)$$

i.e., $(s-b)(2s-2a-b) = 0$ and $(s-a)(2s-2b-a) = 0$

From (1), $s = b$ or $2s = 2a + b$

When $s = b$, from (2) $(b-a)(-a) = 0$ or $b = a$ ($\because a \neq 0$)

When $2s = 2a + b$ from (2) $\frac{b}{2}(a-b) = 0$ or $a = b$ ($\because b \neq 0$)

If we express z as a function of b and c , we similarly get $b = c$.

$$\therefore a = b = c = \frac{2s}{3}$$

$$r = -2s(s-b) = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3} < 0$$

$$s' = s(2b+2a-3s) = s\left(\frac{4s}{3} + \frac{4s}{3} - 3s\right) = s\left(-\frac{s}{3}\right) = -\frac{s^2}{3} < 0$$

$$t = -2s(s-a) = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3}$$

$$rt - s^2 = \left(-\frac{2s^2}{3}\right)\left(-\frac{2s^2}{3}\right) - \left(\frac{s^4}{9}\right) = \frac{4s^4}{9} - \frac{s^4}{9} = \frac{3s^4}{9} = \frac{s^4}{3} > 0$$

Also, $r < 0$

$\therefore z$ and hence Δ is maximum when $a = b = c = \frac{2s}{3}$ i.e., when the triangle is equilateral.

- 115.** Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube. (U.P.T.U., 2007)

Sol. Let x, y, z be the length, breadth and height of the rectangular solid. If V is the volume of the solid, then

$$V = xyz \quad \dots(1)$$

Since each diagonal of the solid passes through the centre of the sphere.

$\therefore$ Each diagonal = diameter of sphere = d (say)

$$\Rightarrow \sqrt{x^2 + y^2 + z^2} = d \text{ or } x^2 + y^2 + z^2 = d^2 \text{ or } z = \sqrt{d^2 - x^2 - y^2}$$

$\therefore$ From (1) $V = xy\sqrt{d^2 - x^2 - y^2}$

$$V^2 = x^2 y^2 (d^2 - x^2 - y^2) \quad \text{or} \quad V^2 = x^2 y^2 d^2 - x^4 y^2 - x^2 y^4 = f(x, y)$$

$$f_x = 2d^2 xy^2 - 4x^3 y^2 - 2xy^4, \quad f_y = 2x^2 y d^2 - 2x^4 y - 4x^2 y^3$$

$$r = f_{xx} = 2d^2 y^2 - 12x^2 y^2 - 2y^4, \quad s = f_{xy} = 4xy d^2 - 8x^3 y - 8xy^3$$

$$t = f_{yy} = 2d^2 x^2 - 2x^4 - 12x^2 y^2$$

$$\text{Now,} \quad f_x = 0 \quad \text{and} \quad f_y = 0$$

$$\Rightarrow \quad d^2 - 2x^2 - y^2 = 0 \quad \dots(2) \quad d^2 - x^2 - 2y^2 = 0 \quad \dots(3)$$

Subtracting (3) from (2),

$$-x^2 + y^2 = 0 \Rightarrow y = x$$

$$\therefore \text{ From (2),} \quad x = y = \frac{d}{\sqrt{3}}$$

$$z = \sqrt{d^2 - x^2 - y^2} = \sqrt{d^2 - \frac{d^2}{3} - \frac{d^2}{3}} = \frac{d}{\sqrt{3}}$$

$$\begin{aligned} \text{At } x = y = \frac{d}{\sqrt{3}}, \quad r &= 2d^2 y^2 - 12x^2 y^2 - 2y^4 = 2d^2 \cdot \frac{d^2}{3} - \frac{12x^2 d^2}{3} - 2 \cdot \frac{d^4}{9} \\ &= \frac{2d^4}{3} - \frac{12d^4}{9} - \frac{2d^4}{9} = -\frac{8d^4}{9} < 0 \end{aligned}$$

$$s = 4d^2 xy - 8x^3 y - 8xy^3 = 4d^2 \cdot \frac{d^2}{3} - \frac{8d^4}{9} - \frac{8d^4}{9} = -\frac{4d^4}{9}$$

$$t = 2d^2 \cdot \frac{d^2}{3} - \frac{2d^4}{9} - \frac{12d^4}{9} = -\frac{8d^4}{9}$$

$$rt - s^2 = 64 \frac{d^8}{81} - \frac{16d^8}{81} = \frac{16d^8}{27} > 0, \quad \text{Also, } r < 0$$

$\therefore V^2$ and hence, V is maximum when $x = y = z$ i.e., when the rectangular solid is a cube.

116. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (\text{M.D.U., Dec., 2005, May 2008})$$

Sol. Let (x, y, z) be a vertex of the parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

Also its dimensions are $2x, 2y, 2z$ so that the volume V is given by $V = 2x \cdot 2y \cdot 2z = 8xyz$

$$\Rightarrow \quad V^2 = 64x^2 y^2 z^2 = 64x^2 y^2 c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = 64c^2 \left(x^2 y^2 - \frac{x^4 y^2}{a^2} - \frac{x^2 y^4}{b^2} \right) = f(x, y)$$

$$\therefore \quad f_x = 64c^2 \left(2xy^2 - \frac{4x^3 y^2}{a^2} - \frac{2xy^4}{b^2} \right), \quad f_y = 64c^2 \left(2x^2 y - \frac{2x^4 y}{a^2} - \frac{4x^2 y^3}{b^2} \right)$$

$$r = f_{xx} = 64c^2 \left(2y^2 - \frac{12x^2 y^2}{a^2} - \frac{2y^4}{b^2} \right), \quad s = f_{xy} = 64c^2 \left(4xy - \frac{8x^3 y}{a^2} - \frac{8xy^3}{b^2} \right)$$

$$t = f_{yy} = 64c^2 \left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2 y^2}{b^2} \right)$$

Now, $f_x = 0$ and $f_y = 0$

$$\Rightarrow 128c^2xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \quad \text{and} \quad 128c^2x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0$$

$$\Rightarrow 1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \dots(1) \quad (\because x \neq 0, y \neq 0)$$

$$1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0 \quad \dots(2)$$

Subtracting (2) from (1),

$$\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{or} \quad y = \frac{bx}{a}$$

$$\therefore \text{From (1), } \frac{3x^2}{a^2} = 1 \quad \text{or} \quad x = \frac{a}{\sqrt{3}}$$

$$\therefore y = \frac{b}{\sqrt{3}} \quad \text{and} \quad z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \quad \text{or} \quad z^2 = c^2\left(1 - \frac{1}{3} - \frac{1}{3}\right) = \frac{c^2}{3}$$

$$\therefore z = \frac{c}{\sqrt{3}}$$

Thus $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$ is a stationary point.

$$\text{At this point, } r = 64c^2\left[\frac{2b^2}{3} - \frac{12}{a^2} \cdot \frac{a^2}{3} \cdot \frac{b^2}{3} - \frac{2}{b^2} \cdot \frac{b^4}{9}\right] \quad \text{or} \quad r = -\frac{512}{9}b^2c^2 < 0$$

$$s = 64c^2\left[4 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} - \frac{8}{a^2} \cdot \frac{a^3}{3\sqrt{3}} \cdot \frac{b}{\sqrt{3}} - \frac{8}{b^2} \cdot \frac{a}{\sqrt{3}} \cdot \frac{b^3}{3\sqrt{3}}\right] \quad \text{or} \quad s = -\frac{256}{9}abc^2$$

$$t = 64c^2\left[2 \cdot \frac{a^2}{3} - \frac{2}{a^2} \cdot \frac{a^4}{9} - \frac{12}{b^2} \cdot \frac{a^2}{3} \cdot \frac{b^2}{3}\right] \quad \text{or} \quad t = -\frac{512}{9}a^2c^2$$

$$rt - s^2 = \left(\frac{512}{9}\right)^2 a^2b^2c^4 - \left(\frac{256}{9}\right)^2 a^2b^2c^4 = \left(\frac{256}{9}\right)^2 a^2b^2c^4 (4 - 1) > 0$$

Also, $r < 0$

$\therefore V^2$ and hence, V is maximum.

$$\text{When } x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$$\text{Maximum Volume} = 8xyz = \frac{8abc}{3\sqrt{3}}.$$

117. In a plane triangle ABC , find the maximum value of $\cos A \cos B \cos C$.

Sol. $\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)] \quad (\because A + B + C = \pi)$

or $\cos A \cos B \cos C = \cos A \cos B \cdot [-\cos (A + B)] = -\cos A \cos B \cos (A + B) = f(A, B)$

$$\frac{\partial f}{\partial A} = -\cos B [-\sin A \cos (A + B) - \cos A \sin (A + B)]$$

$$= \cos B [\sin A \cos (A + B) + \cos A \sin (A + B)] = \cos B \cdot \sin (2A + B)$$

$$\frac{\partial f}{\partial B} = -\cos A [-\sin B \cos (A+B) - \cos B \sin (A+B)]$$

$$= \cos A [\sin (A+2B)] = \cos A \sin (A+2B)$$

$$r = 2\cos B \cos (2A+B)$$

$$s = -\sin A \sin (A+2B) + \cos A \cdot \cos (A+2B)$$

or

$$s = \cos [A + (A+2B)] = \cos (2A+2B)$$

$$t = 2\cos A \cos (A+2B).$$

$$\text{Now, } \frac{\partial f}{\partial A} = 0 \quad \text{and} \quad \frac{\partial f}{\partial B} = 0$$

$$\Rightarrow \cos B \sin (2A+B) = 0 \quad \dots(1) \quad \cos A \sin (A+2B) = 0 \quad \dots(2)$$

$$\text{If } \cos B = 0, \text{ then } B = \frac{\pi}{2}$$

$$\text{From (2), } \cos A \cdot \sin (A+\pi) = 0 \quad \text{or} \quad -\sin A \cos A = 0$$

$$\Rightarrow \text{either } \cos A = 0 \text{ i.e., } A = \frac{\pi}{2} \text{ which is not possible.}$$

$$\text{or } \sin A = 0 \text{ i.e., } A = 0 \text{ or } \pi \text{ which is not possible.}$$

$$\therefore \cos B \neq 0. \text{ Similarly, } \cos A \neq 0.$$

$$\therefore \text{From (1), } \sin (2A+B) = 0 \quad \text{or} \quad 2A+B = \pi$$

$$\text{From (2), } \sin (A+2B) = 0 \quad \text{or} \quad A+2B = \pi$$

$$\text{Solving these equations, } A = B = \frac{\pi}{3}$$

$$\text{At } A = B = \frac{\pi}{3}, \quad r = 2\cos \frac{\pi}{3} \cos \pi = -1, \quad s = \cos \frac{4\pi}{3} = -\frac{1}{2}, \quad t = 2\cos \frac{\pi}{3} \cos \pi = -1$$

$$rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0, \quad \text{Also, } r = -1 < 0$$

$$\Rightarrow f(A, B) \text{ is maximum at } A = B = \frac{\pi}{3}$$

$$\text{Maximum value} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\cos \frac{\pi}{3} \cos \frac{\pi}{3} \cos \frac{2\pi}{3} = \frac{1}{8}.$$

118. Show that of all triangles inscribed in a circle, the one with maximum area, is equilateral.

Sol. Let ABC be a triangle inscribed in a circle of radius R . Let x, y, z be the angles of triangle ABC so that $0 < x < \pi, 0 < y < \pi, 0 < z < \pi$ and $x + y + z = \pi$.

If S denotes the area of $\triangle ABC$, then

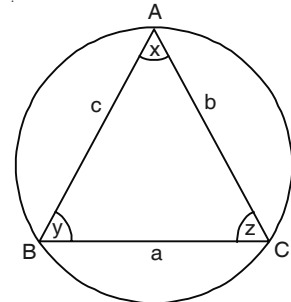
$$S = \frac{1}{2} ab \sin z$$

where a, b denote the lengths of the sides containing the angle z .

$$\text{Since, } R = \frac{a}{2\sin x} = \frac{b}{2\sin y} = \frac{c}{2\sin z}$$

$$\therefore a = 2R \sin x, \quad b = 2R \sin y$$

$$\begin{aligned} \text{and } S &= \frac{1}{2} \cdot 2R \sin x \cdot 2R \sin y \cdot \sin [\pi - (x+y)] \\ &= 2R^2 \sin x \sin y \sin (x+y) \end{aligned}$$



We have to maximize S or equivalently $\frac{S}{2R^2}$.

$$\text{Let } f(x, y) = \frac{S}{2R^2} = \sin x \sin y \sin(x + y)$$

$$f_x = \sin y [\cos x \sin(x + y) + \sin x \cos(x + y)]$$

$$= \sin y [\sin(x + y + x)] = \sin y \sin(2x + y)$$

$$f_y = \sin x [\cos y \sin(x + y) + \sin y \cos(x + y)] = \sin x \sin(x + y + y) = \sin x \sin(x + 2y)$$

$$r = f_{xx} = 2\sin y \cos(2x + y)$$

$$s = f_{xy} = \cos x \sin(x + 2y) + \sin x \cos(x + 2y) = \sin(x + 2y + x) = \sin(2x + 2y)$$

$$t = f_{yy} = 2\sin x \cos(x + 2y)$$

$$\text{Now, } f_x = 0 \text{ and } f_y = 0 \Rightarrow \sin(2x + y) = 0 = \sin(x + 2y) \text{ because}$$

$$\sin x \neq 0, \sin y \neq 0 \text{ since } 0 < x < \pi, 0 < y < \pi$$

$$\Rightarrow 2x + y = \pi \text{ and } x + 2y = \pi$$

$$\text{so that } x = y = \frac{\pi}{3} \quad \left(\text{and hence } z = \pi - x - y = \frac{\pi}{3} \right)$$

$$\text{At } x = \frac{\pi}{3}, y = \frac{\pi}{3}, \quad r = 2\sin \frac{\pi}{3} \cos \pi = -\sqrt{3}$$

$$s = \sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$t = 2\sin x \cos(x + 2y) = 2\sin \frac{\pi}{3} \cos \pi = 2 \cdot \frac{\sqrt{3}}{2} \cdot (-1) = -\sqrt{3}$$

$$\text{Since } rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

and $r < 0$, therefore $f(x, y)$ has a maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3} \right)$.

$$\Rightarrow S \text{ is maximum when } x = y = z = \frac{\pi}{3}.$$

$\Rightarrow$ area of inscribed triangle is maximum when the triangle is equilateral.

119. Test the function $f(x, y) = x^3 y^2 (6 - x - y)$ for maxima and minima for points not at the origin.

$$\text{Sol. Here, } f(x, y) = x^3 y^2 (6 - x - y) = 6x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$\therefore f_x = 18x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$f_y = 12x^3 y - 2x^4 - 3x^3 y^2$$

$$r = f_{xx} = 36xy^2 - 12x^2 y^2 - 6xy^3 = 6xy^2 (6 - 2x - y)$$

$$s = f_{xy} = 36x^2 y - 8x^3 y - 9x^2 y^2 = x^2 y (36 - 8x - 9y)$$

$$t = f_{yy} = 12x^3 - 2x^4 - 6x^3 y = x^3 (12 - 2x - 6y)$$

$$\text{Now, } f_x = 0 \Rightarrow x^2 y^2 (18 - 4x - 3y) = 0 \quad \dots(1)$$

$$\text{and } f_y = 0 \Rightarrow x^3 y (12 - 2x - 3y) = 0 \quad \dots(2)$$

From (1) and (2),

$$4x + 3y = 18, \quad 2x + 3y = 12 \quad \text{and} \quad x = y = 0$$

Solving, we get $x = 3, y = 2$ and $x = 0 = y$

leaving $x = 0 = y$, we get $x = 3, y = 2$.

Hence $(3, 2)$ is the only stationary point under consideration.

$$\begin{aligned}\text{Now,} \quad & rt - s^2 = 6x^4y^2(6 - 2x - y)(12 - 2x - 6y) - x^4y^2(36 - 8x - 9y)^2 \\ \text{At } (3, 2), \quad & rt - s^2 = +ve (> 0)\end{aligned}$$

$$r = 6xy^2(6 - 2x - y) = 6 \cdot 3 \cdot 4(-2) = -144 < 0$$

$\therefore f(x, y)$ has a maximum value at $(3, 2)$.

120. Examine $x^2 + y^2 + 6x + 12$ for extreme values.

Sol. Let $f(x, y) = x^2 + y^2 + 6x + 12$

$$\begin{aligned}\therefore \quad & f_x = 2x + 6, & f_y &= 2y \\ & f_{xx} = 2, & f_{xy} &= 0, & f_{yy} &= 2\end{aligned}$$

For maxima and minima,

$$\begin{aligned}f_x &= 0 \quad \text{and} \quad f_y = 0 \\ 2x + 6 &= 0 \quad \text{and} \quad 2y = 0 \quad \Rightarrow \quad x = -3 \quad \text{and} \quad y = 0\end{aligned}$$

Hence $(-3, 0)$ is the stationary point.

$$\text{At } (-3, 0), \quad rt - s^2 = 4 > 0 \quad \text{Also,} \quad r = 2 > 0$$

Hence $f(x, y)$ is minimum when $x = -3, y = 0$.

$$\text{Minimum Value} = x^2 + y^2 + 6x + 12 = (-3)^2 + 0 + 6(-3) + 12 = 3.$$

121. Test the function $f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}$ for maxima and minima for points not on the circle $x^2 + y^2 = 1$.

Sol. Here, $f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}$

$$f_x = (x^2 + y^2)e^{-(x^2 + y^2)} \cdot (-2x) + 2x \cdot e^{-(x^2 + y^2)} = 2x(1 - x^2 - y^2)e^{-(x^2 + y^2)}$$

$$f_y = (x^2 + y^2)e^{-(x^2 + y^2)} \cdot (-2y) + 2y \cdot e^{-(x^2 + y^2)} = 2y(1 - x^2 - y^2)e^{-(x^2 + y^2)}$$

$$\text{Now,} \quad f_x = 0 \quad \text{and} \quad f_y = 0$$

$$\Rightarrow 2x(1 - x^2 - y^2)e^{-(x^2 + y^2)} = 0 \quad \text{and} \quad 2y(1 - x^2 - y^2)e^{-(x^2 + y^2)} = 0$$

$$\Rightarrow \quad x = 0, \quad y = 0 \quad \text{and} \quad x^2 + y^2 = 1$$

Leaving $x^2 + y^2 = 1$ as given, take $x = 0, y = 0$

Hence $(0, 0)$ is the only stationary point.

$$f_x = (2x - 2x^3 - 2xy^2)e^{-(x^2 + y^2)}$$

$$r = f_{xx} = (2 - 6x^2 - 2y^2)e^{-(x^2 + y^2)} + (2x - 2x^3 - 2xy^2)e^{-(x^2 + y^2)} \cdot (-2x)$$

$$= e^{-(x^2 + y^2)} \cdot (4x^4 - 10x^2 + 4x^2y^2 - 2y^2 + 2)$$

$$s = f_{xy} = (-4xy)e^{-(x^2 + y^2)} + (2x - 2x^3 - 2xy^2)e^{-(x^2 + y^2)} \cdot (-2y)$$

$$= (-8xy + 4x^3y + 4xy^3)e^{-(x^2 + y^2)}$$

$$t = f_{yy} = e^{-(x^2 + y^2)} \cdot (2 - 2x^2 - 6y^2) + e^{-(x^2 + y^2)} \cdot (2y - 2x^2y - 2y^3) \cdot (-2y)$$

$$= e^{-(x^2 + y^2)} \cdot (2 - 2x^2 - 10y^2 - 4x^2y^2 + 4y^4)$$

$$\text{At } (0, 0), \quad r = 2, \quad s = 0, \quad t = 2$$

$$rt - s^2 = 4 > 0$$

$$r = 2 (> 0)$$

$\therefore f(x, y)$ has a minimum value at $(0, 0)$.

$$\text{Minimum Value} = (x^2 + y^2)e^{-(x^2 + y^2)} = (0 + 0) \cdot e^{-0} = 0.$$

122. Examine for extreme values $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Sol. Working rule to find maximum or minimum of any function $u = f(x, y, z)$ is

(i) Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ and equate them to zero.

(ii) Find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial z \partial x}, \frac{\partial^2 u}{\partial x \partial y}$. They are denoted by A, B, C, F, G, H respectively.

(iii) Also find $AB - H^2 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = D_1$ (say) and find $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = D_2$ (say).

The given function will have a

(a) minimum if $A > 0, D_1 > 0, D_2 > 0$

(b) maximum if $A < 0, D_1 > 0, D_2 < 0$

(c) If above conditions are not satisfied, then we have neither maximum nor minimum.

Now, here $u = x^2 + y^2 + z^2 - xy + x - 2z$

$$\frac{\partial u}{\partial x} = 2x - y + 1, \quad \frac{\partial u}{\partial y} = 2y - x, \quad \frac{\partial u}{\partial z} = 2z - 2$$

For extreme values, $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$

$\therefore 2x - y + 1 = 0, 2y - x = 0, 2z - 2 = 0$

Solving these equations, we get

$$x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1$$

Hence $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ is the stationary point.

Now,

$$A = \frac{\partial^2 u}{\partial x^2} = 2 (> 0), \quad B = \frac{\partial^2 u}{\partial y^2} = 2 (> 0)$$

$$C = \frac{\partial^2 u}{\partial z^2} = 2, \quad F = \frac{\partial^2 u}{\partial y \partial z} = 0, \quad G = \frac{\partial^2 u}{\partial z \partial x} = 0, \quad H = \frac{\partial^2 u}{\partial x \partial y} = -1$$

$\therefore D_1 = AB - H^2 = 4 - 1 = 3 (> 0)$

$$D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 (> 0)$$

Since $A > 0, D_1 > 0$, and $D_2 > 0$ hence the given function u will have a minimum at $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

$$\text{Minimum Value} = x^2 + y^2 + z^2 - xy + x - 2z$$

$$= \left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + 1 - \frac{2}{9} - \frac{2}{3} - 2 = \frac{4}{9} + \frac{1}{9} + 1 - \frac{2}{9} - \frac{2}{3} - 2 = -\frac{4}{3}.$$

123. Given $f(x, y, z) = \frac{5xyz}{x + 2y + 4z}$. Find values of x, y, z for which $f(x, y, z)$ is maximum, subject to the condition $xyz = 8$.

Sol.

$$u = f(x, y, z) = \frac{5xyz}{x + 2y + 4z} \quad \dots(1)$$

$$xyz = 8 \quad \dots(2)$$

From (1) and (2),
$$u = \frac{40}{x + 2y + 4z}$$

$$\therefore du = \frac{40}{(x + 2y + 4z)^2} \cdot (dx + 2dy + 4dz)$$

For maximum and minimum of u , $du = 0 \Rightarrow dx + 2dy + 4dz = 0$... (3)

From (2), $\log x + \log y + \log z = \log(8)$

Differentiating, we get $\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$... (4)

From (3) and (4), after introducing λ such that

$$\begin{aligned} dx \left(1 + \frac{\lambda}{x}\right) + dy \left(2 + \frac{\lambda}{y}\right) + dz \left(4 + \frac{\lambda}{z}\right) &= 0 \\ \Rightarrow \quad 1 + \frac{\lambda}{x} &= 0 \quad \Rightarrow \quad x = -\lambda \\ 2 + \frac{\lambda}{y} &= 0 \quad \Rightarrow \quad y = -\frac{\lambda}{2} \\ 4 + \frac{\lambda}{z} &= 0 \quad \Rightarrow \quad z = -\frac{\lambda}{4} \end{aligned}$$

From (2), $xyz = 8 \Rightarrow -\frac{\lambda^3}{8} = 8 \Rightarrow \lambda = -4$

Hence, $x = 4, y = 2, z = 1$

$\therefore u$ is stationary at the point (4, 2, 1).

Now,
$$u = \frac{40}{x + 2y + 4z}$$

$$\frac{\partial u}{\partial x} = -\frac{40}{(x + 2y + 4z)^2} \cdot \left(1 + 4 \frac{\partial z}{\partial x}\right)$$

From (2), $\log x + \log y + \log z = \log(8)$

Differentiation w.r.t. x partially gives us

$$\begin{aligned} \frac{1}{x} + \frac{1}{z} \cdot \frac{\partial z}{\partial x} &= 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x} \\ \therefore \frac{\partial u}{\partial x} &= -\frac{40}{(x + 2y + 4z)^2} \cdot \left(1 - \frac{4z}{x}\right) \\ \therefore \frac{\partial^2 u}{\partial x^2} &= \frac{80}{(x + 2y + 4z)^3} \cdot \left(1 + 4 \frac{\partial z}{\partial x}\right) \left(1 - \frac{4z}{x}\right) - \frac{40}{(x + 2y + 4z)^2} \cdot \left(\frac{4z}{x^2} - \frac{4}{x} \cdot \frac{\partial z}{\partial x}\right) \\ &= \frac{80}{(x + 2y + 4z)^3} \cdot \left(1 - 4 \frac{z}{x}\right)^2 - \frac{40}{(x + 2y + 4z)^2} \cdot \left(\frac{8z}{x^2}\right) \quad \left(\because \frac{\partial z}{\partial x} = -\frac{z}{x}\right) \end{aligned}$$

At the stationary point,

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{80}{(12)^3} (1 - 1)^2 - \frac{40}{144} \left(\frac{1}{2}\right) = -\frac{20}{144} < 0$$

$\therefore u$ is maximum at the point (4, 2, 1).

124. What is Lagrange's Method of Undetermined Multipliers ?

Sol. Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value.

Let the variables (x, y, z) be connected by the relation $\phi(x, y, z) = 0$.

Consider Lagrange's function

$F(x, y, z) = f(x, y, z) + \lambda \cdot \phi(x, y, z)$ where λ is a parameter.

For stationary values of $F(x, y, z)$, $dF = 0$

This implies $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$, $\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$ and $\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

These equations together with $\phi(x, y, z) = 0$, give the values of x, y, z, λ for a maximum or minimum.

This method is called Lagrange's method of undetermined multipliers and used to find maximum or minimum value. However, this method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical considerations of the problem.

- 125.** Find the maximum and minimum distances of the point (3, 4, 12) from the sphere

$$x^2 + y^2 + z^2 = 1. \quad (\text{M.D.U., May 2007})$$

Sol. Let (x, y, z) be any point on the sphere. Distance of the point $A(3, 4, 12)$ from (x, y, z) is given

$$\text{by } \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

If the distance is maximum or minimum, so will be the square of the distance.

$$\text{Let,} \quad f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 \quad \dots(1)$$

Subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \quad \dots(2)$$

Consider Lagrange's function

$$\begin{aligned} F(x, y, z) &= f(x, y, z) + \lambda \cdot \phi(x, y, z) \\ &= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1) \end{aligned}$$

For stationary values, $dF = 0$

$$\Rightarrow 2(x-3)dx + 2(y-4)dy + 2(z-12)dz + 2\lambda xdx + 2\lambda ydy + 2\lambda zdz = 0$$

$$\Rightarrow 2(x-3) + 2\lambda x = 0 \quad \dots(3)$$

$$2(y-4) + 2\lambda y = 0 \quad \dots(4)$$

$$2(z-12) + 2\lambda z = 0 \quad \dots(5)$$

Multiplying (3) by x , (4) by y , (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) - 6x - 8y - 24z + 2\lambda(x^2 + y^2 + z^2) = 0$$

$$\text{or} \quad 2 - 6x - 8y - 24z + 2\lambda = 0$$

$$3x + 4y + 12z = 1 + \lambda \quad \dots(6)$$

From (3), (4) and (5),

$$x = \frac{3}{1+\lambda}, \quad y = \frac{4}{1+\lambda}, \quad z = \frac{12}{1+\lambda}$$

Putting these values of x, y, z in (6), we have

$$\frac{169}{1+\lambda} = 1 + \lambda$$

$$\text{or} \quad (1+\lambda)^2 = 169 \quad \text{or} \quad 1+\lambda = \pm 13 \quad \text{or} \quad \lambda = 12, -14$$

$$\text{When } \lambda = 12, \quad x = \frac{3}{13}, \quad y = \frac{4}{13}, \quad z = \frac{12}{13}$$

$$\text{When } \lambda = -14, \quad x = -\frac{3}{13}, \quad y = -\frac{4}{13}, \quad z = -\frac{12}{13}$$

Thus we get two points $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ on the sphere which are at a maximum or minimum distance from the given point A .

Now,

$$AP = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2} = 12$$

$$AQ = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2} = 14$$

$\therefore P$ is at a minimum distance from A and the min. distance = 12

Q is at a maximum distance from A and the maximum distance = 14.

- 126.** Find the minimum value of $x^2 + y^2 + z^2$ given that $ax + by + cz = p$. [U.P.T.U. (AG) 2005]

Sol. Let $u = x^2 + y^2 + z^2$... (1)

Where $\phi(x, y, z) = ax + by + cz - p = 0$... (2)

Consider Lagrange's function

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$$

For stationary values, $dF = 0$

$$\Rightarrow (2x + \lambda a)dx + (2y + \lambda b)dy + (2z + \lambda c)dz = 0$$

$$\Rightarrow 2x + \lambda a = 0 \quad \dots (3)$$

$$2y + \lambda b = 0 \quad \dots (4)$$

$$2z + \lambda c = 0 \quad \dots (5)$$

Multiplying (3) by x , (4) by y , (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

or $2u + \lambda p = 0$

$$\therefore \lambda = -\frac{2u}{p}$$

From (3), (4) and (5), $x = \frac{au}{p}, y = \frac{bu}{p}, z = \frac{cu}{p}$

$$\therefore \text{From (1), } u = \frac{(a^2 + b^2 + c^2)u^2}{p^2}$$

or $u = \frac{p^2}{a^2 + b^2 + c^2}$

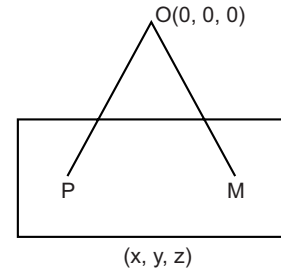
This is the maximum or minimum value of u . Now u is the square of the distance of any point $P(x, y, z)$ on the plane (2) from the origin.

Also, the length of perpendicular from O on the plane is

$$\frac{p}{\sqrt{a^2 + b^2 + c^2}}$$

Clearly, OP is least when P coincides with m , the foot of the perpendicular from O on the plane.

Hence the minimum value of $u = \frac{p^2}{a^2 + b^2 + c^2}$.



- 127.** Find the minimum value of $ax + by + cz$ subject to the condition $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 1$.

Sol. Let $f(x, y, z) = ax + by + cz$... (1)

where $\phi(x, y, z) = \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} - 1 = 0$... (2)

Consider Lagrange's function

$$F(x, y, z) = (ax + by + cz) + \lambda \left(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} - 1 \right)$$

For stationary values, $dF = 0$

$$\Rightarrow \left(a - \frac{\lambda a^2}{x^2} \right) dx + \left(b - \frac{\lambda b^2}{y^2} \right) dy + \left(c - \frac{\lambda c^2}{z^2} \right) dz = 0$$

$$\Rightarrow a - \frac{\lambda a^2}{x^2} = 0 \quad \dots(3) \quad b - \frac{\lambda b^2}{y^2} = 0 \quad \dots(4) \quad c - \frac{\lambda c^2}{z^2} = 0 \quad \dots(5)$$

From (3), (4) and (5), $x = \sqrt{\lambda a}$, $y = \sqrt{\lambda b}$, $z = \sqrt{\lambda c}$

Putting these values in (2), we have

$$\begin{aligned} \frac{a^2}{\sqrt{\lambda a}} + \frac{b^2}{\sqrt{\lambda b}} + \frac{c^2}{\sqrt{\lambda c}} &= 1 \\ \Rightarrow \sqrt{\lambda} &= (a^{3/2} + b^{3/2} + c^{3/2}) \\ \lambda &= (a^{3/2} + b^{3/2} + c^{3/2})^2 \\ \therefore x &= a^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}), \quad y = b^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}) \\ z &= c^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}) \end{aligned}$$

The stationary point is

$$\left[a^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}), b^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}), c^{\frac{1}{2}} (a^{3/2} + b^{3/2} + c^{3/2}) \right] \text{ and the minimum value of } f \text{ is}$$

$$\begin{aligned} ax + by + cz &= a \left(a^{\frac{1}{2}} \right) (a^{3/2} + b^{3/2} + c^{3/2}) + b \left(b^{\frac{1}{2}} \right) (a^{3/2} + b^{3/2} + c^{3/2}) \\ &\quad + c \left(c^{\frac{1}{2}} \right) (a^{3/2} + b^{3/2} + c^{3/2}) \\ &= \left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}} \right)^2. \end{aligned}$$

128. Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.

Sol. Let $f(x, y, z) = x^m y^n z^p$...(1)

where $\phi(x, y, z) = x + y + z - a = 0$...(2)

Consider Lagrange's function

$$F(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a)$$

For stationary values, $dF = 0$

$$\Rightarrow (mx^{m-1}y^n z^p + \lambda)dx + (nx^m y^{n-1} z^p + \lambda)dy + (px^m y^n z^{p-1} + \lambda)dz = 0$$

$$\Rightarrow mx^{m-1}y^n z^p + \lambda = 0 \quad \dots(3)$$

$$nx^m y^{n-1} z^p + \lambda = 0 \quad \dots(4)$$

$$px^m y^n z^{p-1} + \lambda = 0 \quad \dots(5)$$

Multiplying (3), (4) and (5) by x, y, z respectively and adding, we get

$$(m+n+p)x^m y^n z^p + \lambda(x+y+z) = 0$$

$$\Rightarrow (m+n+p)f + \lambda a = 0 \Rightarrow \lambda = -\frac{(m+n+p)f}{a}$$

From (3), multiplying by x , we have

$$mf + \lambda x = 0 \Rightarrow x = -\frac{mf}{\lambda}$$

$$\therefore x = \frac{ma}{m+n+p}$$

$$\text{Similarly, } y = \frac{na}{m+n+p}, \quad z = \frac{pa}{m+n+p}$$

$$\therefore \text{Maximum value of } f = x^m y^n z^p$$

$$= \left(\frac{ma}{m+n+p}\right)^m \left(\frac{na}{m+n+p}\right)^n \left(\frac{pa}{m+n+p}\right)^p = \frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$$

- 129.** Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (M.D.U., Dec., 2008)

Sol. Let x, y, z be the edges of the box.

Capacity of the box = xyz

$$\text{Surface area} = xy + 2yz + 2zx = 432$$

$$\text{Let } f(x, y, z) = xyz \quad \dots(1)$$

$$\text{where } \phi(x, y, z) = xy + 2yz + 2zx - 432 = 0 \quad \dots(2)$$

Consider Lagrange's function

$$F(x, y, z) = xyz + \lambda(xy + 2yz + 2zx - 432)$$

$$\text{For stationary values, } dF = 0$$

$$\Rightarrow [yz + \lambda(y + 2z)]dx + [xz + \lambda(x + 2z)]dy + [xy + \lambda(2y + 2x)]dz = 0$$

$$\Rightarrow yz + \lambda(y + 2z) = 0 \quad \dots(3)$$

$$xz + \lambda(x + 2z) = 0 \quad \dots(4)$$

$$xy + \lambda(2y + 2x) = 0 \quad \dots(5)$$

Multiplying (3) by x , (4) by y and subtracting, we get

$$2z\lambda(x - y) = 0 \Rightarrow x = y$$

Multiplying (4) by y , (5) by z and subtracting, we get

$$x\lambda(y - 2z) = 0 \Rightarrow y = 2z$$

$$\therefore x = y = 2z$$

$$\therefore \text{From (2), } xy + 2yz + 2zx = 432$$

$$x^2 + 2x \cdot \frac{x}{2} + 2x \cdot \frac{x}{2} = 432$$

$$x^2 + x^2 + x^2 = 432$$

$$3x^2 = 432$$

$$x^2 = 144, \quad x = 12$$

$$\therefore y = 12, \quad z = 6$$

Hence the dimensions of the box are 12 cm, 12 cm and 6 cm.

- 130.** Prove that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ where $lx + my + nz = 0$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ are the roots of the equation } \frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0.$$

Sol. Consider Lagrange's function

$$F(x, y, z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \lambda(lx + my + nz) + \mu \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

For stationary values, $dF = 0$

$$\Rightarrow \left(\frac{2x}{a^4} + \lambda + \frac{2\mu x}{a^2} \right) dx + \left(\frac{2y}{b^4} + \lambda + \frac{2\mu y}{b^2} \right) dy + \left(\frac{2z}{c^4} + \lambda + \frac{2\mu z}{c^2} \right) dz = 0$$

$$\Rightarrow \frac{2x}{a^4} + \lambda + \frac{2\mu x}{a^2} = 0 \quad \dots(1) \quad \frac{2y}{b^4} + \lambda + \frac{2\mu y}{b^2} = 0 \quad \dots(2)$$

$$\frac{2z}{c^4} + \lambda + \frac{2\mu z}{c^2} = 0 \quad \dots(3)$$

Multiplying (1), (2), (3) by x, y, z respectively and adding, we get

$$2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right) + \lambda(lx + my + nz) + 2\mu \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 0$$

$$\Rightarrow 2u + \lambda(0) + 2\mu(1) = 0$$

$$\Rightarrow \mu = -u$$

$\therefore$ Equation (1) becomes

$$\frac{2x}{a^4} + \lambda - \frac{2ux}{a^2} = 0 \quad \text{or} \quad \frac{2x}{a^4} (1 - a^2 u) = -\lambda \quad \text{or} \quad x = -\frac{\lambda a^4}{2(1 - a^2 u)}$$

Similarly, $y = -\frac{\lambda m b^4}{2(1 - b^2 u)}, \quad z = -\frac{\lambda n c^4}{2(1 - c^2 u)}$

To eliminate λ between them, multiplying these values by l, m, n respectively and adding, we get

$$lx + my + nz = -\frac{\lambda}{2} \left[\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} \right]$$

Since $lx + my + nz = 0$. We have $\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0$

Which is a quadratic in u and gives two stationary values of u .

131. Find the semi-vertical angle of the cone of maximum volume and of a given slant height.

Sol. Let l be the slant height and θ be the semi-vertical angle of the cone. Then

radius of the base, $r = OC = l \sin \theta$, and height of cone, $h = OA = l \cos \theta$.

Let V be the volume of the cone then

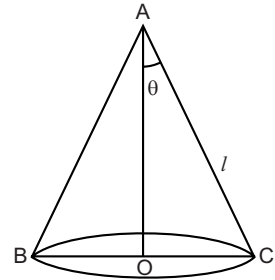
$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (l \sin \theta)^2 (l \cos \theta)$$

$$= \frac{1}{3} \pi l^3 \sin^2 \theta \cos \theta$$

$$\therefore \frac{dV}{d\theta} = \frac{1}{3} \pi l^3 [\sin^2 \theta (-\sin \theta) + \cos \theta (2 \sin \theta \cos \theta)]$$

$$\frac{dV}{d\theta} = \frac{1}{3} \pi l^3 \sin \theta (2 \cos^2 \theta - \sin^2 \theta)$$

$$\frac{d^2 V}{d\theta^2} = \frac{1}{3} \pi l^3 [\sin \theta (-4 \cos \theta \sin \theta) - 2 \sin \theta \cos \theta + (2 \cos^2 \theta - \sin^2 \theta) \cos \theta]$$



$$= \frac{1}{3} \pi l^3 [-6 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - \sin^2 \theta \cos \theta]$$

For maxima and minima, $\frac{dV}{d\theta} = 0$

$\Rightarrow$ Either $\sin \theta = 0$ or $\tan^2 \theta = 2 \Rightarrow \theta = 0$ or $\theta = \pm \tan^{-1} \sqrt{2}$.

When $\theta = 0$, volume of cone becomes zero and the cone becomes a straight line which is not the case.

When $\theta = \tan^{-1} \sqrt{2}$, we have $\frac{d^2V}{d\theta^2} = \frac{1}{3} \pi l^3 \left[-6 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} + 0 \right] = (-)ve$

Hence the volume of cone is maximum when $\theta = \tan^{-1} \sqrt{2}$. When $\theta = -\tan^{-1} \sqrt{2}$, volume of cone becomes negative which is meaningless hence is not the case.

- 132.** The sum of 3 positive numbers is constant. Prove that their product is maximum when they are equal.

Sol. Let x, y, z be the three positive numbers.

$$x + y + z = k \text{ (say)} \quad \dots(1)$$

$$\text{Let } u = xyz \quad \dots(2)$$

Consider Lagrange's function

$$F(x, y, z) = xyz + \lambda(x + y + z - k)$$

For stationary values, $dF = 0$

$$\Rightarrow (yz + \lambda)dx + (zx + \lambda)dy + (xy + \lambda)dz = 0$$

$$\Rightarrow yz + \lambda = 0 \quad \dots(3) \quad zx + \lambda = 0 \quad \dots(4) \quad xy + \lambda = 0 \quad \dots(5)$$

Multiplying (3), (4) and (5) by x, y and z respectively and adding, we get

$$3xyz + \lambda(x + y + z) = 0$$

$$3u + \lambda k = 0$$

$$\lambda = -\frac{3u}{k}$$

$$\text{From (3), (4) and (5), } yz - \frac{3u}{k} = 0 \quad zx - \frac{3u}{k} = 0$$

$$xy - \frac{3u}{k} = 0 \quad \Rightarrow \quad xyz - \frac{3ux}{k} = 0$$

$$u - \frac{3ux}{k} = 0 \quad \Rightarrow \quad x = \frac{k}{3}$$

$$\text{Similarly, } y = \frac{k}{3}, \quad z = \frac{k}{3}$$

Hence $\left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}\right)$ is the stationary point. Now to find whether u is maximum or minimum

Let z be a function of x and y

Then, from $x + y + z = k$, we have $\frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$

Now, $u = xyz$

$$\therefore \frac{\partial u}{\partial x} = y \left(z + x \frac{\partial z}{\partial x} \right) = y(z - x)$$

$$r = \frac{\partial^2 u}{\partial x^2} = y \left[\frac{\partial z}{\partial x} - 1 \right] = y(-1 - 1) = -2y$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = (z - x) \cdot 1 + y(-1) = z - x - y$$

Also,
$$\frac{\partial u}{\partial y} = x \left(z + y \frac{\partial z}{\partial y} \right) = x(z - y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = x \left(\frac{\partial z}{\partial y} - 1 \right) = x(-2) = -2x$$

$$rt - s^2 = (-2y)(-2x) - (z - x - y)^2 = 4xy - (z - x - y)^2$$

$$= 4 \cdot \frac{k}{3} \cdot \frac{k}{3} - \left(-\frac{k}{3} \right)^2 \quad \left[\because x = y = z = \frac{k}{3} \right]$$

$$= \frac{4k^2}{9} - \frac{k^2}{9} = \frac{k^2}{3} (> 0)$$

Also,
$$r = -2y = -2 \frac{k}{3} \text{ (-ve)} < 0$$

$\therefore u$ is maximum at $x = y = z = \frac{k}{3}$

$$\text{Maximum value} = \frac{k}{3} \cdot \frac{k}{3} \cdot \frac{k}{3} = \frac{k^3}{27}$$

133. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$.

Sol. Let (x, y) be any point on the curve.

Distance of the point $A(0, 0)$ from (x, y) is given by $\sqrt{x^2 + y^2}$.

If the distance is maximum or minimum, so will the square of the distance.

Let
$$f(x, y) = x^2 + y^2 \quad \dots(1)$$

subject to the condition
$$\phi(x, y) = x^2 + 4xy + 6y^2 - 140 = 0 \quad \dots(2)$$

Consider Lagrange's function

$$F(x, y) = f(x, y) + \lambda \phi(x, y) = x^2 + y^2 + \lambda(x^2 + 4xy + 6y^2 - 140)$$

For stationary values, $dF = 0$

$$\Rightarrow [2x + \lambda(2x + 4y)]dx + [2y + \lambda(4x + 12y)]dy = 0$$

$$\Rightarrow 2x + \lambda(2x + 4y) = 0 \quad \dots(3) \quad 2y + \lambda(4x + 12y) = 0 \quad \dots(4)$$

Multiplying (3) by x and (4) by y and adding, we get

$$2x^2 + \lambda(2x^2 + 4xy) + 2y^2 + \lambda(4xy + 12y^2) = 0$$

or
$$2(x^2 + y^2) + \lambda(2x^2 + 8xy + 12y^2) = 0$$

$$(x^2 + y^2) + \lambda(x^2 + 4xy + 6y^2) = 0$$

$$x^2 + y^2 + 140\lambda = 0$$

$$\therefore \lambda = -\frac{x^2 + y^2}{140} = -\frac{f}{140}$$

From (3) and (4) $2x - \frac{f}{140}(2x + 4y) = 0$ or $2x - \frac{f}{70}(x + 2y) = 0$

and
$$2y - \frac{f}{35}(x + 3y) = 0 \Rightarrow (140 - f)x - 2fy = 0 \quad \dots(5)$$

$$-fx + (70 - 3f)y = 0 \quad \dots(6)$$

Solving (5) and (6), we get

$$(140 - f)(70 - 3f) = 2f^2 \Rightarrow f^2 - 490f + 9800 = 0$$

$$\begin{aligned} \therefore f &= \frac{490 \pm \sqrt{(490)^2 - 4(9800)}}{2} = \frac{490 \pm 70\sqrt{41}}{2} \\ &= 245 \pm 35\sqrt{41} = 469.1093, 20.8906 \end{aligned}$$

Hence, Maximum distance = $\sqrt{469.1093} = 21.65$

Minimum distance = $\sqrt{20.8906} = 4.57$.

- 134.** Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
[M.D.U., May 2009]

Sol. Consider Lagrange's Function

$$F(x, y) = (x - 0)^2 + (y - 0)^2 + \lambda(5x^2 + 6xy + 5y^2 - 8) = 0$$

For stationary values, $\frac{\partial F}{\partial x} = 2x + 10\lambda x + 6\lambda y = 0$ or $x + \lambda(5x + 3y) = 0$... (1)

$$\frac{\partial F}{\partial y} = 2y + 6\lambda x + 10\lambda y = 0 \quad \text{or} \quad y + \lambda(3x + 5y) = 0 \quad \dots (2)$$

Equations (1) and (2) can be rewritten as

$$(1 + 5\lambda)x + 3\lambda y = 0 \quad \dots (3) \quad (3\lambda)x + (5\lambda + 1)y = 0 \quad \dots (4)$$

Multiplying (3) by 3λ and (4) by $1 + 5\lambda$ and on subtracting, we obtain

$$9\lambda^2 - (5\lambda + 1)^2 = 0 \quad \text{or} \quad 9\lambda^2 - 25\lambda^2 - 10\lambda - 1 = 0$$

$$-16\lambda^2 - 10\lambda - 1 = 0 \quad \text{or} \quad 16\lambda^2 + 10\lambda + 1 = 0$$

$$16\lambda^2 + 8\lambda + 2\lambda + 1 = 0 \quad \text{or} \quad 8\lambda(2\lambda + 1) + 1(2\lambda + 1) = 0$$

$$\Rightarrow 8\lambda + 1 = 0 \text{ and } 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}, -\frac{1}{8}$$

Putting $\lambda = -\frac{1}{2}$ in (1) and (2), we get

$$2x - (5x + 3y) = 0 \Rightarrow x = -y$$

$$2y - (3x + 5y) = 0 \Rightarrow x = -y$$

Putting $x = -y$ in $5x^2 + 6xy + 5y^2 - 8 = 0$, we have

$$5x^2 - 6x^2 + 5x^2 - 8 = 0 \quad \text{or} \quad 4x^2 = 8$$

or $x^2 = 2 \quad \text{or} \quad x = \pm\sqrt{2}$

$\therefore$ Points be $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Putting $\lambda = -\frac{1}{8}$ in (1) and (2), we have

$$8x - 5x - 3y = 0 \Rightarrow x = y$$

$$8y - 3x - 5y = 0 \Rightarrow x = y$$

Putting $x = y$ in $5x^2 + 6xy + 5y^2 - 8 = 0$ or $5x^2 + 6x^2 + 5x^2 - 8 = 0$

or $16x^2 = 8 \quad \text{or} \quad x^2 = \frac{1}{2} \quad \text{or} \quad x = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

Extreme points are $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ value of F at these points are

$$\sqrt{x^2 + y^2} = \sqrt{2 + 2} = 2 \text{ (Maximum)} \quad \text{and} \quad \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 \text{ (Minimum)}$$

- 135.** A tent of given volume has a square base of side $2a$ and has its four sides of height b vertical and is surrounded by a pyramid of height h . Find the values of a and b in terms of h so that the canvas required for its construction be minimum. (M.D.U., Dec. 2007)

Sol. Let V be the volume and S be the surface of the tent.

$$V = 4a^2b + \frac{1}{3}(4a^2)h, \quad S = 8ab + 4a\sqrt{a^2 + h^2}$$

$$\frac{\partial S}{\partial a} + \lambda \frac{\partial V}{\partial a} = 0$$

$$\Rightarrow 8b + 4\sqrt{a^2 + h^2} + \frac{4a^2}{\sqrt{a^2 + h^2}} + \lambda \left(8ab + \frac{8ah}{3} \right) = 0 \quad \dots(1)$$

$$\frac{\partial S}{\partial b} + \lambda \frac{\partial V}{\partial b} = 0 \quad \Rightarrow \quad 8a + 4\lambda a^2 = 0 \quad \dots(2)$$

$$\frac{\partial S}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0 \quad \Rightarrow \quad \frac{4ah}{\sqrt{a^2 + h^2}} + \frac{4}{3}\lambda a^2 = 0 \quad \dots(3)$$

$$\text{From (2),} \quad \lambda a + 2 = 0 \quad \dots(4)$$

$$\text{From (3),} \quad 12ah + 4\lambda a^2 \sqrt{a^2 + h^2} = 0 \quad \text{or} \quad 3h + \lambda a \sqrt{a^2 + h^2} = 0 \quad \dots(5)$$

Using (4), (5),

$$\begin{aligned} \Rightarrow 3h - 2\sqrt{a^2 + h^2} &= 0 \quad \text{or} \quad 9h^2 = 4(a^2 + h^2) \\ 5h^2 &= 4a^2 \quad \text{or} \quad a = \frac{\sqrt{5}h}{2} \end{aligned} \quad \dots(6)$$

From (1), using (4) and (6), we get

$$8b + 4 \cdot \frac{3h}{2} + \frac{2.5h^2}{3h} + (-2) \left(8b + \frac{8h}{3} \right) = 0$$

$$\text{or} \quad 8b + 6h + \frac{10}{3}h - 16b - 16\frac{h}{3} = 0 \quad \text{or} \quad -8b + 6h + \frac{10}{3}h - 16\frac{h}{3} = 0$$

$$-8b = 16\frac{h}{3} - \frac{10h}{3} - 6h = -12\frac{h}{3} = -4h$$

$$\therefore b = \frac{h}{2} \quad \dots(7)$$

Thus, when $a = \frac{\sqrt{5}h}{2}$ and $b = \frac{h}{2}$, we get the minimum value of S (Canvas required for tent construction).

- 136.** The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature a canvas required for text construction the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Sol. Consider Lagrange's function

$$F(x, y, z) = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1) \quad \dots(1)$$

For stationary values, $dF = 0$

$$\Rightarrow [400yz^2 + \lambda(2x)] dx + [400xz^2 + \lambda(2y)] dy + [800xyz + \lambda(2z)] dz = 0$$

$$\Rightarrow 400yz^2 + 2\lambda x = 0 \quad \dots(2)$$

$$400xz^2 + 2\lambda y = 0 \quad \dots(3)$$

$$800xyz + 2\lambda z = 0 \quad \dots(4)$$

Multiplying (2) by x , (3) by y and (4) by z and adding, we get

$$1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0 \quad \text{or} \quad 1600xyz^2 + 2\lambda = 0 \quad \text{or} \quad \lambda = -800xyz^2$$

$$\text{From (2),} \quad 400yz^2 - 1600x^2yz^2 = 0 \quad \text{or} \quad 400 - 1600x^2 = 0 \quad \text{or} \quad x^2 = \frac{1}{4} \quad \text{or} \quad x = \pm \frac{1}{2}$$

$$\text{Similarly,} \quad y = \pm \frac{1}{2}$$

$$\text{From (4),} \quad 800xyz - 1600xyz^3 = 0 \quad \Rightarrow \quad 1 - 2z^2 = 0 \quad \Rightarrow \quad z = \pm \frac{1}{2}$$

Putting values of x, y, z in T , we get

$$T = 400xyz^2 = 400 \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = 50.$$

- 137.** Use Lagrange's method of undetermined multipliers to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$, $xyz + 1 = 0$.

Sol. Here,

$$u = x^2 + y^2 + z^2$$

subject to

$$x + y + z = 1 \quad \text{and} \quad xyz + 1 = 0$$

For a minimum u ,

$$du = 2xdx + 2ydy + 2zdz = 0$$

$$\Rightarrow \quad xdx + ydy + zdz = 0 \quad \dots(1)$$

$$\text{Also} \quad dx + dy + dz = 0 \quad \dots(2) \quad \text{and} \quad yzdx + zxdy + xydz = 0 \quad \dots(3)$$

Multiplying (1) by 1, (2) by λ and (3) by μ and adding, we get

$$x + \lambda + \mu yz = 0 \quad \dots(4) \quad y + \lambda + \mu zx = 0 \quad \dots(5)$$

$$z + \lambda + \mu xy = 0 \quad \dots(6)$$

Subtracting (5) from (4), we get

$$x - y + \mu z(y - x) = 0 \quad \Rightarrow \quad (x - y)(1 - \mu z) = 0$$

$$\Rightarrow \quad x = y \quad \text{or} \quad \mu z = 1$$

$$\text{Similarly,} \quad y = z \quad \text{or} \quad \mu x = 1 \quad \text{[From (5) and (6)]}$$

From these, we choose the solution,

$$x = y \quad \text{and} \quad \mu = \frac{1}{x} \quad \dots(7)$$

Multiplying (4) and (5) by x and y respectively and subtracting (5) from (4), we get

$$x^2 - y^2 + \lambda(x - y) = 0 \quad \Rightarrow \quad (x - y) + (x + y + \lambda) = 0$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x + y + \lambda = 0$$

$$\text{Similarly,} \quad y = z \quad \text{or} \quad y + z + \lambda = 0 \quad \dots(8)$$

$$\text{From (4), (7) and (8),} \quad x + \lambda + \mu yz = 0 \quad \text{or} \quad x - (y + z) + \frac{1}{x}(yz) = 0$$

$$\text{or} \quad x - (1 - x) + \frac{1}{x} \left(-\frac{1}{x} \right) = 0 \quad \text{or} \quad x - 1 + x - \frac{1}{x^2} = 0$$

$$2x - \frac{1}{x^2} - 1 = 0 \quad \text{or} \quad 2x^3 - 1 - x^2 = 0$$

$$(1 - x)(2x^2 + x + 1) = 0$$

Real solution is $x = 1$, $y = x = 1$,

$$z = \frac{xyz}{xy} = \frac{-1}{1 \cdot 1} = -1$$

Hence the minimum

$$u = x^2 + y^2 + z^2 = (1)^2 + (1)^2 + (-1)^2 = 3$$

Since u can increase indefinitely for numerically large x, y, z therefore the above value is minimum.

- 138.** If x and y satisfy the relation $ax^2 + by^2 = ab$, prove that the extreme values of the function $u = x^2 + xy + y^2$ are given by the roots of the equation $4(u - a)(u - b) = ab$.

Sol. Here, $u = x^2 + xy + y^2$

Consider Lagrange's function

$$F(x, y) = x^2 + xy + y^2 + \lambda \left(\frac{x^2}{b} + \frac{y^2}{a} - 1 \right) \quad \dots(1)$$

For stationary values, $dF = 0$

$$\Rightarrow \left[(2x + y) + \lambda \left(\frac{2x}{b} \right) \right] dx + \left[(x + 2y) + \lambda \left(\frac{2y}{a} \right) \right] dy = 0$$

$$\Rightarrow 2x + y + \frac{2\lambda x}{b} = 0 \quad \dots(2) \quad x + 2y + \frac{2\lambda y}{a} = 0 \quad \dots(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$2(x^2 + xy + y^2) + 2\lambda \left(\frac{x^2}{b} + \frac{y^2}{a} \right) = 0 \quad \Rightarrow \quad 2u + 2\lambda = 0 \quad \text{or} \quad \lambda = -u \quad \dots(4)$$

$$\text{From (2),} \quad 2x + y - \frac{2ux}{b} = 0 \quad \Rightarrow \quad 2 \left(1 - \frac{u}{b} \right) x + y = 0 \quad \dots(5)$$

$$\text{From (3),} \quad x + 2y - \frac{2uy}{a} = 0 \quad \Rightarrow \quad 2 \left(1 - \frac{u}{a} \right) y + x = 0 \quad \dots(6)$$

$$\text{From (5),} \quad \frac{x}{y} = \frac{-1}{2 \left(1 - \frac{u}{b} \right)} \quad \text{From (6),} \quad \frac{x}{y} = -2 \left(1 - \frac{u}{a} \right)$$

Equating the values of $\frac{x}{y}$, we get

$$-\frac{1}{2 \left(1 - \frac{u}{b} \right)} = -2 \left(1 - \frac{u}{a} \right) \quad \text{or} \quad 4 \left(1 - \frac{u}{a} \right) \left(1 - \frac{u}{b} \right) = 1$$

$$\Rightarrow 4(u - a)(u - b) = ab, \quad \text{which is the required equation.}$$

- 139.** A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.

Sol. Let x, y, z cm. be the edges of box and S be its surface.

$$S = xy + 2yz + 2zx \quad \dots(1) \quad xyz = 32 \quad \dots(2)$$

$$\therefore z = \frac{32}{xy}, \quad S = xy + \frac{64}{x} + \frac{64}{y}, \quad \frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$$

$$\Rightarrow y = \frac{64}{x^2}, \quad x = \frac{64}{y^2} = \frac{64 \cdot x^4}{(64)^2} = \frac{x^4}{64} \quad \text{or} \quad x^3 = 64 \Rightarrow x = 4, \quad \therefore y = 4$$

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad 's' = \frac{\partial^2 S}{\partial x \partial y} = 1, \quad t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

$$\therefore rt - s^2 = \frac{128}{64} \times \frac{128}{64} - 1 = 3 \text{ (+ve)} > 0.$$

$$r = \frac{128}{64} = 2 (> 0)$$

r is also positive when $rt - s^2$ is positive.

Hence S is minimum for $x = y = 4$.

From equation (2), i.e., $xyz = 32$

$$16z = 32 \quad \text{or} \quad z = 2.$$

Hence $x = 4$ cm, $y = 4$ cm, $z = 2$ cm.

Volume is 32 c.c. and surface S is minimum.

- 140.** Divide $2u$ into three parts such that the continued product of the first, square of the second and cube of the third, may be maximum.

Sol. Let x, y, z be the three parts

$$x + y + z = 2u \quad \dots(1) \quad P = xyz^2z^3 \quad \dots(2)$$

To maximise P subject to condition (1), consider Lagrange's function

$$F(x, y, z) = P + \lambda(x + y + z - 2u) = xyz^2z^3 + \lambda(x + y + z - 2u)$$

$$\frac{\partial F}{\partial x} = y^2z^3 + \lambda = 0 \quad \dots(3) \quad \frac{\partial F}{\partial y} = 2xyz^3 + \lambda = 0 \quad \dots(4)$$

$$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda = 0 \quad \dots(5)$$

Subtracting (4) from (3), we get

$$y^2z^3 - 2xyz^3 = 0 \quad \text{or} \quad yz^3(y - 2x) = 0 \Rightarrow y = 2x$$

Subtracting (5) from (4), we get

$$2xyz^3 - 3xy^2z^2 = 0 \quad \text{or} \quad xyz^2(2z - 3y) = 0$$

$$\Rightarrow \quad 2z - 3y = 0 \quad \text{or} \quad z = \frac{3}{2}y$$

$$\therefore \text{ From (1), } x + y + z = 2u \quad \text{or} \quad x + 2x + 3x = 2u \quad \text{or} \quad 6x = 2u$$

$$x = \frac{u}{3} \quad \dots(6) \quad \therefore \quad y = \frac{2u}{3}, \quad z = u \quad \dots(7)$$

Hence, $x = \frac{u}{3}$, $y = \frac{2u}{3}$, $z = u$ are the three parts.

- 141.** Find points on the surface $z^2 = xy + 1$ nearest to the origin. (M.D.U., 2005, May 2007)

Sol. Let (x, y, z) be any point which is at a least distance from the origin.

$$\text{So} \quad d = \sqrt{x^2 + y^2 + z^2} \quad \text{or} \quad d^2 = x^2 + y^2 + z^2$$

$$\text{Given} \quad z^2 - xy - 1 = 0$$

$$\therefore \quad F = d^2 + \lambda(z^2 - xy - 1)$$

$$F = x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1) \quad \dots(1)$$

$$\frac{\partial F}{\partial x} = 2x - \lambda y = 0 \quad \dots(2) \quad \frac{\partial F}{\partial y} = 2y - \lambda x = 0 \quad \dots(3)$$

$$\frac{\partial F}{\partial z} = 2z + 2z\lambda = 0 \quad \dots(4)$$

Multiplying (2) by y and (3) by x and subtracting (3) from (2), we get

$$2xy - \lambda y^2 - 2xy + \lambda x^2 = 0$$

$$\lambda(x^2 - y^2) = 0 \quad \text{or} \quad x^2 = y^2 \quad \text{i.e.,} \quad x = y$$

$$\text{From (4),} \quad 2z = -2z\lambda \Rightarrow \lambda = -1$$

$$\text{Using (2),} \quad 2x + y = 0 \quad \text{or} \quad 2x + x = 0 \quad \text{or} \quad 3x = 0$$

$$\text{Now,} \quad \begin{matrix} x = 0, & y = 0 \\ z^2 = xy + 1 & \text{or} & z^2 = 1 \quad \text{or} & z = \pm 1 \end{matrix}$$

Thus the required point is $(0, 0, \pm 1)$.

- 142.** Divide 24 into 3 parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (M.D.U., 2008, 2009, U.P.T.U., 2009)

Sol. Let x, y, z be 3 parts.

$$x + y + z = 24 \quad \dots(1) \quad P = xyz^2z^3 \quad \dots(2)$$

$$\therefore F = P + \lambda(x + y + z - 24) \quad \text{or} \quad F = xyz^2z^3 + \lambda(x + y + z - 24)$$

$$\frac{\partial F}{\partial x} = y^2z^3 + \lambda = 0 \quad \dots(3) \quad \frac{\partial F}{\partial y} = 2xyz^3 + \lambda = 0 \quad \dots(4)$$

$$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda = 0 \quad \dots(5)$$

Subtracting (4) from (3), we get

$$y^2z^3 - 2xyz^3 = 0 \quad \text{or} \quad yz^3(y - 2x) = 0 \quad \text{or} \quad y - 2x = 0 \quad \text{or} \quad y = 2x$$

Subtracting (5) from (4), we get

$$2xyz^3 - 3xy^2z^2 = 0 \quad \text{or} \quad xyz^2(2z - 3y) = 0 \quad \text{or} \quad 2z - 3y = 0 \quad \text{or} \quad 2z = 3y \quad \text{or} \quad z = \frac{3}{2}y$$

$$\text{As} \quad x + y + z = 24 \quad \text{or} \quad x + 2x + 3x = 24 \quad \text{or} \quad x = 4$$

$$\therefore y = 8, \quad z = 12.$$

Hence the three parts are 4, 8, 12.

- 143.** Find the minimum value of $x^2 + y^2 + z^2$ when $xyz = a^3$.

$$\text{Sol. Let} \quad u = x^2 + y^2 + z^2 \quad \dots(1) \quad \text{where} \quad \phi(x, y, z) = xyz - a^3 = 0 \quad \dots(2)$$

Consider Lagrange's function

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(xyz - a^3)$$

For stationary values, $dF = 0$

$$\Rightarrow (2x + yz\lambda)dx + (2y + \lambda xz)dy + (2z + \lambda xy)dz = 0$$

$$\Rightarrow 2x + \lambda yz = 0 \quad \dots(3) \quad 2y + \lambda xz = 0 \quad \dots(4) \quad 2z + \lambda xy = 0 \quad \dots(5)$$

Multiplying (3) by x , (4) by y , (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + 3\lambda xyz = 0 \quad \text{or} \quad 2u + 3\lambda a^3 = 0 \quad \text{or} \quad \lambda = -\frac{2u}{3a^3}$$

From (3), (4) and (5),

$$2x + \frac{a^3}{x} \cdot \left(-\frac{2u}{3a^3}\right) = 0 \quad \text{[Using (2)]}$$

$$\text{or} \quad 2x - \frac{2u}{3x} = 0 \quad \text{or} \quad x^2 = \frac{u}{3}$$

$$2y - \frac{2u}{3y} = 0 \quad \text{or} \quad y^2 = \frac{u}{3}$$

$$\Rightarrow x^2 = y^2 \quad \text{or} \quad x = y$$

$$\text{Similarly, } 2z - \frac{2u}{3z} = 0 \quad \text{or} \quad z^2 = \frac{u}{3}$$

$$\therefore z = x = y$$

$$\text{Since } xyz = a^3 \quad \text{or} \quad x^3 = a^3 \quad \text{or} \quad x = a$$

$$\therefore u = x^2 + y^2 + z^2 = 3x^2 \quad \text{or} \quad u = 3a^2$$

$\therefore$ Minimum value of $u = 3a^2$.

144. Evaluate the integral $\int_0^1 \frac{x^\alpha - 1}{\log x} \cdot dx$ by applying differentiation under the integral sign ($\alpha \geq 0$).
(M.D.U., 2005; U.P.T.U., 2009; A.U.U.P., 2007)

Sol. Let
$$F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} \cdot dx \quad \dots(1)$$

Differentiating both sides w.r.t. α ,

$$\begin{aligned} F'(\alpha) &= \int_0^1 \frac{\partial}{\partial \alpha} \left[\frac{x^\alpha - 1}{\log x} \right] dx = \int_0^1 \frac{1}{\log x} \cdot x^\alpha \log x \, dx \quad \left[\because \frac{d}{dx} (a^x) = a^x \log a \right] \\ &= \int_0^1 x^\alpha \cdot dx = \left[\frac{x^{\alpha+1}}{\alpha+1} \right]_0^1 = \frac{1}{1+\alpha} \end{aligned}$$

Integrating both sides w.r.t. α ,

$$F(\alpha) = \log(1 + \alpha) + c \quad \dots(2)$$

From (1), when $\alpha = 0$, $F(0) = 0$.

$$\therefore \text{From (2),} \quad 0 = \log 1 + c = c \Rightarrow c = 0$$

$$F(\alpha) = \log(1 + \alpha) \quad \text{where } \alpha > -1$$

Hence,
$$\int_0^1 \frac{x^\alpha - 1}{\log x} \cdot dx = \log(1 + \alpha).$$

145. Evaluate $\int_0^a \frac{\log(1+ax)}{1+x^2} \cdot dx$ and hence show that $\int_0^1 \frac{\log(1+x)}{1+x^2} \cdot dx = \frac{\pi}{8} \log 2$.

(M.D.U., 2006, May 2009)

Sol. Let
$$F(a) = \int_0^a \frac{\log(1+ax)}{1+x^2} \cdot dx \quad \dots(1)$$

Here the upper limit involves the parameter a . Differentiating w.r.t. a

$$\begin{aligned} F'(a) &= \int_0^a \frac{\partial}{\partial a} \left[\frac{\log(1+ax)}{1+x^2} \right] dx + \frac{\log(1+a^2)}{1+a^2} \cdot \frac{d}{da} (a) - \frac{\log(1+a \cdot 0)}{1+0} \cdot \frac{d}{da} (0) \\ &= \int_0^a \frac{1}{1+x^2} \cdot \frac{1}{1+ax} \cdot x dx + \frac{\log(1+a^2)}{1+a^2} \\ &= \int_0^a \frac{x}{(1+ax)(1+x^2)} dx + \frac{\log(1+a^2)}{1+a^2} \quad \dots(2) \end{aligned}$$

Let
$$\frac{x}{(1+ax)(1+x^2)} = \frac{A}{1+ax} + \frac{Bx+C}{1+x^2} \Rightarrow x = A(1+x^2) + (Bx+C)(1+ax)$$

Putting $1+ax = 0$ i.e., $x = -\frac{1}{a}$,

$$-\frac{1}{a} = A \left(1 + \frac{1}{a^2} \right) \quad \text{or} \quad A = -\frac{a}{1+a^2}$$

Comparing co-efficients of x^2 , we get

$$0 = A + B \cdot a \Rightarrow B = -\frac{A}{a} = \frac{1}{1+a^2}$$

Comparing constant terms, we get

$$0 = A + C \Rightarrow C = -A = \frac{a}{1+a^2}$$

$$\therefore \frac{x}{(1+ax)(1+x^2)} = -\frac{a}{(1+a^2)(1+ax)} + \frac{x+a}{(1+a^2)(1+x^2)} = \frac{1}{1+a^2} \left[\frac{-a}{1+ax} + \frac{x+a}{1+x^2} \right]$$

$$\Rightarrow \int_0^a \frac{x}{(1+ax)(1+x^2)} \cdot dx = \frac{1}{1+a^2} \left[-\log(1+ax) + \frac{1}{2} \log(1+x^2) + a \tan^{-1} x \right]_0^a$$

$$= \frac{1}{1+a^2} \left[-\log(1+a^2) + \frac{1}{2} \log(1+a^2) + a \tan^{-1} a \right]$$

$$= \frac{1}{1+a^2} \left[-\frac{1}{2} \log(1+a^2) + a \tan^{-1} a \right]$$

$$\therefore \text{From (2), we get } F'(a) = \frac{1}{1+a^2} \left[-\frac{1}{2} \log(1+a^2) + a \tan^{-1} a \right] + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2} \left[\frac{1}{2} \log(1+a^2) + a \tan^{-1} a \right]$$

$$\begin{aligned} \text{Integrating w.r.t. } a, \quad F(a) &= \frac{1}{2} \int \log(1+a^2) \cdot \frac{1}{1+a^2} \cdot da + \int \frac{a \tan^{-1} a}{1+a^2} da + c \\ &= \frac{1}{2} \left[\log(1+a^2) \tan^{-1} a - \int \frac{2a}{1+a^2} \tan^{-1} a \cdot da \right] + \int \frac{a \tan^{-1} a}{1+a^2} da + c \end{aligned}$$

$$F(a) = \frac{1}{2} \log(1+a^2) \tan^{-1} a + c \quad \dots(3)$$

$$\text{Putting } a = 0 \text{ in (1), } F(0) = 0$$

$$\text{Putting } a = 0 \text{ in (3), } F(0) = c \text{ or } c = 0$$

$$\therefore \text{From (3), we have } F(a) = \frac{1}{2} \log(1+a^2) \tan^{-1} a.$$

$$\text{Hence } \int_0^a \frac{\log(1+ax)}{1+x^2} dx = \frac{1}{2} \log(1+a^2) \tan^{-1} a$$

Putting $a = 1$, we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{1}{2} \log 2 \cdot \tan^{-1}(1) = \frac{\pi}{8} \log 2.$$

146. Evaluate $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$ by applying differentiation under the integral sign.

(M.D.U., Dec., 2006, May 2008)

Sol. Let $F(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$... (1)

Differentiating both sides w.r.t. a ,

$$\begin{aligned} F'(a) &= \int_0^{\infty} \frac{\partial}{\partial a} \left[\frac{\tan^{-1} ax}{x(1+x^2)} \right] dx = \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2 x^2} \cdot x dx \\ &= \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2 x^2)} \cdot dx = \int_0^{\infty} \frac{1}{(1-a^2)} \cdot \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2 x^2} \right] dx \\ &= \frac{1}{1-a^2} \left[\tan^{-1} x \right]_0^{\infty} - \frac{a^2}{1-a^2} \int_0^{\infty} \frac{dx}{1+a^2 x^2} \\ &= \frac{1}{1-a^2} \left[\tan^{-1} \infty - \tan^{-1} 0 \right] - \frac{a^2}{1-a^2} \cdot \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2 + \frac{1}{a^2}} \\ &= \frac{\pi}{2} \left(\frac{1}{1-a^2} \right) - \frac{1}{1-a^2} \cdot a \left[\tan^{-1} \left(\frac{x}{1/a} \right) \right]_0^{\infty} \\ &= \frac{\pi}{2(1-a^2)} - \frac{a}{1-a^2} [\tan^{-1} \infty - \tan^{-1} 0] \\ &= \frac{\pi}{2(1-a^2)} - \frac{a}{1-a^2} \cdot \frac{\pi}{2} = \frac{\pi}{2(1-a^2)} \{1-a\} = \frac{\pi}{2(1+a)} \\ F'(a) &= \frac{\pi}{2} \cdot \frac{1}{1+a} \end{aligned}$$

Integrating both sides w.r.t. a ,

$$F(a) = \frac{\pi}{2} \log(1+a) + c \quad \dots (2)$$

From (1), when $a = 0$, $F(0) = 0$

$$\therefore \text{From (2),} \quad 0 = \frac{\pi}{2} \log 1 + c = 0 + c \Rightarrow c = 0 \quad \therefore F(a) = \frac{\pi}{2} \log(1+a)$$

Hence, $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} \cdot dx = \frac{\pi}{2} \log(1+a).$

147. Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$.

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$... (1)

Let us first evaluate $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Dividing the numerator, denominator by $\cos^2 x$

$$\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$

Put $\tan x = t$, $\sec^2 x dx = dt$, when $x = 0$, $t = 0$; when $x = \frac{\pi}{2}$, $t = \infty$.

$$\begin{aligned} \therefore \int_0^{\pi/2} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x} &= \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int_0^{\infty} \frac{dt}{t^2 + \frac{a^2}{b^2}} \\ &= \frac{1}{b^2} \cdot \frac{1}{\frac{a}{b}} \left[\tan^{-1} \frac{t}{\frac{a}{b}} \right]_0^{\infty} = \frac{1}{ab} (\tan^{-1} \infty - \tan^{-1} 0) = \frac{\pi}{2ab} \end{aligned}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab} \quad \dots(2)$$

Differentiating both sides w.r.t. a

$$\int_0^{\pi/2} \frac{\partial}{\partial a} [(a^2 \cos^2 x + b^2 \sin^2 x)^{-1}] dx = \frac{\partial}{\partial a} \left(\frac{\pi}{2ab} \right)$$

$$\text{or} \quad \int_0^{\pi/2} -(a^2 \cos^2 x + b^2 \sin^2 x)^{-2} \cdot 2a \cos^2 x dx = \frac{\pi}{2a^2 b}$$

$$\text{or} \quad \int_0^{\pi/2} -(a^2 \cos^2 x + b^2 \sin^2 x)^{-2} \cdot 2a \cos^2 x dx = \frac{\pi}{4a^3 b} \quad \dots(3)$$

Similarly, differentiating (2) partially w.r.t. b , we get

$$\int_0^{\pi/2} \frac{\sin^2 x \cdot dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4ab^3} \quad \dots(4)$$

Adding (3) and (4), we get

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} &= \frac{\pi}{4a^3 b} + \frac{\pi}{4ab^3} \\ \Rightarrow \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} &= \frac{\pi}{4a^3 b^3} (a^2 + b^2) \end{aligned}$$

$$\text{Hence} \quad I = \frac{\pi(a^2 + b^2)}{4a^3 b^3}.$$

$$148. \text{ If } |a| < 1, \text{ prove that } \int_0^{\pi} \log(1 + a \cos x) dx = \pi \log \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 - a^2} \right].$$

$$\text{Sol. Let} \quad F(a) = \int_0^{\pi} \log(1 + a \cos x) dx \quad \dots(1)$$

Differentiating w.r.t. a ,

$$\begin{aligned}
 F'(a) &= \int_0^\pi \frac{\partial}{\partial a} [\log(1 + a \cos x)] dx \\
 F'(a) &= \int_0^\pi \frac{1}{1 + a \cos x} \cdot \cos x \, dx = \frac{1}{a} \int_0^\pi \left(1 - \frac{1}{1 + a \cos x} \right) dx \\
 &= \frac{1}{a} [x]_0^\pi - \frac{1}{a} \int_0^\pi \frac{dx}{1 + a \cos x} = \frac{\pi}{a} - \frac{1}{a} \cdot \frac{1}{\sqrt{1-a^2}} \left[\cos^{-1} \frac{a+1 \cdot \cos x}{1+a \cos x} \right]_0^\pi \quad [\because |a| < 1] \\
 &= \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} \left[\cos^{-1} \frac{a-1}{1-a} - \cos^{-1} \frac{a+1}{1+a} \right] = \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} [\cos^{-1}(-1) - \cos^{-1}(1)] \\
 &= \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} [\pi - 0] = \frac{\pi}{a} \left[1 - \frac{1}{\sqrt{1-a^2}} \right]
 \end{aligned}$$

Integrating w.r.t. a ,

$$F(a) = \pi \log a - \pi \int \frac{da}{a\sqrt{1-a^2}} + c$$

$$\text{Put } a = \frac{1}{t}, da = -\frac{1}{t^2} dt$$

$$\therefore \int \frac{da}{a\sqrt{1-a^2}} = - \int \frac{dt}{\sqrt{t^2-1}} = -\log(t + \sqrt{t^2-1})$$

$$\begin{aligned}
 \therefore F(a) &= \pi \log a + \pi \log \left(\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1} \right) + c = \pi \log \left[a \left(\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1} \right) \right] + c \\
 &= \pi \log (1 + \sqrt{1-a^2}) + c \quad \dots(2)
 \end{aligned}$$

From (1), when $a = 0$, $F(0) = 0$

$$\therefore \text{From (2), } F(0) = \pi \log 2 + c \Rightarrow c = -\pi \log 2$$

$$\therefore F(a) = \pi \log (1 + \sqrt{1-a^2}) - \pi \log 2 = \pi \log \left\{ \frac{1 + \sqrt{1-a^2}}{2} \right\}$$

$$\text{Hence } \int_0^\pi \log(1 + a \cos x) \, dx = \pi \log \left[\frac{1}{2} + \frac{1}{2} \sqrt{1-a^2} \right].$$

149. By successive differentiation of $\int_0^1 x^m \, dx = \frac{1}{m+1}$ w.r.t. m , evaluate $\int_0^1 x^m (\log x)^n \, dx$.

$$\text{Sol. Given } \int_0^1 x^m \cdot dx = \frac{1}{m+1}$$

Differentiating w.r.t. m , we have

$$\int_0^1 \frac{\partial}{\partial m} (x^m) \, dx = \frac{d}{dm} \left(\frac{1}{m+1} \right) \Rightarrow \int_0^1 x^m \log x \, dx = -\frac{1}{(m+1)^2}$$

Differentiating again w.r.t. m , we have

$$\int_0^1 \frac{\partial}{\partial m} (x^m \log x) dx = \frac{d}{dm} \left[-\frac{1}{(m+1)^2} \right] \Rightarrow \int_0^1 x^m (\log x)^2 dx = \frac{(-1)(-2)}{(m+1)^3}$$

Similarly,
$$\int_0^1 x^m (\log x)^3 dx = \frac{(-1)(-2)(-3)}{(m+1)^4}$$

$$\int_0^1 x^m (\log x)^4 dx = \frac{(-1)(-2)(-3)(-4)}{(m+1)^5}$$

$$\int_0^1 x^m (\log x)^n dx = \frac{(-1)(-2)(-3)(-4) \dots (-n)}{(m+1)^{n+1}} = \frac{(-1)^n \cdot n!}{(m+1)^{n+1}}.$$

150. If $y = \int_0^x f(t) \sin [k(x-t)] dt$, prove that y satisfies the differential equation $\frac{d^2 y}{dx^2} + k^2 y = kf(x)$.

Sol. Given $y = \int_0^x f(t) \sin [k(x-t)] dt$...(1)

Here the upper limit involves the parameter x . Differentiating w.r.t. x , we have

$$\begin{aligned} \frac{dy}{dx} &= \int_0^x \frac{\partial}{\partial x} [f(t) \sin [k(x-t)]] dt + f(x) \sin [k(x-x)] \cdot \frac{d}{dx}(x) - f(0) \sin [k(x-0)] \cdot \frac{d}{dx}(0) \\ &= \int_0^x kf(t) \cos [k(x-t)] dt \end{aligned}$$

Differentiating again w.r.t. x , we have

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \int_0^x \frac{\partial}{\partial x} [kf(t) \cos [k(x-t)]] dt + kf(x) \cos [k(x-x)] \cdot \frac{d}{dx}(x) - kf(0) \cos [k(x-0)] \cdot \frac{d}{dx}(0) \\ &= \int_0^x -k^2 f(t) \sin [k(x-t)] dt + kf(x) = -k^2 \int_0^x f(t) \sin [k(x-t)] dt + kf(x) \\ &= -k^2 \cdot y + k \cdot f(x) \end{aligned}$$

[Using (1)]

$$\Rightarrow \frac{d^2 y}{dx^2} + k^2 y = kf(x).$$

Hence proved.

151. Evaluate $\int_0^\infty e^x \frac{\sin ax}{x} \cdot dx$, by the method of differentiation under the integral sign.

(M.D.U., Dec, 2006; U.P.T.U., 2007; A.U.U.P., 2009)

Sol. Let
$$I(a) = \int_0^\infty e^x \frac{\sin ax}{x} \cdot dx$$

By Leibnitz's Rule of differentiation under the integral sign

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \left(e^x \frac{\sin ax}{x} \right) dx = \int_0^\infty \frac{e^x}{x} \cdot \cos ax \cdot x \cdot dx = \int_0^\infty e^x \cos ax \cdot dx$$

$$\therefore \int_0^{\infty} e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$\begin{aligned} \therefore I'(a) &= \int_0^{\infty} e^x \cos ax \, dx = \frac{1}{1+a^2} \left[e^x [\cos ax + a \sin ax] \right]_0^{\infty} \\ &= \frac{1}{1+a^2} \cdot (-1) = -\frac{1}{1+a^2} \end{aligned}$$

Integrating both sides w.r.t. a ,

$$I(a) = -\tan^{-1} a + c \quad \text{as } a \rightarrow \infty, \quad I(\infty) = -\tan^{-1} \infty + c = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\therefore I(a) = \int_0^x e^x \frac{\sin ax}{x} dx = \frac{\pi}{2} - \tan^{-1} a.$$

152. Prove that for $y > 0$, $\int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \cot^{-1} y$.

Sol. Let $I(y) = \int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx$

Differentiating under the integral sign, we have

$$I'(y) = \int_0^{\infty} \frac{\partial}{\partial y} \left(e^{-xy} \cdot \frac{\sin x}{x} \right) dx = \int_0^{\infty} (-x) e^{-xy} \frac{\sin x}{x} dx = - \int_0^{\infty} e^{-xy} \sin x \, dx$$

$$\therefore \int_0^{\infty} e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

Here, $a = -y$, $b = 1$

$$\begin{aligned} \therefore I'(y) &= - \int_0^{\infty} e^{-xy} \sin x \, dx = - \left[\frac{e^{-xy}}{1+y^2} (-y \sin x - \cos x) \right]_0^{\infty} \\ &= - \frac{1}{1+y^2} \cdot [(-1)(-1)] = - \frac{1}{1+y^2} \end{aligned}$$

Integrating both sides w.r.t. y ,

$$I(y) = -\tan^{-1} y + c$$

As $y \rightarrow \infty$, $I(\infty) = 0$

$$I(\infty) = -\tan^{-1} \infty + c = -\frac{\pi}{2} + c = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\therefore I(y) = \frac{\pi}{2} - \tan^{-1} y = \cot^{-1} y.$$

153. Prove that $\int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} dx = \pi \alpha$.

(M.D.U., Dec., 2008)

Sol. Let $F(\alpha) = \int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} \cdot dx$

Differentiating both sides w.r.t. α , we get

$$\begin{aligned} F'(\alpha) &= \int_0^\pi \frac{\cos \alpha \cdot \cos x}{(1 + \sin \alpha \cos x)} \cdot \frac{1}{\cos x} \cdot dx = \int_0^\pi \frac{\cos \alpha}{1 + \sin \alpha \cos x} \cdot dx \\ &= \cos \alpha \int_0^\pi \frac{dx}{1 + \sin \alpha \cos x} = \cos \alpha \cdot \left\{ \frac{1}{\sqrt{1 - \sin^2 \alpha}} \left[\cos^{-1} \frac{\sin \alpha + \cos x}{1 + \sin \alpha \cos x} \right]_0^\pi \right\} \\ &= \cos^{-1} \frac{\sin \alpha + \cos \pi}{1 + \sin \alpha \cos \pi} - \cos^{-1} \frac{\sin \alpha + \cos 0}{1 + \sin \alpha \cos 0} = \cos^{-1}(-1) - \cos^{-1}(1) = \pi \end{aligned}$$

Integrating w.r.t. α , $F(\alpha) = \pi\alpha + C$

Now, at $\alpha = 0$, $F(\alpha) = 0 \Rightarrow C = 0$

$$\therefore \int_0^\pi \frac{\log(1 + \sin \alpha \cdot \cos x)}{\cos x} \cdot dx = \pi\alpha.$$

Hence proved.

154. Comment on Approximation of errors.

Sol. Let $z = f(x, y)$

Let δx , δy , δz be small changes or errors in x , y and z , then

$$\delta z = \frac{\partial f}{\partial x} \cdot \delta x + \frac{\partial f}{\partial y} \cdot \delta y$$

Replacing δx , δy and δz by dx , dy , dz respectively then, we have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Note : If δx is the error in x , then

$$\text{relative error} = \frac{\delta x}{x}; \quad \text{percentage error} = \frac{\delta x}{x} \times 100.$$

155. Find the percentage error in the area of an ellipse when an error of +1 percent is made in measuring the major and minor axes.

Sol. If x and y are semi-major and semi-minor axes of the ellipse, then its area A is given by

$$A = \pi xy$$

Taking logs, $\log A = \log \pi + \log x + \log y$

$$\text{Taking differentials} \quad \frac{dA}{A} = 0 + \frac{dx}{x} + \frac{dy}{y}$$

$$\text{or} \quad \frac{dA}{A} \times 100 = \frac{dx}{x} \times 100 + \frac{dy}{y} \times 100 = 1 + 1 = 2$$

$\therefore$ Error in the area = 2%.

156. The period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to the possible errors upto 1% in l and 2.5% in g . (U.P.T.U., 2009)

$$\text{Sol.} \quad bT = 2\pi\sqrt{\frac{l}{g}} \quad \text{Taking log, } \log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

On differentiating, we get

$$\frac{dT}{T} = \frac{1}{2} \frac{dl}{l} - \frac{1}{2} \frac{dg}{g}$$

$$\frac{dT}{T} \times 100 = \frac{1}{2} \left[\frac{dl}{l} \times 100 - \frac{dg}{g} \times 100 \right] = \frac{1}{2} [1 \pm 2.5]$$

Maximum error in $T = \frac{1}{2} [3.5] = 1.75\%$.

- 157.** The work that must be done to propel a ship of displacement D for a distance 's' in time 't' is proportional to $\frac{S^2 D^{2/3}}{t^2}$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time diminished by 1% and the distance decreased by 2%.

Sol. Let the work done be W , then

$$W \propto \frac{S^2 D^{2/3}}{t^2} \Rightarrow W = \frac{k \cdot S^2 D^{2/3}}{t^2}, \text{ where } k \text{ is a constant.}$$

$$\text{Taking logs, } \log W = \log k + 2 \log S + \frac{2}{3} \log D - 2 \log t$$

Differentiating, we get

$$\frac{dW}{W} = \frac{2dS}{S} + \frac{2}{3} \frac{dD}{D} - \frac{2dt}{t}$$

$$\text{or } \frac{dW}{W} \times 100 = \frac{2dS}{S} \times 100 + \frac{2}{3} \frac{dD}{D} \times 100 - \frac{2dt}{t} \times 100 = 2(-2) + \frac{2}{3}(1) - 2(-1) = -\frac{4}{3}$$

$$\therefore \text{Approximate increase of work} = -\frac{4}{3}\%.$$

- 158.** The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Sol. Let the diameter and altitude of the can be denoted by D and H respectively.

$$\text{Then radius} = \frac{D}{2}$$

(i) The volume V of the can is given by

$$V = \pi r^2 h = \frac{\pi}{4} D^2 H$$

$$dV = \frac{\partial V}{\partial D} dD + \frac{\partial V}{\partial H} dH = \frac{\pi}{4} [2DHdD + D^2 dH] = \frac{\pi}{4} [2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1] = \frac{\pi}{4} (6.4)$$

$$dV = 1.6\pi \text{ cu.cm.}$$

(ii) The lateral surface S of the can is given by

$$S = 2\pi rh = \pi DH$$

$$\therefore dS = \frac{\partial S}{\partial D} dD + \frac{\partial S}{\partial H} dH = \pi H dD + \pi D dH = \pi(6 \times 0.1 + 4 \times .1) = \pi \text{ sq. cm.}$$

- 159.** If the base radius and height of a cone are measured as 4 and 8 inches with a possible error of 0.04 and 0.08 inches respectively, calculate the % error in calculating volume of the cone.

$$\text{Sol. Volume of cone, } V = \frac{1}{3} \pi r^2 h$$

$$\text{Taking log, } \log V = \log \frac{1}{3} + \log \pi + 2 \log r + \log h$$

Differentiating, we get $\frac{\delta V}{V} = 2 \frac{\delta r}{r} + \frac{\delta h}{h} \Rightarrow \frac{\delta V}{V} = 2 \left(\frac{.04}{4} \right) + \frac{.08}{8} = .03$

$\therefore$ Percentage (%) error in volume = $.03 \times 100 = 3\%$.

- 160.** A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surrounded by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of balloon. (U.P.T.U., 2006)

Sol. Here, $r = 1.5$ m, $h = 4$ m.

Let V be the volume of the balloon.

Then,
$$V = \pi r^2 h + \frac{2}{3} \pi r^3 + \frac{2}{3} \pi r^3 = \pi r^2 h + \frac{4}{3} \pi r^3$$

Now,
$$\begin{aligned} \delta V &= \pi \cdot 2r \delta r \cdot h + \pi \cdot r^2 \delta h + \frac{4}{3} \pi \cdot 3r^2 \cdot \delta r \\ &= \pi r [2h \delta r + r \cdot \delta h + 4r \delta r] \\ \frac{\delta V}{V} &= \frac{\pi r [2(h + 2r) \delta r + r \delta h]}{\pi r^2 h + \frac{4}{3} \pi r^3} = \frac{2(h + 2r) \delta r + r \delta h}{rh + \frac{4}{3} r^2} \\ &= \frac{2(7)(0.01) + 1.5 \times .05}{1.5 \times 4 + \frac{4}{3} \times (1.5)^2} = \frac{0.215}{9} \end{aligned}$$

$\therefore \frac{\delta V}{V} \times 100 = \frac{.215}{9} \times 100 = \frac{21.5}{9} = 2.389\%$.

Hence, the percentage change in the volume of the balloon is 2.389.

- 161.** What error in the common logarithm of a number will be produced by an error of 1% in the number?

Sol. Consider x as any number.

Let $y = \log_{10} x$

Then,

$$\begin{aligned} \delta y &= \frac{1}{x} \log_{10} e (\delta x) = \frac{\delta x}{x} \log_{10} e = \left(\frac{\delta x}{x} \times 100 \right) \left(\frac{1}{100} \log_{10} e \right) \\ &= (1) \left(\frac{1}{100} \log_{10} e \right) = \frac{1}{100} \log_{10} e = \frac{.43429}{100} = .0043429 \end{aligned}$$

which is the required error.

- 162.** If the sides and angles of a triangle ABC vary in such a way that its circum-radius remains constant, prove that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$.

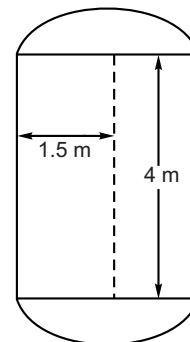
Sol. The circum-radius R of a ΔABC is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$$a = 2R \sin A \quad (R \text{ is constant}), \quad b = 2R \sin B$$

$$c = 2R \sin C, \quad da = 2R \cos A dA \quad \text{or} \quad \frac{da}{\cos A} = 2R dA$$

Similarly, $\frac{db}{\cos B} = 2R dB; \quad \frac{dc}{\cos C} = 2R dC$



Adding, we get $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC)$

As $A + B + C = \pi$, $dA + dB + dC = 0$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

- 163.** The angles of a triangle are calculated from the sides a, b, c . If small changes $\delta a, \delta b, \delta c$ are made in the sides, find $\delta A, \delta B$ and δC approximately, where Δ is the area of the triangle ABC and A, B, C are angles opposite to sides a, b, c respectively. Also show that $\delta A + \delta B + \delta C = 0$.

Sol. We know that $a^2 = b^2 + c^2 - 2bc \cos A$... (1)

Differentiating equation (1), we get

$$\begin{aligned} 2a \delta a &= 2b \delta b + 2c \delta c - 2b \delta c \cos A - 2c \delta b \cos A + 2bc \sin A \delta A \text{ (approx.)} \\ \text{or } bc \sin A \delta A &= a \delta a - (b - c \cos A) \delta b - (c - b \cos A) \delta c \\ \text{or } 2\Delta \delta A &= a \delta a - (a \cos C + c \cos A - c \cos A) \delta b - (a \cos B + b \cos A - b \cos A) \delta c \\ \text{or } \delta A &= \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B) \end{aligned} \quad \dots (2)$$

$$\text{Similarly, we get } \delta B = \frac{b}{2\Delta} (\delta b - \delta c \cos A - \delta a \cos C) \quad \dots (3)$$

$$\delta C = \frac{c}{2\Delta} (\delta c - \delta a \cos B - \delta b \cos A) \quad \dots (4)$$

Adding equations (2), (3) and (4), we get

$$\begin{aligned} \delta A + \delta B + \delta C &= \frac{1}{2\Delta} [(a - b \cos C - c \cos B) \delta a + (b - c \cos A - a \cos C) \delta b \\ &\quad + (c - a \cos B - b \cos A) \delta c] \\ &= \frac{1}{2\Delta} [(a - a) \delta a + (b - b) \delta b + (c - c) \delta c] = 0. \end{aligned}$$

- 164.** If $u = xyz$, find d^2u .

Sol. We know that if $u = f(x, y, z)$ then

$$\begin{aligned} du &= \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) u \\ \therefore d^2u &= d(du) = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) u \\ &= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 u \\ &= \left[(dx)^2 \frac{\partial^2}{\partial x^2} + (dy)^2 \frac{\partial^2}{\partial y^2} + (dz)^2 \frac{\partial^2}{\partial z^2} + 2dxdy \frac{\partial^2}{\partial x \partial y} + 2dydz \frac{\partial^2}{\partial y \partial z} + 2dzdx \frac{\partial^2}{\partial z \partial x} \right] u \\ d^2u &= \frac{\partial^2 u}{\partial x^2} (dx)^2 + \frac{\partial^2 u}{\partial y^2} (dy)^2 + \frac{\partial^2 u}{\partial z^2} (dz)^2 + 2 \frac{\partial^2 u}{\partial x \partial y} dxdy + 2 \frac{\partial^2 u}{\partial y \partial z} dydz + 2 \frac{\partial^2 u}{\partial z \partial x} dzdx \end{aligned} \quad \dots (1)$$

Here, $u = xyz$

$$\therefore \frac{\partial u}{\partial x} = yz, \quad \frac{\partial u}{\partial y} = xz, \quad \frac{\partial u}{\partial z} = xy$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} = 0; \quad \frac{\partial^2 u}{\partial x \partial y} = z, \quad \frac{\partial^2 u}{\partial y \partial z} = x, \quad \frac{\partial^2 u}{\partial z \partial x} = y$$

Substituting these values in (1), we get

$$d^2u = 2zdx dy + 2xdy dz + 2ydz dx.$$

- 165.** In estimating the number of bricks in a pile which is measured to be $(5m \times 10m \times 5m)$, the count of bricks is taken as 100 bricks per m^3 . Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is ₹ 2000 per thousand bricks.

Sol. Let x, y, z m be the length, breadth and height of the pile so that its volume

$$V = xyz$$

Taking log, we get $\log V = \log x + \log y + \log z$

Differentiating, we get $\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$

$$\frac{\delta V}{V} \times 100 = \left(\frac{\delta x}{x} \times 100 \right) + \left(\frac{\delta y}{y} \times 100 \right) + \left(\frac{\delta z}{z} \times 100 \right) = 2 + 2 + 2 = 6$$

$$\therefore \delta V = \frac{6}{100} V = \frac{6}{100} (5 \times 10 \times 5) = 15 m^3$$

$$\therefore \text{Number of bricks in } \delta V = 15 \times 100 = 1500$$

$$\text{Thus, error in the cost} = 1500 \times \frac{2000}{1000} = ₹ 3000$$

Hence, ₹ 3000 is the required error in the cost.

- 166.** In estimating the cost of a pile of bricks measured as $6 m \times 50 m \times 4 m$, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in $1 m^3$ and bricks cost ₹ 100 per 1,000, find the approximate error in the cost. (U.P.T.U., 2005)

Sol. Let x, y and z m be the length, breadth and height of the pile so that its volume

$$V = xyz$$

Taking log, we get $\log V = \log x + \log y + \log z$

Differentiating, we get $\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$

$$\Rightarrow \frac{\delta V}{V} \times 100 = \left(\frac{\delta x}{x} \times 100 \right) + \left(\frac{\delta y}{y} \times 100 \right) + \left(\frac{\delta z}{z} \times 100 \right) = 1 + 1 + 1 = 3$$

$$\Rightarrow \delta V = \frac{3}{100} V = \frac{3}{100} (6 \times 50 \times 4) = 36 m^3$$

$$\therefore \text{Number of bricks in } \delta V = 36 \times 12 = 432$$

$$\text{Thus, error in cost} = 432 \times \frac{100}{1000} = ₹ 43.20.$$

- 167.** Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3

from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, if r_1, r_2, r_3 are each in error by $\pm 1.2\%$.

Sol. We have, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$... (1)

Differentiating, we get

$$-\frac{1}{r^2} \delta r = -\frac{1}{r_1^2} \delta r_1 - \frac{1}{r_2^2} \delta r_2 - \frac{1}{r_3^2} \delta r_3$$

$$\begin{aligned}\Rightarrow \quad \frac{1}{r} \left(\frac{\delta r}{r} \times 100 \right) &= \frac{1}{r_1} \left(\frac{\delta r_1}{r_1} \times 100 \right) + \frac{1}{r_2} \left(\frac{\delta r_2}{r_2} \times 100 \right) + \frac{1}{r_3} \left(\frac{\delta r_3}{r_3} \times 100 \right) \\ &= \frac{1}{r_1} (1.2) + \frac{1}{r_2} (1.2) + \frac{1}{r_3} (1.2) = (1.2) \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) = (1.2) \left(\frac{1}{r} \right) \\ \Rightarrow \quad \frac{\delta r}{r} \times 100 &= 1.2\end{aligned}$$

Hence, % error in $r = 1.2$.

168. If Δ be the area of a triangle, prove that the error in Δ resulting from a small error in c is given by

$$\delta\Delta = \frac{\Delta}{4} \left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c} \right] \delta c.$$

Sol. We know that,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \quad \log \Delta = \frac{1}{2} [\log s + \log(s-a) + \log(s-b) + \log(s-c)]$$

Differentiating w.r.t. c , we get

$$\frac{1}{\Delta} \cdot \frac{\delta\Delta}{\delta c} = \frac{1}{2} \left[\frac{1}{s} \frac{\delta s}{\delta c} + \frac{1}{(s-a)} \frac{\delta(s-a)}{\delta c} + \frac{1}{s-b} \frac{\delta(s-b)}{\delta c} + \frac{1}{s-c} \frac{\delta(s-c)}{\delta c} \right] \quad \dots(1)$$

$$\text{Now,} \quad s = \frac{1}{2}(a+b+c) \quad \therefore \quad \frac{\delta s}{\delta c} = \frac{1}{2}$$

$$\text{Also,} \quad s-c = \frac{1}{2}(a+b-c) \quad \therefore \quad \frac{\delta(s-c)}{\delta c} = -\frac{1}{2}$$

$$\text{Similarly,} \quad s-a = \frac{1}{2}(b+c-a) \Rightarrow \frac{\delta(s-a)}{\delta c} = \frac{1}{2}$$

$$\text{and} \quad s-b = \frac{1}{2}(a+c-b) \Rightarrow \frac{\delta(s-b)}{\delta c} = \frac{1}{2}$$

From (1),

$$\begin{aligned}\frac{1}{\Delta} \cdot \frac{\delta\Delta}{\delta c} &= \frac{1}{2} \left[\frac{1}{s} \cdot \frac{1}{2} + \frac{1}{s-a} \cdot \frac{1}{2} + \frac{1}{s-b} \cdot \frac{1}{2} + \frac{1}{s-c} \cdot \left(-\frac{1}{2}\right) \right] \\ &= \frac{1}{4} \left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c} \right]\end{aligned}$$

$$\text{Hence,} \quad \delta\Delta = \frac{\Delta}{4} \left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c} \right] \delta c.$$

169. Two sides a, b of a triangle and included angle C are measured. Show that the error δc in the computed length of third side c due to a small error in the angle C is given by $\delta c = s \sin B \cdot \delta C$

$$\text{Sol. We know that,} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \dots(1) \quad [\text{By cosine formula}]$$

$$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab \cos C \quad \dots(2)$$

$$\text{Differentiating (2), we get} \quad 2c \delta c = -2ab(-\sin C \delta C)$$

$$\Rightarrow \quad c \delta c = ab \sin C \delta C \quad \dots(3)$$

By sine formula, we know that

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad b \sin C = c \sin B$$

$$\therefore \text{ From (3),} \quad c \delta c = ac \sin B \delta C \quad \text{or} \quad \delta c = a \sin B \delta C.$$

- 170.** If the kinetic energy T is given by $T = \frac{1}{2}mv^2$, find approximate change in T as the mass changes from 49 to 49.5 and the velocity v changes from 1600 to 1590.

Sol. $T = \frac{1}{2}mv^2$

$$\therefore \delta T = \frac{1}{2}(\delta m)v^2 + \frac{1}{2}m(2v\delta v) = \frac{1}{2}v^2\delta m + mv\delta v \quad \dots(1)$$

It is given that m changes from 49 to 49.5

$$\therefore \delta m = 0.5$$

Also, $\delta v = -10$

$$\therefore \text{From (1), } \delta T = \frac{1}{2}(1600)^2(0.5) + (49)(1600)(-10) = -144000$$

Thus, T decreases by 144000 units.

- 171.** In determining the specific gravity by the formula $S = \frac{A}{A-w}$ where A is the weight in air and w is the weight in water; A can be read within 0.01 gm and w within 0.02 gm. Find approximately the maximum error in S if the readings are $A = 1.1$ gm, $w = 0.6$ gm. Find also the maximum relative error.

Sol. $S = \frac{A}{A-w}$

On differentiating, we get $\delta S = \frac{(A-w)\delta A - A(\delta A - \delta w)}{(A-w)^2} \quad \dots(1)$

Maximum error in S can be obtained if we take $\delta A = -.01$ and $\delta w = .02$.

$$\therefore \text{From (1), } \delta S = \frac{(1.1 - .6)(-.01) - (1.1)(-.01 - .02)}{(1.1 - .6)^2} = 0.112$$

Maximum relative error in $S = \frac{(\delta S)_{\max}}{S} = \frac{0.112}{\left(\frac{1.1}{1.1 - .6}\right)} = 0.05091$.

- 172.** Find approximate value of $(1.04)^{3.01}$.

Sol. Let $f(x, y) = x^y$

$$\frac{\partial f}{\partial x} = yx^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \log x$$

Here let $x = 1, dx = 0.04, y = 3, dy = 0.01$

Now, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = yx^{y-1}dx + x^y \log x dy$

$$= 3(1)^2(.04) + (1)^3 \log 1 \cdot (.01) = 0.12$$

$$\therefore (1.04)^{3.01} = f(1, 3) + df = (1)^3 + 0.12 = 1.12.$$

- 173.** Find approximate value of $\left[(0.98)^2 + (2.01)^2 + (1.94)^2\right]^{\frac{1}{2}}$.

Sol. Let $f(x, y, z) = (x^2 + y^2 + z^2)^{\frac{1}{2}} \quad \dots(1)$

Taking $x = 1, y = 2$ and $z = 2$ so that

$$dx = -.02, dy = .01 \text{ and } dz = -.06$$

From (1),

$$\frac{\partial f}{\partial x} = x(x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial f}{\partial y} = y(x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial f}{\partial z} = z(x^2 + y^2 + z^2)^{-1/2}$$

Now,

$$\begin{aligned} df &= \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz \\ &= (x^2 + y^2 + z^2)^{-1/2} (xdx + ydy + zdz) \\ &= \frac{1}{3} \cdot (-.02 + .02 - 2 \times .06) = -.04 \end{aligned}$$

$$\therefore [(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2} = f(1, 2, 2) + df = 3 + (-.04) = 2.96.$$

- 174.** The focal length of a mirror is given by the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. If equal errors δ are made in the determination of u and v , show that the relative error $\frac{\delta f}{f}$ in the focal length is given by $\delta \left(\frac{1}{u} + \frac{1}{v} \right)$.

Sol. We have, $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$... (1)

Differentiating, we get

$$\begin{aligned} -\frac{1}{v^2} \delta v + \frac{1}{u^2} \delta u &= -\frac{2}{f^2} \delta f \\ \Rightarrow -\frac{1}{v^2} \delta + \frac{1}{u^2} \delta &= -\frac{2}{f} \left(\frac{\delta f}{f} \right) \quad [\because \delta v = \delta u = \delta \text{ (given)}] \\ \Rightarrow -\delta \left(\frac{1}{v} - \frac{1}{u} \right) \left(\frac{1}{v} + \frac{1}{u} \right) &= -\frac{2}{f} \left(\frac{\delta f}{f} \right) \\ \Rightarrow -\delta \left(\frac{2}{f} \right) \left(\frac{1}{v} + \frac{1}{u} \right) &= -\frac{2}{f} \left(\frac{\delta f}{f} \right) \\ \frac{\delta f}{f} &= \delta \left(\frac{1}{v} + \frac{1}{u} \right). \end{aligned}$$

- 175.** The area of a ΔABC is determined from the side a and the two angles B and C . If there are small errors in values of B and C , show that the resulting error in the calculated value of the area will be $\frac{1}{2} (b^2 \Delta C + c^2 \Delta B)$.

Sol. Let S be the area of ΔABC . Then,

$$S = \frac{1}{2} ab \sin C \quad \dots (1)$$

and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \dots (2)$

$\therefore$ From (1) and (2), $S = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A} \quad \dots (3)$

Taking log, $\log S = \log \left(\frac{a^2}{2} \right) + \log \sin B + \log \sin C - \log \sin A$

Differentiating, we get

$$\begin{aligned} \frac{\Delta S}{S} &= \frac{\cos B}{\sin B} \Delta B + \frac{\cos C}{\sin C} \Delta C - \frac{\cos A}{\sin A} \Delta A \\ &= \frac{\cos B}{\sin B} \Delta B + \frac{\cos C}{\sin C} \Delta C - \frac{\cos A}{\sin A} (-\Delta B - \Delta C) \quad [\because \Delta A + \Delta B + \Delta C = 0] \\ &= \left(\frac{\cos B}{\sin B} + \frac{\cos A}{\sin A} \right) \Delta B + \left(\frac{\cos C}{\sin C} + \frac{\cos A}{\sin A} \right) \Delta C \\ &= \frac{\sin(A+B)}{\sin A \sin B} \cdot \Delta B + \frac{\sin(A+C)}{\sin A \sin C} \cdot \Delta C \\ &= \frac{\sin C}{\sin A \sin B} \cdot \Delta B + \frac{\sin B}{\sin A \sin C} \cdot \Delta C \quad \dots(4) \end{aligned}$$

Since $S = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B} = \frac{1}{2} c^2 \frac{\sin A \sin B}{\sin C}$ [From (3)]

$\therefore$ From (4), $\frac{\Delta S}{S} = \frac{1}{2} \frac{c^2}{S} \Delta B + \frac{b^2}{2S} \Delta C \Rightarrow \Delta S = \frac{1}{2} (b^2 \Delta C + c^2 \Delta B)$. **Hence shown.**

- 176.** In estimating the cost of a pile of bricks measured as 8 by 50 by 5 ft., the tape is stretched 1% beyond the standard length. If the count is 12 bricks to 1 ft³ and bricks cost ₹ 1400 per thousand, find the approximate error in cost.

Sol. Let x , y and z ft. be the length, breadth and height of the pile of bricks so that its volume

$$V = xyz$$

Taking log, we get $\log V = \log x + \log y + \log z$

Differentiating, we get $\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$

$$\Rightarrow \left(\frac{\delta V}{V} \times 100 \right) = \left(\frac{\delta x}{x} \times 100 \right) + \left(\frac{\delta y}{y} \times 100 \right) + \left(\frac{\delta z}{z} \times 100 \right) = 1 + 1 + 1 = 3$$

$$\Rightarrow \delta V = \frac{3}{100} V = \frac{3}{100} (8 \times 50 \times 5) = 60 \text{ ft}^3$$

$\therefore$ No. of bricks in $\delta V = 12 \times 60 = 720$

Thus, error in cost = $720 \times \frac{1400}{1000} = ₹ 1008$.

- 177.** The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 64 cm respectively. The possible error in each measurement is 5%. Find approximately the maximum possible percentage errors in the computed value for the volume and the lateral surface.

Sol. (i) $V = \pi r^2 h = \frac{\pi}{4} D^2 H$ where D is diameter and H is altitude of the can. $D = 40$ cm,
 $H = 64$ cm, $dD = 5\%$ of 40 = 2, $dH = 5\%$ of 64 = 3.2

$$\begin{aligned} dV &= \frac{\pi}{4} [2DHdD + D^2dH] \\ &= \frac{\pi}{4} [2 \times 40 \times 64 \times 2 + 40 \times 40 \times 3.2] = \pi(2560 + 1280) \end{aligned}$$

$$dV = 3840\pi, \quad V = \frac{\pi}{4} \times 40 \times 40 \times 64 = 25600\pi$$

$$\therefore \frac{dV}{V} \times 100 = \frac{3840\pi}{25600\pi} \times 100 = 15\%$$

$$S = \pi DH, \quad dS = \pi(HdD + DdH)$$

$$\therefore \frac{dS}{S} = \frac{HdD + DdH}{DH} = \frac{64 \times 2 + 40 \times 3.2}{40 \times 64} = \frac{256}{2560} = \frac{1}{10}$$

$$\therefore \frac{dS}{S} \times 100 = \frac{1}{10} \times 100 = 10\%.$$

- 178.** Find the percentage error in the area of a rectangle when an error of 1% is made in measuring its length and breadth.

Sol. Area of a rectangle, $A = xy$ where x is length and y is the breadth.

$$\Rightarrow \log A = \log x + \log y$$

$$\frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y}$$

$$\frac{dA}{A} \times 100 = \frac{dx}{x} \times 100 + \frac{dy}{y} \times 100 = 1 + 1 = 2$$

$\therefore$ Error in the area = 2%.

- 179.** The radius of a circle is found to be 100 cm. Find the relative error in the area of the circle due to an error of 1 mm in measuring the radius.

Sol. Area of the circle, $A = \pi r^2$

$$\Rightarrow \log A = \log \pi + 2 \log r$$

$$\Rightarrow \frac{dA}{A} = 2 \frac{dr}{r} = 2 \times \frac{1}{10} \times \frac{1}{100} = 0.002 \text{ sq. cm.}$$

- 180.** The radius of a sphere is found to be 10 cm with possible error of .02 cm. What is the relative error in computing the volume ?

Sol. $V = \frac{4}{3} \pi r^3$, where r is the radius of the sphere.

$$\text{Taking log,} \quad \log V = \log (4/3) + \log \pi + 3 \log r$$

$$\text{On differentiating, we get} \quad \frac{dV}{V} = 3 \cdot \frac{dr}{r} = \frac{3 \cdot (.02)}{10} = .006 \text{ cm}^3.$$

- 181.** If $z = xy(x + y)$, find d^2z when x and y are the independent variables.

Sol. We know that if $z = f(x, y)$ then

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right) z$$

$$\therefore d^2z = d(dz) = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^2 z = \left[(dx)^2 \frac{\partial^2}{\partial x^2} + (dy)^2 \frac{\partial^2}{\partial y^2} + 2dxdy \frac{\partial^2}{\partial x \partial y} \right] z$$

$$= \frac{\partial^2 z}{\partial x^2} (dx)^2 + \frac{\partial^2 z}{\partial y^2} (dy)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \cdot dxdy$$

Here,

$$z = xy(x + y) = x^2y + xy^2$$

$$\frac{\partial z}{\partial x} = 2xy + y^2, \quad \frac{\partial z}{\partial y} = x^2 + 2xy$$

$$\frac{\partial^2 z}{\partial x^2} = 2y, \quad \frac{\partial^2 z}{\partial y^2} = 2x, \quad \frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

$$\therefore d^2 z = 2y(dx)^2 + 2x(dy)^2 + 2(2x + 2y)dxdy$$

$$\text{or } d^2 z = 2y(dx)^2 + 2x(dy)^2 + 4(x + y)dxdy.$$

- 182.** Find the possible percentage error in computing the parallel resistance r of two resistance r_1 and r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where the error in both r_1 and r_2 is 3% each.

Sol. We have $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$... (1)

Differentiating, we get

$$-\frac{1}{r^2} \delta r = -\frac{1}{r_1^2} \delta r_1 - \frac{1}{r_2^2} \delta r_2$$

$$\Rightarrow \frac{1}{r} \left(\frac{\delta r}{r} \times 100 \right) = \frac{1}{r_1} \left(\frac{\delta r_1}{r_1} \times 100 \right) + \frac{1}{r_2} \left(\frac{\delta r_2}{r_2} \times 100 \right) = \frac{1}{r_1} (3) + \frac{1}{r_2} (3) = 3 \left(\frac{1}{r_1} + \frac{1}{r_2} \right) = 3 \cdot \frac{1}{r}$$

$$\Rightarrow \frac{\delta r}{r} \times 100 = 3$$

Hence % error in $r = 3$.

- 183.** The side a and the opposite angle A of a ΔABC remain constant, show that when the other sides and angles are slightly varied $\frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$.

Sol. We know that, $a = b \cos C + c \cos B$
 $\delta a = \delta b \cos C + \delta c \cos B$

Since a is constant, $\delta a = 0$

$\therefore \delta b \cos C + \delta c \cos B = 0$

Dividing by $\cos B \cdot \cos C$, we get $\frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$.

- 184.** In estimating the cost of a pile of bricks measured as $2 \text{ m} \times 15 \text{ m} \times 1.2 \text{ m}$, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 m^3 and bricks cost ₹ 130 per 1000, find the approximate error in the cost.

Sol. Let x , y and z m be the length, breadth and height of the pile so that its volume

$$V = xyz$$

Taking log, we get $\log V = \log x + \log y + \log z$

Differentiating, we get $\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$

$$\Rightarrow \frac{\delta V}{V} \times 100 = \left(\frac{\delta x}{x} \times 100 \right) + \left(\frac{\delta y}{y} \times 100 \right) + \left(\frac{\delta z}{z} \times 100 \right) = 1 + 1 + 1 = 3$$

$$\Rightarrow \delta V = \frac{3V}{100} = \frac{3}{100} \times 2 \times 15 \times 1.2 = 1.08 \text{ m}^3$$

$\therefore$ No. of bricks in $\delta V = 1.08 \times 450 = 486$

Thus, error in cost = $486 \times \frac{130}{1000} = ₹ 63.18$.

185. The indicated horsepower I of an engine is calculated from the formula $I = \frac{PLAN}{33000}$, where $A = \frac{\pi}{4} d^2$.

Assuming that errors of r percent may have been made in measuring P , L , N and d , find the greatest possible error in I .

Sol. Given horsepower I of the engine is,

$$I = \frac{PLAN}{33000}, \quad A = \frac{\pi}{4} d^2$$

Taking log, we get

$$\begin{aligned} \log I &= \log P + \log L + \log A + \log N - \log 33000 \\ &= \log P + \log L + 2 \log d + \log N - \log 33000 + \log \left(\frac{\pi}{4} \right) \end{aligned}$$

Differentiating, we get

$$\frac{\partial I}{I} \times 100 = \frac{\partial P}{P} + \frac{\partial L}{L} + \frac{2\partial d}{d} + \frac{\partial N}{N}$$

$$\begin{aligned} \therefore \frac{\partial I}{I} \times 100 &= \left(\frac{\partial P}{P} \times 100 \right) + \left(\frac{\partial L}{L} \times 100 \right) + 2 \left(\frac{\partial d}{d} \times 100 \right) + \left(\frac{\partial N}{N} \times 100 \right) \\ &= r + r + 2r + r = 5r \end{aligned}$$

$\therefore$ Maximum error in $I = 5r\%$.

186. The voltage V across a resistor is measured with an error h and the resistance R is measured with an error k . Show that the error in calculating the power $W = \frac{V^2}{R}$ generated in the resistor is

$$\frac{V}{R^2} (2Rh - Vk).$$

Sol. Given $W = \frac{V^2}{R}$

Taking log, $\log W = 2 \log V - \log R$

On Differentiating, we get $\frac{dW}{W} = \frac{2dV}{V} - \frac{dR}{R}$

Given that $dV = h$, $dR = k$

$$\therefore \frac{dW}{W} = \frac{2h}{V} - \frac{k}{R}$$

$$dW = W \left(\frac{2h}{V} - \frac{k}{R} \right) = \frac{V^2}{R} \left(\frac{2h}{V} - \frac{k}{R} \right) = \frac{V^2}{R} \left(\frac{2Rh - Vk}{VR} \right)$$

$$dW = \frac{V}{R^2} (2Rh - Vk)$$

$\therefore$ Error in calculating $W = \frac{V}{R^2} (2Rh - Vk)$.

187. The torsional rigidity of a length of a wire is obtained from the formula $N = \frac{8\pi I l}{t^2 r^4}$. If l is decreased by 2%, t increased by 1.5%, r increased by 2%, show that the value of N is diminished by 13% approximately.

Sol. Given $N = \frac{8\pi I l}{t^2 r^4}$

$$\log N = \log 8\pi I + \log l - 2 \log t - 4 \log r$$

$$\begin{aligned} \text{Differentiating, we get} \quad \frac{\partial N}{N} &= \frac{\partial l}{l} - \frac{2\partial t}{t} - 4 \frac{\partial r}{r} \\ \Rightarrow \quad \frac{\partial N}{N} \times 100 &= \left(\frac{\partial l}{l} \times 100 \right) - 2 \left(\frac{\partial t}{t} \times 100 \right) - 4 \left(\frac{\partial r}{r} \times 100 \right) \\ &= (-2) - 2(1.5) - 4(2) = -2 - 3 - 8 = -13 \\ \therefore N \text{ is diminished by } 13\% \text{ approximately.} \end{aligned}$$

188. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = \frac{v^2 \sin 2\alpha}{g}$.

Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}\%$ in α .

$$\begin{aligned} \text{Sol. Given} \quad R &= \frac{v^2 \sin 2\alpha}{g} \\ \text{Taking log,} \quad \log R &= 2 \log v + \log \sin 2\alpha - \log g \\ \text{Differentiating we get,} \quad \frac{\partial R}{R} &= 2 \frac{\partial v}{v} + 2 \cot 2\alpha \cdot \alpha \\ \text{or} \quad \frac{\partial R}{R} \times 100 &= 2 \left(\frac{\partial v}{v} \times 100 \right) + 2 \cot 2\alpha \cdot (\alpha \times 100) \\ \frac{\partial R}{R} \times 100 &= 2(1) + 2 \cot 2\alpha \cdot \left(\frac{1}{2} \right) = 2 + \cot 2\alpha \\ \text{Percentage error in } R &= \cot 2\alpha + 2. \end{aligned}$$

189. If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$, prove that $\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}$.

$$\begin{aligned} \text{Sol. Given} \quad ax^2 + by^2 + cz^2 &= 1 \\ \text{Differentiating, we get} \quad 2axdx + 2bydy + 2czdz &= 0 \quad \dots(1) \\ \text{Also given} \quad lx + my + nz &= 0 \\ \text{Differentiating, we get} \quad ldx + mdy + ndz &= 0 \quad \dots(2) \\ \text{From (1) and (2), we have} \end{aligned}$$

$$\begin{aligned} \frac{dx}{2by - 2czm} &= -\frac{dy}{(2axn - 2clz)} = \frac{dz}{2axm - 2bly} \\ \therefore \quad \frac{dx}{bny - cmz} &= \frac{dy}{clz - anx} = \frac{dz}{amx - bly} \quad \text{Hence proved.} \end{aligned}$$

190. Find the percentage error in the area of an ellipse when an error of +1% is made in measuring the major and minor axes.

Sol. If x and y are semi-major and semi-minor axes of the ellipse, then its area A is given by

$$\begin{aligned} A &= \pi xy \\ \text{Taking logs,} \quad \log A &= \log \pi + \log x + \log y \\ \text{Taking differentials} \quad \frac{dA}{A} &= 0 + \frac{dx}{x} + \frac{dy}{y} \\ \text{or} \quad \frac{dA}{A} \times 100 &= \frac{dx}{x} \times 100 + \frac{dy}{y} \times 100 = 1 + 1 = 2 \\ \therefore \quad \text{Error in the area} &= 2\%. \end{aligned}$$

- 191.** Find the % error in the area of an ellipse when an error of 7% is made in measuring the major and minor axes. (U.P.T.U., 2006)

Sol. Proceeding as in the previous Q. No. 190, error in the area = 7 + 7 = 14%.

- 192.** Prove that the relative error of a quotient does not exceed the sum of the relative errors of the dividend and divisor. (U.P.T.U., 2006)

Sol. Let the quotient, dividend and divisor be Q , x and y respectively. Then

$$Q = \frac{x}{y}$$

Taking logs, $\log Q = \log x - \log y$

Taking differentials, $\frac{dQ}{Q} = \frac{dx}{x} - \frac{dy}{y}$

$\therefore$ Max. Relative error $\left| \frac{dQ}{Q} \right| \leq \left| \frac{dx}{x} \right| + \left| \frac{dy}{y} \right|$ **Hence the result.**

- 193.** If J be the Jacobian of the system u, v with regard to x, y and J' is the Jacobian of the system x, y with regard to u, v , then prove that $JJ' = 1$. (U.P.T.U., 2006)

Sol. Let $u = u(x, y)$ and $v = v(x, y)$ so that u and v are functions of x, y .

Differentiating partially w.r.t. u and v , we get

$$\begin{aligned} 1 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} = u_x x_u + u_y y_u ; & 0 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\ 0 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} = v_x x_u + v_y y_u ; & 1 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} = v_x x_v + v_y y_v \end{aligned}$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Interchanging rows and columns in the second determinant.

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence, $JJ' = 1$.

□ □ □

Applications of Single Integration

BRIEF DEFINITIONS AND IMPORTANT FORMULAE

1. Revolution about x-axis

The volume of the solid generated by the revolution about x -axis, of the area bounded by the

curve $y = f(x)$, the x -axis and the ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2 dx$.

2. Revolution about y-axis

$$\text{Vol.} = \int_a^b \pi x^2 dy.$$

3. Volume of Revolution (Polar Curves)

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

(i) about the initial line OX ($\theta = 0$) is $\int_{\beta}^{\alpha} \frac{2}{3} \pi r^3 \sin \theta d\theta$.

(ii) About the line OY ($\theta = \frac{\pi}{2}$) is $\int_{\alpha}^{\beta} \frac{2}{3} \pi r^3 \cos \theta d\theta$.

4. Volume Between Two solids

The volume of the solid generated by the revolution about the x -axis of the area bounded by the curves $y = f(x)$, $y = \phi(x)$, and the ordinates $x = a$, $x = b$ is

$$\int_a^b \pi (y_1^2 - y_2^2) dx \quad \text{where } y_1 \text{ is the 'y' of the upper curve and } y_2 \text{ that of the lower curve.}$$

5. Surface Formulae for Cartesian, Parametric and Polar Equations

(a) $\int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} \cdot dx$, where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ (For Cartesian equation).

(b) **Curved surface of solid** generated by the revolution about the x -axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x -axis and the ordinates at the points, where $t = a$, $t = b$ is

$$\int_{t=a}^{t=b} 2\pi y \frac{ds}{dt} \cdot dt, \quad \text{where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(c) **Surface Formula for Polar Equations**

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi y \frac{ds}{d\theta} \cdot d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

and $y = r \sin \theta$, gives curved surface of the solid generated by the revolution, about the initial line, of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$.

SOLVED PROBLEMS

1. Find the volume of a sphere of radius a .

Sol. The sphere is the solid of revolution generated by the revolution of a semicircular area about its bounding diameter.

Let the equation of the circle of radius ' a ' be

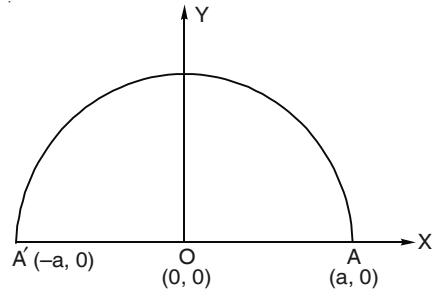
$$x^2 + y^2 = a^2 \quad \dots(1)$$

whose centre is O .

Now for the semicircle above the x -axis, x varies from $-a$ to a .

$\therefore$ Required volume of the sphere

$$\begin{aligned} &= \int_{-a}^a \pi y^2 dx \\ &= \int_{-a}^a \pi(a^2 - x^2) dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{-a}^a \\ &= \pi \left(2a^3 - \frac{2a^3}{3} \right) = \frac{4\pi a^3}{3} \end{aligned}$$



2. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis.

(M.D.U., Dec., 2005)

Sol. Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \text{or} \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad \dots(1)$$

Since the ellipse is symmetrical about y -axis.

$\therefore$ Required volume generated by the ellipse by the revolution about x -axis

$$= 2 \times \text{Volume generated by the arc in first quadrant about the } x\text{-axis.}$$

Now for the arc in the first quadrant, x varies from 0 to a .

$$\therefore \text{ Required volume} = 2 \int_0^a \pi y^2 dx = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx \quad [\text{From (1)}]$$

$$= 2\pi \cdot \frac{b^2}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a = 2\pi \frac{b^2}{a^2} \left[a^3 - \frac{a^3}{3} - 0 \right] = 2\pi \frac{b^2}{a^2} \cdot \frac{2a^3}{3} = \frac{4}{3} \pi a b^2.$$

3. Find the volume of the solid formed by the revolution about the x -axis, of the loop of the curve

$$y^2 = \frac{x^2(a+x)}{a-x}.$$

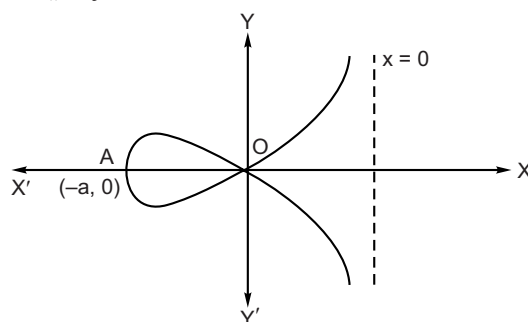
Sol. The equation of the curve is $y^2 = \frac{x^2(a+x)}{a-x}$ or $(a-x)y^2 = x^2(a+x)$... (1)

To trace the curve roughly to find the limits of integration :

- (i) The curve is symmetrical about the x -axis.
- (ii) The curve passes through the origin and the tangent at the origin are given by $y^2 = x^2$ or $y = \pm x$.
- (iii) The curve meets the x -axis at the point $(-a, 0)$. The curve meets the y -axis only at the origin.
- (iv) The curve has an asymptote $a-x=0$ or $x=a$ || to y -axis.

Thus the shape of the curve is roughly as shown in the figure.

For the upper half of the loop, x varies from $-a$ to 0 .



$$\therefore \text{ Required volume} = \int_{-a}^0 \pi y^2 dx$$

$$= \pi \int_{-a}^0 \frac{x^2(a+x)}{a-x} dx$$

$$= \pi \int_{-a}^0 \frac{ax^2 + x^3}{a-x} \cdot dx$$

[Divide the num. by denom.]

$$= \pi \int_{-a}^0 \left[-x^2 - 2ax - 2a^2 + \frac{2a^3}{a-x} \right] dx = \pi \left[-\frac{x^3}{3} - ax^2 - 2a^2x - 2a^3 \log(a-x) \right]_{-a}^0$$

$$= \pi \left[-2a^3 \log a - \left(\frac{a^3}{3} - a^3 + 2a^3 - 2a^3 \log 2a \right) \right]$$

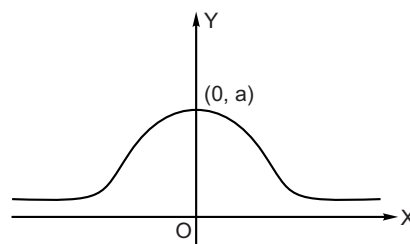
$$= \pi \left[\frac{2a^3}{3} - 2a^3 + 2a^3 \log 2a - 2a^3 \log a \right] = \pi \left[-\frac{4a^3}{3} + 2a^3 \log 2 \right] = 2\pi a^3 \left(\log 2 - \frac{2}{3} \right).$$

4. Find the volume of the solid generated by the revolution of the curve $y(a^2 + x^2) = a^3$ about its asymptote.

Sol. The equation of the curve is $y(x^2 + a^2) = a^3$... (1)

To trace the curve roughly to find limits of integration :

- (i) The curve is symmetrical about y -axis.
- (ii) The curve does not pass through the origin.
- (iii) The curve does not meet x -axis. It meets y -axis at the point $(0, a)$.
- (iv) The asymptote || to x -axis is $y = 0$ i.e., the x -axis itself. The curve has no other asymptote.



The shape of the curve is as shown in the figure.

Required volume = $2 \times$ Volume generated by half of the curve in first quadrant.

For the arc in first quadrant x varies from 0 to ∞

$$\therefore \text{ Required volume} = 2 \int_0^{\infty} \pi y^2 dx = 2 \int_0^{\infty} \pi \left(\frac{a^3}{x^2 + a^2} \right)^2 dx = 2\pi a^6 \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$$

Put $x = a \tan \theta$, $dx = a \sec^2 \theta d\theta$

When $x = 0$, $\theta = 0$; when $x = \infty$, $\theta = \frac{\pi}{2}$

$$\begin{aligned} \therefore \text{ Required volume} &= 2\pi a^6 \int_0^{\pi/2} \frac{a \sec^2 \theta \cdot d\theta}{(a^2 \tan^2 \theta + a^2)^2} = \frac{2\pi a^6}{a^3} \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = 2\pi a^3 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 2\pi a^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2} \pi^2 a^3. \end{aligned}$$

5. Find the volume of the solid formed by the revolution of the cissoid $y^2(2a - x) = x^3$ about its asymptote. (U.P.T.U., 2007)

Sol. Equation of the curve is $y^2 = \frac{x^3}{2a - x}$... (1)

The curve is symmetrical about the x -axis and the asymptote is the line $2a - x = 0$ or $x = 2a$.

If $P(x, y)$ be any point on the curve and $PM \perp$ on the asymptote (the axis of revolution) and $PN \perp OX$, then $PM = NA = OA - NO = 2a - x$ and $AM = NP = y$, where A is the point of intersection of the asymptote and the x -axis.

$$\therefore \text{ Required volume} = 2 \int \pi (PM)^2 d(AM) \quad \dots (2)$$

Now, $AM = y = \frac{x^{3/2}}{\sqrt{2a - x}}$

$$\therefore d(AM) = \frac{\sqrt{x}(3a - x)}{(2a - x)^{3/2}} \cdot dx$$

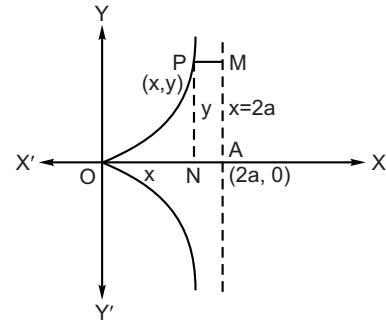
$\therefore$ From (2),

$$\begin{aligned} \text{Required volume} &= 2\pi \int_0^{2a} (2a - x)^2 \cdot \frac{\sqrt{x}(3a - x)}{(2a - x)^{3/2}} \cdot dx \\ &= 2\pi \int_0^{2a} \sqrt{x} (2a - x)^{1/2} (3a - x) \cdot dx \end{aligned}$$

Put $x = 2a \sin^2 \theta$, $dx = 4a \sin \theta \cos \theta d\theta$

When $x = 0$, $\theta = 0$; $x = 2a$, $\theta = \pi/2$.

$$\begin{aligned} &= 2\pi \int_0^{\pi/2} \sqrt{2a \sin^2 \theta} \cdot \sqrt{2a \cos^2 \theta} \cdot (3a - 2a \sin^2 \theta) 4a \sin \theta \cos \theta d\theta \\ &= 16\pi a^3 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \cdot (3 - 2 \sin^2 \theta) d\theta \\ &= 16\pi a^3 \int_0^{\pi/2} (3 \sin^2 \theta \cos^2 \theta - 2 \sin^4 \theta \cos^2 \theta) d\theta \\ &= 16\pi a^3 \left[3 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 2 \cdot \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] = 16\pi a^3 \left[\frac{3\pi}{16} - \frac{\pi}{16} \right] = 2\pi^2 a^3. \end{aligned}$$



6. The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated.

Sol. The equation of the parabola is $y^2 = 4ax$

Let LSL' be the latus rectum and A the vertex of the parabola.

Then the co-ordinates of L are $(a, 2a)$ and of A are $(0, 0)$.

Equation of the line AL is $y - 0 = \frac{2a}{a}(x - 0)$ or $2x - y = 0$

Let $P(x, y)$ be any point on the arc AL of the parabola.

Draw $PM \perp AL$ and join AP .

Then $PM = \perp$ distance of $P(x, y)$ from the line AL i.e., $2x - y = 0$

$$\begin{aligned}\therefore PM &= \frac{2x - y}{\sqrt{4 + 1}} = \frac{2x - y}{\sqrt{5}} \\ &= \frac{2x - \sqrt{4ax}}{\sqrt{5}} = \frac{2\sqrt{x}(\sqrt{x} - \sqrt{a})}{\sqrt{5}}\end{aligned}$$

$$\therefore PM^2 = \frac{4x(x + a - 2\sqrt{ax})}{5}$$

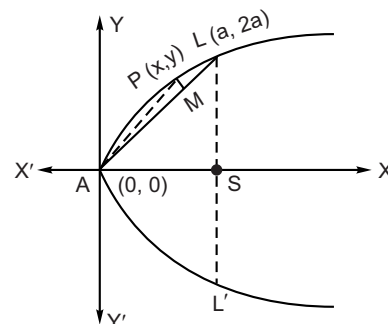
Now, $AM^2 = AP^2 - PM^2$

$$\begin{aligned}&= x^2 + y^2 - \frac{4x(x + a - 2\sqrt{ax})}{5} \\ &= x^2 + 4ax - \frac{4x(x + a - 2\sqrt{ax})}{5} = \frac{1}{5}(x + 4\sqrt{ax})^2\end{aligned}$$

$$\therefore AM = \frac{x + 4\sqrt{ax}}{\sqrt{5}}$$

$$\therefore d(AM) = \frac{1}{\sqrt{5}} \left(1 + 4\sqrt{a} \cdot \frac{1}{2\sqrt{x}} \right) = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{x} + 2\sqrt{a}}{\sqrt{x}} \right) dx$$

$$\begin{aligned}\therefore \text{Required volume} &= \int_0^a \pi (PM)^2 d(AM) = \pi \int_0^a \frac{4x(\sqrt{x} - \sqrt{a})^2}{5} \cdot \frac{1}{\sqrt{5}} \left(\frac{\sqrt{x} + 2\sqrt{a}}{\sqrt{x}} \right) dx \\ &= \frac{4\pi}{5\sqrt{5}} \int_0^a \sqrt{x} (\sqrt{x} - \sqrt{a})^2 (\sqrt{x} + 2\sqrt{a}) dx && \left\{ \begin{array}{l} \because \text{For the arc } AL, \\ x \text{ varies from } 0 \text{ to } a. \end{array} \right. \\ &= \frac{4\pi}{5\sqrt{5}} \int_0^1 \sqrt{a} t (\sqrt{a} t - \sqrt{a})^2 (\sqrt{a} t + 2\sqrt{a}) 2at dt && \left\{ \begin{array}{l} \text{Put } x = at^2, dx = 2at dt \\ \text{When } x = 0, t = 0 \\ \text{When } x = a, t = 1 \end{array} \right. \\ &= \frac{8\pi a^3}{5\sqrt{5}} \int_0^1 t^2(t+2)(t^2-2t+1) dt = \frac{8\pi a^3}{5\sqrt{5}} \int_0^1 t^2(t^3-3t+2) dt \\ &= \frac{8\pi a^3}{5\sqrt{5}} \left[\frac{t^6}{6} - \frac{3t^4}{4} + \frac{2t^3}{3} \right]_0^1 = \frac{8\pi a^3}{5\sqrt{5}} \left(\frac{1}{6} - \frac{3}{4} + \frac{2}{3} \right) \\ &= \frac{8\pi a^3}{5\sqrt{5}} \left(\frac{2-9+8}{12} \right) = \frac{2\pi a^3}{15\sqrt{5}}\end{aligned}$$

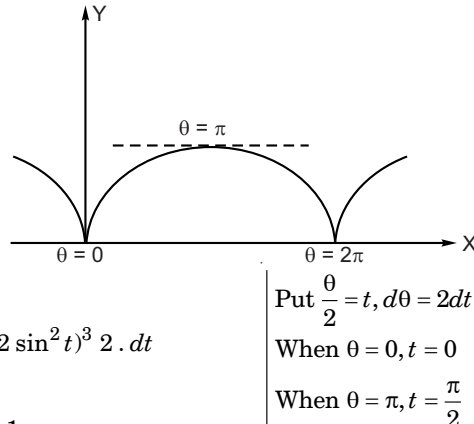


7. Find the volume of the solid formed by the revolution of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about its base.

Sol. The equation of the cycloid are $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

The cycloid is symmetrical about the line through the point, where $\theta = \pi$, $\perp$ to x -axis. For the first half of the cycloid, in the first quadrant, θ varies from 0 to π . The base is x -axis.

$\therefore$ Required volume of the solid formed by the revolution about its base (x -axis)

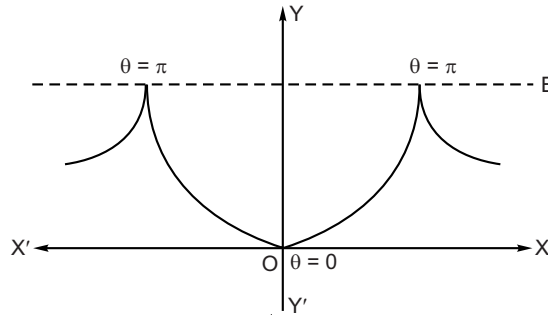
$$\begin{aligned}
 &= 2 \int_0^\pi \pi y^2 \frac{dx}{d\theta} \cdot d\theta \\
 &= 2\pi \int_0^\pi a^2(1 - \cos \theta)^2 \cdot a(1 - \cos \theta) d\theta \\
 &= 2\pi a^3 \int_0^\pi (1 - \cos \theta)^3 d\theta \\
 &= 2\pi a^3 \int_0^\pi \left(2 \sin^2 \frac{\theta}{2}\right)^3 d\theta = 2\pi a^3 \int_0^{\pi/2} (2 \sin^2 t)^3 \cdot 2 \cdot dt \\
 &= 32\pi a^3 \int_0^{\pi/2} \sin^6 t dt = 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 5\pi^2 a^3.
 \end{aligned}$$


Put $\frac{\theta}{2} = t$, $d\theta = 2dt$
 When $\theta = 0$, $t = 0$
 When $\theta = \pi$, $t = \frac{\pi}{2}$

8. Find the volume of the solid formed by the revolution of one arch of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ about the tangent at the vertex.

Sol. The equations of the cycloid are $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$

The cycloid is symmetrical about the y -axis and the tangent at the vertex is x -axis. For half of the curve, θ varies from 0 to π .



$$\begin{aligned}
 \therefore \text{ Required volume} &= 2 \int_0^\pi \pi y^2 \frac{dx}{d\theta} \cdot d\theta = 2\pi \int_0^\pi a^2 (1 - \cos \theta)^2 \cdot a(1 + \cos \theta) d\theta \\
 &= 2\pi a^3 \int_0^\pi (1 - \cos \theta)^2 (1 + \cos \theta) d\theta = 2\pi a^3 \int_0^\pi \left(2 \sin^2 \frac{\theta}{2}\right)^2 \left(2 \cos^2 \frac{\theta}{2}\right) d\theta \\
 &= 2\pi a^3 \int_0^{\pi/2} (2 \sin^2 t)^2 (2 \cos^2 t) 2dt \quad \text{where } \frac{\theta}{2} = t \\
 &= 32\pi a^3 \int_0^{\pi/2} \sin^4 t \cos^2 t dt = 32\pi a^3 \cdot \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi^2 a^3.
 \end{aligned}$$

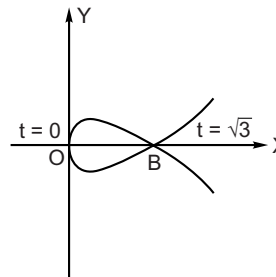
9. Prove that the volume of the solid generated by the revolution about the x -axis of the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$ is $\frac{3\pi}{4}$.

Sol. Equations of the curve are $x = t^2$, $y = t - \frac{t^3}{3}$

The curve is symmetrical about the x -axis and for the loop, putting $y = 0$, we get

$$t \left(1 - \frac{t^2}{3} \right) = 0 \quad \therefore t = 0, \pm \sqrt{3}$$

Thus for the upper half of the loop, t varies from 0 to $\sqrt{3}$.



$$\begin{aligned} \therefore \text{ Required volume} &= \int_0^{\sqrt{3}} \pi y^2 \frac{dx}{dt} dt \\ &= \int_0^{\sqrt{3}} \pi \left(t - \frac{t^3}{3} \right)^2 \cdot 2t dt = 2\pi \int_0^{\sqrt{3}} t \left(t^2 + \frac{t^6}{9} - \frac{2t^4}{3} \right) dt = 2\pi \int_0^{\sqrt{3}} \left(t^3 + \frac{t^7}{9} - \frac{2t^5}{3} \right) dt \\ &= 2\pi \left[\frac{t^4}{4} + \frac{1}{9} \cdot \frac{t^8}{8} - \frac{2}{3} \cdot \frac{t^6}{6} \right]_0^{\sqrt{3}} = 2\pi \left[\frac{9}{4} + \frac{81}{72} - \frac{2}{3} \cdot \frac{27}{6} \right] \\ &= 2\pi \left[\frac{9}{4} + \frac{9}{8} - 3 \right] = 2\pi \left(\frac{18+9-24}{8} \right) = 2\pi \left(\frac{3}{8} \right) = \frac{3\pi}{4}. \quad \text{Hence proved.} \end{aligned}$$

10. Find the volume of the spindle shaped solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x -axis.

Sol. The equation of the curve is

$$x^{2/3} + y^{2/3} = a^{2/3}$$

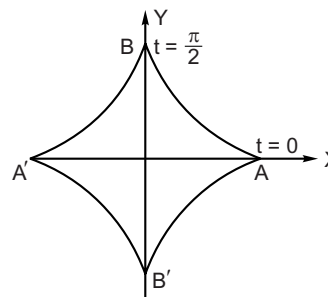
The parametric equations of the curve are

$$x = a \cos^3 t, \quad y = a \sin^3 t$$

The curve is symmetrical about both the axes and for the portion

of the curve in the first quadrant, t varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume = 2 $\times$ Volume generated by the arc in the first quadrant

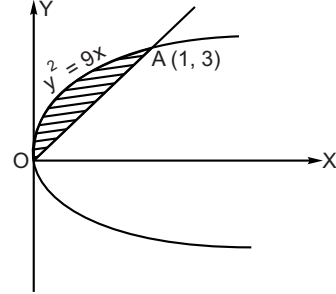


$$\begin{aligned} &= 2 \int_0^{\pi/2} \pi y^2 \frac{dx}{dt} dt = 2\pi \int_0^{\pi/2} a^2 \sin^6 t \cdot (-3a \cos^2 t \cdot \sin t) dt \\ &= -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cdot \cos^2 t \cdot dt = -6\pi a^3 \frac{6.4.2.1}{9.7.5.3.1} = -\frac{32}{105} \pi a^3 \\ &= \frac{32}{105} \pi a^3 \quad (\text{in magnitude}). \end{aligned}$$

11. Find the volume of the solid generated by the revolution of the plane area bounded by $y^2 = 9x$ and $y = 3x$ about the x -axis.

Sol. The line $y = 3x$ cuts the parabola $y^2 = 9x$ at $A(1, 3)$.

$$\begin{aligned} \text{Required volume} &= \int_0^1 \pi (y_1^2 - y_2^2) dx \\ &= \int_0^1 \pi (9x - 9x^2) dx \\ &= 9\pi \int_0^1 (x - x^2) dx = 9\pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\ &= 9\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{3\pi}{2}. \end{aligned}$$



12. The figure bounded by a parabola and the tangents at the extremities of its latus rectum revolves about the axis of the parabola. Find the volume of the solid thus generated.

Sol. Let the equation of the parabola be

$$y^2 = 4ax \quad \dots(1)$$

Let LSL' be its latus rectum and $LT, L'T$ be the tangents at the extremities L, L' .

Equation of the tangent at $L(a, 2a)$ is

$$\begin{aligned} yy_1 &= 2a(x + x_1) \\ y \cdot 2a &= 2a(x + a) \end{aligned}$$

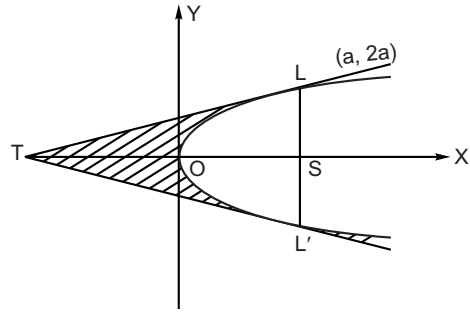
or

$$y = x + a \quad \dots(2)$$

This meets the x -axis, putting $y = 0$, we get $x = -a$.

Thus the point T is $(-a, 0)$. The curve and the two tangents are symmetrical about the x -axis.

$$\begin{aligned} \therefore \text{Required volume} &= \int_{-a}^a \pi y^2 dx - \int_0^a \pi y^2 dx \\ &\quad \text{(for tangent)} \quad \text{(for parabola)} \\ &= \pi \int_{-a}^a (x + a)^2 dx - \pi \int_0^a 4ax dx \\ &= \pi \left[\frac{(x + a)^3}{3} \right]_{-a}^a - 4a\pi \left[\frac{x^2}{2} \right]_0^a \\ &= \frac{\pi}{3} (8a^3) - \frac{4a\pi}{2} \cdot a^2 = \frac{8}{3} \pi a^3 - 2a^3 \pi = \frac{2}{3} \pi a^3. \end{aligned}$$



13. Find the volume of the solid formed by the revolution of the curve $r = a + b \cos \theta$ ($a > b$) about the initial line ($\theta = 0$). Hence or otherwise show that volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos \theta)$ about the initial line is $\frac{8}{3} \pi a^2$.

(M.D.U., Dec., 2007 and U.P.T.U. 2008)

Sol. Equation of the curve is $r = a + b \cos \theta$ ($a > b$)

...(1)

The curve is symmetrical about the initial line ($\theta = 0$) and for the upper half of the curve, θ varies from 0 to π .

$$\therefore \text{Required volume} = \int_0^{\pi/2} \frac{2}{3} \pi r^3 \sin \theta d\theta$$

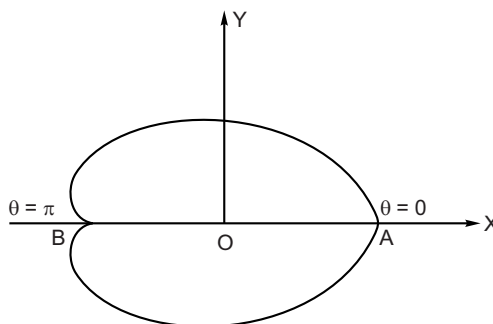
$$= \frac{2}{3} \pi \int_0^{\pi} (a + b \cos \theta)^3 \sin \theta d\theta$$

$$= -\frac{2}{3} \frac{\pi}{b} \int_0^{\pi} (a + b \cos \theta)^3 \cdot (-b \sin \theta) d\theta$$

$$= -\frac{2\pi}{3b} \left[\frac{(a + b \cos \theta)^4}{4} \right]_0^{\pi} = -\frac{\pi}{6b} [(a - b)^4 - (a + b)^4]$$

$$= \frac{\pi}{6b} [(a + b)^4 - (a - b)^4] = \frac{\pi}{6b} [(a + b)^2 + (a - b)^2] [(a + b)^2 - (a - b)^2]$$

$$= \frac{\pi}{6b} [2 \cdot (a^2 + b^2) \cdot 4ab] = \frac{4}{3} \pi a (a^2 + b^2) \quad \dots(2)$$



Putting $b = a$ in (1), the curve becomes $r = a(1 + \cos \theta)$

$$\therefore \text{Required volume} = \frac{4}{3} \pi a (a^2 + a^2) = \frac{8}{3} \pi a^3.$$

- 14.** Find the volume of the solid generated by revolving one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$. (M.D.U., Dec., 2006, May 2009 and U.P.T.U., 2007)

Sol. The given curve is $r^2 = a^2 \cos 2\theta$...(1)

The curve is symmetrical about the initial line. For a loop, putting $r = 0$ in (1), we get

$$\cos 2\theta = 0$$

$$\therefore 2\theta = \pm \frac{\pi}{2}, \quad \theta = \pm \frac{\pi}{4} \quad \text{(Two consecutive values)}$$

Thus as θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, we get a loop. For the upper half of this loop, θ varies from 0 to $\frac{\pi}{4}$.

$\therefore$ Required volume obtained by revolution of the loop about $\theta = \pi/2$.

$$\begin{aligned} &= 2 \int_0^{\pi/4} \frac{2}{3} \pi r^3 \cos \theta d\theta = \frac{4}{3} \pi \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos \theta d\theta \\ &= \frac{4}{3} \pi a^3 \int_0^{\pi/4} (1 - 2 \sin^2 \theta)^{3/2} \cdot \cos \theta d\theta \end{aligned}$$

$$\text{Put, } \sqrt{2} \sin \theta = \sin \phi \Rightarrow \sqrt{2} \cos \theta d\theta = \cos \phi d\phi$$

When $\theta = 0$, $\phi = 0$; and when $\theta = \frac{\pi}{4}$, $\phi = \pi/2$

$$\therefore \text{Required volume} = \frac{4}{3} \pi a^3 \int_0^{\pi/2} (1 - \sin^2 \phi)^{3/2} \frac{\cos \phi}{\sqrt{2}} d\phi$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^4 \phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

15. Find the surface generated by the revolution of arc of the catenary $y = c \cosh \frac{x}{c}$ from $x = 0$ to any point (x, y) about the x -axis.

Sol. Equation of the curve is $y = c \cosh \frac{x}{c}$

$$\frac{dy}{dx} = c \cdot \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c}$$

$$\therefore \frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \sinh^2 \frac{x}{c}} = \cosh \frac{x}{c}$$

If the arc be measured from the vertex (where $x = 0$) to any point (x, y) , then the required surface

$$\begin{aligned} &= \int_{x=0}^x 2\pi y \frac{dS}{dx} \cdot dx = \int_0^x 2\pi c \cosh \frac{x}{c} \cdot \cosh \frac{x}{c} \cdot dx \\ &= 2\pi c \int_0^x \cosh^2 \frac{x}{c} dx = 2\pi c \int_0^x \left(\frac{1 + \cosh \frac{2x}{c}}{2} \right) dx \\ &= \pi c \int_0^x \left[1 + \cosh \left(\frac{2x}{c} \right) \right] dx = \pi c \cdot \left[x + \frac{c}{2} \sinh \left(\frac{2x}{c} \right) \right]_0^x \\ &= \pi c \cdot \left[x + \frac{c}{2} \sinh \left(\frac{2x}{c} \right) \right]. \end{aligned}$$

16. Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.

Sol. Equation of the ellipse is

$$x^2 + 4y^2 = 16$$

$$\text{or} \quad 4y^2 = 16 - x^2 \quad \text{or} \quad y = \frac{\sqrt{16 - x^2}}{2} \quad \dots(1)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{2} (16 - x^2)^{-1/2} (-2x) = \frac{-x}{2\sqrt{16 - x^2}}$$

$$\frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{4(16 - x^2)}} = \sqrt{\frac{64 - 3x^2}{4(16 - x^2)}}.$$

The ellipse $x^2 + 4y^2 = 16$ meets x -axis, where putting $y = 0$, we get $x^2 = 16$ or $x = \pm 4$.

Thus for the upper half of the ellipse in first quadrant, x varies from 0 to 4.

Since ellipse is symmetrical about y -axis,

$\therefore$ Required surface = 2 $\times$ surface generated by the arc in the first quadrant

$$= 2 \int_0^4 2\pi y \frac{dS}{dx} \cdot dx = 4\pi \int_0^4 \frac{\sqrt{16 - x^2}}{2} \cdot \sqrt{\frac{64 - 3x^2}{4(16 - x^2)}} \cdot dx$$

$$\begin{aligned}
 &= \pi \int_0^4 \sqrt{64 - 3x^2} \cdot dx = \sqrt{3}\pi \int_0^4 \sqrt{\frac{64}{3} - x^2} \cdot dx \\
 &= \sqrt{3}\pi \left[\frac{x}{2} \sqrt{\frac{64}{3} - x^2} + \frac{64}{3} \cdot \frac{1}{2} \sin^{-1} \left(\frac{x}{\sqrt{64/3}} \right) \right]_0^4 \\
 &= \sqrt{3}\pi \left[2 \sqrt{\frac{16}{3}} + \frac{32}{3} \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right] = \sqrt{3}\pi \left[\frac{8}{\sqrt{3}} + \frac{32}{3} \cdot \frac{\pi}{3} \right] \\
 &= 8\pi + \frac{32\pi^2\sqrt{3}}{9} = 8\pi \left[1 + \frac{4\pi}{3\sqrt{3}} \right].
 \end{aligned}$$

17. A parabolic reflector of an automobile headlight is 12 cm in diameter and 4 cm deep. Find the cost of plating of the front portion of the reflector if the cost of plating is ₹ 5 per cm^2 .

Sol. Let $y^2 = 4ax$ be the equation of the parabola.

Since, $OA = 4$ cm, $AP = 6$ cm

Co-ordinates of P are $(4, 6)$. As P lies on the parabola

$$6^2 = 4a \times 4 \quad \text{or} \quad a = \frac{9}{4}$$

$\therefore$ Equation of parabola is $y^2 = 9x$.

The surface S of the parabolic reflector is formed by revolving the arc OP of the parabola about x -axis.

For the arc OP , $y = 3\sqrt{x}$, $\frac{dy}{dx} = \frac{3}{2\sqrt{x}}$

$$S = \int_0^4 2\pi y \frac{dS}{dx} \cdot dx = 2\pi \int_0^4 3\sqrt{x} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= 6\pi \int_0^4 \sqrt{x} \cdot \sqrt{1 + \frac{9}{4x}} \cdot dx = 6\pi \int_0^4 \sqrt{x + \frac{9}{4}} \cdot dx$$

$$= 6\pi \left[\frac{2\left(x + \frac{9}{4}\right)^{3/2}}{3} \right]_0^4 = 4\pi \left[\left(4 + \frac{9}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2} \right]$$

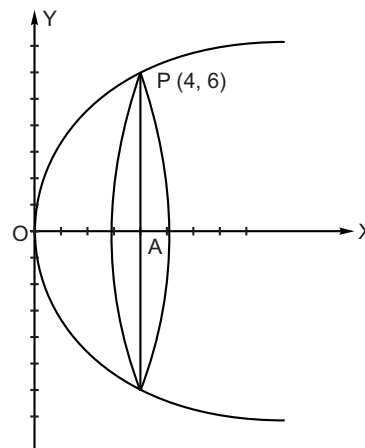
$$= 4\pi \left[\frac{125}{8} - \frac{27}{8} \right] = 49\pi = 154 \text{ cm}^2$$

Cost of plating = ₹ 154×5 = ₹ 770.

18. Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a \cos^3 t$, $y = a \sin^3 t$ about the x -axis.

Sol. Surface formula for parametric equations is given by

$$\int_{t=a}^{t=b} 2\pi y \frac{dS}{dt} \cdot dt \quad \text{where} \quad \frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



The given curve is symmetrical about both the axes and it lies in the square bounded by the lines $x = \pm a$, $y = \pm a$.

For the portion of the curve in the first quadrant t varies from 0 to $\frac{\pi}{2}$.

$$\text{Now,} \quad \frac{dx}{dt} = -3a \cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\therefore \quad \frac{dS}{dt} = 3a \sin t \cos t$$

$$\begin{aligned} \therefore \quad \text{Required surface} &= 2 \int_{t=0}^{t=\pi/2} 2\pi y \frac{dS}{dt} \cdot dt = 4\pi \int_0^{\pi/2} a \sin^3 t \cdot 3a \sin t \cos t \, dt \\ &= 4\pi \cdot 3a^2 \int_0^{\pi/2} \sin^4 t \cos t \, dt = 12\pi a^2 \left[\frac{\sin^5 t}{5} \right]_0^{\pi/2} \\ &= \frac{12}{5} \pi a^2 (1 - 0) = \frac{12}{5} \pi a^2. \end{aligned}$$

19. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about the x -axis.

Sol. The equations of the cycloid are $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

The cycloid is symmetrical about the line through the point $\theta = \pi$ to the x -axis.

For the first half of the cycloid, θ varies from 0 to π .

$$\text{Now,} \quad \frac{dx}{d\theta} = a(1 - \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\begin{aligned} \therefore \quad \frac{dS}{d\theta} &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} \\ &= a\sqrt{2(1 - \cos \theta)} = a\sqrt{4 \sin^2 \frac{\theta}{2}} = 2a \sin \frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{Required surface} &= 2 \int_0^{\pi} 2\pi y \frac{dS}{d\theta} \cdot d\theta = 4\pi \int_0^{\pi} y \frac{dS}{d\theta} \cdot d\theta \\ &= 4\pi \int_0^{\pi} a(1 - \cos \theta) \cdot 2a \sin \frac{\theta}{2} d\theta = 4\pi \int_0^{\pi} a \left(2 \sin^2 \frac{\theta}{2} \right) 2a \sin \frac{\theta}{2} d\theta \\ &= 16\pi a^2 \int_0^{\pi} \sin^3 \frac{\theta}{2} d\theta \end{aligned}$$

$$\text{Put } \frac{\theta}{2} = t, \quad d\theta = 2dt, \quad \text{when } \theta = 0, t = 0 \quad \text{and} \quad \text{when } \theta = \pi, t = \pi/2$$

$$= 16\pi a^2 \int_0^{\pi/2} 2 \sin^3 t \, dt = 32\pi a^2 \int_0^{\pi/2} \sin^3 t \, dt = 32\pi a^2 \cdot \frac{2}{3.1} = \frac{64}{3} \pi a^2.$$

20. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ about the tangent at the vertex.

Sol. Equations of the cycloid are $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

The cycloid is symmetrical about the y -axis and the tangent at the vertex is x -axis. For half of the curve, θ varies from 0 to π .

$$\therefore \text{ Required surface area} = 2 \int_0^{\pi} 2\pi y \frac{dS}{d\theta} \cdot d\theta$$

$$\frac{dx}{d\theta} = a(1 + \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta, \quad \frac{dS}{d\theta} = a\sqrt{2(1 + \cos \theta)} = 2a \cos \frac{\theta}{2}$$

$$\therefore \text{ Required surface area} = 4\pi \int_0^{\pi} a(1 - \cos \theta) \cdot 2a \cos \frac{\theta}{2} d\theta = 4\pi \int_0^{\pi} 2a^2 \cdot 2 \sin^2 \frac{\theta}{2} \cdot \cos \frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^2 \frac{\theta}{2} \cdot \cos \frac{\theta}{2} \cdot d\theta = 16\pi a^2 \left[\frac{\sin^3 \frac{\theta}{2}}{3 \cdot \frac{1}{2}} \right]_0^{\pi} = \frac{32}{3} \pi a^2.$$

- 21.** Determine the area of the surface generated by the revolution of the loops of the curve $r^2 = a^2 \cos 2\theta$ about the initial line.

Sol. The equation of the curve is $r^2 = a^2 \cos 2\theta$... (1)

The curve is symmetrical about the initial line and the line $\theta = \frac{\pi}{2}$.

For a loop, putting $r = 0$, we get $\cos 2\theta = 0$

$$2\theta = \pm \frac{\pi}{2}, \quad \theta = \frac{\pi}{4}, -\frac{\pi}{4} \quad (\text{Two consecutive values})$$

The curve consists of two equal loops and $\theta = -\frac{\pi}{4}, \theta = \frac{\pi}{4}$ are the tangents at the pole. In the first quadrant, for half of the loop, θ varies from 0 to $\frac{\pi}{4}$.

From (1), $2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$ or $\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$

$$\begin{aligned} \therefore \frac{dS}{d\theta} &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}} \\ &= \sqrt{a^2 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{\cos 2\theta}} = \sqrt{\frac{a^2 (\cos^2 2\theta + \sin^2 2\theta)}{\cos 2\theta}} \end{aligned}$$

$$\Rightarrow \frac{dS}{d\theta} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$\begin{aligned} \therefore \text{ Required surface area} &= 2 \int_0^{\pi/4} 2\pi y \frac{dS}{d\theta} \cdot d\theta = 4\pi \int_0^{\pi/4} r \sin \theta \cdot \frac{a}{\sqrt{\cos 2\theta}} \cdot d\theta \\ &= 4\pi a \int_0^{\pi/4} a \sqrt{\cos 2\theta} \cdot \sin \theta \cdot \frac{1}{\sqrt{\cos 2\theta}} \cdot d\theta \\ &= 4\pi a^2 \int_0^{\pi/4} \sin \theta d\theta = 4\pi a^2 [-\cos \theta]_0^{\pi/4} = 4\pi a^2 \left[-\frac{1}{\sqrt{2}} + 1 \right] \\ &= 2\pi a^2 [2 - \sqrt{2}]. \end{aligned}$$

- 22.** Find the volume of the paraboloid generated by the revolution of $y^2 = 4ax$ about x -axis from $x = 0$ to $x = b$.

Sol. Volume of solid generated by the revolution of $y^2 = 4ax$ about x -axis

$$= \int_0^b \pi y^2 dx = \pi \int_0^b 4ax dx = 4\pi a \left[\frac{x^2}{2} \right]_0^b = 4\pi a \cdot \frac{b^2}{2} = 2\pi ab^2.$$

- 23.** Find the volume of the reel-shaped solid formed by the revolution about the y -axis of the part of the parabola $y^2 = 4ax$ cut off by the latus rectum.

Sol. Volume of the solid generated by the revolution about the y -axis of the area bounded by the

curve $x = \frac{y^2}{4a}$, the y -axis and the abscissae $y = -2a$ to $y = 2a$ is $\int_{-2a}^{2a} \pi x^2 dy$. As the curve is

symmetrical about y -axis,

$$\begin{aligned} \text{Volume} &= 2 \int_0^{2a} \pi x^2 dy = 2\pi \int_0^{2a} \left(\frac{y^2}{4a} \right)^2 dy \\ &= \frac{2\pi}{16a^2} \int_0^{2a} y^4 dy = \frac{2\pi}{16a^2} \cdot \left(\frac{y^5}{5} \right)_0^{2a} = \frac{\pi}{40a^2} (2a)^5 = \frac{32}{40} \pi a^3 = \frac{4}{5} \pi a^3. \end{aligned}$$

- 24.** Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus rectum about the axis of the parabola. (M.D.U., May 2006)

Sol.
$$\text{Volume} = \int_0^a \pi y^2 dx = \pi \int_0^a 4ax dx = 4\pi a \left[\frac{x^2}{2} \right]_0^a = 4\pi a \cdot \frac{a^2}{2} = 2\pi a^3$$

- 25.** Area bounded by x -axis, $y^2 = 4ax$ and the ordinate $x = 3a$ is revolved about the x -axis. Find the volume generated.

Sol.
$$\text{Volume} = \int_0^{3a} \pi y^2 dx = 4\pi a \int_0^{3a} x dx = 4\pi a \cdot \frac{(3a)^2}{2} = 18\pi a^3.$$

- 26.** Find the volume formed by the revolution of loop of the curve $y^2(a+x) = x^2(3a-x)$, about the x -axis.

Sol. The equation of the curve is

$$y^2(a+x) = x^2(3a-x)$$

- (i) The curve is symmetrical about the x -axis.
- (ii) The curve passes through the origin.
- (iii) The curve meets x -axis at the point $(3a, 0)$.
- (iv) The curve has an asymptote $a+x=0$ i.e., $x=-a \parallel$ to y -axis.

For the upper half of the loop, x varies from 0 to $3a$.

$$\begin{aligned} \therefore \text{ Required volume} &= \int_0^{3a} \pi y^2 dx = \int_0^{3a} \pi \frac{x^2(3a-x)}{a+x} dx \\ &= \int_0^{3a} \pi \left(\frac{-x^3 + 3ax^2}{x+a} \right) dx = \pi \int_0^{3a} \left(-x^2 + 4xa - 4a^2 + \frac{4a^3}{x+a} \right) dx \end{aligned}$$

$$\begin{aligned}
&= \pi \left[-\frac{x^3}{3} + 4a \cdot \frac{x^2}{2} - 4a^2x + 4a^3 \log(x+a) \right]_0^{3a} \\
&= \pi [-9a^3 + 18a^3 - 12a^3 + 4a^3(\log 4a - \log a)] \\
&= \pi [-3a^3 + 4a^3(\log 4)] = \pi a^3 (8 \log 2 - 3).
\end{aligned}$$

27. Find the volume of the solid obtained by revolving $x = a \cos \theta$, $y = b \sin \theta$ about the y -axis.

Sol. Eliminating θ , we have equation of the curve as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \quad \text{or} \quad x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

Since the ellipse is symmetrical about x -axis,

$\therefore$ Required volume generated by the ellipse by the revolution about y -axis
 $= 2 \times$ Volume generated by the arc in the first quadrant about the x -axis

$$\begin{aligned}
&= 2 \int_0^b \pi x^2 dy = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) dy \\
&= 2\pi \frac{a^2}{b^2} \left(b^2 y - \frac{y^3}{3} \right)_0^b = 2\pi \frac{a^2}{b^2} \left(b^3 - \frac{b^3}{3} \right) = 2\pi \frac{a^2}{b^2} \cdot \frac{2b^3}{3} = \frac{4\pi}{3} a^2 b.
\end{aligned}$$

28. Find the volume when the loop of the curve $y^2 = x(2x-1)^2$ revolves about the x -axis.

Sol. Equation of the curve is $y^2 = x(2x-1)^2$

The curve meets the x -axis at, where putting $y = 0$, $(0, 0)$ and $\left(\frac{1}{2}, 0\right)$.

$\therefore$ For loop of the curve, x varies from 0 to $\frac{1}{2}$

$$\begin{aligned}
\text{Required volume} &= \int_0^{1/2} \pi y^2 dx = \pi \int_0^{1/2} x(2x-1)^2 dx = \pi \int_0^{1/2} x(4x^2 + 1 - 4x) dx \\
&= \pi \int_0^{1/2} (4x^3 + x - 4x^2) dx = \pi \left[\frac{4x^4}{4} + \frac{x^2}{2} - \frac{4x^3}{3} \right]_0^{1/2} = \pi \left[\frac{1}{16} + \frac{1}{8} - \frac{1}{6} \right] = \frac{\pi}{48}.
\end{aligned}$$

29. Show that the volume of the solid obtained by the revolution of the curve $a^2 y^2 = x^2(a^2 - x^2)$ about the x -axis is $\frac{4}{15} \pi a^3$.

Sol. Given curve is $y^2 = \frac{x^2}{a^2}(a^2 - x^2)$

The curve meets the x -axis at $(\pm a, 0)$.

The curve is symmetrical about the y -axis.

$$\begin{aligned}
\therefore \text{ Required volume} &= 2 \int_0^a \pi y^2 dx = 2\pi \int_0^a \frac{x^2}{a^2} (a^2 - x^2) dx = \frac{2\pi}{a^2} \left[\left(a^2 \frac{x^3}{3} - \frac{x^5}{5} \right) \right]_0^a \\
&= \frac{2\pi}{a^2} \cdot \left[\frac{a^2 \cdot a^3}{3} - \frac{a^5}{5} \right] = \frac{2\pi}{a^2} \cdot \frac{2}{15} a^5 = \frac{4\pi}{15} a^3.
\end{aligned}$$

- 30.** Prove that the volume of the reel formed by the revolution of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ about the tangent at the vertex is $\pi^2 a^3$.

Sol. The equations of the cycloid are $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

The cycloid is symmetrical about the y -axis and the tangent at the vertex is x -axis.

For half the curve, θ varies from 0 to π .

$$\begin{aligned} \therefore \text{ Required volume} &= 2 \int_0^\pi \pi y^2 \frac{dx}{d\theta} \cdot d\theta = 2\pi \int_0^\pi a^2(1 - \cos \theta)^2 \cdot a(1 + \cos \theta) d\theta \\ &= 2\pi a^3 \int_0^\pi \left(2 \sin^2 \frac{\theta}{2}\right)^2 \left(2 \cos^2 \frac{\theta}{2}\right) d\theta = 16\pi a^3 \int_0^\pi \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta}{2} d\theta \\ &= 16\pi a^3 \int_0^{\pi/2} 2 \cdot \sin^4 t \cos^2 t dt \quad \left(t = \frac{\theta}{2}\right) \\ &= 32\pi a^3 \cdot \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi^2 a^3. \end{aligned}$$

- 31.** Find the volume generated by revolving about x -axis, the area cut off from the parabola $9y = 4(9 - x^2)$ by the line $4x + 3y = 12$.

Sol. The line meets the parabola at (3, 0).

$$\begin{aligned} \therefore \text{ Required volume} &= \int_0^3 \pi (y_1^2 - y_2^2) dx = \pi \int_0^3 \left\{ \frac{4(9 - x^2)}{9} \right\}^2 - \left\{ \frac{12 - 4x}{3} \right\}^2 \cdot dx \\ &= \pi \int_0^3 \left[\frac{16(81 + x^4 - 18x^2)}{81} - \frac{144 + 16x^2 - 96x}{9} \right] dx \\ &= \frac{\pi}{81} \int_0^3 [16(81 + x^4 - 18x^2) - 9(144 + 16x^2 - 96x)] dx \\ &= \frac{\pi}{81} \int_0^3 (16x^4 - 432x^2 + 864x) dx = \frac{16\pi}{81} \left[\frac{x^5}{5} - 9x^3 + 27x^2 \right]_0^3 \\ &= \frac{16\pi}{81} \left[\frac{243}{5} - 243 + 243 \right] = \frac{48\pi}{5}. \end{aligned}$$

- 32.** The area bounded by the parabola $y^2 = 4x$ and the straight line $4x = 3y - 2$ is rotated about the y -axis. Obtain the volume of the solid formed by this revolution.

Sol. Required volume $= \int_a^b \pi (x_1^2 - x_2^2) dy = \pi \int_a^b \left[\frac{y^4}{16} - \left(\frac{3y - 2}{4} \right)^2 \right] dy$

Line meets the parabola where

$$\begin{array}{ll} y^2 = 3y - 2 & \text{or} \quad y^2 - 3y + 2 = 0 \\ (y - 2)(y - 1) = 0 & \text{or} \quad y = 1, 2 \end{array}$$

$$\begin{aligned} \therefore \text{ Volume} &= \pi \int_1^2 \left[\frac{y^4}{16} - \frac{9y^2 + 4 - 12y}{16} \right] dy = \frac{\pi}{16} \left[\frac{y^5}{5} - \frac{9y^3}{3} - 4y + \frac{12y^2}{2} \right]_1^2 \\ &= \frac{\pi}{16} \left[\frac{32}{5} - 24 - 8 + 24 - \left(\frac{1}{5} - 3 - 4 + 6 \right) \right] = \frac{\pi}{16} \left[-\frac{8}{5} + \frac{4}{5} \right] = \frac{\pi}{20}. \end{aligned}$$

- 33.** Show that the volume of the solid generated by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about x -axis is $\frac{5\pi}{28}$.

Sol. Volume of the solid generated by revolution about x -axis

$$= \int_a^b \pi (y_1^2 - y_2^2) dx \quad \text{where } y_1^2 = x^3 \quad \text{and} \quad y_2^2 = x^{4/3}$$

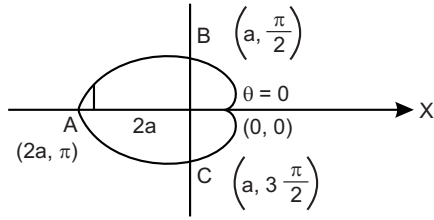
Both the curves meet where $y^3 = x^2$, $y^2 = x^3$

$$\therefore y = \frac{x^2}{x^3} = \frac{1}{x} \Rightarrow \frac{1}{x^2} = x^3 \quad \text{or} \quad x^5 = 1 \Rightarrow x = 1$$

$$\therefore \text{ Required volume} = \pi \int_0^1 \left(x^{\frac{4}{3}} - x^3 \right) dx = \pi \left[\frac{3x^{\frac{7}{3}}}{7} - \frac{x^4}{4} \right]_0^1 = \pi \left[\frac{3}{7} - \frac{1}{4} \right] = \frac{5\pi}{28}.$$

- 34.** The cardioid $r = a(1 - \cos \theta)$ revolves about the initial line. Find the volume of the solid generated.

Sol. The cardioid is symmetrical about the initial line or x -axis. Figure shows the shape of the curve. The curve lies entirely within a circle of radius $2a$.

$$\begin{aligned} \text{Required volume} &= \int_0^\pi \frac{2}{3} \pi r^3 \sin \theta d\theta \\ &= \frac{2}{3} \pi \int_0^\pi a^3 (1 - \cos \theta)^3 \sin \theta d\theta \\ &= \frac{2}{3} \pi a^3 \left[\frac{(1 - \cos \theta)^4}{4} \right]_0^\pi \\ &= \frac{\pi a^3}{6} [(1 - \cos \pi)^4 - (1 - \cos 0)^4] = \frac{\pi a^3}{6} \cdot 16 = \frac{8\pi a^3}{3}. \end{aligned}$$


- 35.** Find the area of the surface generated by revolving the arc of the catenary $y = c \cosh \frac{x}{c}$ from $x = 0$ to $x = c$ about the x -axis.

Sol. Equation of the curve is $y = c \cosh \frac{x}{c}$

$$\therefore \frac{dy}{dx} = \sinh \frac{x}{c}, \quad \frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} = \cosh \frac{x}{c}$$

$$\begin{aligned} \text{Required surface} &= \int_0^c 2\pi y \frac{dS}{dx} \cdot dx = 2\pi c \int_0^c \cosh^2 \frac{x}{c} \cdot dx \\ &= \pi c \int_0^c \left[1 + \cosh \left(\frac{2x}{c} \right) \right] dx \quad \because \cosh^2 \theta = \frac{1 + \cosh 2\theta}{2} \\ &= \pi c \left[x + \frac{c}{2} \sinh \left(\frac{2x}{c} \right) \right]_0^c = \pi c \left[c + \frac{c}{2} \sinh 2 \right] = \frac{\pi c^2}{2} [2 + \sinh 2]. \end{aligned}$$

- 36.** Prove that the surface area of the solid generated by the revolution, about the x -axis, of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is 3π .

Sol. Loop of the curve meets x -axis where $y = 0$ i.e., $t - \frac{1}{3}t^3 = 0$

$$\Rightarrow t \left(1 - \frac{t^2}{3} \right) = 0 \Rightarrow t = 0 \text{ and } t^2 = 3$$

$$\therefore \text{ Required surface area} = \int_0^{\sqrt{3}} 2\pi y \frac{dS}{dt} dt$$

$$\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t)^2 + (1-t^2)^2} = \sqrt{(1+t^2)^2} = 1+t^2$$

$$\begin{aligned} \therefore \text{ Required surface area} &= 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3} \right) (1+t^2) dt = 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3} + t^2 - \frac{t^5}{3} \right) dt \\ &= 2\pi \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^4}{4} - \frac{t^6}{18} \right]_0^{\sqrt{3}} = 2\pi \left[\frac{3}{2} - \frac{9}{12} + \frac{9}{4} - \frac{27}{18} \right] = 2\pi \cdot \frac{54}{36} = 3\pi. \quad \text{Hence proved.} \end{aligned}$$

- 37.** The curve $r = a(1 + \cos \theta)$ revolves about the initial line. Find the area of the surface formed.

(M.D.U., May 2007, Dec., 2008)

Sol. The curve $r = a(1 + \cos \theta)$ is symmetrical about the initial line (x -axis) and for the upper half of the curve, θ varies from 0 to π .

$$\therefore \text{ Required surface area} = \int_0^{\pi} 2\pi y \frac{dS}{d\theta} \cdot d\theta$$

$$\begin{aligned} \text{where, } \frac{dS}{d\theta} &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}, \quad y = r \sin \theta \\ &= \sqrt{a^2(1 + \cos \theta)^2 + (-a \sin \theta)^2} = a\sqrt{1 + \cos^2 \theta + 2 \cos \theta + \sin^2 \theta} \\ &= a\sqrt{2(1 + \cos \theta)} = 2a \cos \frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} \therefore \text{ Required surface area} &= 2\pi \int_0^{\pi} r \sin \theta \cdot 2a \cos \frac{\theta}{2} \cdot d\theta \\ &= 4\pi a \int_0^{\pi} a(1 + \cos \theta) \sin \theta \cos \frac{\theta}{2} d\theta = 8\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} \sin \theta d\theta \\ \therefore \text{ Required surface area} &= 8\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} \sin \theta d\theta = 8\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) d\theta \\ &= 16\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} \cdot \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cdot d\theta \end{aligned}$$

$$\begin{aligned}
 &= 16\pi a^2 \int_0^{\pi/2} 2 \cos^4 t \cdot \sin t \, dt && \left| \begin{array}{l} \text{Put } \frac{\theta}{2} = t, \, d\theta = 2dt \\ \text{When } \theta = \pi, \, t = \pi/2 \end{array} \right. \\
 &= 32\pi a^2 \left[\frac{\cos^5 t}{5} \right]_0^{\pi/2} = 32\pi a^2 \left[\frac{-1}{5} \right] \\
 &= \frac{32}{5} \pi a^2 \quad (\text{Neglecting -ve sign}).
 \end{aligned}$$

- 38.** Find the volume and surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$ about its base.

Sol. The curve meets the x -axis where $y = 0$ i.e., when $\theta = \pm \pi$ i.e., we have $x = \pm a\pi$.

Therefore, the curve meets the x -axis at the point $(\pm a\pi, 0)$.

Curve does not pass through the origin and curve is symmetrical about y -axis. The curve meets the y -axis where $x = 0$ i.e., when $\theta = 0$ or $y = 2a$. Therefore curve meets y -axis at $(0, 2a)$. Least value of y is 0 when $\theta = \pi$ and greatest value of y is $2a$ when $\theta = 0$.

$$\begin{aligned}
 \therefore \text{ Required volume} &= 2 \int_0^{\pi} \pi y^2 \frac{dx}{d\theta} \cdot d\theta = 2\pi \int_0^{\pi} a^2 (1 + \cos \theta)^2 \cdot a(1 + \cos \theta) d\theta \\
 &= 2\pi a^3 \int_0^{\pi} (1 + \cos \theta)^3 d\theta = 2\pi a^3 \int_0^{\pi} \left(2 \cos^2 \frac{\theta}{2} \right)^3 d\theta = 16\pi a^3 \int_0^{\pi} \cos^6 \frac{\theta}{2} d\theta \\
 \Rightarrow \text{ Required volume} &= 16\pi a^3 \int_0^{\pi/2} \cos^6 t \cdot 2dt && \left| \begin{array}{l} \text{Put } \frac{\theta}{2} = t, \, d\theta = 2dt \end{array} \right. \\
 &= 32\pi a^3 \int_0^{\pi/2} \cos^6 t \cdot dt = 32\pi a^3 \left[\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] = 32\pi a^3 \left[\frac{5\pi}{32} \right] = 5\pi^2 a^3
 \end{aligned}$$

$$\text{Required surface area} = 2 \int_0^{\pi} 2\pi y \frac{dS}{d\theta} \cdot d\theta$$

$$\begin{aligned}
 \text{where } \frac{dS}{d\theta} &= \sqrt{\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2} = \sqrt{[a(1 + \cos \theta)]^2 + [-a \sin \theta]^2} \\
 &= a\sqrt{1 + \cos^2 \theta + 2 \cos \theta + \sin^2 \theta} = a\sqrt{2(1 + \cos \theta)} = 2a \cos \frac{\theta}{2}
 \end{aligned}$$

$$\therefore \text{ Required surface area} = 4\pi \int_0^{\pi} a(1 + \cos \theta) \cdot 2a \cos \frac{\theta}{2} \cdot d\theta = 8\pi a^2 \int_0^{\pi} 2 \cos^2 \frac{\theta}{2} \cdot \cos \frac{\theta}{2} d\theta$$

$$\text{Required surface area} = 16\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} d\theta$$

$$\begin{aligned}
 \therefore \text{ Area} &= 16\pi a^2 \int_0^{\pi/2} 2 \cos^3 t \, dt = 32\pi a^2 \left(\frac{2}{3} \right) && \left| \begin{array}{l} \text{Put } \frac{\theta}{2} = t, \, d\theta = 2dt \\ \theta = \pi \Rightarrow t = \frac{\pi}{2} \end{array} \right. \\
 &= \frac{64\pi a^2}{3}.
 \end{aligned}$$

39. Prove that the surface area of the ellipsoid formed by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, round its major axis is $2\pi ab \left[\sqrt{1-e^2} + \frac{1}{e} \sin^{-1} e \right]$.

Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$... (1)

where $b^2 = a^2(1-e^2)$ [$e < 1$]

The required surface area = $2 \times$ Surface area of revolution generated by revolution about the x-axis of the arc in the first quadrant.

From (1) on differentiation, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y} = -\frac{a^2(1-e^2)}{a^2} \cdot \frac{x}{y} = -(1-e^2) \frac{x}{y}$$

$$\begin{aligned} \text{Required surface area} &= 2 \int_0^a 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = 4\pi \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \cdot \sqrt{1 + (1-e^2)^2 \frac{x^2}{y^2}} \cdot dx \\ &= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - x^2} \cdot \sqrt{1 + (1-e^2)^2 \cdot x^2 \cdot \frac{a^2}{b^2(a^2 - x^2)}} \cdot dx \\ &= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - x^2} \cdot \sqrt{1 + (1-e^2)^2 \cdot x^2 \cdot \frac{a^2}{a^2(1-e^2)(a^2 - x^2)}} \cdot dx \\ &= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - x^2} \cdot \sqrt{1 + \frac{(1-e^2)x^2}{a^2 - x^2}} \cdot dx = \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - e^2 x^2} \cdot dx \\ &= \frac{4\pi b e}{a} \int_0^a \sqrt{\frac{a^2}{e^2} - x^2} \cdot dx = \frac{4\pi b e}{a} \left[\frac{x \sqrt{\frac{a^2}{e^2} - x^2}}{2} + \frac{a^2}{2e^2} \sin^{-1} \left(\frac{x}{a/e} \right) \right]_0^a \\ &= \frac{4\pi b e}{a} \left[a \sqrt{\frac{a^2}{e^2} - a^2} + \frac{a^2}{e^2} \sin^{-1}(e) \right] = 2\pi ab \left[\sqrt{1-e^2} + \frac{1}{e} \sin^{-1}(e) \right]. \end{aligned}$$

40. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $\frac{6\pi a^2}{5}$. (U.P.T.U., 2008)

Sol. The parametric equations of the curve are $x = a \cos^3 t$, $y = a \sin^3 t$

$$\therefore \frac{dx}{dt} = -3a \cos^2 t \sin t; \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\begin{aligned} \therefore \text{Required surface area} &= \int_0^{\pi/2} 2\pi y \frac{dS}{dt} \cdot dt = \int_0^{\pi/2} 2\pi a \sin^3 t \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= 2\pi a \int_0^{\pi/2} \sin^3 t \cdot \sqrt{9a^2 [\cos^4 t \sin^2 t + \sin^4 t \cos^2 t]} dt \end{aligned}$$

$$\begin{aligned}
 &= 6\pi a^2 \int_0^{\pi/2} \sin^3 t \cdot \sin t \cos t \, dt \\
 &= 6\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t \, dt = 6\pi a^2 \left[\frac{3}{5} \cdot \frac{1}{3} \right] = \frac{6\pi a^2}{5}. \quad \text{Hence shown.}
 \end{aligned}$$

41. Find the volume of the spindle shaped solid generated by revolution of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ about the y -axis.

Sol. For the part of the curve in the first quadrant t varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}
 \therefore \text{ Required volume} &= 2 \int_0^{\pi/2} \pi x^2 \frac{dy}{dt} \cdot dt = 2\pi \int_0^{\pi/2} a^2 \cos^6 t \cdot 3a \sin^2 t \cos t \, dt \\
 &= 6\pi a^3 \int_0^{\pi/2} \cos^7 t \sin^2 t \, dt = 6\pi a^3 \left[\frac{6 \cdot 4 \cdot 2}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} \right] = \frac{32\pi a^3}{105}.
 \end{aligned}$$

42. The part of the parabola $y^2 = 4ax$, cut off by the latus rectum revolves about the tangent at the vertex. Find the area of the curved surface of the reel thus generated.

Sol. Tangent to the parabola at the origin is y -axis. Required surface area = $2 \times$ surface area generated by the upper half of the parabola.

Equation of the curve is $y^2 = 4ax$

$$\therefore \frac{2ydy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} = \sqrt{\frac{a}{x}}$$

$$\therefore \text{ Required surface area} = 2 \int_0^a 2\pi x \frac{dS}{dx} \cdot dx$$

$$\text{where} \quad \frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{a}{x}} = \sqrt{\frac{x+a}{x}}$$

$\therefore$ Surface area

$$\begin{aligned}
 &= 4\pi \int_0^a x \sqrt{\frac{x+a}{x}} \cdot dx = 4\pi \int_0^a \sqrt{x^2 + ax} \cdot dx = 4\pi \int_0^a \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} \, dx \\
 &= 4\pi \left[\frac{\left(x + \frac{a}{2}\right) \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2}}{2} - \frac{1}{2} \cdot \frac{a^2}{4} \log \left\{ \frac{\left(x + \frac{a}{2}\right) + \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2}}{a/2} \right\} \right]_0^a \\
 &= 4\pi \left[\frac{1}{2} \cdot \frac{3a}{2} \sqrt{\frac{9a^2}{4} - \frac{a^2}{4}} - \frac{a^2}{8} \log \left\{ \frac{\frac{3a}{2} + \sqrt{2a}}{a/2} \right\} \right] \\
 &= 4\pi \left[\frac{3a}{4} \sqrt{2a} - \frac{a^2}{8} \log (3 + 2\sqrt{2}) \right] = 4\pi \left[\frac{3\sqrt{2}}{4} a^2 - \frac{a^2}{8} \log (3 + 2\sqrt{2}) \right] \\
 &= \pi a^2 \left[3\sqrt{2} - \frac{1}{2} \log (3 + 2\sqrt{2}) \right] = \pi a^2 \left[3\sqrt{2} - \frac{1}{2} \log (\sqrt{2} + 1)^2 \right] \\
 &= \pi a^2 [3\sqrt{2} - \log (\sqrt{2} + 1)].
 \end{aligned}$$

43. Find the surface area of the solid generated by revolution of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ about the x -axis.

Sol. The required surface area = $2 \times$ Surface area of revolution about x -axis of the arc in the first quadrant.

As already found out in Q. 40, the same is $2 \times \frac{6\pi a^2}{5} = \frac{12\pi a^2}{5}$.

44. Find the surface area generated if an arc of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ revolves about the line $y = 2a$.

Sol. The shape of the curve is shown in the figure

The required surface area = $2 \times$ Surface area generated by the revolution of the arc OM about the line $y = 2a$

$$x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$$

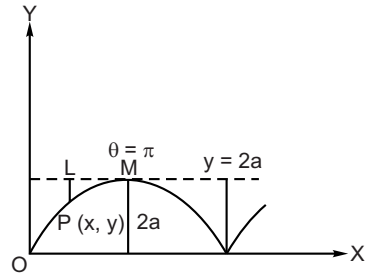
$$\frac{dx}{d\theta} = a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta$$

$\therefore$ The required surface area

$$\begin{aligned} &= 2 \int_0^\pi 2\pi(PL) \frac{ds}{d\theta} \cdot d\theta \\ &= 4\pi \int_0^\pi (2a - y) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta \\ &= 4\pi \int_0^\pi \{2a - a(1 - \cos \theta)\} \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} \cdot d\theta \\ &= 4\pi a^2 \int_0^\pi (1 + \cos \theta) \sqrt{2(1 - \cos \theta)} \cdot d\theta = 4\pi a^2 \int_0^\pi 2 \cos^2 \frac{\theta}{2} \cdot \sqrt{2 \cdot 2 \sin^2 \frac{\theta}{2}} \cdot d\theta \\ &= 16\pi a^2 \int_0^\pi \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} \cdot d\theta \end{aligned}$$

Put $\frac{\theta}{2} = t$, $d\theta = 2 \cdot dt$,

$$= 16\pi a^2 \int_0^{\pi/2} 2 \cdot \cos^2 t \sin t \cdot dt = 32\pi a^2 \left[\frac{\cos^3 t}{-3} \right]_0^{\pi/2} = 32\pi a^2 \left(\frac{1}{3} \right) = \frac{32}{3} \pi a^2.$$



45. Area bounded by x -axis, $y^2 = 4ax$ and the ordinate $x = 3a$ is revolved about x -axis. Find the volume generated. (M.D.U., May 2005)

Sol. We are given the area between x -axis, $y^2 = 4ax$ and $x = 3a$

$\therefore$ At x -axis, $y = 0 \Rightarrow x = 0$

At $x = 3a$, $y^2 = 4a \cdot 3a = 12a^2 \Rightarrow y = 2\sqrt{3}a$

Thus, we have to integrate between 0 and $2\sqrt{3} \cdot a$

The volume is given by

$$= \int_0^{2\sqrt{3} \cdot a} [(x+y)-(x-y)] dx = \int_0^{2\sqrt{3} \cdot a} 2y \, dx = \int_0^{2\sqrt{3} \cdot a} 2 \cdot 2\sqrt{ax} \cdot dx$$

$$\begin{aligned}
 &= 4\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{2\sqrt{3}a} = \frac{8}{3} \sqrt{a} [2\sqrt{3}a]^{3/2} = \frac{8}{3} \cdot a^{1/2} \cdot 2^{3/2} (\sqrt{3})^{3/2} \cdot a^{3/2} \\
 &= \frac{8}{3} \cdot a^2 \cdot (\sqrt{2})^3 \cdot 3 \cdot \frac{1}{2} \cdot \frac{3}{2} = 24a^2 (\sqrt{2})^3 = 48\sqrt{2} a^2 \text{ cubic units.}
 \end{aligned}$$

46. Prove that the area common to the two parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $\frac{16a^2}{3}$.

Sol. The equations of the parabola are

$$x^2 = 4ay \quad \dots(1) \quad y^2 = 4ax \quad \dots(2)$$

To find the points of intersection of the curves (1) and (2), eliminating y , we have

$$\begin{aligned}
 \frac{x^2}{4a} &= y, y^2 = 4ax \Rightarrow \frac{x^4}{16a^2} = 4ax \\
 x^4 &= 64a^3x \quad \text{or} \quad x(x^3 - 64a^3) = 0 \Rightarrow x = 0, 4a
 \end{aligned}$$

When $x = 0$, $y = 0$

When $x = 4a$, $y = 4a$

Hence the parabolas intersect at $(0, 0)$ and $(4a, 4a)$.

$$\begin{aligned}
 \text{Required area} &= \int_0^{4a} (y_2 - y_1) dx \\
 &= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx = \left[2\sqrt{a} \cdot \frac{x^{3/2}}{3/2} - \frac{x^3}{12a} \right]_0^{4a} = \frac{4\sqrt{a}}{3} \cdot (4a)^{3/2} - \frac{(4a)^3}{12a} \\
 &\quad \downarrow \quad \downarrow \\
 &\quad \text{upper} \quad \text{lower} \\
 &\quad \text{curve} \quad \text{curve} \\
 &\quad (2) \quad (1) \\
 &= \frac{32}{3} a^2 - \frac{16a^2}{3} = 16a^2/3.
 \end{aligned}$$

47. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad [\text{M.D.U., May 2008}]$$

Sol. Let (x, y, z) be a vertex of the parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Also its dimensions are $2x, 2y, 2z$ so that the volume V is given by

$$V = 2x \times 2y \times 2z = 8xyz$$

$\Rightarrow$

$$V^2 = 64x^2y^2z^2$$

$$= 64x^2y^2c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = 64c^2 \left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2} = f(x, y) \right)$$

$$f_x = 64c^2 \left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2} \right); \quad f_y = 64c^2 \left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2} \right)$$

$$f_{xx} = 64c^2 \left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2} \right) = r; \quad f_{xy} = 64c^2 \left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2} \right) = s$$

$$f_{yy} = 64c^2 \left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2} \right) = t$$

Now, $f_x = 0$ and $f_y = 0$

$$\Rightarrow 128c^2xy^2 \left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} \right) = 0$$

$$\text{and} \quad 128c^2x^2y \left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} \right) = 0 \quad [\because x \neq 0, y \neq 0]$$

$$\Rightarrow 1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \dots(1)$$

$$\text{and} \quad 1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0 \quad \dots(2)$$

Subtracting (2) from (1), we get

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{or} \quad y = \frac{bx}{a}$$

$$\therefore \text{From (1),} \quad x^2 = \frac{a^2}{3} \quad \text{or} \quad x = \frac{a}{\sqrt{3}} \quad \text{and} \quad y = \frac{b}{\sqrt{3}}$$

$$z^2 = c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \frac{c^2}{3}$$

$$\therefore \quad z = \frac{c}{\sqrt{3}}$$

Thus, $x = \frac{a}{\sqrt{3}}$ and $y = \frac{b}{\sqrt{3}}$ is a stationary point. At this point,

$$\begin{aligned} r = f_{xx} &= 64c^2 \left[\frac{2b^2}{3} - \frac{12}{a^2} \cdot \frac{a^2}{3} \cdot \frac{b^2}{3} - \frac{2}{b^2} \cdot \frac{b^4}{9} \right] = 64c^2 \left[\frac{2b^2}{3} - \frac{4b^2}{3} - \frac{2b^2}{9} \right] \\ &= 64c^2 \left(-\frac{8b^2}{9} \right) = -\frac{512}{9}b^2c^2 < 0 \end{aligned}$$

$$\text{Similarly, we get} \quad s = f_{xy} = -\frac{256}{9}abc^2; \quad t = f_{yy} = -\frac{512}{9}a^2c^2$$

$$\therefore \quad rt - s^2 = \left(\frac{512}{9} \right)^2 a^2b^2c^4 - \left(\frac{256}{9} \right)^2 a^2b^2c^4 = \left(\frac{256}{9} \right)^2 a^2b^2c^4 (4 - 1) > 0$$

Also, $r < 0$ $\therefore V^2$ and hence V is maximum

$$\text{When} \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

$$\text{Maximum volume} = 8xyz = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$$

□ □ □

6

Multiple Integration

BRIEF DEFINITIONS AND IMPORTANT FORMULAE

1. Area by Double Integration

- (a) **Cartesian Co-ordinates:** The area A of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$ is given by

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy dx$$

Similarly, the area A of the region bounded by the curves $x = f_1(y)$, $x = f_2(y)$ and the lines $y = c$, $y = d$ is given by

$$A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx dy .$$

- (b) **Polar Co-ordinates:** The area of the region bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$, is given by

$$A = \int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} r dr d\theta .$$

2. Volume as a Double Integral

- (a) **Cartesian Co-ordinates:**

Consider $z = f(x, y)$, then

$$V = \iint_R z dx dy$$

- (b) **Cylindrical Co-ordinates:**

Consider $z = f(r, \phi)$, then

$$V = \iint_R z r dr d\phi$$

Note 1. Consider
$$\int_a^b \int_c^d f(x, y) dy dx$$

$\underbrace{\quad\quad\quad}_{\text{Limits of } y}$
 $\underbrace{\quad\quad\quad}_{\text{Limits of } x}$

The integrand is to be solved w.r.t. y first, then w.r.t. x .

Note 2. In case of variable limits, the integration is to be performed w.r.t. that variable (having variable limits) **first** and then w.r.t. **the variable with constant limits**.

3. Volume of Solids of Revolution

(a) If the area ***R*** ***revolves about x-axis***, then the required volume is given by

$$V = 2\pi \iint_R y \, dy \, dx$$

(b) If the area ***R*** ***revolves about y-axis***, then the required volume is given by

$$V = 2\pi \iint_R x \, dx \, dy$$

4. Volume as Triple Integral

$$\text{Volume} = \iiint_V dz \, dy \, dx$$

5. Calculation of Mass, C.G.

(a) **For a plane Lamina of area A**

Let ρ be the density at a point P (x, y).

Then, $\text{Mass } M = \text{Density} \times \text{Area}$

$$= \iint_A \rho \, dx \, dy$$

$$\text{In polar co-ordinates, } M = \iint_A \rho r \, dr \, d\theta$$

(b) **For a solid of volume V**

$$M = \iiint \rho \, dx \, dy \, dz$$

$$M = \iiint \rho r^2 \sin \theta \, dr \, d\theta \, d\phi$$

(c) **The centre of gravity (C. G.) (x, y) of a plane lamina of area A, in cartesian co-ordinates,**

$$\bar{x} = \frac{1}{M} \iint \rho x \, dx \, dy$$

$$\bar{y} = \frac{1}{M} \iint \rho y \, dx \, dy$$

In polar co-ordinates

$$\bar{x} = \frac{1}{M} \iint \rho r \cos \theta \, r \, dr \, d\theta$$

$$\bar{y} = \frac{1}{M} \iint \rho r \sin \theta \, r \, dr \, d\theta$$

(d) **The C.G. of a solid of volume V is given by**

$$\bar{x} = \frac{1}{M} \iiint \rho x \, dx \, dy \, dz$$

$$\bar{y} = \frac{1}{M} \iiint \rho y \, dx \, dy \, dz$$

$$\bar{z} = \frac{1}{M} \iiint \rho z \, dx \, dy \, dz$$

SOLVED PROBLEMS

1. Prove that : $\int_1^2 \int_3^4 (xy + e^y) dy dx = \int_3^4 \int_1^2 (xy + e^y) dx dy$

Sol.
$$\begin{aligned} \int_1^2 \int_3^4 (xy + e^y) dy dx &= \int_1^2 \left[\int_3^4 (xy + e^y) dy \right] dx = \int_1^2 \left[\frac{xy^2}{2} + e^y \right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3 \right) dx \\ &= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3 \right) dx = \left[7 \frac{x^2}{4} + (e^4 - e^3)x \right]_1^2 \\ &= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3 \end{aligned} \quad \dots(1)$$

Now,
$$\begin{aligned} \int_3^4 \int_1^2 (xy + e^y) dx dy &= \int_3^4 \left[\int_1^2 (xy + e^y) dx \right] dy = \int_3^4 \left[y \frac{x^2}{2} + xe^y \right]_1^2 dy \\ &= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y \right) dy = \int_3^4 \left(\frac{3y}{2} + e^y \right) dy \\ &= \left[\frac{3}{4}y^2 + e^y \right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3 \end{aligned} \quad \dots(2) \quad \text{Hence proved.}$$

From (1) and (2), L.H.S. = R.H.S.

2. Evaluate : $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$.

Sol.
$$\begin{aligned} \int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}} &= \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}} \right] dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x \right]_0^1 dy \\ &= \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot \frac{\pi}{2} dy = \frac{\pi}{2} \left[\sin^{-1} y \right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}. \end{aligned}$$

3. Show that : $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$.

Sol.
$$\begin{aligned} \text{L.H.S.} &= \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy = \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3} dy \\ &= \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right] dy = \int_0^1 \left[2x \cdot \frac{(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1} \right]_0^1 dx \\ &= \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y} \right]_0^1 dx = \int_0^1 \left[\frac{-x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x} \right] dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{-x + x + 1}{(x+1)^2} dx = \left[-\frac{1}{(x+1)} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2} \\
\text{R.H.S.} &= \int_0^1 dy \int_0^1 \frac{x + y - 2y}{(x+y)^3} \cdot dx \\
&= \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx = \int_0^1 dy \left[\frac{-1}{x+y} - 2y \cdot \frac{(x+y)^{-2}}{-2} \right]_0^1 \\
&= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y} \right] dy = \int_0^1 \left(-\frac{1}{1+y} + \frac{y}{(1+y)^2} \right) dy \\
&= \int_0^1 \left[\frac{-1-y+y}{(1+y)^2} \right] dy = \int_0^1 -\frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}
\end{aligned}$$

Therefore, the two integrals are not equal.

4. Evaluate : $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2} \cdot$ (U.P.T.U., 2006)

Sol.

$$\begin{aligned}
I &= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right] dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx \\
&= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0] dx \\
&= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[\log (x + \sqrt{1+x^2}) \right]_0^1 \\
&= \frac{\pi}{4} [\log (1 + \sqrt{2}) - \log 1] = \frac{\pi}{4} \log (\sqrt{2} + 1)
\end{aligned}$$

5. Evaluate : $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy$

Sol.

$$\begin{aligned}
I &= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2} dx \right] dy \\
&= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy \\
&= \int_0^a \frac{a^2-y^2}{2} \sin^{-1}(1) \cdot dy = \frac{\pi}{4} \int_0^a (a^2-y^2) dy \\
&= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right] = \frac{\pi a^3}{6}
\end{aligned}$$

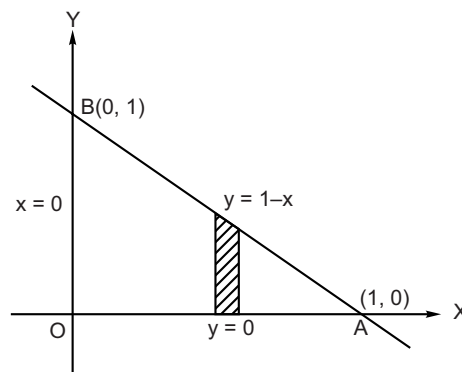
6. Evaluate $\iint e^{2x+3y} dx dy$ over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Sol. The Region R of integration is the triangle OAB . Here x varies from 0 to 1 and y varies from x -axis upto the line $x + y = 1$ i.e., from 0 to $1 - x$.

$\therefore$ The Region R can be expressed as

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x$$

$$\begin{aligned} \therefore \iint e^{2x+3y} dx dy &= \int_0^1 \int_0^{1-x} e^{2x+3y} dy dx \\ &= \int_0^1 \left[\frac{1}{3} e^{2x+3y} \right]_0^{1-x} dx \\ &= \frac{1}{3} \int_0^1 (e^{3-x} - e^{2x}) dx \end{aligned}$$



$$\begin{aligned} &= \frac{1}{3} \left[-e^{3-x} - \frac{1}{2} e^{2x} \right]_0^1 = -\frac{1}{3} \left[\left(e^2 + \frac{1}{2} e^2 \right) - \left(e^3 + \frac{1}{2} \right) \right] \\ &= -\frac{1}{3} \left[-e^2(e-1) + \frac{1}{2} (e^2 - 1) \right] = \frac{1}{6} (e-1) [2e^2 - (e+1)] \\ &= \frac{1}{6} (e-1) [2e^2 - e - 1] = \frac{1}{6} (e-1) (e-1) (2e+1) = \frac{1}{6} (e-1)^2 (2e+1) \end{aligned}$$

7. Evaluate $\iint_R y dx dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol. Solving $y^2 = 4x$ and $x^2 = 4y$, we have $\left(\frac{x^2}{4} \right)^2 = 4x$ or $x(x^3 - 64) = 0$

$\therefore x = 0, 4$. When $x = 4$, $y = 4$

$\therefore$ Both the parabolas meet each other at the points $(0, 0)$ and $(4, 4)$.

The region R can be expressed as $0 \leq x \leq 4$, $\frac{x^2}{4} \leq y \leq 2\sqrt{x}$.

$$\begin{aligned} \therefore \iint_R y dx dy &= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y dy dx = \int_0^4 \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^4 \left(4x - \frac{x^4}{16} \right) dx = \frac{1}{2} \left[2x^2 - \frac{x^5}{80} \right]_0^4 = \frac{1}{2} \left(32 - \frac{1024}{80} \right) = \frac{48}{5} \end{aligned}$$

8. Evaluate $\iint (x+y)^2 dx dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. For the ellipse, $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$

$\therefore$ The region of integration R can be expressed as

$$-a \leq x \leq a, \quad -\frac{b}{a} \sqrt{a^2 - x^2} \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\begin{aligned} \therefore \iint_R (x+y)^2 dx dy &= \iint (x^2 + y^2 + 2xy) dx dy = \int_{-a}^a \int_{(-b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy) dy dx \\ &= \int_{-a}^a \int_{(-b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} (x^2 + y^2) dy dx + \int_{-a}^a \int_{(-b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} 2xy dy dx \end{aligned}$$

Since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y , the second integral is 0.

$$\begin{aligned} \therefore I &= \int_{-a}^a \int_0^{(b/a)\sqrt{a^2-x^2}} 2(x^2 + y^2) dy dx = \int_{-a}^a \left[2 \left(x^2 y + \frac{y^3}{3} \right) \right]_0^{(b/a)\sqrt{a^2-x^2}} dx \\ &= 2 \int_{-a}^a \left[x^2 \cdot \frac{b}{a} \sqrt{a^2-x^2} + \frac{1}{3} \cdot \frac{b^3}{a^3} (a^2-x^2)^{3/2} \right] dx \\ &= 4 \int_0^a \left[\frac{b}{a} x^2 \sqrt{a^2-x^2} + \frac{b^3}{3a^3} (a^2-x^2)^{3/2} \right] dx \end{aligned}$$

Put $x = a \sin \theta$ When $x = a$, $\theta = \pi/2$
 $dx = a \cos \theta$ When $x = 0$, $\theta = 0$

$$\begin{aligned} \sqrt{a^2-x^2} &= a \cos \theta \\ &= 4 \int_0^{\pi/2} \left(\frac{b}{a} a^2 \sin^2 \theta \cdot a \cos \theta + \frac{b^3}{3a^3} \cdot a^3 \cos^3 \theta \right) a \cos \theta d\theta \\ &= 4 \int_0^{\pi/2} \left(a^3 b \sin^2 \theta \cos \theta + \frac{ab^3}{3} \cos^4 \theta \right) d\theta \\ &= 4 \left[a^3 b \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{ab^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{\pi}{4} (a^3 b + ab^3) = \frac{\pi}{4} ab (a^2 + b^2). \end{aligned}$$

9. Evaluate $\iint y dx dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$.

Sol. The line $y = x$ meets the parabola $y = 4x - x^2$ in 2 distinct points (0, 0) and (3, 3).

The region of integration R can be expressed as $0 \leq x \leq 3$, $x \leq y \leq 4x - x^2$.

$$\therefore \iint_R y dx dy = \int_0^3 \int_x^{4x-x^2} y dy \cdot dx = \int_0^3 \left[\frac{y^2}{2} \right]_x^{4x-x^2} dx = \frac{1}{2} \int_0^3 [(4x-x^2)^2 - x^2] dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^3 (15x^2 + x^4 - 8x^3) dx = \frac{1}{2} \left[\frac{15x^3}{3} + \frac{x^5}{5} - \frac{8x^4}{4} \right]_0^3 \\
 &= \frac{1}{2} \left[135 + \frac{243}{5} - 162 \right] = \frac{54}{5}
 \end{aligned}$$

10. When the region R of integration is the triangle given by $y = 0$, $y = x$ and $x = 1$, show that :

$$\iint_R \sqrt{4x^2 - y^2} \, dx dy = \frac{1}{3} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right).$$

Sol. The region of integration R is $0 \leq x \leq 1$, $0 \leq y \leq x$.

$$\begin{aligned}
 \therefore \iint_R \sqrt{4x^2 - y^2} \, dx dy &= \int_0^1 \int_0^x \sqrt{4x^2 - y^2} \, dy dx \\
 &= \int_0^1 \left[\frac{1}{2} y \sqrt{4x^2 - y^2} + \frac{1}{2} \cdot 4x^2 \cdot \sin^{-1} \left(\frac{y}{2x} \right) \right]_0^x dx \\
 &\quad \left\{ \because \int \sqrt{a^2 - x^2} \cdot dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right\} \\
 &= \int_0^1 \frac{1}{2} \left(\sqrt{3}x^2 + 4x^2 \cdot \frac{\pi}{6} \right) dx = \frac{1}{2} \left[\sqrt{3} \cdot \frac{x^3}{3} + \frac{2\pi}{3} \cdot \frac{x^3}{3} \right]_0^1 \\
 &= \frac{1}{2} \left[\sqrt{3} \cdot \frac{1}{3} + \frac{2\pi}{3} \cdot \frac{1}{3} \right] = \left(\frac{1}{2\sqrt{3}} + \frac{\pi}{9} \right) = \frac{1}{3} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)
 \end{aligned}$$

11. Evaluate $\iint_S \sqrt{xy - y^2} \, dx dy$ where S is a triangle with vertices $(0, 0)$, $(10, 1)$ and $(1, 1)$.

Sol. Let OAB be the triangle whose vertices are $(0, 0)$, $(10, 1)$ and $(1, 1)$ as shown in the figure.

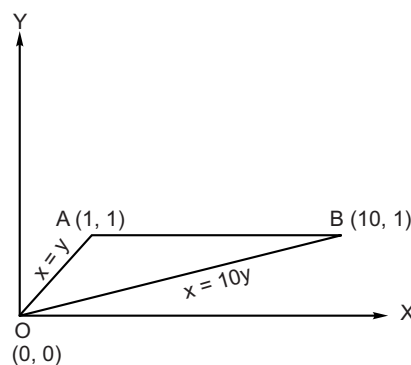
The equation of the line joining $O(0, 0)$ and $A(1, 1)$ is

$$y - 0 = \frac{1-0}{1-0} (x-0) \Rightarrow y = x$$

Equation of line OB is

$$y - 0 = \frac{1-0}{10-0} (x-0)$$

$$\Rightarrow y = \frac{1}{10}x \quad \text{or} \quad x = 10y$$



Hence the region of integration R can be expressed as

$$y \leq x \leq 10y, \quad 0 \leq y \leq 1$$

$$\begin{aligned}
 \therefore \iint_S \sqrt{xy - y^2} \, dx dy &= \int_0^1 \int_y^{10y} \sqrt{xy - y^2} \, dx dy \\
 &= \int_0^1 \left[\frac{(xy - y^2)^{3/2}}{(3/2)y} \right]_y^{10y} dy = \int_0^1 \frac{2}{3y} (9y^2)^{3/2} dy = 18 \int_0^1 y^2 dy = 18 \left[\frac{y^3}{3} \right]_0^1 = \frac{18}{3} = 6
 \end{aligned}$$

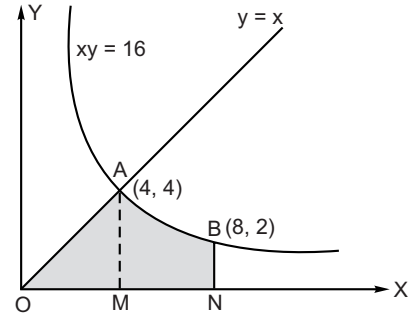
12. Let D be the region in the first quadrant bounded by the curves $xy = 16$, $x = y$, $y = 0$ and $x = 8$.

Sketch the region of integration of the integral $\iint_D x^2 dx dy$ and evaluate it by expressing it as an appropriate repeated integral.

Sol. The straight line $x = y$ intersects the hyperbola $xy = 16$ at $A(4, 4)$. The line $x = 8$ intersects the hyperbola $xy = 16$ at $B(8, 2)$. The shaded portion in the following figure is the region of integration.

To evaluate the given integral, we divide the area $OABNO$ into 2 parts by AM .

$$\begin{aligned} \text{Then } \iint_D x^2 dx dy &= \int_{x=0}^{x=4} \int_{y=0}^{y=x} x^2 dx dy + \int_{x=4}^{x=8} \int_{y=\frac{16}{x}}^{y=\frac{16}{x}} x^2 dx dy \\ &= \int_0^4 x^2 dx \int_0^x dy + \int_4^8 x^2 dx \int_{\frac{16}{x}}^{\frac{16}{x}} dy \\ &= \int_0^4 x^3 dx + \int_4^8 16x dx = \left[\frac{x^4}{4} \right]_0^4 + \left[\frac{16x^2}{2} \right]_4^8 = 64 + 8(64 - 16) = 64 + 384 = 448. \end{aligned}$$



13. Evaluate $\int_0^3 \int_1^2 xy(1+x+y) dy dx$.

$$\begin{aligned} \text{Sol. } \int_0^3 \int_1^2 (xy + x^2y + xy^2) dy \cdot dx &= \int_0^3 \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_1^2 \cdot dx \\ &= \int_0^3 \left(x \cdot 2 + 2x^2 + \frac{8x}{3} - \frac{x}{2} - \frac{x^2}{2} - \frac{x}{3} \right) dx \\ &= \int_0^3 \left(\frac{23x}{6} + \frac{3x^2}{2} \right) dx = \int_0^3 \left(\frac{23}{6}x + \frac{3}{2}x^2 \right) dx \\ &= \left[\frac{23}{6} \cdot \frac{x^2}{2} + \frac{3}{2} \cdot \frac{x^3}{3} \right]_0^3 = \frac{23}{6} \cdot \frac{9}{2} + \frac{3}{2} \cdot \frac{27}{3} = \frac{123}{4} = 30 \frac{3}{4} \end{aligned}$$

14. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy dx$.

$$\begin{aligned} \text{Sol. } \int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy dx &= \int_0^1 \left[x^2y + \frac{y^3}{3} \right]_x^{\sqrt{x}} \cdot dx \\ &= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3} \right) dx = \left[\frac{2}{7} \cdot x^{\frac{7}{2}} + \frac{2}{15} \cdot x^{5/2} - \frac{4}{3} \cdot \frac{x^4}{4} \right]_0^1 \\ &= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30+14-35}{105} = \frac{9}{105} \quad \text{or} \quad \frac{3}{35} \end{aligned}$$

15. Evaluate $\iint x^2 y^2 dx dy$ over the circle $x^2 + y^2 = 1$. [U.P.T.U., (Agr. Engg.) 2005]

Sol. The Region of integration is $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

$$\begin{aligned} I &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y^2 dx dy = \int_{-1}^1 2 \int_0^{\sqrt{1-x^2}} y^2 \cdot dy \cdot x^2 dx \\ &= 2 \int_{-1}^1 \left[\frac{y^3}{3} \right]_0^{\sqrt{1-x^2}} \cdot x^2 dx = \frac{2}{3} \int_{-1}^1 (1-x^2)^{3/2} \cdot x^2 \cdot dx = \frac{4}{3} \int_0^1 x^2 (1-x^2)^{3/2} \cdot dx \end{aligned}$$

Put $x = \sin \theta$, $dx = \cos \theta d\theta$,

When $x = 1$, $\theta = \pi/2$, $x = 0$, $\theta = 0$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta = \frac{4}{3} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{24}$$

16. Evaluate $\int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr d\theta$

Sol.

$$\begin{aligned} I &= \int_0^{\pi/2} \left[\int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr \right] d\theta = \int_0^{\pi/2} \left[\int_0^{a \cos \theta} -\frac{1}{2} (-2r) \sqrt{a^2 - r^2} dr \right] d\theta \\ &= \int_0^{\pi/2} \left[-\frac{1}{2} \cdot \frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^{a \cos \theta} \cdot d\theta \\ &= -\frac{1}{3} \int_0^{\pi/2} (a^3 \sin^3 \theta - a^3) d\theta = -\frac{a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right] = \frac{a^3}{18} (3\pi - 4) \end{aligned}$$

17. Evaluate $\iint r^3 dr d\theta$, over the area bounded between the circles $r = 2 \cos \theta$ and $r = 4 \cos \theta$.

Sol. Here r varies from $2 \cos \theta$ to $4 \cos \theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\begin{aligned} \therefore \int_{-\pi/2}^{\pi/2} \int_{2 \cos \theta}^{4 \cos \theta} r^3 dr d\theta &= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2 \cos \theta}^{4 \cos \theta} d\theta \\ &= \frac{1}{4} \int_{-\pi/2}^{\pi/2} (256 \cos^4 \theta - 16 \cos^4 \theta) d\theta = 60 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta \end{aligned}$$

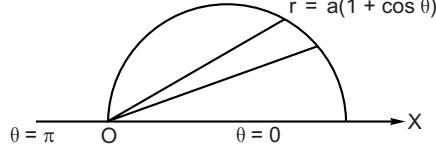
Since $\cos^4 \theta$ is an even function of θ ,

$$= 60 \times 2 \int_0^{\pi/2} \cos^4 \theta d\theta = 120 \cdot \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{45\pi}{2}.$$

18. Evaluate $\iint_R r \sin \theta \, dr \, d\theta$ over the area of the cardioid $r = a(1 + \cos \theta)$ above the initial line.

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a(1 + \cos \theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.

$$\begin{aligned}
 \therefore \iint_R r \sin \theta \, dr \, d\theta &= \int_0^\pi \int_0^{a(1+\cos \theta)} r \sin \theta \, dr \, d\theta \\
 &= \int_0^\pi \sin \theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos \theta)} d\theta \\
 &= \frac{1}{2} a^2 \int_0^\pi \sin \theta \cdot (1 + \cos \theta)^2 d\theta \\
 &= \frac{a^2}{2} \int_0^\pi 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \left(2 \cos^2 \frac{\theta}{2} \right)^2 d\theta = 4a^2 \int_0^\pi \sin \frac{\theta}{2} \cos^5 \frac{\theta}{2} d\theta \\
 &= 4a^2 \int_0^{\pi/2} \sin \phi \cos^5 \phi \cdot 2d\phi \quad \left| \begin{array}{l} \text{Putting } \frac{\theta}{2} = \phi, d\theta = 2d\phi, \\ \text{when } \theta = \pi, \phi = \frac{\pi}{2} \end{array} \right. \\
 &= 8a^2 \int_0^{\pi/2} \cos^5 \phi \sin \phi \, d\phi = -8a^2 \left[\frac{\cos^6 \phi}{6} \right]_0^{\pi/2} = -\frac{4a^2}{3} (0 - 1) = \frac{4a^2}{3}.
 \end{aligned}$$



19. Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the curve $r^2 = a^2 \cos 2\theta$. (M.D.U., Dec. 2006)

Sol. The Region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\frac{\pi}{4}$ and ending at $\theta = \frac{\pi}{4}$.

$$\begin{aligned}
 \therefore \iint_R \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}} &= \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r \, dr}{\sqrt{a^2 + r^2}} \cdot d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2} \cdot (a^2 + r^2)^{-1/2} (2r) \, dr \cdot d\theta \\
 &= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} \frac{(a^2 + r^2)^{1/2}}{1/2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4} \left[(a^2 + a^2 \cos 2\theta)^{\frac{1}{2}} - a \right] d\theta \\
 &= a \int_{-\pi/4}^{\pi/4} \left[(1 + \cos 2\theta)^{\frac{1}{2}} - 1 \right] d\theta = a \int_{-\pi/4}^{\pi/4} \left[(2 \cos^2 \theta)^{\frac{1}{2}} - 1 \right] d\theta \\
 &= a \int_{-\pi/4}^{\pi/4} (\sqrt{2} \cos \theta - 1) d\theta = 2a \int_0^{\pi/4} (\sqrt{2} \cos \theta - 1) d\theta \\
 &= 2a \left[\sqrt{2} \sin \theta - \theta \right]_0^{\pi/4} = 2a \left(\sqrt{2} \cdot \frac{1}{\sqrt{2}} - \frac{\pi}{4} \right) = 2a \left(1 - \frac{\pi}{4} \right)
 \end{aligned}$$

20. Evaluate $\int_0^\pi \int_0^{a(1-\cos \theta)} r^2 \sin \theta \, dr \, d\theta$

Sol. Given
$$I = \int_0^\pi \left[\frac{r^3}{3} \right]_0^{a(1-\cos \theta)} \sin \theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^\pi \sin \theta (1 - \cos \theta)^3 \, d\theta = \frac{a^3}{3} \int_0^2 t^3 \, dt \quad \left\{ \begin{array}{l} \text{When } t = 1 - \cos \theta, dt = \sin \theta \, d\theta \\ \text{When } \theta = 0, t = 0 \\ \text{When } \theta = \pi, t = 2 \end{array} \right.$$

$\therefore I = \frac{a^3}{3} \left[\frac{t^4}{4} \right]_0^2 = \frac{a^3}{12} (16) = \frac{4}{3} a^3$

21. Evaluate $\iint_A r^2 \, dr \, d\theta$ where A is the area between the circles $r = a \cos \theta$ and $r = 2a \cos \theta$.

Sol. Here r varies from $a \cos \theta$ to $2a \cos \theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\therefore \iint_A r^2 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{a \cos \theta}^{2a \cos \theta} (r^2 \, dr) \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_{a \cos \theta}^{2a \cos \theta} d\theta$

$$= \frac{1}{3} \int_{-\pi/2}^{\pi/2} (8a^3 \cos^3 \theta - a^3 \cos^3 \theta) \, d\theta = \frac{7}{3} \times 2a^3 \int_0^{\pi/2} \cos^3 \theta \, d\theta = \frac{14a^3}{3} \left[\frac{2}{3} \right] = \frac{28a^3}{9}.$$

22. Evaluate $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2 \sin \theta$ and $r = 4 \sin \theta$.

Sol. Here r varies from $2 \sin \theta$ to $4 \sin \theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\therefore \int_{-\pi/2}^{\pi/2} \int_{2 \sin \theta}^{4 \sin \theta} r^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2 \sin \theta}^{4 \sin \theta} d\theta = \frac{1}{4} \int_{-\pi/2}^{\pi/2} 240 \sin^4 \theta \, d\theta = 60 \int_{-\pi/2}^{\pi/2} \sin^4 \theta \, d\theta$

$$= 120 \int_0^{\pi/2} \sin^4 \theta \, d\theta = 120 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = \frac{45}{2} \pi$$

23. Evaluate $\int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \cos \theta \, dr \, d\theta$.

Sol.
$$I = \int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \cos \theta \, dr \, d\theta = \int_0^\pi \left[\frac{r^3}{3} \right]_0^{a(1+\cos \theta)} \cos \theta \, d\theta$$

$$= \int_0^\pi \frac{a^3}{3} (1 + \cos \theta)^3 \cos \theta \, d\theta = \frac{a^3}{3} \int_0^\pi \left(2 \cos^2 \frac{\theta}{2} \right)^3 \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) d\theta$$

$$= \frac{8a^3}{3} \int_0^\pi \cos^6 \frac{\theta}{2} \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) d\theta$$

$$\begin{aligned}
&= \frac{8a^3}{3} \left[\int_0^\pi \cos^8 \frac{\theta}{2} d\theta - \int_0^\pi \cos^6 \frac{\theta}{2} \sin^2 \frac{\theta}{2} d\theta \right] \quad \left| \begin{array}{l} \text{Put } \frac{\theta}{2} = \phi, d\theta = 2d\phi \\ \text{When } \theta = \pi, \phi = \frac{\pi}{2} \end{array} \right. \\
&= \frac{8a^3}{3} \left[2 \cdot \int_0^{\pi/2} \cos^8 \phi d\phi - 2 \int_0^{\pi/2} \cos^6 \phi \sin^2 \phi d\phi \right] = \frac{16a^3}{3} \left[\frac{7.5.3.1}{8.6.4.2} \cdot \frac{\pi}{2} - \frac{1.5.3.1}{8.6.4.2} \cdot \frac{\pi}{2} \right] \\
&= \frac{16a^3}{3} \left[\frac{35\pi}{256} - \frac{5\pi}{256} \right] = \frac{16a^3}{3} \times \frac{30\pi}{256} = \frac{5}{8} \pi a^3.
\end{aligned}$$

24. Evaluate $\int_0^{\pi/2} \int_0^{a \cos \theta} r \sin \theta dr d\theta$.

Sol. $\int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{a \cos \theta} \sin \theta d\theta = \frac{1}{2} \int_0^{\pi/2} a^2 \cos^2 \theta \sin \theta d\theta$

$$= -\frac{a^2}{2} \int_0^{\pi/2} \cos^2 \theta (-\sin \theta) d\theta = -\frac{a^2}{2} \left[\frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = -\frac{a^2}{6} (0 - 1) = \frac{a^2}{6}$$

25. Evaluate $\iint r^3 dr d\theta$ over the area included between the circles $r = 2a \cos \theta$, $r = 2b \cos \theta$ ($b < a$).

Sol. θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\begin{aligned}
\iint r^3 dr d\theta &= \int_{-\pi/2}^{\pi/2} \int_{2b \cos \theta}^{2a \cos \theta} r^3 dr d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2b \cos \theta}^{2a \cos \theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{4} [(2a)^4 \cos^4 \theta - (2b)^4 \cos^4 \theta] d\theta = 4(a^4 - b^4) \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta \\
&= 8(a^4 - b^4) \int_0^{\pi/2} \cos^4 \theta d\theta = 8(a^4 - b^4) \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} = \frac{3\pi}{2} (a^4 - b^4)
\end{aligned}$$

26. Evaluate $\iint r \sin \theta dr d\theta$ over the cardioid $r = a(1 - \cos \theta)$ above the initial line.

Sol. θ varies from 0 to π

$$\begin{aligned}
\iint r \sin \theta dr d\theta &= \int_0^\pi \int_0^{a(1 - \cos \theta)} r \sin \theta dr d\theta = \int_0^\pi \left[\frac{r^2}{2} \right]_0^{a(1 - \cos \theta)} \sin \theta d\theta \\
&= \int_0^\pi \frac{a^2}{2} (1 - \cos \theta)^2 \sin \theta d\theta = \frac{a^2}{2} \int_0^\pi (1 - \cos \theta)^2 \sin \theta d\theta = \frac{a^2}{2} \int_0^2 t^2 dt
\end{aligned}$$

$1 - \cos \theta = t, \quad \sin \theta d\theta = dt, \quad t = 2$

When $\theta = \pi, t = 0$, When $\theta = 0$

$$= \frac{a^2}{2} \left[\frac{t^3}{3} \right]_0^2 = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}$$

27. Show that $\iint_R r^2 \sin \theta \, dr d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a \cos \theta$, above the initial line.

Sol. θ varies from 0 to $\pi/2$.

$$\begin{aligned} \int_0^{\pi/2} \int_0^{2a \cos \theta} r^2 \sin \theta \, dr d\theta &= \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_0^{2a \cos \theta} \cdot \sin \theta \, d\theta \\ &= \frac{1}{3} \int_0^{\pi/2} 8a^3 \cos^3 \theta \cdot \sin \theta \, d\theta = \frac{8a^3}{3} \int_0^{\pi/2} \cos^3 \theta \sin \theta \, d\theta \\ &= -\frac{8a^3}{3} \left[\frac{\cos^4 \theta}{4} \right]_0^{\pi/2} = -\frac{8a^3}{3} \left(0 - \frac{1}{4} \right) = \frac{2a^3}{3} \end{aligned}$$

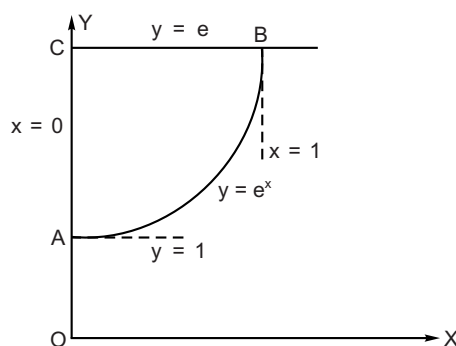
28. Evaluate the following integral by changing the order of integration $\int_0^1 \int_{e^x}^e \frac{dy dx}{\log y}$.

Sol. The given limits show that the region of integration is bounded by the curves $y = e^x$, $y = e$, $x = 0$, $x = 1$.

It is the area shaded in the diagram. It can also be considered as bounded by

$$x = 0, x = \log y, y = 1, y = e$$

$$\begin{aligned} \text{Hence, } \int_0^1 \int_{e^x}^e \frac{dy dx}{\log y} &= \int_1^e \int_0^{\log y} \frac{dx dy}{\log y} \\ &= \int_1^e \left(\frac{x}{\log y} \right)_0^{\log y} dy \\ &= \int_1^e 1 \cdot dy = [y]_1^e = e - 1 \end{aligned}$$



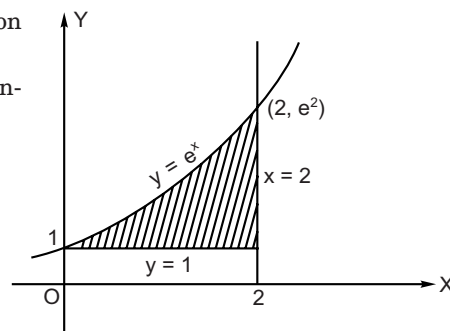
29. Evaluate $\int_0^2 \int_1^{e^x} dy dx$ by changing the order of integration.

Sol. The given limits show that the region of integration is bounded by curves $x = 0$, $x = 2$, $y = 1$ and $y = e^x$.

It is the area shaded in the diagram. It can also be considered as bounded by

$$x = \log y, x = 2, y = 1, y = e^2$$

$$\begin{aligned} \therefore I &= \int_1^{e^2} \int_{\log y}^2 dx dy = \int_1^{e^2} (2 - \log y) dy \\ &= \{2y - y \log y + y\}_1^{e^2} = (3y - y \log y)_1^{e^2} \\ &= (3e^2 - 2e^2) - (3 - 0) = e^2 - 3 \end{aligned}$$



30. Evaluate by changing the order of integration : $\int_0^1 \int_{2y}^2 e^{x^2} dx dy$

Sol. The given limits show that the region of integration is bounded by the lines :

$$x = 2y, \quad x = 2, \quad y = 0, \quad y = 1$$

We can also consider it as bounded by

$$y = 0, \quad y = \frac{x}{2}, \quad x = 0, \quad x = 2$$

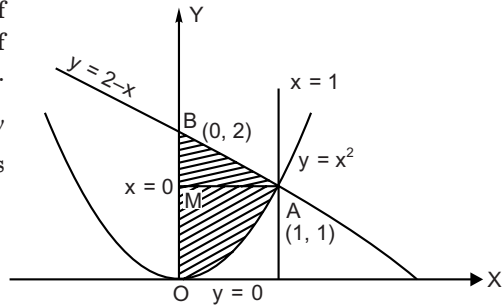
$$\begin{aligned} \therefore \int_0^1 \int_{2y}^2 e^{x^2} dx dy &= \int_0^2 \int_0^{x/2} e^{x^2} dx dy = \int_0^2 e^{x^2} [y]_0^{x/2} dx \\ &= \int_0^2 \frac{x}{2} e^{x^2} dx = \frac{1}{4} \int_0^2 2x e^{x^2} dx = \frac{1}{4} (e^{x^2})_0^2 = \frac{e^4 - 1}{4} \end{aligned}$$

31. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy dy dx$ and evaluate. (U.P.T.U., 2005)

Sol. Solving $y = x^2$ and $y = 2 - x$, the co-ordinate of A are (1, 1). Draw $AM \perp OY$. The region of integration is divided into 2 parts, OAM and MAB.

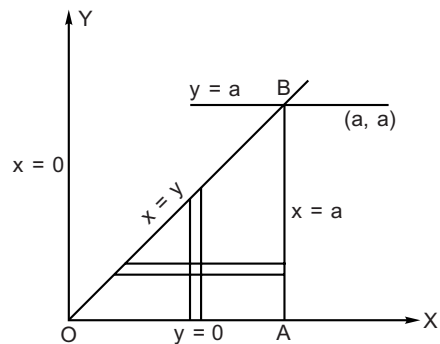
For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $2 - y$ and y varies from 1 to 2.

$$\begin{aligned} \therefore \int_0^1 \int_{x^2}^{2-x} xy dx dy &= \int_0^1 \int_0^{\sqrt{y}} xy dx dy + \int_1^2 \int_0^{2-y} xy dx dy \\ &= \int_0^1 \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} y dy + \int_1^2 \left[\frac{x^2}{2} \right]_0^{2-y} y dy \\ &= \frac{1}{2} \int_0^1 y^2 dy + \int_1^2 \frac{1}{2} \cdot y(2-y)^2 dy = \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1 + \frac{1}{2} \int_1^2 (4y - 4y^2 + y^3) dy \\ &= \frac{1}{6} + \frac{1}{2} \left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2 = \frac{1}{6} + \frac{1}{2} \left(12 - \frac{32}{3} - 2 + \frac{4}{3} - \frac{1}{4} \right) = \frac{3}{8} \end{aligned}$$



32. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2 + y^2} dx dy$ and hence evaluate it.

Sol. From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus the region of integration is the ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become : y varies from 0 to x and x varies from 0 to a .



$$\begin{aligned} \int_0^a \int_y^a \frac{x \, dx \, dy}{x^2 + y^2} &= \int_0^a \int_0^x \frac{x \, dy \, dx}{x^2 + y^2} \\ &= \int_0^a x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx = \int_0^a \frac{\pi}{4} dx = \frac{\pi}{4} \left[x \right]_0^a = \frac{a\pi}{4} \end{aligned}$$

33. Change the order of integration in the integral $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \, dx$ and evaluate it.

(M.D.U., May 2009)

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies

from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and then w.r.t. x which varies from $x = 0$ to $x = 4a$.

Thus the integration is first performed along the vertical strip PQ which extends from a point P on

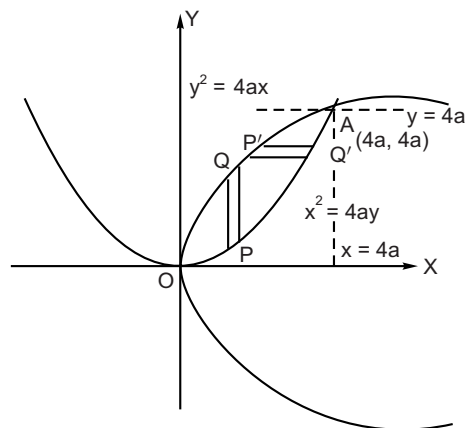
the parabola $y = \frac{x^2}{4a}$ (i.e., $x^2 = 4ay$) to the point Q on

the parabola $y = 2\sqrt{ax}$ ($y^2 = 4ax$). Then the strip slides from O to $A(4a, 4a)$, the point of intersection of the two parabolas. For changing the order of integration, we divide the region of integration $OPAQO$ into horizontal strips $P'Q'$ which extend from

P' on the parabola $y^2 = 4ax$ i.e., $x = \frac{y^2}{4a}$ to Q' on the

parabola $x^2 = 4ay$ i.e., $x = 2\sqrt{ay}$.

Then this strip slides from O to $A(4a, 4a)$ i.e., y varies from 0 to $4a$.



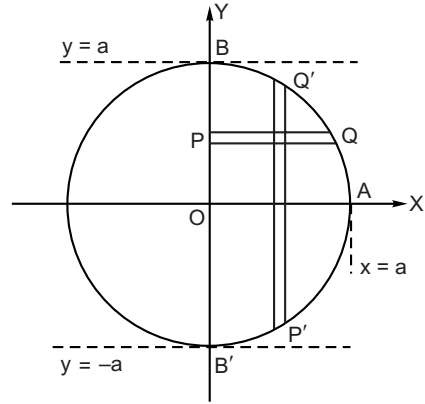
$$\begin{aligned} \therefore \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \, dx &= \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx \, dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy = \left[2\sqrt{a} \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right]_0^{4a} \\ &= \frac{4}{3} \sqrt{a} (4a)^{3/2} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \end{aligned}$$

34. Change the order of integration in the integral $\int_{-a}^a \int_0^{\sqrt{a^2 - y^2}} f(x, y) \, dx \, dy$.

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. x which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and then w.r.t. y which varies from $y = -a$ to $y = a$. Thus integration is first performed along the horizontal strip PQ which extends from a point P on $x = 0$ (i.e., y -axis) to the point Q on the circle $x = \sqrt{a^2 - y^2}$ (i.e., $x^2 + y^2 = a^2$). Then the strip slides from B' to B .

For changing the order of integration, we divide the region of integration $B'AQB'PB'$ into vertical strips $P'Q'$ which extend from P' on the circle $y = -\sqrt{a^2 - x^2}$ to Q' on the circle $y = \sqrt{a^2 - x^2}$.

Then this strip slides from y -axis ($x = 0$) to A , where $x = a$.



$$\therefore \int_{-a}^a \int_0^{\sqrt{a^2 - y^2}} f(x, y) dx dy = \int_0^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} f(x, y) dy dx$$

35. By changing the order of integration, evaluate $\int_0^a \int_{y^2/a}^y \frac{y}{(a-x)\sqrt{ax-y^2}} dx dy$.

Sol. The given limits show that the area of integration lies between $x = \frac{y^2}{a}$, $x = y$, $y = 0$, $y = a$. We can consider it as lying between $y = x$, $y = \sqrt{ax}$, $x = 0$, $x = a$.

Here the given integral,

$$\begin{aligned} &= \int_0^a \int_x^{\sqrt{ax}} \frac{y dy dx}{(a-x)\sqrt{ax-y^2}} = \int_0^a \int_x^{\sqrt{ax}} \frac{1}{(a-x)} \cdot \frac{y dy}{\sqrt{ax-y^2}} dx \\ &= \int_0^a \left[-\frac{(ax-y^2)^{\frac{1}{2}}}{a-x} \right]_x^{\sqrt{ax}} dx = \int_0^a \frac{(ax-x^2)^{\frac{1}{2}}}{a-x} dx = \int_0^a \left(\frac{x}{a-x} \right)^{\frac{1}{2}} dx \end{aligned}$$

$\therefore$ Given integral

$$= \int_0^{\pi/2} \left(\frac{a \sin^2 \theta}{a \cos^2 \theta} \right)^{\frac{1}{2}} \cdot 2a \sin \theta \cos \theta \cdot d\theta$$

$$= \int_0^{\pi/2} 2a \sin^2 \theta d\theta = 2a \int_0^{\pi/2} \sin^2 \theta d\theta = 2a \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a}{2}$$

$$\left| \begin{array}{l} \text{Put } x = a \sin^2 \theta, \quad dx = 2a \sin \theta \cos \theta d\theta, \\ \sqrt{a-x} = \sqrt{a} \cos \theta \end{array} \right.$$

$$\text{When } x = a, \sin^2 \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{When } x = 0, \theta = 0$$

36. Change the order of integration and hence evaluate : $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dx dy}{\sqrt{y^4 - a^2 x^2}}.$

Sol. The given limits show that the area of integration lies between $y^2 = ax$, $y = a$, $x = 0$ and $x = a$. We can consider it as lying between $y = 0$, $y = a$, $x = 0$ and $x = y^2/a$ by changing the order of integration. Hence the given integral,

$$\begin{aligned} \int_{x=0}^a \int_{y=\sqrt{ax}}^a \frac{y^2 dx dy}{\sqrt{y^4 - a^2 x^2}} &= \int_{y=0}^a \int_{x=0}^{y^2/a} \frac{y^2 dy dx}{\sqrt{y^4 - a^2 x^2}} \\ &= \frac{1}{a} \int_0^a \int_0^{y^2/a} \frac{y^2 dy dx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}} = \frac{1}{a} \int_0^a y^2 \left[\sin^{-1} \left(\frac{ax}{y^2} \right) \right]_0^{y^2/a} dy \\ &= \frac{1}{a} \int_0^a y^2 [\sin^{-1}(1) - \sin^{-1}(0)] dy = \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a} \left(\frac{y^3}{3} \right)_0^a = \frac{\pi a^2}{6} \end{aligned}$$

37. By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx dy$, show that: $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}.$

Sol. Let
$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy} dy \right\} dx$$

$$= \int_0^\infty \sin px \cdot \left(\frac{e^{-xy}}{-x} \right)_0^\infty dx = \int_0^\infty \frac{\sin px}{x} dx \quad \dots(1)$$

Again
$$I = \int_0^\infty \left[\int_0^\infty e^{-xy} \sin px \, dx \right] dy = \int_0^\infty \left[-\frac{e^{-xy}}{p^2 + y^2} (p \cos px + y \sin px) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2 + y^2} dy = \left[\tan^{-1} \left(\frac{y}{p} \right) \right]_0^\infty = \frac{\pi}{2} \quad \dots(2)$$

Hence from (1) and (2), $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}.$

38. Show that : $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y) \, dx dy = \frac{\pi^2}{4} - \frac{\pi}{2}.$

Sol. We have
$$I = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \sin^{-1}(\sin x \sin y) \, dy$$

Put $\sin x \sin y = \sin z$
 so that $\sin x \cos y \, dy = \cos z \, dz$

$$\begin{aligned} \therefore I &= \int_0^{\pi/2} dx \int_0^x \frac{z \cos z \, dz}{\cos y} = \int_0^{\pi/2} dx \int_0^x \frac{z \cos z \, dz}{\sqrt{1 - \frac{\sin^2 z}{\sin^2 x}}} \\ &= \int_0^{\pi/2} dx \int_0^x \frac{z \sin x \cos z \cdot dz}{\sqrt{\sin^2 x - \sin^2 z}} = \int_0^{\pi/2} dx \int_0^x \frac{z \sin x \cos z \, dz}{\sqrt{\cos^2 z - \cos^2 x}} \end{aligned}$$

Changing the order of integration, we get

$$\begin{aligned} I &= \int_{z=0}^{\pi/2} \int_{x=z}^{\pi/2} \frac{z \cos z \sin x \, dz \, dx}{\sqrt{\cos^2 z - \cos^2 x}} = \int_{z=0}^{\pi/2} z \cos z \, dz \int_{x=z}^{\pi/2} \frac{\sin x \, dx}{\sqrt{\cos^2 z - \cos^2 x}} \\ &= - \int_0^{\pi/2} z \cos z \, dz \int_z^{\pi/2} \frac{d(\cos x)}{\sqrt{\cos^2 z - \cos^2 x}} = - \int_0^{\pi/2} z \cos z \, dz \left[\sin^{-1} \left(\frac{\cos x}{\cos z} \right) \right]_z^{\pi/2} \\ &= \int_0^{\pi/2} z \cos z \cdot \frac{\pi}{2} \cdot dz = \frac{\pi}{2} \int_0^{\pi/2} z \cos z \, dz = \frac{\pi}{2} [z \sin z + \cos z]_0^{\pi/2} \\ &= \frac{\pi}{2} \left(\frac{\pi}{2} - 1 \right) = \frac{\pi^2}{4} - \frac{\pi}{2}. \end{aligned}$$

39. Evaluate the following integral by changing the order of integration : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$.

(M.D.U., May 2006, 2008)

Sol. The given limits show that the area of integration lies between $y = x$, $y = \infty$, $x = 0$ and $x = \infty$.

We can consider it as lying between $x = 0$, $x = y$, $y = 0$ and $y = \infty$ by changing the order of integration. Hence the given integral,

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy = \int_0^\infty \frac{e^{-y}}{y} \cdot (x)_0^y \, dy = \int_0^\infty e^{-y} \, dy - (-e^{-y})_0^\infty = 1 - 0 = 1.$$

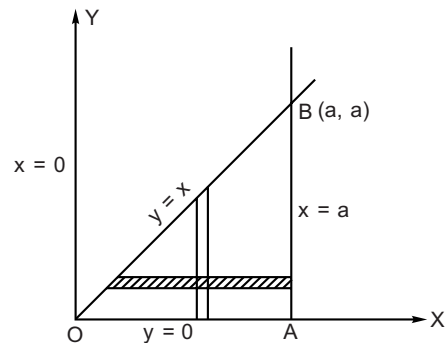
40. Evaluate $\int_0^a \int_0^x \frac{f'(y) \, dy \, dx}{[(a-x)(x-y)]^2}$.

Sol. Here the limits are $x = 0$, $x = a$; $y = 0$, $y = x$

The region of integration clearly, is $OABO$.

By changing the order of integration,

$$I = \int_0^a \int_y^a \frac{f'(y) \, dx \, dy}{\sqrt{(a-x)(x-y)}}$$



$$= \int_0^a f'(y) dy \cdot \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} \quad \dots(1)$$

Put $x = a \cos^2 \theta + y \sin^2 \theta$
 $\therefore a - x = a \sin^2 \theta - y \sin^2 \theta = (a - y) \sin^2 \theta$
 $\Rightarrow -dx = 2(a - y) \sin \theta \cos \theta d\theta$, keeping y constant
 Also $x - y = a \cos^2 \theta + y(\sin^2 \theta - 1) = (a - y) \cos^2 \theta$

When $x = y$, then $x - y = 0$ i.e., $(a - y) \cos^2 \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

When $x = a$, $a - x = 0 \Rightarrow (a - y) \sin^2 \theta = 0 \Rightarrow \theta = 0$

$$\therefore \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} = \int_{\pi/2}^0 \frac{-2(a-y) \sin \theta \cos \theta d\theta}{(a-y) \sin \theta \cos \theta} = 2 \int_0^{\pi/2} d\theta = 2 \left(\frac{\pi}{2} \right) = \pi$$

Now, $I = \int_0^a \pi f'(y) dy = \pi [f(y)]_0^a = \pi [f(a) - f(0)]$

41. Evaluate the integral by changing the order of integration $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x dy dx}{\sqrt{x^2 + y^2}}$.

Sol. Limits for y are $y = x$ and $y = \sqrt{2-x^2}$, $x^2 + y^2 = 2$.

Integration is from $P(y = x)$ to $Q(y = \sqrt{2-x^2})$ and x varies from 0 to 1.

After changing the order of integration, the region of integration is OAB .

Draw $AL \perp OB$ such that we have ΔOAL and region ALB .

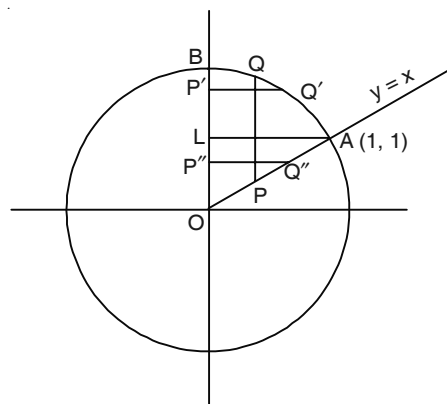
In ΔOAL , x varies from 0 to y , y varies from O to L
 (P'' to Q'')

$$\therefore \int_0^1 \int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx dy \quad \dots(1)$$

In region ALB , x varies from $P'(x = 0)$ to

$Q'(x = \sqrt{2-y^2})$ and y varies from $y = 1$ to $y = \sqrt{2}$.

$$\therefore \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx dy \quad \dots(2)$$

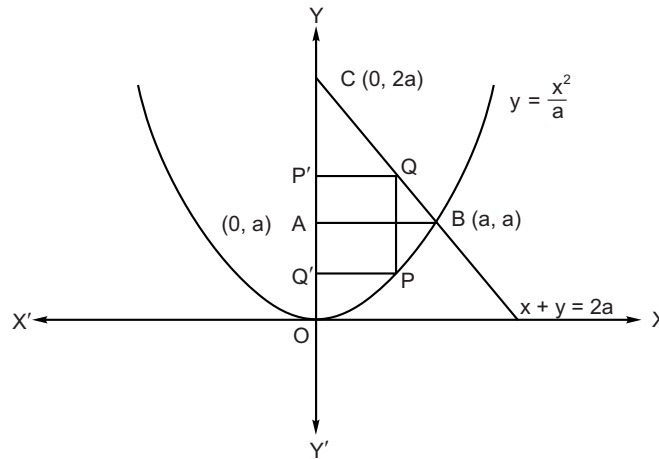


Hence $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x dy dx}{\sqrt{x^2 + y^2}} = \int_0^1 \int_0^y \frac{x dx dy}{\sqrt{x^2 + y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx dy$

$$\begin{aligned}
 &= \int_0^1 \left[\sqrt{x^2 + y^2} \right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2 + y^2} \right]_0^{\sqrt{2-y^2}} dy = \int_0^1 \sqrt{2} \cdot y dy + \int_1^{\sqrt{2}} [\sqrt{2} - y] dy - \int_0^1 y dy \\
 &= \sqrt{2} \cdot \left[\frac{y^2}{2} \right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2} \right]_1^{\sqrt{2}} - \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{\sqrt{2}} + 2 - 1 - \sqrt{2} + \frac{1}{2} - \frac{1}{2} = 1 + \frac{1}{\sqrt{2}} - \sqrt{2} = 1 - \frac{1}{\sqrt{2}}
 \end{aligned}$$

42. Evaluate : $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$ by changing the order of integration.

Sol. y varies from $y = \frac{x^2}{a}$ to $y = 2a - x$ and limits of x are 0 to a .



After changing the order of integration. In region OAB , at $Q'(x = 0)$, at $P(x = \sqrt{ay})$ and $Q'P$ moves for $y = 0$ to $y = a$

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy \quad \dots(1)$$

In region ABC , at $P'(x = 0)$, at $Q(x = 2a - y)$ and $P'Q$ moves for $y = a$ to $y = 2a$.

$$\therefore \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy \quad \dots(2)$$

$$\begin{aligned}
 \therefore \int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx &= \int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy + \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy \\
 &= \int_0^a \left[\frac{x^2}{2} \right]_0^{\sqrt{ay}} y \, dy + \int_a^{2a} \left[\frac{x^2}{2} \right]_0^{2a-y} y \, dy = \frac{1}{2} \int_0^a ay^2 \, dy + \frac{1}{2} \int_a^{2a} (2a - y^2) y \, dy
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a + \frac{1}{2} \int_a^{2a} (4a^2y - 4a^2y + y^3) dy = \frac{a^4}{6} + \frac{1}{2} \left[2a^2y^2 - \frac{4ay^3}{3} + \frac{y^4}{4} \right]_a^{2a} \\
 &= \frac{a^4}{6} + \frac{1}{2} \left[6a^4 - \frac{28a^4}{3} + \frac{15a^4}{4} \right] = \frac{3a^4}{8}.
 \end{aligned}$$

43. Evaluate $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x \, dx \, dy$, changing the order of integration. (M.D.U., May 2005)

Sol. Limits of x are $x = y$ to $x = \sqrt{a^2 - y^2}$

Limits of y are $y = 0$ to $y = a/\sqrt{2}$

$$x = \sqrt{a^2 - y^2} \Rightarrow y = \sqrt{a^2 - x^2}$$

Thus new limits of y are $y = x$ and $y = \sqrt{a^2 - x^2}$

New limits of x are $x = 0$ and $x = \frac{a}{\sqrt{2}}$

Since $y = \frac{a}{\sqrt{2}} \Rightarrow x = \frac{a}{\sqrt{2}}$

Thus the integral becomes

$$\begin{aligned}
 \int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} dy \cdot x \, dx &= \int_0^{a/\sqrt{2}} \left(\sqrt{a^2-x^2} - x \right) x \, dx \\
 &= \int_0^{a/\sqrt{2}} \left(x\sqrt{a^2-x^2} - x^2 \right) dx = \int_0^{a/\sqrt{2}} x\sqrt{a^2-x^2} \cdot dx - \int_0^{a/\sqrt{2}} x^2 \cdot dx
 \end{aligned}$$

Put $a^2 - x^2 = t^2$. When $x^2 = a^2/2$, $t^2 = a^2/2$ or $t = a/\sqrt{2}$

$\therefore xdx = -t dt$

First Integral becomes $-\int_0^{a/\sqrt{2}} t^2 dt = \left[-\frac{t^3}{3} \right]_0^{a/\sqrt{2}} = -\frac{1}{3} \left(\frac{a}{\sqrt{2}} \right)^3$

Hence the given integral is

$$\begin{aligned}
 -\frac{1}{3} \left(\frac{a}{\sqrt{2}} \right)^3 - \left[\frac{x^3}{3} \right]_0^{a/\sqrt{2}} &= -\frac{1}{3} \frac{a^3}{(\sqrt{2})^3} - \frac{1}{3} \left(\frac{a}{\sqrt{2}} \right)^3 = -\frac{2}{3} \frac{a^3}{(\sqrt{2})^3} \\
 &= -\frac{\sqrt{2}a^3}{6} \text{ or } \frac{\sqrt{2} \cdot a^3}{6} \text{ (in magnitude)}
 \end{aligned}$$

44. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) \, dx \, dy$ by changing order of integration.

[M.D.U., May 2007; U.P.T.U., 2008; A.U.U.P., 2009]

Sol. Given that x varies from 1 to $\sqrt{4-y}$ and y varies from 0 to 3.

After changing the order of integration y varies from 0 to $4-x^2$ since $x = \sqrt{4-y}$

$\Rightarrow x^2 = 4-y$ or $y = 4-x^2$. x varies from 1 to 2, since $y = 0 \Rightarrow x = 2$ and $y = 3 \Rightarrow x = 1$.

∴ Given integral becomes after changing order of integration,

$$\begin{aligned}\int_1^2 \int_0^{4-x^2} (x+y) dy dx &= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} dx = \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx \\ &= \int_1^2 \left(4x - x^3 - 4x^2 + \frac{x^4}{2} + 8 \right) dx = \left[2x^2 - \frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^5}{10} + 8x \right]_1^2 \\ &= 10 - \frac{28}{3} + \frac{31}{10} + \frac{1}{4} = \frac{600 - 560 + 186 + 15}{60} = \frac{241}{60}\end{aligned}$$

45. Evaluate $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dy dx$ by changing the order of integration.

(M.D.U., Dec., 2005; U.P.T.U., 2006)

Sol. Given limits of integration are : $x = 0$ to $x = \infty$ and $y = 0$ to $y = x$

Thus after change of integration, x will vary from y to ∞ and y will vary from 0 to ∞ .

Given integral = $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dy dx$ becomes

$$\begin{aligned}&= \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} dx dy = \int_0^\infty \left[\left(-\frac{y}{2} \right) \left(-\frac{2x}{y} \right) e^{-\frac{x^2}{y}} \cdot dx \right] dy \\ &= -\frac{1}{2} \int_0^\infty \left[y e^{-\frac{x^2}{y}} \right]_y^\infty dy = -\frac{1}{2} \int_0^\infty -y e^{-y} dy = \frac{1}{2} \int_0^\infty y e^{-y} dy \\ &= \frac{1}{2} \left[-y e^{-y} + \int 1 \cdot e^{-y} dy \right]_0^\infty = \frac{1}{2} \left[-y e^{-y} - e^{-y} \right]_0^\infty = \frac{1}{2} (1) = \frac{1}{2}\end{aligned}$$

46. Evaluate $\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} xy dx dy$ by changing the order of integration.

Sol. Given limits of integration are :

$$y : 0 \text{ to } b \quad x : 0 \text{ to } \frac{a}{b} \sqrt{b^2 - y^2}$$

$$\text{Now,} \quad x = \frac{a}{b} \sqrt{b^2 - y^2} \Rightarrow x^2 = \frac{a^2}{b^2} (b^2 - y^2) = a^2 - \frac{a^2 y^2}{b^2} \quad \text{or} \quad y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$\therefore y = \frac{b}{a} \sqrt{a^2 - x^2}$$

When $x = 0$, $y = b$. When $y = 0$, $x = a$.

∴ After changing the order of integration, we have y varying from b to $\frac{b}{a} \sqrt{a^2 - x^2}$ and x from a to 0.

∴ Given integral

$$\begin{aligned} \int_0^b \int_{(a/b)\sqrt{b^2-y^2}}^{(b/a)\sqrt{a^2-x^2}} xy \, dx \, dy &= \int_a^0 \int_b^{(b/a)\sqrt{a^2-x^2}} xy \, dx \, dy = \int_a^0 \int_b^{b/a\sqrt{b^2-x^2}} y \, dy \cdot x \, dx \\ &= \int_a^0 \left[\frac{y^2}{2} \right]_b^{b/a\sqrt{b^2-x^2}} x \, dx = \int_a^0 \frac{1}{2} \left[\frac{b^2}{a^2} (a^2 - x^2) - b^2 \right] x \, dx \\ &= \frac{1}{2} \int_a^0 \left(-\frac{b^2}{a^2} x^2 \right) x \, dx = -\frac{b^2}{2a^2} \int_a^0 x^3 \, dx = -\frac{b^2}{2a^2} \left[\frac{x^4}{4} \right]_a^0 = -\frac{b^2}{2a^2} \left(0 - \frac{a^4}{4} \right) = \frac{a^2 b^2}{8} \end{aligned}$$

47. Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$.

Sol. It is clear that here limits of x are 0 to 1 and y varies from 0 to $\sqrt{1-x^2}$.

Now, $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$ or $x = \sqrt{1-y^2}$

Thus, x varies from 0 to $\sqrt{1-y^2}$ after changing the order of integration.

Clearly, y varies between 0 to 1, since $y = 1$ when $x = 0$ and $y = 0$ when $x = 1$.

Given integral becomes $\int_0^1 \int_0^{\sqrt{1-y^2}} dx \cdot y^2 \, dy$

or $I = \int_0^1 [x]_0^{\sqrt{1-y^2}} \cdot y^2 \, dy$ or $I = \int_0^1 y^2 \sqrt{1-y^2} \cdot dy$

Put $\sin \theta = y$, $dy = \cos \theta \, d\theta$, $y = 1 \Rightarrow \theta = \pi/2$, $y = 0 \Rightarrow \theta = 0$

$$\begin{aligned} \therefore I &= \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{4} \sin^2 2\theta \, d\theta = \frac{1}{4} \int_0^{\pi/2} \left(\frac{1 - \cos 4\theta}{2} \right) d\theta \quad \left[\begin{array}{l} \because \cos 2\theta = 1 - 2\sin^2 \theta \\ \text{or } \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \end{array} \right] \\ &= \frac{1}{8} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{1}{8} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{16} \end{aligned}$$

48. Evaluate the integral $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) \, dy \, dx$ by changing the order of integration.

Sol. Given limits of integration are x varies from 0 to a and y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$.

$$y = \frac{x}{a} \Rightarrow x = ay \quad \text{and} \quad y = \sqrt{\frac{x}{a}} \Rightarrow x = ay^2$$

It is clear that we have to integrate first w.r.t. y and then w.r.t. x which varies from 0 to a .

When $x = 0$, $y = 0$

When $x = a$, $y = 1$

$\therefore$ After changing the order of integration, y will vary from 0 to 1 whereas obviously x will vary from ay^2 to ay .

Therefore, the given integral becomes

$$\begin{aligned} \int_0^1 \int_{ay^2}^{ay} (x^2 + y^2) dx \cdot dy &= \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_{ay^2}^{ay} \cdot dy = \int_0^1 \left[\frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4 \right] dy \\ &= \left[\frac{a^3}{3} \cdot \frac{y^4}{4} + \frac{ay^4}{4} - \frac{a^3}{3} \cdot \frac{y^7}{7} - \frac{ay^5}{5} \right]_0^1 = \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \\ &= a^3 \left(\frac{1}{12} - \frac{1}{21} \right) + a \left(\frac{1}{4} - \frac{1}{5} \right) = a^3 \left(\frac{7-4}{84} \right) + a \left(\frac{5-4}{20} \right) = \frac{a^3}{28} + \frac{a}{20} \end{aligned}$$

49. Evaluate the following integral by changing the order of integration :

$$\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2 - y^2}} \log(x^2 + y^2) dx dy \quad (a > 0).$$

Sol. Given limits of integration are $y : 0$ to $\frac{a}{\sqrt{2}}$ and $x : y$ to $\sqrt{a^2 - y^2}$. After changing the order of integration, it is seen that y will vary from x to $\sqrt{a^2 - x^2}$ since $x = y$ and $x = \sqrt{a^2 - y^2} \Rightarrow x^2 = a^2 - y^2$ or $y^2 = a^2 - x^2$ and hence $y = \sqrt{a^2 - x^2}$.

$$\text{When } y = \frac{a}{\sqrt{2}}, \quad x = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}$$

When $y = 0$, $x = 0$

$\therefore$ The given integral becomes

$$\int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2 - x^2}} \log(x^2 + y^2) dy dx = \int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2 - x^2}} 1 \cdot \log(x^2 + y^2) dy \cdot dx \quad \dots(1)$$

$$\text{Now, } \int 1 \cdot \log(x^2 + y^2) dy = y \log(x^2 + y^2) - \int y \cdot \frac{1}{x^2 + y^2} 2y dy$$

$$\begin{aligned} &= y \log(x^2 + y^2) - 2 \int \frac{y^2}{x^2 + y^2} \cdot dy = y \log(x^2 + y^2) - 2 \cdot \int \left[1 - \frac{x^2}{x^2 + y^2} \right] dy \\ &= y \log(x^2 + y^2) - 2y + 2x^2 \int \frac{dy}{x^2 + y^2} = y \log(x^2 + y^2) - 2y + 2x^2 \cdot \frac{1}{x} \cdot \tan^{-1} \frac{y}{x} \\ &= y \log(x^2 + y^2) - 2y + 2x \tan^{-1} \frac{y}{x} \quad \dots(2) \end{aligned}$$

∴ (1) becomes in view of (2);

$$\int_0^{\frac{a}{\sqrt{2}}} \left[y \log(x^2 + y^2) - 2y + 2x \tan^{-1} \frac{y}{x} \right]_{\frac{y}{x}}^{\sqrt{a^2 - x^2}} dx$$

or
$$\int_0^{\frac{a}{\sqrt{2}}} \left[\sqrt{a^2 - x^2} \log a^2 - 2\sqrt{a^2 - x^2} + 2x \tan^{-1} \left(\frac{\sqrt{a^2 - x^2}}{x} \right) - x \log 2x^2 + 2x - 2x \cdot \frac{\pi}{4} \right] dx$$

$$\therefore I = \int_0^{\frac{a}{\sqrt{2}}} \left[\sqrt{a^2 - x^2} (\log a^2 - 2) + 2x \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x} - x \log 2x^2 + 2x - \frac{\pi}{2} x \right] dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \sqrt{a^2 - x^2} (\log a^2 - 2) dx + \int_0^{\frac{a}{\sqrt{2}}} 2x \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x} dx - \int_0^{\frac{a}{\sqrt{2}}} x \log 2x^2 dx + \left[\frac{x^2}{2} - \frac{\pi}{2} \cdot \frac{x^2}{2} \right]_0^{\frac{a}{\sqrt{2}}}$$

$$= \left[(\log a^2 - 2) \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) + \frac{x^2}{2} - \frac{\pi x^2}{4} \right]_0^{\frac{a}{\sqrt{2}}}$$

$$+ \int_0^{\frac{a}{\sqrt{2}}} 2x \tan^{-1} \left(\frac{\sqrt{a^2 - x^2}}{x} \right) dx - \int_0^{\frac{a}{\sqrt{2}}} x \log 2x^2 dx \quad \dots(3)$$

Now,
$$\int x \log 2x^2 dx = \int x \{ \log 2 + 2 \log x \} dx$$

$$= \frac{x^2}{2} \log 2 + 2 \int x \log x dx = \frac{x^2}{2} \log 2 + 2 \left[\frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} \cdot dx \right]$$

$$= \frac{x^2}{2} \log 2 + x^2 \log x - \frac{x^2}{2} \quad \dots(A)$$

Now to evaluate
$$\int_0^{\frac{a}{\sqrt{2}}} 2x \tan^{-1} \left(\frac{\sqrt{a^2 - x^2}}{x} \right) dx$$

Put $x = a \sin \theta$, $dx = a \cos \theta d\theta$

When $x = 0$, $\theta = 0$

When $x = \frac{a}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$

The integral becomes

$$2a^2 \int_0^{\pi/4} \sin \theta \cos \theta \left(\frac{\pi}{2} - \theta \right) d\theta = 2a^2 \cdot \frac{\pi}{2} \int_0^{\pi/4} \sin \theta \cos \theta d\theta - 2a^2 \int_0^{\pi/4} \theta \sin \theta \cos \theta d\theta$$

$$\begin{aligned}
&= \frac{\pi a^2}{2} \int_0^{\pi/4} \sin 2\theta \, d\theta - a^2 \int_0^{\pi/4} \theta \sin 2\theta \, d\theta \\
&= \frac{\pi a^2}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/4} - a^2 \left[-\frac{\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/4} \\
&= \frac{\pi a^2}{2} \cdot \frac{1}{2} - a^2 \cdot \frac{1}{2} \left(0 + \frac{1}{2} \right) = \frac{\pi a^2}{4} - \frac{a^2}{4} \quad \dots(B)
\end{aligned}$$

(A) becomes $\frac{a^2}{4} \log 2 + \frac{a^2}{2} \log \frac{a}{\sqrt{2}} - \frac{a^2}{4}$ and (B) is $\frac{\pi a^2}{4} - \frac{a^2}{4}$

Using (A) and (B), we have from equation (3)

$$\begin{aligned}
&\left[(\log a^2 - 2) \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) + \frac{x^2}{2} - \frac{\pi x^2}{4} \right]_0^{\frac{a}{\sqrt{2}}} + \frac{\pi a^2}{4} - \frac{a^2}{4} - \frac{a^2}{4} \log 2 - \frac{a^2}{2} \log \frac{a}{\sqrt{2}} + \frac{a^2}{4} \\
&= (\log a^2 - 2) \left(\frac{a}{2\sqrt{2}} \cdot \frac{a}{\sqrt{2}} + \frac{a^2}{2} \cdot \frac{\pi}{4} \right) + \frac{a^2}{2} - \frac{\pi a^2}{8} + \frac{\pi a^2}{4} - \frac{a^2}{4} \log 2 - \frac{a^2}{2} \log \frac{a}{\sqrt{2}} \\
&= \left(\frac{a^2}{4} + \frac{\pi a^2}{8} \right) (\log a^2 - 2) + \frac{a^2}{2} - \frac{\pi a^2}{8} + \frac{\pi a^2}{4} - \frac{a^2}{4} \log 2 - \frac{a^2}{2} \log \frac{a}{\sqrt{2}} \\
&= \frac{a^2}{2} \log a - \frac{a^2}{2} + \frac{\pi a^2}{4} \log a - \frac{\pi a^2}{4} + \frac{a^2}{2} - \frac{\pi a^2}{8} + \frac{\pi a^2}{4} - \frac{a^2}{4} \log 2 - \frac{a^2}{4} \log \frac{a^2}{2} \\
&= \frac{\pi a^2}{4} \log a - \frac{\pi a^2}{8} \quad (\text{on simplification}) \\
&= \frac{\pi a^2}{8} \left(\log a - \frac{1}{2} \right)
\end{aligned}$$

50. Evaluate the following by changing to polar co-ordinates : $\int_0^a \int_y^a \frac{x \, dy \, dx}{x^2 + y^2}$.

Sol. Limits are $x = y$, $x = a$ and $y = 0$ to $y = a$.

Put $x = r \cos \theta$, $y = r \sin \theta$

$$x = y \Rightarrow \cos \theta = \sin \theta \quad \text{or} \quad \theta = \frac{\pi}{4}$$

$$x = a \Rightarrow r \cos \theta = a \quad \text{or} \quad r = a \sec \theta$$

$\therefore r$ varies from 0 to $a \sec \theta$

θ varies from 0 to $\frac{\pi}{4}$

$$\therefore \int_0^{\pi/4} \int_0^{a \sec \theta} \frac{r \cos \theta \, dr \, d\theta}{r^2} \big|_J = \int_0^{\pi/4} \int_0^{a \sec \theta} \cos \theta \cdot \frac{r^2}{r^2} \, dr \, d\theta$$

$$\begin{aligned}
 &= \int_0^{\pi/4} \cos \theta \cdot [r]_0^{a \sec \theta} d\theta = a \int_0^{\pi/4} \sec \theta \cdot \cos \theta \cdot d\theta \\
 &= a \left[\theta \right]_0^{\pi/4} = \frac{a\pi}{4}
 \end{aligned}$$

51. Evaluate the integral $\int_0^4 \int_y^4 \frac{x \, dx \, dy}{x^2 + y^2}$.

Sol. Limits are $x = y$, $x = 4$ and $y = 0$ to $y = 4$.

$$\int_0^4 \int_y^4 \frac{x \, dx \, dy}{x^2 + y^2}$$

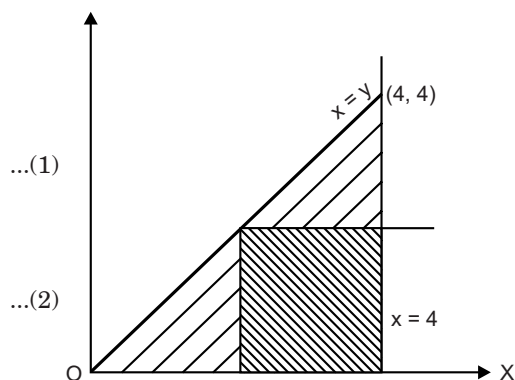
After changing to polar co-ordinates,

$$\Rightarrow \int_0^{\pi/4} \int_0^{4 \sec \theta} \cos \theta \cdot r \, dr \, d\theta$$

When we put $x = r \cos \theta$, $y = r \sin \theta$
 $x = y \Rightarrow \theta = \pi/4$, $r \cos \theta = 4$
 $(\because x = 4)$

or $r = 4 \sec \theta$.

$$\therefore (2) \Rightarrow \int_0^{\pi/4} \cos \theta \cdot 4 \sec \theta \, d\theta = 4 \int_0^{\pi/4} d\theta = 4 \left(\frac{\pi}{4} \right) = \pi$$



52. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x \, dy \, dx}{\sqrt{x^2 + y^2}}$ by changing to polar co-ordinates. (M.D.U., May 2005)

Sol. y varies from 0 to $\sqrt{2x-x^2}$

x varies from 0 to 2

$$\therefore y = \sqrt{2x-x^2} \quad \text{or} \quad x^2 + y^2 = 2x$$

In polar co-ordinates, $x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = r^2$

$$r^2 = 2r \cos \theta \quad \text{or} \quad r = 2 \cos \theta$$

Hence, r varies from 0 to $2 \cos \theta$ and θ varies from 0 to $\pi/2$

In the given integral replacing x by $r \cos \theta$, y by $r \sin \theta$, $dy \, dx$ by $r \, dr \, d\theta$, we have

$$\begin{aligned}
 \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x \, dy \, dx}{\sqrt{x^2 + y^2}} &= \int_0^{\pi/2} \int_0^{2 \cos \theta} \frac{r \cos \theta \cdot r \, dr \, d\theta}{r} \\
 &= \int_0^{\pi/2} \int_0^{2 \cos \theta} r \cos \theta \, dr \, d\theta = \int_0^{\pi/2} \cos \theta \cdot \left(\frac{r^2}{2} \right)_0^{2 \cos \theta} \cdot d\theta \\
 &= \int_0^{\pi/2} 2 \cos^3 \theta \cdot d\theta = 2 \int_0^{\pi/2} \cos^3 \theta \, d\theta = 2 \times \frac{2}{3} = \frac{4}{3}.
 \end{aligned}$$

53. Evaluate $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dy dx$ by changing to polar co-ordinates. (U.P.T.U., 2007, 2009)

Sol. Changing to polar co-ordinates by putting

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

The region of integration is the whole of the first quadrant. To cover this region, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$\begin{aligned} \therefore I \text{ (given integral)} &= \int_0^{\infty} \int_0^{\pi/2} e^{-r^2} r d\theta dr = \int_0^{\infty} r e^{-r^2} [\theta]_0^{\pi/2} dr \\ &= \frac{\pi}{2} \int_0^{\infty} r e^{-r^2} dr = -\frac{\pi}{2} \int_0^{\infty} \frac{(-2r) e^{-r^2} dr}{2} = -\frac{\pi}{4} \left[e^{-r^2} \right]_0^{\infty} = -\frac{\pi}{4} (-1) = \frac{\pi}{4}. \end{aligned}$$

54. Evaluate the integral by changing into polar co-ordinates : $\int_0^a \int_x^{\sqrt{a^2-y^2}} (x^2+y^2) dy dx$.

[M.D.U., May 2008]

Sol. x varies from $x = 0$ to $\sqrt{a^2 - y^2}$ and y varies from $y = 0$ to $y = a$.

Changing into polar co-ordinates, substitute

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow r^2 = x^2 + y^2 = a^2 \quad \text{or} \quad r = a$$

$$dx dy = r dr d\theta$$

θ varies from 0 to $\pi/2$ in the first quadrant and r varies from 0 to a .

$$\therefore I = \int_0^{\pi/2} d\theta \int_0^a r^2 \cdot r dr = \int_0^{\pi/2} d\theta \left[\frac{r^4}{4} \right]_0^a = \frac{a^4}{4} \cdot \frac{\pi}{2} = \frac{\pi a^4}{8}.$$

55. Evaluate the integral $I = \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr d\theta$.

Sol.

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{a \cos \theta} \left(-\frac{1}{2} \right) (-2r) (a^2 - r^2)^{\frac{1}{2}} dr d\theta \\ &= \int_0^{\pi/2} \left[-\frac{1}{2} \cdot \frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^{a \cos \theta} d\theta = -\frac{1}{3} \int_0^{\pi/2} (a^3 \sin^3 \theta - a^3) d\theta \\ &= -\frac{a^3}{3} \int_0^{\pi/2} (\sin^3 \theta - 1) d\theta = -\frac{a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right] = \frac{a^3}{3} \left(\frac{3\pi - 4}{6} \right) = \frac{a^3}{18} (3\pi - 4). \end{aligned}$$

56. Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}} dx dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

(M.D.U., Dec., 2007)

Sol. Changing to polar co-ordinates by putting $x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = 1$ transforms into $r = 1$.

For the region of integration R , r varies from 0 to 1 and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} \cdot r dr d\theta$$

$\therefore dx dy$ is replaced by $r dr d\theta$.

$$\Rightarrow I = \int_0^{\pi/2} \int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}} dr d\theta$$

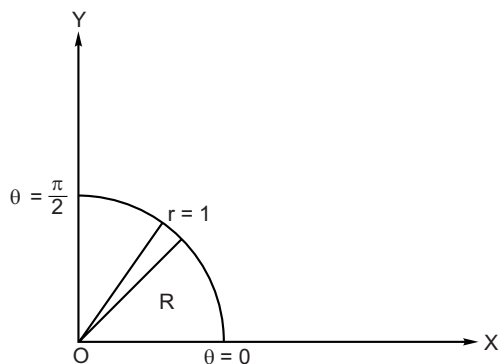
$$\text{Now, } \int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}} dr$$

$$= \int_0^1 \left(\frac{r}{\sqrt{1-r^4}} - \frac{r^3}{\sqrt{1-r^4}} \right) dr$$

$$= \frac{1}{2} \int_0^1 \frac{2r}{\sqrt{1-r^4}} dr + \frac{1}{4} \int_0^1 4r^3 (1-r^4)^{-1/2} \cdot dr = \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{1-t^2}} + \frac{1}{4} \left[\frac{(1-r^4)^{1/2}}{1/2} \right]_0^1, \text{ where } t = r^2.$$

$$= \frac{1}{2} \left[\sin^{-1} t \right]_0^1 + \frac{1}{2} (0-1) = \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore I = \int_0^{\pi/2} \left(\frac{\pi}{4} - \frac{1}{2} \right) d\theta = \left(\frac{\pi}{4} - \frac{1}{2} \right) [\theta]_0^{\pi/2} = \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi^2}{8} - \frac{\pi}{4}.$$



57. Evaluate : $I = \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz dx dy$.

$$\text{Sol. } I = \int_{z=0}^4 \int_{x=0}^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}} dx dz = \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2} dx dz$$

$$= \int_0^4 \left[\frac{1}{2} x \sqrt{4z-x^2} + \frac{1}{2} \cdot 4z \sin^{-1} \left(\frac{x}{\sqrt{4z}} \right) \right]_0^{2\sqrt{z}} dz = \int_0^4 2z \sin^{-1}(1) dz = \pi \int_0^4 z dz = \pi \left[\frac{z^2}{2} \right]_0^4 = 8\pi$$

58. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin \phi}{\sin \theta}} d\phi d\theta$ by the substitution $x = \sin \phi \cos \theta$, $y = \sin \phi \cdot \sin \theta$ and show that its value is π . (U.P.T.U., 2007)

Sol. Obviously, $x^2 + y^2 = \sin^2 \phi$ and $\frac{y}{x} = \frac{\sin \theta}{\cos \theta} = \tan \theta$

θ and ϕ both vary from 0 to $\pi/2$ (given). This indicates that the given integral is taken over the positive quadrant OAB of the circle $x^2 + y^2 = 1$.

Limits of y are from 0 to 1.

Limits of x in terms of y are from 0 to $\sqrt{1-y^2}$.

$$J \left(\frac{x, y}{\theta, \phi} \right) = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} -\sin \phi \sin \theta & \cos \phi \cos \theta \\ \sin \phi \cos \theta & \cos \phi \sin \theta \end{vmatrix} = -\sin \phi \cos \phi$$

$$\begin{aligned} \text{Now, } \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin \phi}{\sin \theta}} d\phi d\theta &= \int_0^1 \int_0^{\sqrt{1-y^2}} \left\{ \sqrt{\frac{\sin \phi}{\sin \theta}} \right\} \cdot \left\{ \frac{1}{|J|} \right\} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{\frac{\sin \phi}{\sin \theta}} \cdot \frac{dy dx}{\sin \phi \cdot \cos \phi} = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{\cos \phi} \cdot \frac{dy dx}{\sqrt{\sin \phi \cdot \sin \theta}} \\ \Rightarrow I &= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-\sin^2 \phi}} \cdot \frac{1}{\sqrt{y}} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{dy dx}{\sqrt{y} \cdot \sqrt{1-(x^2+y^2)}} = \int_0^1 \frac{dy}{\sqrt{y}} \int_0^{\sqrt{1-y^2}} \frac{dx}{\sqrt{(1-y^2)-x^2}} \\ &= \int_0^1 \frac{dy}{\sqrt{y}} \left[\sin^{-1} \left(\frac{x}{\sqrt{1-y^2}} \right) \right]_0^{\sqrt{1-y^2}} = \int_0^1 \frac{\pi}{2} \cdot \frac{dy}{\sqrt{y}} = \frac{\pi}{2} (2\sqrt{y})_0^1 = \pi \end{aligned}$$

59. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z dz dx dy$. (M.D.U., 2009)

Sol.
$$I = \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z dz \right] dx dy$$

Since,
$$\int_1^{e^x} \log z dz = \int_1^{e^x} \log z \cdot 1 \cdot dz$$

Integrating it by parts, we have

$$\left[\log z \cdot z \right]_1^{e^x} - \int_1^{e^x} \frac{1}{z} \cdot z \cdot dz = xe^x - e^x + 1 = (x-1)e^x + 1$$

$$\therefore I = \int_1^e \int_1^{\log y} [(x-1)e^x + 1] dx dy$$

$$\begin{aligned} \text{Now, } \int_1^{\log y} [(x-1)e^x + 1] dx &= \int_1^{\log y} (x-1)e^x dx + [x]_1^{\log y} = \left[(x-1)e^x \right]_1^{\log y} - \int_1^{\log y} 1 \cdot e^x dx + \log y - 1 \\ &= (\log y - 1)e^{\log y} - \left[e^x \right]_1^{\log y} + \log y - 1 \\ &= y(\log y - 1) - (e^{\log y} - e) + \log y - 1 \quad [\because e^{\log y} = y] \\ &= y(\log y - 1) - y + e + \log y - 1 \\ &= (y+1)\log y - 2y + e - 1 \end{aligned}$$

$$\begin{aligned} \therefore I &= \int_1^e [\log y \cdot (y+1) - 2y + e - 1] dy \\ &= \left[\log y \cdot \left(\frac{y^2}{2} + y \right) \right]_1^e - \int_1^e \frac{1}{y} \left(\frac{y^2}{2} + y \right) dy - \left[y^2 \right]_1^e + (e-1)[y]_1^e \\ &= \frac{e^2}{2} + e - \int_1^e \left(\frac{y}{2} + 1 \right) dy - (e^2 - 1) + (e-1)^2 \\ &= \frac{e^2}{2} + e - \left[\frac{y^2}{4} + y \right]_1^e - 2e + 2 = \frac{e^2}{2} + e - \left[\left(\frac{e^2}{4} + e \right) - \left(\frac{1}{4} + 1 \right) \right] - 2e + 2 \\ &= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4}(e^2 - 8e + 13) \end{aligned}$$

60. Evaluate : $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \cdot dz dy dx.$ (M.D.U., Dec. 2008)

$$\begin{aligned} \text{Sol. } I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \cdot dz dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right]_0^{\sqrt{1-x^2-y^2}} dy dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \int_0^{\sqrt{1-x^2}} (\sin^{-1}(1) - \sin^{-1}(0)) dy dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dy dx = \int_0^1 \frac{\pi}{2} [y]_0^{\sqrt{1-x^2}} dx \\
&= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx = \frac{\pi}{2} \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1 = \frac{\pi}{4} [\sin^{-1} 1] = \frac{\pi}{4} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}.
\end{aligned}$$

61. Evaluate : $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{1-x^2-y^2-z^2}}$, by changing to spherical polar co-ordinates.

[M.D.U., May 2007, Dec., 2008]

Sol. Here the region of integration is bounded by

$$z = 0, \quad z = \sqrt{1-x^2-y^2} \quad (\text{i.e., } x^2 + y^2 + z^2 = 1)$$

$$y = 0, \quad y = \sqrt{1-x^2} \quad (\text{i.e., } x^2 + y^2 = 1)$$

$$x = 0, \quad x = 1$$

which is the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Changing to spherical polar co-ordinates by putting $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ so that $x^2 + y^2 + z^2 = r^2$.

For the volume of sphere $x^2 + y^2 + z^2 = 1$ in the positive octant,

$$r \text{ varies from } 0 \text{ to } 1; \theta \text{ varies from } 0 \text{ to } \frac{\pi}{2}; \text{ and } \phi \text{ varies from } 0 \text{ to } \frac{\pi}{2}.$$

Replacing $dz dy dx$ by $r^2 \sin \theta dr d\theta d\phi$, we have

$$\begin{aligned}
I &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin \theta dr d\theta d\phi}{\sqrt{1-r^2}} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin \theta dr d\theta d\phi \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right) \sin \theta dr d\theta d\phi \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2} \sin^{-1} r \right) \right]_0^1 d\theta d\phi \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \left[\frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} \right] d\theta d\phi = \int_0^{\pi/2} \frac{\pi}{4} [-\cos \theta]_0^{\pi/2} d\phi \\
&= \int_0^{\pi/2} \frac{\pi}{4} d\phi = \frac{\pi}{4} [\phi]_0^{\pi/2} = \frac{\pi^2}{8}
\end{aligned}$$

62. Evaluate : $\iint_R \sqrt{a^2 - x^2 - y^2} \, dx dy$ over the semi-circle $x^2 + y^2 = ax$ in the positive quadrant.

Sol. Changing to polar co-ordinates, $x^2 + y^2 = ax$ transforms into $r = a \cos \theta$.

For the region of integration R , r varies from 0 to $a \cos \theta$ and θ varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned} \therefore \iint_R \sqrt{a^2 - x^2 - y^2} \, dx dy &= \int_0^{\pi/2} \int_0^{a \cos \theta} \sqrt{a^2 - r^2} \cdot r \, dr d\theta \\ &= \int_0^{\pi/2} \int_0^{a \cos \theta} -\frac{1}{2} (a^2 - r^2)^{\frac{1}{2}} (-2r) \, dr d\theta = \int_0^{\pi/2} -\frac{1}{2} \left[\frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^{a \cos \theta} d\theta \\ &= \int_0^{\pi/2} \left(-\frac{1}{3} \right) (a^3 \sin^3 \theta - a^3) d\theta = -\frac{a^3}{3} \int_0^{\pi/2} (\sin^3 \theta - 1) d\theta = -\frac{a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right] \\ &= \frac{a^3}{3} \left[\frac{\pi}{2} - \frac{2}{3} \right] = \frac{a^3}{18} (3\pi - 4) \end{aligned}$$

63. Let D be the region in the first quadrant bounded by $x = 0$, $y = 0$ and $x + y = 1$. Change the variables

x, y to u, v where $x + y = u$, $y = uv$ and evaluate $\iint_D xy (1 - x - y)^{\frac{1}{2}} \, dx dy$. (U.P.T.U., 2007)

Sol. $x = u(1 - v)$, $y = uv$

$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\therefore dx dy = |J| \, du dv = u du dv \quad \dots(1)$$

$$xy(1 - x - y)^{1/2} = u(1 - v) uv(1 - u)^{1/2} \quad \dots(2)$$

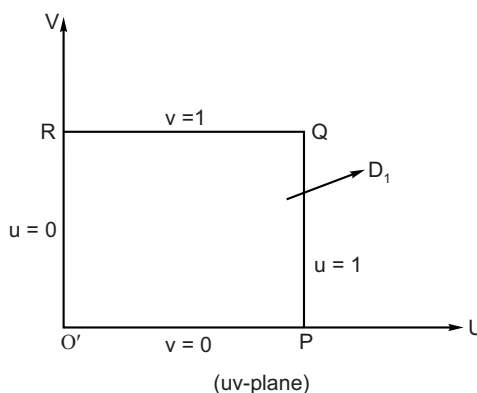
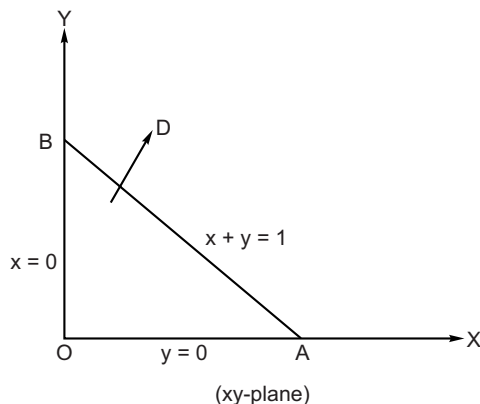
Also,

$$x = 0 \Rightarrow u(1 - v) = 0 \Rightarrow u = 0, v = 1$$

$$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0$$

$$x + y = 1 \Rightarrow u = 1$$

Hence the limits of u are from 0 to 1 and the limits of v are from 0 to 1.



$$\begin{aligned}
\therefore \iint_D xy(1-x-y)^{1/2} dx dy &= \iint_{D_1} u(1-v) uv(1-u)^{1/2} |J| du dv \\
&= \iint_{D_1} u^3(1-u)^{1/2} v(1-v) du dv \\
\therefore \text{Required integral} &= \int_0^1 u^3(1-u)^{1/2} du \int_0^1 v(1-v) dv \\
&= \int_0^1 u^{u-1} (1-u)^{3/2-1} du \int_0^1 v^{2-1} (1-v)^{2-1} dv = \beta\left(4, \frac{3}{2}\right) \cdot \beta(2, 2) \\
&= \frac{\frac{4}{2} \frac{3}{2}}{\frac{11}{2}} \cdot \frac{\frac{2}{2} \frac{2}{2}}{\frac{11}{2}} = \frac{\frac{3}{2} \frac{2}{2}}{\left(\frac{11}{2}\right)} = \frac{\frac{1}{2} \sqrt{\pi} \cdot 1 \cdot 1}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} = \frac{16}{945}.
\end{aligned}$$

64. Using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)} dy dx = \frac{1}{2}(e-1)$.

Sol. Since $x = u(1-v)$, $y = uv$, we have

$\therefore J = u$ (As shown in the previous Q. No. 63)

$$\therefore dx dy = |J| du dv = u du dv$$

Also $x = 0 \Rightarrow u(1-v) = 0 \Rightarrow u = 0, v = 1$

$$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0$$

$$x + y = 1 \Rightarrow u = 1$$

Hence the limits of u are 0 to 1 and the limits of v are 0 to 1.

$$\therefore \int_0^1 \int_0^{1-x} e^{y/(x+y)} dy dx = \int_0^1 \int_0^1 e^{\frac{uv}{u}} \cdot |J| \cdot du dv$$

$$\Rightarrow I = \int_0^1 \int_0^1 u e^v du dv = \left(\frac{u^2}{2}\right)_0^1 (e^v)_0^1 = \frac{1}{2}(e-1)$$

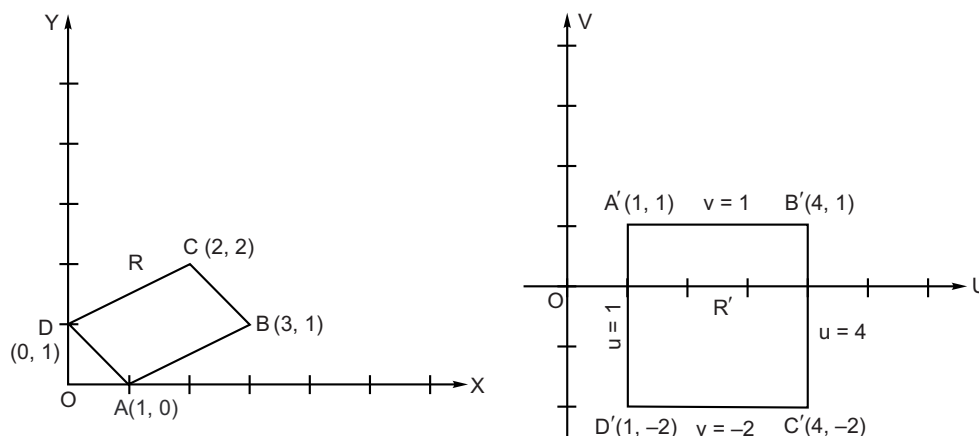
65. Evaluate $\iint_R (x+y)^2 dx dy$, where R is the parallelogram in the xy -plane with vertices $(1, 0)$, $(3, 1)$, $(2, 2)$, $(0, 1)$ using the transformation $u = x + y$ and $v = x - 2y$.

(U.P.T.U., 2008)

Sol. The vertices $A(1, 0)$, $B(3, 1)$, $C(2, 2)$, $D(0, 1)$ of the parallelogram $ABCD$ in the xy -plane become $A'(1, 1)$, $B'(4, 1)$, $C'(4, -2)$, $D'(1, -2)$ in the uv -plane under the given transformation. The Region R in the xy -plane becomes the region R' in the uv -plane which is a rectangle bounded by the line $u = 1$, $u = 4$, $v = -2$, $v = 1$.

Solving the given equations for x and y , we have

$$x = \frac{1}{3}(2u + v), \quad y = \frac{1}{3}(u - v)$$



$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$\begin{aligned} \therefore \iint_R (x+y)^2 dx dy &= \iint_{R'} u^2 |J| du dv = \int_{-2}^1 \int_1^4 u^2 \cdot \frac{1}{3} du dv \\ &= \int_{-2}^1 \frac{1}{3} \left[\frac{u^3}{3} \right]_1^4 dv = \int_{-2}^1 7 dv = 7 \times 3 = 21 \end{aligned}$$

66. Using the transformation $x + y = u$, $y = v$ evaluate : $\int_0^\pi \int_0^\pi |\cos(x+y)| dx dy$.

Sol. Let $I = \int_0^\pi \int_0^\pi |\cos(x+y)| dx dy$... (1)

Consider transformation

$$x + y = u, \quad y = v, \quad \text{then} \quad x = u - v, \quad y = v$$

Now, $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$$\therefore dx dy = |J| du dv = du dv \quad \dots (2)$$

Now, $x = 0 \Rightarrow u = v$
 $x = \pi \Rightarrow u = \pi + v$
 $y = 0 \Rightarrow v = 0$ and $y = \pi \Rightarrow v = \pi$

$$\therefore (1) \text{ becomes } I = \int_0^\pi dv \int_v^{\pi+v} |\cos u| du \quad \dots (3)$$

$$\begin{aligned}
 \int_v^{\pi+v} |\cos u| du &= \int_v^{\pi/2} |\cos u| du + \int_{\pi/2}^{\pi} |\cos u| du + \int_{\pi}^{\pi+v} |\cos u| du \\
 &= \int_v^{\pi/2} \cos u du + \int_{\pi/2}^{\pi} -\cos u du + \int_{\pi}^{\pi+v} -\cos u du \\
 &= (\sin u)_v^{\pi/2} - (\sin u)_{\pi/2}^{\pi} - (\sin u)_{\pi}^{\pi+v} \\
 &= 1 - \sin v - (0 - 1) - [\sin(\pi + v) - 0] = 1 - \sin v + 1 + \sin v = 2
 \end{aligned}$$

$\therefore$ From (3),
$$I = \int_0^{\pi} 2 \cdot dv = 2\pi$$

- 67.** Evaluate $\iiint z(x^2 + y^2 + z^2) dx dy dz$ through the volume of the cylinder $x^2 + y^2 = a^2$ intercepted by the planes $z = 0$ and $z = h$.

Sol. Changing to cylindrical co-ordinates by changing x to $r \cos \phi$, y to $r \sin \phi$ and replacing $dx \cdot dy \cdot dz$ by $r dr d\phi dz$, we have

$$\begin{aligned}
 I &= \int_0^h \int_0^{2\pi} \int_0^a z(r^2 + z^2) r dr d\phi dz = \int_0^h \int_0^{2\pi} \int_0^a (zr^3 + z^3 r) dr d\phi dz \\
 &= \int_0^h \int_0^{2\pi} \left[z \cdot \frac{r^4}{4} + z^3 \cdot \frac{r^2}{2} \right]_0^a d\phi dz = \int_0^h \int_0^{2\pi} \left(\frac{a^4}{4} \cdot z + \frac{a^2}{2} \cdot z^3 \right) d\phi dz \\
 &= \int_0^h \left(\frac{a^4}{4} \cdot z + \frac{a^2}{2} \cdot z^3 \right) [\phi]_0^{2\pi} dz \\
 &= \int_0^h 2\pi \left(\frac{a^4}{4} \cdot z + \frac{a^2}{2} \cdot z^3 \right) dz = 2\pi \left(\frac{a^4 z^2}{8} + \frac{a^2 \cdot z^4}{8} \right)_0^h
 \end{aligned}$$

$\therefore I = \frac{\pi}{4} (a^4 h^2 + a^2 h^4) = \frac{\pi}{4} a^2 h^2 (a^2 + h^2)$

- 68.** Using the transformation $x - y = u$, $x + y = v$, show that : $\iint_R \sin \left(\frac{x-y}{x+y} \right) dx dy = 0$. Where R is

bounded by the co-ordinate axes and $x + y = 1$ in the first quadrant.

Sol. R is triangle OAB in xy -plane as shown in figure 1.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$dudv = |J| dx dy$$

$$dudv = 2 dx dy$$

...(1)

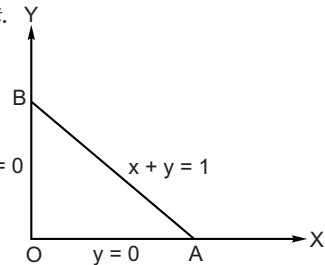


Fig. 1

Since $|J| = 2$

We have the transformation;

$$x - y = u, \quad x + y = v \quad \dots(2)$$

Now (2),

$$\Rightarrow \begin{aligned} 2x &= u + v \\ 2y &= v - u \end{aligned} \quad \dots(3)$$

If $x = 0, y = 0$, then (3) gives $u = -v, u = v$

If $x + y = 1$, then (2) gives $v = 1$

Corresponding domain in uv -plane is bounded by $u = -v, u = v, v = 1$.

The domain in uv -plane is $\triangle PO'Q$ (as shown in figure 2 below).

Take a strip parallel to u -axis. One end of this strip lies on $u = -v$ and the other end on $u = v$. For this strip, v varies from $v = 0$ to $v = 1$.

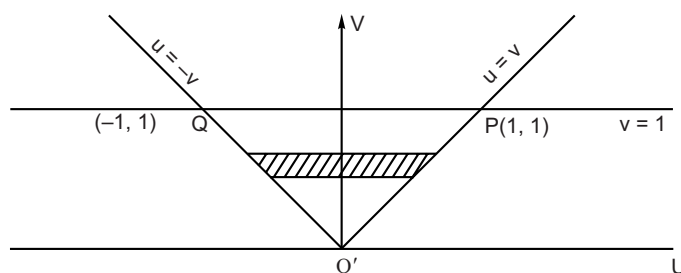


Fig. 2

$$\begin{aligned} \therefore I &= \int_0^1 \int_{-v}^v \sin\left(\frac{u}{v}\right) \frac{1}{2} du dv = \frac{1}{2} \int_0^1 \left\{ \frac{-\cos\left(\frac{u}{v}\right)}{\left(\frac{1}{v}\right)} \right\}_{-v}^v dv \\ &= \frac{1}{2} \int_0^1 v \{-\cos 1 + \cos(-1)\} dv = \frac{1}{2} \int_0^1 0 \cdot dv = 0 \end{aligned}$$

69. Find by triple integration, the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. (U.P.T.U., 2006)

Sol. By symmetry, the required volume is 4 times the volume in the positive octant. The volume in the positive octant is bounded on the sides by the zx and yz -planes; from above by the plane $z = 4$ and below by the curved surface $x^2 + y^2 = 4z$.

The section of the paraboloid by the plane $z = 4$ is the circle $x^2 + y^2 = 16, z = 4$ and its projection on the xy -plane is the circle $x^2 + y^2 = 16, z = 0$.

The volume in the positive octant is bounded by $z = \frac{x^2 + y^2}{4}, z = 4, y = 0, y = \sqrt{16 - x^2}$ and $x = 0, x = 4$.

$$\begin{aligned} \therefore \text{ Required volume} &= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \int_{(x^2+y^2)/4}^4 dz dy dx = 4 \int_0^4 \int_0^{\sqrt{16-x^2}} [z]_{(x^2+y^2)/4}^4 dy dx \\ &= 4 \int_0^4 \left[\left(4 - \frac{x^2}{4} \right) \sqrt{16-x^2} - \frac{1}{12} (16-x^2)^{3/2} \right] dx \end{aligned}$$

$$\begin{aligned}
&= 4 \int_0^4 \left[\frac{1}{4} (16 - x^2) \sqrt{16 - x^2} - \frac{1}{12} (16 - x^2)^{3/2} \right] dx \\
&= 4 \int_0^4 \frac{1}{6} (16 - x^2)^{3/2} dx = \frac{2}{3} \int_0^4 (16 - x^2)^{3/2} dx \\
&= \frac{2}{3} \int_0^{\pi/2} (16)^{3/2} \cdot \cos^3 \theta \cdot 4 \cos \theta d\theta \quad \text{where } x = 4 \sin \theta \\
&= \frac{512}{3} \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{512}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 32\pi.
\end{aligned}$$

70. Find the area in the positive quadrant bounded by the curves $y^2 = 4ax$, $y^2 = 4bx$, $xy = c^2$ and $xy = d^2$.

(U.P.T.U., 2006)

Sol. Take $\frac{y^2}{x} = u$, $xy = v$
so that $y^3 = uv$ or $y = u^{1/3} v^{1/3}$
 $\Rightarrow x = \frac{v}{y}$ or $x = \frac{v}{u^{1/3} v^{1/3}} = \frac{1}{u^{1/3}} \cdot \frac{1}{v^{2/3}}$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} u^{-4/3} \cdot v^{2/3} & \frac{2}{3} u^{-1/3} \cdot v^{-1/3} \\ \frac{1}{3} u^{-2/3} v^{1/3} & \frac{1}{3} u^{1/3} v^{-2/3} \end{vmatrix}$$

$$\Rightarrow J = -\frac{1}{9u} - \frac{2}{9u} = -\frac{3}{9u} \quad \text{or} \quad -\frac{1}{3u}$$

$$|J| = \frac{1}{3u}$$

$$\begin{aligned}
\therefore \text{ Required area} &= \iint_D dx dy = \iint_{D'} |J| du dv = \frac{1}{3} \int_{4a}^{4b} \int_{c^2}^{d^2} \frac{1}{u} du dv = \frac{1}{3} \int_{4a}^{4b} \frac{du}{u} (v)_{c^2}^{d^2} \\
&= \frac{1}{3} (d^2 - c^2) (\log u)_{4a}^{4b} = \frac{1}{3} (d^2 - c^2) \log \left(\frac{b}{a} \right).
\end{aligned}$$

71. Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$.

(U.P.T.U., 2005)

Sol. Required volume = $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx = \int_0^1 \int_0^{1-x} (1-x-y) dy dx$

$$\begin{aligned}
 &= \int_0^1 \left\{ (1-x)y - \frac{y^2}{2} \right\}_0^{1-x} dx = \int_0^1 \left\{ (1-x)^2 - \frac{(1-x)^2}{2} \right\} dx \\
 &= \frac{1}{2} \int_0^1 (1-x)^2 dx = -\frac{1}{6} \left[(1-x)^3 \right]_0^1 = \frac{1}{6}.
 \end{aligned}$$

72. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol. Given $z^2 = a^2 - x^2 \Rightarrow z = \pm \sqrt{a^2 - x^2}$

and $y^2 = a^2 - x^2 \Rightarrow y = \pm \sqrt{a^2 - x^2}$

Also $x^2 = a^2$ (when $y = 0$)

$\Rightarrow x = \pm a$

Hence the required volume common to the given cylinders is given by

$$\begin{aligned}
 V &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz dy dx \\
 \Rightarrow V &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2} \cdot dy dx = 4 \int_{-a}^a (a^2 - x^2) dx = 8 \int_0^a (a^2 - x^2) dx \\
 &= 8 \left(a^2x - \frac{x^3}{3} \right)_0^a = 8 \left[a^3 - \frac{a^3}{3} \right] = 8 \left(\frac{2a^3}{3} \right) \quad \text{or} \quad \frac{16a^3}{3}
 \end{aligned}$$

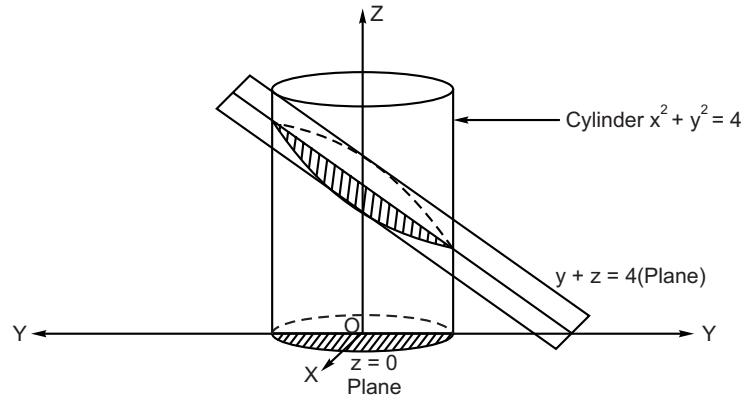
73. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Sol. From the figure, it is clear that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy -plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{4 - y^2}$ while y varies from -2 to 2
 $(\because y^2 = 4 \Rightarrow y = \pm 2)$.

$$\begin{aligned}
 \therefore \text{ Required volume} &= 2 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} z dx dy = 2 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} (4-y) dx dy \quad [\because z = 4-y] \\
 &= 2 \int_{-2}^2 (4-y) \left[x \right]_0^{\sqrt{4-y^2}} dy = 2 \int_{-2}^2 (4-y) \cdot \sqrt{4-y^2} \cdot dy \\
 &= 2 \int_{-2}^2 4\sqrt{4-y^2} \cdot dy - 2 \int_{-2}^2 y\sqrt{4-y^2} \cdot dy = 8 \int_{-2}^2 \sqrt{4-y^2} \cdot dy
 \end{aligned}$$

because the second integral is zero as $y\sqrt{4-y^2}$ is an odd function of y .

$$\Rightarrow \text{ Required volume} = 16 \int_0^2 \sqrt{4-y^2} \cdot dy \quad [\because \sqrt{4-y^2} \text{ is an even function of } y]$$



$$= 16 \left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \frac{y}{2} \right]_0^2 = 16 [2 \sin^{-1} 1] = 16\pi$$

- 74.** Find the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the plane $x = 0$, $y = 0$, $y = 6$, $z = 0$ ($x > 0$). [M.D.U., Dec., 2007]

Sol. The centre of gravity (or centroid) of the region R is $\bar{x}, \bar{y}, \bar{z}$.

Here the region $R = \{(x, y, z) : 0 \leq y \leq 6, 0 \leq z \leq 4 - x^2, 0 \leq x \leq 2\}$

Since when $z = 0$, $4 - x^2 = 0 \Rightarrow x^2 = 4$ or $x = \pm 2 \Rightarrow x = 2, x > 0$.

Then,
$$\bar{x} = \frac{1}{M} \iiint_R \rho x \, dz \, dy \, dx \quad \dots(1)$$

$$\bar{y} = \frac{1}{M} \iiint_R \rho y \, dz \, dy \, dx \quad \dots(2)$$

$$\bar{z} = \frac{1}{M} \iiint_R \rho z \, dz \, dy \, dx \quad \dots(3)$$

Here, M is mass = $\rho(\text{density}) \times \text{volume}$

Now,
$$M = \iiint_R \rho \, dz \, dy \, dx$$

$$= k \int_0^2 \int_0^6 \int_0^{4-x^2} dz \, dy \, dx \quad [\text{As } \rho \text{ is not given } \therefore \rho = k]$$

$$= k \int_0^2 \int_0^6 (4 - x^2) dy \, dx = k \int_0^2 \left[4x - \frac{x^3}{3} \right]_0^6 dx$$

$$= k \int_0^2 \left(8 - \frac{8}{3} \right) dx = k \left[\frac{16x}{3} \right]_0^2 = 32k$$

$$\begin{aligned}\text{Therefore, from (1), } \bar{x} &= \frac{1}{M} \int_0^2 \int_0^6 \int_0^{4-x^2} \rho x \, dz \, dy \, dx = \frac{k}{M} \int_0^2 \int_0^6 x(4-x)^2 dy \, dx \\ \bar{x} &= \frac{k}{M} \int_0^2 \int_0^6 \{x(4-x^2)\} dy = \frac{k}{M} \int_0^2 6x(4-x^2) dx = \frac{k}{M} \left[24 \cdot \frac{x^2}{2} - 6 \cdot \frac{x^4}{4} \right]_0^2 \\ &= \frac{6k}{M} \left[2x^2 - \frac{x^4}{4} \right]_0^2 = \frac{6k}{M} (8-4) = \frac{24k}{M} = \frac{24k}{32k} \text{ or } \frac{3}{4}\end{aligned}$$

$$\text{Similarly from (2), } \bar{y} = \frac{1}{M} \int_0^2 \int_0^6 \int_0^{4-x^2} \rho y \, dz \, dy \, dx$$

$$\begin{aligned}\text{or } \bar{y} &= \frac{k}{M} \int_0^2 \int_0^6 y(4-x^2) dy \, dx = \frac{k}{M} \int_0^2 (4-x^2) \left[\frac{y^2}{2} \right]_0^6 dx \\ \bar{y} &= \frac{k}{M} \int_0^2 18(4-x^2) dx = \frac{18k}{M} \left[4x - \frac{x^3}{3} \right]_0^2 = \frac{18k}{M} \left(8 - \frac{8}{3} \right) = \frac{18k}{32k} \cdot \frac{16}{3} = 3.\end{aligned}$$

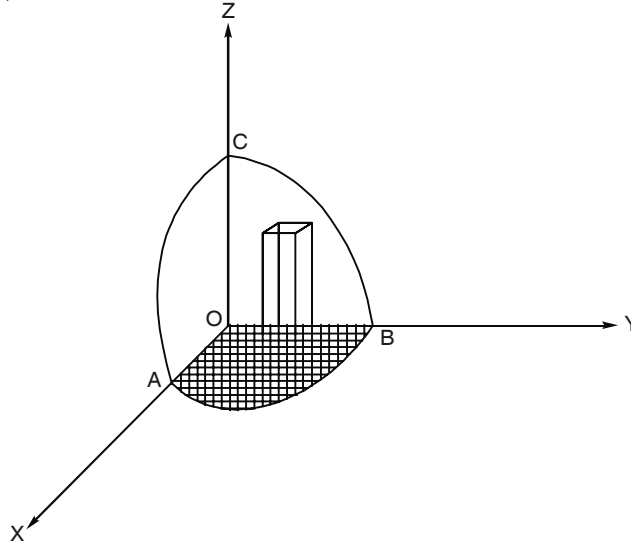
$$\begin{aligned}\text{Finally, from (3), } \bar{z} &= \frac{1}{M} \int_0^2 \int_0^6 \int_0^{4-x^2} \rho z \, dz \, dy \, dx = \frac{k}{M} \int_0^2 \int_0^6 \frac{1}{2} (4-x^2)^2 dy \, dx \\ &= \frac{k}{2M} \int_0^2 \int_0^6 (16+x^4-8x^2) dy \, dx = \frac{k}{2M} \int_0^2 6(16+x^4-8x^2) dx \\ &= \frac{3k}{M} \left[16x + \frac{x^5}{5} - \frac{8x^3}{3} \right]_0^2 = \frac{3k}{M} \left[32 + \frac{32}{5} - \frac{64}{3} \right] = \frac{96k}{M} \cdot \frac{8}{15} \\ &= \frac{8}{5} \quad [\because M = 32k]\end{aligned}$$

Hence the required centroid of the Region R is $(\bar{x}, \bar{y}, \bar{z})$ or $\left(\frac{3}{4}, 3, \frac{8}{5}\right)$.

- 75.** Find by double integration, the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol. On account of symmetry, the required volume is 8 times the volume of the ellipsoid in the positive octant.

The volume $OABC$ in the positive octant lies between the ellipsoid $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ and the plane XOY , and is bounded on the sides by the planes $x = 0, y = 0$ (Figure).



The ellipsoid cuts the plane XOY in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$

$\therefore$ The region OAB above which the volume OABC lies, is bounded by

$$x = 0, \quad x = a \quad \text{and} \quad y = 0, \quad y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Hence the required volume of the ellipsoid

$$\begin{aligned} &= 8 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} z \, dy \, dx = 8 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx \\ &= 8 \int_0^a \int_0^t c \sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}} \, dy \, dx, \quad \text{where } b\sqrt{1 - \frac{x^2}{a^2}} = t \\ &= 8 \int_0^a \int_0^t \frac{c}{b} \sqrt{t^2 - y^2} \, dy \, dx = 8 \int_0^a \frac{c}{b} \left[\frac{y\sqrt{t^2 - y^2}}{2} + \frac{t^2}{2} \sin^{-1}\left(\frac{y}{t}\right) \right]_0^t dx \\ &= \frac{4c}{b} \int_0^a t^2 \sin^{-1}(1) \, dx = \frac{2\pi c}{b} \int_0^a t^2 \, dx = \frac{2\pi c}{b} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx \end{aligned}$$

$$\Rightarrow \text{Required volume} = 2\pi bc \left(x - \frac{x^3}{3a^2} \right)_0^a = 2\pi bc \left(a - \frac{a^3}{3a^2} \right) = 2\pi bc \left(a - \frac{a}{3} \right) = \frac{4\pi abc}{3}.$$

- 76.** A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the planes $z = 0$ and surface $z = c + xy$. (U.P.T.U., 2009)

Sol. Required volume = $\int_0^a \int_{\frac{bx}{a}}^{\frac{bx}{a} + c + xy} dz \, dy \, dx = \int_0^a \int_0^{\frac{bx}{a}} (c + xy) \, dy \, dx$

$$\begin{aligned}
 &= \int_0^a \left(cy + \frac{xy^2}{2} \right)_0^{\frac{bx}{a}} dx = \int_0^a \left(\frac{cbx}{a} + \frac{b^2}{2a^2} x^3 \right) dx \\
 &= \frac{bc}{a} \left(\frac{x^2}{2} \right)_0^a + \frac{b^2}{2a^2} \left(\frac{x^4}{4} \right)_0^a = \frac{abc}{2} + \frac{a^2 b^2}{8}
 \end{aligned}$$

$$\Rightarrow \text{Required volume} = \frac{ab}{8} (4c + ab).$$

- 77.** Find the volume of the cylindrical column standing on the area common to the parabola $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 12 + y - x^2$.

Sol. Required volume =
$$\begin{aligned}
 &\int_0^1 dx \int_{x^2}^{\sqrt{x}} \int_0^{12+y-x^2} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12 + y - x^2) dy \\
 &= \int_0^1 \left(12y + \frac{y^2}{2} - yx^2 \right)_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[12\sqrt{x} + \frac{x}{2} - x^{5/2} - 12x^2 - \frac{x^4}{2} + x^4 \right] dx \\
 &= \left[8x^{3/2} + \frac{x^2}{4} - \frac{2}{7} x^{7/2} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5} \right]_0^1 \\
 &= 8 + \frac{1}{4} - \frac{2}{7} - 4 - \frac{1}{10} + \frac{1}{5} = \frac{569}{140}
 \end{aligned}$$

- 78.** Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol. Let the two curves intersect at points whose abscissae are given by

$$4x - x^2 = x \Rightarrow x = 0, 3$$

Using vertical strips, the required area lies between $x = 0$, $x = 3$ and $y = x$, $y = 4x - x^2$.

$$\begin{aligned}
 \therefore \text{Required area} &= \int_0^3 \int_x^{4x-x^2} dy dx = \int_0^3 [y]_x^{4x-x^2} dx \\
 &= \int_0^3 (4x - x^2 - x) dx = \int_0^3 (3x - x^2) dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = 4.5
 \end{aligned}$$

- 79.** Determine the area of region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

Sol. Required area =
$$\int_1^4 \int_{\frac{2}{y}}^{\frac{2\sqrt{y}}{y}} dx dy \quad \dots(1)$$

Since $x^2 = 4y \Rightarrow x = 2\sqrt{y}$ and $xy = 2 \Rightarrow x = \frac{2}{y} \Rightarrow x$ varies from $\frac{2}{y}$ to $2\sqrt{y}$

Also the curves $xy = 2$ and $x^2 = 4y$ intersect at the point where on solving, we get

$$x^2 = 4y$$

$$x = \frac{2}{y} \quad \text{or} \quad x^2 = \frac{4}{y^2}$$

$$\therefore 4y = \frac{4}{y^2} \Rightarrow y^3 = 1 \quad \text{or} \quad y = 1$$

Hence y varies from $y = 1$ to $y = 4$.

∴ Required area from (1);

$$\begin{aligned}\int_1^4 \left(2\sqrt{y} - \frac{2}{y}\right) dy &= 2 \left(\frac{2}{3} y^{3/2} - \log y \right)_1^4 = 2 \left[\left(\frac{16}{3} - 2 \log 2 \right) - \frac{2}{3} \right] = \frac{32}{3} - \frac{4}{3} - 4 \log 2 \\ &= \frac{28}{3} - 4 \log 2 \quad \text{or} \quad 4 \left(\frac{7}{3} - \log 2 \right)\end{aligned}$$

80. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

[U.P.T.U., (Agr. Engg.) 2005]

Sol. Latus rectum is $x = a$.

$$\begin{aligned}\therefore \text{ Required area} &= 2 \int_0^a \int_0^{2\sqrt{ax}} dy dx = 2 \int_0^a 2\sqrt{ax} dx \\ &= 4\sqrt{a} \left[\frac{2}{3} x^{3/2} \right]_0^a = \frac{8}{3} \sqrt{a} \cdot a^{3/2} = \frac{8a^2}{3}\end{aligned}$$

81. Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.

Sol. Let us find the points where the line intersects the ellipse at the co-ordinate axes.

$$4x^2 + 9y^2 = 36$$

$$2x + 3y = 6$$

$$\text{or} \quad \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \dots(1)$$

$$\text{and} \quad \frac{x}{3} + \frac{y}{2} = 1 \quad \dots(2)$$

Both meet x -axis at $(3, 0)$ and y -axis at $(0, 2)$

$$\text{The required area} = \int_0^2 \int_{x_1}^{x_2} dx dy$$

$$\text{where} \quad x_1 = \frac{6-3y}{2} \quad \text{or} \quad \frac{3}{2}(2-y), \quad [\text{From eq. (2)}]$$

$$x_2 = \frac{\sqrt{36-9y^2}}{2} \quad \text{or} \quad \frac{3}{2}\sqrt{4-y^2}, \quad [\text{From eq. (1)}]$$

$$\text{Hence the required area} = \int_0^2 \int_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dx dy = \int_0^2 \frac{3}{2} \left[\sqrt{4-y^2} - (2-y) \right] dy$$

$$= \frac{3}{2} \left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \left(\frac{y}{2} \right) - 2y + \frac{y^2}{2} \right]_0^2$$

$$= \frac{3}{2} [2 \sin^{-1}(1) - 4 + 2] = \frac{3}{2} \left[2 \cdot \frac{\pi}{2} - 2 \right] = \frac{3}{2} [\pi - 2]$$

82. By double integration, find the whole area of the curve $a^2 x^2 = y^3(2a - y)$.

Sol. Required area = $2 \int_0^{2a} \int_0^x dy dx$ where $x = \sqrt{\frac{y^3(2a - y)}{a^2}}$ or $x = \frac{y^{3/2} \sqrt{2a - y}}{a}$

Hence the required area = $\frac{2}{a} \int_0^{2a} y^{3/2} \sqrt{2a - y} dy$

$$= \frac{2}{a} \int_0^{2a} (2a \sin^2 \theta)^{3/2} \sqrt{2a - 2a \sin^2 \theta} \cdot 4a \sin \theta \cos \theta d\theta$$

[where we put $y = 2a \sin^2 \theta$ and $dy = 4a \sin \theta d\theta$]

$$= 32a^2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta = 32a^2 \left(\frac{\pi}{32} \right) = \pi a^2$$

Note : $I_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{(p-1)(p-3) \dots (q-1)(q-3) \dots 1}{(p+q)(p+q-2)(p+q-4) \dots} \cdot \frac{\pi}{2}$

When p, q are both even.

Using this formula, we obtain the value :

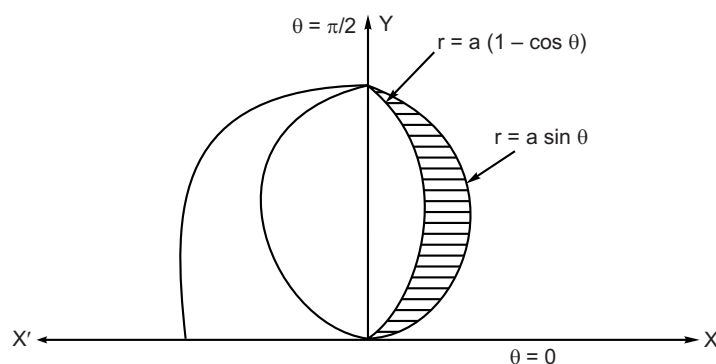
$$I_{4,2} = \int_0^{\pi/2} \sin^4 x \cos^2 x dx \quad (p=4, q=2)$$

$$= \frac{(4-1)(4-3) \dots (2-1)}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{32}$$

83. Find by double integration, the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$. (M.D.U., Dec., 2008)

Sol. For the required area, r varies from $a(1 - \cos \theta)$ to $a \sin \theta$. Eliminating r between the equations of two curves

$$\sin \theta = 1 - \cos \theta \quad \text{or} \quad \sin \theta + \cos \theta = 1$$



Squaring both sides, we get $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ which implies $\theta = 0$ or $\frac{\pi}{2}$.

$\therefore$ Required area = $\int_0^{\pi/2} \int_{a(1-\cos \theta)}^{a \sin \theta} r dr d\theta$

$$\begin{aligned}
 &= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{a^2}{2} \int_0^{\pi/2} (\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta) d\theta \\
 &= \frac{a^2}{2} \int_0^{\pi/2} (-2\cos^2\theta + 2\cos\theta) d\theta = a^2 \left(1 - \frac{\pi}{4} \right).
 \end{aligned}$$

- 84.** Find by double integration the area included between the curves $r = a(\sec\theta + \cos\theta)$ and its asymptotes.

Sol. The given curve is $r = a(\sec\theta + \cos\theta)$

Its asymptote is $r = a\sec\theta$

$$\begin{aligned}
 \therefore \text{ Required area} &= 2 \int_0^{\pi/2} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r dr d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta \\
 &= a^2 \int_0^{\pi/2} [(\sec\theta + \cos\theta)^2 - \sec^2\theta] d\theta \\
 &= a^2 \int_0^{\pi/2} (\cos^2\theta + 2) d\theta = a^2 \int_0^{\pi/2} \cos^2\theta d\theta + 2a^2 \cdot \frac{\pi}{2} \\
 &= a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \pi a^2 = a^2 \left(\frac{\pi}{4} + \pi \right) = \frac{5\pi a^2}{4}.
 \end{aligned}$$

- 85.** Find by double integration the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

$$\begin{aligned}
 \text{Sol. Required area} &= 2 \int_0^{\pi/2} \int_{r=a}^{r=a(1+\cos\theta)} r dr d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_a^{a(1+\cos\theta)} d\theta \\
 &= a^2 \int_0^{\pi/2} [(1 + \cos\theta)^2 - 1] d\theta = a^2 \int_0^{\pi/2} (\cos^2\theta + 2\cos\theta) d\theta \\
 &= a^2 \left(\frac{1}{2} \cdot \frac{\pi}{2} + 2 \right) = \frac{a^2}{4} (\pi + 8)
 \end{aligned}$$

- 86.** Find the area lying inside a cardioid $r = (1 + \cos\theta)$ and outside the parabola $r(1 + \cos\theta) = 1$.

$$\begin{aligned}
 \text{Sol. Required area} &= 2 \int_0^{\pi/2} \int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r dr d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta \\
 &= \int_0^{\pi/2} \left[(1 + \cos\theta)^2 - \frac{1}{(1 + \cos\theta)^2} \right] d\theta \quad \dots(1)
 \end{aligned}$$

$$\text{Now, } \int_0^{\pi/2} (1 + \cos\theta)^2 d\theta = \int_0^{\pi/2} (1 + \cos^2\theta + 2\cos\theta) d\theta = \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} + 2 = \frac{3\pi}{4} + 2$$

$$\text{and } \int_0^{\pi/2} \frac{1}{(1 + \cos\theta)^2} d\theta = \frac{1}{4} \int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta = \frac{1}{4} \int_0^{\pi/4} \sec^4 \phi (2d\phi) \quad \left| \begin{array}{l} \frac{\theta}{2} = \phi \\ d\theta = 2d\phi \end{array} \right.$$

$$= \frac{1}{2} \int_0^{\pi/4} (1 + \tan^2 \phi) \sec^2 \phi d\phi = \frac{1}{2} \int_0^1 (1 + t^2) dt \quad \text{where } t = \tan \phi$$

$$= \frac{1}{2} \left(t + \frac{t^3}{3} \right)_0^1 = \frac{2}{3}$$

$$\therefore \text{From (1) Required area} = \frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3}.$$

87. Find the area included between the curve $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ and its base.

Sol. When $\theta = 0$, $x = 0$ and $y = 0$

When $\theta = 2\pi$, $x = 2a\pi$ and $y = 0$

$\therefore$ The limits for θ are from 0 to 2π

$\therefore$ Required area = area of $OPAO$

$$= \int_0^{2\pi} y \frac{dx}{d\theta} \cdot d\theta = \int_0^{2\pi} a(1 - \cos \theta) a(1 - \cos \theta) d\theta = a^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$$

$$= 4a^2 \int_0^{2\pi} \sin^4 \left(\frac{\theta}{2} \right) d\theta = 8a^2 \int_0^{\pi} \sin^4 \phi d\phi \quad \left(\text{Put } \frac{\theta}{2} = \phi \right)$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 \phi d\phi = 16a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 3\pi a^2$$

88. Find the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$.

(M.D.U., Dec., 2006)

Sol. The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder. On account of the symmetry of the sphere, half of it lies above the plane XOY and half below it.

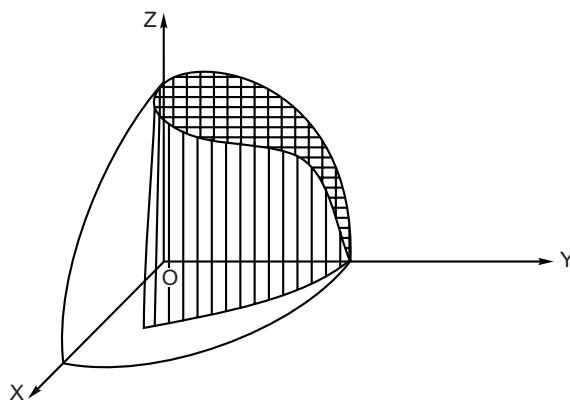
$$\therefore \text{Required volume} = 2 \iiint z dy dx$$

where $z = \sqrt{a^2 - x^2 - y^2}$ and the region of integration is the area inside the circle $x^2 + y^2 = ay$ in the xy -plane. ... (1)

On account of symmetry, the volumes above the two parts of circle (1) in the first and the second quadrants are equal. The figure shows only the part in the first quadrant.

$$\therefore \text{Required volume} = 2 \times 2 \int_R \int \sqrt{a^2 - x^2 - y^2} dy dx$$

where R is the half of the circle (1) lying in the first quadrant.



Changing to polar co-ordinates by putting $x = r \cos \theta$, $y = r \sin \theta$ so that $x^2 + y^2 = r^2$, equation (1) becomes

$$r^2 = ar \sin \theta \quad \text{or} \quad r = a \sin \theta$$

The region of integration is bounded by

$$r^2 = 0, \quad r = a \sin \theta \quad \text{and} \quad \theta = 0, \quad \theta = \frac{\pi}{2}$$

$$\begin{aligned} \therefore \text{ Required volume} &= 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r \, dr \cdot d\theta \\ &= 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \left(-\frac{1}{2} \right) (a^2 - r^2)^{\frac{1}{2}} (-2r) \, dr \cdot d\theta \\ &= 4 \int_0^{\pi/2} \left[\left(-\frac{1}{2} \right) \frac{(a^2 - r^2)^{\frac{3}{2}}}{3/2} \right]_0^{a \sin \theta} \cdot d\theta = -\frac{4}{3} \int_0^{\pi/2} (a^3 \cos^3 \theta - a^3) \, d\theta \\ &= -\frac{4a^3}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta + \frac{4a^3}{3} \cdot \frac{\pi}{2} = -\frac{4a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right] = \frac{2}{9} a^3 (3\pi - 4) \end{aligned}$$

- 89.** Find, by triple integration, the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = R^2$.

Sol. Changing to cylindrical co-ordinates, by putting $x = r \cos \theta$, $y = r \sin \theta$, the equation of the paraboloid becomes $az = r^2$ or $z = \frac{r^2}{a}$ and the equation of the cylinder becomes $r^2 = R^2$ or $r = R$. On account of symmetry, the required volume is 4 times the volume in the positive octant. Thus in the common region z varies from 0 to $\frac{r^2}{a}$, r varies from 0 to R and θ varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned} \therefore \text{ Required volume} &= 4 \int_0^{\pi/2} \int_0^R \int_0^{r^2/a} r \, dz \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^R r \left[z \right]_0^{r^2/a} \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^R \frac{r^3}{a} \, dr \, d\theta \\ \Rightarrow 4 \int_0^{\pi/2} \left[\frac{r^4}{4a} \right]_0^R \, d\theta &= \frac{1}{a} \int_0^{\pi/2} R^4 \cdot d\theta = \frac{R^4}{a} \cdot \frac{\pi}{2} \quad \text{or} \quad \frac{\pi R^4}{2a} \end{aligned}$$

- 90.** Find the mass of the solid surrounded by the ellipsoid with density $\rho = kxyz$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (\text{U.P.T.U., 2006})$$

Sol. Put $\frac{x^2}{a^2} = u, \quad \frac{y^2}{b^2} = v, \quad \frac{z^2}{c^2} = w$

Then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$

Also $x = a\sqrt{u} \Rightarrow dx = \frac{a}{2\sqrt{u}} du$

$$y = b\sqrt{v} \Rightarrow dy = \frac{b}{2\sqrt{v}} dv$$

$$z = c\sqrt{w} \Rightarrow dz = \frac{c}{2\sqrt{w}} dw$$

$$\begin{aligned} \therefore \text{ Required mass} &= 8 \iiint_D kxyz \, dx \, dy \, dz = ka^2b^2c^2 \iiint_{D'} du \, dv \, dw \\ &= ka^2b^2c^2 \cdot \iiint_D u^0v^0w^0 du \, dv \, dw = ka^2b^2c^2 \frac{(1)!(1)!(1)!}{(1+1+1)!} = \frac{ka^2b^2c^2}{6} \end{aligned}$$

- 91.** The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, meets the axes in A, B, C. Find the volume of the tetrahedron OABC. Also find the mass of the tetrahedron, if the density at any point is $kxyz$.

Sol. Put $\frac{x}{a} = u$, $\frac{y}{b} = v$, $\frac{z}{c} = w$, then $u \geq 0$, $v \geq 0$, $w \geq 0$ and $u + v + w \leq 1$.

Also $dx = adu$, $dy = b dv$, $dz = cdw$

$$\begin{aligned} \text{Volume OABC} &= \iiint_D dx \, dy \, dz = \iiint_{D'} abc \, du \, dv \, dw, \text{ where } u + v + w \leq 1 \\ &= abc \iiint_{D'} u^{1-1}v^{1-1}w^{1-1} du \, dv \, dw = abc \frac{\overline{1} \overline{1} \overline{1}}{\overline{1+1+1+1}} = \frac{abc}{3!} = \frac{abc}{6} \\ \therefore \text{ Mass} &= \iiint_D kxyz \, dx \, dy \, dz = \iiint_{D'} k(au)(bv)(cw) abc \, du \, dv \, dw \\ &= ka^2b^2c^2 \iiint_{D'} u^{2-1}v^{2-1}w^{2-1} du \, dv \, dw \\ &= ka^2b^2c^2 \frac{\overline{2} \overline{2} \overline{2}}{\overline{(2+2+2+1)}} = ka^2b^2c^2 \frac{1!1!1!}{6!} = \frac{k \cdot a^2b^2c^2}{720}. \end{aligned}$$

- 92.** Prove that $\int_D x^{l-1}y^{m-1} dx dy = \frac{\overline{l} \overline{m}}{\overline{(l+m+1)}} \cdot h^{l+m}$. Where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq h$.

Sol. Putting $x = hX$ and $y = hY$, the given integral reduces to

$$I = \int_{D'} (hX)^{l-1} (hY)^{m-1} h^2 dX dY$$

where D' is the domain $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$.

$$\begin{aligned} I &= h^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1} dY dX \\ &= h^{l+m} \int_0^1 X^{l-1} \left[\frac{Y^m}{m} \right]_0^{1-X} dX = \frac{h^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m dX \\ &= \frac{h^{l+m}}{m} B(l, m+1) = \frac{h^{l+m}}{m} \cdot \frac{\overline{l} \overline{(m+1)}}{\overline{(l+m+1)}} = h^{l+m} \cdot \frac{\overline{l} \overline{m}}{\overline{(l+m+1)}} \end{aligned}$$

Since $\frac{\overline{(m+1)}}{m} = \overline{m}$

93. State the Dirichlet's theorem for 3 variables. Hence evaluate the integral $\iiint x^{l-1} y^{m-1} z^{n-1} dx dy dz$, where x, y, z are all positive but limited by the condition

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1. \quad (\text{U.P.T.U., 2006})$$

Sol. Statement of Dirichlet's theorem for three variables

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\overline{l} \overline{m} \overline{n}}{\overline{l} + \overline{m} + \overline{n} + 1}$$

where V is the region bounded by

$$x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$$

Put $\left(\frac{x}{a}\right)^p = u, \left(\frac{y}{b}\right)^q = v, \left(\frac{z}{c}\right)^r = w$

Then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$.

Also, $x = au^{\frac{1}{p}}, y = bv^{\frac{1}{q}} \text{ and } z = cw^{\frac{1}{r}}$

so that $dx = \frac{a}{p} u^{\frac{1}{p}-1} du, dy = \frac{b}{q} v^{\frac{1}{q}-1} dv, dz = \frac{c}{r} w^{\frac{1}{r}-1} dw$

$$\begin{aligned} \therefore \iiint_D x^{l-1} y^{m-1} z^{n-1} dx dy dz &= \iiint_{D'} (au^{\frac{1}{p}})^{l-1} (bv^{\frac{1}{q}})^{m-1} (cw^{\frac{1}{r}})^{n-1} \\ &\quad \times \frac{abc}{pqr} \cdot u^{\frac{1}{p}-1} v^{\frac{1}{q}-1} w^{\frac{1}{r}-1} du dv dw \\ &= \frac{a^l b^m c^n}{pqr} \iiint_{D'} u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1} du dv dw \\ &= \frac{a^l b^m c^n}{pqr} \cdot \frac{\overline{l} \overline{m} \overline{n}}{\overline{l} + \overline{m} + \overline{n} + 1} \end{aligned}$$

94. Find the mass of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the variable density being given by $\rho = xyz$.

Sol. Please see the solution of Q. No. 91 where ρ (density) = $kxyz$.

If we put $k = 1$, then we obtain mass of the tetrahedron = $\frac{a^2 b^2 c^2}{720}$.

Else we may independently proceed as follows :

Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$, then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$.

$$dx = a du, dy = b dv, dz = c dw$$

$$\begin{aligned} \text{Mass of the tetrahedron} &= \iiint_D xyz dx dy dz = \iiint_{D'} a^2 b^2 c^2 \cdot uvw du dv dw \\ &= a^2 b^2 c^2 \cdot \frac{\overline{2} \overline{2} \overline{2}}{\overline{(2+2+2+1)}} = a^2 b^2 c^2 \cdot \frac{1!1!1!}{6!} = \frac{a^2 b^2 c^2}{720} \end{aligned}$$

95. Find by double integration volume of the sphere $x^2 + y^2 + z^2 = 9$.

Sol. To find volume of the sphere $x^2 + y^2 + z^2 = 9$.

On account of symmetry of the sphere, half of it lies above the plane and half below it.

$$\therefore \text{ Required volume} = 2 \iint z \, dy \, dx = 2 \iint \sqrt{9 - x^2 - y^2} \, dy \, dx$$

For changing to polar co-ordinates,

We put $x = r \cos \theta$, and $y = r \sin \theta$,

$$\therefore x^2 + y^2 = r^2$$

$$\begin{aligned} \Rightarrow \text{Volume} &= 4 \int_0^{\pi/2} \int_0^3 \sqrt{9 - r^2} \, dr \, d\theta = 8 \int_0^{\pi/2} \int_0^3 \left(-\frac{1}{2}\right) (9 - r^2)^{1/2} (-2r) \, dr \, d\theta \\ &= 8 \int_0^{\pi/2} \left(-\frac{1}{2}\right) \left[\frac{(9 - r^2)^{3/2}}{3/2} \right]_0^3 d\theta = -\frac{8}{3} \int_0^{\pi/2} [0 - (9)^{3/2}] d\theta \\ &= 72 \int_0^{\pi/2} d\theta = 72 \times \frac{\pi}{2} = 36\pi \end{aligned}$$

96. Find, by triple integration, volume in the positive octant bounded by the co-ordinate planes and the plane $x + 2y + 3z = 4$. (M.D.U., May 2008)

Sol. Plane $x + 2y + 3z = 4$ intersects xy -plane (XOY) in the line $x + 2y = 4$, $z = 0$.

$$\therefore \text{ Region } OAB \text{ is bounded by } x = 0, x = 1, y = 0, y = \frac{4 - x}{2}.$$

Hence volume of $OABC$ is,

$$\begin{aligned} V &= \int_0^1 \int_0^{\frac{4-x}{2}} z \cdot dy \, dx = \int_0^1 \int_0^{\frac{4-x}{2}} \left(\frac{4 - x - 2y}{3} \right) dy \, dx \\ &= \frac{1}{3} \int_0^1 \left[4y - xy - y^2 \right]_0^{\frac{4-x}{2}} dx = \frac{1}{3} \int_0^1 \left[2(4 - x) - \frac{x(4 - x)}{2} - \frac{(4 - x)^2}{4} \right] dx \\ &= \frac{1}{3} \left[2 \left(4x - \frac{x^2}{2} \right) - \frac{1}{2} \left(\frac{4x^2}{2} - \frac{x^3}{3} \right) - \frac{1}{4} \left(16x + \frac{x^3}{3} - 4x \right) \right]_0^1 \\ &= \frac{1}{3} \left[8 - 1 - \frac{1}{2} \left(2 - \frac{1}{3} \right) - \frac{1}{4} \left(16 + \frac{1}{3} - 4 \right) \right] \\ &= \frac{1}{3} \left(7 - \frac{5}{6} - \frac{37}{12} \right) = \frac{84 - 10 - 37}{3 \times 12} = \frac{37}{36}. \end{aligned}$$

97. Evaluate $\iiint (x + y + z) \, dx \, dy \, dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. [M.D.U., May 2009]

Sol. Plane $x + y + z = 1$ meets the co-ordinate axes in $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

$\therefore$ Given integral is

$$\int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x + y + z) \, dz \, dy \, dx = \int_{x=0}^1 \int_{y=0}^{1-x} \left(xz + yz + \frac{z^2}{2} \right)_0^{1-x-y} dy \, dx$$

$$\begin{aligned}
&= \int_0^1 \int_0^{1-x} \left[x(1-x-y) + y(1-x-y) + \frac{1}{2}(1+x^2+y^2-2x-2y+2xy) \right] dy \, dx \\
&= \int_0^1 \int_0^{1-x} -\frac{1}{2}[-x^2-y^2-2xy] dy \, dx = \int_0^1 -\frac{1}{2} \int_{y=0}^{1-x} (x+y)^2 dy \, dx \\
&= -\frac{1}{2} \int_{x=0}^1 \left[x^2y + \frac{y^3}{3} + 2x \cdot \frac{y^2}{2} \right]_{y=0}^{1-x} dx = -\frac{1}{2} \int_{x=0}^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} + x(1-x)^2 \right] dx \\
&= -\frac{1}{2} \int_0^1 \left[(x^2-x^3) + \frac{(1-x)^3}{3} + (x+x^3-2x^2) \right] dx = -\frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{(-4) \cdot 3} + \frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3} \right]_0^1 \\
&= -\frac{1}{2} \left[\frac{1}{3} - \frac{1}{4} + 0 + \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right] = \left(-\frac{1}{2} \right) \left(\frac{1}{6} \right) = -\frac{1}{12}
\end{aligned}$$

98. Show that $\beta(p, q) = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx$. (U.P.T.U., 2008, M.D.U., 2009)

Sol. As
$$\beta(p, q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \cdot dx = \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}} \cdot dx + \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \cdot dx$$

For the second integral viz. $\int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \cdot dx$,

Let us put $x = \frac{1}{t}$, $dx = -\frac{1}{t^2} \cdot dt$

so that when $x \rightarrow 1$, $t \rightarrow 1$,
when $x \rightarrow \infty$, $t \rightarrow 0$

$$\begin{aligned}
\therefore \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \cdot dx &= \int_1^0 \frac{\left(\frac{1}{t}\right)^{p-1}}{\left(1+\frac{1}{t}\right)^{p+q}} \cdot \left(-\frac{1}{t^2}\right) dt = \int_1^0 \frac{t^{p+q-2-p+1}}{(t+1)^{p+q}} (-1) dt \\
&= \int_1^0 (-1) \cdot \frac{t^{q-1}}{(1+t)^{p+q}} \cdot dt = \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} \cdot dx
\end{aligned}$$

Hence,
$$\beta(p, q) = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} \cdot dx$$

99. Show that $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$.

Sol. Put $x = \frac{1}{1+y}$, so that $dx = -\frac{1}{(1+y)^2} dy$

$$\beta(p, q) = \int_0^\infty \left(\frac{1}{1+y} \right)^{p-1} \left(\frac{y}{1+y} \right)^{q-1} - \frac{1}{(1+y)^2} \cdot dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} \cdot dy \quad \text{Hence shown.}$$

100. Prove that: $\int_0^{\infty} x^{p-1} e^{-kx} dx = \frac{\sqrt[p]{p}}{k^p} \quad (k > 0).$

Sol. Put $kx = t$ so that $kdx = dt$

$$\therefore \int_0^{\infty} x^{p-1} e^{-kx} dx = \int_0^{\infty} \left(\frac{t}{k}\right)^{p-1} \cdot e^{-t} \cdot \frac{dt}{k} = \frac{1}{k^p} \int_0^{\infty} t^{p-1} e^{-t} dt = \frac{\sqrt[p]{p}}{k^p}$$

101. Show that: (i) $\int_0^{\infty} e^{-ax} x^{n-1} \cos bxdx = \frac{\sqrt[n]{n} \cos n\theta}{(a^2 + b^2)^{n/2}}$

$$(ii) \int_0^{\infty} e^{-ax} x^{n-1} \sin bxdx = \frac{\sqrt[n]{n} \sin n\theta}{(a^2 + b^2)^{n/2}} \text{ where } \tan^{-1} \left(\frac{b}{a} \right) = \theta.$$

Sol. We know that $\int_0^{\infty} e^{-ax} \cdot x^{n-1} dx = \frac{\sqrt[n]{n}}{a^n}$, where a, n are positive

Replacing a by $(a + ib)$, we have

$$\int_0^{\infty} e^{-(a+ib)x} \cdot x^{n-1} dx = \frac{\sqrt[n]{n}}{(a+ib)^n} \quad \dots(1)$$

Now,

$$e^{-(a+ib)x} = e^{-ax} \cdot e^{-ibx} = e^{-ax} (\cos bx - i \sin bx)$$

Putting

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta$$

so that

$$r^2 = a^2 + b^2 \quad \text{and} \quad \theta = \tan^{-1} \left(\frac{b}{a} \right)$$

$$(a+ib)^n = (r \cos \theta + ir \sin \theta)^n = r^n (\cos \theta + i \sin \theta)^n \\ = r^n (\cos n\theta + i \sin n\theta) \quad (\text{De Moivre's Theorem})$$

$\therefore$ From (1),

$$\int_0^{\infty} e^{-ax} (\cos bx - i \sin bx) x^{n-1} \cdot dx = \frac{\sqrt[n]{n}}{r^n (\cos n\theta + i \sin n\theta)} \\ = \frac{\sqrt[n]{n}}{r^n} (\cos n\theta + i \sin n\theta)^{-1} = \frac{\sqrt[n]{n}}{r^n} (\cos n\theta - i \sin n\theta)$$

Now equating real and imaginary parts on both the sides, we get

$$\int_0^{\infty} e^{-ax} x^{n-1} \cos bx dx = \frac{\sqrt[n]{n}}{r^n} \cos n\theta$$

and

$$\int_0^{\infty} e^{-ax} x^{n-1} \sin bx dx = \frac{\sqrt[n]{n}}{r^n} \sin n\theta$$

where,

$$r^2 = a^2 + b^2 \quad \text{and} \quad \theta = \tan^{-1} \left(\frac{b}{a} \right).$$

102. Prove that:
$$\int_0^1 x^4 \left(\log \frac{1}{x} \right)^3 dx = \frac{6}{625}.$$

Sol. Put $\log \frac{1}{x} = t$, $\frac{1}{x} = e^t$ or $x = e^{-t}$, $dx = -e^{-t} dt$

$$\begin{aligned} \therefore \text{ Given integral} &= \int_{\infty}^0 e^{-4t} t^3 (-e^{-t}) dt \\ &= \int_0^{\infty} e^{-5t} t^3 dt = \int_0^{\infty} e^{-y} \left(\frac{y}{5} \right)^3 \cdot \frac{dy}{5} \quad \left(\text{where } y = 5t, dt = \frac{dy}{5} \right) \\ &= \frac{1}{625} \int_0^{\infty} e^{-y} \cdot y^3 dy = \frac{\overline{(4)}}{625} = \frac{3!}{625} \quad \text{or} \quad \frac{6}{625} \end{aligned}$$

103. Show that : $\beta(m, n) = \frac{\overline{m} \overline{n}}{\overline{m+n}}$. (M.D.U., Dec., 2005, May 2007; U.P.T.U., 2009, A.U.U.P., 2008)

Sol. We know that
$$\overline{m} = \int_0^{\infty} e^{-t} t^{m-1} dt$$

Putting $t = x^2$ so that $dt = 2x dx$

$$\overline{m} = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad \dots(1)$$

Similarly,
$$\overline{n} = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dy$$

$$\Rightarrow \overline{m} \overline{n} = 4 \int_0^{\infty} e^{-x^2} \cdot x^{2m-1} dx \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

We have used the following result from double integrals.

If $f(x)$ and $g(y)$ are functions of x and y only, and the limits of integration are constants, then the double integral can be represented as a product of 2 integrals.

Thus,
$$\int_c^d \int_c^d f(x) g(y) dx dy = \int_a^b f(x) dx \cdot \int_c^d g(y) dy$$

$$\therefore \overline{m} \overline{n} = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2 + y^2)} x^{2m-1} \cdot y^{2n-1} dx dy$$

Changing to polar co-ordinates, we have

$$\overline{m} \overline{n} = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} \cos^{2m-1} \theta \sin^{2n-1} \theta dr d\theta$$

$$\Rightarrow \quad \overline{m} \overline{n} = 4 \int_0^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} dr \int_0^{\pi/2} \cos^{2m-1} \theta \cdot \sin^{2n-1} \theta d\theta \quad \dots(2)$$

$$\text{Now,} \quad 2 \int_0^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} dr = \overline{(m+n)}, \quad [\text{From (1)}]$$

and put $\sin^2 \theta = z$ so that

$$2 \sin \theta \cos \theta = dz$$

$$\begin{aligned} 2 \int_0^{\pi/2} \cos^{2m-1} \theta \cdot \sin^{2n-1} \theta d\theta &= \int_0^1 (1-z)^{m-1} z^{n-1} dz = \int_0^1 z^{n-1} (1-z)^{m-1} dz \\ &= \beta(n, m) = \beta(m, n) \quad \text{By symmetry of Beta Function.} \end{aligned}$$

$$\therefore \text{ From (2), we have} \quad \overline{m} \overline{n} = \overline{(m+n)} \beta(m, n)$$

$$\text{Hence} \quad \beta(m, n) = \frac{\overline{m} \overline{n}}{\overline{(m+n)}}.$$

$$\mathbf{104.} \text{ Prove that: } \overline{\frac{1}{2}} = \sqrt{\pi}. \quad [\text{M.D.U. 2005, Dec., 2007; U.P.T.U., 2008}]$$

$$\mathbf{Sol.} \quad \overline{n} = \int_0^{\infty} e^{-x} x^{n-1} dx, \quad n > 0 \quad \Rightarrow \quad \overline{\frac{1}{2}} = \int_0^{\infty} e^{-t} \cdot t^{-1/2} \cdot dt$$

Put $t = x^2$ so that $dt = 2xdx$

$$\overline{\frac{1}{2}} = \int_0^{\infty} e^{-x^2} \cdot \frac{1}{x} \cdot 2xdx = 2 \int_0^{\infty} e^{-x^2} dx \quad \dots(1)$$

Writing y for x , we have

$$\overline{\left(\frac{1}{2}\right)} = 2 \int_0^{\infty} e^{-y^2} dy \quad \dots(2)$$

From (1) and (2), we have

$$\left[\overline{\left(\frac{1}{2}\right)} \right]^2 = 4 \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

Changing to polar co-ordinates with $x = r \cos \theta$, $y = r \sin \theta$, $\Rightarrow dx dy = r dr d\theta$;

the region of integration in this integral is the complete positive quadrant, to cover which, r must

vary from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$.

$$\therefore \left[\frac{1}{2} \right]^2 = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} d\theta = 2 \int_0^{\pi/2} d\theta = \pi$$

Hence $\left[\frac{1}{2} \right] = \sqrt{\pi}$

105. Find the value of $\beta(2.5, 1.5)$.

Sol.
$$\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\left[\frac{5}{2}\right] \left[\frac{3}{2}\right]}{\left[\frac{5}{2} + \frac{3}{2}\right]} \quad \dots(1)$$

We know that if n is a positive fraction, then by using the reduction formula :

$$\overline{(n+1)} = n!, \quad \text{we get } \overline{(n)} = (n-1)(n-2)\dots\dots\dots = \overline{(\text{last +ve factor})}$$

$$\Rightarrow \left[\frac{5}{2}\right] = \frac{3}{2} \cdot \frac{1}{2} \cdot \left[\frac{1}{2}\right] \quad \text{and} \quad \left[\frac{3}{2}\right] = \frac{1}{2} \left[\frac{1}{2}\right]$$

$\therefore$ From (1), we have

$$\begin{aligned} \beta\left(\frac{5}{2}, \frac{3}{2}\right) &= \frac{\left[\frac{3}{2} \cdot \frac{1}{2} \cdot \left[\frac{1}{2}\right]\right] \left[\frac{1}{2} \cdot \left[\frac{1}{2}\right]\right]}{\overline{(4)}} = \frac{\frac{3}{8} \left[\left[\frac{1}{2}\right]\right]^2}{3!} = \frac{\frac{3}{8} \cdot (\sqrt{\pi})^2}{6} \quad \left(\because \left[\frac{1}{2}\right] = \sqrt{\pi}\right) \\ &= \frac{\pi}{16} \end{aligned}$$

106. Evaluate: $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$.

Sol. As explained in the previous Q. No. 105, we have

$$\left[\frac{9}{2}\right] = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \left[\frac{1}{2}\right] \quad \text{and} \quad \left[\frac{7}{2}\right] = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \left[\frac{1}{2}\right]$$

$$\therefore \beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{\left[\frac{9}{2}\right] \cdot \left[\frac{7}{2}\right]}{\left[\frac{9}{2} + \frac{7}{2}\right]} = \frac{\frac{7}{2} \cdot \left[\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \left[\frac{1}{2}\right]\right]^2}{7!}$$

$$\Rightarrow \beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{5\pi}{2048} \quad (\text{after simplification})$$

107. Prove that: $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{\beta(m, n)}{a^n \cdot b^m}$, where m, n, a, b are positive.

Sol. Put $bx = at$ i.e., $x = \frac{at}{b}$ so that $dx = \frac{a}{b} dt$

$$\therefore \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} \cdot \int_0^{\infty} \frac{t^{m-1} \cdot dt}{(1+t)^{m+n}} = \frac{\beta(m, n)}{a^n \cdot b^m}$$

108. Prove that $\overline{(n)} \overline{(1-n)} = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$ using $\int_0^{\infty} \frac{x^{n-1}}{1+x} \cdot dx = \frac{\pi}{\sin n\pi}$.

$$\text{Sol.} \quad \beta(m, n) = \frac{\overline{(m)} \cdot \overline{(n)}}{\overline{(m+n)}} \Rightarrow \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} \cdot dx = \frac{\overline{(m)} \cdot \overline{(n)}}{\overline{(m+n)}}$$

Setting $m+n=1$ so that $m=1-n$, we get

$$\int_0^{\infty} \frac{x^{n-1}}{1+x} \cdot dx = \frac{\Gamma(1-n) \cdot \Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \frac{\pi}{\sin n\pi} = \overline{(n)} \overline{(1-n)} \quad [\because \overline{(1)}=1]$$

Hence proved.

109. Prove that : $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} \cdot \int_0^{\pi/2} \sqrt{\sin \theta} \cdot d\theta = \pi$.

(M.D.U., May 2009; A.U.U.P., 2008; U.P.T.U., 2006, 2007)

$$\text{Sol. We know that} \quad \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{\left(\frac{m+1}{2}\right) \left(\frac{n+1}{2}\right)}{2 \left(\frac{m+n+2}{2}\right)} \quad \dots(1)$$

which is a standard result obtained using the relation

$$\beta(m, n) = \frac{\overline{(m)} \overline{(n)}}{\overline{(m+n)}}$$

$$\text{Given} \quad I = \int_0^{\pi/2} (\sin \theta)^{-\frac{1}{2}} (\cos \theta)^0 d\theta \times \int_0^{\pi/2} (\sin \theta)^{\frac{1}{2}} \cos^0 \theta d\theta$$

On putting $m = -\frac{1}{2}$, $n = 0$, $m = \frac{1}{2}$ in (1) and using that we obtain,

$$I = \frac{\left(\frac{1}{4}\right) \left(\frac{1}{2}\right)}{2 \left(\frac{3}{4}\right)} \times \frac{\left(\frac{3}{4}\right) \cdot \left(\frac{1}{2}\right)}{2 \left(\frac{5}{4}\right)} = \frac{\left(\frac{1}{4}\right) \cdot \sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{\frac{1}{4} \left(\frac{1}{4}\right)}$$

$$= \pi = \text{R.H.S.} \quad (\because \overline{(n+1)} = n \overline{(n!)} \text{ and } \overline{\left(\frac{1}{2}\right)} = \sqrt{\pi})$$

110. Prove that : $\int_0^{\pi/2} \sqrt{\tan \theta} \cdot d\theta = \int_0^{\pi/2} \sqrt{\cot \theta} \cdot d\theta = \frac{1}{2} \left[\left(\frac{1}{4} \right) \left(\frac{3}{4} \right) \right] = \frac{\pi}{\sqrt{2}}.$ (U.P.T.U., 2008)

Sol. $I = \int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \int_0^{\pi/2} \sqrt{\cot \theta} \cdot d\theta \quad \dots(A) \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$

$\Rightarrow I = \int_0^{\pi/2} \cos^{1/2} \theta \sin^{-1/2} \theta d\theta$

or $I = \int_0^{\pi/2} \sin^{-1/2} \theta \cdot \cos^{1/2} \theta d\theta$

As stated in the previous Q. No. 109, equation (1), result for $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \cdot d\theta$ is given in terms of Gamma Function values, we use the same formula and obtain,

$$I = \frac{\left[\left(\frac{1}{4} \right) \left(\frac{3}{4} \right) \right]}{2(1)} = \frac{1}{2} \left[\left(\frac{1}{4} \right) \left(1 - \frac{1}{4} \right) \right] = \frac{1}{2} \cdot \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \quad \left[\because \frac{\Gamma(n) \Gamma(1-n)}{\sin n\pi} = \frac{\pi}{\sin n\pi} \right]$$

Note : Please see solution of Q. 108.

111. Prove that : $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\left(\frac{1}{4} \right) \right]}{\left[\left(\frac{3}{4} \right) \right]}$

Sol. Put $x^4 = t$ i.e., $x = t^{1/4}$ so that $dx = \frac{1}{4} t^{-3/4} dt$

$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4} \int_0^1 t^{-3/4} (1-t)^{-1/2} \cdot dt = \frac{1}{4} \int_0^1 t^{1/4-1} (1-t)^{\frac{1}{2}-1} dt = \frac{1}{4} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4} \cdot \frac{\left[\left(\frac{1}{4} \right) \left(\frac{1}{2} \right) \right]}{\left[\left(\frac{1}{4} + \frac{1}{2} \right) \right]} = \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\left(\frac{1}{4} \right) \right]}{\left[\left(\frac{3}{4} \right) \right]} \quad \left[\because \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \sqrt{\pi} \right]$

Hence proved.

112. Prove that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1).$

[U.P.T.U., (Agr. Engg.) 2005; A.U.U.P., 2008; M.D.U., 2009]

Sol. R.H.S. in the given identity which we are required to prove here, is equal to

$$\begin{aligned} \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1} (1-x)^n dx &= \int_0^1 x^{m-1} (1-x)^{n-1} \{x + (1-x)\} dx \\ &= \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx = \beta(m, n) = \text{L.H.S.} \end{aligned}$$

Hence proved.

113. State and prove duplication formula : $\overline{(m)} \left(\overline{m + \frac{1}{2}} \right) = \frac{\sqrt{\pi}}{(2)^{2m-1}} \cdot \overline{(2m)}$, where m is positive.

(U.P.T.U., 2007; M.D.U., 2006; A.U.U.P., 2008)

Sol. We know that

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{\overline{(m)} \overline{(n)}}{2 \overline{(m+n)}} \quad \dots(1)$$

Putting $2n-1=0$ or $n = \frac{1}{2}$ in (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1} \theta d\theta = \frac{\overline{(m)} \sqrt{\pi}}{2 \overline{\left(m + \frac{1}{2}\right)}} \quad \dots(2)$$

Again putting $n = m$ in equation (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2m-1} \theta \cdot d\theta = \frac{(\overline{(m)})^2}{2 \overline{(2m)}}$$

$$\text{i.e., } \frac{1}{2^{2m-1}} \int_0^{\pi/2} (2 \sin \theta \cos \theta)^{2m-1} d\theta = \frac{(\overline{(m)})^2}{2 \overline{(2m)}}$$

$$\Rightarrow \frac{1}{2^{2m}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} \cdot 2d\theta = \frac{(\overline{(m)})^2}{2 \overline{(2m)}}$$

Putting $2\theta = \phi$, so that $2d\theta = d\phi$, this reduces to

$$\frac{1}{2^{2m}} \int_0^{\pi} \sin^{2m-1} \phi d\phi = \frac{(\overline{(m)})^2}{2 \overline{(2m)}} \quad \text{or} \quad \frac{2}{2^{2m}} \int_0^{\pi/2} \sin^{2m-1} \phi d\phi = \frac{(\overline{(m)})^2}{\overline{(2m)}}$$

Replacing ϕ by θ , we finally obtain

$$\int_0^{\pi/2} \sin^{2m-1} \theta d\theta = \frac{2^{2m-1} (\overline{(m)})^2}{2 \overline{(2m)}} \quad \dots(3)$$

$$\text{From (2) and (3), we get } \frac{\overline{(m)} \sqrt{\pi}}{2 \overline{\left(m + \frac{1}{2}\right)}} = \frac{2^{2m-1} (\overline{(m)})^2}{2 \overline{(2m)}} \Rightarrow \overline{(m)} \cdot \overline{\left(m + \frac{1}{2}\right)} = \frac{\sqrt{\pi} \cdot \overline{(2m)}}{2^{2m-1}}$$

which is duplication formula.

114. Evaluate : $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}} dx.$

Sol.
$$I = \int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}} \cdot dx = \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} \cdot dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}} \cdot dx$$

$$= \beta(9, 15) - \beta(15, 9) = 0 \quad [\because \beta(m, n) = \beta(n, m)]$$

115. Find the value of : $\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}} dx.$

Sol.
$$I = \int_0^{\infty} \frac{x^4}{(1+x)^{15}} \cdot dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}} \cdot dx = \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}} \cdot dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}} \cdot dx$$

$$= \beta(5, 10) + \beta(10, 5) = 2\beta(5, 10) = 2 \cdot \frac{\overline{(5)} \overline{(10)}}{\overline{(15)}} = \frac{1}{5005}.$$

116. Evaluate : $\iiint \log(x+y+z) dx dy dz$, the integral extending over all positive and zero values of x, y, z subject to $x+y+z < 1$. (M.D.U., 2007; U.P.T.U., 2008; A.U.U.P., 2009)

Sol.
$$\iiint \log(x+y+z) dx dy dz = \iiint x^{1-1} y^{1-1} z^{1-1} \log(x+y+z) dx dy dz$$

$$= \frac{\overline{(1)} \overline{(1)} \overline{(1)}}{\overline{(3)}} \int_0^1 t^{1+1+1-1} \log t dt$$

[By Liouville's extension of Dirichlet's Theorem]

$$\Rightarrow I = \frac{1}{2} \int_0^1 t^2 \log t dt = \frac{1}{2} \left[\left(\frac{t^3}{3} \log t \right)_0^1 - \int_0^1 \frac{t^3}{3} \cdot \frac{1}{t} \cdot dt \right] = \frac{1}{2} \left[-\frac{1}{3} \left(\frac{t^3}{3} \right)_0^1 \right] = -\frac{1}{18}$$

117. Prove that : $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1} \cdot \beta(m, m).$ (U.P.T.U., 2009; M.D.U., 2007)

Sol.
$$\beta(m, m) = \frac{\overline{m} \cdot \overline{m}}{\overline{(2m)}} = \frac{(\overline{m})^2}{(2m-1)!} \quad \text{and} \quad \beta\left(m, \frac{1}{2}\right) = \frac{\overline{m} \cdot \overline{\left(\frac{1}{2}\right)}}{\overline{m + \frac{1}{2}}}$$

Now,
$$\overline{\left(m + \frac{1}{2}\right)} = \frac{\overline{m} \cdot \overline{(2m+1)}}{2^{2m} \cdot \overline{(m+1)}}$$

$$\therefore \beta\left(m, \frac{1}{2}\right) = \frac{\overline{m} \cdot \overline{\left(\frac{1}{2}\right)} \cdot 2^{2m} \cdot \overline{m+1}}{\overline{m} \overline{(2m+1)}} = \frac{\sqrt{\pi} \cdot 2^{2m} \cdot m!}{\sqrt{\pi} \cdot (2m)!} = \frac{2^{2m} \cdot m!}{(2m)!}$$

$$\text{Now,} \quad \beta(m, m) = \frac{(m-1)!(m-1)!}{(2m-1)!} \quad \dots(1)$$

$$\text{Hence,} \quad \beta\left(m, \frac{1}{2}\right) = \frac{2^{2m} \cdot m!}{(2m)!} \quad \dots(2)$$

Dividing equation (2) by equation (1), we obtain,

$$\frac{\beta\left(m, \frac{1}{2}\right)}{\beta(m, m)} = \frac{2^{2m} \cdot m! (2m-1)!}{(2m)! (m-1)! (m-1)!} = \frac{2^{2m} \cdot m(m-1)! (2m-1)!}{(2m)(2m-1)! (m-1)! (m-1)!} = 2^{2m-1}$$

$$\text{Hence} \quad \beta\left(m, \frac{1}{2}\right) = 2^{2m-1} \cdot \beta(m, m)$$

118. Evaluate : $\int_0^1 x^5 (1-x^3)^{10} dx.$ [M.D.U., May 2007, 2008]

Sol. Consider the integral $\int_0^1 x^m (1-x^n)^p \cdot dx$

Put $x^n = y \Rightarrow x = (y)^{\frac{1}{n}}, \quad dx = \frac{1}{n} \cdot y^{\frac{1}{n}-1} \cdot dy$

$$\begin{aligned} \therefore \int_0^1 x^m (1-x^n)^p dx &= \int_0^1 y^{\frac{m}{n}} (1-y)^p \cdot \frac{1}{n} \cdot y^{\frac{1}{n}-1} \cdot dy \\ &= \frac{1}{n} \int_0^1 (y)^{\frac{m}{n} + \frac{1}{n} - 1} \cdot (1-y)^p dy = \frac{1}{n} \cdot \beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n} \cdot \frac{\left[\frac{m+1}{n}\right] [p+1]}{\left[\frac{m+1}{n} + p+1\right]} \end{aligned}$$

Now, we have to evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

Compare it with $\int_0^1 x^m (1-x^n)^p dx$, then $m = 5, n = 3, p = 10$.

From the above result, we have

$$\int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{3} \cdot \frac{\left[\frac{6}{3}\right] [11]}{\left[\frac{6}{3} + 11\right]} = \frac{1}{3} \cdot \frac{1 \cdot 10!}{12!} = \frac{1}{3 \times 12 \times 11} = \frac{1}{396}.$$

□ □ □

SOLVED PROBLEMS

1. (a) What is vector function of the scalar variable t . Define derivative of a vector function w.r.t. a scalar and also in terms of its components.

- (b) Show that if $\vec{r} = \vec{a} \sin wt + \vec{b} \cos wt$, where $\vec{a}, \vec{b}, w$ are constants, then

$$\frac{d^2 \vec{r}}{dt^2} = -w^2 \vec{r} \quad \text{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = -w \vec{a} \times \vec{b}$$

Sol. (a) $\vec{r}$ is called a vector function of the scalar variable t and we write $\vec{r} = \vec{f}(t)$, if to each value of t , there corresponds a value of the vector $\vec{r}$.

If $\lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}$ exists, then the value of this limit is denoted by $\frac{d\vec{r}}{dt}$ and is called the derivative of $\vec{r}$ w.r.t. 't'.

Since $\frac{d\vec{r}}{dt}$ is itself a vector function of t , its derivative is denoted by $\frac{d^2 \vec{r}}{dt^2}$ and is called second derivative of $\vec{r}$ w.r.t. t .

Similarly, we can define higher order derivatives. Since every vector can be uniquely expressed as a linear combination of 3 fixed non-coplanar vectors, therefore, we may write

$\vec{f}(t) = f_1(t) \cdot \hat{i} + f_2(t) \cdot \hat{j} + f_3(t) \cdot \hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along the axis of x, y, z respectively. $f_1(t), f_2(t)$ and $f_3(t)$ are called the components of the vector $\vec{f}(t)$ along the co-ordinate axes.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where the components x, y, z are scalar functions of t . Then we have,

$\frac{d\vec{r}}{dt} = \frac{dx}{dt} \cdot \hat{i} + \frac{dy}{dt} \cdot \hat{j} + \frac{dz}{dt} \cdot \hat{k}$, since $\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0}$, $\hat{i}, \hat{j}, \hat{k}$ being fixed unit vectors or constant vectors (since both magnitude and direction are fixed/constant).

If $x = f_1(t), y = f_2(t), z = f_3(t)$; then

$$\begin{aligned} \vec{r} &= f_1(t) \cdot \hat{i} + f_2(t) \cdot \hat{j} + f_3(t) \cdot \hat{k} \\ \Rightarrow \frac{d\vec{r}}{dt} &= f_1'(t) \cdot \hat{i} + f_2'(t) \cdot \hat{j} + f_3'(t) \cdot \hat{k} \end{aligned}$$

Therefore, to differentiate a vector function, we have to differentiate its components to obtain the derivative.

(b) Given that $\vec{r} = \vec{a} \sin wt + \vec{b} \cos wt$, where $\vec{a}, \vec{b}, w$ are constants, we have to show that

$$\frac{d^2 \vec{r}}{dt^2} = -w^2 \vec{r} \quad \text{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = -w \vec{a} \times \vec{b}$$

We know that if $\vec{r} = \phi \vec{f}$ where ϕ is a scalar function of t , then $\frac{d\vec{r}}{dt} = \phi \cdot \frac{d\vec{f}}{dt} + \frac{d\phi}{dt} \cdot \vec{f}$. But if

$\vec{f}$ is a constant vector, then $\frac{d\vec{f}}{dt} = \vec{0}$. $\therefore \frac{d\vec{r}}{dt} = \frac{d\phi}{dt} \cdot \vec{f}$

$$\text{Now, } \vec{r} = \vec{a} \sin wt + \vec{b} \cos wt \Rightarrow \frac{d\vec{r}}{dt} = \vec{a} w \cos wt - \vec{b} w \sin wt$$

$$\therefore \frac{d^2 \vec{r}}{dt^2} = -\vec{a} w^2 \sin wt - \vec{b} w^2 \cos wt = -w^2 (\vec{a} \sin wt + \vec{b} \cos wt) = -w^2 \vec{r}$$

$$\text{Also, } \vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a} \sin wt + \vec{b} \cos wt) \times (\vec{a} w \cos wt - \vec{b} w \sin wt)$$

$$\Rightarrow \vec{r} \times \frac{d\vec{r}}{dt} = w (-\vec{a} \times \vec{b} \sin^2 wt + \vec{b} \times \vec{a} \cos^2 wt) \quad [\because \vec{a} \times \vec{a} = \vec{0} = \vec{b} \times \vec{b}]$$

$$= w (-\vec{a} \times \vec{b} \sin^2 wt - \vec{a} \times \vec{b} \cos^2 wt) = -w (\vec{a} \times \vec{b}).$$

2. Define Gradient of a scalar field. Find gradient ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$.

Sol. Let $\phi(x, y, z)$ be a function defining a scalar field then the vector $\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$ is called the gradient of the scalar field ϕ and is denoted by $\text{grad } \phi$. The gradient is obtained by operating on ϕ by the vector operator $\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ denoted by the symbol $\vec{\nabla}$, read as del (also called nabla). Thus $\text{grad } \phi = \vec{\nabla} \phi$.

$$\begin{aligned} \text{grad } \phi &= \vec{\nabla} \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3x^2y - y^3z^2) \\ &= \hat{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \hat{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2) + \hat{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2) \\ &= \hat{i} (6xy) + \hat{j} (3x^2 - 3y^3z^2) + \hat{k} (-2y^3z) \end{aligned}$$

$$\Rightarrow \text{grad } \phi = -12\hat{i} - 9\hat{j} - 16\hat{k}, \text{ at the point } (1, -2, -1).$$

3. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that:

$$(i) \text{ grad } r = \frac{\vec{r}}{r}$$

$$(ii) \text{ grad } \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3}$$

$$(iii) \vec{\nabla} r^n = nr^{n-2} \vec{r}$$

$$(iv) \vec{\nabla} (\vec{a} \cdot \vec{r}) = \vec{a}, \text{ where } \vec{a} \text{ is a constant vector.}$$

Sol. $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, or $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x , we have

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$(i) \quad \text{grad } r = \bar{V}r = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r$$

$$= \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} = \hat{i} \left(\frac{x}{r} \right) + \hat{j} \left(\frac{y}{r} \right) + \hat{k} \left(\frac{z}{r} \right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = -\frac{\vec{r}}{r}.$$

$$(ii) \quad \text{grad} \left(\frac{1}{r} \right) = \bar{V} \left(\frac{1}{r} \right) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(\frac{1}{r} \right).$$

$$= \hat{i} \left(-\frac{1}{r^2} \cdot \frac{\partial r}{\partial x} \right) + \hat{j} \left(-\frac{1}{r^2} \cdot \frac{\partial r}{\partial y} \right) + \hat{k} \left(-\frac{1}{r^2} \cdot \frac{\partial r}{\partial z} \right)$$

$$= \hat{i} \left(-\frac{1}{r^2} \cdot \frac{x}{r} \right) + \hat{j} \left(-\frac{1}{r^2} \cdot \frac{y}{r} \right) + \hat{k} \left(-\frac{1}{r^2} \cdot \frac{z}{r} \right) = -\frac{1}{r^3} (x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

$$(iii) \quad \bar{V}r^n = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r^n$$

$$= \hat{i} \left(nr^{n-1} \cdot \frac{\partial r}{\partial x} \right) + \hat{j} \left(nr^{n-1} \cdot \frac{\partial r}{\partial y} \right) + \hat{k} \left(nr^{n-1} \cdot \frac{\partial r}{\partial z} \right)$$

$$= \hat{i} \left(nr^{n-1} \cdot \frac{x}{r} \right) + \hat{j} \left(nr^{n-1} \cdot \frac{y}{r} \right) + \hat{k} \left(nr^{n-1} \cdot \frac{z}{r} \right)$$

$$= nr^{n-2} (x\hat{i} + y\hat{j} + z\hat{k}) = n \cdot r^{n-2} \cdot \vec{r}.$$

(iv) Let $\vec{a} = a_1 \cdot \hat{i} + a_2 \cdot \hat{j} + a_3 \cdot \hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\therefore \bar{V}(\vec{a} \cdot \vec{r}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (a_1x + a_2y + a_3z)$$

$$= \hat{i} \cdot \frac{\partial}{\partial x} (a_1x + a_2y + a_3z) + \hat{j} \cdot \frac{\partial}{\partial y} (a_1x + a_2y + a_3z) + \hat{k} \cdot \frac{\partial}{\partial z} (a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$$

4. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.

Sol. Let $\phi = x^3 + y^3 + 3xyz = 3$, then $\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$.

$$\frac{\partial \phi}{\partial y} = 3y^2 + 3xz, \quad \frac{\partial \phi}{\partial z} = 3xy$$

$$\therefore \bar{V}\phi = \hat{i} \cdot \frac{\partial \phi}{\partial x} + \hat{j} \cdot \frac{\partial \phi}{\partial y} + \hat{k} \cdot \frac{\partial \phi}{\partial z}$$

$$\Rightarrow \bar{V}\phi = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At $(1, 2, -1)$, $\bar{V}\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$, which is a vector normal to the given surface at $(1, 2, -1)$.
Hence a unit vector normal to the given surface at $(1, 2, -1)$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{(-3)^2 + (9)^2 + (6)^2}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}} (-\hat{i} + 3\hat{j} + 2\hat{k})$$

5. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$.

In what direction it will be maximum? Find also the magnitude of this maximum (directional derivative).

Sol. We have
$$\begin{aligned}\bar{V}f &= \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} \\ &= 2\hat{i} - 4\hat{j} + 12\hat{k} \quad \text{at } P(1, 2, 3)\end{aligned}$$

Also,
$$\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$$

If $\hat{n}$ is a unit vector in the direction $\vec{PQ}$, then

$$\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$$

$\therefore$ Directional derivative of f in the direction

$$\begin{aligned}\vec{PQ} &= (\bar{V}f) \cdot \hat{n} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) \\ &= \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)] = \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}.\end{aligned}$$

The directional derivative of f is maximum in the direction of the normal to the given surface i.e., in the direction of $\bar{V}f = 2\hat{i} - 4\hat{j} + 12\hat{k}$. The maximum value of this directional derivative

$$= |\bar{V}f| = \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}$$

6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let
$$\phi_1 = x^2 + y^2 + z^2 = 9 \quad \text{and} \quad \phi_2 = x^2 + y^2 - z = 3$$

Then
$$\text{grad } \phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \quad \text{and} \quad \text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Let
$$\vec{n}_1 = \text{grad } \phi_1 \text{ at the point } (2, -1, 2) \text{ and}$$

$$\vec{n}_2 = \text{grad } \phi_2 \text{ at the point } (2, -1, 2).$$

Then
$$\vec{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n}_2 = 4\hat{i} - 2\hat{j} - \hat{k}.$$

The vectors $\vec{n}_1$ and $\vec{n}_2$ are along normals to the two surfaces at the point $(2, -1, 2)$. If θ is the angle between these vectors, then

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16 + 4 + 16} \cdot \sqrt{16 + 4 + 1}} = \frac{16}{6\sqrt{21}} \quad \therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

7. Find the directional derivative of the function $f(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Sol.
$$\bar{V}f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} = y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$$

$$\bar{V}f \text{ at } P(2, -1, 1) = (-1)^2\hat{i} + [2(2)(-1) + 1^3]\hat{j} + 3(-1)(1)^2\hat{k} = \hat{i} - 3\hat{j} - 3\hat{k}$$

Let the vector $\vec{AB}$ is $\hat{i} + 2\hat{j} + 2\hat{k}$.

If $\hat{n}$ is unit vector in direction $\vec{AB}$, then $\hat{n} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

$\therefore$ Directional derivative of f in the direction of $\vec{AB}$ is

$$= \vec{\nabla} f \cdot \hat{n} = (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{3} = \frac{1}{3} [1 - 6 - 6] = -\frac{11}{3}.$$

8. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$. (M.D.U., May 2006, Dec., 2006, 2008)

Sol. $\phi = x^2yz + 4xz^2$

$$\vec{\nabla}\phi = (2xyz + 4z^2)\hat{i} + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

At point $(1, -2, -1)$

$$\vec{\nabla}\phi = \{(4+4)\hat{i} + \hat{j}(-1) + \hat{k}(-2-8)\} = 8\hat{i} - \hat{j} - 10\hat{k}$$

Given vector : $2\hat{i} - \hat{j} - 2\hat{k}$

Directional derivative of ϕ in the direction of vector

$$= \frac{\vec{\nabla}\phi \cdot \vec{a}}{|\vec{a}|} = \frac{(8\hat{i} - \hat{j} - 10\hat{k})(2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{4+1+4}} = \frac{16+1+20}{3} = \frac{37}{3}.$$

9. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$? (M.D.U., May 2007)

Sol. $\phi = xy^2 + yz^3$

$$\vec{\nabla}\phi = \hat{i}(y^2) + \hat{j}(z^3) + \hat{k}(2xy)$$

$$\vec{\nabla}\phi \text{ at } (2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$$

Now, $f = x \log z - y^2 + 4$

$$\vec{\nabla}f = \log z \hat{i} - 2y \hat{j} + \frac{x}{z} \hat{k}$$

$$\vec{\nabla}f \text{ at } (-1, 2, 1) = 0 - 4\hat{j} - \hat{k}$$

Hence the desired unit normal to the surface

$$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{4\hat{j} - \hat{k}}{\sqrt{4^2+1^2}} = \frac{12+3}{\sqrt{17}} \text{ or } \frac{15}{\sqrt{17}}$$

10. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

Sol. Angle between the normals to the surface at 2 points is the angle between the surface at those 2 points.

Let $\phi = xy - z^2$

Then $\text{grad } \phi = \hat{i} \cdot \frac{\partial \phi}{\partial x} + \hat{j} \cdot \frac{\partial \phi}{\partial y} + \hat{k} \cdot \frac{\partial \phi}{\partial z}$

$$\Rightarrow \vec{\nabla}\phi = \hat{i}y + \hat{j}x - 2z\hat{k}$$

Let $\vec{n}_1 = \text{grad } \phi$ at the point $(4, 1, 2)$, $\vec{n}_2 = \text{grad } \phi$ at the 2nd point $(3, 3, -3)$.

Then $\vec{n}_1 = \hat{i} + 4\hat{j} - 4\hat{k}$, $\vec{n}_2 = 3\hat{i} + 3\hat{j} + 6\hat{k}$

If ϕ is the angle between these vectors, then

$$\begin{aligned}\cos \theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{1(3) + 4(3) - 4(6)}{\sqrt{1+16+16} \cdot \sqrt{9+9+36}} = -\frac{9}{\sqrt{33} \cdot \sqrt{54}} = -\frac{9}{\sqrt{11} \cdot \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{27}} \\ &= -\frac{9}{\sqrt{22} \cdot \sqrt{81}} \quad \text{or} \quad -\frac{9}{\sqrt{22} \cdot 9} \quad \text{or} \quad -\frac{1}{\sqrt{22}}\end{aligned}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right).$$

11. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$,

show that $\cos \theta = \frac{3}{7\sqrt{6}}$.

Sol. Let, $\phi_1 \equiv xy^2z - 3x - z^2 = 0$
 $\phi_2 \equiv 3x^2 - y^2 + 2z - 1 = 0$

$$\begin{aligned}\text{Then, } \vec{V}\phi_1 &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2) \\ &= (y^2z - 3)\hat{i} + 2xyz\hat{j} + (xy^2 - 2z)\hat{k} = \hat{i} - 4\hat{j} + 2\hat{k} \quad \text{at } (1, -2, 1)\end{aligned}$$

$$\begin{aligned}\text{and } \vec{V}_2 &= \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z}\right) \times (3x^2 - y^2 + 2z - 1) = 6x\hat{i} - 2y\hat{j} + 2\hat{k} \\ &= 6\hat{i} + 4\hat{j} + 2\hat{k} \quad \text{at } (1, -2, 1)\end{aligned}$$

We know that $\vec{V}\phi_1$ and $\vec{V}\phi_2$ are along normals to the given surfaces at any point.

If ϕ is the angle between these vectors then,

$$\begin{aligned}\cos \phi &= \frac{\vec{V}\phi_1 \cdot \vec{V}\phi_2}{|\vec{V}\phi_1| |\vec{V}\phi_2|} \\ \Rightarrow \cos \phi &= \frac{(1)(6) + (-4)(4) + (2)(2)}{\sqrt{1+16+4} \cdot \sqrt{36+16+4}} = -\frac{6}{\sqrt{21} \cdot \sqrt{56}} = -\frac{3}{7\sqrt{6}}\end{aligned}$$

Let θ be the acute angle such that $\theta + \phi = \pi$, then

$$\cos \theta = \frac{3}{7\sqrt{6}}.$$

12. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude?

$$\begin{aligned}\text{Sol. } \vec{V}\phi &= \hat{i} \cdot \frac{\partial \phi}{\partial x} + \hat{j} \cdot \frac{\partial \phi}{\partial y} + \hat{k} \cdot \frac{\partial \phi}{\partial z} \\ \vec{V}\phi &= 2xy^2z^4 \cdot \hat{i} + 2x^2yz^4 \cdot \hat{j} + \hat{k} \cdot 4x^2y^2z^3 \\ \Rightarrow \text{At the point } (3, 1, -2), \\ \vec{V}\phi &= 96\hat{i} + 288\hat{j} - 288\hat{k} = 96(\hat{i} + 3\hat{j} - 3\hat{k}).\end{aligned}$$

Directional derivative of f is maximum in the direction of the normal to give surface *i.e.*, the direction of $\bar{V}\phi = 96(\hat{i} + 3\hat{j} + 3\hat{k})$.

Maximum value of this directional derivative

$$\begin{aligned} &= |\bar{V}\phi| = \sqrt{(96)^2 + (288)^2 + (288)^2} \\ &= 96\sqrt{1 + 9 + 9} = 96\sqrt{19}. \end{aligned}$$

13. Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a + 2)x$, is orthogonal to the surface $4x^2y + z^3 = 4$, at the point $(1, -1, 2)$. [M.D.U., Dec., 2005]

Sol. Since the surfaces are $\phi = ax^2 - byz - (a + z)x = 0$... (1)

and $\psi = 4x^2y + z^3 - 4 = 0$... (2)

At the point $(1, -1, 2)$ let us find the grad ϕ (or $\bar{V}\phi$) and grad ψ (or $\bar{V}\psi$).

$$\therefore \bar{V}\phi = (2ax - a - 2)\hat{i} - bz\hat{j} - by\hat{k}$$

$$\bar{V}\psi = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

$$\Rightarrow \text{At the point } (1, -1, 2), \bar{V}\phi = (a - 2)\hat{i} - 2b\hat{j} + b\hat{k}$$

$$\text{and } \bar{V}\psi = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

Since the point $P(1, -1, 2)$ lies on the surfaces, therefore, equation (1) becomes

$$a + 2b - (a + 2) = 0 \quad \text{or} \quad 2b - 2 = 0 \quad \text{or} \quad b = 1$$

Since the surfaces (1) and (2) are orthogonal

$$\bar{V}\phi \cdot \bar{V}\psi = 0,$$

$$\therefore -8(a - 2) - 8b + 12b = 0,$$

$$-8a + 16 + 4 = 0$$

$$\Rightarrow a = \frac{5}{2}$$

$$\text{Hence, } a = \frac{5}{2}, b = 1.$$

14. Prove that $A\bar{V}\left(B \cdot \bar{V} \cdot \frac{1}{r}\right) = \frac{3(A \cdot R)(B \cdot R)}{r^5} - \frac{A \cdot B}{r^3}$, where A and B are constant vectors.

[M.D.U., May 2006, 2007]

$$\text{Sol. } \bar{A}\bar{V}\left(\bar{B} \cdot \bar{V} \cdot \frac{1}{r}\right) = \frac{3(\bar{A} \cdot \bar{R})(\bar{B} \cdot \bar{R})}{r^5} - \frac{\bar{A} \cdot \bar{B}}{r^3}$$

$$\bar{B} \cdot \bar{V} \cdot \frac{1}{r} = \bar{B} \cdot \bar{V} \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2 + y^2 + z^2}$$

$$= \bar{B} \left[\hat{i} \cdot \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2 + z^2} \right) + \hat{j} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2 + z^2} \right) + \hat{k} \cdot \frac{\partial}{\partial z} \left(\frac{z}{x^2 + y^2 + z^2} \right) \right]$$

$$= \bar{B} \left[\hat{i} \cdot \frac{(x^2 + y^2 + z^2 - 2x^2)}{(x^2 + y^2 + z^2)^2} + \hat{j} \cdot \frac{x^2 + y^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^2} + \hat{k} \cdot \frac{x^2 + y^2 + z^2 - 2z^2}{(x^2 + y^2 + z^2)^2} \right]$$

$\bar{A}$ and $\bar{B}$ are constant vectors.

$$\bar{A} = A(\hat{i} + \hat{j} + \hat{k}); \bar{B} = B(\hat{i} + \hat{j} + \hat{k})$$

$$\begin{aligned}
B \cdot \bar{V} \cdot \frac{1}{r} &= B \cdot \frac{[y^2 + z^2 - x^2 + x^2 + z^2 - y^2 + x^2 + y^2 - z^2]}{(x^2 + y^2 + z^2)^2} = B \cdot \frac{1}{x^2 + y^2 + z^2} \\
\bar{V} \left(B \cdot \bar{V} \cdot \frac{1}{r} \right) &= \bar{B} \left[\hat{i} \cdot \frac{\partial}{\partial x} \left(\frac{1}{x^2 + y^2 + z^2} \right) + \hat{j} \cdot \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2 + z^2} \right) + \hat{k} \cdot \frac{\partial}{\partial z} \left(\frac{1}{x^2 + y^2 + z^2} \right) \right] \\
&= \bar{B} \left[\frac{-2x\hat{i}}{(x^2 + y^2 + z^2)^2} + \frac{(-2y\hat{j})}{(x^2 + y^2 + z^2)^2} + \frac{(-2z\hat{k})}{(x^2 + y^2 + z^2)^2} \right] \\
\text{or } \bar{A} \cdot \bar{V} \left(B \cdot \bar{V} \cdot \frac{1}{r} \right) &= \bar{A} \cdot \bar{B} \left[\frac{-2(x+y+z)}{(x^2 + y^2 + z^2)^2} \right] = (\text{L.H.S.})
\end{aligned}$$

$$\begin{aligned}
\text{Now, R.H.S.} &= \frac{3(\bar{A} \cdot \bar{R})(\bar{B} \cdot \bar{R})}{r^5} - \frac{\bar{A} \cdot \bar{B}}{r^3} \\
&= \frac{3(x+y+z)(x+y+z)}{(x^2 + y^2 + z^2)^{5/2}} - \frac{\bar{A} \cdot \bar{B}}{(x^2 + y^2 + z^2)^{3/2}} = \bar{A} \bar{B} \left[\frac{-2(x+y+z)}{(x^2 + y^2 + z^2)^2} \right] \\
\text{L.H.S.} &= \text{R.H.S.} \quad \text{Hence proved.}
\end{aligned}$$

15. Show that $\bar{V}r^n = nr^{n-2} \bar{r}$, hence evaluate $\bar{V} \frac{1}{r}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. For solution, please see Part (iii) of Q. 3 given earlier where it has been shown that

$$\bar{V}r^n = nr^{n-2} \bar{r}.$$

Further, to evaluate $\bar{V} \frac{1}{r}$, we put $n = -1$.

$$\therefore \bar{V}r^{-1} = (-1)r^{-1-2} \bar{r} = -\frac{\bar{r}}{r^3} \Rightarrow \bar{V} \left(\frac{1}{r} \right) = -\frac{\bar{r}}{r^3}.$$

16. If $\bar{A}$ is a vector function and ϕ is a scalar function, then show that

$$\bar{V} \cdot (\phi \bar{A}) = \phi \cdot \bar{V} \cdot \bar{A} + \bar{A} \cdot \bar{V} \phi. \quad (\text{U.P.T.U., 2006})$$

Sol. We have,

$$\begin{aligned}
V \cdot (\phi \bar{A}) &= \sum i \cdot \frac{\partial}{\partial x} (\phi \bar{A}) \\
&= \sum i \left\{ \phi \cdot \frac{\partial \bar{A}}{\partial x} + \frac{\partial \phi}{\partial x} \cdot \bar{A} \right\} = \phi \sum i \cdot \frac{\partial \bar{A}}{\partial x} + \left(\sum i \frac{\partial \phi}{\partial x} \right) \cdot \bar{A} \\
&= \phi (\bar{V} \cdot \bar{A}) + \bar{V} \phi \cdot \bar{A} = \phi (\bar{V} \cdot \bar{A}) + \bar{A} \cdot \bar{V} \phi. \quad \text{Hence shown.}
\end{aligned}$$

17. Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1, 1, 1)$ in the direction of

$$\text{the line } \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}. \quad (\text{U.P.T.U., 2007})$$

Sol.

$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\therefore \text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\begin{aligned}\bar{V}\phi &= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k} \\ &= \frac{25}{2}\hat{i} - 5\hat{j} \text{ at } (1, 1, 1)\end{aligned}$$

$$\text{Here } \hat{a} \text{ (Unit normal vector)} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{(2)^2 + (2)^2 + (1)^2}} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$$\begin{aligned}\therefore \text{ Directional derivative} &= (\text{grad } \phi) \cdot \hat{a} \\ &= \left(\frac{25}{2}\hat{i} - 5\hat{j}\right) \cdot \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right) = \frac{25}{3} + \frac{10}{3} \quad \text{or} \quad \frac{35}{3}.\end{aligned}$$

18. If r and R have their usual meaning and A is a constant vector, then prove that

$$\bar{V} \times \frac{A \times R}{r^n} = \frac{2-n}{r^n} \cdot A + \frac{n \cdot A \cdot R}{r^{n+2}} \cdot \bar{R} \quad (\text{M.D.U. Dec., 2006 ; U.P.T.U., 2008, A.U.U.P., 2009})$$

$$\begin{aligned}\text{Sol. } \bar{V} \times [r^{-n} (A \times R)] &= r^{-n} [V \times (A \times R)] + V r^{-n} \times (\bar{A} \times \bar{R}) \\ &= r^{-n} [(V \cdot \bar{R}) A - (\bar{A} \cdot \bar{V}) \bar{R}] + (-nr^{-(n+1)} \cdot \bar{R}/r) \times (\bar{A} \times \bar{R}) \\ &= r^{-n} (3A - A) - nr^{-(n+2)} R \times (\bar{A} \times \bar{R}) \quad \left[\begin{array}{l} \because V\bar{R} = 3 \\ (A \cdot \bar{V})\bar{R} = \bar{A} \end{array} \right] \\ &= 2A r^{-n} - n r^{-(n+2)} [(\bar{R} \cdot \bar{R}) \bar{A} - (\bar{A} \cdot \bar{R}) \bar{R}] \\ &= \frac{2A}{r^n} - \frac{n}{r^{n+2}} [r^2 A - (A \cdot R)R] = \frac{(2-n) \cdot \bar{A}}{r^n} + \frac{n(\bar{A} \cdot \bar{R}) \cdot \bar{R}}{r^{n+2}}.\end{aligned}$$

19. Define divergence and curl of a vector function. Hence find these for the vector $\bar{F}(x, y, z) = e^{xyz} \cdot (xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k})$ at the point $(1, 2, 3)$. (U.P.T.U., 2006)

Sol. The divergence of a differentiable vector point function $\bar{V}$ is denoted by ∇ and is defined as

$$\text{div. } \bar{V} = \bar{V} \cdot \bar{V} = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \cdot \bar{V} = \hat{i} \cdot \frac{\partial \bar{V}}{\partial x} + \hat{j} \cdot \frac{\partial \bar{V}}{\partial y} + \hat{k} \cdot \frac{\partial \bar{V}}{\partial z}$$

Obviously, the divergence of a vector point function is a scalar point function.

If $\bar{V} = V_1 \cdot \hat{i} + V_2 \cdot \hat{j} + V_3 \cdot \hat{k}$, then

$$\text{div. } \bar{V} = \bar{V} \cdot \bar{V} = (V_1 \cdot \hat{i} + V_2 \cdot \hat{j} + V_3 \cdot \hat{k}) \cdot \hat{k} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

Curl of a vector Point Function :

The curl (or rotation) of a differentiable vector point function $\bar{V}$ is denoted by $\text{curl } \bar{V}$ and is defined as

$$\text{Curl } \bar{V} = \bar{V} \times \bar{V} = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \times \bar{V} = \hat{i} \times \frac{\partial \bar{V}}{\partial x} + \hat{j} \times \frac{\partial \bar{V}}{\partial y} + \hat{k} \times \frac{\partial \bar{V}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

$$\text{If } \vec{V} = V_1 \cdot \hat{i} + V_2 \cdot \hat{j} + V_3 \cdot \hat{k}$$

$$\text{Then } \text{Curl } \vec{V} = \vec{V} \times \vec{V} = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \times (V_1 \cdot \hat{i} + V_2 \cdot \hat{j} + V_3 \cdot \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j} \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$\vec{F}(x, y, z) = e^{xyz} \cdot (xy^2 \cdot \hat{i} + yz^2 \cdot \hat{j} + zx^2 \cdot \hat{k})$$

$$\begin{aligned} \therefore \text{div. } \vec{F} &= \frac{\partial}{\partial x} (e^{xyz} xy^2) + \frac{\partial}{\partial y} (e^{xyz} yz^2) + \frac{\partial}{\partial z} (e^{xyz} zx^2) \\ &= y^2 [e^{xyz} \cdot 1 + e^{xyz} \cdot yz \cdot x] + z^2 [e^{xyz} \cdot 1 + e^{xyz} \cdot zx \cdot y] + x^2 [e^{xyz} \cdot 1 + e^{xyz} \cdot xy \cdot z] \\ &= e^{xyz} \cdot (x^2 + y^2 + z^2) + e^{xyz} \cdot xyz (x^2 + y^2 + z^2) = e^{xyz} (x^2 + y^2 + z^2) (1 + xyz) \end{aligned}$$

At the point (1, 2, 3)

$$\text{div. } \vec{F} = e^{(1)(2)(3)} [1^2 + 2^2 + 3^2] [1 + 1 \cdot 2 \cdot 3] = 98e^6$$

$$\begin{aligned} \text{Curl } \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} xy^2 & e^{xyz} yz^2 & e^{xyz} zx^2 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} (e^{xyz} \cdot zx^2) - \frac{\partial}{\partial z} (e^{xyz} \cdot yz^2) \right] \hat{i} + \left[\frac{\partial}{\partial z} (e^{xyz} \cdot xy^2) - \frac{\partial}{\partial x} (e^{xyz} \cdot zx^2) \right] \hat{j} \\ &\quad + \left[\frac{\partial}{\partial x} (e^{xyz} \cdot yz^2) - \frac{\partial}{\partial y} (e^{xyz} \cdot xy^2) \right] \hat{k} \\ &= zx^2 [e^{xyz} \cdot xz] \hat{i} - y [e^{xyz} \cdot xyz^2 + e^{xyz} \cdot 2z] \hat{i} + xy^2 \cdot e^{xyz} \cdot xy \hat{j} - z [e^{xyz} \cdot 2x + e^{xyz} \cdot yzx^2] \hat{j} \\ &\quad + e^{xyz} \cdot yz \cdot yz^2 \cdot \hat{k} - x [e^{xyz} \cdot 2y + e^{xyz} \cdot xz \cdot y^2] \hat{k} \\ &= e^{xyz} \cdot [x^3 z^2 - xy^2 z^2 - 2yz] \hat{i} + e^{xyz} [x^2 z^3 - 2xz - x^2 yz^2] \hat{j} + e^{xyz} [y^2 z^3 - 2xy - x^2 y^2 z] \hat{k} \end{aligned}$$

At the point (1, 2, 3),

$$\text{Curl } \vec{F} = e^6 [9 - 36 - 12] \hat{i} + e^6 (8 - 6 - 18) \hat{j} + e^6 (108 - 4 - 12) \hat{k}$$

$$\Rightarrow \text{Curl } \vec{F} = e^6 (-39) \hat{i} + e^6 (-16) \hat{j} + e^6 (92) \hat{k}$$

$$\text{Hence, } \text{Curl } \vec{F} = -39e^6 \cdot \hat{i} - 16e^6 \cdot \hat{j} + 92e^6 \cdot \hat{k}$$

20. Prove $\text{div.} (\text{grad } r^n) = n(n+1)r^{n-2}$, where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. Hence show that $\vec{V}^2 \left(\frac{1}{r} \right) = 0$. Hence

or otherwise evaluate $\vec{V} \times \left(\frac{\vec{r}}{r^2} \right)$. (U.P.T.U., 2005; A.U.U.P., 2009; M.D.U., 2007)

Sol. Here, $r^2 = x^2 + y^2 + z^2$ so that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \quad \text{grad } r^n = nr^{n-1} \cdot \hat{r} = nr^{n-1} \cdot \frac{\vec{r}}{r} = nr^{n-2} \cdot \vec{r}$$

$$\begin{aligned} \therefore \quad \text{div}(\text{grad } r^n) &= \text{div}(nr^{n-2} \cdot \vec{r}) = nr^{n-2} \cdot \text{div} \cdot \vec{r} + \text{grad}(nr^{n-2}) \cdot \vec{r} \\ &= 3nr^{n-2} + n(n-2)r^{n-3} \hat{r} \cdot \vec{r} \quad [\because \text{div} \cdot \vec{r} = 3] \\ &= 3nr^{n-2} + n(n-2)r^{n-4}(\vec{r} \cdot \vec{r}) = 3nr^{n-2} + n(n-2)r^{n-4}(r^2) \\ &= 3nr^{n-2} + n(n-2)r^{n-2} = \vec{\nabla}^2(r^n) = n(n+1)r^{n-2} \end{aligned}$$

Putting $n = -1$, we get

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = 0.$$

Hence shown.

We know that,

$$\text{curl}(u \vec{a}) = u \text{curl} \vec{a} + (\text{grad } u) \times \vec{a}$$

$$\begin{aligned} \therefore \quad \text{curl}\left(\frac{\vec{r}}{r^2}\right) &= \frac{1}{r^2} \text{curl} \vec{r} + \left(\text{grad} \frac{1}{r^2}\right) \times \vec{r} \\ &= \vec{0} - \left(\hat{r} \cdot \frac{2}{r^3}\right) \times \vec{r} = -\frac{2}{r^4}(\vec{r} \times \vec{r}) = \vec{0}. \end{aligned} \quad \left| \because \text{curl} \vec{r} = \vec{0} \right.$$

21. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that $\text{div}(r^n \cdot \vec{r}) = (n+3)r^n$.

(M.D.U., May 2008)

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\begin{aligned} \text{div}(r^n \cdot \vec{r}) &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x\hat{i} + y\hat{j} + z\hat{k})r^n = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \\ &= \left[x \cdot nr^{n-1} \frac{\partial}{\partial x}(r) + r^n\right] + \left[y \cdot nr^{n-1} \frac{\partial}{\partial y}(r) + r^n\right] + \left[z \cdot nr^{n-1} \frac{\partial}{\partial z}(r) + r^n\right] \\ &= 3r^n + \left(x \cdot nr^{n-1} \cdot \frac{x}{r}\right) + \left(y \cdot nr^{n-1} \cdot \frac{y}{r}\right) + \left(z \cdot nr^{n-1} \cdot \frac{z}{r}\right) \\ &= 3r^n + \frac{n \cdot r^{n-1}}{r}(x^2 + y^2 + z^2) = 3r^n + nr^{n-2} \cdot r^2 = 3r^n + nr^n \end{aligned}$$

$$\text{div}(r^n \cdot \vec{r}) = (n+3)r^n.$$

Hence proved.

22. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that (i) $\text{div} \cdot \vec{r} = 3$ and (ii) $\text{curl} \vec{r} = \vec{0}$.

Sol. (i) $\text{div} \cdot \vec{r} = \vec{\nabla} \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

$$\begin{aligned}
 (ii) \quad \text{curl } \vec{r} &= \vec{V} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
 &= \hat{i} \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y) \right] + \hat{j} \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z) \right] + \hat{k} \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right] \\
 &= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}.
 \end{aligned}$$

Hence shown.

- 23.** A vector field is given by $F = (\sin y)I + x(1 + \cos y)J$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$. (M.D.U., Dec., 2006, May 2007)

Sol.
$$\int_C F \, ds = \iint_S \left(\frac{dF_2}{dx} - \frac{dF_1}{dy} \right) dx dy$$

Given $F = \sin y \cdot \hat{i} + x(1 + \cos y) \hat{j}$, $F_1 = \sin y$, $F_2 = x(1 + \cos y)$

$$\frac{dF_1}{dy} = \cos y, \quad \frac{dF_2}{dx} = 1 + \cos y$$

$$\int_C F \, ds = \iint_S (1 + \cos y - \cos y) dx dy = \iint_S dx dy = \text{area of circle} = \pi a^2.$$

- 24.** Find the divergence and curl of the vector $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2, -1, 1)$. (M.D.U., Dec., 2007)

Sol.
$$\text{div. } \vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z) = yz + 3x^2 + 2xz - y^2$$

$\Rightarrow \text{div. } \vec{V} = -1 + 12 + 4 - 1 = 14$; at the point $(2, -1, 1)$.

Now,
$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$$= (-2yz)\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

$\Rightarrow \text{Curl } \vec{V} = 2\hat{i} - 3\hat{j} - 14\hat{k}$, at $(2, -1, 1)$

- 25.** Find the div. $\vec{F}$ and Curl $\vec{F}$ where $\vec{F} = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$.

Sol. Let $\phi = x^3 + y^3 + z^3 - 3xyz$, then

$$\vec{F} = \text{grad } \phi = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (x^3 + y^3 + z^3 - 3xyz)$$

$\Rightarrow \vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + (3z^2 - 3xy)\hat{k}$

$\therefore \text{div. } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$$= 6x + 6y + 6z = 6(x + y + z)$$

$$\text{and } \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

26. Find $\text{Div.}(\text{grad } \phi)$ and $\text{Curl}(\text{grad } \phi)$ as also $\text{div.}(\text{Curl } \vec{V})$ and $\text{Curl}(\text{Curl } \vec{V})$.

Sol. Let $\phi(x, y, z)$ and $\vec{V}(x, y, z)$ be scalar and vector point functions respectively.

Since $\text{grad } \phi$ and $\text{Curl } \vec{V}$ are also vector point functions, we can find their divergence as well as curl, whereas $\text{div. } \vec{V}$ being a scalar point function, we can find its gradient only.

$$1. \text{ Div.}(\text{grad } \phi) = \vec{\nabla}^2 \cdot \phi \quad \text{where } \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\vec{\nabla}^2$ is called the Laplacian operator and $\vec{\nabla}^2 \phi = 0$, is called Laplace's equation.

$$2. \text{ Curl}(\text{grad } \phi) = \vec{V} \times \vec{\nabla} \cdot \phi = \vec{0}$$

$$3. \text{ Div.}(\text{Curl } \vec{V}) = \vec{V} \cdot (\vec{V} \times \vec{V}) = 0$$

$$4. \text{ Curl}(\text{Curl } \vec{V}) = \text{grad div. } \vec{V} - \vec{\nabla}^2 \cdot \vec{V}$$

Note 1. The above results can be written as

$$\text{grad}(\text{div. } \vec{V}) = \text{Curl}(\text{Curl } \vec{V}) + \vec{\nabla}^2 \cdot \vec{V} \quad \text{or} \quad \vec{\nabla}(\vec{V} \cdot \vec{V}) = \vec{V} \times (\vec{V} + \vec{V}) + \vec{\nabla}^2 \cdot \vec{V}$$

Note 2. Treating $\vec{V}$ as a vector, the results of repeated application of $\vec{\nabla}$ can be easily written down.

$$\begin{aligned} \text{Thus } \vec{\nabla} \cdot \vec{V} \phi &= \vec{\nabla}^2 \cdot \phi & (\because \vec{a} \cdot \vec{a} = a^2) \\ \vec{\nabla} \times \vec{V} \phi &= \vec{0} & (\because \vec{a} \times \vec{a} = \vec{0}) \\ \vec{\nabla} \cdot (\vec{V} \times \vec{V}) &= 0 & [\because \vec{a} \cdot (\vec{a} \times \vec{b}) = \vec{a} \cdot \vec{a} \cdot \vec{b} = 0] \\ \vec{\nabla} \times (\vec{V} \times \vec{V}) &= V(V \cdot \vec{V}) - V^2 \cdot \vec{V} \end{aligned}$$

(by expanding as a vector triple product)

27. Show that $\text{curl}(\text{curl } \vec{V}) = \text{grad div } \vec{V} - \vec{\nabla}^2 \vec{V}$.

(M.D.U., May 2008)

Sol. Let $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\text{Then, } \text{Curl } \vec{V} = \vec{V} \times \vec{\nabla} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j} \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$\text{Curl}(\text{Curl } \vec{V}) = \vec{V} \times (\vec{V} \times \vec{\nabla}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$$

$$\begin{aligned}
&= \sum \hat{i} \left\{ \frac{\partial}{\partial y} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \right\} \\
&= \sum \hat{i} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) - \left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \right\} \\
&= \sum \hat{i} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \right\} \\
&= \sum \hat{i} \left\{ \frac{\partial}{\partial x} (\vec{V} \cdot \vec{V}) - (\vec{V}^2 \cdot V_1) \right\} = \sum \hat{i} \frac{\partial}{\partial x} (\vec{V} \cdot \vec{V}) - \vec{V}^2 \sum \hat{i} V_1 \\
&= \vec{V}(\vec{V} \cdot \vec{V}) - \vec{V}^2 \vec{V} \\
&= \text{grad}(\text{div } \vec{V}) - \vec{V}^2 \cdot \vec{V} = \text{R.H.S.} \quad \text{Hence proved.}
\end{aligned}$$

28. A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Show that the field is irrotational.

Sol. Field $\vec{A}$ is irrotational if $\text{Curl } \vec{A} = \vec{0}$.

$$\begin{aligned}
\text{Now, } \text{Curl } \vec{A} &= \vec{V} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix} \\
\therefore \text{Curl } \vec{A} &= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(2xy-2xy) = \vec{0}
\end{aligned}$$

$\therefore$ Field $\vec{A}$ is irrotational.

Hence shown.

29. Describe the physical interpretation of Divergence and Curl. (M.D.U. 2007, U.P.T.U., 2009)

Sol. (A) $\text{Div. } \vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

If $\text{div. } \vec{V} = 0$ everywhere in some region R of space, then $\vec{V}$ is called *solenoidal vector* point function.

(B) Angular velocity at any point of a rigid body (rotating about a fixed axis) is equal to half the curl of the linear velocity at that point of the body.

If $\text{Curl } \vec{V} = \vec{0}$ (zero vector), then $\vec{V}$ is said to be an irrotational vector, otherwise rotational.

Some properties of divergence and curl are given in earlier two questions viz. Q. No. 26 and 28.

30. If the vector $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ is solenoidal, find the value of a . Find also the curl of this solenoidal vector.

Sol. Here, $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\begin{aligned}
\text{div. } \vec{F} &= \vec{V} \cdot \vec{F} \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left[(ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k} \right] \\
&= \frac{\partial}{\partial x} (ax^2y + yz) + \frac{\partial}{\partial y} (xy^2 - xz^2) + \frac{\partial}{\partial z} (2xyz - 2x^2y^2)
\end{aligned}$$

$$\Rightarrow \operatorname{div.} \vec{F} = 2axy + 2xy + 2xy = 2(a+2)xy$$

Since $\vec{F}$ is solenoidal, $\operatorname{div.} \vec{F} = 0$

$$\Rightarrow 2(a+2)xy = 0 \quad \therefore a = -2$$

Now,
$$\vec{F} = (-2x^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$$

$$\therefore \operatorname{Curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2y + yz & x(y^2 - z^2) & 2xy(z - xy) \end{vmatrix}$$

$$\Rightarrow \operatorname{Curl} \vec{F} = \hat{i}(2xz - 4x^2y + 2xz) - \hat{j}(2yz - 4xy^2 - y) + \hat{k}(y^2 - z^2 + 2x^2 - z)$$

$$\operatorname{Curl} \vec{F} = 4x(z - xy)\hat{i} + (y + 4xy^2 - 2yz)\hat{j} + (2x^2 + y^2 - z^2 - z)\hat{k}.$$

31. Prove that : (i) $\operatorname{div.} \left(\frac{\vec{r}}{r^3} \right) = 0$ (ii) $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. Here, $r^2 = x^2 + y^2 + z^2$ so that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\operatorname{grad} r = \hat{i} \cdot \frac{\partial r}{\partial x} + \hat{j} \cdot \frac{\partial r}{\partial y} + \hat{k} \cdot \frac{\partial r}{\partial z} = \frac{\vec{r}}{r}$$

(i) Since $\operatorname{div.}(\phi \vec{A}) = \phi(\operatorname{div.} \vec{A}) + \vec{A} \cdot \operatorname{grad} \phi$

$$\begin{aligned} \therefore \operatorname{div.} \left(\frac{\vec{r}}{r^3} \right) &= \operatorname{div.}(r^{-3} \vec{r}) = r^{-3}(\operatorname{div.} \vec{r}) + \vec{r} \cdot \operatorname{grad} r^{-3} \\ &= 3r^{-3} + \vec{r} \cdot (-3r^{-4} \operatorname{grad} r) \quad [\because \operatorname{div.} \vec{r} = 3] \\ &= 3r^{-3} + \vec{r} \cdot \left[-3r^{-4} \cdot \frac{\vec{r}}{r} \right] = 3r^{-3} - 3r^{-5}(\vec{r} \cdot \vec{r}) = 3r^{-3} - 3r^{-5}(r^2) = 0 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \nabla^2(r^n) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] r^n = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x} \left(\frac{\partial r^n}{\partial x} \right) \\ &= \sum \frac{\partial}{\partial x} \left(nr^{n-1} \cdot \frac{\partial r}{\partial x} \right) = \sum \frac{\partial}{\partial x} \left(nr^{n-1} \cdot \frac{x}{r} \right) \quad [\because r^2 = x^2 + y^2 + z^2] \\ &= \sum n \cdot \frac{\partial}{\partial x} (r^{n-2} \cdot x) = n \sum \left[(n-2)r^{n-3} \cdot \frac{\partial r}{\partial x} \cdot x + r^{n-2} \right] \\ &= n \sum \left[(n-2)r^{n-3} \cdot \frac{x}{r} \cdot x + r^{n-2} \right] = n \sum \left[(n-2)r^{n-4} \cdot x^2 + r^{n-2} \right] \end{aligned}$$

$$\begin{aligned}
&= n(n-2)r^{n-4}(x^2+y^2+z^2-y^2-z^2)+r^{n-2} \cdot n \\
&= n[(n-2)r^{n-4} \cdot (r^2-y^2-z^2)+r^{n-2}] = n[(n-2)r^{n-2}+3r^{n-2}]
\end{aligned}$$

$$\Rightarrow \bar{V}^2(r^n) = n(n+1)r^{n-2}$$

32. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that : $\bar{V}^2 \left[\bar{V} \left(\frac{\vec{r}}{r^2} \right) \right] = 2r^{-4}$. (M.D.U., Dec., 2007)

Sol.

$$\begin{aligned}
\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\
|r^2| &= x^2 + y^2 + z^2 \\
\frac{\vec{r}}{r^2} &= \frac{x\hat{i}}{r^2} + \frac{y\hat{j}}{r^2} + \frac{z\hat{k}}{r^2} \\
\bar{V} \left(\frac{\vec{r}}{r^2} \right) &= \frac{\partial}{\partial x} \cdot \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^2} \right) \\
&= \left(\frac{r^2 - x \cdot 2r \cdot \frac{x}{r}}{r^4} \right) + \left(\frac{r^2 - y \cdot 2r \cdot \frac{y}{r}}{r^4} \right) + \left(\frac{r^2 - z \cdot 2r \cdot \frac{z}{r}}{r^4} \right) \\
&= \frac{r^2 - 2x^2 + r^2 - 2y^2 + r^2 - 2z^2}{r^4} = \frac{3r^2 - 2r^2}{r^4} = \frac{1}{r^2}
\end{aligned}$$

Now, $\bar{V}^2 \left[\bar{V} \left(\frac{\vec{r}}{r^2} \right) \right] = \bar{V}^2 \left(\frac{1}{r^2} \right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{1}{r^2} \right)$

$$\begin{aligned}
&\Rightarrow \frac{\partial^2}{\partial x^2} \cdot \frac{1}{r^2} + \frac{\partial^2}{\partial y^2} \cdot \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \cdot \frac{1}{r^2} \\
&\Rightarrow \bar{V}^2 \left[\frac{1}{r^2} \right] = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left[\frac{\partial}{\partial x} \cdot \frac{1}{r^2} + \frac{\partial}{\partial y} \cdot \frac{1}{r^2} + \frac{\partial}{\partial z} \cdot \frac{1}{r^2} \right] \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left[\frac{-2x}{r^4} - \frac{2y}{r^4} - \frac{2z}{r^4} \right] \\
&= \left(\frac{-2r^4 + 8x^2r^2}{\partial^8} \right) + \left(\frac{-2r^4 + 8y^2r^2}{\partial^8} \right) + \left(\frac{-2r^4 + 8z^2r^2}{\partial^8} \right) \\
\therefore \bar{V}^2 \left[\bar{V} \left(\frac{\vec{r}}{r^2} \right) \right] &= \frac{-6r^4 + 8r^2(x^2 + y^2 + z^2)}{\partial^8} = \frac{-6r^4 + 8r^4}{\partial^8} = \frac{2r^4}{r^8} = 2r^{-4}.
\end{aligned}$$

33. Prove that the vector $f(r)\vec{r}$ is irrotational.

Sol. The vector $f(r)\vec{r}$ will be irrotational if $\text{curl} [f(r) \cdot \vec{r}] = \vec{0}$.

Since $\text{Curl} (\phi \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{Curl } \vec{A}$

$$\begin{aligned}
\therefore \text{Curl} [f(r) \cdot \vec{r}] &= [\text{grad } f(r)] \times \vec{r} + f(r) \cdot \text{Curl } \vec{r} \\
&= [f'(r) \text{grad } r] \times \vec{r} + f(r) \cdot \vec{0} \quad \{\because \text{Curl } \vec{r} = \vec{0}\}
\end{aligned}$$

$$\Rightarrow \quad \text{Curl} [f(r) \cdot \vec{r}] = \left[f'(r) \cdot \frac{\vec{r}}{r} \right] \times \vec{r} = \frac{f'(r)}{r} (\vec{r} \times \vec{r}) = \vec{0}, \quad \text{since } \vec{r} \times \vec{r} = \vec{0}$$

$\therefore$ The vector $f(r) \vec{r}$ is irrotational.

Hence proved.

34. Prove that : $\vec{\nabla}^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

[M.D.U., Dec., 2005]

Sol. $\vec{\nabla}^2 f(r) = \vec{\nabla} \cdot [\vec{\nabla} \cdot f(r)] = \text{div.} \{ \text{grad } f(r) \}$

$$\begin{aligned} &= \text{div.} \{ f'(r) \text{ grad } r \} = \text{div.} \left\{ f'(r) \cdot \frac{\vec{r}}{r} \right\} = \text{div.} \left\{ \frac{1}{r} f'(r) \cdot \vec{r} \right\} \\ &= \frac{1}{r} f'(r) \text{div } \vec{r} + \vec{r} \cdot \text{grad} \left\{ \frac{1}{r} f'(r) \right\} = \frac{3}{r} f'(r) + \vec{r} \left[\frac{d}{dr} \left(\frac{1}{r} f'(r) \text{grad } r \right) \right] \\ &= \frac{3}{r} f'(r) + \vec{r} \cdot \left[\left\{ -\frac{1}{r^3} f'(r) + \frac{1}{r} f''(r) \right\} \frac{\vec{r}}{r} \right] = \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r) \right] (\vec{r} \cdot \vec{r}) \\ &= \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r) \right] r^2 = \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r). \end{aligned}$$

Hence proved.

35. If $u \vec{F} = \vec{\nabla} v$, where u, v are scalar fields and $\vec{F}$ is a vector field, show that $\vec{F} \cdot \text{Curl } \vec{F} = 0$.

Sol. $\text{Curl } \vec{F} = \vec{\nabla} \times \left(\frac{1}{u} \vec{\nabla} v \right)$

$$= \vec{\nabla} \frac{1}{u} \vec{\nabla} v + \frac{1}{u} \vec{\nabla} \times (\vec{\nabla} v) = \vec{\nabla} \frac{1}{u} \times \vec{\nabla} v \quad (\because \vec{\nabla} \times \vec{\nabla} v = \vec{0})$$

$$\therefore \vec{F} \cdot \text{Curl } \vec{F} = \frac{1}{u} \vec{\nabla} v \cdot \left(\vec{\nabla} \frac{1}{u} \times \vec{\nabla} v \right) = 0$$

being the scalar triple product in which two factors are equal.

36. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that grad u , grad v and grad w are coplanar vectors.

Sol.

$$\text{grad } u = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\begin{aligned} \text{grad } w &= \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (yz + zx + xy) \\ &= \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x) \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \text{grad } u \cdot (\text{grad } v \times \text{grad } w) &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix} \\
 &= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+z+y & y+z+x & z+y+x \\ z+y & z+x & y+x \end{vmatrix} \quad [\text{Applying } R_2 \rightarrow R_2 + R_3] \\
 &= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} \\
 &= 0.
 \end{aligned}$$

Hence $\text{grad } u$, $\text{grad } v$ and $\text{grad } w$ are Coplanar vectors.

37. Show that $\bar{V}(\vec{a}, \vec{r}) = \vec{a}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is a constant vector.

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are constants.

$$\vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\therefore \bar{V}(\vec{a}, \vec{r}) = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (a_1x + a_2y + a_3z) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = \vec{a}.$$

38. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Sol. Let $\phi = x^2y + 2xz - 4$

$$\therefore \text{grad } \phi = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) (x^2y + 2xz - 4) = (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

$$\text{At } (2, -2, 3), \quad \text{grad } \phi = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|\text{grad } \phi| = \sqrt{4 + 16 + 16} = 6.$$

$$\therefore \text{Unit Normal Vector} = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}.$$

39. A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Show that the field is irrotational and find the scalar potential.

Sol. Field $\vec{A}$ is irrotational if $\text{curl } \vec{A} = \vec{0}$

$$\text{Now, } \text{Curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(2xy - 2xy) = \vec{0}$$

$\therefore$ Field $\vec{A}$ is irrotational.

If ϕ is the scalar potential then, $\vec{A} = \text{grad } \phi$

$$\begin{aligned}
 \Rightarrow \vec{A} \cdot d\vec{r} &= (\text{grad } \phi) \cdot d\vec{r} = \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
 &= \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz = d\phi
 \end{aligned}$$

$$\begin{aligned}
 \therefore d\phi &= \vec{A} \cdot d\vec{r} = (x^2 + xy^2)dx + (y^2 + x^2y)dy = x^2dx + y^2dy + xy(ydx + xdy) \\
 &= d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + xy d(xy) = d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2y^2}{2}\right) \\
 \therefore \phi &= \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c,
 \end{aligned}$$

where c is the constant of integration.

40. A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$
 (i) Is this motion irrotational? If so, find the velocity potential.
 (ii) Is the motion possible for an incompressible fluid?

Sol. We have $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

(i) The motion is irrotational if $\text{curl } \vec{V} = \vec{0}$.

$$\begin{aligned}
 \therefore \text{Curl } \vec{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = \hat{i} \left\{ \frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x) \right\} \\
 &\quad - \hat{j} \left\{ \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z) \right\} + \hat{k} \left\{ \frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z) \right\}
 \end{aligned}$$

$$\Rightarrow \text{Curl } \vec{V} = \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1) = \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \vec{0}$$

Hence the motion is *irrotational*.

Now, let $\vec{V} = \text{grad } \phi$, where ϕ is velocity potential.

$$\begin{aligned}
 \Rightarrow \vec{V} \cdot d\vec{r} &= \vec{V} \cdot \vec{\nabla} \phi \\
 &= \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k}) \\
 &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi
 \end{aligned}$$

$$\begin{aligned}
 \therefore d\phi &= \vec{V} \cdot d\vec{r} \\
 &= \{(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\} \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
 &= (y+z)dx + (z+x)dy + (x+y)dz \\
 &= (ydx + xdy) + (zdx + xdz) + (zdy + ydz) \\
 &= d(xy) + d(zx) + d(yz)
 \end{aligned}$$

Integrating, we get $\phi = xy + yz + zx + c$

(ii) For an incompressible fluid,

$$\text{div. } \vec{V} = 0$$

$$\begin{aligned}
 \text{Now, } \text{div. } \vec{V} &= \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \cdot \{(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\} \\
 &= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y) = 0 + 0 + 0 = 0
 \end{aligned}$$

Hence, the motion is possible for an incompressible fluid.

41. If $\vec{a}$ is a constant vector, evaluate $\text{div.}(\vec{r} \times \vec{a})$ and $\text{curl}(\vec{r} \times \vec{a})$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. We know that $\text{div.}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl} \vec{a} - \vec{a} \cdot \text{curl} \vec{b}$
 $\therefore \text{div.}(\vec{r} \times \vec{a}) = \vec{a} \cdot \text{curl} \vec{r} - \vec{r} \cdot \text{curl} \vec{a}$
 $= \vec{a} \cdot \vec{0} - \vec{r} \cdot \vec{0} = \vec{0} - \vec{0} = \vec{0}$

[$\because \text{Curl } \vec{r} = \vec{0}$ and curl of a constant vector is $\vec{0}$]

Also, we know that

$\text{curl}(\vec{a} \times \vec{b}) = \vec{a} \cdot \text{div.} \vec{b} - \vec{b} \cdot \text{div.} \vec{a} + (\vec{b} \cdot \vec{\nabla}) \vec{a} - (\vec{a} \cdot \vec{\nabla}) \vec{b}$
 $\therefore \text{curl}(\vec{r} \times \vec{a}) = \vec{r} \cdot \text{div.} \vec{a} - \vec{a} \cdot \text{div.} \vec{r} + (\vec{a} \cdot \vec{\nabla}) \vec{r} - (\vec{r} \cdot \vec{\nabla}) \vec{a}$
 $\text{curl}(\vec{r} \times \vec{a}) = \vec{0} - 3\vec{a} + \vec{a} - \vec{0}$
 $= -2\vec{a}$

[$\because \text{div.} \vec{a} = 0$, $(\vec{r} \cdot \vec{\nabla}) \vec{a} = \vec{0}$ and $(\vec{a} \cdot \vec{\nabla}) \vec{r} = \vec{a}$]

42. Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

Sol. For the vector field $\vec{F}$ to be irrotational, $\text{curl } \vec{F} = 0$.

We know that, $\text{curl}(u\vec{a}) = u \text{curl } \vec{a} + (\text{grad } u) \times \vec{a}$

$\therefore \text{curl}\left(\frac{1}{r^3} \cdot \vec{r}\right) = \frac{1}{r^3} \text{curl } \vec{r} + \left(\text{grad } \frac{1}{r^3}\right) \times \vec{r}$
 $= \frac{1}{r^3} \vec{0} + \left(-\frac{3}{r^4} \cdot \hat{r}\right) \times \vec{r}$ [$\because \text{Curl } \vec{r} = \vec{0}$]
 $= \vec{0} - \frac{3}{r^5} (\vec{r} \times \vec{r}) = \vec{0} - \vec{0} = \vec{0}$

Hence, vector field $\vec{F}$ is irrotational.

Again, for the vector field $\vec{F}$ to be solenoidal, $\text{div.} \vec{F} = 0$.

We know that $\text{div.}(u\vec{a}) = u \text{div.} \vec{a} + \vec{a} \cdot \text{grad } u$

$\therefore \text{div.}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3} \text{div.} \vec{r} + \vec{r} \cdot \text{grad}\left(\frac{1}{r^3}\right)$
 $= \frac{3}{r^3} + \vec{r} \cdot \left(-\frac{3}{r^4} \cdot \frac{\vec{r}}{r}\right)$ [$\because \text{div.} \vec{r} = 3$]
 $= \frac{3}{r^3} - \frac{3}{r^5} \cdot r^2$ [$\because \vec{r} \cdot \vec{r} = r^2$]
 $= \frac{3}{r^3} - \frac{3}{r^3} = 0$

Hence, vector field $\vec{F}$ is solenoidal.

Now, let $\vec{F} = \bar{V}\phi$ where ϕ is a scalar potential.

To find ϕ we proceed as follows :

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= \bar{V}\phi \cdot d\vec{r} \\ \vec{F} \cdot d\vec{r} &= d\phi \quad [\because \bar{V}\phi \cdot d\vec{r} = d\phi]\end{aligned}$$

$$\begin{aligned}\therefore d\phi &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}} \cdot (dx \cdot \hat{i} + dy \cdot \hat{j} + dz \cdot \hat{k}) \\ &= \frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{3/2}} = d \left\{ -\frac{1}{(x^2 + y^2 + z^2)^{1/2}} \right\} \\ &[\because d\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]\end{aligned}$$

Integrating, we get $\phi = -\frac{1}{\sqrt{x^2 + y^2 + z^2}} + c$ or $\phi = -\frac{1}{r} + c$

where c is the constant of integration.

43. Find $\iint_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ and S is the surface of sphere having centre $(3, -1, 2)$ and radius 3. (U.P.T.U., 2006)

Sol. Let V be the volume enclosed by the surface S . Then by Gauss Divergence theorem, we have

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iiint_V \text{div} \cdot \vec{F} dV \\ &= \iiint_V \left[\frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(-xz - y) + \frac{\partial}{\partial z}(y^2 + 2z) \right] dV \\ &= \iiint_V (2 - 1 + 2) dV = 3 \iiint_V dV = 3V\end{aligned}$$

But V is the volume of a sphere of radius 3.

$$\therefore V = \frac{4}{3} \pi (3)^3 = 36\pi$$

Hence, $\iint_S \vec{F} \cdot d\vec{S} = 3 \times 36\pi = 108\pi$.

44. Verify divergence theorem for $F = (x^2 - yz)I + (y^2 - zx)J + (z^2 - xy)K$ taken over the rectangular parallelepiped $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

(M.D.U., May 2006 and Dec., 2006)

Sol. For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now, $\text{div} \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2(x + y + z)$

$$\begin{aligned}\iiint_V \text{div} \vec{F} dv &= \int_0^c \int_0^b \int_0^a 2(x + y + z) dz dy dx = \int_0^c \int_0^b 2 \left[\frac{x^2}{2} + yx + zx \right]_0^a dy dz \\ &= 2 \int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az \right) dy dz = 2 \int_0^c \left(\frac{a^2}{2} y + \frac{ay^2}{2} + azy \right)_0^b dz\end{aligned}$$

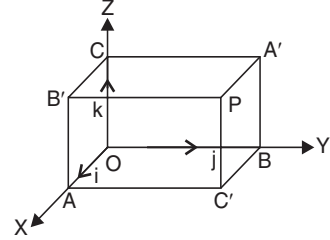
$$\begin{aligned}
&= 2 \int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right) dz = 2 \left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2} \right]_0^c \\
&= a^2 bc + ab^2 c + abc^2 \\
&= abc(a + b + c) \quad \dots(1)
\end{aligned}$$

To evaluate the surface integral, divide the closed "surface S" of the rectangular parallelopiped into 6 parts.

S_1 = the face of $OAC'B$, S_2 = face $CB'PA'$

S_3 = the face $OBA'C$, S_4 = face $AC'PB'$

S_5 = the face $OCB'A$, S_6 = face $BA'PC'$



$$\begin{aligned}
\text{Also, } \iint_S \vec{F} \cdot \hat{n} \, ds &= \iint_{S_1} \vec{F} \cdot \hat{n} \, ds + \iint_{S_2} \vec{F} \cdot \hat{n} \, ds + \iint_{S_3} \vec{F} \cdot \hat{n} \, ds \\
&\quad + \iint_{S_4} \vec{F} \cdot \hat{n} \, ds + \iint_{S_5} \vec{F} \cdot \hat{n} \, ds + \iint_{S_6} \vec{F} \cdot \hat{n} \, ds
\end{aligned}$$

On S_1 ($z = 0$), we have $\hat{n} = -\hat{k}$,

$$\begin{aligned}
\vec{F} &= x^2 \hat{i} + y^2 \hat{j} - xy \hat{k} \\
\vec{F} \cdot \hat{n} &= (x^2 \hat{i} + y^2 \hat{j} - xy \hat{k}) \cdot (-\hat{k}) = xy
\end{aligned}$$

$$\text{so that, } \iint_{S_1} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a xy \, dx \, dy = \int_0^b \left[y \frac{x^2}{2} \right]_0^a dy = \frac{a^2}{2} \left[\frac{y^2}{2} \right]_0^b = \frac{a^2 b^2}{4}$$

On S_2 ($z = c$), we have

$$\begin{aligned}
\hat{n} &= \hat{k} \\
\vec{F} &= (x^2 - cy) \hat{i} + (y^2 - cx) \hat{j} + (c^2 - xy) \hat{k}
\end{aligned}$$

so that $\vec{F} \cdot \hat{n} = c^2 - xy$

$$\begin{aligned}
\iint_{S_2} \vec{F} \cdot \hat{n} \, ds &= \int_0^b \int_0^a (c^2 - xy) \, dx \, dy = \int_0^b \left(c^2 x - \frac{x^2}{2} y \right) \Big|_0^a dy = \int_0^b \left(c^2 a - \frac{a^2}{2} y \right) dy \\
&= \left[c^2 ay - \frac{a^2}{2} \cdot \frac{y^2}{2} \right]_0^b = c^2 ab - \frac{a^2 b^2}{4}
\end{aligned}$$

On S_3 ($x = 0$), we have

$$\hat{n} = -\hat{i}, \vec{F} = -yz \hat{i} + y^2 \hat{j} + z^2 \hat{k}$$

so that $\vec{F} \cdot \hat{n} = yz$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^b yz \, dy \, dz = \int_0^c \frac{b^2}{2} z \, dz = \frac{b^2 c^2}{4}$$

On S_4 ($x = a$), we have

$$\hat{n} = \hat{i}$$

$$\vec{F} = (a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}$$

so that

$$\vec{F} \cdot \hat{n} = a^2 - yz$$

$$\therefore \iint_{S_4} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^b (a^2 - yz) \, dy \, dz = \int_0^c \left(a^2 b - \frac{b^2}{2} z \right) dz = a^2 bc - \frac{b^2 c^2}{4}$$

On S_5 ($y = 0$), we have

$$\hat{n} = -\hat{j}$$

$$F = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$$

so that

$$\vec{F} \cdot \hat{n} = zx$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^c zx \, dx \, dz = \int_0^a \frac{c^2}{2} x \, dx = \frac{a^2 c^2}{4}$$

On S_6 ($y = b$), we have

$$\hat{n} = \hat{j}, \vec{F} = (x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}$$

$$\vec{F} \cdot \hat{n} = b^2 - zx$$

$$\begin{aligned} \iint_{S_6} \vec{F} \cdot \hat{n} \, ds &= \int_0^a \int_0^c (b^2 - zx) \, dx \, dz \\ &= \int_0^a \left[b^2 z - \frac{z^2}{2} x \right]_0^c dz = \int_0^a \left(b^2 c - \frac{c^2}{2} x \right) dx = \left[b^2 cx - \frac{c^2}{2} \cdot \frac{x^2}{2} \right]_0^a = ab^2 c - \frac{a^2 c^2}{4} \end{aligned}$$

Hence surface integral value is sum of all the integrals over S_1, S_2, S_3, S_4, S_5 and S_6

$$\begin{aligned} \text{i.e., } \frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4} \\ = abc^2 + a^2 bc + ab^2 c = abc(a + b + c) \end{aligned}$$

which is same as the volume integral value calculated earlier.

Hence divergence theorem is verified for the given function.

45. Evaluate $\iint_S \vec{A} \cdot \hat{n} \, ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. (M.D.U., Dec., 2007)

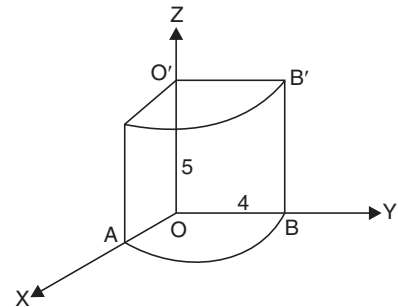
Sol. A vector normal to the surface S is given by

$$\vec{\nabla}(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

and $\hat{n}$ (a unit vector normal to surface S)

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

($\because$ on the surface of the cylinder, $x^2 + y^2 = 16$)



Let R be the projection of S on the yz plane, then

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dydz}{|\hat{i} \cdot \hat{n}|}$$

The region R is OBO' B enclosed by $y = 0$ to $y = 4$ and $z = 0$ to $z = 5$

Now,
$$\hat{i} \cdot \hat{n} = \hat{i} \cdot \left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j} \right) = \frac{1}{4}x$$

$$\vec{A} \cdot \hat{n} = (z\hat{i} + x\hat{j} - 3y^2 \cdot z \cdot \hat{k}) \cdot \left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j} \right) = \frac{1}{4}zx + \frac{1}{4}xy = \frac{1}{4}x(y+z)$$

Hence
$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} \, dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dydz}{|\hat{i} \cdot \hat{n}|} = \int_0^5 \int_0^4 (y+z) \, dy \, dz = \int_0^5 \left(\frac{y^2}{2} + zy \right)_0^4 \, dz \\ &= \int_0^5 (8 + 4z) \, dz = \left(8z + 2z^2 \right)_0^5 = 40 + 50 = 90. \end{aligned}$$

46. A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral over the circular path given by $x^2 + y^2 = a^2$, $z = 0$. (M.D.U., May 2007)

Sol. The parametric equations of the circular path are $x = a \cos t$, $y = a \sin t$, $z = 0$ where t varies from 0 to 2π .

Since the particle moves in the xy -plane ($z = 0$), we can take $\vec{r} = x\hat{i} + y\hat{j}$ so that

$$\begin{aligned} d\vec{r} &= dx\hat{i} + dy\hat{j} \\ \therefore \oint_C \vec{F} \cdot d\vec{r} &= \oint_C [\sin y\hat{i} + x(1 + \cos y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j}) \\ &= \oint_C [\sin y \, dx + x(1 + \cos y) \, dy] = \oint_C [(\sin y \, dx + x \cos y \, dy) + x \, dy] \\ &= \oint_C d(x \sin y) + \oint_C x \, dy = \int_0^{2\pi} d[a \cos t \sin(a \sin t)] \, dt + \int_0^{2\pi} a \cos t \cdot a \cos t \cdot dt \\ &= [a \cos t \cdot \sin(a \sin t)]_0^{2\pi} + a^2 \int_0^{2\pi} \cos^2 t \, dt = \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2t) \, dt = \frac{a^2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi} \end{aligned}$$

$$\therefore \text{Line integral over the circular path is given by } \frac{a^2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi} = \frac{a^2}{2} (2\pi) = \pi a^2.$$

47. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol. Work done
$$\begin{aligned} &= \int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C \{3x^2 dx + (2xz - y)dy + z dz\} \end{aligned} \quad \dots(1)$$

Let $x = t$, then $C : x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$

So that $dx = dt, dy = \frac{t}{2} dt, dz = \frac{9t^2}{8} dt$

From $x = 0$ to $x = 2$, t varies from 0 to 2.

$\therefore$ From (1) we have

$$\begin{aligned} \text{Work done} &= \int_0^2 \left[3t^2 dt + \left\{ 2t \cdot \frac{3t^3}{8} - \frac{t^2}{4} \right\} \left(\frac{t}{2} dt \right) + \frac{3t^3}{8} \left(\frac{9t^2}{8} dt \right) \right] \\ &= \int_0^2 \left(3t^2 + \frac{3}{8}t^5 - \frac{1}{8}t^3 + \frac{27}{64}t^5 \right) dt = \int_0^2 \left(\frac{51}{64}t^5 - \frac{1}{8}t^3 + 3t^2 \right) dt \\ &= \left[\frac{51}{64} \cdot \frac{t^6}{6} - \frac{t^4}{32} + t^3 \right]_0^2 = \frac{17}{2} - \frac{1}{2} + 8 = 16. \end{aligned}$$

48. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \vec{F} dV$, where V is bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. (M.D.U., May 2008)

Sol. $\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x$

$$\begin{aligned} \therefore \iiint_V \vec{\nabla} \cdot \vec{F} dV &= \iiint_V 2x dx dy dz \\ &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x dz dy dx = \int_0^2 \int_0^{2-x} 2x [z]_0^{4-2x-2y} dy dx \\ &= \int_0^2 \int_0^{2-x} 2x (4 - 2x - 2y) dy dx = \int_0^2 \int_0^{2-x} [4x(2-x) - 4xy] dy dx \\ &= \int_0^2 \left[4x(2-x)y - 2xy^2 \right]_0^{2-x} dx = \int_0^2 4x[(2-x)^2 - 2x(2-x)] dx \\ &= \int_0^2 2x(2-x)^2 dx = 2 \int_0^2 x(2-x)^2 dx = 2 \int_0^2 (4x - 4x^2 + 4x^3) dx \\ &= 2 \left[2x^2 - 4 \frac{x^3}{3} + \frac{x^4}{4} \right]_0^2 = 2 \left(8 - \frac{32}{3} + 4 \right) = \frac{8}{3}. \end{aligned}$$

49. Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) d\vec{S}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant. (U.P.T.U., 2005)

Sol. $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) d\vec{S} = \iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot (dy dz \hat{i} + dz dx \hat{j} + dx dy \hat{k})$

$$\begin{aligned}
&= \iint_S (yz \, dy \, dz + zx \, dz \, dx + xy \, dx \, dy) \\
&= \int_0^a \int_0^{\sqrt{a^2 - z^2}} yz \, dy \, dz + \int_0^a \int_0^{\sqrt{a^2 - x^2}} zx \, dz \, dx + \int_0^a \int_0^{\sqrt{a^2 - y^2}} xy \, dx \, dy \\
&= \int_0^a z \cdot \left(\frac{y^2}{2} \right)_0^{\sqrt{a^2 - z^2}} \cdot dz + \int_0^a x \cdot \left(\frac{z^2}{2} \right)_0^{\sqrt{a^2 - x^2}} \cdot dx + \int_0^a y \cdot \left(\frac{x^2}{2} \right)_0^{\sqrt{a^2 - y^2}} \cdot dy \\
&= \frac{1}{2} \int_0^a z (a^2 - z^2) \, dz + \frac{1}{2} \int_0^a x (a^2 - x^2) \, dx + \frac{1}{2} \int_0^a y (a^2 - y^2) \, dy \\
&= \frac{1}{2} \left(\frac{a^2 z^2}{2} - \frac{z^4}{4} \right)_0^a + \frac{1}{2} \left(\frac{a^2 x^2}{2} - \frac{x^4}{4} \right)_0^a + \frac{1}{2} \left(\frac{a^2 y^2}{2} - \frac{y^4}{4} \right)_0^a \\
&= \frac{1}{2} \cdot \frac{a^4}{4} + \frac{1}{2} \cdot \frac{a^4}{4} + \frac{1}{2} \cdot \frac{a^4}{4} = \frac{3a^4}{8}
\end{aligned}$$

50. Use divergence theorem to show that $\oint_S \bar{V} r^2 \cdot d\bar{S} = 6V$, where S is any closed surface enclosing a volume V .

Sol. By the divergence theorem, we have

$$\begin{aligned}
\oint_S \bar{V} r^2 \cdot d\bar{S} &= \int_V \text{div.} (\bar{V} r^2) \, dV \\
&= \int_V \bar{V} \cdot (\bar{V} r^2) \, dV = \int_V \bar{V}^2 \cdot r^2 \, d\bar{V} = \int_V \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (x^2 + y^2 + z^2) \, d\bar{V} \\
&= \int_V 6 \cdot dV = 6V.
\end{aligned}$$

Hence shown.

51. The vector field $\vec{F} = x^2 \hat{i} + z \hat{j} + yz \hat{k}$ is defined over the volume of the cuboid given by $0 \leq x \leq a$,

$0 \leq y \leq b$, $0 \leq z \leq c$ enclosing the surface S , find the value of the surface integral $\iint_S \vec{F} \cdot d\vec{S}$

Sol. By Gauss Divergence theorem

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iiint_V \text{div.} \vec{F} \, dV = \iiint_V \left\{ \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (z) + \frac{\partial}{\partial z} (yz) \right\} dV \\
&= \iiint_V (2x + y) \, dV = \int_0^a \int_0^b \int_0^c (2x + y) \, dz \, dy \, dx \\
&= \int_0^a \int_0^b (2x + y) (z)_0^c \, dy \, dx = c \int_0^a \left(2xy + \frac{y^2}{2} \right)_0^b \, dx
\end{aligned}$$

$$= bc \int_0^a \left(2x + \frac{b}{2} \right) dx = bc \left(x^2 + \frac{b}{2}x \right)_0^a = bc \left(a^2 + \frac{ab}{2} \right)$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = abc \left(a + \frac{b}{2} \right)$$

52. Evaluate the surface integral $\iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy)$, where S is the portion of the plane

$x + 2y + 3z = 6$ which lies in the first octant.

(U.P.T.U., 2009)

Sol. By Gauss' Divergence theorem, we have

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \text{div} \cdot \vec{F} dV \quad \dots(1)$$

Here,

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}, \quad \hat{n} = dy \, dz \hat{i} + dz \, dx \hat{j} + dx \, dy \hat{k}$$

Now,

$$\begin{aligned} \text{div} \cdot \vec{F} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3. \end{aligned}$$

From (1),

$$\begin{aligned} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy) &= \iiint_V 3 dV \\ &= 3 \int_{x=0}^6 \int_{y=0}^{\frac{6-x}{2}} \int_{z=0}^{\frac{6-x-2y}{3}} dz \, dy \, dx = 3 \int_{x=0}^6 \int_{y=0}^{\frac{6-x}{2}} [z]_0^{\frac{6-x-2y}{3}} \cdot dy \, dx \\ &= 3 \int_{x=0}^6 \int_{y=0}^{\frac{6-x}{2}} \left(\frac{6-x-2y}{3} \right) dy \, dx = \int_{x=0}^6 \left\{ (6-x)y - y^2 \right\}_0^{\frac{6-x}{2}} \cdot dx \\ &= \int_{x=0}^6 \left\{ \frac{(6-x)^2}{2} - \frac{(6-x)^2}{4} \right\} dx = \int_0^6 \frac{(6-x)^2}{4} \cdot dx \\ &= \frac{1}{4} \left[-\frac{(6-x)^3}{3} \right]_0^6 = -\frac{1}{12} [(6-x)^3]_0^6 = -\frac{1}{12} (-216) = 18. \end{aligned}$$

53. Evaluate $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} d\vec{S}$ where S is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol. $\vec{F} \cdot \hat{n} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} \quad \dots(1) \quad [\text{given}]$

Let $\phi = ax^2 + by^2 + cz^2 - 1 = 0$

$$\text{grad } \phi = 2ax\hat{i} + 2by\hat{j} + 2cz\hat{k}$$

$$|\text{grad } \phi| = 2\sqrt{a^2x^2 + b^2y^2 + c^2z^2}$$

$$\therefore \hat{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{ax\hat{i} + by\hat{j} + cz\hat{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \quad \dots(2)$$

From (1) and (2), it is clear that

$$\vec{F} = xi + yj + zk$$

$$\therefore \operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

By Divergence theorem,

$$\begin{aligned} \iint_S \vec{F} d\vec{S} &= \iiint_V \operatorname{div} \vec{F} dV = 3 \iiint_V dV = 3V \\ &= 3 \cdot \frac{4\pi}{3} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}} \left[\because \text{Volume of } (ax^2 + by^2 + cz^2 = 1) \text{ is } = \frac{4\pi}{3} \cdot \frac{1}{\sqrt{abc}} \right] \\ &= \frac{4\pi}{\sqrt{abc}}. \end{aligned}$$

54. Evaluate $\iint_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$,

$z = 0$, $z = 3$. Verify Divergence theorem for $\vec{F}$.

(M.D.U., May 2008)

Sol. Since $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\begin{aligned} \therefore \iiint_V \operatorname{div} \vec{F} dV &= \iiint_V (4 - 4y + 2z) dx dy dz \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z) dz dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2 \right]_0^3 dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21 dy dx = \int_{-2}^2 42\sqrt{4-x^2} \cdot dx \\ &= 84 \int_{-2}^2 \sqrt{4-x^2} \cdot dx = 84 \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 \\ &= 84 (2 \sin^{-1} 1) = 84 \cdot 2 \cdot \frac{\pi}{2} = 84\pi \end{aligned} \quad \dots(1)$$

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder, given by the equation $x^2 + y^2 = 4$.

$$\begin{aligned} \text{Also, } \iint_S \vec{F} \cdot \hat{n} dS &= \iint_{S_1} \vec{F} \cdot \hat{n} dS + \iint_{S_2} \vec{F} \cdot \hat{n} dS + \iint_{S_3} \vec{F} \cdot \hat{n} dS \end{aligned}$$

On $S_1 (z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

so that $\vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j}) \cdot (-\hat{k}) = 0$

$$\therefore \iint_{S_1} \vec{F} \cdot \hat{n} dS = 0$$

On $S_2 (z = 3)$, we have $\hat{n} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}) \cdot \hat{k} = 9$

$$\begin{aligned} \therefore \iint_{S_2} \vec{F} \cdot \hat{n} dS &= \iint_{S_2} 9 dx dy = 9 \iint_S dx dy \\ &= 9 \times \text{area of surface } S_2 = 9(\pi \times 2^2) = 36\pi \end{aligned}$$

On S_3 , $x^2 + y^2 = 4$.

A vector normal to the surface S_3 is given by

$$\vec{V}(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

$\therefore \hat{n}$ = a unit vector normal to surface S_3

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4 \times 4}} \quad [\text{Since } x^2 + y^2 = 4]$$

$$\Rightarrow \vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{2} \right) = 2x^2 - y^3$$

Also, on S_3 , i.e., $x^2 + y^2 = 4$, $x = 2 \cos \theta$, $y = 2 \sin \theta$ and $dS = 2 d\theta dz$

To cover the whole surface S_3 , z varies from 0 to 3 and θ varies from 0 to 2π .

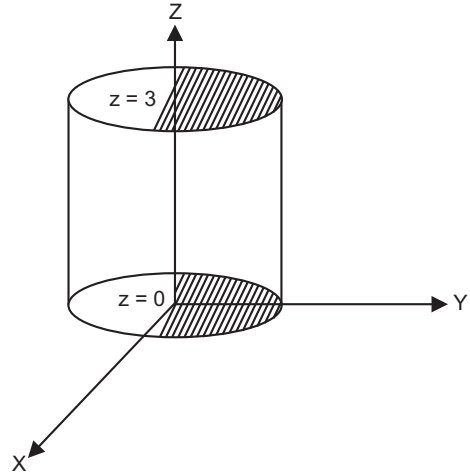
$$\begin{aligned} \therefore \iint_{S_3} \vec{F} \cdot \hat{n} dS &= \int_0^{2\pi} \int_0^3 [2(2 \cos \theta)^2 - (2 \sin \theta)^3] 2 dz d\theta \\ &= \int_0^{2\pi} 16(\cos^2 \theta - \sin^3 \theta) \times 3 d\theta = 48 \int_0^{2\pi} (\cos^2 \theta - \sin^3 \theta) d\theta = 48\pi \end{aligned}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dS = 0 + 36\pi + 48\pi = 84\pi \quad \dots(2)$$

The equality of (1) and (2), verifies divergence theorem.

- 55.** Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$. (M.D.U., May 2009)

Sol. Stoke's Theorem gives relation between line and surface integrals. It states that if S be an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any vector point function having continuous first order partial derivatives, then



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} dS$$

where $\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C .

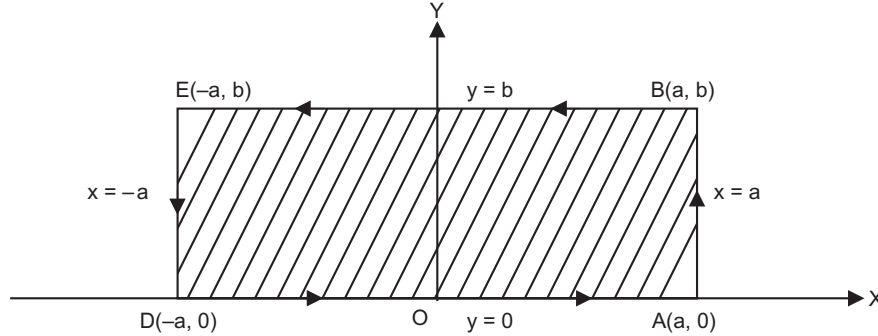
In the given question, let C denote the boundary of the rectangle $ABED$, then

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \oint_C [(x^2 + y^2)\hat{i} - 2xy\hat{j}] \cdot (\hat{i}dx + \hat{j}dy) \\ &= \oint_C [(x^2 + y^2)dx - 2xydy] \end{aligned}$$

The curve 'C' consists of 4 lines AB , BE , ED and DA (see Figure).

Along AB , $x = a$, $dx = 0$ and y varies from 0 to b .

$$\therefore \int_{AB} \{(x^2 + y^2)dx - 2xydy\} = \int_0^b -2aydy = -a[y^2]_0^b = -ab^2 \quad \dots(1)$$



Along BE , $y = b$, $dy = 0$ and x varies from $-a$ to a .

$$\therefore \int_{BE} [(x^2 + y^2)dx - 2xydy] = \int_a^{-a} (x^2 + b^2)dx = \left[\frac{x^3}{3} + b^2 \cdot x \right]_a^{-a} = -\frac{2a^2}{3} - 2ab^2 \quad \dots(2)$$

Along ED , $x = -a$, $dx = 0$ and y varies from b to 0.

$$\therefore \int_{ED} [(x^2 + y^2)dx - 2xydy] = \int_b^0 2aydy = a[y^2]_b^0 = -ab^2 \quad \dots(3)$$

Along DA , $y = 0$, $dy = 0$ and x varies from $-a$ to a .

$$\therefore \int_{DA} [(x^2 + y^2)dx - 2xydy] = \int_{-a}^a x^2dx = \frac{2a^3}{3} \quad \dots(4)$$

Adding (1), (2), (3) and (4), we get

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} \\ &= -4ab^2 \quad \dots(5) \end{aligned}$$

Now,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y) \hat{k} = -4y\hat{k}$$

For the surface S , $\hat{n} = \hat{k}$

$$\begin{aligned} \text{Curl } \vec{F} \cdot \hat{n} &= -4y\hat{k} \cdot \hat{k} = -4y \\ \therefore \iint_S \text{Curl } \vec{F} \cdot \hat{n} dS &= \int_0^b \int_{-a}^a -4y dx dy = \int_0^b -4y [x]_{-a}^a dy \\ &= -8a \int_0^b y dy = -8a \left[\frac{y^2}{2} \right]_0^b = -4ab^2 \end{aligned} \quad \dots(6)$$

The equality of (5) and (6) verifies Stoke's Theorem.

56. If $f = (x^2 + y^2 + z^2)^{-n}$, find $\text{div grad } f$ and determine n if $\text{div grad } f = 0$. (M.D.U. Dec., 2008)

Sol. $f = (x^2 + y^2 + z^2)^{-n}$

$$\begin{aligned} \text{Grad } f &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-n} \\ &= \hat{i} (-n) (x^2 + y^2 + z^2)^{-n-1} \cdot 2x + \hat{j} (-n) (x^2 + y^2 + z^2)^{-n-1} \cdot 2y \\ &\quad + \hat{k} (-n) (x^2 + y^2 + z^2)^{-n-1} \cdot 2z \end{aligned}$$

$$= 2(-n) (x^2 + y^2 + z^2)^{-n-1} [x\hat{i} + y\hat{j} + z\hat{k}]$$

$$\begin{aligned} \text{Div. Grad } f &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) [2(-n) (x^2 + y^2 + z^2)^{-n-1} (x\hat{i} + y\hat{j} + z\hat{k})] \\ &= 2(-n) \left\{ \left[\frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-n-1} \cdot x \right] + \left[\frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-n-1} \cdot y \right] \right. \\ &\quad \left. + \left[\frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-n-1} \cdot z \right] \right\} \\ &= 2(-n) [3(x^2 + y^2 + z^2)^{-n-1} + (x^2 + y^2 + z^2)^{-n-2} \cdot 2(x^2 + y^2 + z^2) (-n-1)] \\ &= 2(-n) (x^2 + y^2 + z^2)^{-n-1} [3 + 2(-n-1)] = 2(-n) (x^2 + y^2 + z^2)^{-n-1} (-2n+1) \\ &= 2n(2n-1) (x^2 + y^2 + z^2)^{-n-1} \end{aligned}$$

$$\text{If grad div } f = 0, 2n - 1 = 0 \Rightarrow n = \frac{1}{2}.$$

57. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ by Stoke's Theorem where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0, 0, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$. (M.D.U., Dec. 2008)

Sol. Since z co-ordinates of each vertex of the triangle is zero, therefore the triangle lies in the xy -plane and $\hat{n} = \hat{k}$,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$

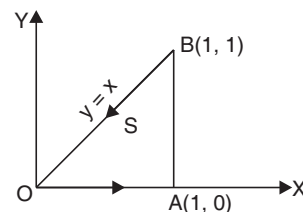
$$\text{Curl } \vec{F} \cdot \hat{n} = [\hat{j} + 2(x-y)\hat{k}] \cdot (\hat{k}) = 2(x-y)$$

Equation of the line OB is $y = x$

By Stoke's theorem
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_C \text{Curl } \vec{F} \cdot \hat{n} \, dS$$

$$= \int_0^1 \int_0^x 2(x-y) \, dy \, dx$$

$$= \int_0^1 \left[2xy - y^2 \right]_0^x dx = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$



58. Verify Stoke's Theorem for the vector field $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy -plane.

(M.D.U., May 2006, 2007, Dec., 2007)

Sol. Let S be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary C of S is a circle in the xy -plane of radius unity and centre O . The equations of C are $x^2 + y^2 = 1, z = 0$, whose parametric form is $x = \cos t, y = \sin t, z = 0, 0 \leq t \leq 2\pi$.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \oint_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz) \\ &= \oint_C [(2x-y)dx - yz^2dy - y^2zdz] \\ &= \oint_C (2x-y)dx, \quad \text{since on } C, z=0, dz=0 \\ &= \int_0^{2\pi} (2\cos t - \sin t) \frac{dx}{dt} \cdot dt = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t) dt \\ &= \int_0^{2\pi} (-\sin 2t + \sin^2 t) dt = \int_0^{2\pi} \left[-\sin 2t + \frac{1 - \cos 2t}{2} \right] dt \\ &= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi \end{aligned}$$

Also,
$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz + 2yz)\hat{i} + (0-0)\hat{j} + (0+1)\hat{k} = \hat{k}$$

$$\Rightarrow \text{Curl } \vec{F} = \hat{k}$$

$$\text{Curl } \vec{F} \cdot \hat{n} = \hat{k} \cdot \hat{n} = \hat{n} \cdot \hat{k}$$

$$\iint_S \text{Curl } \vec{F} \cdot \hat{n} \, dS = \iint_S \hat{n} \cdot \hat{k} \, dS = \iint_R \hat{n} \cdot \hat{k} \cdot \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

where R is the projection of S on xy -plane.

$$= \iint_R dx dy = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx = \int_{-1}^1 2\sqrt{1-x^2} \cdot dx$$

$$= 4 \int_0^1 \sqrt{1-x^2} \cdot dx = 4 \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1 = 4 \left[\frac{1}{2} \cdot \frac{\pi}{2} \right] = \pi$$

Since, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} dS$, **Stoke's Theorem is Verified.**

59. Evaluate the surface integral $\iint_S \text{Curl } \vec{F} \cdot \hat{n} dS$ by transforming it into a line integral, S being that

part of the surface of the paraboloid $z = 1 - x^2 - y^2$, for which $z \geq 0$ and $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = 1, z = 0$ whose parametric equations are $x = \cos t, y = \sin t, z = 0, 0 \leq t \leq 2\pi$.

By Stoke's theorem, we have

$$\begin{aligned} \iint_S \text{Curl } \vec{F} \cdot \hat{n} dS &= \oint_C \vec{F} \cdot d\vec{r} = \oint_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (\hat{i} dx + \hat{j} dy + \hat{k} dz) \\ &= \oint_C (y dx + z dy + x dz) = \oint_C y dx, \text{ since on } C, z = 0, dz = 0 \\ &= \oint_C y \cdot \frac{dx}{dt} \cdot dt = \int_0^{2\pi} \sin t (-\sin t) dt = - \int_0^{2\pi} \sin^2 t dt = -2 \int_0^{\pi} \sin^2 t dt \\ &= -4 \int_0^{\pi/2} \sin^2 t dt = -4 \times \frac{1}{2} \times \frac{\pi}{2} = -\pi. \end{aligned}$$

60. State Green's Theorem in the plane. Verify this in the plane for

$\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the boundary of the region defined by

(a) $y = \sqrt{x}, y = x^2$ (M.D.U., Dec. 2008) (b) $x = 0, y = 0, x + y = 1$.

Sol. "Green's Theorem" States that:

If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives

$\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy -plane bounded by a closed curve C , then

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \text{ where } C \text{ is traversed in the counter clock-wise direction.}$$

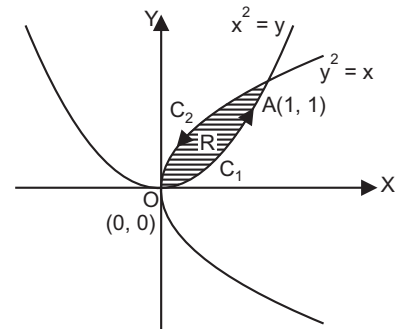
(a) $y = \sqrt{x}$ i.e., $y^2 = x$ and $y = x^2$ are two parabolas intersecting at $O(0, 0)$ and $A(1, 1)$.

Here, $M = 3x^2 - 8y^2$,

$N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y,$$

$$\frac{\partial N}{\partial x} = -6y$$



$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

If R is the region bounded by C , then

$$\begin{aligned} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y dy dx = \int_0^1 5 \left[y^2 \right]_{x^2}^{\sqrt{x}} dx \\ &= 5 \int_0^1 (x - x^4) dx = 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = 5 \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2} \quad \dots(1) \end{aligned}$$

$$\text{Also, } \oint_C (Mdx + Ndy) = \int_{C_1+C_2} (Mdx + Ndy) = \int_{C_1} (Mdx + Ndy) + \int_{C_2} (Mdx + Ndy)$$

Along C_1 , $x^2 = y$, $\therefore 2xdx = dy$ and the limits of x are from 0 to 1.

$$\begin{aligned} \therefore \text{Line integral along } C_1 &= \int_{C_1} (Mdx + Ndy) = \int_0^1 (3x^2 - 8x^4) dx + (4x^2 - 6x \cdot x^2) 2x dx \\ &= \int_0^1 (3x^2 + 8x^3 - 20x^4) dx = \left[x^3 + 2x^4 - 4x^5 \right]_0^1 = -1 \end{aligned}$$

Along C_2 , $y^2 = x$, $\therefore 2ydy = dx$ and the limits of y are from 1 to 0.

$$\begin{aligned} \therefore \text{Line integral along } C_2 &= \int_{C_2} (Mdx + Ndy) = \int_1^0 (3y^4 - 8y^2) 2y dy + (4y - 6y^2 \cdot y) dy \\ &= \int_1^0 (4y - 22y^3 + 6y^5) dy = \left[2y^2 - \frac{11}{2} y^4 + y^6 \right]_1^0 = \frac{5}{2} \end{aligned}$$

$$\therefore \text{Line integral along } C = -1 + \frac{5}{2} = \frac{3}{2} \quad \text{i.e., } \oint_C (Mdx + Ndy) = \frac{3}{2} \quad \dots(2)$$

The equality of (1) and (2) verifies Green's theorem in the plane.

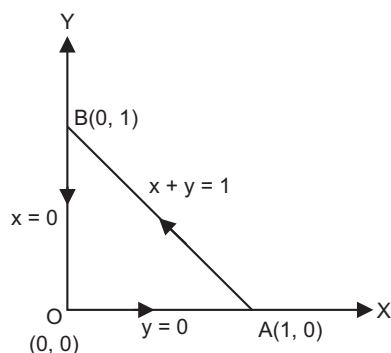
$$\begin{aligned} \text{(b) Here, } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_0^1 \int_0^{1-x} 10y dy dx \\ &= \int_0^1 5 \left[y^2 \right]_0^{1-x} dx = 5 \int_0^1 (1-x)^2 dx = 5 \left[\frac{(1-x)^3}{-3} \right]_0^1 = \frac{5}{3} \quad \dots(1) \end{aligned}$$

Along OA , $y = 0$, $\therefore dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along OA

$$= \int_0^1 3x^2 dx = \left[x^3 \right]_0^1 = 1.$$

Along AB , $y = 1 - x$, $\therefore dy = -dx$ and the limits of x are from 1 to 0.



∴ Line integral along AB

$$\begin{aligned}
 &= \int_1^0 [3x^2 - 8(1-x)^2] dx + [4(1-x) - 6x(1-x)](-dx) \\
 &= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2) dx \\
 &= \int_1^0 (-12 + 26x - 11x^2) dx = \left[-12x + 13x^2 - \frac{11}{3}x^3 \right]_1^0 = -\left[-12 + 13 - \frac{11}{3} \right] = \frac{8}{3}
 \end{aligned}$$

Along BO, $x = 0$, ∴ $dx = 0$ and the limits of y are from 1 to 0.

$$\therefore \text{Line integral along BO} = \int_1^0 4y dy = \left[2y^2 \right]_1^0 = -2$$

$$\therefore \text{Line integral along C (i.e., along OABO)} = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$$\text{i.e., } \oint_C (Mdx + Ndy) = \frac{5}{3} \quad \dots(2)$$

The equality of (1) and (2) verify Green's theorem in the plane.

61. Apply Green's Theorem to evaluate $\oint_C [(y - \sin x) dx + \cos x \cdot dy]$, where C is the plane triangle

enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$.

(M.D.U. Dec., 2005; May 2007)

Sol. Applying Green's Theorem,

$$Mdx + Ndy = (y - \sin x)dx + \cos x \cdot dy$$

Here,

$$M = y - \sin x, \quad N = \cos x.$$

$$\therefore \frac{\partial M}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin x$$

$$\begin{aligned}
 \oint_C (Mdx + Ndy) &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
 &= \iint_R (-\sin x - 1) dx dy = \int_0^{\pi/2} \int_0^{(2/\pi)x} (-\sin x - 1) dx dy \\
 &= \int_0^{\pi/2} [-y \sin x - y]_0^{(2/\pi)x} \cdot dx = \int_0^{\pi/2} \left(-\frac{2}{\pi}x \sin x - \frac{2}{\pi} \cdot x \right) dx \\
 &= -\frac{2}{\pi} \left[x \int_0^{\pi/2} \sin x - \int \left(\int \sin x dx \right) \right] - \frac{2}{\pi} \int x dx \\
 &= \left[\frac{2}{\pi}x \cos x - \frac{2}{\pi} \sin x - \frac{2}{\pi} \cdot \frac{x^2}{2} \right]_0^{\pi/2} = \left[0 - \frac{2}{\pi} - \frac{2}{\pi} \cdot \frac{(\pi/2)^2}{2} \right] = -\left(\frac{\pi}{4} + \frac{2}{\pi} \right).
 \end{aligned}$$

- 62.** Apply Green's Theorem to evaluate $\int_C [(2x^2 - y^2) dx + (x^2 + y^2) dy]$ where C is the boundary of the area enclosed by the x -axis and the upper half of circle $x^2 + y^2 = a^2$.

(U.P.T.U., 2005, 2009; A.U.U.P., 2007; M.D.U., 2008)

Sol.

$$M = 2x^2 - y^2, \quad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = 2x$$

$$\vec{F} \cdot d\vec{r} = (M\hat{i} + N\hat{j})(dx\hat{i} + dy\hat{j}) = Mdx + Ndy$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_C (2x^2 - y^2) dx + (x^2 + y^2) dy \\ &= \iint_S (2x + 2y) dx dy \quad [\text{Using Green's Theorem}] \\ &= 2 \int_{x=-a}^a \int_{y=0}^{\sqrt{a^2-x^2}} (x+y) dx dy = 2 \int_{-a}^a \left(xy + \frac{y^2}{2} \right) \Big|_0^{\sqrt{a^2-x^2}} dx \\ &= 2 \int_{-a}^a \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right) dx = 2 \int_{-a}^a x\sqrt{a^2-x^2} dx + \int_{-a}^a (a^2-x^2) dx \\ &= 0 + 2 \int_0^a (a^2-x^2) dx = 2 \left(a^2x - \frac{x^3}{3} \right) \Big|_0^a = 2 \left(a^3 - \frac{a^3}{3} \right) = \frac{4}{3} a^3. \end{aligned}$$

- 63.** If V_1 and V_2 be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z) then prove that $\text{Curl}(V_1 \times V_2) = 2(V_1 - V_2)$ (M.D.U., May 2009)

Sol. Let $P(x, y, z)$, $Q(x_1, y_1, z_1)$, $R(x_2, y_2, z_2)$

$$QP = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = V_1$$

$$RP = (x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k} = V_2$$

$$\begin{aligned} V_1 \times V_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \end{vmatrix} = \sum \{(y - y_1)(z - z_2) - (z - z_1)(y - y_2)\} \hat{i} \\ &= \sum \{y(z_1 - z_2) - z(y_1 - y_2) + (y_1 z_2 - y_2 z_1)\} \hat{i} \end{aligned}$$

$$\begin{aligned} \text{Curl}(V_1 \times V_2) &= \sum \hat{i} \left[\frac{\partial}{\partial y} \{x(y_1 - y_2) - y(x_1 - x_2) + (x_1 y_2 - x_2 y_1)\} \right. \\ &\quad \left. - \frac{\partial}{\partial z} \{z(x_1 - x_2) - x(z_1 - z_2) + (z_1 x_2 - z_2 x_1)\} \right] \\ &= \sum \{-(x_1 - x_2) - (x_1 - x_2)\} \hat{i} = \sum -2(x_1 - x_2) \hat{i} \\ &= 2 \sum (x_2 - x_1) \hat{i} = 2(V_1 - V_2). \end{aligned}$$

Hence proved.

64. Using divergence theorem, evaluate $\int_S \vec{F} \cdot d\vec{s}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.
(M.D.U., May 2009)

Sol. $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (y^2 z^2 \hat{i} + z^2 x^2 \hat{j} + x^2 y^2 \hat{k}) = 2zy^2$

By using divergent theorem, we have:

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V 2zy^2 dV$$

Changing into polar spherical co-ordinates:

Put $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$
 $dV = r^2 \sin \theta dr d\theta d\phi$

and r varies from 0 to a , θ varies from 0 to $\pi/2$, ϕ varies from 0 to 2π .

$$\begin{aligned} \iiint_V 2zy^2 dV &= 2 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (r \cos \theta) (r^2 \sin^2 \theta \sin^2 \phi) r^2 \sin \theta dr d\theta d\phi \\ &= 2 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^5 \sin^3 \theta \cos \theta \sin^2 \phi dr d\theta d\phi \\ &= 2 \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{r^6}{6} \right]_0^a \sin^3 \theta \cos \theta \sin^2 \phi d\theta d\phi = \frac{2a^6}{6} \int_0^{2\pi} \frac{2}{4} \sin^2 \phi d\phi \\ &= \frac{a^6}{12} \int_0^{2\pi} \sin^2 \phi d\phi = \frac{a^6}{12} \cdot 4 \int_0^{\pi/2} \sin^2 \phi d\phi = \frac{a^6}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^6}{12}. \end{aligned}$$

65. Using Green's theorem, evaluate $\int_C (x^2 y dx + x^2 dy)$ where C is the boundary described counter clockwise of the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.
(U.P.T.U., 2009)

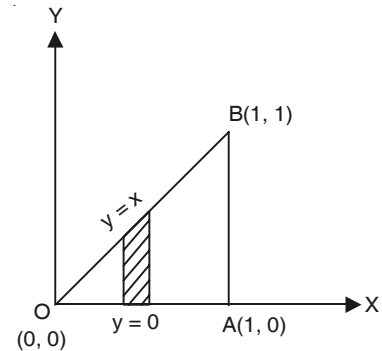
Sol. $\int_C (x^2 y dx + x^2 dy) = \iint_S \left[\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (x^2 y) \right] dx dy$ [Using Green's Theorem]

$$= \int_0^1 \int_0^x (2x - x^2) dx dy$$

$$= \int_0^1 (2x - x^2) (y)_0^x dx$$

$$= \int_0^1 (2x^2 - x^3) dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$



□ □ □

Ordinary Differential Equations and Its Applications

SOLVED PROBLEMS

1. What is a differential equation ? Give examples of an ordinary differential equation. Define order and degree of a differential equation.

Sol. An equation involving the dependent variable, independent variable and the differential coefficient (or coefficients) of the dependent variable with respect to the independent variable (or variables) is known as a differential equation. For example :

$$\frac{dy}{dx} = \cot x, \frac{d^2y}{dx^2} + y = 0; \text{ and } y = x \cdot \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3, \frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$$

A differential equation which involves only one independent variable is called an ordinary differential equation. All the equations given above are ordinary differential equations.

Order of a Differential Equation. It is the order of the highest ordered derivative occurring in

the differential equation e.g., $\frac{dy}{dx} = \cot x$ is of order 1. $\frac{d^2y}{dx^2} + y = 0$ and $\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$,

each are of orders 2.

Degree of a Differential Equation. The degree of a differential equation is the degree of the highest ordered derivative present in the differential equation when it is made free from radical

signs and fractional powers. For example, degree of the equation $\frac{dy}{dx} = \cot x$ is 1.

Consider equation $\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$

However, it involves a radical sign. So to find the degree, we shall remove the radical sign. Squaring it, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3.$$

Clearly its degree is 2.

Again consider the equation $m = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

This involves fractional powers. Therefore, to find the degree of this differential equation we shall remove the fractional power by squaring both sides.

This gives

$$m^2 = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2}$$

Obviously its degree is 2.

2. What is an exact differential equation? What is the necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact? Hence solve :

$$(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0.$$

Sol. A differential equation $x dx + y dy = 0$ is an exact differential equation as it can be obtained from its primitive $x^2 + y^2 = C^2$ **directly by differentiation** without any further operation of multiplication, elimination or reduction etc.

The necessary and sufficient condition for a differential equation $Mdx + Ndy = 0$ of first order and first degree, where M and N are functions of x and y , **to be exact is** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution of an exact differential equation

$$Mdx + Ndy = 0 \text{ is given by } u + F(y) = C \quad \text{where } u = \int_{y \text{ constant}} M dx,$$

$$F(y) = \int (\text{terms of } N \text{ not containing } x) dy \text{ and } C \text{ is an arbitrary constant.}$$

We will illustrate this by solving the given equation ;

$$(1 + 4xy + 2y^2) dx + (1 + 4xy + 2x^2) dy = 0$$

Here,

$$M = 1 + 4xy + 2y^2; \quad N = 1 + 4xy + 2x^2$$

$$\therefore \frac{\partial M}{\partial y} = 4x + 4y, \quad \frac{\partial N}{\partial x} = 4y + 4x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given equation is exact differential equation.

Integrating M w.r.t. x keeping y as constant, we get

$$\int (1 + 4xy + 2y^2) dx = x + 2x^2y + 2y^2x \Rightarrow x + 2x^2y + 2y^2x = u.$$

Integrating w.r.t. y , only those terms of N which do not contain x , we get

$$\int (1 + 4xy + 2x^2) dy = \int 1 \cdot dy = y.$$

Hence the solution is $u + F(y) = C$, where C is an arbitrary constant.

$$\Rightarrow x + 2x^2y + 2y^2x + y = C \quad \text{or} \quad x + y + 2xy(x + y) = C \Rightarrow (x + y)(1 + 2xy) = C.$$

3. Solve : $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$.

Sol. Comparing the given differential equation with $Mdx + Ndy = 0$, we have

$$M = 5x^4 + 3x^2y^2 - 2xy^3; \quad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\text{Also} \quad \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given equation is exact.

Integrating M w.r.t. x , keeping y as constant, we get

$$\int (5x^4 + 3x^2y^2 - 2xy^3) dx = x^5 + x^3y^2 - x^2y^3 = u$$

Integrating N w.r.t. y , only those terms which do not contain x , we get

$$\int (-5y^4)dy = -y^5 = F(y)$$

$$\Rightarrow \text{Solution is } u + F(y) = C \quad \text{or} \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C.$$

4. Solve the differential equation : $(x^2 - ay)dx = (ax - y^2)dy$

$$\text{Sol. We have, } (x^2 - ay)dx = (ax - y^2)dy \quad \text{or} \quad (x^2 - ay)dx + (y^2 - ax)dy = 0 \quad \dots(1)$$

Comparing with $Mdx + Ndy = 0$, we get

$$M = x^2 - ay, N = y^2 - ax$$

$$\therefore \frac{\partial M}{\partial y} = -a = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Equation (1) is exact.

Integrating M w.r.t. x , keeping y as constant, we get

$$\int (x^2 - ay)dx = \frac{x^3}{3} - axy \Rightarrow u = \frac{x^3}{3} - axy$$

Integrating N w.r.t. y , only those terms which do not contain x , we get

$$\int y^2 dy = \frac{y^3}{3} \Rightarrow F(y) = \frac{y^3}{3}$$

$$\therefore \text{Solution is given by } u + F(y) = C \quad \text{or} \quad \frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

$$\Rightarrow x^3 - 3axy + y^3 = C_1, \text{ where } C_1 = 3C.$$

5. Solve the differential equation :

$$[\cos x \tan y + \cos (x + y)] dx + [\sin x \sec^2 y + \cos (x + y)] dy = 0$$

$$\text{Sol. Here, } M = \cos x \tan y + \cos (x + y), \quad N = \sin x \sec^2 y + \cos (x + y)$$

$$\therefore \frac{\partial M}{\partial y} = \cos x \cdot \sec^2 y - \sin (x + y) = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is $u + F(y) = C$, where

$$u = \int [\cos x \tan y + \cos (x + y)] dx = \sin x \tan y + \sin (x + y)$$

Since there is no term of N which does not contain x , $F(y) = 0$

$$\text{Hence solution is } u + F(y) = C \quad \text{or} \quad \sin x \tan y + \sin (x + y) = C.$$

6. Solve : $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$.

Sol. The given equation can be written as

$$(y \cos x + \sin y + y) dx + (\sin x + x \cos y + x) dy = 0$$

$$\text{Here, } M = y \cos x + \sin y + y, \quad N = \sin x + x \cos y + x$$

$$\therefore \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y \text{ constant}} (y \cos x + \sin y + y) dx + \int 0 dy = C$$

$$\text{or} \quad y \sin x + x(\sin y + y) = C.$$

7. Solve : $(1 + e^{x/y}) dx + e^{x/y} \cdot \left(1 - \frac{x}{y}\right) dy = 0$.

Sol. We have the equation,

$$(1 + e^{x/y})dx + e^{x/y} \left(1 - \frac{x}{y}\right) dy = 0 \quad \dots(1)$$

Comparing it with $M dx + N dy = 0$, we get

$$M = 1 + e^{x/y}, N = e^{x/y} \left(1 - \frac{x}{y}\right)$$

$$\begin{aligned} \therefore \quad \frac{\partial M}{\partial y} &= e^{x/y} \cdot \left(-\frac{x}{y^2}\right) \\ \frac{\partial N}{\partial x} &= e^{x/y} \cdot \left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) \cdot e^{x/y} \cdot \frac{1}{y} = \frac{-x}{y^2} \cdot e^{x/y} \\ \Rightarrow \quad \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \end{aligned}$$

Thus the given equation is exact.

Now
$$u = \int_{y \text{ constant}} M \cdot dx = \int (1 + e^{x/y})dx$$

or
$$u = x + \frac{e^{x/y}}{(1/y)} \Rightarrow u = x + y \cdot e^{x/y}$$

Since there is no term in N containing function of y only, the solution may be given by $x + y \cdot e^{x/y} = C$, where C is an arbitrary constant.

8. Solve : $(y^2 \cdot e^{xy^2} + 4x^3) dx + (2xy \cdot e^{xy^2} - 3y^2) dy = 0$.

Sol. Comparing it with $M \cdot dx + N \cdot dy = 0$, we get

$$\begin{aligned} M &= y^2 \cdot e^{xy^2} + 4x^3, \quad N = 2xy \cdot e^{xy^2} - 3y^2 \\ \therefore \quad \frac{\partial M}{\partial y} &= y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y = 2y e^{xy^2} (1 + xy^2) \\ \text{and} \quad \frac{\partial N}{\partial x} &= 2y(x e^{xy^2} \cdot y^2 + e^{xy^2}) = 2y \cdot e^{xy^2} (1 + xy^2) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, given equation is exact.

$\therefore$ Solution of the given equation is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int (y^2 \cdot e^{xy^2} + 4x^3) dx = y^2 \cdot \frac{e^{xy^2}}{y^2} + x^4$$

or
$$u = e^{xy^2} + x^4$$

$$F(y) = \int (-3y^2) dy = -y^3$$

Hence the solution is $u + F(y) = C$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = C, \text{ where } C \text{ is an arbitrary constant.}$$

9. Solve : $x \cos \frac{y}{x} (ydx + xdy) = y \sin \frac{y}{x} (xdy - ydx)$

Sol. Given equation can be re-written as $\frac{xdy + ydx}{xy} = \tan \frac{y}{x} \left(\frac{x dy - y dx}{x^2} \right)$

or
$$d(\log xy) = \tan \frac{y}{x} \cdot d\left(\frac{y}{x}\right) = d\left(\log \sec \frac{y}{x}\right)$$

On integrating both sides, we obtain

$$\log(xy) = \log \sec\left(\frac{y}{x}\right) - \log C \Rightarrow C \cdot xy = \sec\left(\frac{y}{x}\right)$$

where C is an arbitrary constant of integration.

10. Solve : $x dx + y dy + \frac{xdy - ydx}{x^2 + y^2} = 0$.

Sol. Given equation can be re-written as

$$\begin{aligned} \frac{1}{2} (2x dx + 2y dy) + \frac{\frac{xdy - ydx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} &= 0 \\ \Rightarrow d\left(\frac{x^2 + y^2}{2}\right) + \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} &= 0 \quad \Rightarrow d\left(\frac{x^2 + y^2}{2}\right) + d\left(\tan^{-1} \frac{y}{x}\right) = 0 \end{aligned}$$

Each term of this equation is an exact differential. Hence integrating each term separately, the required solution is $\frac{x^2 + y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = C$ where C is an arbitrary constant of integration.

11. Find the value of λ for which the differential equation $(xy^2 + \lambda x^2 y)dx + (x + y)x^2 dy = 0$ is exact. Hence solve.

Sol. We have, $(xy^2 + \lambda x^2 y)dx + (x + y)x^2 dy = 0$

Comparing with $M \cdot dx + N \cdot dy = 0$, we get

$$M = xy^2 + \lambda x^2 y, \quad N = (x + y)x^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2xy + \lambda \cdot x^2 \text{ and } \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

For the given equation to be exact, we should have

$$2xy + \lambda x^2 = 3x^2 + 2xy \quad \text{or} \quad \lambda = 3.$$

$$\Rightarrow M = xy^2 + 3x^2 y, \quad N = (x + y)x^2$$

Now,
$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (xy^2 + 3x^2 y) dx = \frac{x^2}{2} \cdot y^2 + x^3 \cdot y$$

Since there is no term in N containing function of y only hence the solution is

$$u = C \quad \text{or} \quad \frac{x^2}{2} \cdot y^2 + x^3 \cdot y = C, \text{ where C is an arbitrary constant.}$$

12. Solve the equation : $(2x^2 + 3y^2 - 7)xdx - (3x^2 + 2y^2 - 8)ydy = 0$.

(U.P.T.U., 2005)

Sol. Given equation can be written as

$$\frac{x dx}{y dy} = \frac{3x^2 + 2y^2 - 8}{2x^2 + 3y^2 - 7}$$

Applying Componendo and Dividendo rule,

$$\frac{x dx + y dy}{x dx - y dy} = \frac{5x^2 + 5y^2 - 15}{x^2 - y^2 - 1} \quad \text{or} \quad \frac{x dx + y \cdot dy}{x dx - y \cdot dy} = 5 \left[\frac{x^2 + y^2 - 3}{x^2 - y^2 - 1} \right]$$

$$\text{or} \quad \frac{x dx + y dy}{x^2 + y^2 - 3} = \frac{5(xdx - y \cdot dy)}{x^2 - y^2 - 1}$$

$$\Rightarrow \quad \frac{1}{2} \left(\frac{2xdx + 2ydy}{x^2 + y^2 - 3} \right) = \frac{5}{2} \left(\frac{2xdx - 2ydy}{x^2 - y^2 - 1} \right)$$

$$\Rightarrow \quad \frac{1}{2} d \{ \log (x^2 + y^2 - 3) \} = \frac{5}{2} d \{ \log (x^2 - y^2 - 1) \}$$

$$\Rightarrow \quad d \{ \log (x^2 + y^2 - 3) \} = 5 d \{ \log (x^2 - y^2 - 1) \}$$

Integration gives

$$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log C$$

$$\Rightarrow \quad x^2 + y^2 - 3 = C \log (x^2 - y^2 - 1)^5$$

where C is an arbitrary constant of integration.

13. Solve the differential equation : $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y dy = 0$.

Sol. Comparing the given equation with

$M \cdot dx + N \cdot dy = 0$, we have

$$M = \sec x \tan x \tan y - e^x, \quad N = \sec x \cdot \sec^2 y$$

$$\therefore \quad \frac{\partial M}{\partial y} = \sec x \tan x \cdot \sec^2 y, \quad \frac{\partial N}{\partial x} = \sec x \cdot \tan x \cdot \sec^2 y$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the given differential equation is exact.

$\therefore$ Solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (\sec x \tan x \tan y - e^x) dx$$

or

$$u = \sec x \tan y - e^x$$

Since there is no term in N containing function of y only, hence the solution is

$$u = C$$

or

$$\sec x \cdot \tan y - e^x = C, \text{ where } C \text{ is an arbitrary constant of integration.}$$

14. Solve : $(2xy \cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$.

Sol. Comparing the given equation with

$M \cdot dx + N \cdot dy = 0$, we have

$$M = 2xy \cos x^2 - 2xy + 1, \quad N = \sin x^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x \cos x^2 - 2x, \quad \frac{\partial N}{\partial x} = 2x \cdot \cos x^2 - 2x$$

i.e.,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, the given differential equation is exact. Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (2xy \cos x^2 - 2xy + 1) dx$$

or

$$u = y \sin x^2 - x^2y + x$$

Since there is no term in N containing function of y only, hence the solution is given by

$$u = C \quad \text{or} \quad y \sin x^2 - x^2y + x = C, \text{ where } C \text{ is an arbitrary constant of integration.}$$

15. Solve the differential equation : $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)ydy = 0$.

Sol. On comparing with $M \cdot dx + N \cdot dy = 0$, we get

$$M = (x^2 + y^2 - a^2)x, \quad N = (x^2 - y^2 - b^2)y$$

$$\frac{\partial M}{\partial y} = 2yx = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given equation is exact.

$\therefore$ Solution of the given equation is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (x^2 + y^2 - a^2)x \cdot dx = \frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2}$$

and

$$F(y) = \int N \cdot dy = \int (-y^3 - b^2y) dy = -\frac{y^4}{4} - \frac{b^2y^2}{2}$$

Hence solution is given by

$$\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{b^2y^2}{2} = C \Rightarrow x^4 + 2x^2y^2 - 2a^2x^2 - y^4 - 2b^2y^2 = C$$

where C is an arbitrary constant of integration.

16. Solve : $\left[y \left(1 + \frac{1}{x} \right) + \cos y \right] dx + (x + \log x - x \sin y) dy = 0$.

Sol. On comparing with $M \cdot dx + N \cdot dy = 0$, we have

$$M = y \left(1 + \frac{1}{x} \right) + \cos y, \quad N = x + \log x - x \sin y$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y, \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the given equation is exact. Solution is given by

$$u + F(y) = C$$

where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \left\{ y \left(1 + \frac{1}{x} \right) + \cos y \right\} dx$$

or

$$u = y \{x + \log x\} + x \cos y$$

Since there is no term in N containing function of y only, the solution is given by $u = C$

$\Rightarrow y(x + \log x) + x \cos y = C$, where C is an arbitrary constant of integration.

17. Solve the differential equation : $(2xy + y - \tan y)dx + (x^2 - x \tan^2 y + \sec^2 y)dy = 0$.

(M.D.U., May 2007)

Sol. On comparing the given equation with $M \cdot dx + N \cdot dy = 0$, we have

$$M = 2xy + y - \tan y, \quad N = x^2 - x \tan^2 y + \sec^2 y$$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y, \quad \frac{\partial N}{\partial x} = 2x - \tan^2 y = \frac{\partial M}{\partial y}.$$

Therefore, the given differential equation is exact.

Solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (2xy + y - \tan y) dx = yx^2 + y \cdot x - \tan y \cdot x$$

or

$$u = x^2y + x(y - \tan y)$$

and

$$F(y) = \int N(\text{only terms not containing } x) dy = \int \sec^2 y \cdot dy = \tan y$$

Hence the solution of the given differential equation is

$x^2y + xy - x \tan y + \tan y = C$, where C is an arbitrary constant of integration.

18. Solve : $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Sol. On comparing the given equation with $M \cdot dx + N \cdot dy = 0$, we have

$$M = x^2 - 4xy - 2y^2, \quad N = y^2 - 4xy - 2x^2$$

$$\therefore \frac{\partial M}{\partial y} = -4x - 4y, \quad \frac{\partial N}{\partial x} = -4y - 4x = \frac{\partial M}{\partial y}.$$

Therefore, the given differential equation is exact. Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (x^2 - 4xy - 2y^2) dx$$

or

$$u = \frac{x^3}{3} - 2x^2y - 2xy^2$$

and

$$F(y) = \int (\text{terms of } N \text{ not containing } x) dy = \int y^2 dy \text{ or } \frac{y^3}{3}$$

$$\therefore \text{The solution is } \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = C$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = 3C \quad \text{or} \quad x^3 + y^3 - 6xy(x + y) = C_1$$

where C_1 is an arbitrary constant of integration.

19. Solve : $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$.

Sol. On comparing the given equation with $M \cdot dx + N \cdot dy = 0$, we have

$$M = 3x^2 + 6xy^2, \quad N = 6x^2y + 4y^3$$

$$\therefore \frac{\partial M}{\partial y} = 12xy = \frac{\partial N}{\partial x}.$$

Therefore, the given differential equation is exact.

Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (3x^2 + 6xy^2) dx$$

$\Rightarrow$

$$u = x^3 + 3x^2y^2$$

and

$$F(y) = \int (\text{terms of } N \text{ not containing } x) dy = \int 4y^3 \cdot dy = y^4$$

Hence the solution is $x^3 + 3x^2y^2 + y^4 = C$, where C is an arbitrary constant of integration.

20. Solve : $x dx + y dy = \frac{a^2(x dy - y dx)}{x^2 + y^2}$ (U.P.T.U., 2005)

Sol. The given equation is $x dx + y dy - \frac{a^2(x dy - y dx)}{x^2 + y^2} = 0$... (1)

Since $\frac{x dy - y dx}{x^2 + y^2} = d \left(\tan^{-1} \frac{y}{x} \right)$... (2)

We may re-write equation (1) as $x dx + y dy - a^2 \cdot d \left(\tan^{-1} \frac{y}{x} \right) = 0$

On integrating it, we get $\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1} \frac{y}{x} = C$

or $x^2 + y^2 - 2a^2 \tan^{-1} \left(\frac{y}{x} \right) = C_1$, where $C_1 = 2C$ is the solution.

21. Solve : $y dx - x dy + 3x^2 y^2 e^{x^3} dx = 0$.

Sol. Dividing the given equation throughout by y^2 , we get

$$\frac{y dx - x dy}{y^2} + 3x^2 \cdot e^{x^3} \cdot dx = 0 \quad \text{or} \quad d \left(\frac{x}{y} \right) + d(e^{x^3}) = 0$$

This is an exact equation.

On integrating, we obtain $\frac{x}{y} + e^{x^3} = C$, where C is an arbitrary constant of integration.

This is the solution of the given diff. equation which was made exact after multiplying by $\frac{1}{y^2}$ (a suitable factor called the integrating factor).

22. Solve : $x dy - y dx = x \sqrt{x^2 - y^2} \cdot dx$

Sol. The given equation can be written as

$$x dy - y dx = x^2 \sqrt{1 - \left(\frac{y}{x} \right)^2} \cdot dx \quad \text{or} \quad \frac{x dy - y dx}{x^2} = dx \sqrt{1 - \left(\frac{y}{x} \right)^2} \quad \dots (1)$$

We know that, $d(\sin^{-1} \theta) = \frac{1}{\sqrt{1 - \theta^2}} \cdot d\theta$

Let $\theta = \frac{y}{x} \quad \therefore d\theta = d(y/x) = \frac{x dy - y dx}{x^2}$

$\Rightarrow$ L.H.S. of equation (1) is exact differential of $\sin^{-1} (y/x)$.

or $d \left[\sin^{-1} \left(\frac{y}{x} \right) \right] = dx$ which is exact.

On integrating it, we obtain

$\sin^{-1} y/x = x + C$ or $y/x = \sin(x + C) \Rightarrow y = x \sin(x + C)$
is the required solution, where C is an arbitrary constant of integration.

23. Solve the D.E. : $x dy - y dx = (x^2 + y^2) dx$

(M.D.U., Dec., 2007)

Sol.

$$x dy - y dx = (x^2 + y^2) dx$$

$$\frac{x dy - y dx}{(x^2 + y^2)} = dx$$

$$d \left[\tan^{-1} \left(\frac{y}{x} \right) \right] - dx = 0$$

On integration, we get $\tan^{-1} \left(\frac{y}{x} \right) - x = c$.

24. Solve : $(x^3 e^x - m y^2) dx + m xy dy = 0$

Sol. Integrating factor by inspection is $\frac{1}{x^3}$. So multiplying the given equation by $\frac{1}{x^3}$, we obtain

$$\left(e^x - \frac{m y^2}{x^3} \right) dx + \frac{m \cdot y}{x^2} dy = 0 \quad \text{or} \quad e^x - m \left\{ \frac{y^2 - xy \frac{dy}{dx}}{x^3} \right\} = 0$$

$$\text{or} \quad e^x - m \left\{ \frac{xy^2 - x^2 y \frac{dy}{dx}}{x^4} \right\} = 0 \quad e^x + \frac{1}{2} m \left\{ \frac{-2xy^2 + 2x^2 y \frac{dy}{dx}}{x^4} \right\} = 0$$

$$\text{Now,} \quad d \left(\frac{y^2}{x^2} \right) = \frac{x^2 \cdot 2y \frac{dy}{dx} - y^2 \cdot 2x}{x^4}$$

$$\therefore \text{ We have} \quad e^x + \frac{1}{2} m \left\{ \frac{d}{dx} (y^2/x^2) \right\} = 0$$

On integrating this equation, we get the solution of the differential equation as

$$e^x + \frac{1}{2} m \cdot \frac{y^2}{x^2} = C$$

where C is an arbitrary constant of integration.

25. Solve : $y dx - x dy + \log x dx = 0$.

Sol. Given equation is

$$\log x \cdot dx = x dy - y \cdot dx \quad \dots(1)$$

Multiplying by $\frac{1}{x^2}$ we get

$$\frac{1}{x^2} \cdot \log x \cdot dx = \frac{x dy - y \cdot dx}{x^2} \quad \text{or} \quad \frac{1}{x^2} \cdot \log x \cdot dx = d(y/x)$$

$$\Rightarrow \quad d \left(\frac{y}{x} \right) - \frac{1}{x^2} \cdot \log x \cdot dx = 0 \quad \dots(2)$$

$$\text{As} \quad \int (\log x) \left(-\frac{1}{x^2} \right) dx = \frac{1}{x} \log x + \frac{1}{x}$$

$\therefore$ Equation (2) may be re-written as

$$d(y/x) + d \left(\frac{1}{x} \log x + \frac{1}{x} \right) = 0$$

On integration, we get $\frac{y}{x} + \frac{1}{x} \log x + \frac{1}{x} = C \Rightarrow \frac{y}{x} + \frac{1}{x} (1 + \log x) = C$

is the solution of the given differential equation, where C is the constant of integration.

26. Solve : $x dy - y dx = \log y dy$

Sol. Multiplying by $\frac{1}{y^2}$, we get

$$\frac{1}{y^2} \log y dy = \frac{x dy - y dx}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\therefore d\left(\frac{x}{y}\right) + \frac{1}{y^2} \log y \cdot dy = 0 \quad \dots(1)$$

$$\text{As} \quad \int \frac{1}{y^2} \log y dy = -\frac{1}{y} \log y - \int \frac{1}{y} \left(-\frac{1}{y}\right) dy = -\frac{1}{y} \log y - \frac{1}{y} \quad \dots(2)$$

$$\therefore \frac{x}{y} = -\frac{1}{y} (\log y + 1) + C \Rightarrow x = -\log y - 1 + Cy$$

is the solution, where C is the constant of integration.

Hence $x = -\log y - 1 + Cy$.

27. Solve : $x^2 y dx - (x^3 + y^3) dy = 0$

Sol. The given equation is $x^2 y dx - (x^3 + y^3) dy = 0 \quad \dots(1)$

Comparing it with $Mdx + Ndy = 0$, we get

$$M = x^2 y, N = -(x^3 + y^3)$$

$$\therefore \frac{\partial M}{\partial y} = x^2, \frac{\partial N}{\partial x} = -3x^2 \quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ equation (1) is not exact.}$$

But equation (1) is homogeneous in x and y.

Moreover $Mx + Ny = (x^2 y) \cdot x + \{-(x^3 + y^3)\} y = x^3 y - x^3 y - y^4 = -y^4 \neq 0$.

$$\therefore \text{Integrating factor} = \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

$$\text{Multiplying (1) by } -\frac{1}{y^4}, \text{ it becomes } \frac{-x^2}{y^3} \cdot dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0 \quad \dots(2)$$

Now comparing with $Mdx + Ndy = 0$, we get

$$M = -\frac{x^2}{y^3}, N = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{3x^2}{y^4}, \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (2) is an exact differential equation.

$$\text{Now, } u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \left(\frac{-x^2}{y^3}\right) dx = -\frac{x^3}{3y^3}$$

$$\text{and } F(y) = \int (\text{terms in N not containing } x) dy = \int \frac{1}{y} \cdot dy = \log y$$

Hence the solution is $u + F(y) = C \Rightarrow -\frac{x^3}{3y^3} + \log y = C$, where C is an arbitrary constant.

28. Solve : $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Sol. The given equation is homogeneous in x and y with

$$M = x^2y - 2xy^2 \quad \text{and} \quad N = -x^3 + 3x^2y$$

Now, $Mx + Ny = x^3y - 2x^2y^2 - x^2y + 3x^2y^2 = x^2y^2 \neq 0$

$$\therefore \text{ Integrating Factor} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Multiplying the given equation throughout by $\frac{1}{x^2y^2}$, it becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$$\text{Its solution is } \int_{y \text{ constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y} dy = C \quad \text{or} \quad \frac{x}{y} - 2 \log x + 3 \log y = C$$

where C is the constant of integration.

29. Solve : $y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$

Sol. Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = y^3 - 2x^2y, \quad N = 2xy^2 - x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2, \quad \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is not exact, but it is homogeneous in x and y .

$$\text{Also} \quad Mx + Ny = x(y^3 - 2x^2y) + y(2xy^2 - x^3) = 3xy(y^2 - x^2)$$

$$\therefore \text{ I.F. (Integrating Factor)} = \frac{1}{Mx + Ny} = \frac{1}{3xy(y^2 - x^2)}$$

Multiplying the given equation $y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$ by I.F., we get

$$\frac{y(y^2 - 2x^2)}{3xy(y^2 - x^2)} \cdot dx + \frac{x(2y^2 - x^2)}{3xy(y^2 - x^2)} \cdot dy = 0$$

$$\Rightarrow \frac{(y^2 - 2x^2)}{x(y^2 - x^2)} \cdot dx + \frac{2y^2 - x^2}{y(y^2 - x^2)} \cdot dy = 0$$

$$\Rightarrow \frac{1}{x} \cdot dx - \frac{x}{y^2 - x^2} \cdot dx + \frac{1}{y} \cdot dy + \frac{y}{y^2 - x^2} \cdot dy = 0$$

$$\Rightarrow \frac{1}{x} \cdot dx + \frac{1}{y} \cdot dy + \frac{1}{2} \left(\frac{2ydy - 2xdx}{y^2 - x^2} \right) = 0$$

$$\Rightarrow d(\log x) + d(\log y) + \frac{1}{2} d\{\log(y^2 - x^2)\} = 0$$

$$\Rightarrow d(\log x) + d(\log y) + d(\log \sqrt{y^2 - x^2}) = d(\log C)$$

$$\text{On integration, we get} \quad \log x + \log y + \log \sqrt{y^2 - x^2} = \log C$$

$$\text{or} \quad xy\sqrt{y^2 - x^2} = C$$

where C is an arbitrary constant of integration.

30. Solve : $y(xy + 2x^2y^2)dx + x(xy - x^2y^2) dy = 0$

Sol. The given equation is of the form

$$f_1(xy) y dx + f_2(xy) \cdot x dy = 0$$

The rule is that $\frac{1}{Mx - Ny}$ is an I.F. of such differential equation provided $Mx - Ny \neq 0$.

Here,

$$M = xy^2 + 2x^2y^3, N = x^2y - x^3y^2$$

$$Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0.$$

$$\therefore \text{I.F.} = \frac{1}{3x^3y^3}$$

Multiplying the given equation by $\frac{1}{3x^3y^3}$ throughout, the given equation becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x} \right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y} \right) dy = 0$$

which is exact differential equation.

Its solution is $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x} \right) dx = -\frac{1}{3xy} + \frac{2}{3} \log x$$

$$F(y) = \int (\text{Terms in } N \text{ not containing } x) dy = \int \left(-\frac{1}{3y} \right) dy = -\frac{1}{3} \log y$$

$$\Rightarrow -\frac{1}{3xy} + \frac{2}{3} \log x - \frac{1}{3} \log y = C \quad \text{or} \quad -\frac{1}{xy} + 2 \log x - \log y = C_1, \text{ where } C_1 = 3C,$$

is the solution of the given differential equation.

31. Solve : $(y - xy^2) dx - (x + x^2y) dy = 0$.

Sol. The given equation is $(y - xy^2)dx - (x + x^2y)dy = 0$

...(1)

Comparing with $Mdx + Ndy = 0$, we get

$$M = y - xy^2, N = -x - x^2y$$

$$\therefore \frac{\partial M}{\partial y} = 1 - 2xy, \frac{\partial N}{\partial x} = -1 - 2xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ Equation (1) is not exact.}$$

But

$$M = y(1 - xy), N = -x(1 + xy)$$

the given equation (1) is of the form

$$yf_1(xy)dx + x \cdot f_2(xy) dy = 0$$

Also,

$$Mx - Ny = (y - xy^2)x + x(1 + xy)y = 2xy \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{2xy}$$

Multiplying (1) throughout by $\frac{1}{2xy}$, we obtain

$$\frac{y - xy^2}{2xy} \cdot dx - \frac{x + x^2y}{2xy} \cdot dy = 0$$

$$\text{or} \quad \frac{1}{2} \left(\frac{1}{x} - y \right) dx - \frac{1}{2} \left(\frac{1}{y} + x \right) dy = 0 \quad \dots(2)$$

Comparing with $M \cdot dx + N \cdot dy = 0$, we get

$$M = \frac{1}{2} \left(\frac{1}{x} - y \right), N = -\frac{1}{2} \left(\frac{1}{y} + x \right)$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{1}{2} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Equation (2) is an exact differential equation.

Solution of equation (2) is given by

$$u + F(y) = C$$

$$\text{where,} \quad u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \frac{1}{2} \left(\frac{1}{x} - y \right) dx = \frac{1}{2} (\log x - yx)$$

$$\text{and} \quad F(y) = \int -\frac{1}{2y} dy = -\frac{1}{2} \log y.$$

Hence, the solution is

$$\frac{1}{2} (\log x - yx) - \frac{1}{2} \log y = C \Rightarrow \log x - xy - \log y = 2C \Rightarrow \log \left(\frac{x}{y} \right) - xy = 2C.$$

$$\mathbf{32.} \text{ Solve : } (xy^2 - e^{\frac{1}{x^3}}) dx - x^2y dy = 0.$$

Sol. For the equation $Mdx + Ndy = 0$,

$$(i) \text{ If } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x), \text{ a function of } x \text{ only, then } e^{\int f(x) dx} \text{ is an I.F.}$$

$$(ii) \text{ If } \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y), \text{ a function of } y \text{ only, then } e^{\int g(y) dy} \text{ is an I.F.}$$

In the given equation, on comparing with $Mdx + Ndy = 0$, we have

$$M = xy^2 - e^{\frac{1}{x^3}}, N = -x^2y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x}$$

which is a function of x only.

$$\therefore \quad \text{I.F.} = e^{\int -\frac{4}{x} dx} = e^{-4 \log x} = \frac{1}{x^4}$$

$$\text{Multiplying throughout by } \frac{1}{x^4}, \text{ we get } \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx - \frac{y}{x^2} \cdot dy = 0$$

which is an exact differential equation.

The solution of this equation is given by

$$\int_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4} \cdot e^{\frac{1}{x^3}} \right) dx = C$$

or
$$-\frac{y^2}{2x^2} + \int \left(-\frac{1}{x^4} \right) e^{\frac{1}{x^3}} \cdot dx = C$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} \int \left(-\frac{3}{x^4} \right) e^{\frac{1}{x^3}} \cdot dx = C$$

Put $\frac{1}{x^3} = t$, $-\frac{3}{x^4} dx = dt$

Then second term $= \frac{1}{3} \int e^t dt = \frac{1}{3} \cdot e^t$

Hence the solution is given by

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = C \text{ or } -\frac{3y^2}{x^2} + 2e^{\frac{1}{x^3}} = C_1 \text{ where } C_1 = 6C.$$

33. Solve : $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Sol. Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = xy^3 + y, N = 2x^2y^2 + 2x + 2y^4$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y} = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

which is a function of y only.

$$\therefore \text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Multiplying the given equation throughout by y , we have

$$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$$

which is exact differential equation.

Its solution is given by

$$\int_{y \text{ constant}} (xy^4 + y^2) dx + \int 2y^5 \cdot dy = C \text{ or } \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C$$

where C is an arbitrary constant of integration.

34. Solve the D.E. $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$

(M.D.U., Dec., 2008)

Sol. Compare with $Mdx + Ndy = 0$

$$M = y^4 + 2y, N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given D.E. is not exact.

Now,
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3y^3 - 6 = -3(y^3 + 2)$$

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}$$

$$\therefore \text{I.F.} = e^{\int \frac{-3}{y} dy} = e^{-3 \log y} = \frac{1}{y^3}$$

Multiplying the given D.E. with I.F.,

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$$

This is exact D.E. and its solution is given by

$$\int_{y \text{ constant}} \left(y + \frac{2}{y^2}\right) dx + \int 2y dy = c$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right) x + y^2 = c \text{ is the solution of the given D.E.}$$

35. Solve : $(y \log y) dx + (x - \log y) dy = 0$

Sol. The given equation is $(y \log y) dx + (x - \log y) dy = 0$

...(1)

Comparing with $M dx + N dy = 0$, we get

$$M = y \log y, N = x - \log y$$

$$\therefore \frac{\partial M}{\partial y} = y \cdot \frac{1}{y} + \log y = 1 + \log y, \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

$\therefore$ Equation (1) is not exact

$$\text{Now, } \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y \log y} (1 - 1 - \log y) = -\frac{1}{y}$$

$$\therefore \text{I.F.} = e^{\int \left(-\frac{1}{y}\right) dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying equation (1) throughout by $\frac{1}{y}$, we get

$$(\log y) dx + \left(\frac{x - \log y}{y}\right) dy = 0 \quad \dots(2)$$

Again comparing equation (2) with $M dx + N dy = 0$, we get

$$M = \log y, N = \frac{x}{y} - \frac{\log y}{y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{1}{y}, \frac{\partial N}{\partial x} = \frac{1}{y} \quad i.e., \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (2) is an exact differential equation.

Its solution is given by $u = F(y) + C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \log y \cdot dx = x \log y$$

$$F(y) = \int (\text{terms in } N \text{ not containing } x) dy$$

$$= \int \left(-\frac{\log y}{y} \right) dy = -\frac{1}{2} (\log y)^2$$

Hence the solution is $x \log y - \frac{1}{2} (\log y)^2 = C$, where C is an arbitrary constant.

36. Solve : $\left(y + \frac{1}{3} y^3 + \frac{1}{2} x^2 \right) dx + \frac{1}{4} (1 + y^2) x dy = 0$ (M.D.U., May 2007)

Sol. The given equation is

$$\left(y + \frac{1}{3} y^3 + \frac{1}{2} x^2 \right) dx + \frac{1}{4} (1 + y^2) x dy = 0 \quad \dots(1)$$

On comparing with $Mdx + Ndy = 0$, we have

$$M = y + \frac{1}{3} y^3 + \frac{1}{2} x^2, \quad N = \frac{1}{4} (1 + y^2)x$$

$$\frac{\partial M}{\partial y} = 1 + y^2, \quad \frac{\partial N}{\partial x} = \frac{1}{4} (1 + y^2)$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is not exact

Now,

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{(1 + y^2) - \frac{1}{4} (1 + y^2)}{\frac{1}{4} (1 + y^2) x} = \frac{3}{x} = f(x)$$

$$\therefore \text{I.F.} = e^{\int f(x) dx} = e^{\int \frac{3}{x} \cdot dx} = e^{3 \log x} = x^3$$

Multiplying equation (1) by x^3 throughout, we get

$$\left(yx^3 + \frac{1}{3} y^3 x^3 + \frac{1}{2} x^5 \right) dx + \frac{1}{4} (1 + y^2) x^4 dy = 0 \quad \dots(2)$$

Comparing with $Mdx + Ndy = 0$, we have

$$M = yx^3 + \frac{1}{3} y^3 x^3 + \frac{1}{2} x^5, \quad N = \frac{1}{4} (1 + y^2) x^4$$

$$\therefore \frac{\partial M}{\partial y} = x^3 + x^3 y^2 = x^3 (1 + y^2), \quad \frac{\partial N}{\partial x} = x^3 (1 + y^2) = \frac{\partial M}{\partial y}$$

$\therefore$ Equation (2) is exact differential.

Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \left(yx^3 + \frac{1}{3} y^3 x^3 + \frac{1}{2} x^5 \right) dx = \frac{yx^4}{4} + \frac{y^3 x^4}{12} + \frac{x^6}{12}$$

and $F(y) = \int (\text{terms in } N \text{ not containing } x) dy$

Since there is no term in N containing function of y only, the solution is $u = C$

$$\Rightarrow \frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = C, \text{ where } C \text{ is an arbitrary constant of integration.}$$

37. Solve : $2ydx + x(2 \log x - y) dy = 0$.

Sol. The given equation is $2ydx + x(2 \log x - y) dy = 0$... (1)

Here, $M = 2y, N = x(2 \log x - y)$

$$\therefore \frac{\partial M}{\partial y} = 2, \frac{\partial N}{\partial x} = 2 \left(x \cdot \frac{1}{x} + \log x \right) - y = 2(1 + \log x) - y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

$\therefore$ Equation (1) is not exact

$$\text{Now, } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{x(2 \log x - y)} (-2 \log x + y) = -\frac{1}{x} = f(x)$$

$$\therefore \text{I.F.} = e^{\int f(x) dx} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

Multiplying equation (1) by $\frac{1}{x}$, we get $\frac{2y}{x} dx + (2 \log x - y) dy = 0$... (2)

Here, $M = \frac{2y}{x}, N = 2 \log x - y$

$$\frac{\partial M}{\partial y} = \frac{2}{x}, \frac{\partial N}{\partial x} = \frac{2}{x} \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (2) is exact

Solution of equation (2) is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} \frac{2y}{x} \cdot dx = 2y \log x$$

$$F(y) = \int (\text{terms in } N \text{ not containing } x) dy = \int -y \cdot dy = -\frac{y^2}{2}$$

Hence the solution is $2y \log x - \frac{y^2}{2} = C$, where C is an arbitrary constant.

38. Solve : $y(2xy + e^x) dx - e^x dy = 0$

Sol. On comparing with $Mdx + Ndy = 0$, we have

$$M = 2xy^2 + ye^x, \quad N = -e^x$$

$$\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{-2(2xy + e^x)}{y(2xy + e^x)} = -\frac{2}{y} = g(y) \text{ i.e., a function of } y \text{ only.}$$

$$\therefore \text{I.F.} = e^{\int g(y) dy} = e^{\int -\frac{2}{y} dy} = e^{-2 \log y} = \frac{1}{y^2}$$

Multiplying the given differential equation with $\frac{1}{y^2}$ throughout, we get

$$2x \cdot dx + \frac{e^x}{y} dx - \frac{e^x}{y^2} \cdot dy = 0$$

$$d(x^2) + d(e^x/y) = 0 \quad \Rightarrow \quad x^2 + \frac{e^x}{y} = C \text{ is the solution.}$$

39. Solve the D.E. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

(M.D.U., May 2009)

Sol.

$$M = 3x^2y^4 + 2xy, \quad N = 2x^3y^3 - x^2$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Given D.E. is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy} = \frac{-2x(3xy^3 + 2)}{xy(3xy^3 + 2)} = -\frac{2}{y}$$

$$\therefore \text{I.F. is } e^{\int -\frac{2}{y} dy} = e^{-2 \log y} = \frac{1}{y^2}$$

Multiplying the given D.E. with $\frac{1}{y^2}$, we get

$$\frac{3x^2y^4 + 2xy}{y^2} \cdot dx + \frac{2x^3y^3 - x^2}{y^2} \cdot dy = 0$$

which is exact

$$\therefore \text{The solution is } \int_{y \text{ constant}} \left(\frac{3x^2y^4 + 2xy}{y^2} \right) dx + \int 0 = C$$

$$\text{or } \int_{y \text{ constant}} \left(3x^2y^2 + \frac{2x}{y} \right) dx = C$$

$$x^3y^2 + \frac{x^2}{y} = C \text{ is the solution.}$$

40. Solve : $(1 + xy)y dx + (1 - xy)x dy = 0$.

Sol. Given equation is of the form $f_1(xy) y dx + f_2(xy) x dy = 0$.

We know that if $Mdx + Ndy = 0$ is of this form, then $\frac{1}{Mx - Ny}$ is an I.F. provided

$$Mx - Ny \neq 0$$

Comparing the given equation $(1 + xy)y dx + (1 - xy)x dy = 0$ with $M dx + N dy = 0$, we get

$$M = y + xy^2, \quad N = x - x^2y$$

$$\therefore Mx - Ny = xy + x^2y^2 - xy + x^2y^2 = 2x^2y^2 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $\frac{1}{2x^2y^2}$, the given equation becomes

$$\frac{y + xy^2}{2x^2y^2} \cdot dx + \frac{x - x^2y}{2x^2y^2} \cdot dy = 0 \quad \text{or} \quad \left(\frac{1}{2x^2y} + \frac{1}{2x} \right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y} \right) dy = 0$$

Its solution is

$$\begin{aligned} \int \left(\frac{1}{2x^2y} + \frac{1}{2x} \right) dx + \int \left(-\frac{1}{2y} \right) dy = C &\Rightarrow -\frac{1}{2xy} + \frac{1}{2} \log x - \frac{1}{2} \log y = C \\ -\frac{1}{xy} + \log \left(\frac{x}{y} \right) &= 2C \end{aligned}$$

is the solution, where C is an arbitrary constant of integration.

41. Solve : $(x^2y^2 + xy + 1)y dx + (x^2y^2 - xy + 1)x dy = 0$ (M.D.U., May 2008, Dec., 2009)

Sol. On comparing the given equation with

$M dx + N dy = 0$, we have

$$M = (x^2y^2 + xy + 1)y = x^2y^3 + xy^2 + y, \quad N = (x^2y^2 - xy + 1)x = x^3y^2 - x^2y + x$$

Since the given equation $M dx + N dy$ is of the form $f_1(xy) y dx + f_2(xy) x dy = 0$, $\frac{1}{Mx - Ny}$ is an I.F. provided $Mx - Ny \neq 0$.

$$Mx - Ny = (x^3y^3 + x^2y^2 + xy) - (x^3y^3 - x^2y^2 + xy) = 2x^2y^2 \neq 0.$$

$$\text{I.F.} = \frac{1}{2x^2y^2}$$

Multiplying the given equation with $\frac{1}{2x^2y^2}$ throughout, we get

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right) dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2} \right) dy = 0$$

Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right) dx = \frac{xy}{2} + \frac{1}{2} \log x - \frac{1}{2xy}$$

$$F(y) = \int \left(-\frac{1}{2y} \right) dy = -\frac{1}{2} \log y$$

$$\Rightarrow \frac{xy}{2} + \frac{1}{2} \log x - \frac{1}{2xy} - \frac{1}{2} \log y = C \quad \text{or} \quad xy + \log(x/y) - \frac{1}{xy} = 2C = C_1$$

where C is an arbitrary constant.

42. Solve : $(3xy - 2ay^2) dx + (x^2 - 2axy) dy = 0$.

Sol. On comparing with $M dx + N dy = 0$, we get from the given equation

$$M = 3xy - 2ay^2, \quad N = x^2 - 2axy$$

$$\text{Now,} \quad \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{(3x - 4ay) - (2x - 2ay)}{x^2 - 2axy} = \frac{1}{x} \quad \text{which is a function of } x \text{ only.}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{x} \cdot dx} = e^{\log x} = x$$

Multiplying throughout by x the given equation, we get

$$(3x^2y - 2axy^2) dx + (x^3 - 2ax^2y) dy = 0, \text{ which is an exact equation.}$$

Its solution is given by $u + F(y) = C$

$$\text{where} \quad u = \int_{y \text{ constant}} M \cdot dx = \int_{y \text{ constant}} (3x^2y - 2axy^2) dx$$

or

$$u = x^3y - ay^2x^2$$

and since N does not have any term containing y only, $F(y) = 0$

Hence solution is $x^3y - ay^2x^2 = -C \Rightarrow x^2(ay^2 - xy) = C$.

43. Solve : $(x^4e^x - 2mxy^2) dx + 2mx^2y dy = 0$.

Sol. On comparing the given equation with $M dx + N dy = 0$, we have

$$M = x^4e^x - 2mxy^2, \quad N = 2mx^2y$$

$$\therefore \quad \frac{\partial M}{\partial y} = -4mxy, \quad \frac{\partial N}{\partial x} = 4mxy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given differential equation is not exact.

Now,
$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{-8mxy}{2mx^2y} = -\frac{4}{x} = f(x)$$

which is a function of x only

$$\therefore \text{I.F.} = e^{\int f(x) dx} = e^{-4 \int \frac{dx}{x}} = e^{-4 \log x} = \frac{1}{x^4}$$

Multiplying the given differential equation throughout by $\frac{1}{x^4}$, we get

$$\left(e^x - \frac{2my^2}{x^3} \right) dx + \frac{2my}{x^2} dy = 0$$

or
$$e^x - 2m \left\{ \frac{y^2}{x^3} - \frac{y}{x^2} \cdot \frac{dy}{dx} \right\} = 0 \quad \text{or} \quad e^x + m \left\{ \frac{-2y^2x + 2x^2y \frac{dy}{dx}}{x^4} \right\} = 0$$

or
$$e^x + m \left\{ \frac{x^2 \cdot 2y \frac{dy}{dx} - (2x)y^2}{(x^2)^2} \right\} = 0 \Rightarrow e^x + m \cdot \frac{d}{dx} \left(\frac{y^2}{x^2} \right) = 0$$

On integrating term by term it gives, $e^x + m \left(\frac{y}{x} \right)^2 = C$, where C is an arbitrary constant of integration.

This is the solution of the given differential equation.

44. Solve : $(y^2 + 2x^2y)dx + (2x^3 - xy) dy = 0$.

Sol. The given differential equation is $(y^2 + 2x^2y) dx + (2x^3 - xy) dy = 0$

This can be re-written as $x^2(2ydx + 2xdy) + y(ydx - xdy) = 0$

which is of the form $x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$

$$\Rightarrow a = 2, b = 0, m = 2, n = 2, c = 0, d = 1, p = 1, q = -1$$

which are all constants.

I.F. for the equation of this form is $x^h \cdot y^k$, where h and k are so chosen that after multiplication by $x^h \cdot y^k$, the equation becomes exact.

The values of h and k can be determined from the relations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{and} \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

This implies that $\frac{3+h}{2} = \frac{k+1}{2}$ and $\frac{h+1}{1} = \frac{k+2}{-1}$ or $h-k = -2$

$$h+k = -3$$

$$\Rightarrow h = -5/2, k = -1/2$$

$$\therefore \text{I.F. is } x^{-5/2} \cdot y^{-1/2}$$

Now multiplying the given differential equation by $x^{-5/2} \cdot y^{-1/2}$, we get

$$(x^{-5/2} \cdot y^{3/2} + 2 \cdot x^{-1/2} \cdot y^{1/2}) dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}) dy = 0$$

which is an exact equation.

Its solution is given by $u = C$, where

$$\begin{aligned} u &= \int_{y \text{ constant}} (x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2}) dx \\ &= y^{3/2} \left(-\frac{2}{3} \cdot x^{-3/2} \right) + y^{1/2} (4x^{1/2}) = 4\sqrt{xy} - \frac{2}{3} \left(\frac{y}{x} \right)^{3/2} \end{aligned}$$

Here $F(y) = 0$ since N does not contain any term having y only.

Hence, the solution is given by $4\sqrt{xy} - \frac{2}{3} \left(\frac{y}{x} \right)^{3/2} = C$.

45. Solve : $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$.

Sol. Rewriting the given equation as

$$xy^2dx + x^2ydy + 2x^2y^3dx - x^3y^2dy = 0$$

or

$$xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0$$

Comparing this with

$$x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$$

We have, $a = b = 1, m = n = 1, c = d = 2, p = 2, q = -1$ which are all constants.

$$\text{I.F.} = x^h y^k, \text{ where } \frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

$$\Rightarrow h - k = 0, \quad h + 2k + 9 = 0$$

$$\Rightarrow h = k = -3. \quad \therefore \text{I.F.} = 1/x^3 y^3$$

Multiplying the given differential equation by $\frac{1}{x^3 y^3}$, we get

$$\left(\frac{1}{x^2 y} + \frac{2}{x} \right) dx + \left(\frac{1}{xy^2} - \frac{1}{y} \right) dy = 0$$

which is an exact equation.

Its solution is given by $u + F(y) = C$, where

$$u = \int_{y \text{ constant}} \left(\frac{1}{x^2 y} + \frac{2}{x} \right) dx, \quad F(y) = \int -\frac{1}{y} \cdot dy$$

$$\text{i.e.,} \quad u = -\frac{1}{xy} + 2 \log x, \quad F(y) = -\log y$$

$$\Rightarrow \text{Solution is } 2 \log x - \log y - \frac{1}{xy} = C$$

where C is an arbitrary constant of integration.

46. Solve : $(2x^2y^2 + y)dx + (3x - x^3y)dy = 0$.

Sol. The equation can be written as $2x^2y^2dx - x^3y^2dy + ydx + 3x dy = 0$

$$x^2y (2ydx - xdy) + (ydx + 3xdy) = 0$$

Comparing with $x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$

We have,

$$a = 2, b = 1, m = 2, n = -1, c = 0, d = 0$$

$p = 1, q = 3$ which are all constants.

Therefore, it has an I.F. of the form $x^h y^k$, where h and k are determined from the relations

$$\begin{aligned} \frac{a+h+1}{m} &= \frac{b+k+1}{n} \quad \text{and} \quad \frac{c+h+1}{p} = \frac{d+k+1}{q} \\ \Rightarrow \quad \frac{h+3}{2} &= \frac{k+2}{-1} \quad \text{and} \quad \frac{h+1}{1} = \frac{k+1}{3} \\ \Rightarrow \quad h+2k+7 &= 0 \quad \text{and} \quad 3h-k+2=0 \end{aligned}$$

Solving these 2 equations we obtain

$$h = -\frac{11}{7} \quad \text{and} \quad k = -\frac{19}{7} \quad \therefore \text{I.F.} = x^{-\frac{11}{7}} y^{-\frac{19}{7}}$$

Multiplying the given equation by $x^{-\frac{11}{7}} y^{-\frac{19}{7}}$, we have

$$(2x^{3/7} y^{-5/7} + x^{-11/7} y^{-12/7}) dx + (3x^{-4/7} y^{-19/7} - x^{10/7} y^{-12/7}) dy = 0$$

which is exact.

The solution is $\int_{y \text{ constant}} (2x^{3/7} y^{-5/7} + x^{-11/7} y^{-12/7}) dx = c$

$$\Rightarrow \quad 4x^{10/7} y^{-5/7} - 5x^{-4/7} y^{-12/7} = C, \text{ where } C = \frac{20}{7} c.$$

47. Eliminate the arbitrary constants A and B from the equation $y = e^x (A \cos x + B \sin x)$ and obtain the differential equation.

Sol. We have the relation $y = e^x (A \cos x + B \sin x)$... (1)

Differentiating (1) w.r.t. x , we have

$$\begin{aligned} \frac{dy}{dx} &= e^x (A \cos x + B \sin x) + e^x (-A \sin x + B \cos x) \\ &= y + e^x (-A \sin x + B \cos x) \end{aligned} \quad \dots (2)$$

Differentiating again w.r.t. x , we have

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{dy}{dx} + e^x (-A \sin x + B \cos x) + e^x (-A \cos x - B \sin x) \\ &= \frac{dy}{dx} + \left(\frac{dy}{dx} - y \right) - y \end{aligned} \quad [\text{Using (1) and (2)}]$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \text{ is the required differential equation.}$$

48. Solve: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Sol. Separating the variables, we have $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides, it gives $\tan^{-1} y = \tan^{-1} x + \tan^{-1} C$, where $\tan^{-1} C$ is an arbitrary constant of integration

$$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = \tan^{-1} C \Rightarrow \tan^{-1} \left(\frac{y-x}{1+xy} \right) = \tan^{-1} C \Rightarrow \frac{y-x}{1+xy} = C$$

or $y-x = C(1+xy)$ is the required solution of the given differential equation.

49. Solve : $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$.

Sol. Separating the variables, we get $\frac{\sec^2 x}{\tan x} \cdot dx + \frac{\sec^2 y}{\tan y} \cdot dy = 0$

On integrating, we get $\log \tan x + \log \tan y = \log C$

where $\log C$ is an arbitrary constant of integration.

$\Rightarrow \log \tan x \tan y = \log C \Rightarrow \tan x \tan y = C$

which is the required solution.

50. Solve : $\frac{dy}{dx} = e^{x-y} + x^2 \cdot e^{-y}$.

Sol. Given equation is $\frac{dy}{dx} = e^{x-y} + x^2 \cdot e^{-y}$

Multiplying both sides by e^y , we have

$$e^y \frac{dy}{dx} = e^x + x^2 \Rightarrow e^y dy = (e^x + x^2) dx$$

On integration, it gives

$$e^y = e^x + \frac{x^3}{3} + C, \text{ where } C \text{ is an arbitrary constant of integration.}$$

51. Solve : $y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right)$

Sol. We have $y - x \frac{dy}{dx} = ay^2 + a \frac{dy}{dx}$ or $\frac{dy}{dx} (x + a) = y - ay^2 = y(1 - ay)$

Separating the variables, we obtain $\frac{dy}{y(1 - ay)} = \frac{dx}{x + a}$

Resolving the left hand term in partial fractions, $\left(\frac{1}{y} + \frac{a}{1 - ay} \right) dy = \frac{dx}{x + a}$

Now integrating, we get $\log y - \log (1 - ay) = \log (x + a) + \log C$

where $\log C$ is an arbitrary constant of integration.

$\Rightarrow \log \left(\frac{y}{1 - ay} \right) = \log \{C(x + a)\} \Rightarrow \frac{y}{1 - ay} = C(x + a) \Rightarrow y = C(1 - ay)(x + a)$

which gives the required solution.

52. Solve : $\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$ [U.P.T.U., B.Pharm. (C.O.), 2005]

Sol. The given equation can be written as

$$(\sin y + y \cos y) dy = x(2 \log x + 1) dx$$

Integrating both sides, we get

$$-\cos y + y \sin y - \int 1 \cdot \sin y \, dy = 2 \int (\log x) x \, dx + \frac{x^2}{2} + C$$

$$\Rightarrow -\cos y + y \sin y + \cos y = 2 \log x \cdot \left(\frac{x^2}{2} \right) - 2 \int \frac{1}{x} \cdot \frac{x^2}{2} dx + \frac{x^2}{2} + C$$

$$\Rightarrow y \sin y = x^2 \log x + C \text{ which gives the required solution.}$$

53. Solve : $(x + y + 1)^2 \cdot \frac{dy}{dx} = 1$.

Sol. Putting $x + y + 1 = t$

$$\therefore 1 + \frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$\therefore$ The given equation becomes

$$t^2 \left(\frac{dt}{dx} - 1 \right) = 1 \quad \text{or} \quad \frac{dt}{dx} = \frac{1+t^2}{t^2}$$

$$\Rightarrow \frac{t^2}{1+t^2} \cdot dt = dx \quad \text{or} \quad \left(1 - \frac{1}{1+t^2} \right) dt = dx$$

On integrating term by term, we obtain

$$t - \tan^{-1} t = x + C \quad \text{or} \quad (x + y + 1) - \tan^{-1} (x + y + 1) = x + C$$

$$\text{or} \quad y = \tan^{-1} (x + y + 1) + C', \text{ where } C' = C - 1.$$

is the solution of the given differential equation.

54. Solve : $\frac{dy}{dx} = \cos(x + y) + \sin(x + y)$

Sol. Let $x + y = t$ then $1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

The given equation becomes

$$\frac{dt}{dx} - 1 = \cos t + \sin t \Rightarrow \frac{dt}{1 + \cos t + \sin t} = dx$$

| Separating the variables

Writing $\cos t$ and $\sin t$ in terms of $\tan \frac{t}{2}$

$$\frac{dt}{1 + \frac{1 - \tan^2 \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} + \frac{2 \tan \frac{t}{2}}{1 + \tan^2 \frac{t}{2}}} = dx \Rightarrow \frac{1 + \tan^2 \frac{t}{2}}{2 \left(1 + \tan \frac{t}{2} \right)} dt = \frac{\sec^2 \frac{t}{2} \cdot dt}{2 \left(1 + \tan \frac{t}{2} \right)} = dx$$

Now integrating both sides, it gives

$$\log \left(1 + \tan \frac{t}{2} \right) = x + C \Rightarrow \log \left\{ 1 + \tan \left(\frac{x + y}{2} \right) \right\} = x + C$$

where C is an arbitrary constant of integration.

55. Solve: $3e^x \tan y \, dx + (1 + e^x) \sec^2 y \, dy = 0$, given $y = \pi/4$ when $x = 0$

Sol. The given equation can be written as $\frac{3e^x}{1 + e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

On integration, we get

$$3 \log(1 + e^x) + \log \tan y = \log C \Rightarrow \log(1 + e^x)^3 \tan y = \log C$$

$$\Rightarrow (1 + e^x)^3 \tan y = C$$

which is the general solution of the given differential equation.

When $x = 0$, $y = \frac{\pi}{4}$ we have $(1 + 1)^3 \tan \frac{\pi}{4} = C$ or $C = 8$

$\therefore$ The required particular solution is $(1 + e^x)^3 \tan y = 8$.

56. Solve : $x \frac{dy}{dx} + \cot y = 0$ if $y = \frac{\pi}{4}$ when $x = \sqrt{2}$

Sol. Given equation is $x \frac{dy}{dx} = -\cot y$ or $\frac{dx}{x} = -\frac{dy}{\cot y} = -\frac{\sin y}{\cos y} \cdot dy$

Integrating both sides, we get

$$\log x + \log C = \log \cos y$$

$$\Rightarrow \log Cx = \log \cos y \quad \text{or} \quad Cx = \cos y \quad \text{or} \quad x \sec y = \frac{1}{C}$$

is the general solution of the equation. Given $y = \frac{\pi}{4}$ when $x = \sqrt{2}$

$$\Rightarrow \sqrt{2} \sec \frac{\pi}{4} = \frac{1}{C} \quad \text{or} \quad C = \frac{1}{2}. \quad \text{Particular solution is } x \sec y = 2.$$

57. Solve : $\frac{dy}{dx} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\text{Given equation is} \quad \frac{dt}{dx} - 4 = t^2 \quad \text{or} \quad \frac{dt}{dx} = t^2 + 4 \quad \text{or} \quad \frac{dt}{t^2 + 4} = dx$$

On integrating, we get

$$\frac{1}{2} \tan^{-1} \frac{t}{2} = x + C \quad \Rightarrow \quad 2 \tan (2x + 2C) = 4x + y + 1$$

$$\text{or} \quad 4x + y + 1 = 2 \tan \{2(x + C)\} \text{ is the solution.}$$

58. Solve : $\frac{dy}{dx} = x \tan (y - x) + 1$

Sol. Putting $y - x = t$, we get

$$\frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\therefore \text{ Given differential equation is } \frac{dt}{dx} = x \tan t \quad \text{or} \quad \frac{dt}{\tan t} = x dx$$

$$\text{Integrating both sides, we obtain } \log \sin t = \frac{x^2}{2} + C \quad \text{or} \quad \log \sin (y - x) = \frac{x^2}{2} + C$$

is the solution, where C is an arbitrary constant of integration.

59. Solve : $\sin (x + y) dy = dx$

$$\text{Sol. Given equation is} \quad \frac{dy}{dx} = \frac{1}{\sin (x + y)}$$

$$\text{Put } x + y = t, \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dt}{dx} - 1 = \frac{1}{\sin t}, \quad \frac{dt}{1 + \operatorname{cosec} t} = dx$$

$$\frac{\sin t \cdot dt}{1 + \sin t} = dx, \quad \frac{(1 - \sin t) \sin t \, dt}{1 - \sin^2 t} = dx$$

$$(\sec t \cdot \tan t - \tan^2 t) \, dt = dx \quad \text{or} \quad (\sec t \tan t - \sec^2 t + 1) \, dt = dx$$

On integrating both sides, we get

$$\sec t - \tan t + t = x + C \Rightarrow \sec(x + y) - \tan(x + y) + x + y = x + C$$

or $\tan(x + y) - \sec(x + y) = y + C$ is the solution of the given differential equation.

60. Solve : $\frac{dy}{dx} = \cos(x + y)$

Sol. Putting $x + y = t$, we get

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \text{ Given equation becomes } \frac{dt}{dx} - 1 = \cos t$$

$$\frac{dt}{dx} = 1 + \cos t = 2 \cos^2 \frac{t}{2} \quad \text{or} \quad dx = \frac{dt}{2 \cos^2 \frac{t}{2}} = \frac{1}{2} \sec^2 \left(\frac{t}{2} \right) \cdot dt$$

$$\text{Integrating both sides, we get } x + C = \tan \frac{t}{2} = \tan \left(\frac{x + y}{2} \right)$$

which is the solution of the given equation.

61. Solve : $\frac{y}{x} \frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$.

Sol. Putting $x^2 + y^2 = t$, we get

$$2x + 2y \frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad 1 + \frac{y}{x} \frac{dy}{dx} = \frac{1}{2x} \cdot \frac{dt}{dx} \Rightarrow \frac{y}{x} \frac{dy}{dx} = \frac{1}{2x} \frac{dt}{dx} - 1$$

$$\therefore \text{ Given equation becomes } \frac{1}{2x} \frac{dt}{dx} - 1 + \frac{t - 1}{2t + 1} = 0$$

$$\Rightarrow \frac{1}{2x} \cdot \frac{dt}{dx} = 1 - \frac{t - 1}{2t + 1} = \frac{t + 2}{2t + 1} \quad \text{or} \quad 2x \, dx = \frac{2t + 1}{t + 2} \, dt$$

$$\text{or} \quad 2x \, dx = \left(2 - \frac{3}{t + 2} \right) dt$$

On integrating, we obtain $x^2 = 2t - 3 \log(t + 2) + C$

Now, replacing t by $x^2 + y^2$, we get $x^2 = 2x^2 + 2y^2 - 3 \log(x^2 + y^2 + 2) + C$

$$\Rightarrow x^2 + 2y^2 - 3 \log(x^2 + y^2 + 2) + C = 0$$

is the solution of the given differential equation.

62. Solve : $3e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$.

Sol. Given equation is $3e^x \tan y \, dx = (e^x - 1) \sec^2 y \, dy$

$$\frac{3e^x}{e^x - 1} \cdot dx = \frac{\sec^2 y}{\tan y} \, dy$$

On integration, we get $3 \log (e^x - 1) = \log \tan y + \log C$

$\Rightarrow (e^x - 1)^3 = C \tan y$ is the solution of the given equation.

63. Solve : $\frac{dy}{dx} = \cos (x + y + 1).$

Sol. Putting $x + y + 1 = t$, we get

$$\therefore 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - 1 = \frac{dy}{dx}$$

Given equation becomes $\frac{dt}{dx} - 1 = \cos t$

$$\frac{dt}{dx} = 1 + \cos t = 2 \cos^2 t/2$$

$$\therefore dx = \frac{1}{2} \sec^2 t/2 \cdot dt$$

Integrating both sides, we get

$$x + C = \tan t/2 = \tan \left(\frac{x + y + 1}{2} \right)$$

which is the required solution of the given differential equation.

64. Solve : $\cos (x + y) dy = dx$

Sol. $\frac{dx}{dy} = \cos (x + y)$

Putting $x + y = t$, we get

$$\therefore \frac{dx}{dy} + 1 = \frac{dt}{dy} \quad \text{or} \quad \frac{dx}{dy} = \frac{dt}{dy} - 1$$

Given equation becomes

$$\frac{dt}{dy} - 1 = \cos t \quad \text{or} \quad \frac{dt}{dy} = 1 + \cos t \Rightarrow \frac{dt}{1 + \cos t} = dy$$

$$\text{or} \quad \frac{dt}{2 \cos^2 \frac{t}{2}} = dy \quad \text{or} \quad \frac{1}{2} \sec^2 \frac{t}{2} dt = dy$$

On integrating both sides, we get

$$y = \tan \frac{t}{2} + C \Rightarrow y = \tan \left(\frac{x + y}{2} \right) + C \text{ which is the required solution of the given equation.}$$

65. Solve : $(x + 1) \frac{dy}{dx} + 1 = 2e^{-y}$

Sol. Given equation is $(x + 1) \frac{dy}{dx} = 2e^{-y} - 1$ or $\frac{dy}{2e^{-y} - 1} = \frac{dx}{x + 1} \Rightarrow \frac{e^y}{2 - e^y} dy = \frac{dx}{x + 1}$

Integrating both sides, we get

$$\log (x + 1) = -\log (2 - e^y) + \log C$$

$$\Rightarrow \log \{(x + 1)(2 - e^y)\} = \log C \quad \text{or} \quad (x + 1)(2 - e^y) = C \text{ is the required solution.}$$

66. Solve : $(x^2 - y^2) dx - xy dy = 0$

Sol. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$

which is homogeneous equation since numerator and denominator are homogeneous functions of the same degree (2) in x and y .

Put $y = vx$ then, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

Given equation becomes $v + x \frac{dv}{dx} = \frac{1 - v^2}{v}$ or $x \frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}$

On separating the variables, we have

$$\frac{v}{1 - 2v^2} dv = \frac{dx}{x}$$

Integrating both sides,

$$\begin{aligned} \int \frac{v}{1 - 2v^2} dv &= \int \frac{dx}{x} + C \\ -\frac{1}{4} \int \frac{-4v}{1 - 2v^2} dv &= \log x + C \\ -\frac{1}{4} \log(1 - 2v^2) &= \log x + C \quad \text{or} \quad 4 \log x + \log(1 - 2v^2) = -4C \\ x^4(1 - 2y^2/x^2) &= e^{-4C} = C' \quad (\text{say}) \end{aligned}$$

Hence the required solution is $x^2(x^2 - 2y^2) = C'$

where C' is an arbitrary constant of integration.

67. Solve : $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$.

Sol. The given equation may be re-written as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}} \quad \dots(1)$$

which is a homogeneous equation.

Putting $x = vy$ in equation (1), we get

$$v + y \frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} \quad \text{or} \quad y \frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} - v = -\frac{v + e^v}{1 + e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1 + e^v}{v + e^v} dv$$

Integrating both sides, $-\log y = \log(v + e^v) + C$

or $y(v + e^v) = e^{-C} = C' \quad (\text{say})$ where $v = x/y$,

is the required solution.

68. Solve : $x^2y dx - (x^3 + y^3) dy = 0$

Sol. Given equation is $(x^3 + y^3) \frac{dy}{dx} = x^2y$ or $\frac{dx}{dy} = \frac{x^3 + y^3}{x^2y}$

Putting $x = vy$, $v + y \frac{dv}{dy} = \frac{v^3 + 1}{v^2}$ or $y \frac{dv}{dy} = \frac{v^3 + 1}{v^2} - v = \frac{1}{v^2}$ or $\frac{dy}{y} = v^2 dv$

Integrating both sides, we get $\log y = \frac{v^3}{3} + C$ or $\log y = \frac{x^3}{3y^3} + C = \frac{x^3}{3y^3} + \log C'$

$$\left(\frac{x}{y}\right)^3 = 3 \log C'y$$

is the required solution of the given differential equation.

69. Solve : $y dx - x dy = \sqrt{x^2 + y^2} dx$.

Sol. Given equation is $y - x \frac{dy}{dx} = \sqrt{x^2 + y^2}$

$$x \frac{dy}{dx} = y - \sqrt{x^2 + y^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{y}{x} - \frac{1}{x} \sqrt{x^2 + y^2} = \frac{y}{x} - \sqrt{1 + \frac{y^2}{x^2}}$$

Putting $y = vx$,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = v - \sqrt{1 + v^2} \quad \text{or} \quad x \frac{dv}{dx} = -\sqrt{1 + v^2}$$

Separating the variables, $\frac{dx}{x} = -\frac{dv}{\sqrt{1 + v^2}}$

Integrating both sides, we get

$$\log x = -\log \{v + \sqrt{1 + v^2}\} + \log C = -\log \left\{ \frac{y + \sqrt{x^2 + y^2}}{x} \right\} + \log C$$

$$= -\log \{y + \sqrt{x^2 + y^2}\} + \log x + \log C$$

$$\Rightarrow \log \{y + \sqrt{x^2 + y^2}\} = \log C$$

or $y + \sqrt{x^2 + y^2} = C$ is the solution of the given differential equation.

70. Solve : $(x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0$.

Sol. Given equation is $(x^3 - 3x^2y) dy = (x^2y - 2xy^2) dx$

$$\frac{dy}{dx} = \frac{x^2y - 2xy^2}{x^3 - 3x^2y} = \frac{xy - 2y^2}{x^2 - 3xy} \quad \dots(1)$$

This is homogeneous in x and y

Put $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore (1) \text{ becomes } v + x \frac{dv}{dx} = \frac{v - 2v^2}{1 - 3v}$$

$$\therefore x \frac{dv}{dx} = \frac{v - 2v^2}{1 - 3v} - v = \frac{v^2}{1 - 3v} \quad \text{or} \quad \frac{dx}{x} = \frac{(1 - 3v) dv}{v^2}$$

Integrating both sides, we get

$$\log x = -\frac{1}{v} - 3 \log v$$

$$\begin{aligned} \log x + 3 \log \left(\frac{y}{x} \right) &= -\frac{x}{y} - \log C \quad \text{or} \quad \log C + \log x + \log (y^3/x^3) = -\frac{x}{y} \\ \Rightarrow \quad -\frac{x}{y} &= \log (Cx \cdot y^3/x^3) = \log (Cy^3/x^2) \\ \text{i.e.,} \quad \frac{Cy^3}{x^2} &= e^{-x/y} \quad \text{or} \quad Cy^3 = x^2 e^{-x/y} \end{aligned}$$

is the required solution of the given differential equation.

71. Solve : $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$.

Sol. Given equation is $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

Dividing by x^2 throughout, we have

$$\frac{y^2}{x^2} + \frac{dy}{dx} = \frac{y}{x} \cdot \frac{dy}{dx} \quad \text{or} \quad \frac{y^2}{x^2} = \frac{dy}{dx} \cdot \left(\frac{y}{x} - 1 \right) \quad \text{or} \quad \frac{dy}{dx} = \frac{y^2}{x^2} \cdot \frac{x}{y-x} = \frac{y^2}{x(y-x)}$$

Put $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{v^2 x^2}{(v-1)x^2} = \frac{v^2}{v-1}$$

$$x \frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v}{v-1} \quad \text{or} \quad \frac{v-1}{v} \cdot dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \left(1 - \frac{1}{v} \right) dv = \log x + \log C$$

$$v - \log v = \log x + \log C$$

$$v = \log v + \log x + \log C$$

$$\frac{y}{x} = \log (y/x) + \log x + \log C$$

$$\frac{y}{x} = \log Cy$$

or

$y = x \log Cy$ is the solution of the given differential equation.

72. Solve : $(x^2 - y^2) dx = 2xy dy$.

Sol. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ which is homogeneous in x and y .

Putting $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = \frac{x^2(1-v^2)}{2vx^2} = \frac{1-v^2}{2v} \quad \text{or} \quad x \frac{dv}{dx} = \frac{1-v^2}{2v} - v$$

$$x \frac{dv}{dx} = \frac{1-3v^2}{2v}$$

Separating the variables, $\frac{2v}{1-3v^2} dv = \frac{dx}{x}$

$$\text{or} \quad -\frac{1}{3} \log(1-3v^2) = \log x + C \quad \Rightarrow \quad \log(1-3v^2)^{-1/3} - \log x = C$$

$$\Rightarrow \quad \log \left(1 - 3 \frac{y^2}{x^2} \right)^{-1/3} - \log x = C \quad \Rightarrow \quad \log \left(\frac{x^2 - 3y^2}{x^2} \right)^{-1/3} \cdot \frac{1}{x} = C$$

$$\Rightarrow \quad \left(\frac{x^2 - 3y^2}{x^2} \right)^{-1/3} = Cx \quad \Rightarrow \quad \left(\frac{x^2 - 3y^2}{x^2} \right)^{-1} = C^3 x^3$$

$$\frac{x^2}{x^2 - 3y^2} = C^3 x^3$$

$$1 = C^3 x(x^2 - 3y^2)$$

or $x(x^2 - 3y^2) = C'$ is the required solution of the given differential equation.

73. Solve : $\frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x}$.

Sol. Given equation is $\frac{dy}{dx} = \frac{y}{x} + \sin \left(\frac{y}{x} \right)$

Put $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = v + \sin v \quad \text{or} \quad x \frac{dv}{dx} = \sin v$$

Separating the variables, $\frac{dv}{\sin v} = \frac{dx}{x}$

On integrating both sides, we get

$$\log(\operatorname{cosec} v - \cot v) = \log x + C \quad \text{or} \quad \log \left(\frac{1 - \cos v}{\sin v} \right) = \log x + C$$

$$\log \left(\frac{1 - \cos v}{x \sin v} \right) = C \quad \Rightarrow \quad \frac{1 - \cos v}{x \sin v} = e^C$$

$$1 - \cos v = x \sin v \cdot e^C$$

$$1 - \cos(y/x) = x \sin(y/x) \cdot e^C$$

is the required solution of the given differential equation.

Note. We can simplify the solution further as follows :

$$1 - \cos 2\theta = 2 \sin^2 \theta$$

$$\therefore 1 - \cos(y/x) = 2 \sin^2(y/2x)$$

Also, $\sin(y/x) = 2 \sin y/2x \cos y/2x$

$$\therefore 1 - \cos y/x = x \sin y/x \cdot e^C$$

or $2 \sin^2(y/2x) = x \cdot 2 \sin y/2x \cdot \cos y/2x \cdot e^C$

$$\sin(y/2x) = x \cdot \cos(y/2x) \cdot e^C$$

or $\tan(y/2x) = xC' \quad \text{or} \quad \frac{y}{2x} = \tan^{-1}(C'x)$

i.e., $y = 2x \tan^{-1}(C'x)$ is the solution.

74. Solve : $\left(x \tan \frac{y}{x} - y \sec^2 \frac{y}{x} \right) dx + x \sec^2 \frac{y}{x} \cdot dy = 0.$

Sol. Given equation is

$$\left(x \tan \frac{y}{x} - y \sec^2 \frac{y}{x} \right) + x \sec^2 \frac{y}{x} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad x \sec^2 \frac{y}{x} \cdot \frac{dy}{dx} = y \sec^2 \frac{y}{x} - x \tan \frac{y}{x}$$

or
$$\frac{dy}{dx} = \frac{y}{x} - \frac{\tan \frac{y}{x}}{\sec^2 \frac{y}{x}}$$

Let $y/x = v, \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$$

$$x \frac{dv}{dx} = - \frac{\tan v}{\sec^2 v}$$

Separating the variables, we have

$$\frac{-\sec^2 v}{\tan v} \cdot dv = \frac{dx}{x}$$

Now integrating both sides,

$$\log C - \log \tan v = \log x \quad \text{or} \quad \log C = \log x + \log \tan v$$

or $\log C = \log [x \tan (y/x)] \quad \text{or} \quad x \tan y/x = C$ is the solution.

75. Solve : $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}.$

Sol. Given equation is
$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad \dots(1)$$

which is not homogeneous but can be reduced to homogeneous form.

Put $x = X + h, y = Y + k$ (h, k being constants) so that $dx = dX, dy = dY$.

Equation (1) becomes
$$\frac{dY}{dX} = \frac{Y+X+(k+h-2)}{Y-X+(k-h-4)} \quad \dots(2)$$

Put $h+k-2=0$ and $k-h-4=0$ so that

$$h = -1, k = 3$$

$\therefore$ Equation (2) becomes
$$\frac{dY}{dX} = \frac{Y+X}{Y-X} \quad \text{which is homogeneous in } X \text{ and } Y. \quad \dots(3)$$

Put $Y = vX$, then
$$\frac{dY}{dX} = v + X \frac{dv}{dX}$$

$$v + X \frac{dv}{dX} = \frac{v+1}{v-1} \quad \text{or} \quad X \frac{dv}{dX} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$$

or
$$\frac{v-1}{1+2v-v^2} dv = \frac{dX}{X}$$

Integrating both sides, we get

$$-\frac{1}{2} \int \frac{2-2v}{1+2v-v^2} \cdot dv = \int \frac{dX}{X} + C \quad \text{or} \quad -\frac{1}{2} \log (1+2v-v^2) = \log X + C$$

$$\log(1 + 2v - v^2) + 2 \log X = -2C \quad \text{or} \quad \log \left(1 + 2 \frac{Y}{X} - \frac{Y^2}{X^2} \right) + \log X^2 = -2C$$

$$\log(X^2 + 2XY - Y^2) = -2C \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2C} = C' \text{ (say)} \quad \dots(4)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (4) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = C'$$

or $x^2 + 2xy - y^2 - 4x + 8y - 14 = C'$ which is the required solution.

76. Solve : $(3y + 2x + 4)dx - (4x + 6y + 5)dy = 0$.

Sol. Given equation is $\frac{dy}{dx} = \frac{2x + 3y + 4}{2(2x + 3y) + 5} \quad \dots(1)$

Put $2x + 3y = t$ so that $2 + 3 \frac{dy}{dx} = \frac{dt}{dx}$

Equation (1) becomes $\frac{1}{3} \left(\frac{dt}{dx} - 2 \right) = \frac{t + 4}{2t + 5}$

or $\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22} \cdot dt = dx$

Integrating both sides $\int \frac{2t + 5}{7t + 22} \cdot dt = \int dx + C$

or $\int \left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t + 22} \right) dt = x + C \quad \text{or} \quad \frac{2}{7} t - \frac{9}{49} \log(7t + 22) = x + C$

Putting $t = 2x + 3y$, we have

$$14(2x + 3y) - 9 \log(14x + 21y + 22) = 49x + 49C$$

or $21x - 42y + 9 \log(14x + 21y + 22) = C'$

which is the required solution.

77. Solve : $(2x + y - 3)dy = (x + 2y - 3)dx$.

Sol. The given equation is $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

Put $x = X + h$, $y = Y + k$ (h, k being constants) so that $dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3}$$

Put $h + 2k - 3 = 0$ and $2h + k - 3 = 0$ so that $h = k = 1$

$\therefore \frac{dY}{dX} = \frac{X + 2Y}{2X + Y}$ which is homogeneous in X and Y .

Put $Y = vX$, then $\frac{dY}{dX} = v + X \frac{dv}{dX}$

$$v + X \frac{dv}{dX} = \frac{1 + 2v}{2 + v}$$

or $X \frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v} \quad \text{or} \quad \frac{2 + v}{1 - v^2} \cdot dv = \frac{dX}{X}$

Integrating both sides, we get

$$\int \frac{2 + v}{1 - v^2} \cdot dv = \int \frac{dX}{X} + C \quad \text{or} \quad \int \frac{2 + v}{(1 + v)(1 - v)} \cdot dv = \log X + C$$

Now,
$$\frac{2+v}{(1+v)(1-v)} = \frac{A}{1+v} + \frac{B}{1-v} \quad (\text{say})$$

(by Resolving into partial fractions)

$$\therefore \quad 2+v = A(1-v) + B(1+v)$$

$$2 = A + B$$

$$1 = -A + B \quad \text{or} \quad A = \frac{1}{2}, B = \frac{3}{2}$$

$$\therefore \quad \frac{2+v}{(1+v)(1-v)} = \frac{1}{2} \left[\frac{1}{1+v} + \frac{3}{1-v} \right]$$

Now integrating term by term,

$$\frac{1}{2} \int \left(\frac{1}{1+v} + \frac{3}{1-v} \right) dv = \frac{1}{2} [\log(1+v) - 3 \log(1-v)] = \log \left\{ \frac{(1+v)^{1/2}}{(1-v)^{3/2}} \right\}$$

Hence
$$\log \left\{ \frac{(1+v)^{1/2}}{(1-v)^{3/2}} \right\} = \log X + C \quad \text{or} \quad \log \left\{ \frac{\left(1 + \frac{Y}{X}\right)^{1/2}}{\left(1 - \frac{Y}{X}\right)^{3/2}} \right\} = \log X + C$$

or
$$\log \left\{ \frac{(X+Y)^{1/2} \cdot X^{3/2}}{X^{1/2} (X-Y)^{3/2}} \cdot \frac{1}{X} \right\} = C \quad \text{or} \quad \log \left\{ \frac{(X+Y)^{1/2}}{(X-Y)^{3/2}} \right\} = C$$

$$\frac{(X+Y)^{1/2}}{(X-Y)^{3/2}} = e^C \quad \therefore \quad (X+Y)^{1/2} = e^C \cdot (X-Y)^{3/2}$$

Squaring both sides, we get

$$X+Y = C' (X-Y)^3$$

As
$$X = x-1, \quad Y = y-1$$

$$x+y-2 = C'(x-y)^3$$

is the required solution of the given differential equation.

78. Solve : $(3y-7x+7)dx + (7y-3x+3)dy = 0$.

Sol. The given equation can be written as
$$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3} \quad \dots(1)$$

which is not homogeneous in this form. $\left[\text{case } \frac{a}{a'} \neq \frac{b}{b'} \right]$

To reduce equation (1) to be homogeneous

Put
$$x = X + h, \quad y = Y + k \quad (h, k \text{ are constants})$$

so that
$$dx = dX, \quad dy = dY$$

Equation (1) becomes
$$\frac{dY}{dX} = -\frac{3Y-7X+(-7h+3k+7)}{7Y-3X+(-3h+7k+3)} \quad \dots(2)$$

Now choosing h, k such that $-7h+3k+7=0$ and $-3h+7k+3=0$

Solving these equations we have $h=1, k=0$

With these values of h and k equation (2) reduces to

$$\frac{dY}{dX} = -\frac{3Y-7X}{7Y-3X} \quad \dots(3)$$

Putting $Y = vX$ so that

$$\frac{dY}{dX} = v + X \frac{dv}{dX}$$

Equation (3) becomes $v + X \frac{dv}{dX} = \frac{7-3v}{7v-3}$ or $X \frac{dv}{dX} = \frac{7-3v}{7v-3} - v = \frac{7-7v^2}{7v-3}$

Separating the variables $\frac{7v-3}{7-7v^2} \cdot dv = \frac{dX}{X}$

$$\frac{7v-3}{1-v^2} \cdot dv = 7 \cdot \frac{dX}{X}$$

Resolving L.H.S. into partial fractions, we have

$$\left(\frac{2}{1-v} - \frac{5}{1+v} \right) dv = 7 \cdot \frac{dX}{X}$$

Integrating both sides, we get

$$-2 \log(1-v) - 5 \log(1+v) = 7 \log X + C$$

$$\text{or } 7 \log X + 2 \log(1-v) + 5 \log(1+v) = -C$$

$$\text{or } \log [X^7 (1-v)^2 (1+v)^5] = -C$$

$$X^7 \left(1 - \frac{Y}{X}\right)^2 \left(1 + \frac{Y}{X}\right)^5 = e^{-C}$$

$$\text{or } (X-Y)^2 (X+Y)^5 = C$$

where $C = e^{-C}$

...(4)

Putting $X = x - h = x - 1$, $Y = y - k = y$

$$(x-y-1)^2 (x+y-1)^5 = C$$

which is the required solution of the given differential equation.

79. Solve : $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$

Sol. Given equation is

$$\frac{dy}{dx} = \frac{2(3x-2y)+3}{3x-2y+1} \quad \dots(1)$$

Putting $3x-2y = t$ so that $3-2 \frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{1}{2} \left(3 - \frac{dt}{dx}\right)$

$$\therefore \text{Equation (1) becomes, } \frac{1}{2} \left(3 - \frac{dt}{dx}\right) = \frac{2t+3}{t+1}$$

$$\frac{dt}{dx} = 3 - \frac{2(2t+3)}{t+1}$$

$$\frac{dt}{dx} = \frac{3t+3-4t-6}{t+1} = -\frac{t+3}{t+1} \quad \text{or} \quad -\frac{t+1}{t+3} \cdot dt = dx$$

$$-\left[1 - \frac{2}{t+3}\right] dt = dx$$

Integrating both sides, we get

$$-t + 2 \log(t+3) = x + C$$

$$\text{or } -3x + 2y + 2 \log(3x-2y+3) = x + C$$

$$\text{or } 4x - 2y + C = 2 \log(3x-2y+3)$$

$$\text{or } 2x - y + C = \log(3x-2y+3)$$

$$\text{i.e., } 2x - y = \log(3x-2y+3) + C'$$

is the required solution of the given differential equation.

80. Solve : $(2x + y + 1)dx + (4x + 2y - 1) dy = 0$.

Sol. Given equation is $\frac{dy}{dx} = -\frac{2x + y + 1}{4x + 2y - 1}$ or $\frac{dy}{dx} = -\frac{2x + y + 1}{2(2x + y) - 1}$... (1)

Putting $2x + y = t$, $2 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Equation (1) becomes $\frac{dt}{dx} - 2 = -\frac{t + 1}{2t - 1}$

$$\frac{dt}{dx} = 2 - \frac{t + 1}{2t - 1} = \frac{3t - 3}{2t - 1} = \frac{3(t - 1)}{2t - 1}$$

$$\text{or} \quad \frac{2t - 1}{3(t - 1)} \cdot dt = dx \Rightarrow \frac{1}{3} \left[2 + \frac{1}{t - 1} \right] dt = dx$$

On integrating both sides, we get

$$\frac{2t}{3} + \frac{1}{3} \log(t - 1) = x + C \quad \text{or} \quad 2t + \log(t - 1) = 3x + 3C$$

$$2(2x + y) + \log(2x + y - 1) = 3x + 3C \quad \text{or} \quad x + 2y + \log(2x + y - 1) = 3C$$

is the required solution of the given differential equation.

81. Solve : $(x + 1) \frac{dy}{dx} - y = e^{3x} (x + 1)^2$.

Sol. Given equation can be rewritten as

$$\frac{dy}{dx} - \frac{y}{x + 1} = e^{3x} \cdot (x + 1)$$

This is the standard form of a linear equation of the first order (commonly known as Leibnitz's

linear equation) as $\frac{dy}{dx} + Py = Q$

where P, Q are the functions of x only. Solution of this standard form is given by

$$y(\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + C, \quad \text{where} \quad \text{I.F.} = e^{\int P \cdot dx}$$

Here,

$$P = -1/(x + 1), \quad Q = e^{3x} (x + 1)$$

Thus,

$$\text{I.F.} = e^{\int -\frac{1}{x+1} \cdot dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

$$\text{and solution is} \quad y \cdot \left(\frac{1}{x+1} \right) = \int e^{3x} \cdot (x+1) \cdot \frac{1}{x+1} \cdot dx + C$$

$$= \int e^{3x} \cdot dx + C = \frac{1}{3} \cdot e^{3x} + C$$

or

$$y = \left(\frac{1}{3} e^{3x} + C \right) (x + 1)$$

82. Solve : $(1 + y^2)dx = (\tan^{-1} y - x)dy$.

$$\text{Sol.} \quad (1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$$

$$\text{or} \quad (1 + y^2) \frac{dx}{dy} + x = \tan^{-1} y \quad \text{or} \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

This is the standard form of linear equation as

$$\frac{dx}{dy} + P'x = Q'$$

where P' and Q' are functions of y alone.

$$P' = \frac{1}{1+y^2}, Q' = \frac{\tan^{-1}y}{1+y^2}$$

$$\text{I.F.} = e^{\int P' dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

Solution is therefore given by

$$x(\text{I.F.}) = \int Q' (\text{I.F.}) dy + C$$

$$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y} \cdot dy + C$$

Put $\tan^{-1}y = t$ then $\frac{dy}{1+y^2} = dt$

$\therefore$ Integral on the right hand side is $\int t e^t dt$

Integrating this integral “by parts” method we have

$$t \cdot e^t - \int 1 \cdot e^t \cdot dt = te^t - e^t = (t-1)e^t$$

Hence the solution is given by

$$xe^{\tan^{-1}y} = (\tan^{-1}y - 1) e^{\tan^{-1}y} + C \quad \text{or} \quad x = \tan^{-1}y - 1 + C \cdot e^{-\tan^{-1}y}.$$

83. Solve : $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right) \frac{dx}{dy} = 1.$

Sol. Given equation can be written as $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.$

Comparing it with $\frac{dy}{dx} + Py = Q$, we have

$$P = \frac{1}{\sqrt{x}}, Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$\therefore$ $\text{I.F.} = e^{\int P \cdot dx} = e^{\int \frac{1}{\sqrt{x}} \cdot dx} = e^{2\sqrt{x}}$

$\therefore$ Solution is $ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} \cdot dx + C = \int \frac{1}{\sqrt{x}} \cdot dx + C$

or $ye^{2\sqrt{x}} = 2\sqrt{x} + C$ is the solution.

84. Solve : $x(1-x^2) \frac{dy}{dx} + (2x^2-1)y = x^3.$

Sol. Dividing the given equation by

$x(1-x^2)$ throughout, we have $\frac{dy}{dx} + Py = Q,$

where $P = \frac{2x^2 - 1}{x(1 - x^2)}$, $Q = \frac{x^3}{x(1 - x^2)}$ or $\frac{x^2}{(1 - x^2)}$

Thus, $\text{I.F.} = e^{\int P \cdot dx} = e^{\int \frac{2x^2 - 1}{x(1 - x^2)} \cdot dx}$

Now resolving into partial fractions,

$$\begin{aligned} \int \frac{2x^2 - 1}{x(1 - x^2)} \cdot dx &= \int \left\{ -\frac{1}{x} + \frac{1}{2(1 - x)} - \frac{1}{2(1 + x)} \right\} dx \\ &= -\log x + \frac{1}{2} \log(1 - x)^{-1} - \frac{1}{2} \log(1 + x) = -\log [x(1 - x)^{1/2} (1 + x)^{1/2}] \end{aligned}$$

$\therefore \text{I.F.} = e^{-\log [x\sqrt{1 - x^2}]} = \frac{1}{x\sqrt{1 - x^2}}$

Thus the solution is $y (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dx + C$

$$\begin{aligned} \frac{y}{x\sqrt{1 - x^2}} &= \int \frac{x^2}{1 - x^2} \cdot \frac{1}{x\sqrt{1 - x^2}} \cdot dx + C \\ &= \int \frac{x}{(1 - x^2)^{3/2}} dx + C = -\frac{1}{2} \int \frac{-2x}{(1 - x^2)^{3/2}} \cdot dx + C \\ &= -\frac{1}{2} [-2(1 - x^2)^{-1/2}] + C \quad \text{or} \quad \frac{y}{x\sqrt{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} + C \end{aligned}$$

or $y = x + Cx\sqrt{1 - x^2}$

is the required solution of the given differential equation.

85. Solve : $\frac{dy}{dx} + \frac{x}{1 - x^2} y = x\sqrt{y}$

Sol. Given equation is of the form

$$\frac{dy}{dx} + Py = Q \cdot y^n, \text{ where } P \text{ and } Q \text{ are functions of } x \text{ only or constants.}$$

This is known as Bernoulli's equation. Though not linear, it can be made linear.

Dividing both sides by $\sqrt{y}$, we have

$$y^{-1/2} \frac{dy}{dx} + P\sqrt{y} = x \quad \text{i.e.,} \quad y^{-1/2} \frac{dy}{dx} + \frac{x}{1 - x^2} \cdot \sqrt{y} = x \quad \dots(1)$$

Putting $\sqrt{y} = z$ so that $\frac{1}{2\sqrt{y}} \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Equation (1) becomes

$$2 \frac{dz}{dx} + \frac{x}{1 - x^2} \cdot z = x \quad \text{or} \quad \frac{dz}{dx} + \frac{x}{2(1 - x^2)} \cdot z = \frac{x}{2} \quad \dots(2)$$

This is linear in z .

$$\text{I.F.} = e^{\int \frac{x}{2(1 - x^2)} \cdot dx} = e^{-\frac{1}{4} \int \frac{-2x}{1 - x^2} \cdot dx} = e^{-\frac{1}{4} \log(1 - x^2)} = (1 - x^2)^{-1/4}$$

Solution of (2) is z (I.F.) = $\int \frac{x}{2(1-x^2)^{1/4}} \cdot dx + C$

$$\text{or } z(1-x^2)^{-1/4} = -\frac{1}{4} \int \frac{-2x dx}{(1-x^2)^{1/4}} + C = -\frac{1}{4} \left[\frac{(1-x^2)^{3/4}}{3/4} \right] + C$$

$$z(1-x^2)^{-1/4} = -\frac{1}{3} (1-x^2)^{3/4} + C$$

$$\text{or } z = -\frac{1}{3} (1-x^2) + C(1-x^2)^{1/4} \Rightarrow \sqrt{y} = -\frac{1}{3} (1-x^2) + C(1-x^2)^{1/4}$$

is the required solution of the given differential equation.

86. Solve : $xy(1+xy^2) \frac{dy}{dx} = 1$.

Sol. The given equation can be written as $\frac{dx}{dy} - yx = y^3x^2$

Dividing throughout by x^2 , we have

$$x^{-2} \frac{dx}{dy} - \frac{y}{x} = y^3$$

Putting $x^{-1} = z$ so that $-\frac{1}{x^2} \frac{dx}{dy} = \frac{dz}{dy}$

$$-\frac{dz}{dy} - yz = y^3 \quad \text{or} \quad \frac{dz}{dy} + yz = -y^3 \text{ which is linear in } z.$$

$$\text{I.F.} = e^{\int y dy} = e^{\frac{1}{2}y^2}$$

$$\therefore \text{ Solution is } z \text{ (I.F.)} = \int (-y^3)(\text{I.F.}) dy + C = - \int y^3 \cdot e^{\frac{1}{2}y^2} dy + C$$

Putting $\frac{1}{2} y^2 = t$, $y dy = dt$

Integral on the R.H.S. is

$$- \int 2t e^t dt = -2 [te^t - e^t] = -2 (t-1)e^t$$

$$\therefore z e^{\frac{1}{2}y^2} = -2 \left(\frac{1}{2} y^2 - 1 \right) e^{\frac{1}{2}y^2} + C$$

$$z \cdot e^{\frac{1}{2}y^2} = (-y^2 + 2) e^{\frac{1}{2}y^2} + C$$

$$\text{or } z = 2 - y^2 + C e^{-\frac{1}{2}y^2} \text{ is the solution, where } z = \frac{1}{x}.$$

87. Solve : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Sol. Dividing by $\cos^2 y$, we have $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$... (1)

Putting $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx} + 2xz = x^3$ which is linear in z .

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

$$\therefore \text{ The solution is } z \cdot e^{x^2} = \int x^3 e^{x^2} dx + C = \frac{1}{2} \int x^2 e^{x^2} 2x dx + C$$

Putting $x^2 = t$; $2x dx = dt$ in the integral on right hand side we have

$$\frac{1}{2} \int t e^t dt = \frac{1}{2} (t - 1) e^t = \frac{1}{2} (x^2 - 1) e^{x^2}$$

$$\therefore z = \frac{1}{2} (x^2 - 1) + C e^{-x^2}$$

$$\text{or } \tan y = \frac{1}{2} (x^2 - 1) + C e^{-x^2} \text{ is the required solution.}$$

88. Solve : $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x} (\log z)^2$.

Sol. Given equation on dividing by z becomes $\frac{1}{z} \cdot \frac{dz}{dx} + \frac{1}{x} \cdot \log z = \frac{1}{x} (\log z)^2$... (1)

Put $\log z = t$ so that $\frac{1}{z} \cdot \frac{dz}{dx} = \frac{dt}{dx}$

$\therefore$ Equation (1) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2} \frac{dt}{dx} + \frac{1}{x} \cdot \frac{1}{t} = \frac{1}{x} \quad \dots (2)$$

This is a Bernoulli's equation

Put $\frac{1}{t} = v$ so that $-\frac{1}{t^2} \frac{dt}{dx} = \frac{dv}{dx}$

Equation (2) becomes,

$$\therefore -\frac{dv}{dx} + \frac{1}{x} \cdot v = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{1}{x}$$

This is Leibnitz's linear equation in v .

$$\therefore \text{ I.F.} = e^{-\int \frac{1}{x} \cdot dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore \text{ Solution is } v \cdot \frac{1}{x} &= -\int \frac{1}{x} \cdot \frac{1}{x} \cdot dx + C \\ v \cdot \frac{1}{x} &= \frac{1}{x} + C \end{aligned}$$

Replacing now v by $\frac{1}{\log z}$, we obtain

$$\frac{1}{x \log z} = \frac{1}{x} + C \quad \text{or} \quad \frac{1}{\log z} = 1 + Cx$$

which is the required solution of the given differential equation.

89. Solve : $x \frac{dy}{dx} + y = x^3 y^6$.

Sol. Given equation is $\frac{dy}{dx} + \frac{1}{x} \cdot y = x^2 y^6$

Dividing both sides by y^6

$$y^{-6} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y^5} = x^2$$

Put $y^{-5} = z$, so that

$$-5y^{-6} \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad -\frac{1}{5} \frac{dz}{dx} + \frac{1}{x} \cdot z = x^2$$

$\Rightarrow$

$$\frac{dz}{dx} - \frac{z}{x} = -5x^2$$

This is linear in z

$$\text{I.F.} = e^{\int \frac{-5}{x} \cdot dx} = e^{-5 \log x} = \frac{1}{x^5}$$

Solution is

$$z \left(\frac{1}{x^5} \right) = \int -5x^2 \cdot \frac{1}{x^5} dx + C = \frac{+5}{2x^2} + C \quad \text{or} \quad z = + \frac{5x^3}{2} + Cx^5$$

Replacing z by y , we have

$$y^{-5} = \frac{5x^3}{2} + Cx^5 \quad \text{or} \quad \frac{1}{y^5} = \frac{5x^3}{2} + Cx^5 \text{ is the required solution of the given}$$

differential equation.

90. Solve : $2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol.

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{y^2}{2x^2}$$

Dividing by y^2 we have

$$\frac{1}{y^2} \frac{dy}{dx} - \frac{1}{2xy} = \frac{1}{2x^2}$$

Put $-\frac{1}{y} = z$, so that

$$\frac{1}{y^2} \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$

$$\frac{dz}{dx} + \frac{z}{2x} = \frac{1}{2x^2}$$

This is linear in z .

$$\text{I.F.} = e^{\int \frac{1}{2x} \cdot dx} = e^{\frac{1}{2} \log x} = x^{1/2}$$

Solution is

$$z(x^{1/2}) = \int \frac{1}{2x^2} \cdot x^{1/2} \cdot dx + C = \frac{1}{2} \int x^{-3/2} \cdot dx + C = \frac{1}{2} \cdot \frac{x^{-1/2}}{(-1/2)} + C$$

$$z \cdot \sqrt{x} = -\frac{1}{\sqrt{x}} + C \quad \text{or} \quad zx = -1 + C\sqrt{x}$$

Replacing z by $-1/y$

$$-\frac{x}{y} = -1 + C\sqrt{x} \quad \text{or} \quad \frac{x}{y} = 1 - C\sqrt{x}$$

is the required solution of the given differential equation.

91. Solve : $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$.

Sol.

$$\frac{dy}{dx} - \frac{\sin y}{(1+x) \cos y} = (1+x) e^x \cdot \frac{1}{\cos y}$$

$$\cos y \frac{dy}{dx} - \frac{\sin y}{(1+x)} = (1+x) e^x$$

Put $\sin y = z$, so that $\cos y \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x) e^x$$

This is linear in z .
$$\text{I.F.} = e^{\int \frac{-1}{1+x} \cdot dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

Solution is
$$z \left(\frac{1}{1+x} \right) = \int (1+x) e^x \cdot \frac{1}{1+x} + C$$

$$z \cdot \frac{1}{1+x} = e^x + C \quad \text{or} \quad z = (1+x)(e^x + C)$$

Hence solution is
$$\sin y = (1+x)(e^x + C).$$

92. Solve : $(x^3 y^2 + xy) dx = dy$.

Sol.
$$\frac{dy}{dx} = xy + x^3 y^2 \quad \text{or} \quad \frac{dy}{dx} - xy = x^3 y^2$$

Dividing by y^2 on both sides, we get

$$\frac{1}{y^2} \frac{dy}{dx} - \frac{x}{y} = x^3$$

Put $-\frac{1}{y} = z$,
$$\frac{1}{y^2} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \frac{dz}{dx} + zx = x^3, \text{ which is linear in } z.$$

$$\text{I.F.} = e^{\int x dx} = e^{\frac{1}{2}x^2}$$

$$\therefore \text{Solution is} \quad z \cdot \left(e^{\frac{1}{2}x^2} \right) = \int x^3 \cdot e^{\frac{1}{2}x^2} \cdot dx + C = 2 \int \left(\frac{1}{2} x^2 \right) e^{\frac{1}{2}x^2} x dx + C$$

Putting $\frac{1}{2} x^2 = t$ in the integral on the R.H.S., we have $dt = x dx$

$$\therefore \text{Integral} = 2 \int t e^t dt = 2 [(t-1)e^t]$$

$$\therefore z \left(e^{\frac{1}{2}x^2} \right) = 2 \left(\frac{1}{2} x^2 - 1 \right) e^{\frac{1}{2}x^2} + C$$

or
$$z = (x^2 - 2) + C e^{-\frac{1}{2}x^2} \quad \text{or} \quad -\frac{1}{y} = x^2 - 2 + C e^{-\frac{1}{2}x^2}$$

$$\frac{1}{y} = -x^2 + 2 - C e^{-\frac{1}{2}x^2}$$

is the required solution of the given differential equation.

93. Solve : $\tan y \frac{dy}{dx} + \tan x = \cos y \cdot \cos^2 x$.

Sol. Dividing throughout by $\cos y$, we get the differential equation as

$$\frac{\tan y}{\cos y} \frac{dy}{dx} + \sec y \tan x = \cos^2 x$$

or
$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$$

Put $\sec y = z$, so that $\sec y \tan y \frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + z \tan x = \cos^2 x$$

which is linear in z .

$$\text{I.F.} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

$$\therefore \text{Solution is} \quad z \sec x = \int \cos^2 x \cdot \sec x \, dx + C$$

$$z \cdot \sec x = \sin x + C$$

or

$$z = \cos x (C + \sin x)$$

i.e.,

$$\sec y = (C + \sin x) \cos x \text{ is the solution.}$$

94. Solve : $\frac{dy}{dx} + y \tan x = y^3 \cos x$.

Sol. Given equation is $\frac{dy}{dx} + y \tan x = y^3 \cos x$

Dividing throughout by y^3 , we get

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^2} \tan x = \cos x$$

Put $\frac{1}{y^2} = z$, so that $-2y^{-3} \frac{dy}{dx} = \frac{dz}{dx}$

$$-\frac{1}{2} \frac{dz}{dx} + z \tan x = \cos x \quad \text{or} \quad \frac{dz}{dx} - 2z \tan x = -2 \cos x$$

which is linear in z .

$$\text{I.F.} = e^{\int -2 \tan x \, dx} = e^{2 \log \cos x} = \cos^2 x.$$

$$\therefore \text{Solution is} \quad z(\cos^2 x) = \int -2 \cos x \cos^2 x \, dx + C$$

$$\text{or} \quad z(\cos^2 x) = -2 \int \cos^3 x \, dx + C = -2 \left[\int (1 - \sin^2 x) \cos x \, dx \right] + C$$

To find the integral on the right hand side,

Put $\sin x = t$ so that $\cos x \, dx = dt$

$$\int (1 - t^2) \, dt = t - \frac{t^3}{3} = \sin x - \frac{1}{3} \sin^3 x$$

$$\therefore z(\cos^2 x) = -2 \left[\sin x - \frac{1}{3} \sin^3 x \right] + C$$

As

$$z = \frac{1}{y^2}$$

$$\cos^2 x = -2y^2 \left[\sin x - \frac{1}{3} \sin^3 x \right] + Cy^2$$

$$\cos^2 x = y^2 \left[-2 \sin x + \frac{2}{3} \sin^3 x + C \right]$$

is the required solution of the given differential equation.

95. Solve : $x(x-y)dy + y^2dx = 0$.

Sol. Given equation is $x(x-y)dy = -y^2dx$ or $\frac{dy}{dx} = \frac{-y^2}{x(x-y)}$

Let

$$y = vx$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}, \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx} = -\frac{v^2x^2}{x^2(1-v)}$$

or

$$v + x \frac{dv}{dx} = \frac{v^2}{v-1}$$

$$x \frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v}{v-1} \quad \text{or} \quad \left(\frac{v-1}{v}\right) dv = \frac{dx}{x}$$

On integrating both sides, we get

$$v - \log v = \log x + C \quad \text{or} \quad v = \log xv + C$$

As

$$v = \frac{y}{x}$$

$$\frac{y}{x} = \log y + C \quad \text{or} \quad y = x \log y + Cx \text{ is the solution.}$$

96. Solve : $e^y \left(\frac{dy}{dx} + 1 \right) = e^x$.

Sol.

$$e^y \frac{dy}{dx} + e^y = e^x$$

Put $e^y = z$,

$$e^y \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dz}{dx} + z = e^x \text{ is linear in } z$$

$$\text{I.F.} = e^{\int 1 \cdot dx} = e^x$$

$\therefore$ Solution is

$$z(e^x) = \int e^x \cdot e^x \cdot dx + C = \frac{1}{2} e^{2x} + C \quad \text{or} \quad e^{x+y} = \frac{1}{2} e^{2x} + C.$$

97. Solve : $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$.

Sol.

$$2xy \frac{dy}{dx} - y^2 = x^2 + 1 \quad \text{or} \quad 2y \frac{dy}{dx} - \frac{y^2}{x} = x + \frac{1}{x}$$

Put $y^2 = z$, so that

$$2y \frac{dy}{dx} = \frac{dz}{dx} \quad \therefore \frac{dz}{dx} - \frac{z}{x} = x + \frac{1}{x} \text{ is linear in } z.$$

$$\text{I.F.} = e^{\int -\frac{1}{x} \cdot dx} = e^{-\log x} = \frac{1}{x}$$

Solution is

$$z \left(\frac{1}{x} \right) = \int \left(x + \frac{1}{x} \right) \cdot \frac{1}{x} dx + C$$

$$z \left(\frac{1}{x} \right) = x + \left(-\frac{1}{x} \right) + C$$

or

$$y^2 = x^2 - 1 + Cx \quad \text{i.e.,} \quad y^2 = x^2 + Cx - 1 \text{ is the solution.}$$

98. Solve : $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$.

Sol. Given equation is

$$\frac{dx}{dy} = \frac{x + \sqrt{xy}}{y} \quad \text{or} \quad \frac{dx}{dy} = \frac{x}{y} + \sqrt{\frac{x}{y}}$$

$$\frac{dx}{dy} - \frac{x}{y} = \sqrt{\frac{x}{y}} \quad \dots(1)$$

Dividing throughout by $\sqrt{x}$, we get

$$\frac{1}{\sqrt{x}} \frac{dx}{dy} - \frac{\sqrt{x}}{y} = \frac{1}{\sqrt{y}} \quad \dots(2)$$

Put $\sqrt{x} = z$, so that $\frac{1}{2\sqrt{x}} \frac{dx}{dy} = \frac{dz}{dy}$

Equation (2) becomes $2 \cdot \frac{dz}{dy} - \frac{z}{y} = \frac{1}{\sqrt{y}}$

$$\frac{dz}{dy} - \frac{z}{2y} = \frac{1}{2\sqrt{y}} \quad \dots(3)$$

Equation (3) is linear in z .

$$\text{I.F.} = e^{\int -\frac{1}{2y} \cdot dy} = e^{-1/2(\log y)} = \frac{1}{\sqrt{y}}$$

Solution is $z \left(\frac{1}{\sqrt{y}} \right) = \int \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{y}} \cdot dy + C = \frac{1}{2} \log y + C$

$$\frac{\sqrt{x}}{\sqrt{y}} = \log \sqrt{y} + C \quad \text{or} \quad \sqrt{x} = \sqrt{y} (\log \sqrt{y} + C).$$

99. Solve : $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$.

Sol. Putting $\tan y = z$, we get

$$\sec^2 y \cdot \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dz}{dx} + zx = x^3 \text{ is linear in } z.$$

$$\text{I.F.} = e^{\int x \, dx} = e^{x^2/2}$$

Solution is $z(e^{x^2/2}) = \int x^3 \cdot e^{x^2/2} \cdot dx = 2 \int \frac{x^2}{2} e^{x^2/2} \cdot x \, dx$

$$z(e^{x^2/2}) = 2 \left[\left(\frac{x^2}{2} - 1 \right) e^{x^2/2} \right] + C$$

or $z = x^2 - 2 + C e^{-\frac{x^2}{2}} \quad \text{or} \quad \tan y = x^2 - 2 + C e^{-\frac{x^2}{2}}$

is the solution of the given differential equation.

100. What is a linear differential equation with constant coefficients ? What is an operator ? What is a complete solution ? What are complementary functions and what is an auxiliary equation ?

Sol. The equation of the form $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q$

where $a_0, a_1, a_2, \dots, a_n$ are all constants and Q is a function of x alone, is called a linear differential equation of n^{th} order with constant coefficients. The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may

be regarded as an operator denoted by symbol D . Similarly $\frac{d^2}{dx^2} = D^2, \frac{d^3}{dx^3} = D^3, \dots, \frac{d^n}{dx^n} = D^n$.

Written in symbolic form, the above equation becomes

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = Q \quad \text{or} \quad f(D)y = Q$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution of the equation $f(D)y = Q$, then the complete solution of the equation $f(D)y = Q$ is $y = u + v$.

The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)y = Q$.

The complete solution is $y = \text{C.F.} + \text{P.I.}$

Thus in order to solve the equation $f(D)y = Q$, we first find the C.F. *i.e.*, C.S. of equation $f(D)y = 0$ and then the P.I. *i.e.*, a particular solution of equation $f(D)y = Q$.

To find C.F. we first find auxiliary equation (A.E.). by following steps :

Step 1 : Replace y by 1

Step 2 : Replace $\frac{dy}{dx}$ by m , $\frac{d^2y}{dx^2}$ by m^2 and so on replace $\frac{d^ny}{dx^n}$ by m^n .

By doing so, we get an algebraic equation in m of degree n called A.E. While finding out A.E., we assume Q as zero. The complementary function depends upon the nature of roots of the A.E.

Consider the equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0 \quad \dots(1)$$

where all a_i 's are constants.

$$\text{Its A.E. is} \quad m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \dots(2)$$

It is an algebraic equation in m of degree n . So, it will give n values of m on solving.

Let $m = m_1, m_2, m_3, \dots, m_n$ be the roots of the A.E. The C.F. of equation (1) is given by

C.F. = $C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$ when the roots of A.E. are real and distinct.

If 2 roots are equal, C.F. = $(C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$

If 3 roots are equal, C.F. = $(C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \dots + C_n e^{m_n x}$

101. Solve: $(D^3 - 3D^2 + 4)y = 0$, where $D = \frac{d}{dx}$.

Sol. The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0 \Rightarrow (m+1)(m-2)^2 = 0 \Rightarrow m = -1, 2, 2$$

$$\therefore \text{C.F.} = C_1 e^{-x} + (C_2 + C_3 x) e^{2x}$$

As $Q = 0$, P.I. = 0

$$\therefore \text{The complete solution is } y = \text{C.F.} + \text{P.I.} = C_1 e^{-x} + (C_2 + C_3 x) e^{2x}$$

where C_1, C_2 and C_3 are arbitrary constants of integration.

102. Solve : $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

Sol. The auxiliary equation is $m^3 - 7m - 6 = 0$

By inspection $m = -1$ satisfies it, so $m + 1$ is a factor.

$$m^3 - 7m - 6 \equiv (m+1)(m^2 - m - 6) \equiv (m+1)(m+2)(m-3)$$

$$\Rightarrow m = -1, -2, 3$$

The roots are real and distinct.

$$\therefore \text{C.F.} = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{+3x}$$

$$\therefore \quad Q = 0, \text{ P.I.} = 0$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x}$

where C_1, C_2, C_3 are arbitrary constants of integration.

103. Solve: $(D^3 - 4D^2 + 4D)y = 0$; $D \equiv \frac{d}{dx}$.

Sol. The auxiliary equation is $m^3 - 4m^2 + 4m = 0$

$$\Rightarrow m(m^2 - 4m + 4) = 0 \Rightarrow m = 0, 2, 2.$$

$$\therefore \text{C.F.} = C_1 e^{0x} + (C_2 + C_3 x)e^{2x} = C_1 + (C_2 + C_3 x)e^{2x}$$

$$\therefore Q = 0, \text{ P.I.} = 0$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$ or $y = C_1 + (C_2 + C_3 x)e^{2x}$

where C_1, C_2, C_3 are arbitrary constants of integration.

104. Solve: $\frac{d^4 y}{dx^4} + 13 \frac{d^2 y}{dx^2} + 36y = 0$.

Sol. The auxiliary equation is

$$m^4 + 13m^2 + 36 = 0 \Rightarrow (m^2 + 9)(m^2 + 4) = 0 \Rightarrow m = \pm 3i, \pm 2i$$

$$\therefore \text{C.F.} = e^{0x} (C_1 \cos 3x + C_2 \sin 3x) + e^{0x} (C_3 \cos 2x + C_4 \sin 2x)$$

$$\therefore Q = 0, \text{ P.I.} = 0$$

Complete solution is $y = \text{C.F.} + \text{P.I.}$

$$\text{or } y = C_1 \cos 3x + C_2 \sin 3x + C_3 \cos 2x + C_4 \sin 2x$$

where C_1, C_2, C_3 and C_4 are arbitrary constants of integration.

105. Solve: $(D^2 - 2D + 4)^2 y = 0$; $D \equiv \frac{d}{dx}$.

Sol. The auxiliary equation is $(m^2 - 2m + 4)^2 = 0$ or $m = 1 \pm \sqrt{3}i, 1 \pm \sqrt{3}i$

The roots are repeated imaginary

$$\text{C.F.} = e^x [(C_1 + C_2 x) \cos \sqrt{3}x + (C_3 + C_4 x) \sin \sqrt{3}x]$$

$$\text{P.I.} = 0$$

$$(\because Q = 0)$$

Hence the complete solution is $y = e^x [(C_1 + C_2 x) \cos \sqrt{3}x + (C_3 + C_4 x) \sin \sqrt{3}x]$

where C_1, C_2, C_3, C_4 are arbitrary constants of integration.

106. Solve: $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$, where $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0 \Rightarrow (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i, 1 \pm i$$

$$\therefore \text{C.F.} = e^x [(C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x]$$

$$\text{P.I.} = 0$$

$$(\because Q = 0)$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$ or $y = e^x [(C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x]$

where C_1, C_2, C_3 and C_4 are arbitrary constants of integration.

107. Solve: $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$, where $D \equiv \frac{d}{dx}$

Sol. Auxiliary equation of the given differential equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$

$$\Rightarrow (m^2 + 1)^3 = 0 \text{ gives } m = \pm i, \pm i, \pm i$$

and $m^2 + m + 1 = 0$ gives $m = \frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}$

Hence C.F. = $e^{0x} [(C_1 + C_2x + C_3x^2) \cos x + (C_4 + C_5x + C_6x^2) \sin x]$

$$+ e^{-x/2} \left[(C_7 + C_8x) \cos \frac{\sqrt{3}}{2} x + (C_9 + C_{10}x) \sin \frac{\sqrt{3}}{2} x \right]$$

$$= (C_1 + C_2x + C_3x^2) \cos x + (C_4 + C_5x + C_6x^2) \sin x$$

$$+ e^{-x/2} \left\{ (C_7 + C_8x) \cos \frac{\sqrt{3}}{2} x + (C_9 + C_{10}x) \sin \frac{\sqrt{3}}{2} x \right\}$$

P.I. = 0, since Q = 0

Therefore the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= (C_1 + C_2x + C_3x^2) \cos x + (C_4 + C_5x + C_6x^2) \sin x$$

$$+ e^{-x/2} \left\{ (C_7 + C_8x) \cos \frac{\sqrt{3}}{2} x + (C_9 + C_{10}x) \sin \frac{\sqrt{3}}{2} x \right\}$$

108. Solve : $\frac{d^4x}{dt^4} + 4x = 0$.

Sol. The given differential equation is $(D^4 + 4)x = 0$, where $D = \frac{d}{dt}$

Its auxiliary equation is $D^4 + 4 = 0$ or $D^4 + 4D^2 + 4 - 4D^2 = 0$

$$(D^2 + 2)^2 - (2D)^2 = 0 \quad \text{or} \quad (D^2 + 2D + 2)(D^2 - 2D + 2) = 0$$

$$\Rightarrow D = \frac{-2 \pm \sqrt{-4}}{2} \text{ and } \frac{2 \pm \sqrt{-4}}{2} \quad \text{i.e., } D = -1 \pm i \quad \text{and } 1 \pm i$$

The complete solution is $x = e^{-t} (C_1 \cos t + C_2 \sin t) + e^t (C_3 \cos t + C_4 \sin t)$

$$\text{P.I.} = 0 \quad (\because Q = 0)$$

C_1, C_2, C_3, C_4 are arbitrary constants of integration.

109. Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ given $y(0) = 4$, and $y'(0) = 1$.

Sol. The given equation in symbolic form is $(D^2 - 2D + 10)y = 0$

Its auxiliary equation is

$$D^2 - 2D + 10 = 0 \quad \Rightarrow \quad D = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i$$

The complete solution is $y = e^x(C_1 \cos 3x + C_2 \sin 3x)$... (1)

Now $y(0) = 4 \Rightarrow y = 4$ when $x = 0$

$$\therefore 4 = C_1$$

Equation (1) becomes $y = e^x (4 \cos 3x + C_2 \sin 3x)$... (2)

so that $y' = e^x(4 \cos 3x + C_2 \sin 3x) + e^x(-12 \sin 3x + 3C_2 \cos 3x)$

Since $y' = 1$ when $x = 0$

$$\therefore 1 = 4 + 3C_2 \quad \text{or} \quad C_2 = -1$$

Equation (2) becomes $y = e^x (4 \cos 3x - \sin 3x)$

which is the required particular solution.

110. Solve : $(D^4 - n^4)y = 0$, where $D \equiv \frac{d}{dx}$.

Sol. The auxiliary equation is

$$m^4 - n^4 = 0 \quad \Rightarrow \quad (m^2 + n^2)(m^2 - n^2) = 0$$

$\Rightarrow$

$$m = \pm n, \pm ni$$

$\therefore$

$$\text{C.F.} = C_1 e^{nx} + C_2 e^{-nx} + (C_3 \cos nx + C_4 \sin nx) e^{0x}$$

$$\text{P.I.} = 0$$

$\therefore$ Complete solution is $y = \text{C.F.} + \text{P.I.} = C_1 e^{nx} + C_2 e^{-nx} + (C_3 \cos nx + C_4 \sin nx)$

where C_1, C_2, C_3, C_4 are arbitrary constants of integration.

111. Solve : $\frac{d^4 y}{dx^4} + m^4 y = 0$.

Sol. Given equation in symbolic form is

$$(D^4 + m^4)y = 0, \text{ where } D \equiv d/dx$$

Auxiliary equation is $M^4 + m^4 = 0$

$\Rightarrow$

$$M^4 + m^4 + 2M^2 m^2 = 2M^2 m^2 \quad \text{or} \quad (M^2 + m^2)^2 = 2M^2 m^2$$

$\Rightarrow$

$$M^2 + m^2 = \pm \sqrt{2} Mm$$

Case I. Taking (+)ve sign :

$$M^2 + m^2 - \sqrt{2} Mm = 0$$

$\therefore$

$$M = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{\sqrt{2}m \pm i\sqrt{2}m}{2} = \frac{m}{\sqrt{2}} \pm i \frac{m}{\sqrt{2}}$$

Case II. Taking (-)ve sign :

$$M^2 + m^2 + \sqrt{2} Mm = 0$$

or

$$M = \frac{m}{\sqrt{2}} \pm i \frac{m}{\sqrt{2}}; -\frac{m}{\sqrt{2}} \pm i \frac{m}{\sqrt{2}}$$

$$\text{C.F.} = e^{\frac{m}{\sqrt{2}}x} \left(C_1 \cos \frac{m}{\sqrt{2}}x + C_2 \sin \frac{m}{\sqrt{2}}x \right) + e^{-\frac{m}{\sqrt{2}}x} \left(C_3 \cos \frac{m}{\sqrt{2}}x + C_4 \sin \frac{m}{\sqrt{2}}x \right)$$

$$\text{P.I.} = 0$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{\frac{m}{\sqrt{2}}x} \left(C_1 \cos \frac{m}{\sqrt{2}}x + C_2 \sin \frac{m}{\sqrt{2}}x \right) + e^{-\frac{m}{\sqrt{2}}x} \left(C_3 \cos \frac{m}{\sqrt{2}}x + C_4 \sin \frac{m}{\sqrt{2}}x \right)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants of integration.

112. Solve : $(D^2 + 1)^2 (D - 1)y = 0$, where $D \equiv \frac{d}{dx}$

[U.P.T.U., B.Pharm. (C.O.) 2005]

Sol. Auxiliary equation is

$$(m^2 + 1)^2 (m - 1) = 0$$

$\Rightarrow$

$$m = 1, \pm i, \pm i$$

Thus,

$$m_1 = m_2 = 0 + i, m_3 = m_4 = 0 - i, m_5 = 1$$

$$\text{C.F.} = (C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x + C_5 e^x$$

$$\text{P.I.} = 0$$

$$(\because Q = 0)$$

The complete solution is $y = (C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x + C_5 e^x$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants of integration.

113. Solve : $\frac{d^3 y}{dx^3} + y = 0$.

Sol. The differential equation is

$$(D^3 + 1)y = 0$$

The auxiliary equation is $m^3 + 1 = 0$

$$\Rightarrow (m + 1)(m^2 + 1 - m) = 0$$

$$\Rightarrow m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore \text{C.F.} = e^{\frac{1}{2}x} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right) + C_3 e^{-x}$$

$$\text{P.I.} = 0 \quad (\because Q = 0)$$

$$\text{The complete solution is } y = e^{\frac{x}{2}} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right) + C_3 e^{-x}$$

where C_1, C_2, C_3 are arbitrary constants of integration.

114. Find the P.I. of $(4D^2 + 4D - 3)y = e^{2x}$

$$\text{Sol.} \quad \text{P.I.} = \frac{1}{4D^2 + 4D - 3} e^{2x}$$

$$\text{Replacing D by 2} \quad \text{P.I.} = \frac{1}{4(2)^2 + 4(2) - 3} \cdot e^{2x} \quad \text{or} \quad \text{P.I.} = \frac{1}{21} e^{2x}.$$

115. Find the P.I. of $(D^2 + 3D + 2)y = 5$.

$$\begin{aligned} \text{Sol.} \quad \text{P.I.} &= \frac{1}{D^2 + 3D + 2} (5e^{0x}) = 5 \cdot \frac{1}{0 + 0 + 2} \cdot e^{0x} \quad (\text{Replacing D by 0}) \\ &= \frac{5}{2}. \end{aligned}$$

116. Find the P.I. of $(D^3 - 3D^2 + 4)y = e^{2x}$

$$\text{Sol.} \quad \text{P.I.} = \frac{1}{D^3 - 3D^2 + 4} e^{2x}$$

Here the denominator vanishes, when D is replaced by 2. It is a case of failure.

We multiply the numerator by x and differentiate the denominator w.r.t. D

$$\therefore \text{P.I.} = x \cdot \frac{1}{3D^2 - 6D} \cdot e^{2x}$$

It is again a case of failure.

We multiply the numerator by x and differentiate the denominator w.r.t. D

$$\therefore \text{P.I.} = x^2 \cdot \frac{1}{6D - 6} \cdot e^{2x} = \frac{x^2}{6} \cdot e^{2x} \quad (\text{Replacing D by 2})$$

$$\text{Thus,} \quad \text{P.I.} = \frac{x^2}{6} \cdot e^{2x}$$

117. Find P.I. of $(D^3 + 1)y = \sin(2x + 3)$.

Sol.
$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 + 1} \sin(2x + 3) = \frac{1}{D(-2^2) + 1} \sin(2x + 3) \\ & \quad \text{(Putting } D^2 = -2^2) \\ &= \frac{1}{1 - 4D} \sin(2x + 3) \end{aligned}$$

Multiplying and dividing by $(1 + 4D)$, we get

$$\begin{aligned} \text{P.I.} &= \frac{1 + 4D}{(1 - 4D)(1 + 4D)} \sin(2x + 3) = \frac{1 + 4D}{1 - 16D^2} \sin(2x + 3) \\ &= \frac{1 + 4D}{1 - 16(-2^2)} \sin(2x + 3) \quad \text{(Putting } D^2 = -2^2) \\ &= \frac{1}{65} [\sin(2x + 3) + 4D \sin(2x + 3)] \\ &= \frac{1}{65} [\sin(2x + 3) + 8 \cos(2x + 3)] \quad \left(\because D = \frac{d}{dx} \right) \end{aligned}$$

118. Find the P.I. of $(D^2 + 4)y = \cos 2x$.

Sol.
$$\text{P.I.} = \frac{1}{D^2 + 4} \cos 2x$$

Here the denominator vanishes when D^2 is replaced by $-2^2 = -4$. It is a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D .

$$\begin{aligned} \therefore \text{P.I.} &= x \cdot \frac{1}{2D} \cos 2x = \frac{x}{2} \cdot \frac{1}{D} (\cos 2x) \\ &= \frac{x}{2} \cdot \int \cos 2x \, dx \quad \left(\because \frac{1}{D} f(x) = \int f(x) \, dx \right) \end{aligned}$$

or
$$\text{P.I.} = \frac{x}{4} \sin 2x$$

119. Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Sol.
$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 5D + 4} (x^2 + 7x + 9) \\ &= \frac{1}{4 \left(1 + \frac{5D}{4} + \frac{D^2}{4} \right)} (x^2 + 7x + 9) = \frac{1}{4} \left(1 + \frac{5D}{4} + \frac{D^2}{4} \right)^{-1} (x^2 + 7x + 9) \\ \text{P.I.} &= \frac{1}{4} \left[1 - \left(\frac{5D}{4} + \frac{D^2}{4} \right) + \left(\frac{5D}{4} + \frac{D^2}{4} \right)^2 - \dots \right] (x^2 + 7x + 9) \\ &= \frac{1}{4} \left[1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} - \dots \right] (x^2 + 7x + 9) \\ &= \frac{1}{4} \left[1 - \frac{5D}{4} + \frac{21D^2}{16} - \dots \right] (x^2 + 7x + 9) \\ &= \frac{1}{4} \left[(x^2 + 7x + 9) - \frac{5}{4} D(x^2 + 7x + 9) + \frac{21}{16} D^2(x^2 + 7x + 9) \right] \\ &= \frac{1}{4} \left[(x^2 + 7x + 9) - \frac{5}{4} (2x + 7) + \frac{21}{16} (2) \right] = \frac{1}{4} \left[x^2 + \frac{9}{2}x + \frac{23}{8} \right] \end{aligned}$$

120. Find the P.I. of $(D^2 - 4D + 3)y = e^x \cos 2x$.

Sol. We know that when $X = e^{ax} \cdot V$, where V is a function of x

$$\frac{1}{f(D)}(e^{ax} V) = e^{ax} \cdot \frac{1}{f(D+a)} V$$

$$\begin{aligned} \therefore \text{P.I.} &= \frac{1}{D^2 - 4D + 3} e^x \cos 2x \\ &= e^x \cdot \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x \quad (\text{Replacing } D \text{ by } D+1) \\ &= e^x \frac{1}{D^2 - 2D} \cos 2x = e^x \cdot \frac{1}{-2^2 - 2D} \cos 2x \quad (\text{Putting } D^2 = -2^2) \\ &= -\frac{1}{2} e^x \cdot \frac{1}{2+D} \cos 2x = -\frac{1}{2} e^x \cdot \frac{2-D}{(2+D)(2-D)} \cos 2x \\ &= -\frac{1}{2} e^x \cdot \frac{2-D}{4-D^2} \cos 2x = -\frac{1}{2} e^x \cdot \frac{2-D}{4-(-2^2)} \cos 2x \quad (\because D^2 = -2^2) \\ &= -\frac{1}{16} e^x (2 \cos 2x - D \cos 2x) = -\frac{1}{16} e^x (2 \cos 2x + 2 \sin 2x) \\ &= -\frac{1}{8} e^x (\cos 2x + \sin 2x). \end{aligned}$$

121. Solve : $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. Auxiliary equation is $D^3 - 6D^2 + 11D - 6 = 0$

By inspection $D = 1$ satisfies it.

$$D^3 - 6D^2 + 11D - 6 = (D-1)(D^2 - 5D + 6) = (D-1)(D-2)(D-3)$$

$$\therefore (D-1)(D-2)(D-3) = 0 \quad \Rightarrow \quad D = 1, 2, 3$$

$$\therefore \text{C.F.} = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 6} (e^{-2x} + e^{-3x})$$

$$\begin{aligned} \text{or} \quad \text{P.I.} &= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x} \\ &= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} \cdot e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} \cdot e^{-3x} \\ &= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} = -\frac{1}{120} (2e^{-2x} + e^{-3x}) \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} \quad \text{or} \quad y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x} - \frac{1}{120} (2e^{-2x} + e^{-3x}).$$

122. Solve : $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

(M.D.U., May 2007)

Sol. Auxiliary equation is

$$(D-2)^2 = 0$$

$$\therefore D = 2, 2$$

$$\text{C.F.} = (C_1 + C_2 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} [8(e^{2x} + \sin 2x + x^2)]$$

$$\text{P.I.} = 8 \left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \right]$$

Now $\frac{1}{(D-2)^2} \cdot e^{2x} = x \cdot \frac{1}{2(D-2)} \cdot e^{2x} = x^2 \cdot \frac{1}{2} e^{2x}$ (Case of failure)

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x \quad (\text{Putting } D^2 = -2^2)$$

$$= -\frac{1}{4D} \sin 2x = -\frac{1}{4} \int \sin 2x \, dx = -\frac{1}{4} \left(-\frac{\cos 2x}{2} \right) = \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4 \left(1 - \frac{D}{2} \right)^2} x^2 = \frac{1}{4} \left(1 - \frac{D}{2} \right)^{-2} x^2$$

$$= \frac{1}{4} \left[1 - 2 \left(-\frac{D}{2} \right) + \frac{(-2)(-3)}{2} \left(\frac{D}{2} \right)^2 \dots \right] x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3}{4} D^2 + \dots \right] x^2 = \frac{1}{4} \left[x^2 + D(x^2) + \frac{3}{4} D^2 (x^2) \right]$$

$$\therefore \text{P.I.} = 8 \left[\frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right) \right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the complete solution is

$$y = (C_1 + C_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$$

123. Solve the D.E. $(D^3 - D)z = 2y + 1 + 4 \cos y + 2e^y$, where $D \equiv \frac{d}{dy}$. (M.DU., May, 2009)

Sol. $(D^3 - D)z = 2y + 1 + 4 \cos y + 2e^y$

Its A.E. is $D^3 - D = 0 \Rightarrow D = 0, \pm 1.$

$$\text{C.F.} = c_1 + c_2 e^y + c_3 e^{-y}$$

$$\text{P.I.} = \frac{1}{D^3 - D} (2y + 1 + 4 \cos y + 2e^y)$$

Solving separately, we get

$$\frac{2y}{D^3 - D} = \frac{2}{D(D^2 - 1)} \cdot y = \frac{-2}{D} \cdot y = -y^2$$

$$\frac{1}{D^3 - D} = \frac{1}{D^3 - D} \cdot e^{0y} = \frac{y \cdot e^{0y}}{3D^2 - 1} = -y$$

$$\frac{1}{D^3 - D} \cdot 4 \cos y = -2 \int \cos y \, dy = -2 \sin y$$

$$\frac{1}{D^3 - D} \cdot 2e^y = \frac{2y e^y}{3D^2 - 1} = \frac{2}{2} y e^y = y e^y$$

Hence complete solution is

$$z = c_1 + c_2 e^y + c_3 e^{-y} - y^2 - y - 2 \sin y + y e^y.$$

124. Solve : $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.

Sol. Auxiliary equation is

$$(D + 2)(D - 1)^2 = 0 \text{ so that } D = -2, 1, 1$$

$$\therefore \text{C.F.} = C_1 e^{-2x} + (C_2 + C_3 x) e^x$$

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^2} (e^{-2x} + 2 \sinh x)$$

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^2} (e^{-2x} + e^x - e^{-x}) \quad \left[\because \sinh x = \frac{e^x - e^{-x}}{2} \right]$$

$$\begin{aligned} \text{Now, } \frac{1}{(D + 2)(D - 1)^2} \cdot e^{-2x} &= \frac{1}{D + 2} \left[\frac{1}{(D - 1)^2} \cdot e^{-2x} \right] \\ &= \frac{1}{D + 2} \left[\frac{1}{(-2 - 1)^2} e^{-2x} \right] = \frac{1}{9} \cdot \frac{1}{D + 2} e^{-2x} \quad (\text{Case of failure}) \\ &= \frac{1}{9} \cdot x \cdot \frac{1}{1} \cdot e^{-2x} = \frac{x}{9} e^{-2x} \end{aligned}$$

$$\begin{aligned} \frac{1}{(D + 2)(D - 1)^2} e^x &= \frac{1}{(D - 1)^2} \left[\frac{1}{D + 2} e^x \right] = \frac{1}{(D - 1)^2} \left(\frac{1}{1 + 2} e^x \right) \\ &= \frac{1}{3} \cdot \frac{1}{(D - 1)^2} \cdot e^x \quad (\text{Case of failure}) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \cdot \frac{x}{2(D - 1)} \cdot e^x \quad (\text{Case of failure}) \\ &= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x = \frac{1}{6} x^2 e^x \end{aligned}$$

$$\frac{1}{(D + 2)(D - 1)^2} \cdot e^{-x} = \frac{1}{(-1 + 2)(-1 - 1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$$\therefore \text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$$

Hence the complete solution is

$$y = C_1 e^{-2x} + (C_2 + C_3 x) e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}.$$

125. Solve : $\frac{d^2 y}{dx^2} - 4y = x \sinh x$.

(M.D.U., May 2008)

Sol. Given equation in symbolic form is $(D^2 - 4)y = x \sinh x$

Auxiliary equation is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$$\therefore \text{C.F.} = C_1 e^{2x} + C_2 e^{-2x}.$$

$$\text{P.I.} = \frac{1}{D^2 - 4} x \sinh x$$

$$\begin{aligned} &= \frac{1}{D^2 - 4} x \left(\frac{e^x - e^{-x}}{2} \right) = \frac{1}{2} \left[\frac{1}{D^2 - 4} e^x \cdot x - \frac{1}{D^2 - 4} \cdot e^{-x} \cdot x \right] \\ &= \frac{1}{2} \left[e^x \cdot \frac{1}{(D + 1)^2 - 4} \cdot x - e^{-x} \cdot \frac{1}{(D - 1)^2 - 4} \cdot x \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[e^x \cdot \frac{1}{D^2 + 2D - 3} x - e^{-x} \cdot \frac{1}{D^2 - 2D - 3} \cdot x \right] \\
&= \frac{1}{2} \left[e^x \cdot \frac{1}{-3 \left(1 - \frac{2D}{3} - \frac{D^2}{3} \right)} x - e^{-x} \cdot \frac{1}{-3 \left(1 + \frac{2D}{3} - \frac{D^2}{3} \right)} \cdot x \right] \\
&= -\frac{1}{6} \left[e^x \left\{ 1 - \left(\frac{2D}{3} + \frac{D^2}{3} \right) \right\}^{-1} x - e^{-x} \left\{ 1 + \left(\frac{2D}{3} - \frac{D^2}{3} \right) \right\}^{-1} \cdot x \right] \\
&= -\frac{1}{6} \left[e^x \left(1 + \frac{2D}{3} \dots \right) x - e^{-x} \left(1 - \frac{2D}{3} \dots \right) x \right] \\
&= -\frac{1}{6} \left[e^x \left(x + \frac{2}{3} \right) - e^{-x} \left(x - \frac{2}{3} \right) \right] = -\frac{x}{3} \left(\frac{e^x - e^{-x}}{2} \right) - \frac{2}{9} \left(\frac{e^x + e^{-x}}{2} \right) \\
&= -\frac{x}{3} \sinh x - \frac{2}{9} \cosh x
\end{aligned}$$

Hence the complete solution is $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x}{3} \sinh x - \frac{2}{9} \cosh x$.

126. Solve the D.E. $\frac{d^2 y}{dx^2} + 4y = x \sin x$. (M.D.U., Dec., 2007)

Sol.

$$\frac{d^2 y}{dx^2} + 4y = x \sin x$$

Its A.E. is $D^2 + 4 = 0$ or $D^2 = -4$ or $D = \pm 2i$

$$\begin{aligned}
\text{C.F. is } y &= e^{0x} [A \cos \sqrt{2}x + B \sin \sqrt{2}x] + e^{0x} [C \cos \sqrt{2}x + D \sin \sqrt{2}x] \\
&= c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x
\end{aligned}$$

$$\text{P.I. } y = \frac{x \sin x}{D^2 + 4}$$

$$\begin{aligned}
&= x \cdot \frac{\text{I.P. of } e^{ix}}{D^2 + 4} = \text{I.P. of } \frac{e^{ix} \cdot x}{D^2 + 4} = \text{I.P. of } \left(e^{ix} \cdot \frac{x}{(D+i)^2 + 4} \right) \\
&= \text{I.P. of } \left[\frac{e^{ix}}{3} \left(\frac{x}{1 + \frac{D^2 + 2Di}{3}} \right) \right] = \text{I.P. of } \left[\frac{e^{ix}}{3} \left\{ x \left(1 + \frac{D^2 + 2Di}{3} \right)^{-1} \right\} \right] \\
&= \text{I.P. of } \left[\frac{e^{ix}}{3} \left\{ x \left(1 - \frac{D^2 + 2Di}{3} \right) \right\} \right] = \text{I.P. of } \left[\frac{e^{ix}}{3} \left(x - \frac{0 + 2i}{3} \right) \right] \\
&= \text{I.P. of } \left[\frac{(\cos 3x + i \sin 3x)}{3} \left(x - \frac{2i}{3} \right) \right] \\
&= \frac{1}{3} \left[-\frac{2}{3} \cos 3x + x \sin 3x \right] = \frac{1}{9} (3x \sin x - 2 \cos 3x)
\end{aligned}$$

$\therefore$ Complete solution is $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + \frac{1}{9} (3x \sin x - 2 \cos 3x)$

127. Solve : $\frac{d^4 y}{dx^4} - y = \cos x \cosh x$.

Sol. Given equation in symbolic form is $(D^4 - 1)y = \cos x \cosh x$

Auxiliary equation is $D^4 - 1 = 0$ or $(D^2 - 1)(D^2 + 1) = 0 \Rightarrow D = \pm 1, \pm i$

$$\begin{aligned}\therefore \text{C.F.} &= c_1 e^x + c_2 e^{-x} + e^{0x} (c_3 \cos x + c_4 \sin x) \\ &= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x\end{aligned}$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^4 - 1} \cos x \cosh x = \frac{1}{D^4 - 1} \cos x \left(\frac{e^x + e^{-x}}{2} \right) \\ &= \frac{1}{2} \left[\frac{1}{D^4 - 1} e^x \cos x + \frac{1 \cdot e^{-x}}{D^4 - 1} \cos x \right] \\ &= \frac{1}{2} \left[e^x \cdot \frac{1}{(D+1)^4 - 1} \cos x + e^{-x} \cdot \frac{1}{(D-1)^4 - 1} \cos x \right] \\ &= \frac{1}{2} \left[e^x \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x + e^{-x} \cdot \frac{1}{D^4 - 4D^3 + 6D^2 - 4D} \cos x \right] \\ &= \frac{1}{2} \left[e^x \cdot \frac{1}{(-1^2)^2 + 4D(-1^2) + 6(-1^2) + 4D} \cos x \right. \\ &\quad \left. + e^{-x} \cdot \frac{1}{(-1^2)^2 - 4D(-1^2) + 6(-1^2) - 4D} \cos x \right] \\ &= \frac{1}{2} \left[e^x \cdot \frac{1}{-5} \cos x + e^{-x} \cdot \frac{1}{-5} \cos x \right] \\ &= -\frac{1}{5} \left(\frac{e^x + e^{-x}}{2} \right) \cos x = -\frac{1}{5} \cosh x \cos x\end{aligned}$$

Hence the complete solution is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x$.

128. Solve : $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x e^x \sin x$.

Sol. The given equation in symbolic form is $(D^2 - 2D + 1)y = x e^x \sin x$

Auxiliary equation is

$$D^2 - 2D + 1 = 0 \text{ or } (D - 1)^2 = 0 \text{ i.e., } D = 1, 1.$$

$$\therefore \text{C.F.} = (C_1 + C_2 x) e^x$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{(D - 1)^2} e^x \cdot x \sin x = e^x \cdot \frac{1}{(D + 1 - 1)^2} x \sin x \\ &= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \\ &= e^x \frac{1}{D} [-x \cos x + \sin x] \quad (\text{Integrating by parts})\end{aligned}$$

$$\begin{aligned}
 &= e^x \int (-x \cos x + \sin x) dx = e^x [-\{x \sin x - \int 1 \cdot \sin x dx\} - \cos x] \\
 &= e^x [-x \sin x - \cos x - \cos x] = -e^x (x \sin x + 2 \cos x)
 \end{aligned}$$

Hence the complete solution is $y = (C_1 + C_2 x)e^x - e^x (x \sin x + 2 \cos x)$.

129. Solve : $\frac{d^2 y}{dx^2} - 4y = \cosh(2x - 1) + 3^x$.

Sol. Given equation is $(D^2 - 4)y = \cosh(2x - 1) + 3^x$

Auxiliary equation is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$\therefore$ C.F. = $C_1 e^{2x} + C_2 e^{-2x}$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 4} [\cosh(2x - 1) + 3^x] \\
 &= \frac{1}{D^2 - 4} \left[\frac{e^{2x-1} + e^{-(2x-1)}}{2} + e^{\log 3^x} \right] \quad \left(\because \cosh t = \frac{e^t + e^{-t}}{2} \text{ and } u = e^{\log u} \right) \\
 &= \frac{1}{2} \left[\frac{1}{D^2 - 4} e^{2x-1} + \frac{1}{D^2 - 4} e^{-2x+1} \right] + \frac{1}{D^2 - 4} e^{x \log 3} \\
 \text{or } \text{P.I.} &= \frac{1}{2} \left[x \cdot \frac{1}{2D} e^{2x-1} + x \cdot \frac{1}{2D} e^{-(2x-1)} \right] + \frac{1}{(\log 3)^2 - 4} e^{x \log 3} \\
 &= \frac{x}{4} \left[\int e^{2x-1} dx + \int e^{-(2x-1)} dx \right] + \frac{3^x}{(\log 3)^2 - 4} \\
 &= \frac{x}{4} \left[\frac{e^{2x-1}}{2} + \frac{e^{-(2x-1)}}{-2} \right] + \frac{3^x}{(\log 3)^2 - 4} = \frac{x}{4} \left[\frac{e^{2x-1} - e^{-(2x-1)}}{2} \right] + \frac{3^x}{(\log 3)^2 - 4} \\
 &= \frac{x}{4} \sinh(2x - 1) + \frac{3^x}{(\log 3)^2 - 4}
 \end{aligned}$$

Hence the complete solution is $y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{4} \sinh(2x - 1) + \frac{3^x}{(\log 3)^2 - 4}$.

130. Solve : $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$.

Sol. Given equation in symbolic form is $(D^2 + 1)y = \operatorname{cosec} x$

Auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. = $C_1 \cos x + C_2 \sin x$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + 1} \operatorname{cosec} x = \frac{1}{(D + i)(D - i)} \operatorname{cosec} x \\
 &= \frac{1}{2i} \left(\frac{1}{D - i} - \frac{1}{D + i} \right) \operatorname{cosec} x \quad (\text{Partial Fractions}) \\
 &= \frac{1}{2i} \left(\frac{1}{D - i} \operatorname{cosec} x - \frac{1}{D + i} \operatorname{cosec} x \right)
 \end{aligned}$$

Now, $\frac{1}{D - i} \operatorname{cosec} x = e^{ix} \int \operatorname{cosec} x \cdot e^{-ix} dx$

$\therefore \frac{1}{D - a} X = e^{ax} \int X e^{-ax} dx$

$$\begin{aligned}\therefore \frac{1}{D-i} \operatorname{cosec} x &= e^{ix} \int \operatorname{cosec} x (\cos x - i \sin x) dx \\ &= e^{ix} \int (\cot x - i) dx = e^{ix} (\log \sin x - ix)\end{aligned}$$

Changing i to $-i$, we have

$$\begin{aligned}\frac{1}{D+i} \operatorname{cosec} x &= e^{-ix} (\log \sin x + ix) \\ \therefore \text{P.I.} &= \frac{1}{2i} [e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix)] \\ &= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i} \right) - x \left(\frac{e^{ix} + e^{-ix}}{2} \right) = \log \sin x \cdot \sin x - x \cos x\end{aligned}$$

Hence the complete solution is $y = C_1 \cos x + C_2 \sin x + \sin x \log \sin x - x \cos x$.

131. Solve the D.E. $(D^2 + 4)y = \sin 3x + \cos 2x$. (M.D.U., Dec. 2008)

Sol. $(D^2 + 4)y = \sin 3x + \cos 2x$

Its A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

Hence, C.F. is $c_1 \cos 2x + c_2 \sin 2x$

$$\begin{aligned}\text{P.I. is } \frac{\sin 3x + \cos 2x}{D^2 + 4} &= \frac{\sin 3x}{D^2 + 4} + \frac{\cos 2x}{D^2 + 4} \\ \frac{\sin 3x}{D^2 + 4} &= \frac{\sin 3x}{-9 + 4} = \frac{\sin 3x}{-5} \\ \frac{\cos 2x}{D^2 + 4} &= \frac{x(2 \sin 2x)}{2D} = x \int \sin 2x \cdot dx = x \left(-\frac{\cos 2x}{2} \right) \text{ or } -\frac{x \cos 2x}{2}\end{aligned}$$

$$\text{Hence, P.I.} = -\frac{1}{5} \sin 3x - \frac{x \cos 2x}{2}$$

$\therefore$ Complete solution is

$$\text{C.S.} = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \sin 3x - \frac{x \cos 2x}{2}.$$

132. Solve : $\frac{d^2 y}{dx^2} + a^2 y = \tan ax$.

Sol. Given equation in symbolic form is $(D^2 + a^2)y = \tan ax$

Auxiliary equation is $D^2 + a^2 = 0 \Rightarrow D = \pm ia$

$\therefore$ C.F. = $C_1 \cos ax + C_2 \sin ax$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + a^2} \tan ax = \frac{1}{(D+ia)(D-ia)} \tan ax \\ &= \frac{1}{2ia} \left[\frac{1}{D-ia} - \frac{1}{D+ia} \right] \tan ax = \frac{1}{2ia} \left[\frac{1}{D-ia} \tan ax - \frac{1}{D+ia} \tan ax \right]\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{D-ia} \tan ax &= e^{iax} \int \tan ax \cdot e^{-iax} dx = e^{iax} \int \tan ax (\cos ax - i \sin ax) dx \\ &= e^{iax} \int \left\{ \sin ax - i \left(\frac{1 - \cos^2 ax}{\cos ax} \right) \right\} dx = e^{iax} \int \{ \sin ax - i (\sec ax - \cos ax) \} dx\end{aligned}$$

$$\begin{aligned}
&= e^{iax} \left[-\frac{\cos ax}{a} - \frac{i}{a} \log (\sec ax + \tan ax) + i \cdot \frac{\sin ax}{a} \right] \\
&= -\frac{1}{a} e^{iax} [(\cos ax - i \sin ax) + i \log (\sec ax + \tan ax)] \\
&= -\frac{1}{a} e^{iax} [e^{-iax} + i \log (\sec ax + \tan ax)] = -\frac{1}{a} [1 + ie^{iax} \log (\sec ax + \tan ax)]
\end{aligned}$$

Changing i to $-i$, we have

$$\begin{aligned}
\frac{1}{D+ia} \tan ax &= -\frac{1}{a} [1 - ie^{-iax} \log (\sec ax + \tan ax)] \\
\therefore \text{P.I.} &= \frac{1}{2ia} \left[-\frac{1}{a} \{1 + ie^{iax} \log (\sec ax + \tan ax)\} + \frac{1}{a} \{1 - ie^{-iax} \log (\sec ax + \tan ax)\} \right] \\
&= -\frac{1}{a^2} \log (\sec ax + \tan ax) \left(\frac{e^{iax} + e^{-iax}}{2} \right) \\
&= -\frac{1}{a^2} \log (\sec ax + \tan ax) \cdot \cos ax
\end{aligned}$$

Hence the complete solution is $y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \cos ax \log (\sec ax + \tan ax)$.

133. Find P.I. of $(D^2 - 4D + 3)y = e^{3x}$.

$$\begin{aligned}
\text{Sol.} \quad \text{P.I.} &= \frac{1}{D^2 - 4D + 3} e^{3x} = x \cdot \frac{1}{2D - 4} e^{3x} && \text{(Case of failure)} \\
&= x \cdot \frac{1}{2(3) - 4} e^{3x} = \frac{x}{2} e^{3x}.
\end{aligned}$$

134. Solve : $(D^2 - a^2)y = e^{ax} - e^{-ax}$; $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is $m^2 - a^2 = 0$ or $m = \pm a$

$$\text{C.F.} = C_1 e^{ax} + C_2 e^{-ax}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - a^2} (e^{ax} - e^{-ax}) = \frac{1}{D^2 - a^2} (e^{ax}) - \frac{1}{D^2 - a^2} (e^{-ax}) \\
&= x \cdot \frac{1}{2D} (e^{ax}) - x \cdot \frac{1}{2D} (e^{-ax}) = \frac{x}{2} \int e^{ax} dx - \frac{x}{2} \int e^{-ax} dx \\
&= \frac{x}{2} \cdot \frac{e^{ax}}{a} - \frac{x}{2} \cdot \frac{e^{-ax}}{(-a)} = \frac{x}{a} \left[\frac{e^{ax} + e^{-ax}}{2} \right] = \frac{x}{a} \cosh ax
\end{aligned}$$

The complete solution is $y = \text{C.F.} + \text{P.I.} = C_1 e^{ax} + C_2 e^{-ax} + \frac{x}{a} \cosh ax$

where C_1 and C_2 are arbitrary constants of integration.

135. Solve : $(D^2 + D + 1)y = (1 + e^x)^2$; $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is $m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$\text{C.F.} = e^{-\frac{1}{2}x} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + D + 1} (1 + e^x)^2 = \frac{1}{D^2 + D + 1} (1 + e^{2x} + 2e^x) \\ \text{P.I.} &= \frac{1}{D^2 + D + 1} (e^{0x}) + \frac{1}{D^2 + D + 1} e^{2x} + \frac{1}{D^2 + D + 1} (2e^x) \\ &= \frac{1}{(0)^2 + (0) + 1} e^{0x} + \frac{1}{(2)^2 + (2) + 1} e^{2x} + \frac{2}{(1)^2 + (1) + 1} e^x = 1 + \frac{1}{7} e^{2x} + \frac{2}{3} e^x \end{aligned}$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{\frac{-x}{2}} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right) + 1 + \frac{1}{7} e^{2x} + \frac{2}{3} e^x$$

where C_1 and C_2 are arbitrary constants of integration.

136. Solve the differential equation : $\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - y = e^x + 2$.

Sol. The given equation is $(D^3 - 3D^2 + 3D - 1)y = e^x + 2$ or $(D - 1)^3 y = e^x + 2$

Auxiliary equation is $(m - 1)^3 = 0 \Rightarrow m = 1, 1, 1$

$\therefore$ C.F. = $(C_1 + C_2 x + C_3 x^2) e^x$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - 1)^3} (e^x + 2) = \frac{1}{(D - 1)^3} e^x + \frac{1}{(D - 1)^3} (2e^{0x}) = x \cdot \frac{1}{3(D - 1)^2} e^x + \frac{1}{(0 - 1)^3} (2e^{0x}) \\ &= x^2 \cdot \frac{1}{6(D - 1)} e^x - 2 = x^3 \cdot \frac{1}{3 \cdot 2 \cdot 1} (e^x) - 2 = \frac{x^3}{6} e^x - 2 \end{aligned}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.} = (C_1 + C_2 x + C_3 x^2) e^x + \frac{x^3}{6} e^x - 2$

where C_1, C_2 and C_3 are arbitrary constants of integration.

137. Find P.I. of $(D^2 + 4D + 8)y = \sin(2x + 3)$.

$$\begin{aligned} \text{Sol.} \quad \text{P.I.} &= \frac{1}{D^2 + 4D + 8} \sin(2x + 3) \\ &= \frac{1}{-4 + 4D + 8} \sin(2x + 3) = \frac{1}{4(D + 1)} \sin(2x + 3) \\ &= \frac{1}{4} \cdot \frac{D - 1}{(D + 1)(D - 1)} \sin(2x + 3) = \frac{1}{4} \cdot \frac{D - 1}{D^2 - 1} \cdot \sin(2x + 3) \\ &= \frac{1}{4} \cdot \frac{D - 1}{(-4 - 1)} \sin(2x + 3) = -\frac{1}{20} [2 \cos(2x + 3) - \sin(2x + 3)] \\ &= \frac{1}{20} \{\sin(2x + 3) - 2 \cos(2x + 3)\} \end{aligned}$$

138. Solve : $\frac{d^4 y}{dx^4} - m^4 y = \cos mx$.

Sol. Auxiliary equation is $M^4 - m^4 = 0$ or $(M^2 - m^2)(M^2 + m^2) = 0 \Rightarrow M = \pm m, \pm mi$

$\therefore$ C.F. = $c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^4 - m^4} (\cos mx) = x \cdot \frac{1}{4D^3} \cdot \cos mx \\ &= \frac{x}{4} \cdot \frac{1}{D^2} \left(\frac{\sin mx}{m} \right) = \frac{x}{4m} \frac{1}{D} \left(-\frac{\cos mx}{m} \right) = -\frac{x}{4m^2} \left(\frac{\sin mx}{m} \right) = -\frac{x}{4m^3} \sin mx \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx - \frac{x}{4m^3} \sin mx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

139. Solve : $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$; $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is $m^3 - 3m^2 + 4m - 2 = 0$

$$\Rightarrow (m-1)(m^2 - 2m + 2) = 0 \quad \text{or} \quad m = 1, 1 \pm i$$

$$\therefore \text{C.F.} = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 3D^2 + 4D - 2} \cdot e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x \\ &= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{-D + 3 + 4D - 2} (\cos x) \quad (\because D^2 = -1) \\ &= x \cdot \frac{1}{(7-6)} e^x + \frac{1}{3D+1} (\cos x) \\ &= x e^x + \frac{3D-1}{9D^2-1} (\cos x) = x e^x - \frac{1}{10} (-3 \sin x - \cos x) \end{aligned}$$

$\therefore$ The complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + x e^x + \frac{1}{10} (3 \sin x + \cos x) \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants of integration.

140. Solve : $(D^3 - D^2 - 6D)y = 1 + x^2$; $D = \frac{d}{dx}$.

Sol. Auxiliary equation is $m^3 - m^2 - 6m = 0 \Rightarrow m(m-3)(m+2) = 0 \Rightarrow m = 0, -2, 3$

$$\therefore \text{C.F.} = c_1 + c_2 e^{-2x} + c_3 e^{3x}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - D^2 - 6D} (1 + x^2) = \frac{1}{-6D - D^2 + D^3} (1 + x^2) = -\frac{1}{6D} \left[1 + \left(\frac{D - D^2}{6} \right) \right]^{-1} (1 + x^2) \\ &= -\frac{1}{6D} \left[1 - \left(\frac{D - D^2}{6} \right) + \left(\frac{D - D^2}{6} \right)^2 - \dots \right] (1 + x^2) \\ &= -\frac{1}{6D} \left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} \right] (1 + x^2) = -\frac{1}{6D} \left[1 + x^2 - \frac{1}{6} (2x) + \frac{7}{36} (2) \right] \\ &= -\frac{1}{6D} \left(1 + x^2 - \frac{x}{3} + \frac{7}{18} \right) = -\frac{1}{6D} \left(x^2 - \frac{x}{3} + \frac{25}{18} \right) \\ &= -\frac{1}{6} \left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18} x \right) = -\frac{x}{18} \left(x^2 - \frac{x}{2} + \frac{25}{6} \right) \end{aligned}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x}{18} \left(x^2 - \frac{x}{2} + \frac{25}{6} \right)$

where c_1, c_2 and c_3 are arbitrary constants of integration.

141. Solve the D.E. $(D^2 - 4D + 3)z = 2ye^{3y} + 3e^y \cos 2y$, where $D = \frac{d}{dy}$. (M.D.U., Dec., 2009)

Sol. A.E. is $D^2 - 4D + 3 = 0 \Rightarrow D = 3, 1$ C.F. = $e^y (c_1 \cos \sqrt{3}z + c_2 \sin \sqrt{3}z)$

Assume P.I. as $y = a_1 z^2 + a_2 z + a_3 + a_4 e^{-z}$

$$Dy = 2a_1 z + a_2 - a_4 e^{-z}$$

$$D^2 y = 2a_1 + a_4 e^{-z}$$

Substituting these in the given equation, we get

$$(2a_1 z + a_4 e^{-z} - 8a_1 z - 4a_2 + 4a_4 e^{-z} + 3)z = 2ye^{3y} + 3e^y \cos 2y$$

Equating corresponding coefficients on both sides, we get $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$, $a_3 = 0$, $a_4 = 1$

Thus,
$$\text{P.I.} = \frac{1}{2}z^2 - \frac{1}{2}z + e^{-z}$$

$\therefore$ Complete solution is $y = e^y (c_1 \cos \sqrt{3}z + c_2 \sin \sqrt{3}z) + \frac{1}{2}z^2 - \frac{1}{2}z + e^{-z}$.

142. If $\frac{d^2 x}{dt^2} + \frac{g}{b}(x - a) = 0$; a, b and g are positive numbers and $x = a'$, $\frac{dx}{dt} = 0$, when $t = 0$, show that

$$x = a + (a' - a) \cos \sqrt{\frac{g}{b}} t.$$

Sol. We have,
$$\frac{d^2 x}{dt^2} + \frac{g}{b}x = \frac{ag}{b}$$

Auxiliary equation is $m^2 + \frac{g}{b} = 0 \Rightarrow m = \pm \sqrt{\frac{g}{b}} i$

$\therefore$ C.F. = $C_1 \cos \sqrt{\frac{g}{b}} t + C_2 \sin \sqrt{\frac{g}{b}} t$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{b}} \left(\frac{ag}{b} \right) = \frac{ag}{b} \cdot \frac{1}{\frac{g}{b} \left(1 + \frac{bD^2}{g} \right)} (1) = a \left(1 + \frac{bD^2}{g} \right)^{-1} (1) = a$$

$\therefore$ General solution is $x = \text{C.F.} + \text{P.I.}$ or $x = C_1 \cos \sqrt{\frac{g}{b}} t + C_2 \sin \sqrt{\frac{g}{b}} t + a$... (1)

At $t = 0$, $x = a'$

$\therefore$ From (1), $a' = C_1 + a \Rightarrow C_1 = a' - a$... (2)

Now,
$$\frac{dx}{dt} = \sqrt{\frac{g}{b}} \left(-C_1 \sin \sqrt{\frac{g}{b}} t + C_2 \cos \sqrt{\frac{g}{b}} t \right)$$
 ... (3)

At $t = 0$,
$$\frac{dx}{dt} = 0$$

$\therefore$ From (3), $0 = C_2 \sqrt{\frac{g}{b}} \Rightarrow C_2 = 0$... (4)

$\therefore$ From (1), (2) and (4), the complete solution is

$$x = (a' - a) \cos \sqrt{\frac{g}{b}} t + a, \text{ which is the required solution.}$$

- 143.** Find the solution of the equation $(D^2 - 1)y = 1$ which vanishes when $x = 0$ and tends to a finite limit as $x \rightarrow -\infty$ and $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\therefore \text{C.F.} = C_1 e^x + C_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1} (1) = -(1 - D^2)^{-1} (1) = -(1 + D^2 + \dots) (1) = -1$$

$$\therefore \text{General solution is } y = C_1 e^x + C_2 e^{-x} - 1. \quad \dots(1)$$

When, $x = 0, y = 0$

$$\therefore \text{From (1), } 0 = C_1 + C_2 - 1 \Rightarrow C_1 + C_2 = 1 \quad \dots(2)$$

Also, y tends to a finite limit as $x \rightarrow -\infty$

This condition will be satisfied only when $C_2 = 0$

$$\therefore \text{From (2), } C_1 = 1$$

Hence from (1), particular solution is $y = e^x - 1$.

- 144.** Obtain the general solution of the differential equation : $\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 12y = (x - 1)e^{2x}$.

Sol. The given equation is $(D^2 + 4D - 12)y = (x - 1)e^{2x} \quad \dots(1)$

Auxiliary equation is $m^2 + 4m - 12 = 0 \Rightarrow (m - 2)(m + 6) = 0$ or $m = 2, -6$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{-6x}$$

$$\text{P.I.} = \frac{1}{D^2 + 4D - 12} (x - 1) e^{2x}$$

$$= e^{2x} \cdot \frac{1}{[(D + 2)^2 + 4(D + 2) - 12]} (x - 1) = e^{2x} \cdot \frac{1}{D^2 + 8D} (x - 1)$$

$$\text{P.I.} = e^{2x} \cdot \frac{1}{8D} \left(1 + \frac{D}{8}\right)^{-1} (x - 1)$$

$$= e^{2x} \cdot \frac{1}{8D} \left(1 - \frac{D}{8}\right) (x - 1) \quad (\text{Ignoring higher power terms})$$

$$= e^{2x} \cdot \frac{1}{8D} \left(x - 1 - \frac{1}{8}\right) = e^{2x} \cdot \frac{1}{8D} \left(x - \frac{9}{8}\right) = \frac{e^{2x}}{8} \left(\frac{x^2}{2} - \frac{9x}{8}\right)$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = C_1 e^{2x} + C_2 e^{-6x} + e^{2x} \left(\frac{x^2}{16} - \frac{9x}{64}\right)$

where C_1 and C_2 are arbitrary constants of integration.

- 145.** Solve : $y'' - 2y' + 2y = x + e^x \cos x$.

Sol. The given equation is $(D^2 - 2D + 2)y = x + e^x \cos x$; $D \equiv \frac{d}{dx}$

Auxiliary equation is $m^2 - 2m + 2 = 0$, $m = \frac{2 \pm \sqrt{4 - 8}}{2}$ or $m = 1 \pm i$

$$\therefore \text{C.F.} = e^x (C_1 \cos x + C_2 \sin x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2} (x + e^x \cos x)$$

$$\begin{aligned}
&= \frac{1}{D^2 - 2D + 2} (x) + \frac{1}{D^2 - 2D + 2} (e^x \cos x) \\
&= \frac{1}{2 - 2D + D^2} (x) + e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} (\cos x) \\
&= \frac{1}{2} \left[1 - \left(\frac{2D - D^2}{2} \right) \right]^{-1} (x) + e^x \cdot \frac{1}{D^2 + 1} \cos x \\
&= \frac{1}{2} \left[1 + \left(\frac{2D - D^2}{2} \right) \right] (x) + e^x \cdot x \cdot \frac{1}{2D} \cos x \\
&= \frac{1}{2} [1 + D](x) + e^x \cdot \frac{x}{2} \sin x \quad (\text{Ignoring higher powers}) \\
&= \frac{1}{2} (x + 1) + x \frac{e^x}{2} \sin x
\end{aligned}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = e^x (C_1 \cos x + C_2 \sin x) + \frac{1}{2} (x + 1) + x \frac{e^x}{2} \sin x$

where C_1 and C_2 are arbitrary constants of integration.

146. Solve : $(D^2 + 1)y = x^2 \sin 2x$.

Sol. A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. = $C_1 \cos x + C_2 \sin x$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 1} x^2 \sin 2x = \text{Imaginary part of } \frac{1}{D^2 + 1} x^2 e^{2ix} \\
&= \text{I.P. of } e^{2ix} \cdot \frac{1}{(D + 2i)^2 + 1} \cdot x^2 = \text{I.P. of } e^{2ix} \cdot \frac{1}{D^2 + 4iD - 3} x^2 \\
&= \text{I.P. of } e^{2ix} \cdot \frac{1}{-3 \left(1 - \frac{4}{3} iD - \frac{D^2}{3} \right)} \cdot x^2 = \text{I.P. of } \frac{e^{2ix}}{-3} \left[1 - \left(\frac{4iD + D^2}{3} \right) \right]^{-1} \cdot x^2 \\
&= \text{I.P. of } -\frac{1}{3} e^{2ix} \left[1 + \left(\frac{4iD + D^2}{3} \right) + \left(\frac{4iD + D^2}{3} \right)^2 + \dots \right] x^2 \\
&= \text{I.P. of } -\frac{1}{3} e^{2ix} \left[1 + \frac{4iD}{3} + \left(\frac{1}{3} - \frac{16}{9} \right) D^2 + \dots \right] x^2 \\
&= \text{I.P. of } -\frac{1}{3} e^{2ix} \left[x^2 + \frac{4i}{3} (2x) - \frac{13}{9} (2) \right] \\
&= \text{I.P. of } -\frac{1}{3} (\cos 2x + i \sin 2x) \left[\left(x^2 - \frac{26}{9} \right) + \left(\frac{8x}{3} \right) i \right] \\
&= -\frac{1}{3} \left[\frac{8x}{3} \cos 2x + \left(x^2 - \frac{26}{9} \right) \sin 2x \right] = -\frac{1}{27} [24x \cos 2x + (9x^2 - 26) \sin 2x]
\end{aligned}$$

Hence the complete solution is $y = C_1 \cos x + C_2 \sin x - \frac{1}{27} [24x \cos 2x + (9x^2 - 26) \sin 2x]$.

147. Solve : $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

Sol. Auxiliary equation is $(D^2 + 1)^2 = 0 \Rightarrow D = \pm i, \pm i$

$\therefore$ C.F. = $(C_1x + C_2) \cos x + (C_3x + C_4) \sin x$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2 + 1)^2} x^2 \cos x = \text{Real part of } \frac{1}{(D^2 + 1)^2} x^2 (\cos x + i \sin x) \\ &= \text{R.P. of } \frac{1}{(D^2 + 1)^2} x^2 e^{ix} = \text{R.P. of } e^{ix} \cdot \frac{1}{[(D + i)^2 + 1]^2} \cdot x^2 \\ &= \text{R.P. of } e^{ix} \cdot \frac{1}{(D^2 + 2iD)^2} \cdot x^2 = \text{R.P. of } e^{ix} \cdot \frac{1}{\left[2iD \left(1 + \frac{D}{2i}\right)\right]^2} \cdot x^2 \\ &= \text{R.P. of } e^{ix} \cdot \frac{1}{-4D^2 \left(1 - \frac{iD}{2}\right)^2} \cdot x^2 = \text{R.P. of } \frac{e^{ix}}{-4} \cdot \frac{1}{D^2} \left(1 - \frac{iD}{2}\right)^{-2} \cdot x^2 \\ &= \text{R.P. of } -\frac{1}{4} e^{ix} \cdot \frac{1}{D^2} \left[1 + 2\left(\frac{iD}{2}\right) + 3\left(\frac{iD}{2}\right)^2 + \dots\right] x^2 \\ &= \text{R.P. of } -\frac{1}{4} e^{ix} \cdot \frac{1}{D^2} \left[x^2 + i(2x) - \frac{3}{4}(2)\right] = \text{R.P. of } -\frac{1}{4} e^{ix} \cdot \frac{1}{D} \left[\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right] \\ &= \text{R.P. of } -\frac{1}{4} e^{ix} \left[\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4}\right] \\ &= \text{R.P. of } -\frac{1}{48} (\cos x + i \sin x) [(x^4 - 9x^2) + (4x^3)i] \\ \text{P.I.} &= -\frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x] \end{aligned}$$

Hence complete solution is $y = (C_1x + C_2) \cos x + (C_3x + C_4) \sin x - \frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x]$.

148. Solve : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

Sol. Auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow (m - 1)(m - 2) = 0 \Rightarrow m = 1, 2$

$\therefore$ C.F. = $C_1e^x + C_2e^{2x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 3D + 2} \cdot (xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2} (e^{3x} \cdot x) + \frac{1}{D^2 - 3D + 2} (\sin 2x) \\ \text{P.I.} &= e^{3x} \cdot \frac{1}{[(D + 3)^2 - 3(D + 3) + 2]} (x) + \frac{1}{-4 - 3D + 2} (\sin 2x) \\ &= e^{3x} \cdot \frac{1}{D^2 + 3D + 2} (x) + \frac{1}{-3D - 2} (\sin 2x) \\ &= e^{3x} \cdot \frac{1}{2 \left[1 + \left(\frac{3D + D^2}{2}\right)\right]} (x) - \frac{3D - 2}{9D^2 - 4} (\sin 2x) \\ &= \frac{e^{3x}}{2} \left[1 + \left(\frac{3D + D^2}{2}\right)\right]^{-1} (x) - \frac{3D - 2}{-40} (\sin 2x) \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{3x}}{2} \left(1 - \frac{3D}{2} \right) (x) + \frac{1}{40} (6 \cos 2x - 2 \sin 2x) \\
&= \frac{e^{3x}}{2} \left(x - \frac{3}{2} \right) + \frac{1}{20} (3 \cos 2x - \sin 2x)
\end{aligned}$$

Hence the complete solution is $y = C_1 e^x + C_2 e^{2x} + \frac{e^{3x}}{2} \left(x - \frac{3}{2} \right) + \frac{1}{20} (3 \cos 2x - \sin 2x)$

where C_1 and C_2 are arbitrary constants of integration.

149. Solve : $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = \frac{e^{-x}}{x+2}$.

Sol. Auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$ or $m = -1, -1$

$\therefore$ C.F. = $(C_1 + C_2 x) e^{-x}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D+1)^2} \cdot \left(\frac{e^{-x}}{x+2} \right) = e^{-x} \cdot \frac{1}{(D-1+1)^2} \left(\frac{1}{x+2} \right) \\
&= e^{-x} \cdot \frac{1}{D^2} \left(\frac{1}{x+2} \right) = e^{-x} \cdot \frac{1}{D} \log(x+2) = e^{-x} \cdot \left[x \log(x+2) - \int \frac{x dx}{x+2} \right] \\
&= e^{-x} \left[x \log(x+2) - \int \left(1 - \frac{2}{x+2} \right) dx \right] = e^{-x} [x \log(x+2) - x + 2 \log(x+2)] \\
&= e^{-x} [(x+2) \log(x+2) - x]
\end{aligned}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = (C_1 + C_2 x) e^{-x} + e^{-x} [(x+2) \log(x+2) - x]$

where C_1 and C_2 are arbitrary constants of integration.

150. Solve : $(D^2 + 2D + 1)y = x \cos x$.

Sol. Auxiliary equation is $m^2 + 2m + 1 = 0$ or $m = -1, -1$

C.F. = $(C_1 + C_2 x) e^{-x}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 2D + 1} (x \cos x) = \text{Real part of } \frac{1}{D^2 + 2D + 1} (x e^{ix}) \\
&= \text{R.P. of } e^{ix} \cdot \frac{1}{(D+i)^2 + 2(D+i) + 1} (x) = \text{R.P. of } e^{ix} \cdot \frac{1}{D^2 + 2D(1+i) + 2i} (x) \\
&= \text{R.P. of } \frac{e^{ix}}{2i} \left[1 + \frac{1+i}{i} D + \frac{D^2}{2i} \right]^{-1} (x) \\
&= \text{R.P. of } \frac{e^{ix}}{2i} \left[1 - \frac{1+i}{i} D \right] x, \text{ leaving higher powers} \\
&= \text{R.P. of } \frac{e^{ix}}{2i} \left(x - \frac{1+i}{i} \right) = \text{R.P. of } \frac{1}{2} (\cos x + i \sin x) (-ix + 1 + i) \\
&= \frac{1}{2} \cos x + \frac{1}{2} (x-1) \sin x
\end{aligned}$$

$\therefore$ The complete solution is given by $y = \text{C.F.} + \text{P.I.} = (C_1 + C_2 x) e^{-x} + \frac{1}{2} \cos x + \frac{1}{2} (x-1) \sin x$

where C_1 and C_2 are the arbitrary constants of integration.

151. Solve : $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

[U.P.T.U., (B.Pharm. 2005)]

Sol. The auxiliary equation is $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$\therefore$ C.F. $= (C_1 + C_2 x)e^{2x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4D + 4} (8x^2 e^{2x} \sin 2x) = \frac{1}{(D - 2)^2} (8x^2 e^{2x} \sin 2x) \\ &= 8e^{2x} \frac{1}{(D + 2 - 2)^2} (x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D^2} (x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx \\ \text{P.I.} &= 8e^{2x} \cdot \frac{1}{D} \left[x^2 \left(-\frac{\cos 2x}{2} \right) - \int 2x \cdot \left(-\frac{\cos 2x}{2} \right) dx \right] \\ &= 8e^{2x} \cdot \frac{1}{D} \left[-\frac{x^2}{2} \cos 2x + x \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx \right] \\ &= 8e^{2x} \cdot \frac{1}{D} \left[-\frac{x^2}{2} \cos 2x + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right] \\ &= 8e^{2x} \cdot \left[\left(-\frac{x^2}{2} \right) \frac{\sin 2x}{2} - \int (-x) \frac{\sin 2x}{2} dx + \int \frac{x \sin 2x}{2} dx + \frac{\sin 2x}{8} \right] \\ &= 8e^{2x} \left[-\frac{x^2}{4} \sin 2x + \frac{\sin 2x}{8} + \int x \sin 2x \, dx \right] \\ &= 8e^{2x} \left[\left(\frac{1}{8} - \frac{x^2}{4} \right) \sin 2x + x \left(-\frac{\cos 2x}{2} \right) - \int 1 \cdot \left(-\frac{\cos 2x}{2} \right) dx \right] \\ &= 8e^{2x} \left[\left(\frac{1}{8} - \frac{x^2}{4} \right) \sin 2x - \frac{x}{2} \cos 2x + \frac{\sin 2x}{4} \right] \\ &= 8e^{2x} \left[\left(\frac{3}{8} - \frac{x^2}{4} \right) \sin 2x - \frac{x}{2} \cos 2x \right] = e^{2x} [(3 - 2x^2) \sin 2x - 4x \cos 2x] \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} \quad \text{or} \quad y = e^{2x} [C_1 + C_2 x + (3 - 2x^2) \sin 2x - 4x \cos 2x]$$

where C_1 and C_2 are arbitrary constants of integration.

152. Solve the differential equation : $(D^2 - 1)y = xe^x \sin x$; $D \equiv \frac{d}{dx}$.

Sol. Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ C.F. $= C_1 e^x + C_2 e^{-x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 1} (xe^x \sin x) = e^x \cdot \frac{1}{(D + 1)^2 - 1} (x \sin x) = e^x \cdot \frac{1}{D^2 + 2D} (x \sin x) \\ &= e^x \cdot \left[\text{I.P. of } \frac{1}{D^2 + 2D} (x e^{ix}) \right] = e^x \left[\text{I.P. of } e^{ix} \cdot \frac{1}{(D + i)^2 + 2(D + i)} (x) \right] \\ &= e^x \left[\text{I.P. of } e^{ix} \cdot \frac{1}{D^2 + 2(1 + i)D + 2i - 1} (x) \right] \\ &= e^x \left[\text{I.P. of } \frac{e^{ix}}{2i - 1} \left(1 + \left\{ \frac{2(1 + i)D + D^2}{2i - 1} \right\}^{-1} \right) (x) \right] \end{aligned}$$

$$\begin{aligned}
&= e^x \left[\text{I.P. of } \frac{e^{ix}}{2i-1} \left\{ x - \frac{2(1+i)}{2i-1} \right\} \right] = e^x \left[\text{I.P. of } \frac{e^{ix} \cdot (2i+1)}{(2i)^2-1} \left\{ x - \frac{2(1+i)(2i+1)}{(-5)} \right\} \right] \\
&= e^x \left[\text{I.P. of } \left\{ -\frac{1}{5} e^{ix} (2i+1) \left(x - \frac{2-6i}{5} \right) \right\} \right] = -\frac{e^x}{5} \left[\text{I.P. of } e^{ix} (2i+1) \left\{ \frac{5x-2(1-3i)}{5} \right\} \right] \\
&= -\frac{e^x}{25} [\text{I.P. of } (\cos x + i \sin x) \{(10i+5)x - 14 + 2i\}] \\
&= -\frac{e^x}{25} [\text{I.P. of } (\cos x + i \sin x) \{(5x-14) + i(10x+2)\}] \\
&= -\frac{e^x}{25} [(10x+2) \cos x + (5x-14) \sin x]
\end{aligned}$$

Hence the complete solution is $y = C_1 e^x + C_2 e^{-x} - \frac{e^x}{25} [2(1+5x) \cos x + (5x-14) \sin x]$

where C_1 and C_2 are arbitrary constants of integration.

153. A body executes damped forced vibrations given by the equation:

$$\frac{d^2x}{dt^2} + 2k \frac{dx}{dt} + b^2x = e^{-kt} \sin wt. \text{ Solve the equation for both the cases, when } w^2 = b^2 - k^2 \text{ and } w^2 \neq b^2 - k^2. \quad [\text{U.P.T.U., (C.O.), 2005}]$$

Sol. The auxiliary equation is $m^2 + 2km + b^2 = 0 \Rightarrow m = -k \pm \sqrt{k^2 - b^2}$

For damped forced vibrations $k^2 < b^2$

$$\therefore m = -k \pm i\sqrt{b^2 - k^2}$$

Case I. When $w^2 \neq b^2 - k^2$

$$\begin{aligned}
\therefore \text{C.F.} &= e^{-kt} [C_1 \cos \sqrt{b^2 - k^2} t + C_2 \sin \sqrt{b^2 - k^2} t] \\
\text{P.I.} &= \frac{1}{D^2 + 2kD + b^2} (e^{-kt} \sin wt) = e^{-kt} \cdot \frac{1}{(D-k)^2 + 2k(D-k) + b^2} \sin wt. \\
\text{P.I.} &= e^{-kt} \cdot \frac{1}{D^2 + (b^2 - k^2)} \sin wt = e^{-kt} \cdot \frac{1}{-w^2 + b^2 - k^2} \sin wt \\
&= \frac{e^{-kt}}{b^2 - k^2 - w^2} \sin wt
\end{aligned}$$

Hence complete solution is

$$x = e^{-kt} (C_1 \cos \sqrt{b^2 - k^2} t + C_2 \sin \sqrt{b^2 - k^2} t) + \frac{e^{-kt}}{b^2 - k^2 - w^2} \sin wt$$

where C_1 and C_2 are arbitrary constants of integration.

Case II. When $w^2 = b^2 - k^2$

$$\begin{aligned}
\text{C.F.} &= e^{-kt} (C_1 \cos wt + C_2 \sin wt) \\
\text{P.I.} &= \frac{1}{D^2 + 2kD + b^2} (e^{-kt} \sin wt) = e^{-kt} \cdot \frac{1}{(D-k)^2 + 2k(D-k) + b^2} \sin wt \\
\text{P.I.} &= e^{-kt} \cdot \frac{1}{D^2 + b^2 - k^2} \sin wt = e^{-kt} \cdot t \cdot \frac{1}{2D} \sin wt \quad (\text{Case of failure}) \\
&= \frac{te^{-kt}}{2} \left(\frac{-\cos wt}{w} \right) = -\frac{t}{2w} e^{-kt} \cos wt
\end{aligned}$$

Hence complete solution is $x = \text{C.F.} + \text{P.I.} = e^{-kt} (C_1 \cos wt + C_2 \sin wt) - \frac{t}{2w} e^{-kt} \cos wt$

where C_1 and C_2 are arbitrary constants of integration.

154. Find the P.I. of $(D^2 + a^2)y = \sec ax$.

Sol.
$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{(D - ia)(D + ia)} \sec ax$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2ia} \left[\frac{1}{D - ia} - \frac{1}{D + ia} \right] \sec ax = \frac{1}{2ia} \left[\frac{1}{D - ia} (\sec ax) - \frac{1}{D + ia} (\sec ax) \right] \\ &= \frac{1}{2ia} (P_1 - P_2) \end{aligned}$$

where
$$P_1 = \frac{1}{D - ia} (\sec ax) = e^{iax} \int e^{-iax} \sec ax \, dx = e^{iax} \int (\cos ax - i \sin ax) \sec ax \, dx$$

$$= e^{iax} \int (1 - i \tan ax) \, dx = e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\}$$

$$P_2 = \frac{1}{D + ia} (\sec ax) = e^{-iax} \int e^{iax} \sec ax \, dx = e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\}$$

$$\begin{aligned} \therefore \text{P.I.} &= \frac{1}{2ia} \left[e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\} - e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\} \right] \\ &= \frac{1}{2ia} \left[x(e^{iax} - e^{-iax}) + i \left(\frac{\log \cos ax}{a} \right) (e^{iax} + e^{-iax}) \right] \\ &= \frac{1}{2ia} \left[2ix \sin ax + \frac{i}{a} (\log \cos ax) (2 \cos ax) \right] \end{aligned}$$

Hence,
$$\text{P.I.} = \frac{1}{a} \left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right].$$

155. Solve : $\frac{d^2 y}{dx^2} + y = x - \cot x$.

Sol. Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F. = $C_1 \cos x + C_2 \sin x$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 1} (x - \cot x) = \frac{1}{D^2 + 1} (x) - \frac{1}{(D - i)(D + i)} \cot x \\ &= (1 + D^2)^{-1} (x) - \frac{1}{2i} \left[\frac{1}{D - i} - \frac{1}{D + i} \right] \cot x \end{aligned}$$

$$\text{P.I.} = (1 - D^2)(x) - \frac{1}{2i} \left[\frac{1}{D - i} (\cot x) - \frac{1}{D + i} (\cot x) \right] = x - \frac{1}{2i} (P_1 - P_2) \quad \dots(1)$$

where
$$P_1 = \frac{1}{D - i} (\cot x) = e^{ix} \int e^{-ix} \cot x \, dx$$

$$\begin{aligned} &= e^{ix} \int (\cos x - i \sin x) \cot x \, dx = e^{ix} \int \left(\frac{\cos^2 x}{\sin x} - i \cos x \right) dx \\ &= e^{ix} \int \left(\frac{1 - \sin^2 x}{\sin x} - i \cos x \right) dx = e^{ix} \int (\operatorname{cosec} x - \sin x - i \cos x) \, dx \\ &= e^{ix} [\log (\operatorname{cosec} x - \cot x) + \cos x - i \sin x] = e^{ix} [\log (\operatorname{cosec} x - \cot x) + e^{-ix}] \end{aligned}$$

Similarly, $P_2 = e^{-ix} [\log (\operatorname{cosec} x - \cot x) + e^{ix}]$

$$\begin{aligned}\therefore \text{ From (1), P.I.} &= x - \frac{1}{2i} [e^{ix} \{\log (\operatorname{cosec} x - \cot x) + e^{-ix}\} - e^{-ix} \{\log (\operatorname{cosec} x - \cot x) + e^{ix}\}] \\ &= x - \frac{1}{2i} [(e^{ix} - e^{-ix}) \log (\operatorname{cosec} x - \cot x)] = x - \sin x \log (\operatorname{cosec} x - \cot x)\end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = C_1 \cos x + C_2 \sin x + x - \sin x \log (\operatorname{cosec} x - \cot x)$$

where C_1 and C_2 are arbitrary constants of integration.

156. Solve : $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Sol. Auxiliary equation is $D^2 - 4D + 3 = 0$ or $(D - 1)(D - 3) = 0 \Rightarrow D = 1, 3$

$$\therefore \text{C.F.} = C_1 e^x + C_2 e^{3x}$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{(D-1)(D-3)} [\sin 3x \cos 2x] \\ &= \frac{1}{D^2 - 4D + 3} \cdot \frac{1}{2} [\sin 5x + \sin x] \quad \{\because 2 \sin A \cos B = \sin (A+B) + \sin (A-B)\} \\ &= \frac{1}{D^2 - 4D + 3} \left(\frac{1}{2} \sin 5x \right) + \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3} (\sin x) \\ &= \frac{1}{2} \cdot \frac{1}{(-25 + 3 - 4D)} \sin 5x + \frac{1}{2} \cdot \frac{1}{(-1 - 4D + 3)} \sin x \\ &= -\frac{1}{4} \cdot \frac{1}{(2D + 11)} \sin 5x + \frac{1}{4} \cdot \frac{1}{(1 - 2D)} \sin x = -\frac{1}{4} \cdot \frac{2D - 11}{4D^2 - 121} \sin 5x + \frac{1}{4} \cdot \frac{1 + 2D}{1 - 4D^2} \sin x\end{aligned}$$

Put $D^2 = -25$ and $D^2 = -1$

$$\begin{aligned}\text{P.I.} &= -\frac{1}{4} \cdot \frac{2D - 11}{-221} \sin 5x + \frac{1}{4} \left(\frac{1 + 2D}{5} \right) \sin x = \frac{2D - 11}{884} (\sin 5x) + \frac{1}{20} (\sin x + 2 \cos x) \\ &= \frac{10 \cos 5x - 11 \sin 5x}{884} + \frac{1}{20} (\sin x + 2 \cos x)\end{aligned}$$

$$\therefore y = C_1 e^x + C_2 e^{3x} + \frac{(\sin x + 2 \cos x)}{20} + \frac{10 \cos 5x - 11 \sin 5x}{884}$$

where C_1 and C_2 are arbitrary constants of integration.

157. Solve : $\frac{d^2 y}{dx^2} + 4y = e^x + \sin 2x$.

Sol. Auxiliary equation is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ C.F. is C.F. = $C_1 \cos 2x + C_2 \sin 2x$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 4} [e^x + \sin 2x] = \frac{1}{D^2 + 4} e^x + \frac{1}{D^2 + 4} \sin 2x \\ &= \frac{e^x}{5} + \frac{x}{2D} (\sin 2x) = \frac{e^x}{5} + \frac{x}{2} \cdot \int \sin 2x \, dx = \frac{e^x}{5} + \frac{x}{2} \left(-\frac{\cos 2x}{2} \right) = \frac{e^x}{5} - \frac{x \cos 2x}{4}\end{aligned}$$

$\therefore$ The complete solution is given by $y = C_1 \cos 2x + C_2 \sin 2x + \frac{e^x}{5} - \frac{x \cos 2x}{4}$.

158. Solve: $\frac{d^3 y}{dx^3} - 2 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = e^{2x} + \sin 2x$.

Sol. The equation in symbolic form is $(D^3 - 2D^2 + 4D)y = e^{2x} + \sin 2x$

Auxiliary equation is $D^3 - 2D^2 + 4D = 0$ or $D(D^2 - 2D + 4) = 0 \Rightarrow D = 0, 1 \pm i\sqrt{3}$

$$\begin{aligned}
 \therefore \text{C.F.} &= C_1 + e^x (C_2 \cos \sqrt{3} x + C_3 \sin \sqrt{3} x) \\
 \text{P.I.} &= \frac{1}{D(D^2 - 2D + 4)} (e^{2x} + \sin 2x) = \frac{1}{D(D^2 - 2D + 4)} e^{2x} + \frac{1}{D(D^2 - 2D + 4)} \sin 2x \\
 &= \frac{1}{8} e^{2x} + \frac{1}{D^3 - 2D^2 + 4D} \cdot \sin 2x \\
 \text{P.I.} &= \frac{1}{8} e^{2x} + \frac{1}{D^3 - 2D^2 + 4D} \sin 2x = \frac{1}{8} e^{2x} + \frac{1}{D(D^2 - 2D + 4)} \sin 2x \\
 &= \frac{1}{8} e^{2x} + \frac{1}{D(-4 - 2D + 4)} \sin 2x \quad (\because D^2 = -4) \\
 &= \frac{1}{8} e^{2x} + \frac{1}{-2D^2} \sin 2x = \frac{1}{8} e^{2x} + \frac{1}{8} \sin 2x = \frac{1}{8} (e^{2x} + \sin 2x)
 \end{aligned}$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\text{or } y = C_1 + (C_2 \cos \sqrt{3} x + C_3 \sin \sqrt{3} x)e^x + \frac{1}{8} (e^{2x} + \sin 2x)$$

where C_1 , C_2 and C_3 are arbitrary constants of integration.

159. Solve : $(D - 1)^2 (D + 1)^2 y = \sin^2 \left(\frac{x}{2} \right) + e^x + x$. (M.D.U., 2005)

Sol. $(D - 1)^2 (D + 1)^2 y = \sin^2 \left(\frac{x}{2} \right) + e^x + x$

Auxiliary equation is $(D^2 - 1)^2 = 0$ i.e., $D = \pm 1, \pm 1$

$\therefore$ C.F. = $(C_1 + C_2 x) e^x + (C_3 + C_4 x) e^{-x}$

Now, P.I. = $\frac{1}{(D^2 - 1)^2} \left[\sin^2 \left(\frac{x}{2} \right) + e^x + x \right] = P_1 + P_2 + P_3$

where $P_1 = \frac{1}{(D^2 - 1)^2} \sin^2 \left(\frac{x}{2} \right) = \frac{1}{(D - 1)^2 (D + 1)^2} \left(\frac{1 - \cos x}{2} \right)$

$$= \frac{1}{2} - \frac{1}{(D^2 - 1)^2} \cdot \frac{\cos x}{2} = \frac{1}{2} - \frac{1}{(D - 1)^2 (D^2 + 2D + 1)} \cdot \frac{\cos x}{2}$$

$$\begin{aligned}
 P_1 &= \frac{1}{2} - \frac{1}{(D - 1)^2} \cdot \frac{1}{4} \sin x = \frac{1}{2} - \frac{1}{D^2 + 1 - 2D} \cdot \left(\frac{1}{4} \sin x \right) = \frac{1}{2} + \frac{1}{2D} \cdot \left(\frac{1}{4} \sin x \right) \\
 &= \frac{1}{2} + \frac{1}{8} (-\cos x) = \frac{1}{2} - \frac{1}{8} \cos x
 \end{aligned}$$

$$P_2 = \frac{1}{(D - 1)^2 (D + 1)^2} e^x = \frac{1}{4(D - 1)^2} e^x = \frac{1}{4} x \frac{1}{2(D - 1)} e^x = \frac{1}{8} x^2 e^x$$

$$P_3 = \frac{1}{(D^2 - 1)^2} \cdot x \quad \text{or} \quad \frac{1}{(1 - D^2)^2} \cdot x = x$$

$\therefore$ P.I. = $\frac{1}{2} - \frac{1}{8} \cos x + \frac{1}{8} x^2 e^x + x$

Complete solution is $y = (C_1 + C_2 x) e^x + (C_3 + C_4 x) e^{-x} + \frac{1}{2} - \frac{1}{8} \cos x + \frac{1}{8} x^2 e^x + x$.

160. Solve the differential equation : $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$. (M.D.U., 2005)

Sol. The auxiliary equation is

$$D^3 + 2D^2 + D = 0 \quad \text{or} \quad D(D^2 + 2D + 1) = 0$$

$$D(D + 1)^2 = 0 \quad \text{i.e., } D = 0, -1, -1$$

$$\therefore \text{C.F.} = C_1 + (C_2 + C_3 x) e^{-x}$$

$$\begin{aligned} \text{P.I.} &= \frac{x^2 e^{2x}}{D^3 + 2D^2 + D} + \frac{1 - \cos 2x}{2(D^3 + 2D^2 + D)} \\ &= e^{2x} \left[\frac{x^2}{(D+2)^3 + 2(D+2)^2 + (D+2)} \right] + \frac{1}{2(D^3 + 2D^2 + D)} - \frac{\cos 2x}{2(D^3 + 2D^2 + D)} \\ &= e^{2x} \cdot \frac{x^2}{D^3 + 8D^2 + 21D + 18} + \frac{(1 + D^2 + 2D)^{-1}}{2D} - \frac{\cos 2x}{2[D(-4) + 2(-4) + D]} \\ &= \frac{e^{2x}}{18} \left[\frac{x^2}{1 + \left(\frac{D^3 + 8D^2 + 21D}{18} \right)} \right] + \frac{1 - D^2 - 2D}{2D} + \frac{\cos 2x (3D - 8)}{2(9D^2 - 64)} \\ &= \frac{e^{2x}}{18} \left[x^2 \left(1 + \frac{D^3 + 8D^2 + 21D}{18} \right)^{-1} \right] + \frac{1}{2D} + \frac{-6 \sin 2x - 8 \cos 2x}{2(-100)} \\ &= \frac{e^{2x}}{18} \left[x^2 - \frac{x^2}{18} (D^3 + 8D^2 + 21D) \right] + \frac{x}{2} + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) \\ &= \frac{e^{2x}}{18} \left[x^2 - \frac{1}{18} \{D^3(x^2) + 8D^2(x^2) + 21D(x^2)\} \right] + \frac{x}{2} + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) \\ &= \frac{e^{2x}}{18} \left[x^2 - \frac{1}{18} \{0 + 8(2) + 21(2x)\} \right] + \frac{x}{2} + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) \\ &= \frac{e^{2x}}{18} \left[x^2 - \frac{1}{18} (42x + 16) \right] + \frac{x}{2} + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) \\ \Rightarrow \text{P.I.} &= \frac{e^{2x}}{18} \left[x^2 - \frac{21x}{9} - \frac{8}{9} \right] + \frac{x}{2} + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) \end{aligned}$$

$$\therefore \text{The solution is } y = C_1 + (C_2 + C_3 x) e^{-x} + \frac{e^{2x}}{18} \left[x^2 - \frac{7x}{3} - \frac{8}{9} \right] + \frac{1}{100} (3 \sin 2x + 4 \cos 2x) + \frac{x}{2}$$

where C_1 , C_2 and C_3 are arbitrary constants of integration.

- 161.** Describe the method of variation of parameters to find P.I. of a linear differential equation of second order. Apply this method to solve $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$.

Sol. Consider the linear equation of second order with constant co-efficients

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$$

Let its C.F. be $y = C_1 y_1 + C_2 y_2$

$$\text{Then, P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2 .

Now we will use this method to solve the given equation $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

The auxiliary equation is $D^2 - 6D + 9 = 0$ or $(D - 3)^2 = 0$ or $D = 3, 3$

$\therefore$ C.F. is $y = (C_1 + C_2 x) e^{3x}$

Thus, $y_1 = e^{3x}, y_2 = xe^{3x}, X = \frac{e^{3x}}{x^2}$

$$\therefore W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$$

$$\begin{aligned} \therefore \text{P.I.} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = -e^{3x} \int xe^{3x} \cdot \frac{e^{3x}}{x^2} \cdot e^{-6x} dx + xe^{3x} \int e^{3x} \cdot \frac{e^{3x}}{x^2} e^{-6x} dx \\ &= -e^{3x} \int \frac{1}{x} dx + xe^{3x} \int \frac{1}{x^2} dx = -e^{3x} \log x + xe^{3x} \left(-\frac{1}{x} \right) \\ &= e^{3x} [-\log x - 1] \text{ or } -(1 + \log x) e^{3x} \end{aligned}$$

Hence complete solution is

$$y = (C_1 + C_2 x) e^{3x} - (1 + \log x) e^{3x} = (C_1 + C_2 x - 1 - \log x) e^{3x}$$

or $y = (C_1' + C_2 x - \log x) e^{3x}$, where $C_1' = C_1 - 1$.

162. Apply the method of variation of parameters to solve $\frac{d^2 y}{dx^2} + 4y = 4 \sec^2 2x$.

(M.D.U., May 2009)

Sol. Given equation in symbolic form is $(D^2 + 4)y = 4 \sec^2 2x$.

Its A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ C.F. is $y = C_1 \cos 2x + C_2 \sin 2x$

Thus, $y_1 = \cos 2x, y_2 = \sin 2x, X = 4 \sec^2 2x$.

$$\therefore W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2$$

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx \\ &= -\cos 2x \int \frac{\sin 2x \cdot 4 \sec^2 2x}{2} dx + \sin 2x \int \frac{\cos 2x \cdot 4 \sec^2 2x}{2} dx \\ &= -2 \cos 2x \int \sec 2x \tan 2x dx + 2 \sin 2x \int \sec 2x \cdot dx \\ &= -2 \cos 2x \cdot \frac{\sec 2x}{2} + 2 \sin 2x \cdot \frac{1}{2} \log (\sec 2x + \tan 2x) \\ &= -1 + \sin 2x \log (\sec 2x + \tan 2x) \end{aligned}$$

Hence the complete solution is given by

$$y = C_1 \cos 2x + C_2 \sin 2x - 1 + \sin 2x \log (\sec 2x + \tan 2x).$$

163. Solve by the method of variation of parameters $\frac{d^2 y}{dx^2} - y = e^{-x} \cdot \sin(e^{-x}) + \cos(e^{-x})$.

Sol. Given equation in symbolic form is $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

Its A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore$ C.F. is $y = C_1 e^x + C_2 e^{-x}$

Here, $y_1 = e^x, y_2 = e^{-x}, X = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

$$\therefore W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$\begin{aligned} \therefore \text{P.I.} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx \\ &= -e^x \int \frac{e^{-x} \cdot [e^{-x} \sin(e^{-x}) + \cos(e^{-x})] dx}{-2} + e^{-x} \int \frac{e^x \cdot [e^{-x} \sin(e^{-x}) + \cos(e^{-x})] dx}{-2} \\ \text{P.I.} &= \frac{1}{2} e^x \int e^{-x} [e^{-x} \sin(e^{-x}) + \cos(e^{-x})] dx - \frac{1}{2} e^{-x} \int e^x [e^{-x} \sin(e^{-x}) + \cos(e^{-x})] dx \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Now, } \int e^{-x} [e^{-x} \sin(e^{-x}) + \cos(e^{-x})] dx \\ &= - \int (t \sin t + \cos t) dt, \text{ where } t = e^{-x}, dt = -e^{-x} dx \\ &= - \left[t(-\cos t) - \int 1 \cdot (-\cos t) dt + \sin t \right] \\ &= - [-t \cos t + \sin t + \sin t] = t \cos t - 2 \sin t = e^{-x} \cos(e^{-x}) - 2 \sin(e^{-x}) \end{aligned}$$

$$\begin{aligned} \text{Also, } \int e^x [\cos(e^{-x}) + e^{-x} \sin(e^{-x})] dx \\ &= e^x \cos(e^{-x}) \quad \left| \text{Form } \int e^x [f(x) + f'(x)] dx = e^x f(x) \right. \end{aligned}$$

$\therefore$ From equation (1), we have

$$\begin{aligned} \text{P.I.} &= \frac{1}{2} e^x [e^{-x} \cos(e^{-x}) - 2 \sin(e^{-x})] - \frac{1}{2} e^{-x} \cdot e^x \cdot \cos(e^{-x}) \\ &= \frac{1}{2} \cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2} \cos(e^{-x}) = -e^x \sin(e^{-x}) \end{aligned}$$

Hence, C.S. is $y = C_1 e^x + C_2 e^{-x} - e^x \sin(e^{-x})$.

164. Solve by the method of variation of parameters : $y'' - 2y' + y = e^x \log x$.

Sol. Given equation is $(D^2 - 2D + 1)y = e^x \log x$

Its A.E. is $D^2 - 2D + 1 = 0$ or $(D - 1)^2 = 0$ or $D = 1, 1$

C.F. is $y = (C_1 + C_2 x)e^x$

Here, $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}$$

$$\begin{aligned} \text{Thus, P.I.} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx \\ \text{P.I.} &= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}} dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx = -e^{+x} \int x \log x dx + x e^x \int \log x \cdot dx \\ &= -e^{+x} \left[\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} \cdot dx \right] + xe^x \{x \log x - x\} \\ &= -e^{+x} \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right] + x^2 e^x (\log x - 1) = \frac{1}{4} [2x^2 e^x \log x - 3x^2 e^x] = \frac{1}{4} x^2 e^x [2 \log x - 3]. \end{aligned}$$

165. Solve the D.E. by the method of variation of parameters : $(D^2 - 2D)y = e^x \sin x$.

(M.D.U., Dec., 2009)

Sol. $(D^2 - 2D)y = e^x \sin x$

Its A.E. is $D^2 - 2D = 0 \Rightarrow D = 0, 2$

C.F. = $(C_1 + C_2 x)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D} \cdot e^x \sin x = e^x \cdot \frac{1}{D(D-2)} \cdot \sin x = e^x \cdot \frac{1}{D} \cdot \sin x \quad (\text{when } D = 2)$$

$$= e^x \cdot (-\cos x) \quad \text{or} \quad -e^x \cos x$$

C.S. is $y = (C_1 + C_2 x)e^x - e^x \cos x$.

166. Solve by using the method of variation of parameters the equation : $y'' - 2y' + 2y = e^x \tan x$.

(M.D.U., 2005)

Sol. The equation is $(D^2 - 2D + 2)y = e^x \tan x$

The auxiliary equation is $D^2 - 2D + 2 = 0$ or $D = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

C.F. is $y = e^x [C_1 \cos x + C_2 \sin x]$

For P.I. $y_1 = e^x \cdot \cos x, y_2 = e^x \cdot \sin x, X = e^x \tan x$

$$y_1' = -e^x \sin x + e^x \cos x$$

$$y_2' = e^x \cos x + e^x \sin x$$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = |y_1 y_2' - y_2 y_1'| \\ &= e^x \cos x (e^x \cos x + e^x \sin x) - e^x \sin x (-e^x \sin x + e^x \cos x) \\ &= (e^x)^2 [\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x] = (e^x)^2 \end{aligned}$$

P.I. is given by

$$\begin{aligned} \text{P.I.} &= -e^x \cos x \int \frac{e^x \sin x}{(e^x)^2} e^x \tan x \, dx + e^x \sin x \int \frac{e^x \cos x}{(e^x)^2} e^x \tan x \, dx \\ &= -e^x \cos x \int \frac{\sin^2 x}{\cos x} \, dx + e^x \sin x \int \sin x \, dx \\ &= -e^x \cos x \int \frac{1 - \cos^2 x}{\cos x} \, dx + e^x \sin x (-\cos x) \\ &= -e^x \cos x \log (\sec x + \tan x) + e^x \cos x \sin x - e^x \sin x \cos x \\ &= -e^x \cos x \cdot \log (\sec x + \tan x) \end{aligned}$$

$\therefore$ The complete solution is given by $y = e^x (C_1 \cos x + C_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$.

167. By using the method of variation of parameters, solve $\frac{d^2 y}{dx^2} + 4y = \tan 2x$. (M.D.U., 2005)

Sol. Equation in symbolic form is $(D^2 + 4)y = \tan 2x$

Its auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm 2i$

$\therefore$ C.F. is given by $y = C_1 \cos 2x + C_2 \sin 2x, y_1 = \cos 2x, y_2 = \sin 2x, X = \tan 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} \, dx + y_2 \int \frac{y_1 X}{W} \, dx$$

$$\begin{aligned}
\text{P.I.} &= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \cdot \tan 2x}{2} dx \\
&= -\frac{\cos 2x}{2} \int \frac{1 - \cos^2 2x}{\cos 2x} dx + \sin 2x \int \frac{\sin 2x}{2} dx \\
&= -\frac{\cos 2x}{2} \int (\sec 2x - \cos 2x) dx - \frac{\sin 2x \cos 2x}{4} \\
&= -\frac{\cos 2x}{2} \left[\frac{\log (\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2} \right] - \frac{\sin 2x \cos 2x}{4} \\
&= -\frac{\cos 2x}{4} \log (\sec 2x + \tan 2x)
\end{aligned}$$

∴ The complete solution is given by

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \cos 2x \cdot \log (\sec 2x + \tan 2x).$$

168. Solve : $\frac{d^2 y}{dx^2} + y = x \sin x$ by the method of variation of parameters.

Sol. Given equation is $(D^2 + 1)y = x \sin x$

Its auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

∴ C.F. is given by C.F. = $C_1 \cos x + C_2 \sin x$

For P.I., $y_1 = \cos x, y_2 = \sin x, X = x \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\begin{aligned}
\therefore \text{P.I.} &= -\cos x \int \sin x \cdot x \sin x dx + \sin x \int \cos x \cdot x \sin x dx \\
&= -\cos x \int x \left(\frac{1 - \cos 2x}{2} \right) dx + \frac{\sin x}{2} \int x \sin 2x dx \\
&= -\cos x \left[\frac{x^2}{4} - \frac{1}{2} \int x \cos 2x dx \right] + \frac{\sin x}{2} \left(\frac{x(-\cos 2x)}{2} + \int \frac{\cos 2x}{2} dx \right) \\
&= -\frac{x^2 \cos x}{4} + \frac{\cos x}{2} \left[x \frac{(\sin 2x)}{2} - \int \frac{\sin 2x}{2} dx \right] - \frac{\sin x (x \cos 2x)}{4} + \frac{\sin x}{2} \cdot \frac{\sin 2x}{4} \\
&= -\frac{x^2 \cos x}{4} + \frac{x}{4} (\sin 2x \cos x) + \frac{\cos x \cos 2x}{8} - \frac{x \sin x \cos 2x}{4} + \frac{\sin x \cdot \sin 2x}{8} \\
&= -\frac{x^2}{4} \cos x + \frac{x}{4} (\cos x \sin 2x - \sin x \cos 2x) + \frac{1}{8} (\cos x \cos 2x + \sin x \cdot \sin 2x) \\
&= -\frac{x^2}{4} \cos x + \frac{x}{4} \sin x + \frac{1}{8} \cos x
\end{aligned}$$

∴ The complete solution is

$$y = C_1 \cos x + C_2 \sin x - \frac{x^2}{4} \cos x + \frac{x}{4} \sin x + \frac{1}{8} \cos x.$$

169. Solve by the method of variation of parameters: $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} = e^x \cdot \sin x$. (M.D.U., May 2007)

Sol. $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} = e^x \cdot \sin x$

Given equation in symbolic form is $(D^2 - 2D)y = e^x \cdot \sin x$

Its A.E. is $D^2 - 2D = 0 \Rightarrow D(D - 2) = 0$ or $D = 0, 2$

C.F. is $y = c_1 + c_2 e^{2x}$

Here, $y_1 = 1$, $y_2 = e^{2x}$ and $X = e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

$$\begin{aligned} \therefore \text{P.I.} &= -y_1 \int \frac{y_2 X}{W} \cdot dx + y_2 \int \frac{y_1 X}{W} \cdot dx = - \int \frac{e^{2x} \cdot e^x \sin x}{2e^{2x}} \cdot dx + e^{2x} \int \frac{e^x \sin x}{2e^{2x}} dx \\ &= - \frac{1}{2} \int e^x \sin x dx + \frac{e^{2x}}{2} \int \frac{\sin x}{e^x} \cdot dx = - \frac{1}{2} \int e^x \sin x dx + \frac{e^{2x}}{2} \int e^{-x} \sin x dx \end{aligned}$$

Solving, $\text{P.I.} = -\frac{1}{2} e^x \sin x$

$\therefore$ Complete solution is $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$.

170. Solve the D.E. $\frac{d^2 y}{dx^2} + y = \tan x$, by the method of variation of parameters. (M.D.U., Dec., 2007)

Sol. Given equation in symbolic form is, $(D^2 + 1)y = \tan x$

A.E. is $D^2 + 1 = 0, \Rightarrow D = \pm i$

C.F. $y = e^{0x} (A \cos x + B \sin x) + e^{0x} (C \cos x - D \sin x) = c_1 \cos x + c_2 \sin x$

For P.I., $y_1 = \cos x$, $y_2 = \sin x$, $y_1' = -\sin x$, $y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\begin{aligned} \therefore \text{P.I.} \quad y &= -\cos x \int \frac{\sin x}{1} \tan x dx + \sin x \int \frac{\cos x}{1} \tan x dx \\ &= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x dx \\ &= -\cos x \int \frac{1 - \cos^2 x}{\cos x} \cdot dx + \sin x \int \sin x dx \\ &= -\cos x \int \sec x dx + \cos x \int \cos x dx + \sin x \int \sin x dx \\ &= -\cos x \cdot \log(\sec x + \tan x) + \cos x \sin x - \sin x \cos x \\ &= -\cos x \cdot \log(\sec x + \tan x) \end{aligned}$$

$\therefore$ Complete solution is, $y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$.

171. Solve by the method of variation of parameters : $\frac{d^2 y}{dx^2} + y = \sec x$.

Sol. The equation in symbolic form is $(D^2 + 1)y = \sec x$

Auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

Hence, C.F. = $C_1 \cos x + C_2 \sin x$

$$y_1 = \cos x, y_2 = \sin x, X = \sec x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = |y_1 y_2' - y_2 y_1'|$$

$$\text{or } W = \cos x \cdot \cos x - \sin x (-\sin x) = \cos^2 x + \sin^2 x = 1$$

$$\begin{aligned} \therefore \text{P.I.} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx \\ &= -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx = -\cos x \int \tan x dx + x \sin x \end{aligned}$$

$$\text{P.I.} = \cos x \int (-\tan x) dx + x \sin x = \cos x \cdot \log \cos x + x \sin x$$

$\therefore$ The complete solution is $y = C_1 \cos x + C_2 \sin x + x \sin x + \cos x \cdot \log \cos x$.

172. Solve the D.E. by the method of variation of parameters: $y'' + y = \sec^2 x$. (M.D.U., May, 2008)

Sol. The symbolic form of given D.E. is $D^2 y + y = \sec^2 x$

Its A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. is $C_1 \cos x + C_2 \sin x$

$$y_1 = \cos x, y_2 = \sin x, y_1' = -\sin x, y_2' = \cos x, X = \sec^2 x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 X dx}{W} + y_2 \int \frac{y_1 X}{W} \cdot dx \\ &= -\cos x \int \sin x \sec^2 x dx + \sin x \int \cos x \sec^2 x dx \\ &= -\cos x \int \sec x \tan x dx + \sin x \int \sec x dx \\ &= -\cos x \sec x + \sin x \log (\sec x + \tan x) \end{aligned}$$

$$\text{or } \text{P.I.} = -1 + \sin x \log (\sec x + \tan x)$$

$$\text{C.S.} = \text{C.F.} + \text{P.I.}$$

$$= C_1 \cos x + C_2 \sin x - 1 + \sin x \log (\sec x + \tan x).$$

173. Solve the D.E. by variation of parameters: $(D^2 - 2D + 2)y = e^x \tan x$. (M.D.U., Dec., 2008)

Sol. $(D^2 - 2D + 2)y = e^x \tan x$

Its A.E. is $D^2 - 2D + 2 = 0 \Rightarrow D = 1 \pm i$

Its C.F. is $CF = e^x [C_1 \cos x + C_2 \sin x]$

For P.I., $y_1 = e^x \cos x, y_2 = e^x \sin x, y_1' = -e^x \sin x + e^x \cos x, y_2' = e^x \cos x + e^x \sin x$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix} \\ &= e^{2x} \{ \cos x \sin x + \cos^2 x \} - e^{2x} \{ \cos x \sin x - \sin^2 x \} \\ &= e^{2x} \{ \cos x \sin x + \cos^2 x - \cos x \sin x + \sin^2 x \} = e^{2x}. \end{aligned}$$

$$\begin{aligned}
\therefore \text{P.I.} &= -e^x \cos x \int \frac{e^x \sin x}{e^{2x}} \cdot e^x \tan x \, dx + e^x \sin x \int \frac{e^x \cos x}{e^{2x}} \times e^x \tan x \, dx \\
&= -e^x \cos x \int \frac{\sin^2 x}{\cos x} \, dx + e^x \sin x \int \sin x \, dx \\
&= -e^x \cos x \int \frac{1 - \cos^2 x}{\cos x} \cdot dx + e^x \sin x (-\cos x) \\
&= -e^x \cos x \left[\int \frac{dx}{\cos x} - \int \cos x \, dx \right] - e^x \sin x \cos x \\
&= -e^x \cos x [\log (\sec x + \tan x) - \sin x] - e^x \sin x \cos x \\
&= -e^x \cos x \log (\sec x + \tan x) + e^x \cos x \sin x - e^x \sin x \cos x \\
&= -e^x \cos x \log (\sec x + \tan x)
\end{aligned}$$

Hence complete solution is, $y = e^x [C_1 \cos x + C_2 \sin x] - e^x \cos x \log (\sec x + \tan x)$.

174. Solve by the method of variation of parameters : $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$.

Sol. The equation is $(D^2 + 1)y = \operatorname{cosec} x$

Its auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$\text{C.F.} = C_1 \cos x + C_2 \sin x$$

For P.I. $y_1 = \cos x, y_2 = \sin x, X = \operatorname{cosec} x$

$$W = |y_1 y_2' - y_2 y_1'| = \cos^2 x + \sin^2 x = 1.$$

$$\text{P.I.} = -\cos x \int \sin x \cdot \operatorname{cosec} x \, dx + \sin x \int \cos x \cdot \operatorname{cosec} x \, dx$$

or $\text{P.I.} = -x \cos x + \sin x \log (\sin x)$

$\therefore$ The complete solution is $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log (\sin x)$.

175. Solve : $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = x^2 \log x$.

Sol. Given equation is called a form of Cauchy's homogeneous linear equation, because coefficient of $x \frac{dy}{dx}$ and y are constants (-1 and -3 respectively) and $X = x^2 \log x$ is a function of x alone (on the R.H.S.)

In this case we put $x = e^z$ so that $z = \log x, x \frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dz}$

$$\begin{aligned}
D(D-1)y - Dy - 3y &= ze^{2z} \quad \text{or} \quad [D(D-1) - D - 3]y = z \cdot e^{2z} \\
[D^2 - 2D - 3]y &= z \cdot e^{2z}
\end{aligned}$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 2D - 3 = 0$ or $(D-3)(D+1) = 0$

$\therefore D = 3, -1$

$$\text{C.F.} = C_1 e^{3z} + C_2 e^{-z} = C_1 x^3 + \frac{C_2}{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3} (z \cdot e^{2z}) = e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3} \cdot z \equiv e^{2z} \cdot \frac{1}{D^2 + 2D - 3} z.$$

$$\begin{aligned} \text{P.I.} &= e^{2z} \frac{1}{-3 \left(1 - \frac{2D}{3} - \frac{D^2}{3} \right)} \cdot z = -\frac{1}{3} e^{2z} \left[1 + \frac{2D}{3} + \frac{D^2}{3} \right] \cdot z \\ &= -\frac{1}{3} e^{2z} \left(z + \frac{2}{3} \right) = -\frac{x^2}{3} \left(\log x + \frac{2}{3} \right) \end{aligned}$$

Hence the complete solution is $y = C_1 x^3 + \frac{C_2}{x} - \frac{x^2}{3} \left(\log x + \frac{2}{3} \right)$.

- 176.** Solve the D.E. $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$. (M.D.U., Dec., 2007)

Sol. This is Cauchy's homogeneous linear equation.

Put $x = e^t \Rightarrow t = \log x$.

$$\frac{x dy}{dx} = Dy, \quad \frac{x^2 d^2 y}{dx^2} = D(D-1)y$$

$\therefore$ Equation becomes $D(D-1)y - 2Dy - 4y = x^2 2 \log x$ or $(D^2 - 3D - 4)y = x^2 + 2 \log x$

A.E. is $D^2 - 3D - 4 = 0 \Rightarrow D = -1, 4$

$\therefore$ C.F. is $y = C_1 e^{4t} + C_2 e^{-t}$

$$\begin{aligned} \text{For P.I.,} \quad y &= \frac{x^2 + 2 \log x}{D^2 - 3D - 4} = \frac{e^{2t} + 2t}{D^2 - 3D - 4} = \frac{e^{2t}}{D^2 - 3D - 4} + \frac{2t}{D^2 - 3D - 4} \\ &= \frac{e^{2t}}{4 - 6 - 4} + \frac{2}{4} \cdot \frac{t}{\left(\frac{D^2 - 3D}{4} - 1 \right)} = \frac{-e^{2t}}{6} - \frac{1}{2} t \left(1 - \frac{D^2 - 3D}{4} \right)^{-1} \\ &= \frac{-e^{2t}}{6} - \frac{1}{2} t \left(1 + \frac{D^2 - 3D}{4} \right) \\ y &= \frac{-e^{2t}}{6} - \frac{1}{2} t - \frac{1}{8} (0) + \frac{3}{8} (1) \end{aligned}$$

$\therefore$ Complete solution is $y = C_1 e^{4t} + C_2 e^{-t} - \frac{e^{2t}}{6} - \frac{t}{2} + \frac{3}{8}$

Changing t to x ($t = \log x$, $x = e^t$), we get $y = C_1 x^4 + C_2 \left(\frac{1}{x} \right) - \frac{x^2}{6} - \left(\frac{\log x}{2} \right) + \frac{3}{8}$.

- 177.** Solve the D.E. : $(x^2 D^2 - xD - 3)y = x^2 (\log x)^2$. (M.D.U., May 2008)

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$, $z = \log x$.

$$\text{So that } \frac{xdy}{dx} = Dy, \quad \frac{x^2 d^2 y}{dx^2} = D(D-1)y$$

Given equation becomes $[D(D-1) - D - 3]y = z^2 e^{2z}$ or $(D^2 - 2D - 3)y = z^2 e^{2z}$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 2D - 3 = 0$ or $(D-3)(D+1) = 0$ or $D = 3, -1$

C.F. is $C_1 e^{3z} + C_2 \cdot e^{-z}$ or $C_1 x^3 + \frac{C_2}{x}$

$$\text{P.I. is } \frac{1}{D^2 - 2D - 3} (e^{2z} \cdot z^2) = e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3} \cdot z^2 = e^{2z} \cdot \frac{z^2}{D^2 + 2D - 3}$$

$$\begin{aligned} \text{P.I. is} &= \frac{e^{2z}}{-3} \left[\frac{z^2}{1 - \frac{2D}{3} - \frac{D^2}{3}} \right] = -\frac{1}{3} e^{2z} \left[1 - \left(\frac{2D}{3} + \frac{D^2}{3} \right) \right]^{-1} \cdot z^2 \\ &= -\frac{1}{3} e^{2z} \left[1 + \frac{2D}{3} + \frac{D^2}{3} + \left(\frac{2D}{3} + \frac{D^2}{3} \right)^2 + \dots \right] \cdot z^2 \\ &= -\frac{1}{3} e^{2z} \left[1 + \frac{2D}{3} + \frac{D^2}{3} + \frac{4D^2}{9} + \dots \right] \cdot z^2 = -\frac{1}{3} e^{2z} \left[1 + \frac{2D}{3} + \frac{7D^2}{9} + \dots \right] \cdot z^2 \\ &= -\frac{1}{3} e^{2z} \left[z^2 + \frac{2}{3} \cdot 2z + \frac{7}{9} \cdot 2 \right] = -\frac{1}{3} e^{2z} \left[z^2 + \frac{4}{3} z + \frac{14}{9} \right] \end{aligned}$$

$$\text{or } \text{P.I.} = -\frac{1}{3} x^2 \left[(\log x)^2 + \frac{4}{3} \log x + \frac{14}{9} \right]$$

Hence complete solution is

$$\text{C.S.} = \text{C.F.} + \text{P.I.} = C_1 x^3 + \frac{C_2}{x} - \frac{x^2}{3} \left[(\log x)^2 + \frac{4}{3} \log x + \frac{14}{9} \right].$$

178. Solve : $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$. [U.P.T.U., (C.O.) 2005]

Sol. Given equation is a Cauchy's homogeneous linear equation.

$$\text{Put } x = e^z \text{ i.e., } z = \log x, \quad x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y, \quad \text{where } D = d/dz.$$

The given equation then becomes $(D^2 + 3D + 2)y = e^{e^z}$

Its A.E. is $D^2 + 3D + 2 = 0$ or $(D+1)(D+2) = 0$

$$\therefore D = -1, -2$$

$$\text{C.F.} = C_1 e^{-z} + C_2 e^{-2z} = \frac{C_1}{x} + \frac{C_2}{x^2}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 3D + 2} \cdot e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^{e^z} = \frac{1}{D-(-1)} e^{e^z} - \frac{1}{D-(-2)} e^{e^z} \\ &= e^{-z} \int e^{e^z} \cdot e^z dz - e^{-2z} \int e^{e^z} \cdot e^{2z} dz \quad \left[\because \frac{1}{D-a} X = e^{ax} \int X \cdot e^{-ax} \cdot dx \right] \end{aligned}$$

Now put $e^z = t$

$$\begin{aligned} \text{P.I.} &= e^{-z} \int e^{e^z} \cdot e^z dz - e^{-2z} \int e^{e^z} \cdot e^{2z} dz = e^{-z} \int e^t dt - e^{-2z} \int t e^t dt \\ &= e^{-z} \cdot e^t - e^{-2z} (t-1)e^t \quad | \text{ Integrating by parts} \\ &= e^{-z} \cdot e^{e^z} - e^{-2z} (e^z - 1)e^{e^z} = [e^{-z} - e^{-2z} (e^z - 1)] e^{e^z} \\ &= [e^{-z} - e^{-z} + e^{-2z}] e^{e^z} = e^{-2z} \cdot e^{e^z} = x^{-2} \cdot e^x \end{aligned}$$

Hence the complete solution is $y = (C_1 x + C_2 + e^x) x^{-2}$.

179. Solve : $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$.

Sol. Put $x = e^z$ so that $z = \log x$ and

Let $D \equiv \frac{d}{dz}$. Then $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]y = x + \log x$$

or

$$(D^3 + 1)y = e^z + z$$

Auxiliary equation is $D^3 + 1 = 0$ or $(D+1)(D^2 - D + 1) = 0$

$$\Rightarrow D = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \text{C.F.} = C_1 e^{-z} + e^{z/2} \left(C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1} \cdot (e^z + z) = \frac{1}{D^3 + 1} \cdot e^z + \frac{1}{1 + D^3} \cdot z = \frac{1}{2} e^z + (1 + D^3)^{-1} \cdot z$$

$$= \frac{1}{2} e^z + (1 - D^3) (z).$$

| Ignoring higher terms

$$= \frac{1}{2} e^z + z - D^3 (z) = \frac{1}{2} e^z + z$$

$\therefore$ The complete solution is

$$y = C_1 e^{-z} + e^{z/2} \left(C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right) + \frac{1}{2} e^z + z$$

As

$$e^z = x, \quad z = \log x$$

or

$$y = \frac{C_1}{x} + \sqrt{x} \left[C_2 \cos \frac{\sqrt{3}}{2} (\log x) + C_3 \sin \frac{\sqrt{3}}{2} (\log x) \right] + \frac{x}{2} + \log x.$$

180. Solve : $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10 \left(x + \frac{1}{x} \right)$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ i.e., $z = \log x$

so that $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y, \text{ where } D = \frac{d}{dz}$$

The given equation thus becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z}) \quad \text{or} \quad (D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

This is a linear equation with constant coefficients.

Its A.E. is $D^3 - D^2 + 2 = 0$ or $(D+1)(D^2 - 2D + 2) = 0$

$$\Rightarrow D = -1, 1 \pm i$$

$$\therefore \text{C.F.} = C_1 e^{-z} + e^z (C_2 \cos z + C_3 \sin z)$$

$$= \frac{C_1}{x} + x [C_2 \cos (\log x) + C_3 \sin (\log x)]$$

$$\begin{aligned}
 \text{P.I.} &= 10 \cdot \frac{1}{D^3 - D^2 + 2} (e^z + e^{-z}) = 10 \left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} \cdot e^{-z} \right] \\
 &= 10 \left[\frac{1}{2} e^z + z \cdot \frac{1}{3D^2 - 2D} \cdot e^{-z} \right] = 10 \left[\frac{1}{2} e^z + z \cdot \frac{1}{3(-1)^2 - 2(-1)} \cdot e^{-z} \right] \\
 &= 5e^z + 2z \cdot e^{-z} = 5x + \frac{2}{x} \log x
 \end{aligned}$$

Hence the complete solution is

$$y = \frac{C_1}{x} + x [C_2 \cos (\log x) + C_3 \sin (\log x)] + 5x + \frac{2}{x} \log x.$$

181. Solve : $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin (\log x)$.

Sol. Given equation is a Cauchy homogeneous linear equation.

Put $x = e^z$ i.e., $z = \log x$ so that $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dz}$.

Substituting these values in the given equation, we have

$$[D(D-1) + D + 1]y = z \sin z \quad \text{or} \quad (D^2 + 1)y = z \sin z$$

Its A.E. is $(D^2 + 1) = 0$ so that $D = \pm i$

$$\begin{aligned}
 \text{C.F.} &= C_1 \cos z + C_2 \sin z \\
 &= C_1 \cos (\log x) + C_2 \sin (\log x)
 \end{aligned}$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + 1} z \sin z = \text{Imaginary part of } \frac{1}{D^2 + 1} \cdot ze^{iz} \\
 &= \text{I.P. of } e^{iz} \cdot \frac{1}{(D+i)^2 + 1} \cdot z = \text{I.P. of } e^{iz} \cdot \frac{1}{D^2 + 2iD} \cdot z \\
 &= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD \left(1 + \frac{D^2}{2iD}\right)} \cdot z = \text{I.P. of } e^{iz} \cdot \frac{1}{2iD \left(1 + \frac{D}{2i}\right)} \cdot z \\
 &= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD} \left[1 + \frac{D}{2i}\right]^{-1} \cdot z = \text{I.P. of } e^{iz} \cdot \frac{1}{2iD} \left[1 - \frac{D}{2i}\right] z \\
 &= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD} \left[1 + \frac{iD}{2} + \dots\right] z = \text{I.P. of } e^{iz} \cdot \frac{1}{2iD} \left[z + \frac{i}{2}\right] \\
 &= \text{I.P. of } e^{iz} \cdot \frac{1}{2i} \cdot \frac{1}{D} \left[z + \frac{i}{2}\right] = \text{I.P. of } \frac{e^{iz}}{2i} \left[\frac{z^2}{2} + \frac{i}{2} z\right] \\
 &= \text{I.P. of } \frac{e^{iz}}{2i} \left[\frac{z^2}{2} + \frac{i}{2} z\right] = \text{I.P. of } e^{iz} \left[-\frac{z^2 i}{4} + \frac{z}{4}\right] \\
 &= \text{I.P. of } (\cos z + i \sin z) \left(-\frac{i}{4} z^2 + \frac{z}{4}\right) = -\frac{z^2}{4} \cos z + \frac{z \sin z}{4}
 \end{aligned}$$

$$\text{or} \quad \text{P.I.} = -\frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} \log x \sin (\log x)$$

Hence the complete solution is

$$y = C_1 \cos (\log x) + C_2 \sin (\log x) - \frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} \log x \sin (\log x).$$

182. Solve the D.E. $x^2 y'' - 4xy' + 4y = 4x^2 + 6x^3$, $y(2) = 4$, $y'(2) = -1$. (M.D.U., Dec., 2009)

Sol. $x^2 y'' - 4xy' + 4y = 4x^2 + 6x^3$

Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$, $z = \log x$.

$$x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Substituting these values in the given equation, it reduces to $D(D-1)y - 4Dy + y = 2(2e^{2z} + 3e^{3z})$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 5D + 1 = 0$

$$D = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 1}}{2} = \frac{5 \pm \sqrt{21}}{2} = \frac{5 \pm 4.58}{2} \quad \text{or} \quad D = 4.79, 0.21$$

$$\text{C.F.} = C_1 e^{2z} + e^{3z} (C_2 \cos z + C_3 \sin z)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 5D + 1} (2e^{2z} + 3e^{3z}) = \frac{2e^{2z}}{D^2 - 5D + 1} + \frac{3e^{3z}}{D^2 - 5D + 1} \\ &= \frac{2e^{2z}}{-5.9 \times 10^{-3}} + \frac{3e^{3z}}{-5.9 \times 10^{-3}} = -\frac{2x^2}{5.9 \times 10^{-3}} - \frac{3x^3}{5.9 \times 10^{-3}} \end{aligned}$$

$$\text{Hence, C.S. is } C_1 e^{2z} + e^{3z} (C_2 \cos z + C_3 \sin z) - \frac{2x^2}{5.9 \times 10^{-3}} - \frac{3x^3}{5.9 \times 10^{-3}}.$$

183. Solve : $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$. (M.D.U., May 2009)

Sol. Put $x = e^z$ so that $z = \log x$, $D = \frac{d}{dz}$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$ and $x \frac{dy}{dx} = Dy$

Given differential equation reduces to

$$[D(D-1) - 3D + 5]y = e^{2z} \cdot \sin z \quad \text{or} \quad (D^2 - 4D + 5)y = e^{2z} \cdot \sin z$$

Auxiliary equation is $D^2 - 4D + 5 = 0 \Rightarrow D = 2 \pm i$

$\therefore$ C.F. = $e^{2z} (C_1 \cos z + C_2 \sin z)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4D + 5} (e^{2z} \sin z) \\ &= e^{2z} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 5} \cdot (\sin z) = e^{2z} \cdot \frac{1}{D^2 + 1} (\sin z) \\ &= e^{2z} \cdot \frac{z}{2D} \cdot \sin z = e^{2z} \cdot \frac{z}{2} (-\cos z) = -\frac{z}{2} e^{2z} \cdot \cos z. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = e^{2z} (C_1 \cos z + C_2 \sin z) - \frac{z}{2} e^{2z} \cdot \cos z \\ &= x^2 \{C_1 \cos(\log x) + C_2 \sin(\log x)\} - \frac{x^2}{2} \log x \cdot \cos(\log x). \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration.

184. Solve : $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 24x^2$.

Sol. Put $x = e^z$ so that $z = \log x$ and let $D = \frac{d}{dz}$ then the given equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D]y = 24e^{2z} \quad \text{or} \quad D^3 y = 24e^{2z}$$

Its auxiliary equation is $D^3 = 0$ or $D = 0, 0, 0$

$$\therefore \text{C.F.} = (C_1 + C_2 z + C_3 z^2) e^{0z} = C_1 + C_2 z + C_3 z^2$$

$$\text{P.I.} = \frac{1}{D^3} (24 e^{2z}) = 24 \frac{1}{D^3} (e^{2z}) = 24 \cdot \frac{1}{8} e^{2z} = 3e^{2z}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.} = C_1 + C_2 z + C_3 z^2 + 3e^{2z}$

$$\Rightarrow y = C_1 + C_2 \log x + C_3 (\log x)^2 + 3x^2$$

where C_1, C_2 and C_3 are arbitrary constants of integration.

- 185.** Solve the Legendre's linear differential equation: $(3x+2)^2 \frac{d^2 y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

Sol. Put $3x+2 = e^z$ i.e., $z = \log(3x+2)$ so that

$$(3x+2) \frac{dy}{dx} = 3Dy \quad \text{or} \quad (3x+2)^2 \frac{d^2 y}{dx^2} = 3^2 D(D-1)y, \text{ where } D = \frac{d}{dz}$$

The given equation (after substituting these values) becomes

$$[9D(D-1) + 9D - 36] y = 3 \left(\frac{e^z - 2}{3} \right)^2 + 4 \left(\frac{e^z - 2}{3} \right) + 1$$

$$\text{or} \quad 9(D^2 - 4)y = \frac{1}{3} e^{2z} - \frac{1}{3} \quad \text{or} \quad (D^2 - 4)y = \frac{1}{27} (e^{2z} - 1)$$

which is a linear equation with constant coefficients.

Its auxiliary equation is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$$\therefore \text{C.F.} = C_1 e^{2z} + C_2 e^{-2z} = C_1 (3x+2)^2 + C_2 (3x+2)^{-2}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{27} \cdot \frac{1}{D^2 - 4} (e^{2z} - 1) = \frac{1}{27} \left[\frac{1}{D^2 - 4} e^{2z} - \frac{1}{D^2 - 4} \cdot e^{0z} \right] \\ &= \frac{1}{27} \left[z \cdot \frac{1}{2D} e^{2z} - \frac{1}{0-4} \cdot e^{0z} \right] = \frac{1}{27} \left[\frac{z}{2} \int e^{2z} dz + \frac{1}{4} \right] \\ &= \frac{1}{27} \left[\frac{z}{4} e^{2z} + \frac{1}{4} \right] = \frac{1}{108} (ze^{2z} + 1) = \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1] \end{aligned}$$

Hence the complete solution is

$$y = C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1]$$

where C_1 and C_2 are arbitrary constants of integration.

- 186.** Solve the equation : $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \{\log(1+x)\}$. (M.D.U., 2005)

Sol. Put $1+x = e^z$ so that $z = \log(1+x)$ and let $D = \frac{d}{dz}$, then the given equation reduces to

$$[D(D-1) + D + 1]y = 4 \cos z \Rightarrow (D^2 + 1)y = 4 \cos z$$

Its auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$\therefore \text{C.F.} = C_1 \cos z + C_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1} (4 \cos z) = 4 \cdot z \cdot \frac{1}{2D} (\cos z) = 2z \sin z$$

Hence the complete solution is

$$\begin{aligned} y &= C_1 \cos z + C_2 \sin z + 2z \sin z \\ &= C_1 \cos [\log(1+x)] + C_2 \{\sin \log(1+x)\} + 2 \log(1+x) \sin \{\log(1+x)\}. \end{aligned}$$

187. Solve the D.E. $(1+x)^2 y'' + (1+x)y' + y = 2 \sin [\log (1+x)]$.

(M.D.U., Dec., 2008)

Sol. This is a Legendre's Linear D.E.

Put $1+x = e^t$ or $\log (1+x) = t$

So that $(1+x) \frac{dy}{dx} = Dy$ or $(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$ where, $D = \frac{d}{dt}$

Given D.E. becomes, $D(D-1)y + Dy + y = 2 \sin t$ or $(D^2 + 1)y = 2 \sin t$

which is a linear D.E. with constant coefficients.

Its A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

C.F. = $C_1 \cos t + C_2 \sin t$

$$\text{P.I.} = \frac{2 \sin t}{D^2 + 1} = 2 \cdot t \cdot \frac{1}{2D} \cdot \sin t = t \int \sin t dt = -t \cos t$$

Replacing t by $\log (1+x)$, we get

$$\text{P.I.} = -\log (1+x) \cos \{\log (1+x)\}$$

$\therefore$ Complete solution of the given D.E. is

$$y = C_1 \cos \{\log (1+x)\} + C_2 \sin \{\log (1+x)\} - \log (1+x) \cos \{\log (1+x)\}.$$

188. Solve : $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$.

Sol. Given equation is a Cauchy's homogeneous linear equation

Put $x = e^z$ i.e., $z = \log x$, so that $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dz}$

Substituting these values the given equation reduces to

$$[D(D-1) - D + 2]y = ze^z \text{ or } [D^2 - 2D + 2]y = ze^z$$

Its auxiliary equation is $D^2 - 2D + 2 = 0 \Rightarrow D = 1 \pm i$

$\therefore$ C.F. = $(C_1 \cos z + C_2 \sin z) e^z = [C_1 \cos (\log x) + C_2 \sin (\log x)] x$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2D + 2} \cdot ze^z = e^z \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} \cdot (z) \\ &= e^z \cdot \frac{1}{D^2 + 1} (z) = e^z \cdot (1 + D^2)^{-1} (z) = e^z \cdot (1 - D^2) (z) = e^z \cdot z = x \log x \end{aligned}$$

$\therefore$ Complete solution is $y = [C_1 \cos (\log x) + C_2 \sin (\log x)] x + x \log x$

where C_1 and C_2 are arbitrary constants of integration.

189. Solve : $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x+1)^2$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $e^z = x$, $z = \log x$

$$x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

The given equation therefore reduces to

$$[D(D-1) + 2D - 20]y = (e^z + 1)^2 \text{ or } (D^2 + D - 20)y = (1 + e^z)^2$$

or $(D+5)(D-4)y = (1 + e^z)^2$ i.e., $(D+5)(D-4)y = e^{2z} + 2e^z + 1$

This is a linear equation with constant coefficients.

$\therefore$ A.E. is $(D+5)(D-4) = 0 \Rightarrow D = 4, -5$

$$\therefore \text{C.F.} = C_1 e^{4z} + C_2 e^{-5z} = C_1 x^4 + C_2 x^{-5}$$

$$\text{P.I.} = \frac{1}{(D+5)(D-4)} [e^{2z} + 2e^z + 1]$$

$$\text{P.I.} = \frac{e^{2z}}{(5+2)(2-4)} + \frac{2e^z}{(5+1)(1-4)} + \frac{e^{0z}}{5(-4)} = \frac{e^{2z}}{-14} - \frac{e^z}{9} - \frac{1}{20} = -\frac{x^2}{14} - \frac{x}{9} - \frac{1}{20}$$

$$\therefore \text{Complete solution is } y = C_1 x^4 + C_2 \cdot \frac{1}{x^5} - \frac{x^2}{14} - \frac{x}{9} - \frac{1}{20}.$$

190. Solve : $x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 4y = x \log x.$

Sol. Put $x = e^z$ or $z = \log x$

$$x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$[D(D-1) + 5D + 4]y = z \cdot e^z \quad \text{or} \quad (D^2 + 4D + 4)y = z e^z$$

Its auxiliary equation is $D^2 + 4D + 4 = 0 \Rightarrow D = -2, -2$

$$\therefore \text{C.F.} = (C_1 + C_2 z) e^{-2z}$$

or $\text{C.F.} = \frac{C_1 + C_2 \log x}{x^2}$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4} \cdot z e^z = e^z \cdot \frac{1}{(D+1)^2 + 4(D+1) + 4} \cdot (z) = e^z \cdot \frac{1}{D^2 + 6D + 9} (z)$$

$$\text{P.I.} = e^z \cdot \frac{1}{9 \left(1 + \frac{6}{9} D + \frac{D^2}{9} \right)} (z) = \frac{e^z}{9} \cdot \left(1 + \frac{2D}{3} + \frac{D^2}{9} \right)^{-1} (z) = \frac{e^z}{9} \cdot \left(1 - \frac{2D}{3} - \frac{D^2}{9} \right) (z)$$

$$= \frac{e^z}{9} \left[z - \frac{2}{3} D(z) \right] = \frac{e^z}{9} \left[z - \frac{2}{3} \right] = \frac{z \cdot e^z}{9} - \frac{2}{27} e^z = \frac{e^z}{9} \left(z - \frac{2}{3} \right)$$

$$\therefore \text{Complete solution is } y = \frac{C_1 + C_2 \log x}{x^2} + \frac{x}{9} \left(\log x - \frac{2}{3} \right)$$

where C_1 and C_2 are arbitrary constants of integration.

191. Solve : $\frac{dx}{dt} + 4x + 3y = t, \quad \frac{dy}{dt} + 2x + 5y = e^t.$ (M.D.U., May 2007)

Sol. Writing D for $\frac{d}{dt}$, the given equations become

$$(D+4)x + 3y = t \quad \dots(1) \quad \text{and} \quad (D+5)y + 2x = e^t \quad \dots(2)$$

To eliminate y , operating on both sides of (1) by $(D+5)$ and on both sides of (2) by 3 and subtracting, we get

$$[(D+4)(D+5) - 6] x = (D+5)t - 3e^t$$

or $(D^2 + 9D + 14)x = 1 + 5t - 3e^t$

Its A.E. is $D^2 + 9D + 14 = 0 \Rightarrow D = -2, -7$

$$\text{C.F.} = C_1 e^{-2t} + C_2 e^{-7t}$$

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14} (1 + 5t - 3e^t) = \frac{1}{D^2 + 9D + 14} \cdot e^{0t} + \frac{5 \cdot 1 \cdot t}{D^2 + 9D + 14} - \frac{3 \cdot 1}{D^2 + 9D + 14} \cdot e^t$$

$$\begin{aligned}
&= \frac{1}{14} + \frac{5}{14} \left[1 + \left(\frac{9D}{14} + \frac{D^2}{14} \right) \right]^{-1} t - \frac{1}{8} e^t = \frac{1}{14} + \frac{5}{14} \left[1 - \left(\frac{9D}{14} + \frac{D^2}{14} \right) + \dots \right] t - \frac{1}{8} e^t \\
&= \frac{1}{14} + \frac{5}{14} \left(t - \frac{9}{14} \right) - \frac{1}{8} e^t = \frac{5}{14} t - \frac{1}{8} e^t - \frac{31}{196}
\end{aligned}$$

$$\therefore x = C_1 e^{-2t} + C_2 e^{-7t} + \frac{5}{14} t - \frac{1}{8} e^t - \frac{31}{196}$$

$$\text{Now, } \frac{dx}{dt} = -2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{5}{14} - \frac{1}{8} e^t$$

Substituting the values of x and $\frac{dx}{dt}$ in (1), we have

$$\begin{aligned}
3y &= t - \frac{dx}{dt} - 4x \\
&= t + 2C_1 e^{-2t} + 7C_2 e^{-7t} - \frac{5}{14} + \frac{1}{8} e^t - 4C_1 e^{-2t} - 4C_2 e^{-7t} - \frac{10}{7} t + \frac{31}{49} + \frac{1}{2} e^t
\end{aligned}$$

$$\therefore y = \frac{1}{3} \left[-2C_1 e^{-2t} + 3C_2 e^{-7t} - \frac{3}{7} t + \frac{27}{98} + \frac{5}{8} e^t \right]$$

$$\text{Hence } x = C_1 e^{-2t} + C_2 e^{-7t} + \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t$$

$$y = -\frac{2}{3} C_1 e^{-2t} + C_2 e^{-7t} - \frac{1}{7} t + \frac{9}{98} + \frac{5}{24} e^t.$$

192. Solve the simultaneous equations : $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, given that $x = 0 = y$ when $t = 0$.
(M.D.U., 2005)

Sol. Taking $\frac{d}{dt} = D$, the given equations become

$$(D + 5)x - 2y = t \quad \dots(1) \quad 2x + (D + 1)y = 0 \quad \dots(2)$$

To eliminate x we multiply equation (1) by (2) and operating on equation (2) by $(D + 5)$ and then subtracting, we get

$$[-4 - (D + 5)(D + 1)]y = 2t \quad \text{or} \quad (D^2 + 6D + 9)y = -2t$$

$$\text{Its A.E. is } D^2 + 6D + 9 = 0 \quad \text{or} \quad (D + 3)^2 = 0 \Rightarrow D = -3, -3$$

$$\text{C.F.} = (C_1 + C_2 t)e^{-3t}$$

$$\text{P.I.} = \frac{1}{(D + 3)^2} (-2t) = -\frac{2}{9} \left(1 + \frac{D}{3} \right)^{-2} t$$

$$\therefore y = -\frac{2}{9} \left(1 - 2\frac{D}{3} + \dots \right) t + (C_1 + C_2 t)e^{-3t} = \left[-\frac{2t}{9} + \frac{4}{27} + (C_1 + C_2 t)e^{-3t} \right] \quad \dots(3)$$

Now to find x , from eqn. (2), we have

$$\begin{aligned}
x &= -\frac{1}{2} [Dy + y] \\
&= -\frac{1}{2} \left[-\frac{2}{9} + C_2 e^{-3t} + (C_1 + C_2 t)(-3)e^{-3t} + (C_1 + C_2 t)e^{-3t} - \frac{2t}{9} + \frac{4}{27} \right]
\end{aligned}$$

[Using equation (3)]

$$\therefore x = \left[\left(C_1 - \frac{1}{2} C_2 \right) + C_2 t \right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \quad \dots(4)$$

Hence equations (3) and (4) constitute the solutions of the given equations.

Since $x = y = 0$, when $t = 0$, equations (3) and (4) give

$$0 = C_1 + \frac{4}{27} \quad \text{and} \quad 0 = C_1 - \frac{1}{2} C_2 + \frac{1}{27} \quad \Rightarrow \quad C_1 = -\frac{4}{27}, C_2 = -\frac{2}{9}$$

Hence the desired solutions are

$$x = -\frac{1}{27} (1 + 6t) e^{-3t} + \frac{1}{27} (1 + 3t)$$

$$y = -\frac{2}{27} (2 + 3t) e^{-3t} + \frac{2}{27} (2 - 3t).$$

193. Solve the system of simultaneous equations : $\frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = \sin t$, $\frac{dx}{dt} + x - 3y = 0$.

(M.D.U., 2005)

Sol. The equations can be written as

$$\left. \begin{aligned} (D^2 + D - 2)y &= \sin t \\ (D + 1)x - 3y &= 0 \end{aligned} \right\} \quad \dots(1)$$

Now, A.E. is $D^2 + D - 2 = 0$ or $(D + 2)(D - 1) = 0 \Rightarrow D = 1, -2$

$\therefore$ C.F. = $C_1 e^t + C_2 e^{-2t}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + D - 2} \sin t = \frac{1}{-1 + D - 2} \sin t = \frac{1}{D - 3} \sin t \\ &= \frac{D + 3}{D^2 - 9} \sin t = \frac{D + 3}{-10} \sin t = -\frac{1}{10} (\cos t + 3 \sin t) \end{aligned}$$

Complete solution is

$$y = C_1 e^t + C_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t) \quad \dots(2)$$

From equation (1), we get

$$(D + 1)x = \frac{3 \sin t}{D^2 + D - 2} \quad \text{or} \quad (D^2 + D - 2)(D + 1)x = 3 \sin t$$

$$(D + 2)(D - 1)(D + 1)x = 3 \sin t$$

A.E. is $(D + 2)(D - 1)(D + 1) = 0 \Rightarrow D = 1, -1, -2$

$\therefore$ C.F. = $C_1 e^t + C_2 e^{-t} + C_3 e^{-2t}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2 - 1)(D + 2)} \cdot 3 \sin t = \frac{1}{(-2)(D + 2)} 3 \sin t = \frac{(D - 2)}{D^2 - 4} \left(-\frac{3}{2} \right) \sin t \\ &= (D - 2) \cdot \frac{3}{10} \cdot \sin t = \frac{3}{10} [\cos t - 2 \sin t] \end{aligned}$$

$\therefore$ Complete solution is $x = \frac{3}{10} (\cos t - 2 \sin t) + C_1 e^t + C_2 e^{-t} + C_3 e^{-2t} \quad \dots(3)$

Hence equations (2) and (3) are complete solutions of the given system of differential equations.

194. Solve : $\frac{d^2x}{dt^2} + 4x + 5y = t^2$, $\frac{d^2y}{dt^2} + 5x + 4y = t + 1$.

Sol. Writing D for $\frac{d}{dt}$, the given equations become

$$(D^2 + 4)x + 5y = t^2 \quad \dots(1)$$

$$\text{and} \quad 5x + (D^2 + 4)y = t + 1 \quad \dots(2)$$

To eliminate y , operating on both sides of (1) by $(D^2 + 4)$ and on both sides of (2) by 5 and then subtracting, we get

$$[(D^2 + 4)^2 - 25]x = (D^2 + 4)t^2 - 5t - 5$$

$$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5 \quad \text{or} \quad (D^4 + 8D^2 - 9)x = 4t^2 - 5t - 3$$

$$\text{Its A.E. is } D^4 + 8D^2 - 9 = 0 \quad \text{or} \quad (D^2 + 9)(D^2 - 1) = 0$$

$$\therefore D = \pm 1, \pm 3i$$

$$\text{C.F.} = C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^4 + 8D^2 - 9} (4t^2 - 5t - 3) = -\frac{1}{9} \left[1 - \left\{ \frac{8D^2}{9} + \frac{D^4}{9} \right\} \right]^{-1} (4t^2 - 5t - 3) \\ &= -\frac{1}{9} \left[1 + \frac{8D^2}{9} + \frac{D^4}{9} \right] (4t^2 - 5t - 3) \\ &= -\frac{1}{9} \left[4t^2 - 5t - 3 + \frac{8}{9} D^2 (4t^2 - 5t - 3) \right] \quad (\text{Ignoring higher terms}) \\ &= -\frac{1}{9} \left[4t^2 - 5t - 3 + \frac{8}{9} (8) \right] = -\frac{1}{9} \left(4t^2 - 5t + \frac{37}{9} \right) \end{aligned}$$

$$\therefore x = C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t - \frac{4}{9} t^2 + \frac{5}{9} t - \frac{37}{81}$$

$$\text{Now,} \quad \frac{dx}{dt} = C_1 e^t - C_2 e^{-t} - 3C_3 \sin 3t + 3C_4 \cos 3t - \frac{8}{9} t + \frac{5}{9}$$

$$\frac{d^2x}{dt^2} = C_1 e^t + C_2 e^{-t} - 9C_3 \cos 3t - 9C_4 \sin 3t - \frac{8}{9}$$

Substituting these values in equation (1), we get

$$\begin{aligned} 5y &= t^2 - 4x - \frac{d^2x}{dt^2} \\ &= t^2 - 4C_1 e^t - 4C_2 e^{-t} - 4C_3 \cos 3t - 4C_4 \sin 3t + \frac{16}{9} t^2 \\ &\quad - \frac{20}{9} t + \frac{148}{81} - C_1 e^t - C_2 e^{-t} + 9C_3 \cos 3t + 9C_4 \sin 3t + \frac{8}{9} \end{aligned}$$

$$\therefore y = \frac{1}{5} \left[-5 \cdot C_1 e^t - 5 \cdot C_2 e^{-t} + 5 \cdot C_3 \cos 3t + 5 \cdot C_4 \sin 3t + \frac{25}{9} t^2 - \frac{20}{9} t + \frac{220}{81} \right]$$

$$\text{Hence,} \quad x = C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t - \frac{1}{9} \left(4t^2 - 5t + \frac{37}{9} \right)$$

$$y = -C_1 e^t - C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{9} \left(5t^2 - 4t + \frac{44}{9} \right).$$

195. Solve the simultaneous equations: $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 2z$, $\frac{dz}{dt} = 2x$.

Sol. The given equations are

$$\frac{dx}{dt} = 2y \quad \dots(1) \quad \frac{dy}{dt} = 2z \quad \dots(2) \quad \frac{dz}{dt} = 2x \quad \dots(3)$$

Differentiating equation (1) w.r.t. t ,

$$\frac{d^2x}{dt^2} = 2 \cdot \frac{dy}{dt} = 2(2z)$$

Again differentiating w.r.t. t ,

$$\frac{d^3x}{dt^3} = 4 \cdot \frac{dz}{dt} = 4(2x)$$

$$(D^3 - 8)x = 0, \text{ where } D = \frac{d}{dt}$$

Its A.E. is $D^3 - 8 = 0$ or $(D - 2)(D^2 + 2D + 4) = 0 \Rightarrow D = 2, -1 \pm i\sqrt{3}$

$$\therefore x = C_1 e^{2t} + C_2 \cdot e^{-t} \cos(\sqrt{3}t - C_3)$$

From equation (1),

$$\begin{aligned} y &= \frac{1}{2} \cdot \frac{dx}{dt} = \frac{1}{2} [2C_1 e^{2t} - C_2 e^{-t} \cos(\sqrt{3}t - C_3) - C_2 \sqrt{3} e^{-t} \sin(\sqrt{3}t - C_3)] \\ &= C_1 e^{2t} + C_2 e^{-t} \left[\cos \frac{2\pi}{3} \cdot \cos(\sqrt{3}t - C_3) - \sin \frac{2\pi}{3} \cdot \sin(\sqrt{3}t - C_3) \right] \\ y &= C_1 e^{2t} + C_2 e^{-t} \cos \left(\sqrt{3}t - C_3 + \frac{2\pi}{3} \right) \end{aligned}$$

From equation (2),

$$\begin{aligned} z &= \frac{1}{2} \cdot \frac{dy}{dt} = \frac{1}{2} \left[2C_1 e^{2t} - C_2 e^{-t} \cos \left(\sqrt{3}t - C_3 + \frac{2\pi}{3} \right) - C_2 \sqrt{3} e^{-t} \sin \left(\sqrt{3}t - C_3 + \frac{2\pi}{3} \right) \right] \\ &= C_1 e^{2t} + C_2 e^{-t} \left[\cos \frac{2\pi}{3} \cos \left(\sqrt{3}t - C_3 + \frac{2\pi}{3} \right) - \sin \frac{2\pi}{3} \sin \left(\sqrt{3}t - C_3 + \frac{2\pi}{3} \right) \right] \end{aligned}$$

$$\text{or } z = C_1 e^{2t} + C_2 e^{-t} \cos \left(\sqrt{3}t - C_3 + \frac{4\pi}{3} \right).$$

Note. $C_1 \cos \beta x + C_2 \sin \beta x$ can be replaced by $C_1 \cos(\beta x - C_2)$.

196. Solve the simultaneous equations: $t \frac{dx}{dt} + y = 0$, $t \frac{dy}{dt} + x = 0$ given $x(1) = 1$, $y(-1) = 0$

Sol. The given equations are

$$t \frac{dx}{dt} + y = 0 \quad \dots(1) \quad t \frac{dy}{dt} + x = 0 \quad \dots(2)$$

Differentiating equation (1) w.r.t. t , we have

$$t \frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

Multiplying throughout by t ,

$$t^2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} + t \frac{dy}{dt} = 0 \quad \text{or} \quad t^2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} - x = 0 \quad \dots(3)$$

which is Cauchy's homogeneous linear equation.

Putting, $t = e^u$, $u = \log t$, $t \frac{dx}{dt} = \frac{dx}{du} = Dx$ and $t^2 \frac{d^2x}{dt^2} = D(D-1)x$

$$D(D-1)x + Dx - x = 0$$

$$[D(D-1) + D-1]x = 0$$

$$(D^2-1)x = 0$$

Its A.E. is $D^2-1=0 \Rightarrow D=\pm 1$

$$\therefore x = C_1 e^u + C_2 e^{-u} = C_1 t + \frac{C_2}{t} \quad \dots(4)$$

$$\text{From (1), } y = -t \frac{dx}{dt} = -t \left(C_1 - \frac{C_2}{t^2} \right) = -C_1 t + \frac{C_2}{t} \quad \dots(5)$$

Since, $x(1) = 1$, $\therefore$ From (4), we have $1 = C_1 + C_2$

Also $y(-1) = 0$, $\therefore$ From (5), we have $0 = C_1 - C_2$

Solving $C_1 = C_2 = \frac{1}{2}$

$$\text{Hence } x = \frac{1}{2} \left(t + \frac{1}{t} \right), y = \frac{1}{2} \left(-t + \frac{1}{t} \right)$$

197. Solve the simultaneous equations: $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2 \cos t - 7 \sin t$, $\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4 \cos t - 3 \sin t$.

Sol. Given equations may be written as

$$Dx + (D-2)y = 2 \cos t - 7 \sin t \quad \dots(1)$$

$$(D+2)x - Dy = 4 \cos t - 3 \sin t \quad \dots(2)$$

Operating (1) by D and (2) by $(D-2)$ to eliminate y , we get

$$D^2x + D(D-2)y = -2 \sin t - 7 \cos t$$

$$(D^2-4)x - D(D-2)y = [-4 \sin t - 3 \cos t - 2(4 \cos t - 3 \sin t)]$$

On adding, we get $(2D^2-4)x = -18 \cos t$

$$(D^2-2)x = -9 \cos t$$

Its A.E. is $D^2-2=0 \Rightarrow D=\pm \sqrt{2}$

$$\therefore \text{C.F.} = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$$

$$\text{P.I.} = \frac{1}{D^2-2} (-9 \cos t) = 3 \cos t$$

$$\therefore x = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t} + 3 \cos t \quad \dots(3)$$

Putting the value of x in equation (2), we get

$$\frac{dy}{dt} = 2x + \frac{dx}{dt} - 4 \cos t + 3 \sin t$$

$$\frac{dy}{dt} = 2 \{C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t} + 3 \cos t\} + \sqrt{2} C_1 e^{\sqrt{2}t} - \sqrt{2} C_2 e^{-\sqrt{2}t} - 3 \sin t - 4 \cos t + 3 \sin t$$

$$\text{or } \frac{dy}{dt} = (2 + \sqrt{2})C_1 e^{\sqrt{2}t} + (2 - \sqrt{2})C_2 e^{-\sqrt{2}t} + 2 \cos t$$

Integrating w.r.t. t , we get

$$y = (\sqrt{2} + 1)C_1 e^{\sqrt{2}t} - (\sqrt{2} - 1)C_2 e^{-\sqrt{2}t} + 2 \sin t + C_3 \quad \dots(4)$$

Hence equations (3) and (4) taken together, give the complete solution of the given simultaneous system of equations.

198. Solve : $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t$ and $\frac{dx}{dt} + y - x = \cos t$.

Sol. Let $\frac{d}{dt} \equiv D$, then the given system becomes,

$$(D + 3)x + Dy = \sin t \quad \dots(1) \quad (D - 1)x + y = \cos t \quad \dots(2)$$

Operating (2) by D and then subtracting from equation (1), we get

$$[(D + 3) - D(D - 1)]x = 2 \sin t$$

$$\Rightarrow (D^2 - 2D - 3)x = -2 \sin t$$

Its A.E. is $D^2 - 2D - 3 = 0$ or $D = -1, 3$

$$\therefore \text{C.F.} = C_1 e^{-t} + C_2 e^{3t}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3} (-2 \sin t) = \frac{-2}{-2D - 4} \sin t = \frac{1}{D + 2} \sin t = \frac{D - 2}{D^2 - 4} \sin t = -\frac{1}{5} (\cos t - 2 \sin t)$$

$$\therefore x = C_1 e^{-t} + C_2 e^{3t} - \frac{1}{5} (\cos t - 2 \sin t) \quad \dots(3)$$

But $y = x + \cos t - \frac{dx}{dt}$

$$\begin{aligned} \text{or } y &= C_1 e^{-t} + C_2 e^{3t} - \frac{1}{5} (\cos t - 2 \sin t) + \cos t - \left[-C_1 e^{-t} + 3C_2 e^{3t} + \frac{1}{5} \sin t + \frac{2}{5} \cos t \right] \\ &= 2C_1 e^{-t} - 2C_2 e^{3t} + \frac{2}{5} \cos t + \frac{1}{5} \sin t \end{aligned} \quad \dots(4)$$

Equations (3) and (4) when taken together, give the complete solution.

199. Solve the system $\dot{x}(t) = y, \dot{y}(t) = -x, x(0) = 0, y(0) = 0$.

Sol. We have $\dot{x} = y \quad \dots(1) \quad \dot{y} = -x \quad \dots(2)$

So that $\ddot{x} = \dot{y} = -x \Rightarrow \ddot{x} + x = 0$

A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$\therefore \text{C.F.} = C_1 \cos t + C_2 \sin t$$

$$\text{P.I.} = 0$$

$$\therefore x(t) = C_1 \cos t + C_2 \sin t \quad \dots(3)$$

From (1) and (3), $y(t) = \dot{x}(t) = -C_1 \sin t + C_2 \cos t \quad \dots(4)$

$$x(0) = 0 \Rightarrow C_1 = 0$$

$$y(0) = 0 \Rightarrow C_2 = 0$$

$$\therefore x(t) = 0 \text{ and } y(t) = 0$$

which is the required solution of the system.

200. Solve the simultaneous differential equations :

$$\frac{d^2 x}{dt^2} - 4 \cdot \frac{dx}{dt} + 4x = y \quad \text{and} \quad \frac{d^2 y}{dt^2} + 4 \cdot \frac{dy}{dt} + 4y = 25x + 16e^t.$$

Sol. Let $\frac{d}{dt} \equiv D$.

$$\text{Given equations are } (D^2 - 4D + 4)x - y = 0 \quad \dots(1)$$

$$- 25x + (D^2 + 4D + 4)y = 16e^t \quad \dots(2)$$

Operating (1) by $D^2 + 4D + 4$ and adding to (2), we get

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - 25x = 16e^t \quad \text{or} \quad (D^4 - 8D^2 - 9)x = 16e^t$$

Its A.E. is $D^4 - 8D^2 - 9 = 0$

$$\Rightarrow (D^2 - 9)(D^2 + 1) = 0 \Rightarrow D = \pm i, \pm 3$$

$$\therefore \text{C.F.} = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t$$

$$\text{P.I.} = \frac{1}{D^4 - 8D^2 - 9} (16 e^t) = 8e^t$$

$$\therefore x = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t + 8e^t \quad \dots(3)$$

$$\frac{dx}{dt} = 3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t + 8e^t$$

$$\frac{d^2x}{dt^2} = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t + 8e^t$$

$$\text{From (1), } y = \frac{d^2x}{dt^2} - 4 \frac{dx}{dt} + 4x$$

$$\Rightarrow y = C_1 e^{3t} + 25C_2 e^{-3t} + (3C_3 - 4C_4) \cos t + (4C_3 + 3C_4) \sin t + 8e^t \quad \dots(4)$$

Equations (3) and (4) when taken together give the complete solution.

201. Solve : $(D - 1)x + Dy = 2t + 1$, $(2D + 1)x + 2Dy = t$.

Sol. Given equations are

$$(D - 1)x + Dy = 2t + 1 \quad \dots(1) \quad (2D + 1)x + 2Dy = t \quad \dots(2)$$

Multiplying (1) by 2 throughout and then subtracting (2) so as to eliminate y , we get

$$[2(D - 1) - (2D + 1)]x = 3t + 2 \quad \text{or} \quad 3x = -3t - 2 \quad \text{or} \quad x = -t - \frac{2}{3}$$

$$\begin{aligned} \text{Now, } Dy &= 2t + 1 - (D - 1)x = 2t + 1 - D \left\{ -t - \frac{2}{3} \right\} + x = 2t + 1 + 1 + x \\ &= 2t + 2 + \left(-t - \frac{2}{3} \right) = t + \frac{4}{3} \end{aligned}$$

$$\text{Integrating above, we get } y = \frac{t^2}{2} + \frac{4}{3}t + C$$

$$\text{Hence the solution is given by } x = -t - \frac{2}{3}, y = \frac{1}{2}t^2 + \frac{4}{3}t + C$$

where C is any arbitrary constant of integration.

202. Solve : $\frac{d^2x}{dt^2} - 3x - 4y = 0$, $\frac{d^2y}{dt^2} + x + y = 0$. (U.P.T.U., 2005)

$$\text{Sol. Given equations are } (D^2 - 3)x - 4y = 0 \quad \dots(1)$$

$$(D^2 + 1)y + x = 0 \quad \dots(2)$$

Operating (2) by $(D^2 - 3)$ and then subtracting from (1) to eliminate x ,

$$-4y - (D^2 - 3)(D^2 + 1)y = 0 \quad \text{or} \quad (D^2 - 3)(D^2 + 1)y = -4y$$

$$\{(D^2 - 3)(D^2 + 1) + 4\}y = 0$$

$$\{D^4 + D^2 - 3D^2 - 3 + 4\}y = 0$$

$$\{D^4 - 2D^2 + 1\}y = 0$$

$$(D^2 - 1)^2 y = 0$$

$$\text{A.E. is } (D^2 - 1)^2 = 0 \quad \text{or} \quad D = \pm 1, \pm 1$$

$$\text{C.F.} = (C_1 + C_2 t)e^{-t} + (C_3 + C_4 t)e^t$$

$$\text{P.I.} = 0$$

$$\therefore y = (C_1 + C_2 t)e^{-t} + (C_3 + C_4 t)e^t$$

From equation (2), we have

$$(D^2 + 1)y = -x$$

$$x = -(D^2 + 1)y = -D^2(y) - y$$

$$x = -[D^2(y)] - (C_1 + C_2 t)e^{-t} - (C_3 + C_4 t)e^t$$

$$\text{or } x = -[D\{(C_1 + C_2 t)e^{-t} + e^{-t}(C_2) + (C_3 + C_4 t)e^t + C_4 e^t\}] - (C_1 + C_2 t)e^{-t} - (C_3 + C_4 t)e^t$$

$$x = -[(C_1 + C_2 t)e^{-t} + e^{-t}(-C_2) + e^{-t} \cdot C_2(-1) + (C_3 + C_4 t)e^t + e^t(C_4) + C_4 e^t - (C_1 + C_2 t)e^{-t} - (C_3 + C_4 t)e^t]$$

$$x = -\frac{1}{2} [C_1 + C_2(1+t)]e^{-t} + \frac{1}{2} [C_4(1-t) - C_3]e^t$$

Hence the complete solution is

$$y = (C_1 + C_2 t)e^{-t} + (C_3 + C_4 t)e^t$$

$$x = -\frac{1}{2} [C_1 + C_2(1+t)]e^{-t} + \frac{1}{2} [C_4(1-t) - C_3]e^t$$

203. Solve: $\frac{d^2 x}{dt^2} + y = \sin t$, $\frac{d^2 y}{dt^2} + x = \cos t$.

Sol. Writing D for $\frac{d}{dt}$, the given equations become

$$D^2 x + y = \sin t \quad \dots(1)$$

$$D^2 y + x = \cos t \quad \dots(2)$$

Operating D^2 on equation (2) and then subtracting it from equation (1) to eliminate x , we get

$$y - D^4(y) = \sin t + \cos t$$

$$D^4 y - y = -(\sin t + \cos t)$$

$$(D^4 - 1)y = -(\sin t + \cos t)$$

$$\text{Its A.E. is } D^4 - 1 = 0 \Rightarrow D = \pm 1, \pm i$$

$$\text{C.F.} = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$$

$$\text{P.I.} = \frac{1}{D^4 - 1} \{-(\sin t + \cos t)\}$$

$$= -\frac{1}{D^4 - 1} (\sin t) - \frac{1}{D^4 - 1} (\cos t) = -\frac{t}{4D^3} (\sin t) - \frac{1}{4D^3} (\cos t)$$

$$= -\frac{t}{-4D} (\sin t) - \frac{1}{-4D} (\cos t) \quad (\text{Putting } D^2 = -1^2)$$

$$= \frac{t}{4} \int \sin t \, dt + \frac{t}{4} \int \cos t \, dt = \frac{t}{4} (\sin t - \cos t)$$

$$\text{Hence, } y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4} (\sin t - \cos t)$$

$$\text{Now, } D(y) = C_1 e^t + C_2(-e^{-t}) - C_3 \sin t + C_4 \cos t + \frac{1}{4} (\sin t - \cos t) + \frac{t}{4} (\sin t + \cos t)$$

$$D^2(y) = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{1}{4} (\cos t + \sin t) + \frac{t}{4} (\cos t - \sin t) + \frac{1}{4} (\sin t + \cos t).$$

∴ From equation (2), we have

$$x = \cos t - D^2 y$$

$$\text{or } x = \cos t - \left[C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{1}{2} (\sin t + \cos t) + \frac{t}{4} (\cos t - \sin t) \right]$$

$$= \frac{1}{2} (\cos t - \sin t) - C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t - \frac{t}{4} (\cos t - \sin t)$$

$$= -C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{1}{4} (2-t)(\cos t - \sin t)$$

$$\text{or } x = -C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{1}{4} (t-2)(\sin t - \cos t)$$

Hence the complete solution is

$$x = -C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{1}{4} (t-2)(\sin t - \cos t)$$

$$y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4} (\sin t - \cos t).$$

- 204.** Solve the simultaneous equations : $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$, when $t = 0$. (M.D.U., Dec., 2007)

Sol. The given equations in symbolic form are

$$Dx + y = \sin t \quad \dots(1)$$

$$x + Dy = \cos t \quad \dots(2)$$

To eliminate y , operating D on equation (1) and then subtracting equation (2), we get

$$D^2 x - x = 0 \quad \text{or} \quad (D^2 - 1)x = 0 \Rightarrow D = \pm 1$$

$$\therefore \text{C.F.} = C_1 e^t + C_2 \cdot e^{-t}, \text{P.I.} = 0$$

$$\therefore x = C_1 e^t + C_2 e^{-t} \quad \dots(3)$$

$$\text{As } x = 2 \text{ when } t = 0, \therefore C_1 + C_2 = 2 \quad \dots(4)$$

From equation (3), we have

$$\frac{dx}{dt} = C_1 e^t - C_2 e^{-t} \quad \dots(5)$$

Substituting for x and $\frac{dx}{dt}$ in equation (1) viz $Dx + y = \sin t$, we have

$$\begin{aligned} C_1 e^t - C_2 e^{-t} + y &= \sin t \\ y &= \sin t + C_2 e^{-t} - C_1 e^t \end{aligned} \quad \dots(6)$$

When, $t = 0, y = 0$

$$\therefore 0 = 0 + C_2 - C_1 \quad \text{or} \quad C_1 = C_2$$

$$\therefore \text{From (4); } C_1 = C_2 = 1.$$

$$\therefore y = e^{-t} - e^t + \sin t \quad \text{and} \quad x = e^t + e^{-t}$$

constitute the complete solution of the given system of simultaneous equations.

- 205.** A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half of its theoretical maximum in $(L \log 2)/R$ seconds.

Sol. Let i be the current in the circuit at any time t .

By Kirchhoff's law, we have

$$L \frac{di}{dt} + Ri = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \quad \dots(1)$$

which is Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$$

$\therefore$ Solution of equation (1) is

$$i e^{\frac{R}{L} t} = \int \frac{E}{L} \cdot e^{\frac{R}{L} t} dt = \frac{E}{R} \cdot e^{\frac{R}{L} t} + C$$

$$\therefore i = \frac{E}{R} + C e^{-\frac{Rt}{L}} \quad \dots(2)$$

Initially when $t = 0$, $i = 0$ so that

$$0 = \frac{E}{R} + C \quad \text{or} \quad C = -\frac{E}{R}$$

Thus equation (2) becomes

$$i = \frac{E}{R} + \left(-\frac{E}{R}\right) e^{-\frac{Rt}{L}} \quad \text{or} \quad i = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right) \quad \dots(3)$$

This equation gives the current in the circuit at any time t .

i increases with t and attains the maximum value $\frac{E}{R}$.

Let the current in the circuit be half its theoretical maximum after a time T seconds.

$$\text{Then,} \quad \frac{1}{2} \cdot \frac{E}{R} = \frac{E}{R} \left(1 - e^{-\frac{R}{L} \cdot T}\right)$$

$$\text{or} \quad e^{-\frac{RT}{L}} = \frac{1}{2} \quad \text{or} \quad -\frac{RT}{L} = \log \left(\frac{1}{2}\right) = -\log 2$$

$$\therefore T = \frac{L}{R} \log 2.$$

Hence shown.

206. The equation of e.m.f. in terms of current i for an electrical circuit having resistance R and a condenser of capacity C , in series, is $E = Ri + \int \frac{i}{C} dt$. Find the current i at any time t , when $E = E_0 \sin wt$.

Sol. The given equation can be written as $Ri + \int \frac{i}{C} dt = E_0 \sin wt$

Differentiating both sides w.r.t. t , we have

$$R \frac{di}{dt} + \frac{i}{C} = w E_0 \cos wt \quad \text{or} \quad \frac{di}{dt} + \frac{i}{RC} = \frac{w E_0}{R} \cos wt \quad \dots(1)$$

which is Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

∴ The solution of equation (1) is

$$ie^{\frac{t}{RC}} = \int \frac{wE_0}{R} \cos wt \cdot e^{\frac{t}{RC}} dt + k = \frac{wE_0}{R} \int e^{\frac{t}{RC}} \cdot \cos wt dt + k$$

Since we know that

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left(bx - \tan^{-1} \frac{b}{a} \right)$$

$$\int e^{\frac{t}{RC}} \cdot \cos wt dt = \frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)^2 + w^2}} \cos (wt - \tan^{-1} (RCw))$$

$$\therefore ie^{\frac{t}{RC}} = \frac{wE_0}{R} \cdot \frac{RC}{\sqrt{1 + R^2 C^2 w^2}} \cdot e^{\frac{t}{RC}} \cos (wt - \phi) + k \quad \text{where, } \tan \phi = RCw$$

$$\text{or} \quad i = \frac{wCE_0}{\sqrt{1 + R^2 C^2 w^2}} \cdot \cos (wt - \phi) + k \cdot e^{-\frac{t}{RC}}$$

which gives the current at any time.

207. The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation: $R \frac{dQ}{dt} + \frac{Q}{C} = V$. If $Q = 0$ at $t = 0$, show that

$$Q = CV \left[1 - e^{-\frac{t}{RC}} \right]. \quad (\text{M.D.U., May 2007})$$

Sol. $R \frac{dQ}{dt} + \frac{Q}{C} = V \quad \dots(1) \quad \text{or} \quad \frac{dQ}{dt} + \frac{1}{RC} \cdot Q = \frac{V}{R}$

which is Leibnitz's linear equation, I.F. = $e^{\int \frac{1}{RC} dt} = e^{\frac{1}{RC} \cdot t}$

∴ The solution of (1) is,

$$Q \cdot \text{IF} = \int \frac{V}{R} (\text{IF}) \cdot dt + k$$

$$Q \cdot e^{\frac{1}{RC} \cdot t} = \int \frac{V}{R} \cdot e^{\frac{t}{RC}} \cdot dt + k$$

$$Q \cdot e^{\frac{t}{RC}} = \frac{V}{R} \left[\frac{e^{\frac{t}{RC}}}{\frac{1}{RC}} \right] + k \quad \text{or} \quad Q \cdot e^{\frac{t}{RC}} = CV \cdot e^{\frac{t}{RC}} + k$$

When $Q = 0$, $t = 0$
 $0 = CV + k \quad \text{or} \quad k = -CV$

$$\therefore Q \cdot e^{\frac{t}{RC}} = CV \cdot e^{\frac{t}{RC}} - CV \quad \text{or} \quad Q = CV (1 - e^{-t/RC}).$$

208. Determine the current $i(t)$ in an LCR circuit with e.m.f. $E(t) = E_0 \sin wt$, in case the circuit is tuned to resonance so that $w^2 = \frac{1}{LC}$ and $\frac{R}{L}$ is so small that second and higher degree terms can be rejected. Assume that at $t = 0$, $I(0) = I'(0) = 0$. (M.D.U., May 2008)

Sol. In an LCR circuit, the charge q on a plate of the condenser is given by

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E_0 \sin wt$$

The symbolic form of given D.E. is $\left(LD^2 + RD + \frac{1}{C} \right) q = E_0 \sin wt$

Its A.E. is $LD^2 + RD + \frac{1}{C} = 0$ or $D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$

which gives $D = \frac{1}{2L} \left[-R \pm \sqrt{R^2 - \frac{4L}{C}} \right]$ or $D = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

As $\frac{R}{L}$ is small, therefore

$$D = -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC}} = -\frac{R}{2L} \pm iw$$

$\therefore$ C.F. = $e^{\frac{-Rt}{2L}} (C_1 \cos wt + C_2 \sin wt) = \left(1 - \frac{Rt}{2L} \right) (C_1 \cos wt + C_2 \sin wt)$

$$\begin{aligned} \text{P.I.} &= \frac{E_0 \sin wt}{LD^2 + RD + \frac{1}{C}} = E_0 \cdot \frac{\sin wt}{-Lw^2 + RD + \frac{1}{C}} \\ &= E_0 \cdot \frac{\sin wt}{RD} \quad \left(\because w^2 = \frac{1}{LC} \right) \\ &= \frac{E_0}{R} \int \sin wt \, dt = -\frac{E_0 \cos wt}{Rw} \end{aligned}$$

Thus the complete solution is given by

$$\begin{aligned} q &= \left(1 - \frac{Rt}{2L} \right) (C_1 \cos wt + C_2 \sin wt) - \frac{E_0 \cos wt}{Rw} \\ i = \frac{dq}{dt} &= \left(1 - \frac{Rt}{2L} \right) (-C_1 \sin wt + C_2 \cos wt)w - \frac{R}{2L} (C_1 \cos wt + C_2 \sin wt) \\ &\quad + \frac{E_0}{R} \sin wt \end{aligned}$$

Initially when $t = 0, q = 0, i = 0$

$$0 = C_1 - E_0/Rw \quad \text{or} \quad C_1 = \frac{E_0}{Rw} \quad \text{and} \quad 0 = C_2 W - \frac{R}{2L} C_1$$

$$\therefore C_2 = \frac{RC_1}{2Lw} = \frac{R}{2Lw} \left(\frac{E_0}{Rw} \right) = \frac{E_0}{2Lw^2}$$

$$\text{Thus, } i = \left(1 - \frac{Rt}{2L} \right) \left(-\frac{E_0}{Rw} \sin wt + \frac{E_0}{2Lw^2} \cos wt \right) w - \frac{R}{2L} \left(\frac{E_0}{Rw} \cos wt + \frac{E_0}{2Lw^2} \sin wt \right) + \frac{E_0}{R} \sin wt$$

$$\therefore \frac{R}{L} \text{ is small, we have } i = \frac{E_0 t}{2L} \sin wt.$$

- 209.** When a resistance R ohms is connected in series with an inductance L henries, an e.m.f. of E volts, the current i amperes at time t is given by $L \frac{di}{dt} + Ri = E$. If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t . (M.D.U., 2005)

Sol.
$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \quad \dots(1)$$

is a linear differential equation I.F. = $e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

Solution of (1) is
$$ie^{\frac{Rt}{L}} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}} dt + C_1 = \int \frac{10 \sin t}{L} \cdot e^{\frac{Rt}{L}} dt + C_1 \quad \dots(2)$$

Let
$$\begin{aligned} I &= \int \sin t \cdot e^{\frac{Rt}{L}} dt = \sin t \cdot \frac{L}{R} e^{\frac{Rt}{L}} - \int \cos t \cdot \frac{L}{R} \cdot e^{\frac{Rt}{L}} \cdot dt \\ &= \frac{L}{R} \sin t \cdot e^{\frac{Rt}{L}} - \frac{L}{R} \left[\cos t \cdot \frac{L}{R} e^{\frac{Rt}{L}} + \int \sin t \cdot \frac{L}{R} \cdot e^{\frac{Rt}{L}} \cdot dt \right] \\ &= \frac{L}{R} \sin t \cdot e^{\frac{Rt}{L}} - \frac{L^2}{R^2} \cos t \cdot e^{\frac{Rt}{L}} - \frac{L^2}{R^2} \int \sin t e^{\frac{Rt}{L}} \cdot dt \\ \left(1 + \frac{L^2}{R^2} \right) I &= \frac{L}{R} \sin t \cdot e^{\frac{Rt}{L}} - \frac{L^2}{R^2} \cos t \cdot e^{\frac{Rt}{L}} \end{aligned}$$

or

$$I = \frac{R^2}{R^2 + L^2} \left[\frac{L}{R} \sin t \cdot e^{\frac{Rt}{L}} - \frac{L^2}{R^2} \cos t \cdot e^{\frac{Rt}{L}} \right]$$

Hence

$$ie^{\frac{Rt}{L}} = \frac{10R^2}{L(R^2 + L^2)} \left[\frac{L}{R} \sin t e^{\frac{Rt}{L}} - \frac{L^2}{R^2} \cos t \cdot e^{\frac{Rt}{L}} \right] + C_1$$

$\therefore$

$$i = \frac{10R^2}{L(R^2 + L^2)} \left[\frac{L}{R} \sin t - \frac{L^2}{R^2} \cos t \right] + C_1 \cdot e^{-\frac{Rt}{L}}$$

When $t = 0, i = 0$. So $0 = \frac{10R^2}{L(R^2 + L^2)} \left[-\frac{L^2}{R^2} \cdot \right] + C_1 \Rightarrow C_1 = \frac{10L}{R^2 + L^2}$

$\therefore$

$$i = \frac{10R^2}{L(R^2 + L^2)} \left[\frac{L}{R} \sin t - \frac{L^2}{R^2} \cos t \right] + \frac{10L}{R^2 + L^2} \cdot e^{-\frac{Rt}{L}}$$

or

$$i = \frac{10}{(L^2 + R^2)} [R \sin t - L \cos t + L e^{-\frac{Rt}{L}}].$$

210. Determine Q and I in the LCR circuit with $L = 0.5 \text{ H}$, $R = 6 \text{ } \Omega$, $C = 0.02 \text{ F}$, $E(t) = 24 \sin 10t$ and initial conditions $Q = I = 0$ at $t = 0$, $q = 0.05$ when $t = 0$. (M.D.U. Dec., 2009)

Sol. $L = 0.5 \text{ H}$, $R = 6 \text{ } \Omega$, $C = 0.02 \text{ F}$, $E(t) = 24 \sin 10t$.

The differential equation for the circuit can be written as

$$\begin{aligned} L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} &= 0 \\ 0.5 \frac{d^2q}{dt^2} + 6 \frac{dq}{dt} + \frac{q}{0.02} &= 0 \\ \frac{d^2q}{dt^2} + 12 \frac{dq}{dt} + 100q &= 0 \end{aligned} \quad \dots(1)$$

Its A.E. is $D^2 + 12D + 100 = 0 \Rightarrow D = -6 \pm 8i$

Its solution is $q = e^{-6t} [C_1 \cos 8t + C_2 \sin 8t]$... (2)

Differentiating w.r.t. t , we have

$$\frac{dq}{dt} = -6e^{-6t} [C_1 \cos 8t + C_2 \sin 8t] + e^{-6t} [-8C_1 \sin 8t + 8C_2 \cos 8t]$$

... (3)

Since, $q = 0.05$ when $t = 0$

From (2), $C_1 = .05$

Also, $\frac{dq}{dt} = 0$ when, $t = 0$

$\therefore$ From (3), $0 = -6C_1 + 8C_2$ or $6C_1 = 8C_2$ or $8C_2 = 6 \times .05$ or $C_2 = .0375$

Hence, $q = e^{-6t} [.05 \cos 8t + .0375 \sin 8t]$

$$i = \frac{dq}{dt} = e^{-6t} [.05 \times 8 (-\sin 8t) + .0375 \times 8 \cos 8t] + (-6) e^{-6t} [.05 \cos 8t + .0375 \sin 8t]$$

$$= -6e^{-6t} \sin 16t.$$

211. A voltage $E.e^{-at}$ is applied at $t = 0$ to a circuit containing inductance L and resistance R . Show that

$$\text{the current at any time } t \text{ is } \frac{E}{R - aL} \left(e^{-at} - e^{-\frac{Rt}{L}} \right). \quad (\text{M.D.U., May 2008})$$

Sol. Let i ampere be the current in circuit at any time t . Then applying Kirchhoff's law, we have

$$L \frac{di}{dt} + Ri = E \cdot e^{-at} \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L} \cdot i = \frac{E}{L} \cdot e^{-at}$$

This is a linear differential equation I.F. = $e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

$$\text{Solution is } ie^{\frac{Rt}{L}} = \int \frac{E}{L} e^{-at} \cdot e^{\frac{Rt}{L}} dt + C = \frac{E}{L} \int e^{\left(\frac{R}{L} - a\right)t} dt + C = \frac{E}{L} \cdot \frac{L}{R - aL} \cdot e^{\left(\frac{R}{L} - a\right)t} + C$$

$$\text{or } i = \frac{E}{R - aL} \cdot e^{-\frac{Rt}{L}} \cdot e^{\left(\frac{R}{L} - a\right)t} + Ce^{-\frac{Rt}{L}}$$

Initially $i = 0$, $t = 0$

$$\therefore C = -\frac{E}{R - aL}$$

$$\therefore i = \frac{E}{R - aL} \left[e^{-at} - e^{-\frac{Rt}{L}} \right]. \quad \text{Hence shown.}$$

212. A capacitor $C = .01F$ in series with a resistor $R = 20$ ohms is charged from a battery $E_0 = 10$ V. Assuming that initially the capacitor is completely uncharged, determine the charge $Q(t)$, voltage $V(t)$ and current $I(t)$ in the circuit. (M.D.U., Dec., 2008)

Sol. The D.E. of the above circuit are $iR + \frac{q}{C} = E_0$ or $R \frac{dq}{dt} + \frac{q}{C} = E_0$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E_0}{R}$$

... (1)

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

Solution of equation (1) is

$$q e^{\frac{t}{RC}} = \int \frac{E_0}{R} \cdot e^{\frac{t}{RC}} \cdot dt = \frac{E_0}{R} \cdot \frac{e^{\frac{t}{RC}}}{1/RC} + k \quad \text{or} \quad q = CE_0 + k \cdot e^{-\frac{t}{RC}} \quad \dots(2)$$

When, $t = 0, q = 0, k = -CE_0$

$$\text{Putting in equation (2), we have } q = CE_0 - CE_0 \cdot e^{-\frac{t}{RC}} \quad \text{or} \quad q = CE_0 \left[1 - e^{-\frac{t}{RC}} \right] \quad \dots(3)$$

When, $C = 0.01F, R = 20 \text{ Ohms}, E_0 = 10 \text{ Volt}$

$$Q(t) = 0.01 \times 10 [1 - e^{-5t}]$$

$$V(t) = \frac{q}{C} = E_0 \left[1 - e^{-\frac{t}{RC}} \right]$$

$$V(t) = 10 [1 - e^{-5t}]$$

Differentiating (3) w.r.t. t , we get

$$i = \frac{dq}{dt} = CE_0 \left[-e^{-\frac{t}{RC}} \left(-\frac{1}{RC} \right) \right] = \frac{E_0}{R} \cdot e^{-\frac{t}{RC}} = \frac{1}{2} e^{-5t}.$$

- 213.** Determine the charge on the capacitor at any time $t > 0$ in a circuit in series having an e.m.f. given by $E(t) = 100 \sin 60 t$, a resistor of 2Ω , and inductor of $0.1 H$ and a capacitor of $\frac{1}{260}$ farads if the initial current and initial charge on the capacitor are both zero. Find the steady state solution.
(M.D.U., May 2009)

Sol. The D.E. of L-R-C circuit is given by $L \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{q}{C} = E \sin wt$

$$\Rightarrow \left(LD^2 + RD + \frac{1}{C} \right) q = E \sin wt$$

$$\text{Its A.E. is} \quad LD^2 + RD + \frac{1}{C} = 0$$

$$D = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\frac{R}{2L} \pm i P$$

$$\text{where, } P^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$\text{C.F. is } e^{\frac{-Rt}{2L}} [C_1 \cos pt + C_2 \sin pt]$$

$$\text{P.I.} = \frac{E \sin wt}{LD^2 + RD + \frac{1}{C}} = \frac{E \sin wt}{L(-w^2) + RD + \frac{1}{C}} = -\frac{E}{LR} \cos wt$$

$$\text{C.S. is} \quad q = e^{\frac{-Rt}{2L}} [C_1 \cos pt + C_2 \sin pt] - \frac{E}{Rw} \cos wt$$

$$\text{Initially, } q = 0, \quad t = 0, \quad C_1 = \frac{E}{Rw}, \quad C_2 = \frac{E}{2LPw} \quad \left(\because \frac{dq}{dt} = 0, t = 0 \right)$$

Putting the values of C_1 and C_2 , we get

$$\begin{aligned} q &= e^{\frac{-Rt}{2L}} \left[\frac{E}{Rw} \cos pt + \frac{E}{2LPw} \sin pt \right] - \frac{E}{Rw} \cos wt \\ &= \frac{E}{Rw} \left[e^{\frac{-Rt}{2L}} \left(\cos pt + \frac{R}{2Lp} \sin pt \right) - \cos wt \right] \end{aligned}$$

Here, $w = 60$, $E = 100$, $R = 2$, $L = 0.1$, $C = \frac{1}{260}$

$$\therefore q = \frac{100}{2 \times 60} \left[e^{\frac{-2t}{2 \times 0.1}} \left(\cos pt + \frac{1}{0.1p} \sin pt \right) - \cos 60t \right]$$

$$\text{where, } p^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{0.1 \times \frac{1}{260}} - \frac{4}{4(.1)^2} = 2600 - 100 = 2500 \Rightarrow p = 50.$$

$$\Rightarrow q = \frac{5}{6} \left[e^{-10t} \left(\cos 50t + \frac{1}{5} \sin 50t \right) - \cos 60t \right], \text{ which is the required steady state solution.}$$

- 214.** If the air temperature is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C .

Sol. By Newton's law of cooling $\frac{dT}{dt} = -k(T - 30)$ gives the temperature T of the substance at any instant t , where k is the constant.

$$\Rightarrow \frac{dT}{T - 30} = -k dt$$

Integrating $\log(T - 30) = -kt + C$, where C is a constant. ... (1)

Initially, when $t = 0$, $T = 100$

$$\therefore \text{ From (1), } C = \log 70 \Rightarrow \log(T - 30) = -kt + \log 70$$

$$\text{or } kt = \log 70 - \log(T - 30) \quad \dots (2)$$

When, $t = 15$, $T = 70$

$$\therefore \text{ From (2), } 15k = \log 70 - \log 40 \quad \dots (3)$$

Dividing equation (2) by (3), we have

$$\frac{t}{15} = \frac{\log 70 - \log(T - 30)}{\log 70 - \log 40} \quad \dots (4)$$

Now, when $T = 40$, from equation (4), we have

$$\frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40} = \frac{\log 7}{\log 7/4} = \frac{\log 7}{\log 1.75} = 3.4773$$

$$\Rightarrow t = 52.16 \text{ minutes.}$$

Hence the temperature will be 40°C after 52.16 minutes.

- 215.** A body is heated to 110°C and is placed in air at 10°C . After 1 hour, its temperature is 60°C . How much additional time is required for it to cool to 30°C . (M.D.U., May 2009)

Sol. Let the unit of time be a minute and T the temperature of the body at any instance. Thus by Newton's Law of cooling we have

$$\frac{dT}{dt} = -k(T - 10) \text{ where } k \text{ is a constant.}$$

$$\Rightarrow \frac{dT}{T - 10} = -k dt \quad \text{or} \quad \log(T - 10) = -kt + C \quad \dots(1)$$

Initially when $t = 0$, $T = 110$, $C = \log 100$

Putting the value of C in (1), we have

$$\log(T - 10) = -kt + \log 100 \quad \text{or} \quad kt = \log 100 - \log(T - 10)$$

Also when $t = 60$, $T = 60^\circ$...(2)

$$60k = \log 100 - \log 50 \quad \dots(3)$$

Dividing (2) by (3) we get

$$\frac{t}{60} = \frac{\log 100 - \log(T - 10)}{\log 100 - \log 50} \quad \text{or} \quad \frac{t}{60} = \frac{\log 100 - \log(T - 10)}{\log 2}$$

Now, when $T = 30^\circ$

$$\frac{t}{60} = \frac{\log 100 - \log 20}{\log 2} = \frac{\log 5}{\log 2}$$

$$\text{or} \quad t = 60 \times \frac{\log 5}{\log 2} = \frac{2.2513}{1.8718} \times 60 \quad \text{or} \quad t = 72.16 \text{ min.}$$

- 216.** A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C . What will be the temperature of the body after 40 minutes from the original.

Sol. Let the unit of time be a minute and T the temperature of the body at any instant t . Then by

Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - 40)$, where k is a constant.

$$\Rightarrow \frac{dT}{T - 40} = -k dt$$

On integrating, $\log(T - 40) = -kt + C$ where C is a constant. ...(1)

Initially, when $t = 0$, $T = 80$

$\therefore$ From (1), $C = \log 40$

Substituting the value of C in (1), we have

$$\log(T - 40) = -kt + \log 40 \quad \text{or} \quad kt = \log 40 - \log(T - 40) \quad \dots(2)$$

Also when $t = 20$, $T = 60$

$$\therefore \text{ From (2), } 20k = \log 40 - \log 20 \quad \dots(3)$$

Dividing equation (2) by (3), we have

$$\frac{t}{20} = \frac{\log 40 - \log(T - 40)}{\log 40 - \log 20} \quad \dots(4)$$

Now, when $t = 40$ we have from equation (4)

$$\frac{40}{20} = \frac{\log\left(\frac{40}{T - 40}\right)}{\log 2} \Rightarrow 2 \log 2 = \log\left(\frac{40}{T - 40}\right)$$

$$\text{or} \quad \log 2^2 = \log\left(\frac{40}{T - 40}\right) \quad \text{i.e.,} \quad \frac{40}{T - 40} = 4 \quad \text{or} \quad T - 40 = 10$$

$$\therefore T = 50$$

Hence, the temperature of the body after 40 minutes will be 50°C .

- 217.** If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes. (M.D.U. Dec., 2007)

Sol. By Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - 30), \text{ where } k \text{ is a constant.}$$

$$\Rightarrow \frac{dT}{T - 30} = -k dt$$

$$\log(T - 30) = -kt + C \quad \dots(1)$$

Initially when $t = 0$, $T = 80$

$$\therefore C = \log 50$$

$$\Rightarrow \log(T - 30) = -kt + \log 50 \quad \text{or} \quad kt = \log 50 - \log(T - 30) \quad \dots(2)$$

Also when $t = 12$, $T = 60$

$$12k = \log 50 - \log 30 \quad \dots(3)$$

Dividing (2) by (3), we have

$$\frac{t}{12} = \frac{\log 50 - \log(T - 30)}{\log 50 - \log 30} \quad \dots(4)$$

Now, when $t = 24$,

$$2 = \frac{\log\left(\frac{50}{T - 30}\right)}{\log\left(\frac{50}{30}\right)} \quad \text{or} \quad 2 \log\left(\frac{50}{30}\right) = \log\left(\frac{50}{T - 30}\right)$$

or

$$\frac{25}{9} = \frac{50}{T - 30} \quad \text{or} \quad 25T - 750 = 450$$

$\therefore$

$$T = 48. \text{ Hence temperature will be } 48^\circ\text{C.}$$

- 218.** A long hollow pipe has an inner diameter of 5 cm and outer diameter of 10 cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 metres long? Find the temperature at a distance $x = 7.5$ cm from the centre of the pipe.

Sol. The fundamental principle involved in the problems of heat conduction is that the quantity of heat Q flowing per second across a slab of area A and thickness δx , whose faces are at temperatures T and $T + \delta T$, is given by $Q = -kA \frac{dT}{dx}$, where k is the coefficient of thermal conductivity

and depends on the material of the body. Negative sign is prefixed because T decreases as x increases. This is known as Fourier's law.

Here the isothermal surfaces are cylinders, the axis of each one of them is the axis of the pipe. Consider one such cylinder of radius x cm and length 1 cm. The surface area of this cylinder is $A = 2\pi x$ sq. cm.

Let Q cal/sec be the quantity of heat flowing across this surface, then

$$Q = -kA \frac{dT}{dx} = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{Q}{2\pi k} \cdot \frac{dx}{x}$$

$$\text{Integrating, we have} \quad T = -\frac{Q}{2\pi k} \cdot \log x + C \quad \dots(1)$$

Since $T = 200$ when $x = 5$

$$\therefore 200 = -\frac{Q}{2\pi k} \cdot \log 5 + C \quad \dots(2)$$

Also, $T = 50$ when $x = 10$

$$\therefore 50 = -\frac{Q}{2\pi k} \log 10 + C \quad \dots(3)$$

Subtracting (3) from (2), we have

$$150 = +\frac{Q}{2\pi k} (\log 10 - \log 5) \quad \text{or} \quad 150 = \frac{Q}{2\pi k} \log 2 \quad \dots(4)$$

$$\therefore Q = \frac{300 \pi k}{\log 2} = \frac{300 \times \pi \times 0.12}{\log 2} = 163 \text{ cal/sec.} \quad (\because k = 0.12)$$

Hence the heat lost **per minute** through 20 metre length of pipe

$$= 60 \times 2000 Q = 60 \times 2000 \times 163 = 1956,0000 \text{ cal.}$$

Now, let $T = t$, when $x = 7.5$

$$\therefore \text{From (1),} \quad t = -\frac{Q}{2\pi k} \log 7.5 + C \quad \dots(5)$$

Subtracting (2) from (5),

$$t - 200 = -\frac{Q}{2\pi k} (\log 7.5 - \log 5) \quad \text{or} \quad t - 200 = -\frac{Q}{2\pi k} \log 1.5 \quad \dots(6)$$

Dividing (6) by (4), we have

$$\frac{t - 200}{150} = -\frac{\log 1.5}{\log 2} \quad \text{or} \quad t = 200 - 150 \times 0.58 = 113$$

$\therefore$ When, $x = 7.5$ cm, $T = 113^\circ\text{C}$.

- 219.** A tank contains 5000 litres of fresh water. Salt water which contains 100 gm of salt per litre flows into it at the rate of 10 litres per minute and the mixture kept uniform by stirring, runs out at the same rate. When will the tank contain 200000 gm of salt? How long will it take for the quantity of salt in the tank to increase from 150000 gm to 250000 gm?

Sol. Let Q gm be the quantity of salt present in the tank at time t , then

$$\frac{dQ}{dt} = \text{rate at which salt content is changing}$$

$$= \text{rate of salt entering the tank} - \text{rate of salt leaving the tank.}$$

$$\text{Rate at which the salt increases due to the inflow} = 100 \times 10 = 1000 \text{ gm/min.}$$

$$\text{Let } C \text{ gm be the concentration of salt at time } t.$$

$$\text{Rate at which the salt content decreases due to the outflow} = C \times 10 = 10 \text{ gm/min.}$$

Since the rate of inflow is the same as the rate of outflow, there is no change in the volume of water at any instant.

$$\Rightarrow C = \frac{Q}{5000}$$

The rate of decrease of salt content

$$= 10 \times \frac{Q}{5000} = \frac{Q}{500} \text{ gm/min}$$

$$\therefore \frac{dQ}{dt} = 1000 - \frac{Q}{500}$$

$$\text{or} \quad \frac{dQ}{dt} = \frac{500000 - Q}{500} \quad \text{or} \quad \frac{dQ}{500000 - Q} = \frac{dt}{500}$$

$$\text{Integrating, } -\log (500000 - Q) = \frac{t}{500} + C \quad \dots(1)$$

Initially, when $t = 0$, $Q = 0$

$$\therefore C = -\log 500000$$

$$\text{From (1), we have } \frac{t}{500} = \log 500000 - \log (500000 - Q) \text{ or } t = 500 \log \frac{500000}{500000 - Q} \quad \dots(2)$$

Let $t = T$ when $Q = 200000$

$$\begin{aligned} \text{From (2), } T &= 500 \log_e \left(\frac{5}{3} \right) = 500 \times 2.303 \times \log_{10} \left(\frac{5}{3} \right) = 500 \times 2.303 \times 0.2219 \\ &= 255.5 \text{ minutes} = 4 \text{ hours } 15.52 \text{ minutes} \end{aligned}$$

Let $t = T_1$ when $Q = 150000$ and $t = T_2$, when $Q = 250000$

$$\text{From (2) we have } T_1 = 500 \log \frac{500000}{350000} = 500 \log \frac{10}{7}$$

$$T_2 = 500 \log \frac{500000}{250000} = 500 \log 2$$

$$\begin{aligned} \therefore \text{ Required time} &= T_2 - T_1 \\ &= 500 \left(\log 2 - \log \frac{10}{7} \right) = 500 \log_e \left(\frac{7}{5} \right) = 500 \times 2.303 \times \log_{10} \left(\frac{7}{5} \right) \\ &= 500 \times 2.303 \times 0.1461 = 168.23 \text{ minutes} \\ &= 2 \text{ hours } 48.23 \text{ minutes.} \end{aligned}$$

- 220.** A tank contains 100 litres of fresh water. Two litres of brine, each containing 1 gm of dissolved salt, run into the tank per minute, and the mixture kept uniform by stirring runs out at the rate of 1 litre per minute. Find the amount of salt present when the tank contains 150 litres of brine.

Sol. Let Q gm be the quantity of salt present in the brine at time t , then $\frac{dQ}{dt}$ is the rate at which salt content is changing. The rate at which the salt content increases due to the inflow $= 2 \times 1 = 2$ gms/min.

Let C gm be the concentration of brine at time t .

The rate at which the salt content decreases due to the outflow $= C \times 1 = C$ gm/min.

$$\therefore \frac{dQ}{dt} = 2 - C \quad \dots(1)$$

Now, the initial volume of liquid is 100 litres. In one minute, 2 litres of brine enter the tank and 1 litre of brine leaves the tank so that the volume of liquid in the tank increases at the rate of $(2 - 1) = 1$ litre/min.

$\therefore$ The volume of liquid at time t is $(100 + t)$ litres containing Q gm of salt.

$$\therefore C = \frac{Q}{100 + t}$$

$$\text{From (1), we have } \frac{dQ}{dt} = 2 - \frac{Q}{100 + t} \text{ or } \frac{dQ}{dt} + \frac{Q}{100 + t} = 2 \quad \dots(2)$$

which is a linear equation in Q and t .

$$\text{I.F.} = e^{\int \frac{dt}{100+t}} = e^{\log(100+t)} = 100 + t.$$

$\therefore$ The solution of equation (2) is

$$Q(100 + t) = \int 2(100 + t) dt + C \quad \text{or} \quad (100 + t)Q = 2 \left(100t + \frac{t^2}{2} \right) + C \quad \dots(3)$$

Initially, when $t = 0$, $Q = 0$ so that $C = 0$

$$\therefore \text{ From (3), } Q = \frac{2 \left(100t + \frac{t^2}{2} \right)}{100 + t} \quad \dots(4)$$

Now, if V is the volume of liquid at time t , then $V = 100 + t$

$\therefore$ When $V = 150$ litres, $t = 150 - 100 = 50$ minutes.

$$\text{and salt content } Q = \frac{2 \left(100 \times 50 + \frac{(50)^2}{2} \right)}{100 + 50} = 83.3 \text{ gm.}$$

- 221.** A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?

Sol. Let the salt content at time t be u lb so that its rate of change is $\frac{du}{dt} = 2 \text{ gal} \times 2 \text{ lb} = 4 \text{ lb/min.}$

If C be the concentration of the brine at time t , the rate at which the salt content decreases due to out-flow

$$= 2 \text{ gal} \times C \text{ lb} = 2C \text{ lb/min.}$$

$$\therefore \frac{du}{dt} = 4 - 2C \quad \dots(1)$$

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

$$\therefore (1) \text{ becomes } \frac{du}{dt} = 4 - 2 \cdot \frac{u}{50}$$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log(100 - u) + k \quad \dots(2)$$

Initially when $t = 0$, $u = 0$

$$\therefore 0 = -25 \log_e 100 + k \quad \dots(3)$$

Eliminating k from (2) and (3), we get

$$t = 25 \log_e \frac{100}{100 - u}$$

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$,

$$\text{We have, } t_1 = 25 \log_e \frac{100}{60}, t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time

$$(t_2 - t_1) = 25 \log_e 5 - 25 \log_e \left(\frac{5}{3} \right) = 25 \log_e 3 = 25 \times 1.0986 = 27 \text{ min. } 28 \text{ sec.}$$

- 222.** Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, show that the half life of uranium is

$$\frac{(T_2 - T_1) \log 2}{\log (M_1/M_2)}.$$

Sol. Let M grams of uranium be present at any time t . Then the equation of disintegration of uranium is

$$\frac{dM}{dt} = -kM, \text{ where } k \text{ is a constant.}$$

or
$$\frac{dM}{M} = -k \cdot dt$$

Integrating, $\log M = -kt + C$... (1)

Initially, when $t = 0$, $M = M_0$ (say)

$\therefore$ From (1), $C = \log M_0$

Substituting the value of C in (1), we have

$$\log M = \log M_0 - kt \quad \dots (2)$$

Now, when $t = T_1$, $M = M_1$ and when $t = T_2$, $M = M_2$

$\therefore$ From (2), we have

$$kT_1 = \log M_0 - \log M_1 \quad \dots (3)$$

and $kT_2 = \log M_0 - \log M_2 \quad \dots (4)$

Subtracting (3) from (4), we get

$$k(T_2 - T_1) = \log M_1 - \log M_2 \quad \text{or} \quad k = \frac{\log \frac{M_1}{M_2}}{T_2 - T_1}$$

Let T be the half-life of uranium i.e. ;

When $t = T$, $M = \frac{1}{2} M_0$

$\therefore$ From (2), we get $kT = \log M_0 - \log \frac{M_0}{2} = \log 2$

$\therefore T = \frac{\log 2}{k} = \frac{(T_2 - T_1) \log 2}{\log \left(\frac{M_1}{M_2} \right)}$

Hence shown.

223. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$, where λ is a parameter.

Sol. The equation of the family of given curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1 \quad \dots (1)$$

Differentiating (1) w.r.t. x

$$\frac{2x}{a^2} + \frac{2y}{b^2 + \lambda} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{x}{a^2} + \frac{y}{b^2 + \lambda} \cdot \frac{dy}{dx} = 0 \quad \dots (2)$$

To eliminate the parameter λ , we equate the values of $b^2 + \lambda$ from (1) and (2).

From (1), $\frac{y^2}{b^2 + \lambda} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \Rightarrow b^2 + \lambda = \frac{a^2 y^2}{a^2 - x^2}$

From (2), $b^2 + \lambda = -\frac{a^2 y}{x} \cdot \frac{dy}{dx}$

$\therefore \frac{a^2 y^2}{a^2 - x^2} = -\frac{a^2 y}{x} \cdot \frac{dy}{dx}$

or
$$\frac{y}{a^2 - x^2} = -\frac{1}{x} \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} + \frac{xy}{a^2 - x^2} = 0 \quad \dots(3)$$

which is the differential equation of the given family (1).

Now to find the orthogonal trajectories

(i) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get the differential equation of the orthogonal trajectory.

(ii) Solve the above equation to get the equation of the required orthogonal trajectory.

So, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (3), we get

$$\frac{xy}{a^2 - x^2} - \frac{dx}{dy} = 0 \quad \text{or} \quad y \, dy - \left(\frac{a^2 - x^2}{x} \right) dx = 0 \quad \dots(4)$$

which is the differential equation of the orthogonal trajectories.

Integrating (4), we get

$$\int y \, dy - \int \left(\frac{a^2}{x} - x \right) dx = C \quad \text{or} \quad \frac{y^2}{2} - a^2 \log x + \frac{x^2}{2} = C$$

or

$$x^2 + y^2 = 2a^2 \log x + C$$

which is the equation of the required orthogonal trajectories of (1).

224. Find the orthogonal trajectory of the cardioids $r = a(1 - \cos \theta)$.

Sol. The equation of the family of given cardioids is $r = a(1 - \cos \theta)$...(1)

Differentiating (1) w.r.t. θ , $\frac{dr}{d\theta} = a \sin \theta$...(2)

Dividing (2) by (1) [to eliminate a]

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\sin \theta}{1 - \cos \theta} \quad \text{or} \quad \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \cot \frac{\theta}{2} \quad \dots(3)$$

which is the differential equation of the given family (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (3), we get

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \cot \frac{\theta}{2} \quad \text{or} \quad r \frac{d\theta}{dr} + \cot \frac{\theta}{2} = 0 \quad \text{or} \quad \frac{dr}{r} + \tan \frac{\theta}{2} = 0 \quad \dots(4)$$

which is the differential equation of the family of orthogonal trajectories.

Integrating (4), $\log r - 2 \log \left(\cos \frac{\theta}{2} \right) = \log C$

or
$$\log r = \log C \cos^2 \left(\frac{\theta}{2} \right) \quad \text{or} \quad r = C \cdot \cos^2 \frac{\theta}{2} = \frac{C}{2} (1 + \cos \theta)$$

or

$$r = C(1 + \cos \theta)$$

which is the required equation of orthogonal trajectories of (1).

225. Find the orthogonal trajectories of $r^2 = a \sin 2\theta$. (M.D.U. Dec., 2009)

Sol. $r^2 = a \sin 2\theta$... (1)

Differentiating (1) w.r.t. θ $2r \frac{\partial r}{\partial \theta} = 2a \cos 2\theta$... (2)

Dividing (2) by (1) to eliminate 'a'

$$\frac{2}{r} \cdot \frac{\partial r}{\partial \theta} = 2 \cot 2\theta$$
 ... (3)

which is the D.E. of the given family (1).

Replacing $\frac{\partial r}{\partial \theta}$ by $-r^2 \frac{\partial \theta}{\partial r}$ in (3), we get

$$\begin{aligned} \frac{2}{r} \left(-r^2 \frac{\partial \theta}{\partial r} \right) &= 2 \cot 2\theta \quad \text{or} \quad -2r \frac{\partial \theta}{\partial r} = 2 \cot 2\theta \\ \frac{-1}{2r} \frac{dr}{d\theta} &= \frac{1}{2} \tan 2\theta \quad \text{or} \quad \frac{1}{2r} \frac{dr}{d\theta} + \frac{1}{2} \tan 2\theta = 0 \end{aligned}$$
 ... (4)

which is the D.E. of the family of orthogonal trajectories.

Integrating (4), $\frac{1}{2} \log r - \frac{1}{2} \log \cos 2\theta = \log C$

$$\log r = \log c^2 + \log \cos 2\theta$$

$$\log r = \log c^2 \cos 2\theta$$

$$r = c^2 \cos 2\theta.$$

226. Prove that the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal.

(M.D.U. Dec., 2009)

Sol. The equation of the family of given parabolas is

$$y^2 = 4a(x + a)$$
 ... (1)

Differentiating equation (1) w.r.t. x

$$2y \frac{dy}{dx} = 4a \quad \text{or} \quad y \frac{dy}{dx} = 2a$$
 ... (2)

Eliminating a between (1) and (2), we have

$$y^2 = 2y \frac{dy}{dx} \left(x + \frac{y}{2} \cdot \frac{dy}{dx} \right) \quad \text{or} \quad y^2 = 2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx} \right)^2$$

$$\text{or} \quad y \left(\frac{dy}{dx} \right)^2 + 2x \cdot \frac{dy}{dx} - y = 0$$
 ... (3)

which is the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$y \left(\frac{dx}{dy} \right)^2 - 2x \left(\frac{dx}{dy} \right) - y = 0 \quad \text{or} \quad y - 2x \cdot \frac{dy}{dx} - y \left(\frac{dy}{dx} \right)^2 = 0$$

$$\text{or} \quad y \left(\frac{dy}{dx} \right)^2 + 2x \cdot \frac{dy}{dx} - y = 0$$
 ... (4)

which is the differential equation of the orthogonal trajectories.

Since equation (4) is the same as (3), the system of confocal and coaxial parabolas is self orthogonal, i.e., each member of (1) cuts every other member orthogonally, at right angles.

- 227.** A tank contains 1000 litres of fresh water. Salt water which contains 150 gm of salt per litre, runs into it at the rate of 5 litres per minute and well stirred mixture runs out of it at the same rate. When will the tank contain 5,000 gm of salt ? (M.D.U., 2005)

Sol. Let Q gm be the quantity of salt present in the tank at time t , then

$\frac{dQ}{dt}$ is the rate at which the salt content is changing

= rate of salt entering the tank - rate of salt leaving the tank.

Now, the rate at which the salt increases due to inflow = $150 \times 5 = 750$ gm/min

Let C gm be the concentration of salt at time t . The rate at which the salt content decreases due to outflow = $C \times 5 = 5$ gm/minute.

Since the rate of inflow is the same as the rate of outflow, there is no change in the volume of

water at any instant. $\Rightarrow C = \frac{Q}{1000}$

The rate of decrease of salt content = $5 \times \frac{Q}{1000} = \frac{Q}{200}$ gm/min.

$$\therefore \frac{dQ}{dt} = 750 - \frac{Q}{200} \quad \text{or} \quad \frac{dQ}{dt} = \frac{150000 - Q}{200} \quad \text{or} \quad \frac{dQ}{150000 - Q} = \frac{dt}{200}$$

Integrating it, we have

$$-\log(150000 - Q) = \frac{t}{200} + C \quad \dots(1)$$

Initially, when $t = 0$, $Q = 0$

$$C = -\log 150000$$

From equation (1), we have

$$\frac{t}{200} = \log 150000 - \log(150000 - Q) \quad \text{or} \quad t = 200 \log \left(\frac{150000}{150000 - Q} \right) \quad \dots(2)$$

Let $t = T$ when $Q = 5000$.

From equation (2),

$$\begin{aligned} T &= 200 \log_e \left(\frac{150000}{150000 - 5000} \right) = 200 \log_e \left(\frac{150000}{145000} \right) \\ &= 200 \log_e \left(\frac{150}{145} \right) = 200 \times 2.303 \log_{10} \left(\frac{30}{29} \right) \\ &= 460.6 \times \log_{10}(1.035) = 460.6 \times 0.0147 = 6.77 \text{ minutes.} \end{aligned}$$

- 228.** Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

Sol. The equation of the family of given parabolas is $y = ax^2$...(1)

Differentiating (1) w.r.t. x $\frac{dy}{dx} = 2ax$...(2)

Eliminating ' a ' between (1) and (2), we have

$$\frac{dy}{dx} = 2 \left(\frac{y}{x^2} \right) x \quad \text{or} \quad \frac{dy}{dx} = 2 \cdot \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{2}{x} \cdot y = 0 \quad \dots(3)$$

which is the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$-\frac{dx}{dy} - \frac{2}{x} \cdot y = 0 \quad \text{or} \quad \frac{dx}{dy} = -\frac{2y}{x} \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{2y} \quad \dots(4)$$

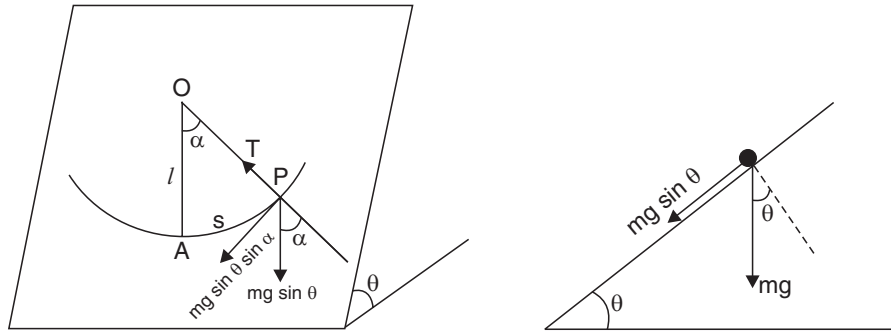
$$2y \cdot dy = -x \, dx.$$

Integrating, we get $y^2 = -\frac{x^2}{2} + C \Rightarrow x^2 + 2y^2 = C.$

- 229.** A pendulum of length l hangs against a wall inclined at an angle θ to the horizontal. Show that the time of complete oscillation is $2\pi \sqrt{\frac{l}{g \sin \theta}}.$

Sol. Let the position of the bob of mass m , at any point t , be P and O be the point of suspension so that $OP = l$, $\angle AOP = \alpha$, where OA is the line of greatest slope through O.

The component of weight of the bob along the plane is $mg \sin \theta$.



The equation of motion of the bob along the tangent at P is given by

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta \sin \alpha.$$

$$\text{or} \quad \frac{d^2(l\alpha)}{dt^2} = -g \sin \theta \left(\alpha - \frac{\alpha^3}{3!} + \dots \right) \quad [\because s = l\alpha]$$

$$\text{or} \quad l \frac{d^2 \alpha}{dt^2} = -g \sin \theta \cdot \alpha \quad (\text{to a first approximation})$$

$$\text{or} \quad \frac{d^2 \alpha}{dt^2} = -\frac{g}{l} \sin \theta \cdot \alpha \quad \text{or} \quad \frac{d^2 \alpha}{dt^2} = -w^2 \alpha, \text{ where } w^2 = \frac{g \sin \theta}{l}$$

$\therefore$ The motion of the bob is simple harmonic and the time of one oscillation is $= \frac{2\pi}{w} = 2\pi \sqrt{\frac{l}{g \sin \theta}}.$

- 230.** Find how many seconds a clock would lose per day, if the length of its pendulum were increased in the ratio 900 : 901?

Sol. Let l be the original length and $l + dl$ the increased length of the pendulum, then

$$\frac{l}{l + dl} = \frac{900}{901} \quad \text{or} \quad \frac{l + dl}{l} = \frac{901}{900}, \quad \therefore \frac{dl}{l} = \frac{1}{900}$$

Let n be the number of beats per day, then $n = 86,400.$

If dn is the change in the number of beats, then $dn = -\frac{n}{2} \cdot \frac{dl}{l} = -\frac{86400}{2} \cdot \frac{1}{900} = -48$

Since dn is negative, the clock will lose 48 seconds per day.

- 231.** A second's pendulum which gains 10 seconds per day at one place loses 10 seconds at another place. Compare the accelerations due to gravity at the two places.

Sol. Let g be the acceleration due to gravity when the pendulum beats seconds.

Let $g + g_1$ be the acceleration due to gravity at the place, where it gains 10 seconds per day, then

$$dn = 10, n = 86400$$

Since

$$dn = \frac{n}{2} \cdot \frac{g_1}{g}$$

$$10 = \frac{86400}{2} \cdot \frac{g_1}{g} \Rightarrow \frac{g_1}{g} = \frac{1}{4320}$$

Adding 1 to both sides

$$\frac{g + g_1}{g} = \frac{4321}{4320} \quad \dots(1)$$

Let $g + g_2$ be the acceleration due to gravity at the place, where it loses 10 seconds per day, then

$$dn = -10, -10 = \frac{86400}{2} \cdot \frac{g_2}{g}$$

$$\text{or} \quad \frac{g_2}{g} = -\frac{1}{4320} \quad \text{or} \quad \frac{g + g_2}{g} = \frac{4319}{4320} \quad \dots(2)$$

Dividing (1) by (2), we have

$$\frac{g + g_1}{g + g_2} = \frac{4321}{4319} \text{ which is the required ratio.}$$

- 232.** A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.

Sol. Let the position of the bob (of mass m), at any time t be P and O be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and therefore arc $AP = s = l\theta$

$\therefore$ The equation of motion along the tangent PT is

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta - \lambda \frac{ds}{dt},$$

where λ is a constant.

$$\text{or} \quad \frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m} \cdot \frac{d(l\theta)}{dt} + g \sin \theta = 0$$

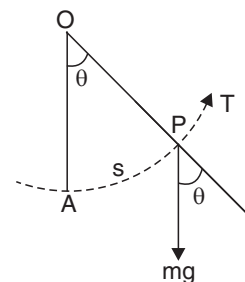
Replacing $\sin \theta$ by θ since it is small and writing $\frac{\lambda}{m} = 2k$, we get

$$\frac{d^2(\theta)}{dt^2} + 2k \frac{d\theta}{dt} + \frac{g \cdot \theta}{l} = 0 \quad \dots(1)$$

which is the required differential equation.

Its auxiliary equation has roots $D = k \pm \sqrt{k^2 - w^2}$, where $w = g/l$

The oscillatory motion of the bob is only possible when $k < w$.



Then the roots of the auxiliary equation are $-k \pm i \sqrt{w^2 - k^2}$.

$\therefore$ The solution of (1) is $\theta = e^{-kt} [C_1 \cos \sqrt{w^2 - k^2} t + C_2 \sin \sqrt{w^2 - k^2} t]$

which gives a vibratory motion of period $\frac{2\pi}{\sqrt{w^2 - k^2}}$.

- 233.** Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{LC}}$ per minute.

Sol. Let i be the current and q the charge in the condenser plate at any time t .

The voltage drops across L and C are $L \frac{di}{dt} = L \cdot \frac{d^2q}{dt^2}$ and $\frac{q}{C}$ respectively.

Since there is no applied E.M.F. in the circuit, we have by Kirchhoff's law,

$$L \cdot \frac{d^2q}{dt^2} + \frac{q}{C} = 0 \quad \text{or} \quad \frac{d^2q}{dt^2} = -\frac{1}{LC} \cdot q$$

Writing $\frac{1}{LC} = w^2$, it becomes $\frac{d^2q}{dt^2} = -w^2q$

It represents oscillatory, current with period $\frac{2\pi}{w} = 2\pi\sqrt{LC}$.

$\therefore$ Frequency = $\frac{1}{T}$ per second = $\frac{60}{2\pi\sqrt{LC}}$ per minute = $\frac{30}{\pi\sqrt{LC}}$ per minute.

- 234.** An electric circuit consists of an inductance of 0.1 henry, a resistance of 20 ohms and a condenser of capacitance 25 micro-farads. Find the charge q and the current i at any time t , given that at

$t = 0$, $q = 0.05$ coulomb, $i = \frac{dq}{dt} = 0$, when $t = 0$.

Sol. The differential equation for the circuit can be written as

$$L \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{q}{C} = 0$$

$$0.1 \frac{d^2q}{dt^2} + 20 \frac{dq}{dt} + \frac{q}{25 \times 10^{-6}} = 0 \quad (\because 1 \text{ micro-farad} = 10^{-6} \text{ farads})$$

$$\text{or} \quad \frac{d^2q}{dt^2} + 200 \cdot \frac{dq}{dt} + 400,000 q = 0 \quad \dots(1)$$

Its A.E. is $D^2 + 200 D + 400,000 = 0$

$$\text{so that} \quad D = \frac{-200 \pm \sqrt{40000 - 1600000}}{2} = -100 \pm 100\sqrt{39} i$$

$$\therefore \text{ Its solution is } q = e^{-100t} [C_1 \cos(100\sqrt{39} t) + C_2 \sin(100\sqrt{39} t)] \quad \dots(2)$$

Differentiating w.r.t. t , we have

$$\begin{aligned} \frac{dq}{dt} &= -100 e^{-100t} [C_1 \cos(100\sqrt{39} t) + C_2 \sin(100\sqrt{39} t)] \\ &\quad + e^{-100t} [-100\sqrt{39} C_1 \sin(100\sqrt{39} t) + 100\sqrt{39} C_2 \cos(100\sqrt{39} t)] \quad \dots(3) \end{aligned}$$

Since, $q = 0.05$, when $t = 0$

$\therefore$ From (2), $C_1 = 0.05$

Also, $\frac{dq}{dt} = 0$, when $t = 0$

$\therefore$ From (3), $0 = -100 C_1 + 100 \sqrt{39} C_2$ or $C_2 = \frac{C_1}{\sqrt{39}} = \frac{0.05}{\sqrt{39}} = 0.008$

Hence, $q = e^{-100t} [0.05 \cos (624.5 t) + .008 \sin (624.5 t)]$

and $i = \frac{dq}{dt} = -100e^{-100t} [(C_1 - \sqrt{39} C_2) \cos 624.5 t + (\sqrt{39} C_1 + C_2) \sin 624.5 t]$
 $= -40 C_2 \cdot e^{-100t} \sin 624.5 t$

or $i = -0.32 e^{-100t} \sin 624.5 t$.

235. An uncharged condenser of capacity C is charged by applying an e.m.f. $E \sin \frac{t}{\sqrt{LC}}$, through leads of self-inductance L and negligible resistance. Prove that at time t , the charge on one of the plates is

$$\frac{EC}{2} \left[\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} \right].$$

Sol. Let q be the charge on the condenser at any time t . The differential equation for the circuit is

$$L \frac{d^2 q}{dt^2} + \frac{q}{C} = E \sin \frac{t}{\sqrt{LC}} \quad \dots(1)$$

Its A.E. is $LD^2 + \frac{1}{C} = 0$ or $D^2 = -\frac{1}{LC}$ so that $D = \pm \frac{i}{\sqrt{LC}}$

$$\text{C.F.} = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}}$$

and $\text{P.I.} = \frac{1}{LD^2 + \frac{1}{C}} E \sin \frac{t}{\sqrt{LC}} \quad (\text{Case of failure})$

$$= Et \cdot \frac{1}{2LD} \sin \frac{t}{\sqrt{LC}} = \frac{Et}{2L} \cdot \int \sin \frac{t}{\sqrt{LC}} dt$$

$$= \frac{Et}{2L} \left(-\sqrt{LC} \cos \frac{t}{\sqrt{LC}} \right) = -\frac{Et}{2} \cdot \sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}}$$

$\therefore$ Complete solution of (1) is

$$q = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}} - \frac{E.t}{2} \cdot \sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}} \quad \dots(2)$$

Initially, when $t = 0$, $q = 0 \therefore C_1 = 0$

Differentiating (2) w.r.t. t , we have

$$\frac{dq}{dt} = -\frac{C_1}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}} + \frac{C_2}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} - \frac{E}{2} \sqrt{\frac{C}{L}} \left[\cos \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}} \right]$$

Initially, $\frac{dq}{dt} = i = 0$, when $t = 0$

$$\therefore \frac{C_2}{\sqrt{LC}} - \frac{E}{2} \sqrt{\frac{C}{L}} = 0 \text{ or } C_2 = \frac{EC}{2}.$$

Substituting the values of C_1 and C_2 in equation (2), the charge q on the condenser plate, at any time t , is given by

$$q = \frac{EC}{2} \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2} \sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}}$$

or

$$q = \frac{EC}{2} \left[\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} \right]. \quad \text{Hence proved.}$$

236. In an L - C - R circuit, the charge q on a plate of the condenser is given by

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E \sin wt, \text{ where } i = \frac{dq}{dt}. \text{ The circuit is tuned to resonance so that } w^2 = \frac{1}{LC}.$$

$$\text{If } R^2 < \frac{4L}{C} \text{ and } q = 0, i = 0, \text{ when } t = 0, \text{ show that } q = \frac{E}{Rw} \left[-\cos wt + e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2LP} \sin pt \right) \right]$$

$$\text{and } i = \frac{E}{R} \left[\sin wt - \frac{1}{p\sqrt{LC}} e^{-\frac{Rt}{2L}} \sin pt \right], \text{ where } p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}.$$

Sol. The given differential equation is

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E \sin wt$$

$$\text{or } \left(LD^2 + RD + \frac{1}{C} \right) q = E \sin wt, \text{ where } D = \frac{d}{dt} \quad \dots(1)$$

$$\text{Its A.E. is } LD^2 + RD + \frac{1}{C} = 0$$

$$\begin{aligned} \text{so that } D &= \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R \pm i \sqrt{\frac{4L}{C} - R^2}}{2L}, \text{ since } R^2 < \frac{4L}{C} \\ &= -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\frac{R}{2L} \pm ip, \text{ since } \frac{1}{LC} - \frac{R^2}{4L^2} = p^2 \quad (\text{Given}) \end{aligned}$$

$$\text{Its C.F.} = e^{-\frac{R}{2L}t} (C_1 \cos pt + C_2 \sin pt)$$

$$\begin{aligned} \text{and P.I.} &= \frac{1}{LD^2 + RD + \frac{1}{C}} E \sin wt = E \cdot \frac{1}{-Lw^2 + RD + \frac{1}{C}} \sin wt \\ &= \frac{E}{R} \cdot \frac{1}{D} \sin wt, \text{ since } w^2 = \frac{1}{LC} = -\frac{E}{RW} \cos wt \end{aligned}$$

$\therefore$ The complete solution of equation (1) is

$$q = e^{-\frac{Rt}{2L}} (C_1 \cos pt + C_2 \sin pt) - \frac{E}{RW} \cos wt \quad \dots(2)$$

$$\text{Initially, when } t = 0, q = 0 \quad \therefore C_1 = \frac{E}{RW}$$

Differentiating (2) w.r.t. t

$$\frac{dq}{dt} = e^{-\frac{Rt}{2L}} (-p C_1 \sin pt + p C_2 \cos pt) - \frac{R}{2L} e^{-\frac{Rt}{2L}} (C_1 \cos pt + C_2 \sin pt) + \frac{E}{R} \sin wt$$

Initially, when $t = 0$, $\frac{dq}{dt} = i = 0$

$$\therefore pC_2 - \frac{R}{2L} C_1 = 0 \quad \text{or} \quad C_2 = \frac{R}{2Lp} C_1 = \frac{R}{2Lp} \cdot \frac{E}{RW}$$

$$\therefore C_2 = \frac{E}{2LpW}$$

Substituting the values of C_1 and C_2 in equation (2), we get

$$q = e^{-\frac{Rt}{2L}} \left(\frac{E}{Rw} \cos pt + \frac{E}{2pLw} \sin pt \right) - \frac{E}{Rw} \cos wt$$

$$\text{or} \quad q = \frac{E}{Rw} \left[-\cos wt + e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2pL} \sin pt \right) \right] \quad \dots(3)$$

Differentiating equation (3) w.r.t. t ,

$$\frac{dq}{dt} = \frac{E}{Rw} \left[w \sin wt - \frac{R}{2L} e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2Lp} \sin pt \right) + e^{-\frac{Rt}{2L}} \left(-p \sin pt + \frac{R}{2L} \cos pt \right) \right]$$

$$\text{or } i = \frac{E}{Rw} \left[w \sin wt - e^{-\frac{Rt}{2L}} \left(\frac{R^2}{4L^2 p} + p \right) \sin pt \right] = \frac{E}{Rw} \left[w \sin wt - e^{-\frac{Rt}{2L}} \cdot \frac{R^2 + 4L^2 p^2}{4L^2 p} \cdot \sin pt \right]$$

$$= \frac{E}{R} \left[\sin wt - e^{-\frac{Rt}{2L}} \cdot \frac{1}{LCpw} \sin pt \right] \quad \text{since } \frac{R^2 + 4L^2 p^2}{4L^2} = \frac{1}{LC}$$

$$= \frac{E}{R} \left[\sin wt - e^{-\frac{Rt}{2L}} \cdot \frac{\sqrt{LC}}{LCp} \sin pt \right] \quad \text{since } w^2 = \frac{1}{LC}$$

$$\text{or } i = \frac{E}{R} \left[\sin wt - \frac{1}{p\sqrt{LC}} \cdot e^{-\frac{Rt}{2L}} \cdot \sin pt \right].$$

Hence shown.

237. In an L - C - R circuit, the charge q on a plate of a condenser is given by $L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E$

$\sin pt$. The circuit is tuned to resonance so that $p^2 = \frac{1}{LC}$. If initially the current i and the charge

q be zero, show that, for small values of $\frac{R}{L}$, the current in the circuit at time t is given by $\left\{ \frac{Et}{2L} \right\} \sin$

pt .

(M.D.U., 2005)

Sol. The given equation is $\left(LD^2 + RD + \frac{1}{C} \right) q = E \sin pt$; $D \equiv \frac{d}{dt}$...(1)

Its A.E. is $LD^2 + RD + \frac{1}{C} = 0$

$$\begin{aligned}
 \Rightarrow D &= \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \pm \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \\
 &= -\frac{R}{2L} \pm \frac{1}{2} \sqrt{-\frac{4}{LC}} \quad \left\{ \text{Ignoring the term } \frac{R^2}{L^2} \right\} \\
 &= -\frac{R}{2L} \pm \frac{i}{\sqrt{LC}} = -\frac{R}{2L} \pm pi
 \end{aligned}$$

Since, $p^2 = \frac{1}{LC}$ (given)

$$\therefore \text{C.F.} = e^{-\frac{Rt}{2L}} (C_1 \cos pt + C_2 \sin pt) = \left(1 - \frac{Rt}{2L}\right) (C_1 \cos pt + C_2 \sin pt)$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{LD^2 + RD + \frac{1}{C}} (E \sin pt) \\
 &= E \cdot \frac{1}{-Lp^2 + RD + \frac{1}{C}} \cdot \sin pt = E \cdot \frac{1}{RD} \cdot \sin pt \quad \left(\because p^2 = \frac{1}{LC} \right) \\
 &= -\frac{E}{Rp} \cos pt
 \end{aligned}$$

Complete solution of equation (1) is

$$q = \left(1 - \frac{Rt}{2L}\right) (C_1 \cos pt + C_2 \sin pt) - \frac{E}{Rp} \cos pt \quad \dots(2)$$

$$\therefore i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right) (-p C_1 \sin pt + p C_2 \cos pt) - \frac{R}{2L} (C_1 \cos pt + C_2 \sin pt) + \frac{E}{R} \sin pt \quad \dots(3)$$

When $t = 0$, $q = 0$ and $i = 0$, then

From equation (2), $0 = C_1 - \frac{E}{Rp} \Rightarrow C_1 = \frac{E}{Rp}$

and from (3), $0 = C_2 p - \frac{R}{2L} C_1 \Rightarrow C_2 = \frac{E}{2Lp^2}$

Putting the values of C_1 and C_2 in (3), we get

$$\begin{aligned}
 i &= \left(1 - \frac{Rt}{2L}\right) \left(-\frac{E}{R} \sin pt + \frac{E}{2Lp} \cos pt\right) - \frac{R}{2L} \left(\frac{E}{Rp} \cos pt + \frac{E}{2Lp^2} \sin pt\right) + \frac{E}{R} \sin pt \\
 &= \frac{Et}{2L} \sin pt - \frac{ERt}{4pL^2} \cos pt - \frac{ER}{4L^2p^2} \sin pt = \frac{Et}{2L} \sin pt; \quad \frac{R}{L} \text{ being small.}
 \end{aligned}$$

238. The voltage V and the current i at a distance x from the sending end of the transmission line satisfy the equations $-\frac{dV}{dx} = Ri$ and $-\frac{di}{dx} = GV$, where R and G are constants. If $V = V_0$ at the

sending end ($x = 0$) and $V = 0$ at the receiving end ($x = l$); show that $V = V_0 \left[\frac{\sinh n(l-x)}{\sinh nl} \right]$, where

$$n^2 = RG.$$

Sol. $-\frac{di}{dx} = GV$

$$\Rightarrow -\frac{d}{dx} \left(-\frac{1}{R} \frac{dV}{dx} \right) = GV \Rightarrow \frac{d^2V}{dx^2} - RGV = 0$$

$$\Rightarrow (D^2 - RG)V = 0, \text{ where } D \equiv \frac{d}{dx}$$

Auxiliary equation is

$$D^2 - RG = 0 \quad \text{or} \quad D^2 - n^2 = 0 \quad [\because n^2 = RG \text{ (given)}]$$

$$\Rightarrow D = \pm n$$

$$\therefore \text{C.F.} = C_1 e^{nx} + C_2 e^{-nx}$$

$$\text{P.I.} = 0$$

$$\therefore \text{Solution is} \quad V = C_1 e^{nx} + C_2 e^{-nx} \quad \dots(1)$$

$$\text{It is given that at } x = 0, V = V_0 \quad \therefore V_0 = C_1 + C_2 \quad \dots(2)$$

$$\text{at } x = l, V = 0, \quad 0 = C_1 e^{nl} + C_2 e^{-nl} \quad \dots(3)$$

Solving equations (2) and (3), we have

$$C_1 = \frac{V_0}{1 - e^{2nl}}, \quad C_2 = \frac{-V_0 \cdot e^{2nl}}{1 - e^{2nl}}$$

$$\begin{aligned} \text{From equation (1),} \quad V &= \frac{V_0}{1 - e^{2nl}} [e^{nx} - e^{2nl-nx}] = V_0 \cdot \frac{e^{nx} - e^{2nl-nx}}{1 - e^{2nl}} = V_0 \cdot \frac{e^{2nl-nx} - e^{nx}}{e^{2nl} - 1} \\ &= V_0 \cdot \frac{e^{nl-nx} - e^{nx-nl}}{e^{nl} - e^{-nl}} \quad (\text{On dividing the numerator and denom. by } e^{nl}) \\ &= V_0 \cdot \frac{e^{nl-nx} - e^{-(nl-nx)}}{e^{nl} - e^{-nl}} \end{aligned}$$

$$\text{or} \quad V = V_0 \cdot \left\{ \frac{\sinh n(l-x)}{\sinh nl} \right\}. \quad \text{Hence shown.}$$

239. When a particle is said to execute a simple harmonic motion ? What is amplitude of the motion ?

Sol. A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

Let O be the fixed point in the line A'A. Let P be the position of the particle at any time t , where $OP = x$. Since the acceleration is always directed towards O, i.e., the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu^2 x.$$

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \quad \text{or} \quad (D^2 + \mu^2)x = 0, \text{ where } D \equiv \frac{d}{dt} \quad \dots(1)$$

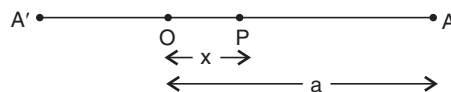
This is a linear differential equation with constant coefficients.

Its A.E. is $D^2 + \mu^2 = 0$ so that $D = \pm i\mu$

$$\therefore \text{Solution of (1) is} \quad x = C_1 \cos \mu t + C_2 \sin \mu t \quad \dots(2)$$

$$\text{Velocity of particle at P} = \frac{dx}{dt} = -C_1 \mu \sin \mu t + C_2 \mu \cos \mu t \quad \dots(3)$$

If the particle starts from rest at A, where $OA = a$ then from equation (2),



$$(\text{at } t = 0, x = a) ; C_1 = a$$

$$\text{and from (3), } \left(\text{at } t = 0, \frac{dx}{dt} = 0 \right) ; C_2 = 0$$

$$\therefore x = a \cos \mu t \quad \dots(4)$$

$$\text{and } \frac{dx}{dt} = -a\mu \sin \mu t \quad \dots(5)$$

$$\frac{dx}{dt} = -a\mu \sqrt{1 - \cos^2 \mu t} = -a\mu \sqrt{1 - \frac{x^2}{a^2}} = -\mu \sqrt{a^2 - x^2} \quad \dots(6)$$

Equation (4) gives the displacement of the particle from the fixed point O at any time t . Equation (6) gives the velocity of the particle at any time t , when its displacement from the fixed point O is x . Equation (6) also shows that the velocity is directed towards O and decreases as x increases.

Equations (4) and (5) remain unchanged when t is replaced by $t + \frac{2\pi}{\mu}$ i.e., when t is increased by

$\frac{2\pi}{\mu}$ showing thereby that the particle occupies the same position and has the same velocity after

a time $\frac{2\pi}{\mu}$. The quantity $\frac{2\pi}{\mu}$, usually denoted by T, is called the time of complete oscillation. The

point O is called the centre of motion or the mean position. **The maximum distance 'a'** which the particle covers on either side of the mean position is called the **amplitude** of the motion. The number of complete oscillations per second is called the frequency of motion. If n is the frequency,

$$\text{then } n = \frac{1}{T} = \frac{\mu}{2\pi}.$$

- 240.** The distances of a point moving with Simple Harmonic Motion (S.H.M.) at the end of 3 successive seconds from its mean position are x_1, x_2, x_3 respectively. Show that the time of a complete oscillation is $2\pi/\cos^{-1} \left(\frac{x_1 + x_3}{2x_2} \right)$.

Sol. Let the moving point be at distances x_1, x_2, x_3 from the mean position at the end of $t, t + 1, t + 2$ seconds respectively.

Using $x = a \cos \mu t$

$$\text{We have } x_1 = a \cos \mu t \quad \dots(1) \quad x_2 = a \cos \mu(t + 1) \quad \dots(2) \quad x_3 = a \cos \mu(t + 2) \quad \dots(3)$$

Adding (1) and (3), we get

$$\begin{aligned} x_1 + x_3 &= a [\cos \mu(t + 2) + \cos \mu t] \\ &= a \left[2 \cos \frac{\mu(t + 2) + \mu t}{2} \cos \frac{\mu(t + 2) - \mu t}{2} \right] \\ &= 2a \cos \mu(t + 1) \cos \mu = 2x_2 \cos \mu \quad \text{[Using (2)]} \end{aligned}$$

$$\Rightarrow \mu = \cos^{-1} \left(\frac{x_1 + x_3}{2x_2} \right)$$

$$\text{Hence the time of a complete oscillation} = \frac{2\pi}{\mu} \quad \text{or} \quad 2\pi/\cos^{-1} \left(\frac{x_1 + x_3}{2x_2} \right).$$

- 241.** A particle moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from

the centre are x_1 and x_2 respectively. Show that the period of motion is $2\pi \sqrt{\frac{x_1^2 - x_2^2}{v_1^2 - v_2^2}}$.

Sol. The velocity v of the particle, when it is at a distance x from the mean position is given by $v^2 = \mu^2(a^2 - x^2)$, where a is the amplitude.

$$\therefore v_1^2 = \mu^2(a^2 - x_1^2) \quad \dots(1)$$

$$\text{and } v_2^2 = \mu^2(a^2 - x_2^2) \quad \dots(2)$$

Subtracting (1) from (2), we get

$$v_2^2 - v_1^2 = \mu^2(x_1^2 - x_2^2) \quad \text{or} \quad \mu^2 = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$$

$$\text{Periodic time} \quad T = \frac{2\pi}{\mu} = 2\pi \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}.$$

- 242.** A particle is executing S.H.M. with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.

Sol. Here $a = 20$ cm $T = 4$ seconds

$$\text{Since } T = \frac{2\pi}{\mu} \quad \therefore \mu = \frac{\pi}{2}$$

Let t_1 and t_2 seconds be the times when the particle is at distances 15 cm and 5 cm respectively from the centre of force.

Using $x = a \cos \mu t$, we have

$$15 = 20 \cos \frac{\pi}{2} t_1 \Rightarrow t_1 = \frac{2}{\pi} \cos^{-1} \frac{3}{4}$$

$$\text{and } 5 = 20 \cos \frac{\pi}{2} t_2 \Rightarrow t_2 = \frac{2}{\pi} \cos^{-1} \frac{1}{4}$$

$$\text{Required time} = t_2 - t_1 = \frac{2}{\pi} \left(\cos^{-1} \frac{1}{4} - \cos^{-1} \frac{3}{4} \right) = 0.38 \text{ sec.}$$

- 243.** A point moves in a straight line towards a centre of force $\frac{\mu}{(\text{distance})^3}$, starting from rest at a distance 'a' from the centre of force. Show that the time of reaching a point distant 'b' from the centre of force is $\frac{a}{\sqrt{\mu}} \sqrt{a^2 - b^2}$, and that its velocity then is $\frac{\sqrt{\mu}}{ab} \cdot \sqrt{a^2 - b^2}$.

Sol. O is the centre of force and the point starts from P, where $OP = a$. We have to find out the time of reaching from P to Q and velocity at Q, where $OQ = b$

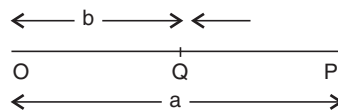
$$\text{Given } \frac{d^2x}{dt^2} = -\frac{\mu}{x^3} \quad \dots(1)$$

Multiplying both sides of (1) by $2 \frac{dx}{dt}$, we get

$$2 \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} = -\frac{\mu}{x^3} \cdot 2 \frac{dx}{dt}$$

Integrating both sides, we get

$$\left(\frac{dx}{dt} \right)^2 = \int \left(-\frac{\mu}{x^3} \right) \cdot 2 \frac{dx}{dt} \cdot dt + C \Rightarrow \left(\frac{dx}{dt} \right)^2 = \frac{\mu}{x^2} + C \quad \dots(2)$$



Also, velocity $\frac{dx}{dt} = 0$, when $x = a$ (given)

$\therefore$ By (2), $C = -\mu a^{-2}$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu(x^{-2} - a^{-2}) = \mu \left(\frac{a^2 - x^2}{a^2 x^2}\right) \quad \dots(3)$$

$\therefore$ Velocity towards O at Q i.e., at $x = b$ is

$$\left(\frac{dx}{dt}\right)_{x=b} = \frac{\sqrt{\mu}}{ab} \sqrt{a^2 - b^2} \quad [\text{From (3)}]$$

Again, separating the variables in equation (3)

$$\frac{-ax}{\sqrt{a^2 - x^2}} \cdot dx = \sqrt{\mu} \cdot dt \text{ where } (-) \text{ sign is taken because point is moving towards O.}$$

Integrating both sides, we get

$$\int \frac{-ax}{\sqrt{a^2 - x^2}} \cdot dx = \int \sqrt{\mu} \cdot dt + A \Rightarrow a \sqrt{a^2 - x^2} = \sqrt{\mu} \cdot t + A \quad \dots(4)$$

Now, $t = 0$ when $x = a$ (given)

$$\therefore \text{ From (4), } A = 0. \therefore \text{ At Q (when } x = b), t = \frac{a\sqrt{a^2 - b^2}}{\sqrt{\mu}}$$

Hence velocity towards O at Q (at $x = b$) is $\frac{\sqrt{\mu}}{ab} \cdot \sqrt{a^2 - b^2}$ and time of reaching a point distant 'b'

from the centre of force is $\frac{a}{\sqrt{\mu}} \cdot \sqrt{a^2 - b^2}$.

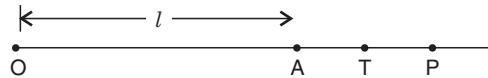
- 244.** In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to a weight mg . Also find the displacements of the particle when initially $S = S_0$, $v = 0$.

Sol. Let OA = l be the elastic horizontal string with the end O fixed and a particle of mass m attached at A.

Let P be the position of the particle at any time t , where OP = S , so that the elongation

$$AP = S - l.$$

Now, for the elongation e , tension = mg



$$\therefore \text{ For the elongation } (S - l), \text{ tension} = \frac{mg(S - l)}{e}$$

Since tension is the only horizontal force acting on the particle, its equation of motion is

$$m \frac{d^2 S}{dt^2} = -T \text{ or } m \frac{d^2 S}{dt^2} = -\frac{mg(S - l)}{e} \quad \dots(1)$$

From (1), we have $\frac{d^2 S}{dt^2} + \frac{g}{e} \cdot S = \frac{gl}{e}$

Its A.E. is $m^2 + \frac{g}{e} = 0 \Rightarrow m = \pm i \sqrt{\frac{g}{e}}$

$\therefore$ C.F. = $C_1 \cos \left(\sqrt{\frac{g}{e}} t \right) + C_2 \sin \left(\sqrt{\frac{g}{e}} t \right)$

and P.I. = $\frac{1}{D^2 + \frac{g}{e}} \cdot \frac{gl}{e}$ or P.I. = $\frac{gl}{e} \cdot \frac{1}{D^2 + \frac{g}{e}} \cdot e^{0t} = \frac{gl}{e} \cdot \frac{1}{0 + \frac{g}{e}} \cdot e^{0t} = l$

$\therefore$ The complete solution of (1) is

$$S = C_1 \cos \left(\sqrt{\frac{g}{e}} t \right) + C_2 \sin \left(\sqrt{\frac{g}{e}} t \right) + l \quad \dots(2)$$

When $t = 0$, $S = S_0$, so that from (2),

$$S_0 = C_1 + l \quad \text{or} \quad C_1 = S_0 - l$$

Also from (2) $\frac{dS}{dt} = -C_1 \sqrt{\frac{g}{e}} \sin \left(\sqrt{\frac{g}{e}} t \right) + C_2 \sqrt{\frac{g}{e}} \cos \left(\sqrt{\frac{g}{e}} t \right)$

When $t = 0$, $\frac{dS}{dt} = v = 0 \quad \therefore C_2 = 0$

$\therefore$ From (2), we get $S = (S_0 - l) \cos \left(\sqrt{\frac{g}{e}} t \right) + l$.

- 245.** A particle moves with S.H.M. in a straight line. In the first second, starting from rest, it travels a distance 'a' and in the next second, it travels a distance 'b' in the same direction. Prove that the amplitude of motion is $\frac{2a^2}{3a-b}$.

Sol. Let A be the amplitude of motion.

According to the problem $A - a = A \cos \mu \quad \dots(1)$

$$A - a - b = A \cos 2\mu \quad \dots(2)$$

Now, $A - a - b = A (2 \cos^2 \mu - 1)$

$$= A \left[2 \cdot \left(\frac{A-a}{A} \right)^2 - 1 \right] = \frac{2}{A} (A^2 + a^2 - 2Aa) - A \quad \text{or} \quad A - a - b = 2A + \frac{2a^2}{A} - 4a - A$$

$$3a - b = \frac{2a^2}{A} \Rightarrow A = \frac{2a^2}{3a-b}$$

Hence proved.

- 246.** A particle moves with S.H.M. in a straight line under the action of a force which is proportional to the distance of the particle from $x = 0$. If it starts at $x = 5$ cm with a velocity of 10 cm/sec and it reaches an extreme position $x = 10$ cm, at what speed does it pass through the origin?

Sol. $v^2 = \mu(a^2 - x^2) \quad \dots(1)$

Given, $x = 5$, $v = 10$

$\therefore (10)^2 = \mu(a^2 - 25) \quad \dots(2)$

Dividing (1) by (2), we get

$$\frac{v^2}{100} = \frac{a^2 - x^2}{a^2 - 25} \Rightarrow \frac{v^2}{100} = \frac{100 - x^2}{100 - 25} \quad | \text{ Here } a = 10$$

$$\Rightarrow v^2 = \frac{4}{3} (100 - x^2)$$

$$\text{At } x = 0, \quad v^2 = \frac{400}{3} = 133.33$$

$$\therefore v = \sqrt{133.33} = 11.546 \text{ cm/sec.}$$

Hence the required speed is 11.546 cm/sec.

- 247.** Show that if the displacement of a particle in a straight line is expressed by the equation $x = a \cos \mu t + b \sin \mu t$, it describes a S.H.M. whose amplitude is $\sqrt{a^2 + b^2}$ and time period is $\frac{2\pi}{\mu}$.

$$\text{Sol.} \quad x = a \cos \mu t + b \sin \mu t \quad \dots(1)$$

$$\therefore \frac{dx}{dt} = -a \mu \sin \mu t + b \mu \cos \mu t \quad \dots(2)$$

$$\frac{d^2x}{dt^2} = -a \mu^2 \cos \mu t - b \mu^2 \sin \mu t = -\mu^2 [a \cos \mu t + b \sin \mu t]$$

$$\text{or} \quad \frac{d^2x}{dt^2} = -\mu^2 \cdot x$$

which represents simple harmonic motion with centre at origin.

$$\text{Time period, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\mu}$$

Also, amplitude is the value of x , when $\frac{dx}{dt} = 0$

$$\text{From (2), } 0 = -a \mu \sin \mu t + b \mu \cos \mu t \Rightarrow \tan \mu t = \frac{b}{a}$$

$$\therefore \sin \mu t = \frac{b}{\sqrt{a^2 + b^2}} \text{ and } \cos \mu t = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\text{From (1), } x = a \cdot \frac{a}{\sqrt{a^2 + b^2}} + b \cdot \frac{b}{\sqrt{a^2 + b^2}} \Rightarrow x = \sqrt{a^2 + b^2} \quad \text{Hence the result.}$$

- 248.** In a system, the amplitude of motion is 5 metres and the period is 4 seconds. Find the time required by the particle in passing between the points which are at distances of 4 metres and 2 metres from the centre of force and are on same side of it. Also find the velocities at these points.

Sol. Equation of Simple Harmonic Motion is

$$\frac{d^2x}{dt^2} = -\mu^2 \cdot x \quad \dots(1)$$

$$\text{Time period} \quad T = \frac{2\pi}{\mu} = 4 \quad (\text{given})$$

$$\therefore \mu = \frac{\pi}{2}$$

$$\text{Solution to (1) is, } x = a \cos \mu t = 5 \cos \frac{\pi}{2} t \quad \dots(2) \quad (\because a = 5 \text{ m})$$

Let t_1 sec and t_2 sec be the times when the particle is at a distance of 4 metres and 2 metres respectively from the centre of force.

Then from equation (2) $4 = 5 \cos \left(\frac{\pi}{2} t_1 \right) \Rightarrow t_1 = \frac{2}{\pi} \cos^{-1} \left(\frac{4}{5} \right)$

$$2 = 5 \cos \left(\frac{\pi}{2} t_2 \right) \Rightarrow t_2 = \frac{2}{\pi} \cos^{-1} \left(\frac{2}{5} \right)$$

$\therefore$ Time required in passing through these points

$$= t_2 - t_1 = \frac{2}{\pi} \left[\cos^{-1} \left(\frac{2}{5} \right) - \cos^{-1} \left(\frac{4}{5} \right) \right] = 0.33 \text{ secs.}$$

To find the velocities at $x = 4$ m and $x = 2$ m

Differentiating equation (2) w.r.t. ' t ', we get

$$\frac{dx}{dt} = -\frac{5\pi}{2} \sin \frac{\pi t}{2} = -\frac{5\pi}{2} \sqrt{1 - \cos^2 \frac{\pi t}{2}} = -\frac{5\pi}{2} \sqrt{1 - \frac{x^2}{25}} = -\frac{\pi}{2} \sqrt{25 - x^2}$$

When, $x = 4$ m, $v = -\frac{\pi}{2} \sqrt{25 - 16} = -\frac{3\pi}{2}$ m/sec.

When, $x = 2$ m, $v = -\frac{\pi}{2} \sqrt{21}$ m/sec.

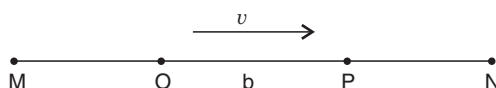
Negative sign indicates that it is directed towards centre of force.

- 249.** A particle is performing a S.H.M. of period T about a centre O and it passes through a point P , where $OP = b$ with velocity v in the direction OP . Prove that the time which elapses before it

returns to P is $\frac{T}{\pi} \tan^{-1} \left(\frac{vT}{2\pi b} \right)$.

Sol. Let the amplitude be a .

Required time = time taken from P to N + time taken from N to P
 $= 2$ (time taken from N to P) ...(1)



For the motion from N to P , we have

$$\frac{dx}{dt} = -\mu \sqrt{a^2 - x^2} \Rightarrow dt = -\frac{dx}{\mu \sqrt{a^2 - x^2}} \quad \dots(2)$$

Let t_1 be the time taken from N to P . Then at N , $t = 0$, $x = a$ and at P , $t = t_1$, $x = b$

$\therefore$ Integrating (2), we get

$$\int_0^{t_1} dt = \frac{1}{\mu} \int_a^b \frac{-dx}{\sqrt{a^2 - x^2}}$$

$$t_1 = \frac{1}{\mu} \left[\cos^{-1} \left(\frac{x}{a} \right) \right]_a^b = \frac{1}{\mu} \cos^{-1} \left(\frac{b}{a} \right)$$

$\therefore$ From equation (1),

Required time $= \frac{2}{\mu} \cos^{-1} \left(\frac{b}{a} \right) = \frac{2}{\mu} \tan^{-1} \left(\frac{\sqrt{a^2 - b^2}}{b} \right)$

Since $\frac{dx}{dt} = -\mu \sqrt{a^2 - x^2}$

At P, $x = b, \frac{dx}{dt} = v \quad \therefore v = -\mu \sqrt{a^2 - b^2}$

$\Rightarrow$ Required time $= \frac{2}{\mu} \tan^{-1} \left(\frac{v}{b\mu} \right),$ (Ignore (-)ve sign of v)

Since $T = \frac{2\pi}{\mu} \text{ or } \mu = \frac{2\pi}{T}$

Required time $= \frac{T}{\pi} \tan^{-1} \left(\frac{vT}{2\pi b} \right).$

Hence proved.

- 250.** Discuss free oscillations, damped free oscillations, forced oscillatory motions (without damping and with damping) for a mechanical system when a spring is suspended vertically from a fixed support and a body of mass m be suspended from the other end. Derive equations of motion of the body in each case.

Sol. (a) Free oscillations

Consider a spring OA suspended vertically from a fixed support at O.

Let a body of mass m be suspended from the end A, the mass of the body being so large in comparison with the mass of the spring that the latter may be neglected. Let $e (= AB)$ be the elongation produced by the mass m hanging in equilibrium, then B is called the position of **static equilibrium** and e is called the **static extension**.

Let k be the stiffness of the spring, i.e., the restoring force per unit stretch of the spring due to elasticity.

For equilibrium at B, $mg = T = k, e$... (1)

Let the mass be displaced through a further distance x from the equilibrium position. The acceleration of the mass m at this position is $\frac{d^2x}{dt^2}$

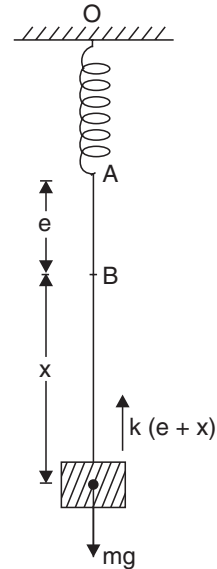
and the forces acting upon it are its weight mg downwards and the restoring force $k(e + x)$ upwards.

$\therefore$ The equation of motion of mass m is

$$m \frac{d^2x}{dt^2} = mg - k(e + x) = -kx \quad (\because ke = mg)$$

or $\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$

Writing $\omega^2 = \frac{k}{m}$, it becomes $\frac{d^2x}{dt^2} + \omega^2 x = 0$... (2)



(b) Damped free oscillations

If the motion of the mass m be subjected to an additional force of resistance, proportional to the instantaneous velocity of the mass, say $\lambda \frac{dx}{dt}$, produced by a damper, the oscillations are said to be damped.

The equation of motion of the mass m is

$$m \frac{d^2x}{dt^2} = mg - k(e + x) - \lambda \frac{dx}{dt} = -kx - \lambda \frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{\lambda}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

Writing $\frac{\lambda}{m} = 2p$ and $\frac{k}{m} = w^2$, it becomes $\frac{d^2x}{dt^2} + 2p \frac{dx}{dt} + w^2x = 0$... (1)

Equation (1) is a linear differential equation with constant coefficients.

Its auxiliary equation is $D^2 + 2pD + w^2 = 0 \Rightarrow D = -p \pm \sqrt{p^2 - w^2}$

Thus the general solution of equation (1) depends upon the relative values of p and w .

Case I. When $p > w$, roots of A.E. are real and distinct. The solution of (1) is

$$x = C_1 \cdot e^{(-p + \sqrt{p^2 - w^2})t} + C_2 \cdot e^{(-p - \sqrt{p^2 - w^2})t}$$

or

$$x = e^{-pt} (C_1 e^{\sqrt{p^2 - w^2} t} + C_2 \cdot e^{-\sqrt{p^2 - w^2} t})$$

where C_1, C_2 are arbitrary constants to be determined from the initial conditions of the given problem. The solution shows that x is always positive and hence the motion is non-oscillatory. Further, as $t \rightarrow \infty, x \rightarrow 0$; the restoring force in this case is so great that the non-oscillatory motion is called **over-damped** or dead-beat motion.

Case II. When $p = w$, roots of the auxiliary equation are equal, each being equal to $-p$.

The solution of (1) is $x = (C_1 + C_2 t) e^{-pt}$

Here also x is always positive and the motion is non-oscillatory. Further as $t \rightarrow \infty, x \rightarrow 0$. The motion of the mass in this case is called critically damped motion, because it separates the non-oscillatory motion of case I from the oscillatory motion of case III.

Case III. When $p < w$, the roots of the auxiliary equation are imaginary i.e.,

$$D = -p \pm i \sqrt{w^2 - p^2}$$

The solution of equation (1) is $x = e^{-pt} (C_1 \cos \sqrt{w^2 - p^2} t + C_2 \sin \sqrt{w^2 - p^2} t)$

where C_1 and C_2 are constants.

The variation of x is now oscillatory due to the presence of the trigonometric factor. The periodic

time of the oscillation is $T = \frac{2\pi}{\sqrt{w^2 - p^2}}$ which is greater than the periodic time in case of freed

oscillations (without damping) which is $\frac{2\pi}{w}$. Thus the effect of damping is to increase the period of oscillation and the motion ultimately dies away. Such a motion is called **damped oscillatory motion**.

(c) Forced oscillations (without damping)

If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called **forced oscillatory motion**.

Consider the motion of the mass m as in (a) with an external periodic force $Q \cos nt$ impressed upon the system. This force is applied to the support of the spring so that now the support is not steady :

The equation of motion of mass m

$$m \frac{d^2x}{dt^2} = mg - k(e + x) + Q \cos nt = -kx + Q \cos nt$$

or $\frac{d^2x}{dt^2} + \frac{k}{m} x = \frac{Q}{m} \cos nt$

Writing $w^2 = \frac{k}{m}$ and $E = \frac{Q}{m}$, it becomes $\frac{d^2x}{dt^2} + w^2x = E \cos nt$... (1)

This is a linear differential equation with constant coefficients.

Its A.E. is $D^2 + w^2 = 0 \Rightarrow D = \pm iw$
 C.F. = $C_1 \cos wt + C_2 \sin wt$

and $P.I. = E \cdot \frac{1}{D^2 + w^2} \cos nt$

Evaluation of P.I. depends upon the relative values of w and n . Thus we have to consider two cases :

$$w \neq n \text{ and } w = n.$$

Case I. When $w \neq n$

$$P.I. = \frac{E}{w^2 - n^2} \cos nt$$

$\therefore$ Complete solution of (1) is

$$x = C_1 \cos wt + C_2 \sin wt + \frac{E}{w^2 - n^2} \cos nt$$

which may be written as $x = A \cos (wt + B) + \frac{E}{w^2 - n^2} \cdot \cos nt$... (2)

This shows that the variation of x consists of two oscillatory motions : the one due to the C.F. gives free oscillations of period $\frac{2\pi}{w}$ and the second due to the P.I. gives forced oscillations of

period $\frac{2\pi}{n}$. Also, if the frequency of free oscillations is very high *i.e.*, w is large, then the amplitude $\frac{E}{w^2 - n^2}$ of the forced oscillations is small.

Case II. When $w = n$

$$P.I. = E.t. \cdot \frac{1}{2D} \cos wt = \frac{E.t}{2} \int \cos wt \cdot dt = \frac{E.t}{2w} \sin wt$$

$\therefore$ The complete solution of (1) is $x = C_1 \cos wt + C_2 \sin wt + \frac{E.t}{2w} \sin wt$

or $x = C_1 \cos wt + \left(C_2 + \frac{E.t}{2w} \right) \sin wt$... (3)

Let $C_1 = r \sin \phi$ and $C_2 + \frac{E.t}{2w} = r \cos \phi$

then $r = \sqrt{C_1^2 + \left(C_2 + \frac{E.t}{2w} \right)^2}$ and $\phi = \tan^{-1} \left\{ \frac{C_1}{C_2 + \frac{E.t}{2w}} \right\}$

Equation (3) reduces to $x = r \sin (wt + \phi)$ which shows that motion is oscillatory with period $\frac{2\pi}{w}$

and amplitude = $\sqrt{C_1^2 + \left(C_2 + \frac{E.t}{2w} \right)^2}$.

Clearly, amplitude increases with time, so that after a long time the amplitude becomes abnormally large. This phenomenon in which the impressed frequency due to external periodic force becomes equal to the natural frequency of the system is known as resonance. A practical instance of the case of resonance is the soldiers breaking step while marching over a bridge, for the simple reason that, if by chance the steps be in rhythm with the natural frequency of the bridge, it may collapse due to resonance.

(d) **Forced oscillations (with damping).** In the above case (c), if in addition, there is a damping force, which is proportional to the instantaneous velocity of the mass, say $\lambda \frac{dx}{dt}$, then the

equation of motion of mass m is $m \frac{d^2x}{dt^2} = mg - k(e + x) - \lambda \frac{dx}{dt} + Q \cos nt$

Since $mg = ke$, it becomes

$$m \frac{d^2x}{dt^2} = -kx - \lambda \frac{dx}{dt} + Q \cos nt$$

$$\text{or } \frac{d^2x}{dt^2} + \frac{\lambda}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{Q}{m} \cos nt$$

Writing $\frac{\lambda}{m} = 2p$, $\frac{k}{m} = w^2$ and $\frac{Q}{m} = E$, it becomes

$$\frac{d^2x}{dt^2} + 2p \frac{dx}{dt} + w^2 x = E \cos nt \quad \dots(1)$$

This is a linear differential equation with constant coefficients.

$$\text{Its A.E. is } D^2 + 2pD + w^2 = 0 \Rightarrow D = -p \pm \sqrt{p^2 - w^2}$$

$$\therefore \text{C.F.} = e^{-pt} (C_1 e^{\sqrt{p^2 - w^2} t} + C_2 e^{-\sqrt{p^2 - w^2} t})$$

$$\begin{aligned} \text{P.I.} &= E \cdot \frac{1}{D^2 + 2pD + w^2} \cos nt = E \cdot \frac{1}{-n^2 + 2pD + w^2} \cdot \cos nt \\ &= E \cdot \frac{1}{(w^2 - n^2) + 2pD} \cos nt \end{aligned}$$

$$\text{or } \text{P.I.} = E \cdot \frac{(w^2 - n^2) - 2pD}{(w^2 - n^2)^2 - 4p^2 D^2} \cos nt = E \cdot \frac{(w^2 - n^2) \cos nt + 2pn \sin nt}{(w^2 - n^2)^2 + 4p^2 n^2}$$

$$\begin{aligned} \text{Now, let } w^2 - n^2 &= r \cos \phi \\ 2pn &= r \sin \phi \end{aligned}$$

$$\text{so that } r^2 = (w^2 - n^2)^2 + 4p^2 n^2 \quad \text{and} \quad \phi = \tan^{-1} \left(\frac{2pn}{w^2 - n^2} \right)$$

$$\therefore \text{P.I.} = E \cdot \frac{r \cos (nt - \phi)}{r^2} = \frac{E}{r} \cdot \cos (nt - \phi)$$

$\therefore$ The complete solution of equation (1) is

$$x = e^{-pt} (C_1 e^{\sqrt{p^2 - w^2} t} + C_2 e^{-\sqrt{p^2 - w^2} t}) + \frac{E}{r} \cos (nt - \phi)$$

The C.F. represents free oscillations of the system which die out as $t \rightarrow \infty$ due to the presence of the factor e^{-pt} .

The P.I. represents the forced oscillations of the system having a constant amplitude

$$= \frac{E}{\sqrt{(w^2 - n^2)^2 + 4p^2 n^2}}$$

and the period = $\frac{2\pi}{w}$ which is the same as that of the impressed force.

Thus as t increases, the free oscillations (given by the C.F.) die out while the forced oscillations (given by the P.I.) persist giving the steady state of motion.

- 251.** A body weighing 4.9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.

Sol. Let O be the fixed end and A the free end of the spring.

Let k be the stiffness of the spring (or the spring constant).

Since a pull of 10 kg stretches the spring by 0.05 metres,

$$\therefore 10 = k \times 0.05 \Rightarrow k = 200 \text{ kg/m.}$$

Let e (= AB) be the elongation produced by the body weighing 4.9 kg hanging in equilibrium, then $4.9 = ke = 200e$

$$\therefore e = \frac{4.9}{200} = 0.0245 \text{ m}$$

Now the body is pulled down to C, where BC = 0.06 m.

After t sec of its release from C, let the body be at P, where BP = x metres.

The forces acting on the body are

(i) its weight $W = 4.9$ kg downwards.

(ii) the restoring force $T = k(e + x)$ or $T = 200(0.0245 + x)$ kg upwards.

The equation of motion of the body is

$$m \frac{d^2x}{dt^2} = W - T$$

$$\text{or} \quad \frac{4.9}{g} \frac{d^2x}{dt^2} = 4.9 - 200(0.0245 + x)$$

$$\text{or} \quad \frac{4.9}{9.8} \frac{d^2x}{dt^2} = -200x$$

$$\text{or} \quad \frac{d^2x}{dt^2} + 400x = 0$$

Its A.E. is $D^2 + 400 = 0 \Rightarrow D = \pm 20i$

$$\therefore \text{Its solution is } x = C_1 \cos 20t + C_2 \sin 20t \quad \dots(1)$$

Initially, when $t = 0$, $x = 0.06$

$\therefore$ From equation (1), $C_1 = 0.06$

$$\text{Also, } \frac{dx}{dt} = -20C_1 \sin 20t + 20C_2 \cos 20t$$

$$\text{Initially, when } t = 0, \frac{dx}{dt} = 0$$

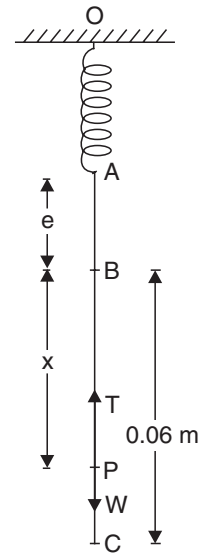
$$\therefore C_2 = 0$$

Substituting the values of C_1 and C_2 in equation (1), we get

$x = 0.06 \cos 20t$ which gives the displacement of the body from its equilibrium position at time t .

$$\begin{aligned} \text{Maximum velocity} &= w \text{ (amplitude)} \\ &= 20 \times 0.06 = 1.2 \text{ m/sec} \end{aligned}$$

$$\text{Period of oscillation} = \frac{2\pi}{w} = \frac{2\pi}{20} = \frac{\pi}{10} \text{ or } 0.314 \text{ sec.}$$



- 252.** A spring for which the spring constant $k = 700 \text{ Nm}^{-1}$ hangs in a vertical position with its upper end fixed to a support. A mass of 20 kg is attached to the lower end and system brought to rest. Find the position of the mass at time t , if a force $70 \sin 2t \text{ N}$ is applied to the support.

Sol. Equation of motion is

$$m \frac{d^2x}{dt^2} = -kx + 70 \sin 2t$$

$$\Rightarrow 20 \frac{d^2x}{dt^2} = -700x + 70 \sin 2t \Rightarrow \frac{d^2x}{dt^2} + 35x = \frac{7}{2} \sin 2t$$

Auxiliary equation is $D^2 + 35 = 0 \Rightarrow D = \pm i\sqrt{35}$

$$\therefore \text{C.F.} = C_1 \cos \sqrt{35} t + C_2 \sin \sqrt{35} t$$

$$\text{P.I.} = \frac{1}{D^2 + 35} \left(\frac{7}{2} \sin 2t \right) = \frac{7}{62} \sin 2t \quad (D^2 = -4)$$

$$\therefore x = C_1 \cos \sqrt{35} t + C_2 \sin \sqrt{35} t + \frac{7}{62} \sin 2t \quad \dots(1)$$

At $t = 0, x = 0$

$$\therefore \text{From (1), } C_1 = 0$$

$$\therefore x = C_2 \sin \sqrt{35} t + \frac{7}{62} \sin 2t \quad \dots(2)$$

$$\frac{dx}{dt} = \sqrt{35} C_2 \cos \sqrt{35} t + \frac{7}{31} \cos 2t \quad \dots(3)$$

When, $t = 0, v = \frac{dx}{dt} = 0$

Substituting in (3), we get

$$0 = \sqrt{35} C_2 + \frac{7}{31} \Rightarrow C_2 = -\frac{7}{31\sqrt{35}}$$

$$\therefore \text{From equation (2), } x = -\frac{7}{31\sqrt{35}} \sin \sqrt{35} t + \frac{7}{62} \sin 2t.$$

- 253.** A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin wt$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S , prove that the displacement of M at time t from

the commencement of motion is given by $x = \frac{F}{M(p^2 - w^2)} \left\{ \sin wt - \frac{w}{p} \sin pt \right\}$, where $p^2 = \frac{S}{M}$ and

damping effects are neglected.

Sol. Let x be the displacement from the equilibrium position at any time t , then the equation of motion of the mass M is

$$M \frac{d^2x}{dt^2} = -Sx + F \sin wt \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{S}{M} x + \frac{F}{M} \sin wt$$

$$\text{or} \quad \frac{d^2x}{dt^2} + p^2 x = \frac{F}{M} \sin wt \quad \dots(1) \quad \left(\because \frac{S}{M} = p^2 \right)$$

Its A.E. is $D^2 + p^2 = 0 \Rightarrow D = \pm ip$

$$\text{C.F.} = C_1 \cos pt + C_2 \sin pt$$

$$\text{P.I.} = \frac{F}{M} \cdot \frac{1}{D^2 + p^2} \sin wt$$

$$\text{P.I.} = \frac{F}{M} \cdot \frac{1}{p^2 - w^2} \sin wt$$

∴ Complete solution of equation (1) is

$$x = C_1 \cos pt + C_2 \sin pt + \frac{F}{M} \cdot \frac{1}{p^2 - w^2} \sin wt \quad \dots(2)$$

Initially, when $t = 0$, $x = 0$ ∴ $C_1 = 0$

Differentiating equation (2) w.r.t. t ,

$$\frac{dx}{dt} = -pC_1 \sin pt + pC_2 \cos pt + \frac{F}{M} \cdot \frac{w}{p^2 - w^2} \cos wt$$

Since $\frac{dx}{dt} = 0$, when $t = 0$

$$\therefore pC_2 + \frac{F}{M} \cdot \frac{w}{p^2 - w^2} = 0 \quad \text{or} \quad C_2 = -\frac{w}{p} \cdot \frac{F}{M(p^2 - w^2)}$$

Substituting the values of C_1 and C_2 in equation (2), the displacement of the mass at any time t is given by

$$x = -\frac{w}{p} \cdot \frac{F}{M(p^2 - w^2)} \sin pt + \frac{F}{M} \cdot \frac{1}{(p^2 - w^2)} \sin wt$$

$$\text{or} \quad x = \frac{F}{M(p^2 - w^2)} \left(\sin wt - \frac{w}{p} \sin pt \right), \quad \text{where } p^2 = \frac{S}{M}.$$

- 254.** A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of w lb at the other. It is found that resonance occurs when an axial periodic force $2 \cos 2t$ lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$ and find the values of w and c .

Sol. When a weight of 2 lbs is attached to A, spring stretches by $\frac{1}{12}$ ft.

$$\therefore 2 = k \cdot \frac{1}{12} \Rightarrow k = 24 \text{ lb/ft.}$$

Let B be the position of the weight w attached to A, then

$$w = k \times AB \Rightarrow AB = \frac{w}{24} \text{ ft.}$$

At any time t , let the weight be at P, where BP = x .

$$\text{Tension at P,} \quad T_p = k \times AP = 24 \left(\frac{w}{24} + x \right) = w + 24x$$

Its equation of motion is

$$\frac{w}{g} \frac{d^2x}{dt^2} = -T + w + 2 \cos 2t = -w - 24x + w + 2 \cos 2t = 2 \cos 2t - 24x$$

$$\Rightarrow w \frac{d^2x}{dt^2} + 24gx = 2g \cos 2t \quad \dots(1)$$

The phenomenon of resonance occurs when the period of free oscillations is equal to the period of forced oscillations,

From equation (1),

$$\frac{d^2x}{dt^2} + \mu^2x = \frac{2g}{w} \cos 2t \quad \left| \text{where } \mu^2 = \frac{24g}{w} \right.$$

$\therefore$ The period of free oscillations is found as $\frac{2\pi}{\mu}$ and the period of

the force $\left(\frac{2g}{w}\right) \cos 2t$ is π .

$$\therefore \quad \frac{2\pi}{\mu} = \pi \Rightarrow \mu = 2$$

$$\text{Hence,} \quad 4 = \frac{24g}{w} \Rightarrow w = 6g \quad \dots(2)$$

$$\text{From equation (1),} \quad \frac{d^2x}{dt^2} + 4x = \frac{1}{3} \cos 2t \quad \dots(3)$$

We know that the free oscillations are given by the C.F. and the forced oscillations are given by P.I. Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (3).

$$\therefore \quad \text{P.I.} = \frac{1}{3} \left(\frac{1}{D^2 + 4} \cos 2t \right) = \frac{1}{3} \cdot t \cdot \frac{1}{2D} \cos 2t \quad \text{or} \quad \text{P.I.} = \frac{t}{12} \sin 2t$$

$$\text{Hence,} \quad C = \frac{1}{12}.$$

- 255.** A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm. and released. Find the displacement, t seconds after release, if there be a damping force of 2000 dynes per cm per second. What should be the damping force for the dead beat motion?

Sol. Let O be the fixed end and A be the free end of the spring.

Since a force $\frac{196000}{980} = 200$ gm wt stretches the spring by 4 cm.

$$\therefore \quad 200 = k \times 4 \quad \text{or} \quad k = 50 \text{ gm/cm.}$$

where k is the restoring force.

Let B be the equilibrium position when a mass of 200 gm is attached to A, then

$$200 = k \times AB \quad \text{or} \quad 200 = 50 \times AB$$

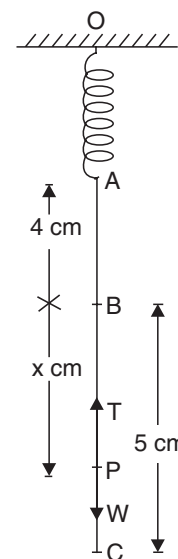
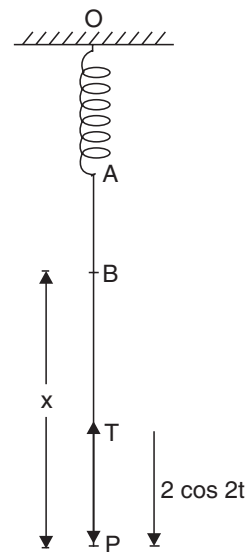
$$\therefore \quad AB = 4 \text{ cm.}$$

Now the 200 gm weight is pulled down to C, where $BC = 5$ cm. If t seconds after its release from C, the weight be at P, where $BP = x$ cm, then the forces acting on it are

- (i) its weight $W = 200$ gm acting downwards.
- (ii) the tension $T = k \times AP = 50(4 + x)$ gm upwards.
- (iii) the damping force $\frac{2000}{g} \cdot \frac{dx}{dt}$ gm upwards.

$\therefore$ The equation of motion is

$$\frac{200}{g} \cdot \frac{d^2x}{dt^2} = W - T - \frac{2000}{g} \cdot \frac{dx}{dt} = 200 - 50(4 + x) - \frac{2000}{g} \cdot \frac{dx}{dt}$$



$$\frac{200}{g} \cdot \frac{d^2x}{dt^2} = -50x - \frac{2000}{g} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + 10 \cdot \frac{dx}{dt} + \frac{g}{4} \cdot x = 0$$

$$\text{or} \quad \frac{d^2x}{dt^2} + 10 \cdot \frac{dx}{dt} + 245x = 0 \quad \dots(1) \quad (\because g = 980 \text{ cm/sec})$$

This is a linear differential equation with constant coefficients.

$$\text{Its A.E. is } D^2 + 10D + 245 = 0 \quad \text{This} \Rightarrow D = -5 \pm i\sqrt{220}$$

$$\therefore \text{Solution of equation (1) is } x = e^{-5t} (C_1 \cos \sqrt{220} t + C_2 \sin \sqrt{220} t) \quad \dots(2)$$

$$\text{Initially, when } t = 0, x = 5 \Rightarrow C_1 = 5$$

$$\begin{aligned} \text{Also,} \quad \frac{dx}{dt} &= -5e^{-5t} (C_1 \cos \sqrt{220} t + C_2 \sin \sqrt{220} t) \\ &\quad + e^{-5t} (-\sqrt{220} C_1 \sin \sqrt{220} t + \sqrt{220} C_2 \cos \sqrt{220} t) \end{aligned}$$

$$\text{Since, } \frac{dx}{dt} = 0 \text{ when } t = 0, \text{ we have } C_2 = \frac{25}{\sqrt{220}}$$

$\therefore$ From equation (2), the displacement, t seconds after release, is given by

$$x = e^{-5t} \left(5 \cos \sqrt{220} t + \frac{25}{\sqrt{220}} \sin \sqrt{220} t \right)$$

Let the damping force for the dead beat motion be p dynes per cm per second, then the equation of motion becomes

$$\frac{200}{g} \cdot \frac{d^2x}{dt^2} = -50x - \frac{p}{g} \cdot \frac{dx}{dt} \quad \text{or} \quad 200 \cdot \frac{d^2x}{dt^2} + p \cdot \frac{dx}{dt} + 49000x = 0$$

$$\text{Its A.E. is} \quad 200D^2 + pD + 49000 = 0$$

For dead beat motion, roots of A.E. must be equal.

$$\therefore p^2 = 4 \times 200 \times 49000 \Rightarrow p = 2800\sqrt{5} = 6261.$$

- 256.** A particle is projected with velocity u making an angle α with the horizontal. Neglecting air resistance, show that the equation of its path is the parabola, $y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$. Find the time of flight, greatest height attained and range on the horizontal plane.

Sol. Let a particle of mass m be projected from a point O with velocity u in a direction making an angle α with the horizontal. Let the horizontal and the vertical lines through O in the plane of motion of the particle be taken as the axes of x and y respectively. Let $P(x, y)$ be the position of the particle at time t .

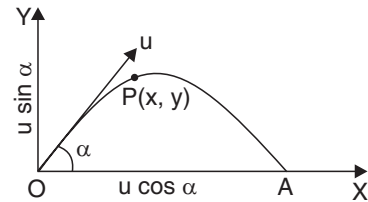
The horizontal and vertical components of u are $u \cos \alpha$ and $u \sin \alpha$ respectively.

The only force acting on the particle as its weight mg acting vertically downwards.

$\therefore$ The equations of motion are

$$\text{Parallel to } x\text{-axis} \quad m \frac{d^2x}{dt^2} = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} = 0 \quad \dots(1)$$

$$\text{Parallel to } y\text{-axis} \quad m \frac{d^2y}{dt^2} = -mg \quad \text{or} \quad \frac{d^2y}{dt^2} = -g \quad \dots(2)$$



$$\begin{aligned} \text{Integrating (1) w.r.t. } t, \quad & \frac{dx}{dt} = C_1 \\ \text{Initially when } t = 0, \quad & \frac{dx}{dt} = u \cos \alpha \quad \therefore C_1 = u \cos \alpha \\ \therefore \quad & \frac{dx}{dt} = u \cos \alpha \end{aligned} \quad \dots(3)$$

$$\begin{aligned} \text{Integrating w.r.t. } t \quad & x = (u \cos \alpha)t + C_2 \\ \text{Initially when } t = 0, x = 0 \quad & \therefore C_2 = 0 \\ & x = (u \cos \alpha)t \end{aligned} \quad \dots(4)$$

$$\begin{aligned} \text{Integrating (2) w.r.t. } t, \quad & \frac{dy}{dt} = -gt + C_3 \\ \text{Initially when } t = 0, \frac{dy}{dt} = u \sin \alpha \text{ so that} \quad & C_3 = u \sin \alpha \\ \therefore \quad & \frac{dy}{dt} = u \sin \alpha - gt \end{aligned} \quad \dots(5)$$

$$\begin{aligned} \text{Integrating w.r.t. } t, \quad & y = (u \sin \alpha)t - \frac{1}{2} gt^2 + C_4 \\ \text{Initially, when } t = 0, y = 0 \quad & \therefore C_4 = 0 \\ \therefore \quad & y = (u \sin \alpha)t - \frac{1}{2} gt^2 \end{aligned} \quad \dots(6)$$

Equations (4) and (6) give the position of the particle at any time t . The equation of the path described by the particle is obtained by eliminating the parameter t between equations (4) and (6).

$$\text{From (4),} \quad t = \frac{x}{u \cos \alpha}$$

Substituting this value of t in equation (6), we get

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \dots(7)$$

which is the equation of the path of the projectile. Clearly, the path is a parabola.

The **time of flight** of the projectile is the time taken by the particle to reach the horizontal plane through O.

At the point A, $y = 0$

$$\therefore \text{ From equation (6),} \quad 0 = (u \sin \alpha)t - \frac{1}{2} gt^2 \quad \text{or} \quad t = \frac{2u \sin \alpha}{g}$$

At the highest point, the vertical component of velocity vanishes, i.e., $\frac{dy}{dt} = 0$

$$\text{so that } u \sin \alpha - gt = 0 \quad \text{or} \quad t = \frac{u \sin \alpha}{g}$$

$$\begin{aligned} \therefore \text{ The greatest height} &= y \left(\text{when } \frac{dy}{dx} = 0 \right) \\ &= (u \sin \alpha)t - \frac{1}{2} gt^2 \left(\text{when } t = \frac{u \sin \alpha}{g} \right) \\ &= \frac{u^2 \sin^2 \alpha}{2g} \end{aligned}$$

The Range R(= OA) is the horizontal distance covered by the particle in the time of flight.

$$\therefore \quad R = u \cos \alpha \times \frac{2u \sin \alpha}{g} = \frac{2u^2 \sin \alpha \cos \alpha}{g} \quad \text{or} \quad R = \frac{u^2 \sin 2\alpha}{g} .$$

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Laplace Transforms and Its Applications

SOLVED PROBLEMS

1. Let $f(t)$ be a function of t defined for all $t \geq 0$. What is the Laplace transform of $f(t)$? Write the Laplace transforms of $1, t^n, e^{at}, \sin at, \cos at, \sinh at$ and $\cosh at$.

Sol. Laplace transform of $f(t)$ denoted by

$$L\{f(t)\}, \text{ is defined as } L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

provided that the integral exists, 's' is a parameter which may be real or complex. **$L\{f(t)\}$ is a function of s and is briefly written as $\bar{f}(s)$.** Some authors use the letter 'p' for the parameter instead of 's'.

Laplace transforms of $1, t^n, e^{at}, \sin at, \cos at, \sinh at$ and $\cosh at$:

$$(i) L(1) = \int_0^{\infty} e^{-st} \cdot 1 \cdot dt = \left[-\frac{e^{-st}}{s} \right]_0^{\infty} = \frac{1}{s}, \text{ if } s > 0$$

$$(ii) L(t^n) = \int_0^{\infty} e^{-st} t^n \cdot dt$$

Put $st = x, dt = dx/s$

$$\therefore L(t^n) = \int_0^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^{\infty} x^n e^{-x} dx = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ provided } s > 0 \text{ and } n+1 > 0$$

If n is a positive integer, $\Gamma(n+1) = n!$

$$\text{Hence, } L(t^n) = \frac{n!}{s^{n+1}}.$$

$$(iii) L(e^{at}) = \int_0^{\infty} e^{-st} \cdot e^{at} \cdot dt = \int_0^{\infty} e^{-(s-a)t} \cdot dt = \frac{1}{s-a}, \text{ if } s > a$$

$$(iv) L(\sin at) = \int_0^{\infty} e^{-st} \sin at \cdot dt = \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at + a \cos at) \right]_0^{\infty} = \frac{a}{s^2 + a^2}$$

$$(v) L(\cos at) = \int_0^{\infty} e^{-st} \cdot \cos at \cdot dt = \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at - a \sin at) \right]_0^{\infty} = \frac{s}{s^2 + a^2}$$

$$(vi) L(\sinh at) = L\left\{\frac{1}{2}(e^{at} - e^{-at})\right\} = \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at}) = \frac{1}{2} \left(\frac{1}{s-a}\right) - \frac{1}{2} \left(\frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}$$

$$(vii) L(\cosh at) = L\left\{\frac{1}{2}(e^{at} + e^{-at})\right\} = \frac{s}{s^2 - a^2}.$$

2. State first shifting theorem. Applying this theorem, write the Laplace transforms of $e^{at} \cdot t^n$, $e^{at} \sin bt$, $e^{at} \cos bt$, $e^{at} \sinh at$ and $e^{at} \cosh at$.

Sol. First shifting theorem states that if $L\{f(t)\} = \bar{f}(s)$, then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

Similarly $L\{e^{-at} f(t)\} = \bar{f}(s+a)$

Applying this theorem, we get the useful results :

$$\begin{aligned} L(e^{at} \cdot t^n) &= \frac{n!}{(s-a)^{n+1}} & \therefore L(t^n) &= \frac{n!}{s^{n+1}} \\ L(e^{at} \sin bt) &= \frac{b}{(s-a)^2 + b^2}; & L(e^{at} \cos bt) &= \frac{s-a}{(s-a)^2 + b^2} \\ L(e^{at} \sinh bt) &= \frac{b}{(s-a)^2 - b^2}; & L(e^{at} \cosh bt) &= \frac{s-a}{(s-a)^2 - b^2}. \end{aligned}$$

3. What is change of scale property ? Prove it.

Sol. If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a} \cdot \bar{f}\left(\frac{s}{a}\right)$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \bar{f}(s)$$

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty e^{-st} f(at) dt = \int_0^\infty e^{-\frac{s}{a}u} f(u) \cdot \frac{du}{a} \quad | \text{ Put } at = u; a dt = du \\ &= \frac{1}{a} L\{f(u)\} = \frac{1}{a} \cdot \bar{f}\left(\frac{s}{a}\right). \end{aligned}$$

4. Find the Laplace transforms of

(i) $\sin 2t \cos 3t$

(ii) $e^{-3t} (\cos 4t + 3 \sin 4t)$.

Sol. (i) Since $\sin 2t \cos 3t = \frac{1}{2} (2 \cos 3t \sin 2t) = \frac{1}{2} (\sin 5t - \sin t)$

$$\begin{aligned} \therefore L\{\sin 2t \cos 3t\} &= L\left\{\frac{1}{2} (\sin 5t - \sin t)\right\} = \frac{1}{2} [L(\sin 5t) - L(\sin t)] \\ &= \frac{1}{2} \left[\frac{5}{s^2 + 5^2} - \frac{1}{s^2 + 1^2} \right] = \frac{2(s^2 - 5)}{(s^2 + 25)(s^2 + 1)}. \end{aligned}$$

(ii) $L\{e^{-3t} (\cos 4t + 3 \sin 4t)\}$

$$\begin{aligned} &= L\{e^{-3t} \cos 4t\} + 3L\{e^{-3t} \sin 4t\} \\ &= \frac{s+3}{(s+3)^2 + 4^2} + 3 \cdot \frac{4}{(s+3)^2 + 4^2} = \frac{s+15}{s^2 + 6s + 25}. \end{aligned}$$

5. Find the Laplace transform of $t^2 \sin at$.

Sol. Since $L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$

$\therefore$ By first shifting theorem, we have

$$L(t^2 e^{iat}) = \frac{2}{(s-ia)^3} = \frac{2(s+ia)^3}{(s-ia)^3 (s+ia)^3}$$

$$\text{or } L\{t^2(\cos at + i \sin at)\} = \frac{2[(s^3 - 3a^2s) + i(3as^2 - a^3)]}{(s^2 + a^2)^3}$$

Equating the imaginary parts on both sides, we get

$$L\{t^2 \sin at\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

6. Find the Laplace transform of $te^{-4t} \sin 3t$.

Sol. Since $L(t) = \frac{1}{t^2}$

$\therefore$ By first shifting theorem, we have

$$L\{te^{3it}\} = \frac{1}{(s - 3i)^2} = \frac{(s + 3i)^2}{\{(s - 3i)(s + 3i)\}^2}$$

$$\text{or } L\{t(\cos 3t + i \sin 3t)\} = \frac{(s^2 - 9) + 6is}{(s^2 + 9)^2}$$

Equating the imaginary parts on both sides, we get

$$L\{t \sin 3t\} = \frac{6s}{(s^2 + 9)^2}$$

Again applying the first shifting theorem, we have

$$L\{e^{-4t} \cdot t \sin 3t\} = \frac{6(s + 4)}{[(s + 4)^2 + 9]^2} = \frac{6(s + 4)}{(s^2 + 8s + 25)^2}.$$

7. Find the Laplace transform of $f(t)$ defined as $f(t) = \begin{cases} \frac{t}{T}, & \text{when } 0 < t < T \\ 1, & \text{when } t > T. \end{cases}$

Sol. $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^T e^{-st} \cdot \frac{t}{T} dt + \int_T^\infty e^{-st} \cdot 1 \cdot dt$

$$= \frac{1}{T} \left[\left[t \cdot \frac{e^{-st}}{-s} \right]_0^T - \int_0^T 1 \cdot \frac{e^{-st}}{-s} \cdot dt \right] + \left[\frac{e^{-st}}{-s} \right]_T^\infty$$

$$= \frac{1}{T} \left[-\frac{T}{s} e^{-sT} + \frac{1}{s} \left\{ \frac{e^{-st}}{-s} \right\}_0^T \right] + \frac{e^{-sT}}{s} = -\frac{e^{-sT}}{s} - \frac{e^{-sT} - 1}{Ts^2} + \frac{e^{-sT}}{s} = \frac{1 - e^{-sT}}{Ts^2}.$$

8. Find the Laplace transform of $7e^{2t} + 9e^{-2t} + 5 \cos t + 7t^3 + 5 \sin 3t + 2$.

Sol. $L(7e^{2t} + 9e^{-2t} + 5 \cos t + 7t^3 + 5 \sin 3t + 2)$

$$= 7L(e^{2t}) + 9L(e^{-2t}) + 5L(\cos t) + 7L(t^3) + 5L(\sin 3t) + 2L(1)$$

$$= 7 \cdot \frac{1}{s - 2} + 9 \cdot \frac{1}{s + 2} + 5 \cdot \frac{s}{s^2 + 1} + 7 \cdot \frac{3!}{s^4} + 5 \cdot \frac{3}{s^2 + 9} + 2 \cdot \frac{1}{s}$$

$$= \frac{7}{s - 2} + \frac{9}{s + 2} + \frac{5s}{s^2 + 1} + \frac{42}{s^4} + \frac{15}{s^2 + 9} + \frac{2}{s}.$$

9. Find the Laplace transforms of the following :

(i) $t^{7/2} e^{5t}$

(ii) $\int_0^x \frac{\sin x}{x} dx$

(M.D.U. Dec., 2009)

Sol. (i) $L(e^{5t}) = \frac{1}{s-5}$

$$L(t^{7/2} e^{5t}) = (-1)^{7/2} \frac{d^3}{ds^3} \left(\frac{1}{s-5} \right) = \frac{(-1)^{7/2} \cdot 5!}{(s-5)^4} = \frac{120}{(s-5)^4}.$$

(ii) $f(x) = \int_0^x \frac{\sin x}{x} \cdot dx$

$$L(\sin x) = \frac{1}{s^2+1}, \quad L\left(\frac{\sin x}{x}\right) = \int_s^\infty \frac{1}{s^2+1} \cdot ds = [\tan^{-1} s]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$L \int_0^x \frac{\sin x}{x} \cdot dx = \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s \right].$$

10. Find the Laplace transform of

(i) $\sin^3 2t$

(ii) $\cosh^3 2t$

(iii) $(1 + te^{-t})^3$

Sol. (i) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

$$\Rightarrow \sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$$

$$\begin{aligned} \therefore L(\sin^3 2t) &= \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t) \\ &= \frac{3}{4} \cdot \frac{2}{s^2+4} - \frac{1}{4} \cdot \frac{6}{s^2+36} = \frac{48}{(s^2+4)(s^2+36)}. \end{aligned}$$

(ii) Since $\cosh 6t = 4 \cosh^3 2t - 3 \cosh 2t$

$$\Rightarrow \cosh^3 2t = \frac{3}{4} \cosh 2t + \frac{1}{4} \cosh 6t$$

$$\begin{aligned} \therefore L(\cosh^3 2t) &= \frac{3}{4} L(\cosh 2t) + \frac{1}{4} L(\cosh 6t) \\ &= \frac{3}{4} \cdot \frac{s}{s^2-4} + \frac{1}{4} \cdot \frac{s}{s^2-36} = \frac{s(s^2-28)}{(s^2-4)(s^2-36)}. \end{aligned}$$

(iii) $(1 + te^{-t})^3 = 1 + t^3 e^{-3t} + 3te^{-t}(1 + te^{-t}) = 1 + t^3 e^{-3t} + 3te^{-t} + 3t^2 e^{-2t}$

$$L\{(1 + te^{-t})^3\} = L(1) + L(t^3 e^{-3t}) + 3L(te^{-t}) + 3L(t^2 e^{-2t}) \quad \dots(1)$$

Determination of $L(t^3 e^{-3t})$

$$L(t^3) = \frac{3!}{s^4}$$

$$\therefore L(t^3 e^{-3t}) = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4} \quad \text{(Using first shifting property)}$$

Determination of $L(te^{-t})$

$$L(t) = \frac{1}{s^2}$$

$$\therefore L(te^{-t}) = \frac{1}{(s+1)^2} \quad \text{(Using first shifting property)}$$

12. Find the Laplace transform of

(i) $t \sin^3 t$

(ii) $e^{-t} \cdot \frac{\sin t}{t}$

(M.D.U. May, 2007)

Sol. (i) $L\{f(t)\} = L\{t \sin^3 t\}$

Let $F(t) = \sin^3 t = \frac{3 \sin t - \sin^3 t}{4}$

$$L\{F(t)\} = L\left\{\frac{3}{4} \sin t\right\} - L\left\{\frac{1}{4} \sin^3 t\right\} = \frac{3}{4} \cdot \frac{1}{s^2 + 1} - \frac{1}{4} \cdot \frac{3}{s^2 + 9}$$

$$\begin{aligned} L\{t F(t)\} &= (-1) \frac{d}{ds} \left[\frac{3}{4} \cdot \frac{1}{s^2 + 1} - \frac{1}{4} \cdot \frac{3}{s^2 + 9} \right] = -\frac{3}{4} \cdot \frac{-2s}{(s^2 + 1)^2} + \frac{1}{4} \cdot \frac{-2s}{(s^2 + 9)^2} \\ &= \frac{3s}{2(s^2 + 1)^2} - \frac{s}{2(s^2 + 9)^2} = \frac{s}{2} \left[\frac{3(s^2 + 9)^2 - (s^2 + 1)^2}{(s^2 + 1)^2 (s^2 + 9)^2} \right] \\ &= \frac{s}{2} \left[\frac{3(s^4 + 81 + 18s^2) - (s^4 + 1 + 2s^2)}{(s^2 + 1)^2 (s^2 + 9)^2} \right] = \frac{s}{2} \left[\frac{2s^4 + 242 + 52s^2}{(s^2 + 1)^2 (s^2 + 9)^2} \right] \\ &= s \left[\frac{s^4 + 26s^2 + 121}{(s^2 + 1)^2 (s^2 + 9)^2} \right]. \end{aligned}$$

(ii) $L(e^{-t} \sin t) = \frac{1}{(s + 1)^2 + 1}$

$$L\left(e^{-t} \frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{(s + 1)^2 + 1} ds = \left[\tan^{-1}(s + 1) \right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s + 1) = \cot^{-1}(s + 1).$$

13. Find the Laplace transform of (i) $te^{-t} \sin 2t$ (ii) $t^2 e^t \sin 4t$.

Sol. (i) $L(t) = \frac{1}{s^2}$

By first shifting property,

$$L(t \cdot e^{2it}) = \frac{1}{(s - 2i)^2} = \frac{(s + 2i)^2}{(s^2 + 4)^2} \quad \text{or} \quad L(t \cos 2t + it \sin 2t) = \frac{s^2 - 4 + 4is}{(s^2 + 4)^2}$$

Equating imaginary parts, we get

$$L(t \sin 2t) = \frac{4s}{(s^2 + 4)^2}$$

Again applying first shifting property, we get

$$L(e^{-t} \cdot t \sin 2t) = \frac{4(s + 1)}{\{(s + 1)^2 + 4\}^2} = \frac{4s + 4}{(s^2 + 2s + 5)^2}.$$

(ii) $L(t^2) = \frac{2}{s^3}$

By first shifting property

$$L(t^2 e^{4it}) = \frac{2}{(s - 4i)^3} = \frac{2(s + 4i)^3}{(s^2 + 16)^3}$$

$$\text{or } L(t^2 \cos 4t + it^2 \sin 4t) = \frac{2[s^3 - 64i + 12is(s + 4i)]}{(s^2 + 16)^3} = \frac{2[(s^3 - 48s) + i(12s^2 - 64)]}{(s^2 + 16)^3}$$

Equating imaginary parts on both sides,

$$L(t^2 \sin 4t) = \frac{8(3s^2 - 16)}{(s^2 + 16)^3}$$

Again applying first shifting property, we get

$$L(e^t \cdot t^2 \sin 4t) = \frac{8[3(s-1)^2 - 16]}{[(s-1)^2 + 16]^3} = \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}.$$

14. Find the Laplace transform of

$$(i) f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 0, & t > \pi \end{cases} \quad (ii) f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & 1 \leq t < 2 \\ t^2, & 2 \leq t < \infty \end{cases}.$$

$$\begin{aligned} \text{Sol. (i)} L\{f(t)\} &= \int_0^\infty e^{-st} \cdot f(t) dt = \int_0^\pi e^{-st} \cdot \cos t \cdot dt + \int_\pi^\infty e^{-st} \cdot 0 dt \\ &= \left[\frac{e^{-st}}{s^2 + 1} (-s \cos t + \sin t) \right]_0^\pi = \left[\frac{e^{-s\pi}}{s^2 + 1} \cdot s - \frac{1}{s^2 + 1} (-s) \right] = \frac{s(1 + e^{-s\pi})}{s^2 + 1} \end{aligned}$$

$$\begin{aligned} (ii) L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} \cdot dt + \int_1^2 t \cdot e^{-st} \cdot dt + \int_2^\infty t^2 e^{-st} \cdot dt \\ &= \left(\frac{e^{-st}}{-s} \right)_0^1 + \left\{ t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right\}_1^2 + \left(t^2 \frac{e^{-st}}{-s} \right)_2^\infty - \int_2^\infty 2t \cdot \frac{e^{-st}}{-s} dt \end{aligned}$$

$$\begin{aligned} \text{or } L\{f(t)\} &= \left(\frac{1 - e^{-s}}{s} \right) + \left(-\frac{2}{s} e^{-2s} - \frac{e^{-2s}}{s^2} \right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} \right) + \frac{4}{s} e^{-2s} + \frac{2}{s} \int_2^\infty t e^{-st} \cdot dt \\ &= \frac{1}{s} + \frac{2}{s} e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s} \left[\frac{2}{s} e^{-2s} + \frac{1}{s} \left(\frac{e^{-st}}{-s} \right)_2^\infty \right] \\ &= \frac{1}{s} + \frac{2}{s} e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} (0 - e^{-2s}) \\ &= \frac{1}{s} + \frac{2}{s} e^{-2s} + \frac{e^{-s}}{s^2} + \frac{3}{s^2} e^{-2s} + \frac{2}{s^3} e^{-2s}. \end{aligned}$$

15. Find the Laplace transform of $e^{3t} + 5t^4 - 2 \cos t + 3 \sin t$

(M.D.U. Dec., 2007)

Sol. $L(e^{3t}) + L(5t^4) - L(2 \cos t) + 3L(\sin t)$

$$= \frac{1}{s-3} + 5 \cdot \frac{4!}{s^5} - 2 \cdot \frac{s}{s^2 + 1} + 3 \cdot \frac{1}{s^2 + 1} = \frac{1}{s-3} + \frac{120}{s^5} - \frac{2s}{s^2 + 1} + \frac{3}{s^2 + 1}.$$

16. Find the Laplace transform of the function $f(t)$, where

$$(i) f(t) = \begin{cases} \sin \left(t - \frac{\pi}{3} \right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases} \quad (ii) f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases} \quad (\text{U.P.T.U., 2006})$$

Sol. (i) $L\{f(t)\} = e^{-s \frac{\pi}{3}} \cdot L(\sin t) = e^{-s \frac{\pi}{3}} \cdot \frac{1}{s^2 + 1}$

(ii) $L\{f(t)\} = e^{-s} \cdot L(t^2) = e^{-s} \cdot \frac{2}{s^3}$

17. Find the Laplace transform of function $f(t)$ given by $f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$

(M.D.U. May, 2007)

Sol. $L f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$

Since $f(t)$ is periodic with period $2c$

$$\begin{aligned} L[f(t)] &= \frac{\int_0^{2c} e^{-st} f(t) dt}{1 - e^{-2cs}} = \int_0^c e^{-st} f(t) dt + \int_c^{2c} e^{-st} f(t) dt \\ &= \int_0^c e^{-st} \cdot t dt + \int_c^{2c} e^{-st} (2c - t) dt \\ &= \left[\frac{te^{-st}}{-s} \right]_0^c - \int_0^c \frac{e^{-st}}{-s} \cdot dt + \left[(2c - t) \frac{e^{-st}}{(-s)} \right]_c^{2c} - \int_c^{2c} (-1) \frac{e^{-st}}{(-s)} dt \\ &= \left(\frac{ce^{-cs}}{-s} - 0 \right) - \left(\frac{e^{-st}}{s^2} \right)_0^c + \left(0 - \frac{ce^{-cs}}{-s} \right) + \left(\frac{e^{-st}}{s^2} \right)_c^{2c} \\ &= -\frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} + \frac{1}{s^2} + \frac{ce^{-cs}}{s} + \frac{e^{-2cs}}{s^2} - \frac{e^{-cs}}{s^2} - \frac{2e^{-cs}}{s^2} + \frac{1}{s^2} + \frac{e^{-2cs}}{s^2} \\ &= \frac{1}{s^2} [e^{-2cs} - 2e^{-cs} + 1] = \frac{1}{s^2} \frac{(1 - e^{-cs})^2}{(1 - e^{-2cs})} \\ &= \frac{1}{s^2} \frac{(1 - e^{-cs})^2}{(1 + e^{-cs})(1 - e^{-cs})} = \frac{1}{s^2} \left(\frac{1 - e^{-cs}}{1 + e^{-cs}} \right) = \frac{1}{s^2} \left(\frac{e^{\frac{cs}{2}} - e^{-\frac{cs}{2}}}{e^{\frac{cs}{2}} + e^{-\frac{cs}{2}}} \right) = \frac{1}{s^2} \tanh \frac{cs}{2} \end{aligned}$$

18. Given that $L\left(\frac{\sin t}{t}\right) = \tan^{-1}\left(\frac{1}{s}\right)$, find $L\left(\frac{\sin at}{t}\right)$. (M.D.U. Dec., 2009; U.P. May, 2005)

Sol. By change of scale property

$$\begin{aligned} L\left(\frac{\sin at}{at}\right) &= \frac{1}{a} \tan^{-1}\left(\frac{1}{s/a}\right) \\ \frac{1}{a} L\left(\frac{\sin at}{t}\right) &= \frac{1}{a} \tan^{-1}\left(\frac{a}{s}\right) \quad \Rightarrow \quad L\left(\frac{\sin at}{t}\right) = \tan^{-1}\left(\frac{a}{s}\right) \end{aligned}$$

19. Find the Laplace transform of $\sin kt \cdot \sinh kt$.

Sol. As $\sinh kt = \frac{e^{kt} - e^{-kt}}{2}$

$$\therefore L[\sin kt \cdot \sinh kt] = L\left[\frac{(e^{kt} - e^{-kt}) \sin kt}{2}\right]$$

$$\begin{aligned}
 \text{R.H.S.} &= \mathcal{L} \left[\frac{(e^{kt} - e^{-kt})}{2} \sin kt \right] = \mathcal{L} \left[\frac{e^{kt} \cdot \sin kt}{2} \right] - \mathcal{L} \left[\frac{e^{-kt} \cdot \sin kt}{2} \right] \\
 &= \frac{k}{2[(s-k)^2 + k^2]} - \frac{k}{2[(s+k)^2 + k^2]} = \frac{k}{2} \left[\frac{(s+k)^2 + k^2 - (s-k)^2 - k^2}{\{(s-k)^2 + k^2\} \{(s+k)^2 + k^2\}} \right] \\
 &= \frac{k}{2} \left[\frac{4sk}{s^4 + 4k^4} \right] = \frac{2k^2 \cdot s}{s^4 + 4k^4}.
 \end{aligned}$$

20. Find the Laplace transform of

$$\begin{aligned}
 (i) \quad &e^{-t} \sin^2 t \\
 (ii) \quad &f(t) = \begin{cases} t^2, & 0 < t \leq 2 \\ t-1, & 2 < t \leq 3 \\ 7, & t > 0 \end{cases} \quad (\text{U.P., 2007, 2009})
 \end{aligned}$$

Sol. (i) $\sin^2 t = \frac{1 - \cos 2t}{2}$

$$\begin{aligned}
 \therefore \quad \mathcal{L} \left[\frac{e^{-t} (1 - \cos 2t)}{2} \right] &= \mathcal{L} \left(\frac{e^{-t}}{2} \right) - \mathcal{L} \left(\frac{e^{-t} \cdot \cos 2t}{2} \right) \\
 &= \frac{1}{2(s+1)} - \frac{1}{2} \frac{(s+1)}{(s+1)^2 + 4} = \frac{1}{2} \left[\frac{(s+1)^2 + 4 - (s+1)^2}{(s+1)(s^2 + 2s + 5)} \right] \\
 &= \frac{2}{(s+1)(s^2 + 2s + 5)} = \frac{2}{s^3 + 3s^2 + 7s + 5}
 \end{aligned}$$

$$(ii) \quad f(t) = \begin{cases} t^2, & 0 < t \leq 2 \\ t-1, & 2 < t \leq 3 \\ 7, & t > 0 \end{cases}$$

$$\begin{aligned}
 \mathcal{L} \{f(t)\} &= \int_0^\infty e^{-st} f(t) dt = \int_0^2 t^2 e^{-st} dt + \int_2^3 (t-1)e^{-st} dt + \int_0^\infty 7 \cdot e^{-st} dt \\
 &= 4 \cdot \frac{e^{-2s}}{-s} + \frac{2}{s} \left[\left\{ \frac{t e^{-st}}{-s} \right\}_0^2 - \int_0^2 \frac{1 \cdot e^{-st}}{(-s)} dt \right] + \left(\frac{(t-1)e^{-st}}{-s} \right)_2^3 - \int_2^3 1 \cdot \frac{e^{-st}}{(-s)} dt + 7 \left(\frac{e^{-st}}{-s} \right)_3^\infty \\
 &= -\frac{4}{s} e^{-2s} + \frac{2}{s} \left[\frac{2 \cdot e^{-2s}}{-s} + \frac{1}{s} \left(\frac{e^{-st}}{-s} \right)_0^2 \right] + 2 \cdot \frac{e^{-3s}}{(-s)} - \frac{e^{-2s}}{(-s)} + \frac{1}{s} \left(\frac{e^{-st}}{-s} \right)_2^3 + 7 \cdot \left(\frac{e^{-3s}}{s} \right) \\
 &= \left[-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3} \right] - 2 \cdot \frac{e^{-3s}}{s} + \frac{e^{-2s}}{s} + \frac{1}{s} \left[\frac{e^{-3s}}{(-s)} - \frac{e^{-2s}}{(-s)} \right] + 7 \cdot \frac{e^{-3s}}{s} \\
 &= \frac{2}{s^3} - \frac{e^{-2s}}{s^3} (2 + 4s + 4s^2) - 2 \cdot \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + 7 \cdot \frac{e^{-3s}}{s} \\
 &= \frac{2}{s^3} - \frac{e^{-2s}}{s^3} (2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2} (5s - 1).
 \end{aligned}$$

21. Find Laplace transform of each of the following:

(i) $e^{4t} \sin 2t \cos t$

(ii) $\sin h(t) \cos^2 t$

(iii) $e^{-t} \cos^2 t$.

(M.D.U. May, 2008; U.P.T.U., 2006)

Sol. (i) $\sin 2t \cos t = \frac{1}{2} [\sin 3t + \sin t]$

$$L(\sin 2t \cos t) = \frac{1}{2} \left[\frac{3}{s^2 + 9} + \frac{1}{s^2 + 1} \right]$$

Now, $L(e^{4t} \sin 2t \cos t) = \frac{1}{2} \left[\frac{3}{(s-4)^2 + 9} + \frac{1}{(s-4)^2 + 1} \right] = \frac{1}{2} \left[\frac{3}{s^2 - 8s + 25} + \frac{1}{s^2 - 8s + 17} \right]$.

(ii) $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right]$

$$L(\sin h t \cos^2 t) = L\left(\frac{e^t - e^{-t}}{2} \cdot \cos^2 t\right) = \frac{1}{2} L\{e^t \cos^2 t\} - \frac{1}{2} L\{e^{-t} \cos^2 t\}$$

$$= \frac{1}{4} \left[\frac{1}{s-1} + \frac{s-1}{(s-1)^2 + 4} \right] - \frac{1}{4} \left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2 + 4} \right]$$

$$= \frac{1}{4} \left[\frac{1}{s-1} - \frac{1}{s+1} \right] + \frac{1}{4} \left[\frac{s-1}{(s-1)^2 + 4} - \frac{s+1}{(s+1)^2 + 4} \right]$$

$$= \frac{1}{4} \left[\frac{1}{s-1} - \frac{1}{s+1} \right] + \frac{1}{4} \left[\frac{s-1}{s^2 - 2s + 5} - \frac{s+1}{s^2 + 2s + 5} \right]$$

(iii) $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 4} \right\}$

$\therefore L(e^{-t} \cos^2 t) = \frac{1}{2} \left\{ \frac{1}{s+1} + \frac{s+1}{(s+1)^2 + 4} \right\} = \frac{1}{2} \left\{ \frac{1}{s+1} + \frac{s+1}{s^2 + 2s + 5} \right\}$.

22. Find the Laplace transform of the function $f(t)$ defined as

$$f(t) = |t-1| + |t+1| + |t+2| + |t-2|, f \geq 0. \quad (\text{M.D.U., 2005})$$

Sol. For $0 \leq t \leq 1$, $|t-1| = -t+1$, $|t+1| = t+1$

$$|t+2| = t+2, \quad |t-2| = -t+2$$

$\therefore f(t) = 6$

For $t > 1$, $|t-1| = t-1$, $|t+1| = t+1$, $|t+2| = t+2$,

$$|t-2| = t-2, f(t) = 4t.$$

$\therefore L\{f(t)\} = \int_0^1 e^{-st} \cdot 6 \cdot dt + \int_1^\infty e^{-st} 4t dt = 6 \left[\frac{e^{-st}}{-s} \right]_0^1 + 4 \left[\frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s} \cdot dt \right]_1^\infty$

$$L\{f(t)\} = \frac{6}{s} (1 - e^{-s}) + 4 \left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right] = \frac{6}{s} - \frac{6}{s} e^{-s} + \frac{4e^{-s}}{s} - 4 \cdot \frac{e^{-s}}{s^2}$$

$$= \frac{6}{s} - \frac{2e^{-s}}{s} - \frac{4e^{-s}}{s^2} = \frac{2}{s} \left[3 - e^{-s} - 2 \frac{e^{-s}}{s} \right]$$

23. Find the Laplace transform of the following functions :

$$(i) g(t) = \begin{cases} 0, & 0 < t < 5 \\ t-3, & t > 5 \end{cases} \quad (ii) \frac{e^{-at} - e^{-bt}}{t}. \quad (\text{M.D.U. Dec., 2008})$$

$$\begin{aligned} \text{Sol. (i) } L[g(t)] &= \int_0^{\infty} e^{-st} \cdot g(t) dt = \int_0^5 0 \cdot e^{-st} \cdot dt + \int_5^{\infty} (t-3) e^{-st} dt \\ &= (t-3) \left[\frac{e^{-st}}{-s} \right] - (1) \left(\frac{e^{-st}}{s^2} \right) \Bigg|_5^{\infty} = 0 - 0 - \left\{ \frac{2e^{-5s}}{-s} - \frac{e^{-5s}}{s^2} \right\} \\ &= \frac{2e^{-5s}}{s} + \frac{e^{-5s}}{s^2} = \left(\frac{2}{s} + \frac{1}{s^2} \right) e^{-5s}. \end{aligned}$$

$$(ii) L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\begin{aligned} L \left[\frac{e^{-at} - e^{-bt}}{t} \right] &= \int_s^{\infty} \left(\frac{1}{s+a} - \frac{1}{s+b} \right) ds = [\log(s+a) - \log(s+b)]_s^{\infty} \\ &= \left[\log \frac{s+a}{s+b} \right]_s^{\infty} = \left[\log \left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right) \right]_s^{\infty} = \log 1 - \log \left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right) = -\log \frac{s+a}{s+b} = \log \left(\frac{s+b}{s+a} \right). \end{aligned}$$

24. Write the inverse Laplace transforms of

$$\begin{array}{llll} (i) \frac{1}{s} & (ii) \frac{1}{s-a} & (iii) \frac{1}{s^n} & (iv) \frac{1}{(s-a)^n} \\ (v) \frac{1}{s^2+a^2} & (vi) \frac{s}{s^2+a^2} & (vii) \frac{1}{s^2-a^2} & (viii) \frac{s}{s^2-a^2} \\ (ix) \frac{1}{(s-a)^2+b^2} & (x) \frac{s-a}{(s-a)^2+b^2} & (xi) \frac{s}{(s^2+a^2)^2} & (xii) \frac{1}{(s^2+a^2)^2} \end{array}$$

Sol. If $L\{f(t)\} = \bar{f}(s)$, then $f(t)$ is called the inverse Laplace transform of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\} = f(t)$

Here L^{-1} denotes the inverse Laplace transform.

For example, Since $L(e^{5t}) = \frac{1}{s-5}$, $L^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}$

From solution of Question 1, it follows that inverse Laplace transforms of various functions are as given below :

$$\begin{array}{ll} (i) L^{-1}\left(\frac{1}{s}\right) = 1 & (ii) L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \\ (iii) L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}, \text{ if } n \text{ is a positive integer and } = \frac{t^{n-1}}{\Gamma n}, \text{ otherwise.} & \\ (iv) L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at} \cdot \frac{t^{n-1}}{(n-1)!} & (v) L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \sin at \\ (vi) L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at & (vii) L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a} \sinh at \end{array}$$

$$\begin{aligned}
(viii) \quad L^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} &= \cosh at & (ix) \quad L^{-1} \left\{ \frac{1}{(s-a)^2 + b^2} \right\} &= \frac{1}{b} e^{at} \sin bt \\
(x) \quad L^{-1} \left\{ \frac{s-a}{(s-a)^2 + b^2} \right\} &= e^{at} \cdot \cos bt & (xi) \quad L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} &= \frac{1}{2a} t \sin at \\
(xii) \quad L^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} &= \frac{1}{2a^3} (\sin at - at \cos at).
\end{aligned}$$

25. Find the inverse Laplace transforms of the following functions :

$$\begin{aligned}
(i) \quad & \frac{3(s^2 - 1)^2}{2s^5} & (ii) \quad & \frac{3s + 5\sqrt{2}}{s^2 + 8} \\
(iii) \quad & \frac{4s + 15}{16s^2 - 25} & (iv) \quad & \frac{s + 1}{s^2 + s + 1}.
\end{aligned}$$

Sol. (i) $\frac{3(s^2 - 1)^2}{2s^5} = \frac{3s^4 - 6s^2 + 3}{2s^5} = \frac{3}{2} \cdot \frac{1}{s} - 3 \cdot \frac{1}{s^3} + \frac{3}{2} \cdot \frac{1}{s^5}$

$$\therefore L^{-1} \left\{ \frac{3(s^2 - 1)^2}{2s^5} \right\} = \frac{3}{2} L^{-1} \left\{ \frac{1}{s} \right\} - 3 L^{-1} \left\{ \frac{1}{s^3} \right\} + \frac{3}{2} L^{-1} \left\{ \frac{1}{s^5} \right\} = \frac{3}{2} (1) - 3 \left\{ \frac{t^2}{2!} \right\} + \frac{3}{2} \left\{ \frac{t^4}{4!} \right\}$$

or $L^{-1} \left\{ \frac{3(s^2 - 1)^2}{2s^5} \right\} = \frac{3}{2} - \frac{3}{2} t^2 + \frac{1}{16} t^4.$

(ii) Since $\frac{3s + 5\sqrt{2}}{s^2 + 8} = 3 \cdot \frac{s}{s^2 + (2\sqrt{2})^2} + 5\sqrt{2} \cdot \frac{1}{s^2 + (2\sqrt{2})^2}$

$$\begin{aligned}
\therefore L^{-1} \left\{ \frac{3s + 5\sqrt{2}}{s^2 + 8} \right\} &= 3 L^{-1} \left\{ \frac{s}{s^2 + (2\sqrt{2})^2} \right\} + 5\sqrt{2} \cdot L^{-1} \left\{ \frac{1}{s^2 + (2\sqrt{2})^2} \right\} \\
&= 3 \cos (2\sqrt{2} t) + 5\sqrt{2} \cdot \frac{1}{2\sqrt{2}} \sin (2\sqrt{2} t) \\
&= 3 \cos (2\sqrt{2} t) + \frac{5}{2} \sin (2\sqrt{2} t).
\end{aligned}$$

(iii) Since $\frac{4s + 15}{16s^2 - 25} = \frac{4s + 15}{16 \left(s^2 - \frac{25}{16} \right)} = \frac{1}{4} \cdot \frac{s}{s^2 - \left(\frac{5}{4} \right)^2} + \frac{15}{16} \cdot \frac{1}{s^2 - \left(\frac{5}{4} \right)^2}$

$$\begin{aligned}
\therefore L^{-1} \left\{ \frac{4s + 15}{16s^2 - 25} \right\} &= \frac{1}{4} L^{-1} \left[\frac{s}{s^2 - \left(\frac{5}{4} \right)^2} \right] + \frac{15}{16} \cdot L^{-1} \left(\frac{1}{s^2 - \left(\frac{5}{4} \right)^2} \right) \\
&= \frac{1}{4} \cosh \left(\frac{5}{4} t \right) + \frac{15}{16} \cdot \frac{1}{\frac{5}{4}} \cdot \sinh \left(\frac{5}{4} t \right) = \frac{1}{4} \cosh \left(\frac{5}{4} t \right) + \frac{3}{4} \sinh \left(\frac{5}{4} t \right).
\end{aligned}$$

$$\begin{aligned}
 \text{(iv) Since } \frac{s+1}{s^2+s+1} &= \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2} \cdot \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
 \therefore \mathcal{L}^{-1} \left\{ \frac{s+1}{s^2+s+1} \right\} &= \mathcal{L}^{-1} \left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right] + \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \right] \\
 &= e^{-\frac{1}{2}t} \cdot \cos \left(\frac{\sqrt{3}}{2}t \right) + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \cdot e^{-\frac{1}{2}t} \cdot \sin \left(\frac{\sqrt{3}}{2}t \right) = \frac{1}{\sqrt{3}} \cdot e^{-\frac{1}{2}t} \left(\sqrt{3} \cos \frac{\sqrt{3}}{2}t + \sin \frac{\sqrt{3}}{2}t \right).
 \end{aligned}$$

26. Find the inverse Laplace transform of $\frac{1}{s^2(s^2+a^2)}$. (M.D.U. Dec., 2007)

Sol. $\mathcal{L}^{-1} \left[\frac{1}{s^2(s^2+a^2)} \right]$

Compare with $\left[\frac{1}{s^2} \bar{f}(s) \right]$, $\bar{f}(s) = \frac{1}{s^2+a^2}$, $f(t) = \frac{1}{a} \sin at$

$$\begin{aligned}
 \mathcal{L}^{-1} \left[\frac{1}{s^2} \bar{f}(s) \right] &= \int_0^t \left[\int_0^t \frac{1}{a} \sin at \, dt \right] = \int_0^t \left(\frac{1}{a} \cdot \frac{(-\cos at)}{a} \right)_0^t dt \\
 &= \int_0^t \frac{1}{a^2} \cdot (1 - \cos at) dt = \frac{1}{a^2} \int_0^t (1 - \cos at) dt \\
 &= \frac{1}{a^2} \left[t - \frac{\sin at}{a} \right]_0^t = \frac{1}{a^2} \left[t - \frac{\sin at}{a} \right].
 \end{aligned}$$

27. Find the inverse Laplace transforms of

$$\begin{aligned}
 \text{(i)} \quad & \frac{2s^2-4}{(s+1)(s-2)(s-3)} & \text{(ii)} \quad & \frac{2s^2-1}{(s^2+1)(s^2+4)} & \text{(iii)} \quad & \frac{21s-33}{(s+1)(s-2)^3}.
 \end{aligned}$$

Sol. (i) Here the denominator has non-repeated linear factors.

Let $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

Multiplying both sides by $(s+1)(s-2)(s-3)$, we have

$$2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$$

Putting $s = -1$, we get $-2 = A(-3)(-4) \Rightarrow A = -\frac{1}{6}$

Putting $s = 2$, we get $4 = B(3)(-1) \Rightarrow B = -\frac{4}{3}$

Putting $s = 3$, we get $14 = C(4)(1) \Rightarrow C = \frac{7}{2}$

Thus
$$\frac{2s^2 - 4}{(s+1)(s-2)(s-3)} = -\frac{1}{6(s+1)} - \frac{4}{3(s-2)} + \frac{7}{2(s-3)}$$

$$\begin{aligned} \therefore L^{-1} \left\{ \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \right\} &= -\frac{1}{6} L^{-1} \left(\frac{1}{s+1} \right) - \frac{4}{3} L^{-1} \left(\frac{1}{s-2} \right) + \frac{7}{2} L^{-1} \left(\frac{1}{s-3} \right) \\ &= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t}. \end{aligned}$$

(ii) Put $s^2 = p$. The given fraction becomes, $\frac{2p-1}{(p+1)(p+4)}$ which has non-repeated linear factors in the denominator.

Resolving it into partial fractions ;

Let
$$\frac{2p-1}{(p+1)(p+4)} = \frac{A}{p+1} + \frac{B}{p+4} \quad \text{or} \quad 2p-1 = A(p+4) + B(p+1)$$

Putting $p = -1, -4$, we get $A = -1, B = 3$

Substituting the values of A and B and replacing p by s^2 , we get

$$\begin{aligned} \frac{2s^2 - 1}{(s^2+1)(s^2+4)} &= -\frac{1}{s^2+1} + \frac{3}{s^2+4} \\ \therefore L^{-1} \left\{ \frac{2s^2 - 1}{(s^2+1)(s^2+4)} \right\} &= (-1) L^{-1} \left(\frac{1}{s^2+1^2} \right) + 3 L^{-1} \left(\frac{1}{s^2+2^2} \right) = -\sin t + \frac{3}{2} \sin 2t. \end{aligned}$$

(iii) Here the denominator has a linear factor repeated thrice.

Let
$$\frac{21s-33}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

Multiplying both sides by $(s+1)(s-2)^3$, we get

$$21s - 33 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Putting $s = -1$, we get $-54 = A(-27) \Rightarrow A = 2$

Putting $s = 2$, we get $9 = D(3) \Rightarrow D = 3$

Equating co-efficients of s^3 , $0 = A + B \Rightarrow B = -A = -2$

Putting $s = 0$, we get $-33 = -8A + 4B - 2C + D$

or $-33 = -16 - 8 - 2C + 3 \quad \text{or} \quad C = 6$

Thus
$$\frac{21s-33}{(s+1)(s-2)^3} = \frac{2}{s+1} - \frac{2}{s-2} + \frac{6}{(s-2)^2} + \frac{3}{(s-2)^3}$$

$$\begin{aligned} \therefore L^{-1} \left\{ \frac{21s-33}{(s+1)(s-2)^3} \right\} &= 2L^{-1} \left\{ \frac{1}{s+1} \right\} - 2L^{-1} \left\{ \frac{1}{s-2} \right\} + 6L^{-1} \left\{ \frac{1}{(s-2)^2} \right\} + 3L^{-1} \left\{ \frac{1}{(s-2)^3} \right\} \\ &= 2e^{-t} - 2e^{2t} + 6e^{2t} \cdot \frac{t}{1!} + 3 \cdot e^{2t} \cdot \frac{t^2}{2!} \\ &\quad \left(\text{Since } L^{-1} \left\{ \frac{1}{(s-a)^n} \right\} = e^{at} \cdot \frac{t^{n-1}}{(n-1)!} \right) \\ &= 2e^{-t} - 2e^{2t} + 6te^{2t} + \frac{3}{2} t^2 e^{2t}. \end{aligned}$$

28. Find the inverse Laplace transforms of

$$(i) \frac{5s+3}{(s-1)(s^2+2s+5)}$$

$$(ii) \frac{s}{s^4+s^2+1}$$

$$(iii) \frac{s}{s^4+4a^4}$$

$$(iv) \frac{s^3}{s^4-a^4}.$$

Sol. (i) Here the denominator has a non-factorisable quadratic factor s^2+2s+5 .

$$\text{Let } \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

Multiplying both sides by $(s-1)(s^2+2s+5)$, we get

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$

Putting $s=1$, we get $8 = 8A \Rightarrow A=1$

Equating the coefficients of s^2 , $0 = A+B \Rightarrow B = -A = -1$

Putting $s=0$, we get $3 = 5A - C \Rightarrow C = 5A - 3 = 2$

$$\begin{aligned} \text{Thus } \frac{5s+3}{(s-1)(s^2+2s+5)} &= \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5} \\ &= \frac{1}{s-1} - \frac{(s+1)-3}{(s+1)^2+4} = \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + 3 \cdot \frac{1}{(s+1)^2+2^2} \end{aligned}$$

$$\begin{aligned} \therefore L^{-1} \left\{ \frac{5s+3}{(s-1)(s^2+2s+5)} \right\} &= L^{-1} \left\{ \frac{1}{s-1} \right\} - L^{-1} \left\{ \frac{s+1}{(s+1)^2+2^2} \right\} + 3L^{-1} \left\{ \frac{1}{(s+1)^2+2^2} \right\} \\ &= e^t - e^{-t} \cos 2t + \frac{3}{2} e^{-t} \sin 2t \end{aligned}$$

(ii) Since $s^4+s^2+1 = (s^4+2s^2+1) - (s^2) = (s^2+1)^2 - s^2 = (s^2+1+s)(s^2+1-s)$

$$\therefore \frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1+s)(s^2+1-s)}$$

$$\text{Let } \frac{s}{(s^2+1+s)(s^2+1-s)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

Multiplying both sides by $(s^2+s+1)(s^2-s+1)$, we get

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

Equating coefficients of s^3 , $0 = A+C$ or $C = -A$... (1)

Equating coefficients of s^2 , $0 = -A+B+C+D$... (2)

Equating coefficients of s , $1 = A-B+C+D$... (3)

Putting $s=0$, $0 = B+D$ or $D = -B$... (4)

From (2), $0 = -A+B-A-B$

$\Rightarrow A=0$, $\therefore$ From (1), $C=0$

From (3), $1 = A-B-A-B \Rightarrow B = -\frac{1}{2}$

$\therefore$ From (4), $D = \frac{1}{2}$

$$\begin{aligned} \text{Thus } \frac{s}{s^4+s^2+1} &= \frac{-\frac{1}{2}}{s^2+s+1} + \frac{\frac{1}{2}}{s^2-s+1} = \frac{1}{2} \left[\frac{1}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} \right] \end{aligned}$$

$$\begin{aligned}
\therefore \quad \mathcal{L}^{-1} \left[\frac{s}{s^4 + s^2 + 1} \right] &= \frac{1}{2} \left[\mathcal{L}^{-1} \left\{ \frac{1}{\left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} \right] \\
&= \frac{1}{2} \left[\frac{1}{\sqrt{3}/2} e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t - \frac{1}{\sqrt{3}/2} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right] \\
&= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(\frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2} \right) = \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \cdot \sinh \frac{t}{2}.
\end{aligned}$$

(iii) Since $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$

$$\begin{aligned}
\therefore \quad \frac{s}{s^4 + 4a^4} &= \frac{s}{(s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)} \\
&= \frac{1}{4a} \left(\frac{1}{s^2 - 2as + 2a^2} - \frac{1}{s^2 + 2as + 2a^2} \right) \\
&= \frac{1}{4a} \left[\frac{1}{(s-a)^2 + a^2} - \frac{1}{(s+a)^2 + a^2} \right] \\
\mathcal{L}^{-1} \left[\frac{s}{s^4 + 4a^4} \right] &= \frac{1}{4a} \left[\mathcal{L}^{-1} \left\{ \frac{1}{(s-a)^2 + a^2} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+a)^2 + a^2} \right\} \right] \\
&= \frac{1}{4a} \left\{ \frac{1}{a} e^{at} \sin at - \frac{1}{a} e^{-at} \sin at \right\} = \frac{1}{2a^2} \sin at \left\{ \frac{e^{at} - e^{-at}}{2} \right\} \\
&= \frac{1}{2a^2} \sin at \sinh at.
\end{aligned}$$

$$(iv) \quad \frac{s^3}{s^4 - a^4} = s \left[\frac{s^2}{(s^2 - a^2)(s^2 + a^2)} \right] = \frac{1}{2} s \left[\frac{1}{s^2 - a^2} + \frac{1}{s^2 + a^2} \right] = \frac{1}{2} \left[\frac{s}{s^2 - a^2} + \frac{s}{s^2 + a^2} \right]$$

$$\therefore \quad \mathcal{L}^{-1} \left\{ \frac{s^3}{s^4 - a^4} \right\} = \frac{1}{2} \left[\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} \right] = \frac{1}{2} (\cosh at + \cos at).$$

29. Find the inverse Laplace transform of $\frac{s^2}{(s+1)^3}$. (M.D.U. May, 2007)

Sol. Put $s+1 = u$ or $s = u-1$ or $s^2 = (u-1)^2 = u^2 - 2u + 1$

$$\frac{s^2}{(s+1)^3} = \frac{(u-1)^2}{u^3} = \frac{u^2 - 2u + 1}{u^3} = \frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3} = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}$$

$$\therefore \quad \mathcal{L}^{-1} \left[\frac{s^2}{(s+1)^3} \right] = \mathcal{L}^{-1} \left[\frac{1}{s+1} \right] - \mathcal{L}^{-1} \left[\frac{2}{(s+1)^2} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^3} \right] = e^{-2t} - 2te^{-2t} + \frac{t^2}{2} e^{-3t}.$$

30. Find the inverse Laplace transforms of

$$(i) \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}$$

$$(ii) \frac{1}{s^{3/2}} + \frac{4}{s-2} + \frac{2s-18}{s^2-9}$$

$$(iii) \frac{2s+9}{s^2+9} + \frac{12}{4-3s} + \frac{1}{\sqrt{s}}.$$

Sol. (i) $L^{-1} \left(\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right)$

$$\begin{aligned} &= 3L^{-1} \left(\frac{1}{s-3/2} \right) - \frac{1}{3} L^{-1} \left\{ \frac{1}{s^2 - \left(\frac{4}{3}\right)^2} \right\} - \frac{4}{9} L^{-1} \left\{ \frac{s}{s^2 - \left(\frac{4}{3}\right)^2} \right\} \\ &\quad + \frac{1}{2} L^{-1} \left\{ \frac{1}{s^2 + \left(\frac{3}{4}\right)^2} \right\} - \frac{3}{8} L^{-1} \left\{ \frac{s}{s^2 + \left(\frac{3}{4}\right)^2} \right\} \\ &= 3e^{\frac{3}{2}t} - \frac{1}{3} \cdot \frac{3}{4} \sinh \frac{4}{3}t - \frac{4}{9} \cdot \cosh \frac{4}{3}t + \frac{1}{2} \cdot \frac{4}{3} \sin \frac{3}{4}t - \frac{3}{8} \cos \frac{3}{4}t \\ &= 3e^{\frac{3}{2}t} - \frac{1}{4} \sinh \frac{4}{3}t - \frac{4}{9} \cosh \frac{4}{3}t + \frac{2}{3} \sin \frac{3}{4}t - \frac{3}{8} \cos \frac{3}{4}t. \end{aligned}$$

$$\begin{aligned} (ii) \quad L^{-1} \left(\frac{1}{s^{3/2}} + \frac{4}{s-2} + \frac{2s-18}{s^2-9} \right) &= L^{-1} \left(\frac{1}{s^{3/2}} \right) + 4L^{-1} \left(\frac{1}{s-2} \right) + 2L^{-1} \left(\frac{s}{s^2-9} \right) - 18L^{-1} \left(\frac{1}{s^2-9} \right) \\ &= \frac{t^{1/2}}{\Gamma(3/2)} + 4e^{2t} + 2 \cosh 3t - 6 \sinh 3t \\ &= 2 \sqrt{\frac{t}{\pi}} + 4e^{2t} + 2 \cosh 3t - 6 \sinh 3t \quad \left(\because \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \right) \end{aligned}$$

$$\begin{aligned} (iii) \quad L^{-1} \left\{ \frac{2s+9}{s^2+9} + \frac{12}{4-3s} + \frac{1}{\sqrt{s}} \right\} &= L^{-1} \left\{ \frac{2s+9}{s^2+9} \right\} - L^{-1} \left(\frac{12}{3s-4} \right) + L^{-1} \left(\frac{1}{\sqrt{s}} \right) \\ &= 2L^{-1} \left(\frac{s}{s^2+9} \right) + 3L^{-1} \left(\frac{3}{s^2+9} \right) - 4L^{-1} \left(\frac{1}{s-\frac{4}{3}} \right) + L^{-1} \left(\frac{1}{\sqrt{s}} \right) \\ &= 2 \cos 3t + 3 \sin 3t - 4e^{\frac{4}{3}t} + \frac{1}{\sqrt{\pi t}}. \end{aligned}$$

31. Find the inverse Laplace transforms of

$$(i) \frac{2s+1}{s(s+1)}$$

$$(ii) \left(\frac{\sqrt{s-1}}{s} \right)^2$$

$$(iii) \frac{1}{\sqrt{2s+3}}.$$

$$\text{Sol. (i)} \quad \mathcal{L}^{-1} \left[\frac{2s+1}{s(s+1)} \right] = \mathcal{L}^{-1} \left[\frac{s+(s+1)}{s(s+1)} \right] = \mathcal{L}^{-1} \left[\frac{1}{s+1} + \frac{1}{s} \right] = e^{-t} + 1$$

$$\begin{aligned} \text{(ii)} \quad \mathcal{L}^{-1} \left(\frac{\sqrt{s}-1}{s} \right)^2 &= \mathcal{L}^{-1} \left(\frac{s+1-2\sqrt{s}}{s^2} \right) = \mathcal{L}^{-1} \left(\frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^{3/2}} \right) \\ &= \mathcal{L}^{-1} \left(\frac{1}{s} \right) + \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) - 2\mathcal{L}^{-1} \left(\frac{1}{s^{3/2}} \right) = 1 + t - 2 \cdot \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} = 1 + t - 4\sqrt{\frac{t}{\pi}}. \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \mathcal{L}^{-1} \left(\frac{1}{\sqrt{2s+3}} \right) &= \mathcal{L}^{-1} \left(\frac{1}{\sqrt{2} \cdot \sqrt{s+3/2}} \right) = \frac{1}{\sqrt{2}} \cdot \mathcal{L}^{-1} \left(\frac{1}{\sqrt{s+\frac{3}{2}}} \right) \\ &= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \cdot \mathcal{L}^{-1} \left(\frac{1}{\sqrt{s}} \right) = \frac{1}{\sqrt{2\pi t}} \cdot e^{-\frac{3}{2}t}. \end{aligned}$$

32. Find the inverse Laplace transforms of

$$\text{(i)} \quad \frac{1}{\sqrt{s+a}} \qquad \text{(ii)} \quad \frac{s}{2s^2+8} \qquad \text{(iii)} \quad \frac{15}{s^2+4s+13} \qquad \text{(iv)} \quad \frac{s+8}{s^2+4s+5}.$$

$$\text{Sol. (i)} \quad \mathcal{L}^{-1} \left(\frac{1}{\sqrt{s+a}} \right) = e^{-at} \cdot \mathcal{L}^{-1} \left(\frac{1}{\sqrt{s}} \right) = e^{-at} \cdot \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{e^{-at}}{\sqrt{\pi t}}.$$

$$\text{(ii)} \quad \mathcal{L}^{-1} \left(\frac{s}{2s^2+8} \right) = \frac{1}{2} \mathcal{L}^{-1} \left(\frac{s}{s^2+4} \right) = \frac{1}{2} \cos 2t$$

$$\text{(iii)} \quad \mathcal{L}^{-1} \left(\frac{15}{s^2+4s+13} \right) = \mathcal{L}^{-1} \left\{ \frac{15}{(s+2)^2+9} \right\} = e^{-2t} \cdot \mathcal{L}^{-1} \left(\frac{15}{s^2+9} \right)$$

[By first shifting property]

$$= e^{-2t} \cdot 15 \cdot \frac{1}{3} \cdot \sin 3t = 5 \cdot e^{-2t} \cdot \sin 3t$$

$$\begin{aligned} \text{(iv)} \quad \mathcal{L}^{-1} \left(\frac{s+8}{s^2+4s+5} \right) &= \mathcal{L}^{-1} \left\{ \frac{(s+2)+6}{(s+2)^2+1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+1} \right\} + 6 \cdot \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2+1} \right\} \\ &= e^{-2t} \cdot \mathcal{L}^{-1} \left(\frac{s}{s^2+1} \right) + 6e^{-2t} \cdot \mathcal{L}^{-1} \left(\frac{1}{s^2+1} \right) \\ &= e^{-2t} \cos t + 6e^{-2t} \cdot \sin t = e^{-2t} \cdot (\cos t + 6 \sin t). \end{aligned}$$

33. Find the inverse Laplace transforms of

$$\text{(i)} \quad \frac{s^2+2s-3}{s(s-3)(s+2)}$$

$$\text{(ii)} \quad \frac{1+2s}{(s+2)^2(s-1)^2}$$

$$\text{(iii)} \quad \frac{1}{(s-\alpha)(s-\beta)}$$

$$\text{(iv)} \quad \frac{s}{(s^2-1)^2}.$$

Sol. (i) Let, $\frac{s^2 + 2s - 3}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$

Multiplying both sides by $s(s-3)(s+2)$, we have

$$s^2 + 2s - 3 = A(s-3)(s+2) + B(s+2)s + Cs(s-3)$$

Comparing the coefficients, we get

$$A + B + C = 1, -A + 2B - 3C = 2, -6A = -3$$

Solving these, we get $A = \frac{1}{2}, B = \frac{4}{5}, C = -\frac{3}{10}$

$$\therefore \frac{s^2 + 2s - 3}{s(s-3)(s+2)} = \frac{1}{2s} + \frac{4}{5(s-3)} - \frac{3}{10(s+2)}$$

$$\therefore L^{-1} \left\{ \frac{s^2 + 2s - 3}{s(s-3)(s+2)} \right\} = \frac{1}{2} L^{-1} \left(\frac{1}{s} \right) + \frac{4}{5} L^{-1} \left(\frac{1}{s-3} \right) - \frac{3}{10} L^{-1} \left(\frac{1}{s+2} \right) = \frac{1}{2} + \frac{4}{5} e^{3t} - \frac{3}{10} e^{-2t}$$

$$(ii) \quad \frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$\Rightarrow \begin{aligned} 1+2s &= A(s+2)(s-1)^2 + B(s-1)^2 + C(s+2)^2(s-1) + D(s+2)^2 \\ &= A(s^3 - 3s + 2) + B(s^2 - 2s + 1) + C(s^3 + 3s^2 - 4) + D(s^2 + 4s + 4) \end{aligned}$$

Comparing the coefficients of s^3, s^2, s and constants on both sides, we have

$$A + C = 0, B + 3C + D = 0, -3A - 2B + 4D = 2, 2A + B - 4C + 4D = 1$$

On solving these, we get $A = 0, C = 0, B = -\frac{1}{3}, D = \frac{1}{3}$

$$\therefore \frac{1+2s}{(s+2)^2(s-1)^2} = -\frac{1}{3(s+2)^2} + \frac{1}{3(s-1)^2}$$

$$\begin{aligned} \therefore L^{-1} \left\{ \frac{1+2s}{(s+2)^2(s-1)^2} \right\} &= -\frac{1}{3} L^{-1} \left(\frac{1}{(s+2)^2} \right) + \frac{1}{3} L^{-1} \left(\frac{1}{(s-1)^2} \right) \\ &= -\frac{1}{3} \cdot e^{-2t} \cdot t + \frac{1}{3} e^t \cdot t = \frac{t}{3} (e^t - e^{-2t}) \end{aligned}$$

$$(iii) \text{ We have } \frac{1}{(s-\alpha)(s-\beta)} = \frac{1}{\alpha-\beta} \left(\frac{1}{s-\alpha} - \frac{1}{s-\beta} \right)$$

$$\therefore L^{-1} \left[\frac{1}{(s-\alpha)(s-\beta)} \right] = \frac{1}{\alpha-\beta} \cdot L^{-1} \left(\frac{1}{s-\alpha} - \frac{1}{s-\beta} \right) = \frac{1}{\alpha-\beta} \cdot (e^{\alpha t} - e^{\beta t})$$

$$(iv) \quad \frac{1}{(s^2-1)^2} = \frac{1}{(s+1)^2(s-1)^2} = \frac{1}{4s} \left\{ \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2} \right\}$$

$$\frac{s}{(s^2-1)^2} = \frac{1}{4} \left\{ \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2} \right\}$$

$$\therefore L^{-1} \left\{ \frac{s}{(s^2-1)^2} \right\} = \frac{1}{4} \left[L^{-1} \left\{ \frac{1}{(s-1)^2} \right\} - L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \right] = \frac{1}{4} \cdot (e^t \cdot t - e^{-t} \cdot t) = \frac{t}{2} \cdot \sinh t$$

34. Find the inverse Laplace transform of the function $f(s) = \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$.

Sol. We observe that denominator,

$$s^3 - 6s^2 + 11s - 6 = (s - 1)(s - 2)(s - 3)$$

$$\text{Let } \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s - 3}$$

$$\therefore 2s^2 - 6s + 5 = A(s - 2)(s - 3) + B(s - 1)(s - 3) + C(s - 1)(s - 2)$$

$$\text{Putting } s = 1, \text{ we get } 2A = 1 \text{ or } A = \frac{1}{2}$$

$$\text{Putting } s = 2, \text{ we get } -B = 1 \text{ or } B = -1$$

$$\text{Putting } s = 3, \text{ we get } 2C = 5 \text{ or } C = \frac{5}{2}$$

$$\text{Hence } \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{1}{2(s - 1)} - \frac{1}{s - 2} + \frac{5}{2(s - 3)}$$

$$\therefore L^{-1} \left[\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} \right] = L^{-1} \left[\frac{1}{2(s - 1)} \right] - L^{-1} \left[\frac{1}{s - 2} \right] + \frac{5}{2} L^{-1} \left[\frac{1}{s - 3} \right] = \frac{1}{2} e^t - e^{2t} + \frac{5}{2} e^{3t}.$$

35. If $L^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} = \frac{1}{2} t \sin t$, find $L^{-1} \left\{ \frac{32s}{(16s^2 + 1)^2} \right\}$.

$$\text{Sol. Replacing } s \text{ by } as, \text{ we get } L^{-1} \left\{ \frac{as}{(a^2s^2 + 1)^2} \right\} = \frac{1}{2} \cdot \frac{t}{a} \sin \frac{t}{a}$$

Now, writing $a = 4$

$$L^{-1} \left\{ \frac{4s}{(16s^2 + 1)^2} \right\} = \frac{1}{8} t \sin \frac{t}{4} \quad \text{or} \quad \frac{1}{8} L^{-1} \left\{ \frac{32s}{(16s^2 + 1)^2} \right\} = \frac{1}{8} t \sin \frac{t}{4}$$

$$\therefore L^{-1} \left\{ \frac{32s}{(16s^2 + 1)^2} \right\} = t \sin \frac{t}{4}.$$

36. Find the inverse Laplace transform of $\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)}$. (U.P.T.U., 2005)

$$\text{Sol. Let } \frac{5s + 3}{(s - 1)(s^2 + 2s + 5)} = \frac{5(1) + 3}{(s - 1)(1^2 + 2 + 5)} + \frac{As + B}{s^2 + 2s + 5}$$

Multiplying both sides by $(s - 1)(s^2 + 2s + 5)$,

$$5s + 3 = 1 \cdot (s^2 + 2s + 5) + (As + B)(s - 1)$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A \quad \therefore A = -1$$

Similarly comparing coefficient of s from both sides,

$$5 = 2 - A + B \quad \text{or} \quad B = 3 + A = 2$$

$$\therefore L^{-1} \left[\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)} \right] = L^{-1} \left[\frac{1}{s - 1} \right] + L^{-1} \left[\frac{-s + 2}{s^2 + 2s + 5} \right] = L^{-1} \left[\frac{1}{s - 1} \right] + L^{-1} \left[\frac{-(s + 1) + 3}{(s + 1)^2 + 4} \right]$$

$$\begin{aligned}
&= L^{-1} \left(\frac{1}{s-1} \right) - L^{-1} \left[\frac{s+1}{(s+1)^2+4} \right] + 3L^{-1} \left[\frac{1}{(s+1)^2+4} \right] \\
&= e^t - e^t \cos 2t + \frac{3}{2} e^{-t} \sin 2t.
\end{aligned}$$

37. Find the inverse Laplace transforms of

$$(i) \frac{s+2}{s^2-4s+13} \quad (ii) \frac{4s+5}{(s-1)^2(s+2)} \quad (iii) \frac{s+3}{s^2-4s+13} \quad (iv) \frac{s^2}{(s-2)^3}.$$

Sol. (i) $L^{-1} \left(\frac{s+2}{s^2-4s+13} \right) = L^{-1} \left(\frac{s+2}{(s-2)^2+9} \right) = L^{-1} \left(\frac{s-2+4}{(s-2)^2+9} \right)$

$$= L^{-1} \left(\frac{s-2}{(s-2)^2+9} \right) + 4L^{-1} \left(\frac{1}{(s-2)^2+3^2} \right) = e^{2t} \cos 3t + \frac{4}{3} e^{2t} \sin 3t.$$

(ii) $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$

Multiplying both sides by $(s-1)^2(s+2)$,

$$4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$$

Putting $s=1$, we get $9=3B \Rightarrow B=3$

Equating the coefficients of s^2 from both sides, $0 = A - \frac{1}{3}$, $\therefore A = \frac{1}{3}$

$$\therefore L^{-1} \left[\frac{4s+5}{(s-1)^2(s+2)} \right] = \frac{1}{3} L^{-1} \left(\frac{1}{s-1} \right) + 3L^{-1} \left[\frac{1}{(s-1)^2} \right] - \frac{1}{3} L^{-1} \left(\frac{1}{s+2} \right) = \frac{1}{3} e^t + 3te^t - \frac{1}{3} e^{-2t}.$$

(iii) $\frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$

$$\therefore L^{-1} \left\{ \frac{s+3}{s^2-4s+13} \right\} = L^{-1} \left\{ \frac{s-2}{(s-2)^2+3^2} \right\} + \frac{5}{3} L^{-1} \left\{ \frac{3}{(s-2)^2+3^2} \right\} \quad (\text{Using shifting property})$$

$$= e^{2t} \cos 3t + \frac{5}{3} e^{2t} \sin 3t$$

(iv) $\frac{s^2}{(s-2)^3} = \frac{(s-2)^2+4(s-2)+4}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$

$$\therefore L^{-1} \left[\frac{s^2}{(s-2)^3} \right] = L^{-1} \left(\frac{1}{s-2} \right) + 4L^{-1} \left\{ \frac{1}{(s-2)^2} \right\} + 4L^{-1} \left\{ \frac{1}{(s-2)^3} \right\}$$

$$= e^{2t} + 4e^{2t} \cdot t + 2e^{2t} \cdot t^2 \quad (\text{Using shifting property})$$

38. State the theorems regarding Laplace transforms of integrals and derivatives.

Sol. (a) Theorem on Laplace transform of derivatives is :

If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$,

then $L\{f'(t)\} = s\bar{f}(s) - f(0)$, provided $\lim_{t \rightarrow \infty} \{e^{-st} \cdot f(t)\} = 0$

(b) Theorem on Laplace transforms of integrals states :

$$\text{If } L\{f(t)\} = \bar{f}(s), \text{ then } L\left\{\int_0^t f(u) du\right\} = \frac{1}{s} \bar{f}(s) \quad \text{or} \quad L^{-1}\left\{\frac{1}{s} \bar{f}(s)\right\} = \int_0^t f(u) du.$$

39. If $L\{t \sin wt\} = \frac{2ws}{(s^2 + w^2)^2}$, evaluate

(i) $L\{wt \cos wt + \sin wt\}$ (ii) $L\{2 \cos wt - wt \sin wt\}$. (U.P.T.U., 2005, 2006)

Sol. Let

$$f(t) = t \sin wt$$

Then

$$f'(t) = wt \cos wt + \sin wt$$

$$f''(t) = 2w \cos wt - w^2 t \sin wt$$

Also

$$f(0) = 0, f'(0) = 0, f''(0) = 2w$$

$$\bar{f}(s) = \frac{2ws}{(s^2 + w^2)^2}$$

Since

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - s^{n-3} \cdot f''(0) \dots$$

(i)

$$L\{f'(t)\} = s \bar{f}(s) - f(0)$$

$\Rightarrow$

$$L\{wt \cos wt + \sin wt\} = s \cdot \frac{2ws}{(s^2 + w^2)^2} - 0 = \frac{2ws^2}{(s^2 + w^2)^2}$$

(ii)

$$L\{f''(t)\} = s^2 \bar{f}(s) - s \cdot f(0) - f'(0)$$

$\Rightarrow$

$$L\{2w \cos wt - w^2 t \sin wt\} = s^2 \cdot \frac{2ws}{(s^2 + w^2)^2} - s \cdot 0 - 0 = \frac{2ws^3}{(s^2 + w^2)^2}$$

$\therefore$

$$L\{2 \cos wt - wt \sin wt\} = \frac{2s^3}{(s^2 + w^2)^2}.$$

40. Find the inverse Laplace transforms of

$$(i) \frac{1}{s^3(s^2 + a^2)}$$

$$(ii) \frac{1}{s(s+1)^3}.$$

Sol. (i) Since, $L^{-1}\left[\frac{1}{s^2 + a^2}\right] = \frac{1}{a} \sin at$

therefore by "Laplace transform of integral" Theorem, we have

$$L^{-1}\left[\frac{1}{s(s^2 + a^2)}\right] = \int_0^t \frac{1}{a} \sin au \, du = -\frac{1}{a^2} \left[\cos au\right]_0^t = \frac{1}{a^2} (1 - \cos at)$$

$$L^{-1}\left[\frac{1}{s^2(s^2 + a^2)}\right] = \int_0^t \frac{1}{a^2} (1 - \cos au) \, du = \frac{1}{a^2} \left[u - \frac{\sin au}{a}\right]_0^t = \frac{1}{a^2} \left[t - \frac{\sin at}{a}\right]$$

$$L^{-1}\left[\frac{1}{s^3(s^2 + a^2)}\right] = \int_0^t \frac{1}{a^2} \left(u - \frac{\sin au}{a}\right) du = \frac{1}{a^2} \left[\frac{u^2}{2} + \frac{\cos au}{a^2}\right]_0^t = \frac{1}{a^2} \left[\frac{t^2}{2} + \frac{\cos at}{a^2} - \frac{1}{a^2}\right]$$

(ii) Since, $L^{-1}\left[\frac{1}{(s+1)^3}\right] = \frac{e^{-t} \cdot t^2}{2!} = \frac{1}{2} t^2 \cdot e^{-t}$

$$\begin{aligned} \therefore L^{-1}\left[\frac{1}{s(s+1)^3}\right] &= \frac{1}{2} \int_0^t t^2 e^{-t} \cdot dt = \frac{1}{2} \left[t^2 (-e^{-t}) - 2t (e^{-t}) + 2(-e^{-t})\right]_0^t \\ &= \frac{1}{2} [(-t^2 - 2t - 2)e^{-t} + 2] = 1 - e^{-t} \left(1 + t + \frac{t^2}{2}\right). \end{aligned}$$

41. Find the Laplace transforms of (i) $t \sin^2 3t$, (ii) $t^3 \cdot e^{-3t}$. (M.D.U. 2009, U.P.T.U., 2005, 2007, 2008)

Sol. (i) Since $L(\sin^2 3t) = L\left\{\frac{1 - \cos 6t}{2}\right\} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 36}\right) = \frac{18}{s(s^2 + 36)}$

$$\begin{aligned}\therefore L\{t \sin^2 3t\} &= (-1) \frac{d}{ds} \left[\frac{18}{s(s^2 + 36)} \right] \\ &= -18(-1)(s^3 + 36s)^{-2} (3s^2 + 36) = \frac{54(s^2 + 12)}{s^2(s^2 + 36)^2}.\end{aligned}$$

(ii) Since $L(e^{-3t}) = \frac{1}{s+3}$

$$\therefore L\{t^3 \cdot e^{-3t}\} = (-1)^3 \cdot \frac{d^3}{ds^3} \left(\frac{1}{s+3} \right) = -\frac{(-1)^3 \cdot 3!}{(s+3)^4} = \frac{6}{(s+3)^4}.$$

42. Find the inverse Laplace transform of $\log \frac{s+1}{s-1}$.

Sol. Let $L^{-1} \left\{ \log \frac{s+1}{s-1} \right\} = f(t)$

Then $L\{f(t)\} = \bar{f}(s) = \log \frac{s+1}{s-1}$.

$$\begin{aligned}\therefore L\{t \cdot f(t)\} &= (-1) \frac{d}{ds} \left\{ \log \frac{s+1}{s-1} \right\} = -\frac{d}{ds} [\log(s+1) - \log(s-1)] \\ &= -\left[\frac{1}{s+1} - \frac{1}{s-1} \right] = \frac{1}{s-1} - \frac{1}{s+1} = L\{e^t - e^{-t}\} = L\{2 \sinh t\}\end{aligned}$$

$$\Rightarrow t \cdot f(t) = 2 \sinh t \text{ or } f(t) = \frac{2 \sinh t}{t}. \text{ Hence } L^{-1} \left\{ \log \frac{s+1}{s-1} \right\} = \frac{2 \sinh t}{t}.$$

Note. In questions 41 and 42, we have used the result :

If $L\{f(t)\} = \bar{f}(s)$ then $L\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n} [\bar{f}(s)]$, where $n = 1, 2, 3, \dots$

(M.D.U., U.P.T.U., 2009)

Thus this is useful to find the transform and inverse Laplace transform of the function $f(t)$ when

multiplied by t^n . Also when there is a division of $f(t)$ by t , then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s) ds$, provided the integral exists.

43. Find the Laplace transforms of

(i) $\frac{e^{-at} - e^{-bt}}{t}$

(ii) $\frac{\cos at - \cos bt}{t}$

(iii) $\frac{e^{-t} \cdot \sin t}{t}$

(iv) $\frac{1 - \cos 2t}{t}$.

(U.P.T.U., 2006, A.U.U.P., 2009 Uttarakhand June, 2007)

Sol. (i) Since $L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = [\log(s+a) - \log(s+b)]_s^\infty$$

$$\begin{aligned}
&= \left[\log \frac{s+a}{s+b} \right]_s^\infty = \left[\log \frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right]_s^\infty = \log 1 - \log \frac{1+\frac{a}{s}}{1+\frac{b}{s}} = 0 - \log \frac{s+a}{s+b} \\
&= \log \frac{s+b}{s+a}
\end{aligned}$$

$$(ii) \text{ Since } L(\cos at - \cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\begin{aligned}
\therefore L \left\{ \frac{\cos at - \cos bt}{t} \right\} &= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right) ds \\
&= \left[\frac{1}{2} \log(s^2+a^2) - \frac{1}{2} \log(s^2+b^2) \right]_s^\infty = \frac{1}{2} \left[\log \frac{s^2+a^2}{s^2+b^2} \right]_s^\infty \\
&= \frac{1}{2} \left[\log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} \right]_s^\infty = \frac{1}{2} \left[\log 1 - \log \frac{s^2+a^2}{s^2+b^2} \right] = \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right).
\end{aligned}$$

$$(iii) \quad L(e^{-t} \cdot \sin t) = \frac{1}{(s+1)^2+1}$$

$$\begin{aligned}
\Rightarrow L \left(\frac{e^{-t} \cdot \sin t}{t} \right) &= \int_s^\infty \frac{1}{(s+1)^2+1} \cdot ds = \left[\tan^{-1}(s+1) \right]_s^\infty \\
&= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)
\end{aligned}$$

$$(iv) \quad L(1 - \cos 2t) = \frac{1}{s} - \frac{s}{s^2+2^2}$$

$$\begin{aligned}
\Rightarrow L \left(\frac{1 - \cos 2t}{t} \right) &= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4} \right) ds = \left[\log s - \frac{1}{2} \log(s^2+4) \right]_s^\infty \\
&= \frac{1}{2} \left[\log \frac{s^2}{s^2+4} \right]_s^\infty = \frac{1}{2} \left[\log 1 - \log \frac{s^2}{s^2+4} \right] = \frac{1}{2} \log \left(\frac{s^2+4}{s^2} \right).
\end{aligned}$$

44. Find inverse Laplace transform of the function $\log \frac{s(s+1)}{s^2+4}$. (M.D.U. May, 2009)

$$\text{Sol. } L^{-1} \left[\log \frac{s(s+1)}{s^2+4} \right] = f(t) \quad \Rightarrow L\{f(t)\} = \log \left(\frac{s(s+1)}{s^2+4} \right) = f(s)$$

$$L[t f(t)] = -\frac{d}{ds} [\log s + \log(s+1) - \log(s^2+4)] = -\left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4} \right]$$

$$t f(t) = -L^{-1} \left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4} \right]$$

$$t \cdot f(t) = -[1 + e^{-t} - 2 \cos 2t] = -1 - e^{-t} + 2 \cos 2t$$

$$\therefore f(t) = \frac{-1 - e^{-t} + 2 \cos 2t}{t}.$$

45. Find the Laplace transforms of

$$(i) t^2 \sin at \qquad (ii) \frac{1-e^t}{t}.$$

Sol. (i) Since, $L(\sin at) = \frac{a}{s^2 + a^2}$

$$\therefore L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2} \left(\frac{a}{s^2 + a^2} \right) = \frac{d}{ds} \left\{ \frac{-2as}{(s^2 + a^2)^2} \right\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}$$

(ii) Since, $L(1 - e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1}$

$$\begin{aligned} \therefore L\left(\frac{1-e^t}{t}\right) &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \left[\log s - \log(s-1) \right]_s^\infty \\ &= \left[\log\left(\frac{s}{s-1}\right) \right]_s^\infty = -\log\left[\frac{1}{1-\frac{1}{s}}\right] = \log\left(\frac{s-1}{s}\right). \end{aligned}$$

46. Evaluate :

$$(i) \int_0^\infty te^{-2t} \sin t \, dt \qquad (ii) \int_0^\infty \frac{\sin mt}{t} dt \quad (\text{U.P. May, 2005}) \qquad (iii) L\left\{\int_0^t \frac{e^t \sin t}{t} dt\right\}.$$

(A.U.U.P., 2008)

Sol. (i) $\int_0^\infty te^{-2t} \cdot \sin t \, dt = \int_0^\infty e^{-st} (t \sin t) \, dt$, where $s = 2$
 $= L(t \sin t)$ by definition
 $= (-1) \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2} = \frac{2 \times 2}{(2^2 + 1)^2} = \frac{4}{25}$

(ii) Since, $L(\sin mt) = \frac{m}{s^2 + m^2} = f(s)$, say

$$L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s) \, ds = \int_s^\infty \frac{m \cdot ds}{s^2 + m^2} = \left[\tan^{-1} \frac{s}{m} \right]_s^\infty$$

or $\int_0^\infty e^{-st} \cdot \frac{\sin mt}{t} \, dt = \frac{\pi}{2} - \tan^{-1} \frac{s}{m}$

Now $\lim_{s \rightarrow 0} \tan^{-1}\left(\frac{s}{m}\right) = 0$ if $m > 0$ or π if $m < 0$

Thus taking limits as $s \rightarrow 0$, we get

$$\int_0^\infty \frac{\sin mt}{t} \, dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2} \text{ if } m < 0.$$

(iii) Since, $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = [\tan^{-1} s]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

$\therefore L\left\{e^t \left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1)$, by shifting property

Thus, $L\left\{\int_0^t \frac{e^t \sin t}{t} \cdot dt\right\} = \frac{1}{s} \cot^{-1}(s-1).$

47. Evaluate : (i) $\int_0^\infty t^3 e^{-t} \sin t \, dt$ (ii) $\int_0^\infty \frac{e^{-2t} \sin^2 t}{t} \, dt$.

Sol. (i) $\int_0^\infty t^3 e^{-t} \sin t \, dt = \int_0^\infty e^{-st} (t^3 \sin t) \, dt$, where $s = 1$
 $= L(t^3 \sin t)$ by definition
 $= (-1)^3 \frac{d^3}{ds^3} \left(\frac{1}{s^2 + 1} \right) = \frac{d^2}{ds^2} \left(\frac{2s}{(s^2 + 1)^2} \right) = \frac{d}{ds} \left[\frac{(s^2 + 1)^2 \times 2 - 2s \times 2(s^2 + 1) \times 2s}{(s^2 + 1)^4} \right]$
 $= \frac{d}{ds} \left[\frac{2(s^2 + 1) - 8s^2}{(s^2 + 1)^3} \right] = \frac{d}{ds} \left[\frac{2(1 - 3s^2)}{(s^2 + 1)^3} \right] = 2 \cdot \frac{(s^2 + 1)^3 \times (-6s) - (1 - 3s^2) \times 3(s^2 + 1)^2 \times 2s}{(s^2 + 1)^6}$
 $= 2 \cdot \frac{-6s(s^2 + 1) - 6s(1 - 3s^2)}{(s^2 + 1)^4} = \frac{24s(s^2 - 1)}{(s^2 + 1)^4} = 0$ as $s \rightarrow 1$.

(ii) $\int_0^\infty \frac{e^{-2t} \sin^2 t}{t} \, dt = \int_0^\infty e^{-st} \left(\frac{\sin^2 t}{t} \right) \, dt$, where $s = 2$.
 $= L \left(\frac{\sin^2 t}{t} \right)$ by definition
 $= L \left(\frac{1 - \cos 2t}{t} \right) = \frac{1}{2} \log \left(\frac{s^2 + 4}{s^2} \right)$ as in Q. 35 part (iv).
 $= \frac{1}{2} \log 2$ as $s \rightarrow 2$.

48. Evaluate : $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt$ (M.D.U. Dec., 2007)

Sol. $\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-3t}}{t} \right) \, dt = L \left(\frac{1}{t} \right)_{s=1} - L \left(\frac{1}{t} \right)_{s=3}$
 $= \int_1^\infty \frac{1}{s} \, ds - \int_3^\infty \frac{1}{s} \, ds = \int_1^3 \frac{1}{s} \, ds = (\log s)_1^3 = \log 3 - \log 1 = \log 3$.

49. Evaluate : (i) $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt$ (ii) $\int_0^\infty e^{-st} t^3 \cos t \, dt$ (iii) $\int_0^\infty e^{-3t} t \sin t \, dt$.

Sol. (i) $L(\sin^2 t) = \frac{1}{2} L(1 - \cos 2t) = \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 4} \right)$
 $L \left(\frac{\sin^2 t}{t} \right) = \frac{1}{2} \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4} \right) \, ds = \frac{1}{4} \log \frac{s^2 + 4}{s^2}$

By definition

$$\int_0^\infty e^{-st} \frac{\sin^2 t}{t} \, dt = \frac{1}{4} \log \left(\frac{s^2 + 4}{s^2} \right)$$

Put $s = 1$. $\int_0^\infty e^{-st} \frac{\sin^2 t}{t} \, dt = \frac{1}{4} \log 5$.

$$(ii) \quad L(\cos t) = \frac{s}{s^2 + 1}$$

$$\begin{aligned} L(t^3 \cos t) &= (-1)^3 \frac{d^3}{ds^3} \left(\frac{s}{s^2 + 1} \right) = - \frac{d^2}{ds^2} \left(\frac{1 - s^2}{(s^2 + 1)^2} \right) = - \frac{d}{ds} \left(\frac{2s^3 - 6s}{(s^2 + 1)^3} \right) \\ &= - \left[\frac{(1 + s^2)^3 \cdot (6s^2 - 6) - (2s^3 - 6s) \cdot 3(1 + s^2)^2 \cdot 2s}{(s^2 + 1)^6} \right] = \frac{6(s^4 - 6s^2 + 1)}{(1 + s^2)^4} \end{aligned}$$

$$\Rightarrow \int_0^\infty e^{-st} \cdot t^3 \cos t \, dt = \frac{6(s^4 - 6s^2 + 1)}{(1 + s^2)^4} \quad (\text{By definition})$$

$$(iii) \quad L(\sin t) = \frac{1}{s^2 + 1}$$

$$L(t \sin t) = - \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

$$\int_0^\infty e^{-st} \cdot t \sin t \, dt = \frac{2s}{(s^2 + 1)^2} \quad (\text{By definition})$$

$$\text{Putting } s = 3, \text{ we obtain } \int_0^\infty e^{-3t} \cdot t \sin t \, dt = \frac{6}{100} \text{ or } \frac{3}{50}.$$

50. Find the Laplace transform of the following functions :

$$\begin{array}{lll} (i) \quad t^2 e^{-2t} \cos t & (ii) \quad \int_0^t e^{-t} \cos t \, dt & (iii) \quad \sin \sqrt{t} \\ (iv) \quad \frac{\cos \sqrt{t}}{\sqrt{t}} & (v) \quad \int_0^{t/2} \frac{1 - e^{-2x}}{x} \cdot dx. & \end{array}$$

$$\text{Sol. (i)} \quad L(e^{-2t} \cos t) = \frac{s + 2}{(s + 2)^2 + 1} = \frac{s + 2}{s^2 + 4s + 5}$$

$$\begin{aligned} \Rightarrow L(t^2 e^{-2t} \cos t) &= (-1)^2 \frac{d^2}{ds^2} \left[\frac{s + 2}{s^2 + 4s + 5} \right] \\ &= \frac{d}{ds} \left[\frac{(s^2 + 4s + 5) - (s + 2)(2s + 4)}{(s^2 + 4s + 5)^2} \right] = \frac{d}{ds} \left[- \frac{s^2 + 4s + 3}{(s^2 + 4s + 5)^2} \right] \\ &= - \frac{(s^2 + 4s + 5)^2 (2s + 4) - (s^2 + 4s + 3) \cdot 2(s^2 + 4s + 5)(2s + 4)}{(s^2 + 4s + 5)^4} \\ &= \frac{2(s^3 + 6s^2 + 9s + 2)}{(s^2 + 4s + 5)^3} \end{aligned}$$

$$(ii) \quad L(\cos t) = \frac{s}{s^2 + 1}$$

$$L(e^{-t} \cos t) = \frac{s + 1}{(s + 1)^2 + 1} = f(s), \text{ say}$$

$$\therefore L\left(\int_0^t e^{-t} \cos t \, dt\right) = \frac{1}{s} f(s) = \frac{s + 1}{s(s^2 + 2s + 2)}$$

$$(iii) \sin \sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \dots$$

$$\begin{aligned} \therefore L(\sin \sqrt{t}) &= L\left(t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \dots\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3!s^{5/2}} + \frac{\Gamma\left(\frac{7}{2}\right)}{5!s^{7/2}} - \dots \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} \left\{ 1 - \left(\frac{1}{2^2 \cdot s}\right) + \frac{1}{2!} \left(\frac{1}{2^2 \cdot s}\right)^2 - \dots \right\} \end{aligned}$$

$$\Rightarrow L(\sin \sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-\left(\frac{1}{4s}\right)}$$

$$(iv) \text{ Now, } L\left[\frac{d}{dt}(\sin \sqrt{t})\right] = s[L(\sin \sqrt{t})] - 0 \quad [\text{See Q. 38}]$$

$$L\left(\frac{\cos \sqrt{t}}{2\sqrt{t}}\right) = \frac{\sqrt{\pi}}{2\sqrt{s}} \cdot e^{-\left(\frac{1}{4s}\right)} \quad [\text{See part (iii) above}]$$

$$\therefore L\left(\frac{\cos \sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}} \cdot e^{-\left(\frac{1}{4s}\right)}$$

$$(v) \int_0^{t/2} \frac{1 - e^{-2x}}{x} \cdot dx$$

$$\text{Put } 2x = u, 2dx = du$$

$$\therefore \text{ Integral becomes } \int_0^t \left(\frac{1 - e^{-u}}{u}\right) du$$

$$\text{Now, } L(1 - e^{-u}) = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left(\frac{1 - e^{-u}}{u}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds = \left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty = 0 - \log\left(\frac{s}{s+1}\right) = \log\left(\frac{s+1}{s}\right)$$

$$\therefore L\left[\int_0^t \left(\frac{1 - e^{-u}}{u}\right) du\right] = \frac{1}{s} \log\left(\frac{s+1}{s}\right) = \frac{1}{s} \log\left(1 + \frac{1}{s}\right).$$

Note. (For Part iii) Q. No. 50.

The values for $\Gamma\left(\frac{3}{2}\right)$, $\Gamma\left(\frac{5}{2}\right)$ and $\Gamma\left(\frac{7}{2}\right)$ have been used to calculate Laplace transform of $t^{1/2}$, $t^{3/2}$ and $t^{5/2}$. Here,

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} ; \text{ if } n > -1 \text{ and } s > 0.$$

$$L(t^{1/2}) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}$$

To calculate $\Gamma\left(\frac{3}{2}\right)$, $\Gamma\left(\frac{5}{2}\right)$, $\Gamma\left(\frac{7}{2}\right)$, we may use an important result of **Gamma Function**

(Reduction Formula) which is $\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1) \cdot \sqrt{\pi}}{2^{2n} \cdot \Gamma(n+1)} = n!$

$$\text{Put } n = 1 \quad \Gamma\left(\frac{3}{2}\right) = \frac{\Gamma(3) \cdot \sqrt{\pi}}{2^2 \cdot \Gamma(2)} = \frac{2! \sqrt{\pi}}{2^2 \cdot 1!} = \frac{1}{2} \sqrt{\pi}$$

$$\text{Put } n = 2 \quad \Gamma\left(\frac{5}{2}\right) = \frac{\Gamma(5) \cdot \sqrt{\pi}}{2^4 \cdot \Gamma(3)} = \frac{4! \sqrt{\pi}}{2^4 \cdot 2!} = \frac{12\sqrt{\pi}}{2^4} = \frac{3\sqrt{\pi}}{2^2}$$

$$\text{Put } n = 3 \quad \Gamma\left(\frac{7}{2}\right) = \frac{\Gamma(7) \cdot \sqrt{\pi}}{2^6 \cdot \Gamma(4)} = \frac{6! \sqrt{\pi}}{2^6 \cdot 3!} = \frac{120\sqrt{\pi}}{2^6}$$

On using these values and simplification gives the final results.

51. Find the inverse Laplace transforms of

$$\begin{aligned} (i) \frac{1}{s(s^2 + a^2)} \quad (ii) \frac{s+2}{s^2(s+1)(s-2)} \quad (iii) \frac{s+2}{(s^2+4s+5)^2} \quad (iv) \frac{s}{(s^2+a^2)^2} \\ (v) \frac{s^2}{(s^2+a^2)^2} \quad (vi) \frac{1}{(s^2+a^2)^2} \quad (vii) \log \frac{s^2+1}{s(s+1)} \end{aligned}$$

Sol. (i) Since $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$

$$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t \frac{1}{a} \sin at \, dt = \frac{1}{a^2} [-\cos at]_0^t = \frac{1 - \cos at}{a^2}$$

(ii) Let $\frac{s+2}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$

$$s+2 = A(s-2) + B(s+1)$$

Putting $s = 2, -1$, we get $A = -1/3, B = 4/3$

$$\therefore \frac{s+2}{(s+1)(s-2)} = -\frac{1}{3(s+1)} + \frac{4}{3(s-2)}$$

$$L^{-1}\left\{\frac{s+2}{(s+1)(s-2)}\right\} = \frac{4}{3} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+1}\right) = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}$$

Now, $L^{-1}\left\{\frac{s+2}{s(s+1)(s-2)}\right\} = \int_0^t L^{-1}\left(\frac{s+2}{(s+1)(s-2)}\right) dt$

$$= \int_0^t \left(\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}\right) dt = \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1$$

Again, $L^{-1}\left\{\frac{s+2}{s^2(s+1)(s-2)}\right\} = \int_0^t L^{-1}\left(\frac{s+2}{s(s+1)(s-2)}\right) dt$

$$= \int_0^t \left(\frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1\right) dt = \frac{1}{3} (e^{2t} - e^{-t} - t)$$

(iii) $L^{-1}\left(\frac{1}{s^2+4s+5}\right) = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t} \cdot \sin t$

Now, $L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\} = (-1)^1 \cdot t \cdot e^{-2t} \cdot \sin t$

i.e., $L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\} = -t \cdot e^{-2t} \sin t$ or $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{1}{2} t \cdot e^{-2t} \cdot \sin t$

(iv) If $f(t) = L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\}$, then

$$L \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \frac{s}{(s^2 + a^2)^2} ds = \frac{1}{2} \int_s^\infty \frac{2s}{(s^2 + a^2)^2} \cdot ds = -\frac{1}{2} \left(\frac{1}{s^2 + a^2} \right)_s^\infty = \frac{1}{2} \cdot \frac{1}{s^2 + a^2}$$

$$\therefore \frac{f(t)}{t} = \frac{1}{2} L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{\sin at}{2a}, \text{ Hence } f(t) = \frac{1}{2a} t \sin at.$$

(v) In part (iv) we have proved that

$$L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at = f(t), \text{ say}$$

Since, $f(0) = 0$

$$L^{-1} \left[\frac{s^2}{(s^2 + a^2)^2} \right] = L^{-1} \left\{ s \cdot \frac{s}{(s^2 + a^2)^2} \right\} = \frac{d}{dt} \{f(t)\} = \frac{d}{dt} \left(\frac{1}{2a} t \sin at \right) = \frac{1}{2a} (\sin at + at \cos at)$$

(vi) In part (iv) we have shown that

$$L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at = f(t), \text{ say}$$

$$\begin{aligned} L^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} &= L^{-1} \left\{ \frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2} \right\} = \int_0^t f(t) dt \\ &= \int_0^t \frac{1}{2a} t \sin at dt = \frac{1}{2a} \left[t \left(\frac{-\cos at}{a} \right) \Big|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a} \right) dt \right] \\ &= \frac{1}{2a} \left\{ \frac{-t \cos at}{a} + \frac{\sin at}{a^2} \right\} = \frac{1}{2a^3} \{ \sin at - at \cos at \} \end{aligned}$$

(vii) If $f(t) = L^{-1} \log \frac{s^2 + 1}{s(s+1)}$, then

$$\begin{aligned} t f(t) &= L^{-1} \left\{ -\frac{d}{ds} \log \left(\frac{s^2 + 1}{s(s+1)} \right) \right\} \\ &= -L^{-1} \left\{ \frac{d}{ds} \log(s^2 + 1) \right\} + L^{-1} \left\{ \frac{d}{ds} \log s \right\} + L^{-1} \left\{ \frac{d}{ds} \log(s+1) \right\} \\ &= -L^{-1} \left(\frac{2s}{s^2 + 1} \right) + L^{-1} \left(\frac{1}{s} \right) + L^{-1} \left(\frac{1}{s+1} \right) = -2 \cos t + 1 + e^{-t} \end{aligned}$$

Thus, $f(t) = \frac{1}{t} (1 + e^{-t} - 2 \cos t)$.

52. Apply convolution theorem to evaluate :

$$(i) L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\}$$

$$(ii) L^{-1} \left\{ \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right\}.$$

Sol. Convolution theorem states that

$$\text{If } L^{-1} \{ \bar{f}(s) \} = f(t) \text{ and } L^{-1} \{ \bar{g}(s) \} = g(t)$$

$$\text{Then } L^{-1} \{ \bar{f}(s) \cdot \bar{g}(s) \} = \int_0^t f(u) g(t-u) du = F * G$$

$F * G$ is called the convolution of F and G .

$$(i) \text{ Since } L^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at \quad \text{and} \quad L^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a} \sin at$$

$\therefore$ By convolution theorem, we get

$$L^{-1} \left(\frac{s}{s^2 + a^2} \cdot \frac{1}{s^2 + a^2} \right) = \int_0^t \cos a(t-u) \frac{\sin au}{a} \cdot du = \frac{1}{2a} \int_0^t \{2 \cos a(t-u) \sin au\} du$$

Using formula ;

$$2 \sin A \cos B = [\sin (A + B) + \sin (A - B)]$$

$$\begin{aligned} \text{R.H.S.} &= \frac{1}{2a} \int_0^t [\sin at + \sin (2au - at)] du \\ &= \frac{1}{2a} \left[u \sin at - \frac{1}{2a} \cos (2au - at) \right]_0^t \end{aligned}$$

$$\text{Hence } L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at$$

$$(ii) \text{ Since } L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at, L^{-1} \left(\frac{s}{s^2 + b^2} \right) = \cos bt$$

$$\therefore \text{ By convolution theorem, we have } L^{-1} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] = L^{-1} \left[\frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + b^2} \right].$$

53. Find the inverse Laplace transform of

$$(i) \log \left(1 + \frac{1}{s^2} \right) \quad (ii) \cot^{-1} \left(\frac{s+3}{2} \right) \quad (\text{M.D.U. May, 2009})$$

$$(iii) \tan^{-1} \left(\frac{2}{s^2} \right) \quad (\text{M.D.U. May, 2008}) \quad (iv) \cot^{-1} \left(\frac{s}{2} \right) \quad (v) \log \left(\frac{s+a}{s+b} \right).$$

Sol. (i) Let $f(s) = \log \left(1 + \frac{1}{s^2} \right) = L\{F(t)\}$, then

$$f'(s) = \frac{1}{\left(1 + \frac{1}{s^2} \right)} \left(\frac{-2}{s^3} \right) = \frac{-2}{s(s^2 + 1)} = -2 \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

$$\Rightarrow L^{-1}\{f'(s)\} = -2L^{-1} \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) = -2(1 - \cos t)$$

But $L^{-1}\{f'(s)\} = -tF(t)$

$$\therefore -tF(t) = -2(1 - \cos t) \Rightarrow F(t) = \frac{2(1 - \cos t)}{t}$$

(ii) Let $f(s) = \cot^{-1} \left(\frac{s+3}{2} \right) = L\{F(t)\}$,

then
$$f'(s) = \frac{-1}{1 + \left(\frac{s+3}{2} \right)^2} \cdot \frac{1}{2} = \frac{-2}{(s+3)^2 + 4}$$

$$\therefore \quad L^{-1}\{f'(s)\} = -2 L^{-1}\left(\frac{1}{(s+3)^2+4}\right) = -2e^{-3t} L^{-1}\left(\frac{1}{s^2+4}\right) = -e^{-3t} \cdot \sin 2t.$$

But $L^{-1}\{f'(s)\} = -t F(t)$

$$\therefore \quad -tF(t) = -e^{-3t} \cdot \sin 2t \Rightarrow F(t) = \frac{e^{-3t} \sin 2t}{t}$$

(iii) Let,
$$f(s) = \tan^{-1}\left(\frac{2}{s^2}\right) = \tan^{-1}\left(\frac{2}{1+s^2-1}\right)$$

$$= \tan^{-1}\left\{\frac{2}{1+(s-1)(s+1)}\right\} = \tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}$$

$$\therefore \quad L^{-1}\{f(s)\} = L^{-1}\left\{\tan^{-1}\left(\frac{1}{s-1}\right)\right\} - L^{-1}\left\{\tan^{-1}\frac{1}{s+1}\right\}$$

$$= e^t L^{-1}\left(\tan^{-1}\frac{1}{s}\right) - e^{-t} \cdot L^{-1}\left(\tan^{-1}\frac{1}{s}\right)$$

$$= 2 \sinh t L^{-1}\left(\tan^{-1}\frac{1}{s}\right) = 2 \sinh t \cdot \frac{\sin t}{t}$$

$$\Rightarrow \quad F(t) = \frac{2}{t} \sin t \sinh t$$

(iv) Let, $f(s) = \cot^{-1}\left(\frac{s}{2}\right) = L\{F(t)\}$, then $f'(s) = \frac{-1}{1+\frac{s^2}{4}} \cdot \frac{1}{2} = \frac{-2}{s^2+4}$

$$\therefore \quad L^{-1}\{f'(s)\} = -2 L^{-1}\left(\frac{1}{s^2+4}\right) = -\sin 2t$$

But, $L^{-1}\{f'(s)\} = -t F(t)$

$$\therefore \quad F(t) = \frac{\sin 2t}{t}$$

(v) Let, $f(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b) = L\{F(t)\}$, then

$$f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

Now, $L^{-1}\{f'(s)\} = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right) = e^{-at} - e^{-bt}$

But, $L^{-1}\{f'(s)\} = -t F(t)$

$$\therefore \quad -t F(t) = e^{-at} - e^{-bt} \Rightarrow F(t) = \frac{e^{-bt} - e^{-at}}{t}.$$

54. Find inverse Laplace transforms of the following functions :

(i) $\frac{s}{(s^2+1)(s^2+4)}$

(ii) $\frac{1}{s(s^2-a^2)}$

(iii) $\frac{s}{(s^2+4)^2}$

(iv) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

(M.D.U., Dec., 2008, 2009)

Sol. (i) Let $f(s) = \frac{1}{s^2 + 4}$ $g(s) = \frac{s}{s^2 + 1}$

$\therefore F(t) = \frac{1}{2} \sin 2t, G(t) = \cos t$

$\therefore F(u) = \frac{1}{2} \sin 2u, G(t - u) = \cos(t - u)$

$\therefore L^{-1} \left\{ \frac{s}{(s^2 + 1)(s^2 + 4)} \right\} = \int_0^t \frac{1}{2} \sin 2u \cdot \cos(t - u) du = \frac{1}{4} \int_0^t [\sin(u + t) + \sin(3u - t)] du$

$$= \frac{1}{4} \left[-\cos(u + t) - \frac{\cos(3u - t)}{3} \right]_0^t$$

$$= \frac{1}{4} \left[\left(-\cos 2t - \frac{\cos 2t}{3} \right) - \left(-\cos t - \frac{\cos t}{3} \right) \right]$$

$$= \frac{1}{4} \left[-\frac{4}{3} \cos 2t + \frac{4}{3} \cos t \right] = \frac{1}{3} [\cos t - \cos 2t]$$

(ii) Let $f(s) = \frac{1}{s^2 - a^2}$ and $g(s) = \frac{1}{s}$

$\therefore F(t) = \frac{1}{a} \sinh at, G(t) = 1$

$F(u) = \frac{1}{a} \sinh au, G(t - u) = 1$

$\therefore L^{-1} \left\{ \frac{1}{s(s^2 - a^2)} \right\} = \int_0^t \frac{1}{a} \sinh au \cdot 1 \cdot du = \frac{1}{a} \left(\frac{\cosh au}{a} \right)_0^t$

or $L^{-1} \left\{ \frac{1}{s(s^2 - a^2)} \right\} = \frac{1}{a^2} (\cosh at - 1)$

(iii) $\frac{s}{(s^2 + 4)^2} = \frac{1}{s^2 + 4} \cdot \frac{s}{s^2 + 4}$

Let $f(s) = \frac{1}{s^2 + 4}, g(s) = \frac{s}{s^2 + 4}$

$\therefore F(t) = L^{-1} \{f(s)\} = L^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} \sin 2t$

and $G(t) = L^{-1} \{g(s)\} = L^{-1} \left\{ \frac{s}{s^2 + 4} \right\} = \cos 2t$

Now, $F(u) = \frac{1}{2} \sin 2u, G(t - u) = \cos 2(t - u)$

$\therefore$ By convolution theorem, we have

$$L^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\} = \int_0^t \frac{1}{2} \sin 2u \cdot \cos 2(t - u) du = \frac{1}{4} \int_0^t [\sin 2t + \sin(4u - 2t)] du$$

$$= \frac{1}{4} \left[u \sin 2t - \frac{\cos(4u - 2t)}{4} \right]_0^t = \frac{1}{4} t \sin 2t.$$

$$(iv) \quad \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} = \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + b^2}$$

$$\text{Let} \quad f(s) = \frac{s}{s^2 + a^2} \cdot g(s) = \frac{s}{s^2 + b^2}$$

$$\therefore \quad F(t) = L^{-1} \{f(s)\} = L^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at$$

$$\text{and} \quad G(t) = L^{-1} \{g(s)\} = L^{-1} \left\{ \frac{s}{s^2 + b^2} \right\} = \cos bt$$

$$\text{Now} \quad F(u) = \cos au, \quad G(t - u) = \cos b(t - u)$$

$\therefore$ By convolution theorem, we have

$$\begin{aligned} L^{-1} \left\{ \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right\} &= \int_0^t \cos au \cdot \cos b(t - u) du \\ &= \frac{1}{2} \int_0^t [\cos \{(a - b)u + bt\} + \cos \{(a + b)u - bt\}] du \\ &= \frac{1}{2} \left[\frac{\sin \{(a - b)u + bt\}}{a - b} + \frac{\sin \{(a + b)u - bt\}}{a + b} \right]_0^t = \frac{a \sin at - b \sin bt}{a^2 - b^2}. \end{aligned}$$

55. Using the convolution theorem, evaluate $\int_0^t \sin u \cos (t - u) du$.

Sol. By convolution theorem, we have

$$\int_0^t \sin u \cos (t - u) du = \sin t * \cos t \quad \dots(1)$$

$$\text{Here,} \quad F(t) = \sin t, \quad G(t) = \cos t.$$

$$\therefore \quad f(s) = L \{F(t)\} = L (\sin t) = \frac{1}{s^2 + 1}$$

$$\text{and} \quad g(s) = L \{G(t)\} = L (\cos t) = \frac{s}{s^2 + 1}$$

$$\sin t * \cos t = L^{-1} \{f(s), g(s)\} = L^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} \quad \dots(2)$$

$$\text{Now,} \quad L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at$$

Differentiating this w.r.t. a , we get

$$L^{-1} \left\{ \frac{-2as}{(s^2 + a^2)^2} \right\} = -t \sin at$$

$$\text{If } a = 1, \quad L^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} = \frac{t}{2} \sin t$$

$$\text{From (1) and (2),} \quad \int_0^t \sin u \cdot \cos (t - u) du = \frac{1}{2} t \sin t.$$

56. Prove that : $L^{-1} \left\{ \frac{1}{(s^2 + 1)^3} \right\} = \frac{1}{8} [(3 - t^2) \sin t - 3t \cos t].$

Sol. We have $L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{1}{a} \sin at$

Differentiating w.r.t. a , we get

$$L^{-1} \left\{ \frac{d}{da} \left(\frac{1}{s^2 + a^2} \right) \right\} = \frac{d}{da} \left(\frac{\sin at}{a} \right) \Rightarrow L^{-1} \left\{ \frac{-2a}{(s^2 + a^2)^2} \right\} = -\frac{1}{a^2} \sin at + \frac{t}{a} \cos at$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} = \frac{1}{2a^3} \sin at - \frac{t}{2a^2} \cos at$$

Differentiating again w.r.t. a , we get

$$L^{-1} \left\{ \frac{-4a}{(s^2 + a^2)^3} \right\} = -\frac{3}{2a^4} \sin at + \frac{t}{2a^3} \cos at + \frac{t}{a^3} \cos at + \frac{t^2}{2a^2} \sin at.$$

Putting $a = 1$, this yields

$$L^{-1} \left\{ \frac{1}{(s^2 + 1)^3} \right\} = \frac{1}{8} [3(\sin t - t \cos t) - t^2 \sin t] = \frac{1}{8} [(3 - t^2) \sin t - 3t \cos t].$$

57. Apply convolution theorem to evaluate : $L^{-1} \left\{ \frac{s^2}{(s^2 + 4)^2} \right\}.$

Sol. Since $L^{-1} \left[\frac{s}{s^2 + 4} \right] = \cos 2t$

$\therefore$ By convolution theorem

$$\begin{aligned} L^{-1} \left[\frac{s}{s^2 + 4} \cdot \frac{s}{s^2 + 4} \right] &= \int_0^t \cos 2(t-u) \cos 2u \, du \\ &= \int_0^t \cos 2u \cdot [\cos 2t \cdot \cos 2u + \sin 2t \sin 2u] \, du \\ &= \cos 2t \int_0^t \cos^2 2u \, du + \sin 2t \int_0^t \cos 2u \cdot \sin 2u \, du \\ &= \frac{1}{2} \cos 2t \int_0^t (1 + \cos 4u) \, du + \frac{1}{2} \sin 2t \int_0^t \sin 4u \, du \\ &= \frac{\cos 2t}{2} \left[u + \frac{\sin 4u}{4} \right]_0^t + \frac{\sin 2t}{2} \left[-\frac{\cos 4u}{4} \right]_0^t \\ &= \frac{\cos 2t}{2} \left[t + \frac{\sin 4t}{4} \right] + \frac{\sin 2t}{8} [1 - \cos 4t] \\ &= \frac{t \cos 2t}{2} + \frac{\sin 2t}{8} + \frac{1}{8} [\sin 4t \cos 2t - \sin 2t \cos 4t] \\ &= \frac{1}{2} t \cos 2t + \frac{1}{8} \sin 2t + \frac{1}{8} \sin 2t = \frac{[2t \cos 2t + \sin 2t]}{4}. \end{aligned}$$

58. Find the Laplace transform of the function $\frac{\cos 2t - \cos 3t - e^t + 1}{t}$. (M.D.U., 2005)

Sol. Let $f(t) = \cos 2t - \cos 3t - e^t + 1$

$$L\{f(t)\} = L[\cos 2t - \cos 3t - e^t + 1] = \frac{s}{s^2 + 4} - \frac{s}{s^2 + 9} - \frac{1}{s-1} + \frac{1}{s}$$

$$\begin{aligned} L\left\{\frac{1}{t} \cdot f(t)\right\} &= \int_s^\infty \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9} - \frac{1}{s-1} + \frac{1}{s} \right) ds \\ &= \left[\frac{1}{2} \log(s^2 + 4) - \frac{1}{2} \log(s^2 + 9) - \log(s-1) + \log s \right]_s^\infty \\ &= \frac{1}{2} \left[\log \left(\frac{s^2 + 4}{s^2 + 9} \right) \right]_s^\infty - \left[\log \left(\frac{s-1}{s} \right) \right]_s^\infty = \frac{1}{2} \log \left(\frac{1 + \frac{4}{s^2}}{1 + \frac{9}{s^2}} \right) \Bigg|_s^\infty - \left[\log \left(1 - \frac{1}{s} \right) \right]_s^\infty \\ &= \frac{1}{2} \left[\log 1 - \log \left(\frac{1 + \frac{4}{s^2}}{1 + \frac{9}{s^2}} \right) \right] - \left[\log 1 - \log \left(1 - \frac{1}{s} \right) \right] \\ &= -\frac{1}{2} \log \left(\frac{s^2 + 4}{s^2 + 9} \right) + \log \left(\frac{s-1}{s} \right) \quad \text{or} \quad \frac{1}{2} \log \left(\frac{s^2 + 9}{s^2 + 4} \right) + \log \left(\frac{s-1}{s} \right). \end{aligned}$$

59. If $F(t) = \frac{e^{at} - \cos bt}{t}$, find the Laplace transform of $F(t)$.

Sol. $L(e^{at}) = \frac{1}{s-a}, L(\cos bt) = \frac{s}{s^2 + b^2}$

$\therefore L(e^{at} - \cos bt) = \frac{1}{s-a} - \frac{s}{s^2 + b^2}$

Now, $L\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2 + b^2} \right) ds = \left[\log(s-a) - \frac{1}{2} \log(s^2 + b^2) \right]_s^\infty$

$$= \frac{1}{2} \left[\log(s-a)^2 - \log(s^2 + b^2) \right]_s^\infty = \frac{1}{2} \left[\log \frac{(s-a)^2}{s^2 + b^2} \right]_s^\infty = \frac{1}{2} \left[\log \frac{\left(1 - \frac{a}{s}\right)^2}{1 + \frac{b^2}{s^2}} \right]_s^\infty$$

$$= -\frac{1}{2} \left[\log \frac{(s-a)^2}{s^2 + b^2} \right] = \frac{1}{2} \log \left\{ \frac{s^2 + b^2}{(s-a)^2} \right\}.$$

60. Find the Laplace transform of $\frac{\sin at}{t}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

(U.P.T.U., 2005)

Sol. Since $\lim_{t \rightarrow 0} \frac{\sin at}{t} = a$, the Laplace transform of $\frac{\sin at}{t}$ exists.

Now, $L(\sin at) = \frac{a}{s^2 + a^2}$

$$\therefore L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

The function $\frac{\cos at}{t}$ is discontinuous at $t = 0$, so that the Laplace transform does not exist.

61. Apply the method of Laplace transform to solve differential equation :

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t} \sin t, \quad x(0) = 0, \quad x'(0) = 1.$$

Sol. The given equation is $x'' + 2x' + 5x = e^{-t} \cdot \sin t$

Taking the Laplace transform of both sides, we get

$$[s^2 \bar{x} - s x(0) - x'(0)] + 2[s \bar{x} - x(0)] + 5 \bar{x} = \frac{1}{(s+1)^2 + 1}$$

Since, $L\{f^n(t)\} = s^n \cdot \bar{f}(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) \dots \dots f^{n-1}(0)$

Using the given conditions $x(0) = 0$, $x'(0) = 1$, it reduces to

$$(s^2 + 2s + 5)\bar{x} - 1 = \frac{1}{s^2 + 2s + 2}$$

or

$$\begin{aligned} \bar{x} &= \frac{1}{(s^2 + 2s + 2)(s^2 + 2s + 5)} + \frac{1}{s^2 + 2s + 5} \\ &= \frac{1}{3} \left[\frac{1}{s^2 + 2s + 2} - \frac{1}{s^2 + 2s + 5} \right] + \frac{1}{s^2 + 2s + 5} \\ &= \frac{1}{3} \left[\frac{1}{s^2 + 2s + 2} + \frac{2}{s^2 + 2s + 5} \right] = \frac{1}{3} \left[\frac{1}{(s+1)^2 + 1} + \frac{2}{(s+1)^2 + 4} \right] \end{aligned}$$

Taking the inverse Laplace transform of both sides, we get

$$\begin{aligned} x &= \frac{1}{3} L^{-1} \left[\frac{1}{(s+1)^2 + 1} + 2 \cdot \frac{1}{(s+1)^2 + 2^2} \right] \\ &= \frac{1}{3} \cdot \left[e^{-t} \sin t + 2 \cdot \frac{1}{2} \cdot e^{-t} \cdot \sin 2t \right] \quad \text{or} \quad x = \frac{1}{3} e^{-t} (\sin t + \sin 2t) \end{aligned}$$

which is the required solution.

62. Solve the equation : $y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1$, $y'(0) = -1$.

Sol. Taking the Laplace transform of both sides, we have

$$[s^2 \bar{y} - s y(0) - y'(0)] - 3[s \bar{y} - y(0)] + 2 \bar{y} = \frac{4}{s^2} + \frac{1}{s-3}$$

Using the given conditions, it reduces to

$$(s^2 - 3s + 2)\bar{y} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3} \quad \text{or} \quad (s^2 - 3s + 2)\bar{y} = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$\therefore \bar{y} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)}$$

or
$$\bar{y} = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2(s-1)} - \frac{2}{s-2} + \frac{1}{2(s-3)} \quad (\text{Resolving into partial fractions})$$

Taking inverse Laplace transform of both sides, we get

$$y = 3L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - 2L^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$y = 3 + 2t - \frac{1}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$$

or
$$y = 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}, \text{ which is the required solution.}$$

63. Solve the D.E. by the method of Laplace transformation :

$$y'' - 8y' + 15y = 9te^{2t}, y(0) = 5, y'(0) = 10. \quad (\text{M.D.U. Dec., 2009})$$

Sol. Taking L.T. of both sides of the D.E., we get

$$[s^2\bar{y} - sy(0) - y'(0)] - 8[s\bar{y} - y(0)] + 15\bar{y} = \frac{9}{s^2(s-2)}$$

Under the given conditions, it reduces to

$$[s^2\bar{y} - 5s - 10] - 8(s\bar{y} - 5) + 15\bar{y} = \frac{9}{s^2(s-2)}$$

$$(s^2 - 8s + 15)\bar{y} = \frac{9}{s^2(s-2)} + 5s - 30$$

$$\begin{aligned} (s-5)(s-3)\bar{y} &= \frac{9}{s^2(s-2)} + 5s - 30 = \frac{9 + s^2(5s-30)(s-2)}{s^2(s-2)} \\ &= \frac{9 + s^2[5s^2 - 10s - 30s + 60]}{s^2(s-2)} = \frac{9 + s^2[5s^2 - 40s + 60]}{s^2(s-2)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \bar{y} &= \frac{9 + 5s^4 - 40s^3 + 60s^2}{s^2(s-2)(s-3)(s-5)} = \frac{5s^4 - 40s^3 + 60s^2 + 9}{s^2(s-2)(s-3)(s-5)} \\ &= \frac{3}{s} + \frac{2}{s^2} + \frac{3}{4(s-2)} + \frac{7}{s-3} - \frac{2.44}{s-5} \end{aligned}$$

(Resolving by Partial Fractions)

Taking the inverse L.T. of both sides, we get

$$y = 3L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) + \frac{3}{4}L^{-1}\left(\frac{1}{s-2}\right) + 7L^{-1}\left(\frac{1}{s-3}\right) - 2.44L^{-1}\left(\frac{1}{s-5}\right)$$

or
$$y = 3 + 2t + \frac{3}{4}e^{2t} + 7e^{3t} - 2.44e^{5t}$$

which is the required solution.

64. Solve the equation :

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0, \text{ where } y = 1, \frac{dy}{dt} = 2, \frac{d^2y}{dt^2} = 2 \text{ at } t = 0.$$

Sol. The given equation is $y''' + 2y'' - y' - 2y = 0$

Taking the Laplace transform of both sides, we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0 \quad \dots(1)$$

Using the given conditions $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$, equation (1) reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} - s^2 - 2s - 2 - 2s - 4 + 1 = 0 \quad \text{or} \quad (s^3 + 2s^2 - s - 2)\bar{y} = s^2 + 4s + 5$$

$$\therefore \bar{y} = \frac{s^2 + 4s + 5}{s^3 + 2s^2 - s - 2} = \frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)} = \frac{5}{3(s-1)} - \frac{1}{s+1} + \frac{1}{3(s+2)}$$

(After resolving into partial fractions)

Taking the inverse Laplace transforms of both sides, we get

$$y = \frac{5}{3} L^{-1} \left\{ \frac{1}{s-1} \right\} - L^{-1} \left\{ \frac{1}{s+1} \right\} + \frac{1}{3} L^{-1} \left\{ \frac{1}{s+2} \right\} = \frac{5}{3} e^t - e^{-t} + \frac{1}{3} e^{-2t}$$

or $y = \frac{1}{3} (5e^t + e^{-2t}) - e^{-t}$ which is the required solution.

65. Using Laplace transformation method, solve the differential equation :

$$\frac{d^2x}{dt^2} + 9x = \cos 2t, \text{ if } x(0) = 1, x\left(\frac{\pi}{2}\right) = -1.$$

Sol. The given equation is $x'' + 9x = \cos 2t$

Taking the Laplace transform of both sides, we get

$$L(x'') + 9L(x) = L(\cos 2t)$$

$$[s^2\bar{x} - s \cdot x(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2 + 4} \Rightarrow (s^2 + 9)\bar{x} - s - A = \frac{s}{s^2 + 4} \quad | \text{ where } A = x'(0)$$

$$\Rightarrow \bar{x} = \frac{s}{(s^2 + 4)(s^2 + 9)} + \frac{s}{s^2 + 9} + \frac{A}{s^2 + 9}$$

$$= \frac{1}{5} \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9} \right) + \frac{s}{s^2 + 9} + \frac{A}{s^2 + 9}$$

Taking the inverse Laplace transform of both sides, we get

$$x = \frac{1}{5} (\cos 2t - \cos 3t) + \cos 3t + \frac{A}{3} \sin 3t$$

But $x\left(\frac{\pi}{2}\right) = -1$ (given)

$$\Rightarrow x = -1 = \frac{1}{5} (-1 - 0) + 0 + \frac{A}{3} (-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \quad \text{or} \quad A = \frac{12}{5}$$

$$\therefore x(t) = \frac{1}{5} (\cos 2t - \cos 3t) + \cos 3t + \frac{4}{5} \sin 3t$$

or $x = \frac{1}{5} (\cos 2t + 4 \cos 3t + 4 \sin 3t)$

This is the required solution.

66. Solve the following D.E. by using Laplace transformation $y'' + y = t \cos 2t$, $y(0) = 0$, $y'(0) = 0$.

(M.D.U. Dec., 2008)

Sol. $y'' + y = t \cos 2t$,

Taking L.T. on both sides,

$$L(y'') + L(y) = L(t \cos 2t)$$

$$s^2 - s(y(0) - y'(0)) + \bar{y} = -\frac{d}{ds} \left[\frac{s}{s^2 + 4} \right] \quad \text{or} \quad (s^2 + 1)\bar{y} = \frac{s^2 - 4}{(s^2 + 4)^2} \quad \text{or} \quad \bar{y} = \frac{s^2 - 4}{(s^2 + 1)(s^2 + 4)^2}$$

Now,
$$\frac{s^2 - 4}{(s^2 + 1)(s^2 + 4)^2} = \frac{A}{s^2 + 1} + \frac{B}{s^2 + 4} + \frac{C}{(s^2 + 4)^2}$$

Let,
$$s^2 = P$$

$$\frac{P - 4}{(P + 1)(P + 4)^2} = \frac{A}{P + 1} + \frac{B}{P + 4} + \frac{C}{(P + 4)^2} \quad \dots(1)$$

$$P - 4 = (P + 4)^2 A + (P + 1)(P + 4) B + (P + 1) C$$

Putting $P = -1$,
$$A = -\frac{5}{9}$$

Putting $P = -4$,
$$C = \frac{8}{3}$$

Putting $P = 0$,
$$-4 = 16A + 4B + C$$

$$-4 = -\frac{80}{9} + 4B + \frac{8}{3} \quad \text{or} \quad -4 + \frac{80}{9} - \frac{8}{3} = 4B$$

$$4B = \frac{-36 + 80 - 24}{9} = \frac{20}{9} \quad \text{or} \quad B = \frac{5}{9}$$

Put these values of A, B and C in (1)

$$\frac{P - 4}{(P + 1)(P + 4)^2} = -\frac{5}{9(P + 1)} + \frac{5}{9(P + 4)} + \frac{8}{3(P + 4)^2}$$

Putting $P = s^2$,
$$\bar{y} = -\frac{5}{9(s^2 + 1)} + \frac{5}{9(s^2 + 4)} + \frac{8}{3(s^2 + 4)^2}$$

Taking L.T. on both sides,

$$y = -\frac{5}{9} \sin t + \frac{5}{9} \cdot \frac{1}{2} \sin 2t + \frac{8}{3} \left[\frac{1}{2 \times 8} (\sin 2t - 2 \cos 2t) \right]$$

which on simplification gives
$$y = \frac{1}{9} [4 \sin 2t - 5 \sin t - 3 t \cos 2t].$$

67. Using Laplace transformation, solve the differential equation :

$$(D^2 + n^2)x = a \sin (nt + \alpha); x = Dx = 0 \text{ at } t = 0.$$

Sol. The given equation is $(D^2 + n^2)x = a \sin (nt + \alpha)$

Taking Laplace transform of both sides, we get

$$L(x'') + n^2 L(x) = L\{a \sin (nt + \alpha)\}$$

$$[s^2 \cdot \bar{x} - s \cdot x(0) - x'(0)] + n^2 \bar{x} = a \cos \alpha \cdot \frac{n}{s^2 + n^2} + a \sin \alpha \cdot \frac{s}{s^2 + n^2}$$

$$(s^2 + n^2) \bar{x} = \frac{an \cos \alpha}{s^2 + n^2} + \frac{as \sin \alpha}{s^2 + n^2}$$

$$\Rightarrow \bar{x} = \frac{an \cos \alpha}{(s^2 + n^2)^2} + \frac{as \sin \alpha}{(s^2 + n^2)^2} \quad \dots(1)$$

Taking the Inverse Laplace transform of both sides, we get

$$x = (a \cos \alpha) L^{-1} \left[\frac{n}{(s^2 + n^2)^2} \right] + (a \sin \alpha) L^{-1} \left[\frac{s}{(s^2 + n^2)^2} \right] \quad \dots(2)$$

We know that
$$L^{-1} \left(\frac{1}{s^2 + n^2} \right) = \frac{1}{n} \sin nt \quad \dots(3)$$

$$L^{-1} \left[\frac{d}{dn} \left\{ \frac{1}{s^2 + n^2} \right\} \right] = \frac{nt \cos nt - \sin nt}{n^2} \quad \text{or} \quad L^{-1} \left[\frac{-2n}{(s^2 + n^2)^2} \right] = \frac{nt \cos nt - \sin nt}{n^2}$$

$$L^{-1} \left[\frac{n}{(s^2 + n^2)^2} \right] = \frac{1}{2n^2} (\sin nt - nt \cos nt)$$

$$\text{Again, from (3), } L^{-1} \left[\frac{d}{ds} \left(\frac{1}{s^2 + n^2} \right) \right] = -t \cdot \frac{1}{n} \sin nt$$

$$\Rightarrow L^{-1} \left[\frac{-2s}{(s^2 + n^2)^2} \right] = -\frac{t}{n} \sin nt \quad \Rightarrow L^{-1} \left[\frac{s}{(s^2 + n^2)^2} \right] = \frac{t}{2n} \sin nt$$

$$\therefore \text{ From (2), } x = (a \cos \alpha) \cdot \frac{1}{2n^2} (\sin nt - nt \cos nt) + (a \sin \alpha) \cdot \frac{t}{2n} \cdot \sin nt$$

$$\text{or } x = \frac{a}{2n^2} [\cos \alpha \sin nt - nt \cos (\alpha + nt)]$$

This is the required solution.

68. Solve the differential equation by Laplace transform

$$y''' - 2y'' + 5y' = 0; y = 0, y' = 1 \text{ at } t = 0 \text{ and } y = 1 \text{ at } t = \frac{\pi}{8}.$$

Sol. Taking Laplace transform on both sides, we get

$$L(y''') - 2L(y'') + 5 \cdot L(y') = L(0)$$

$$\Rightarrow [s^3 \bar{y} - s^2 y(0) - s \cdot y'(0) - y''(0)] - 2[s^2 \bar{y} - s \cdot y(0) - y'(0)] + 5[s\bar{y} - y(0)] = 0$$

$$\Rightarrow (s^3 - 2s^2 + 5s) \bar{y} - s - A + 2 = 0 \quad |y''(0) = A, \text{ say}$$

$$\Rightarrow \bar{y} = \frac{A - 2 + s}{s(s^2 - 2s + 5)} = \left(\frac{A - 2}{5} \right) \left(\frac{1}{s} - \frac{s - 2}{s^2 - 2s + 5} \right) + \frac{1}{s^2 - 2s + 5}$$

$$= \left(\frac{A - 2}{5} \right) \frac{1}{s} - \left(\frac{A - 2}{5} \right) \left\{ \frac{(s - 1) - 1}{(s - 1)^2 + 4} \right\} + \frac{1}{(s - 1)^2 + 4}$$

$$\Rightarrow \bar{y} = \left(\frac{A - 2}{5} \right) \frac{1}{s} - \left(\frac{A - 2}{5} \right) \left[\frac{s - 1}{(s - 1)^2 + 4} \right] + \left(\frac{A + 3}{10} \right) \left(\frac{2}{(s - 1)^2 + 4} \right)$$

Taking inverse Laplace transform on both sides, we get

$$y = \frac{A - 2}{5} - \left(\frac{A - 2}{5} \right) [e^t \cos 2t] + \left(\frac{A + 3}{10} \right) e^t \sin 2t$$

Since $y = 1$ when $t = \pi/8$

$$\therefore 1 = \frac{A - 2}{5} - \left(\frac{A - 2}{5} \right) e^{\pi/8} \cdot \frac{1}{\sqrt{2}} + \left(\frac{A + 3}{10} \right) e^{\pi/8} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow A = 7 \quad (\text{On simplification})$$

The required solution therefore is $y = 1 + e^t (\sin 2t - \cos 2t)$.

69. Solve by the method of Laplace transforms, the differential equation

$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$

Sol. Taking the Laplace transform of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2 \bar{y} - s y(0) - y'(0)] - [s \bar{y} - y(0)] - 2 \bar{y} = 0$$

Using the given conditions, it reduces to $(s^3 + 2s^2 - s - 2)\bar{y} = 6$

$$\therefore \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)6} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On taking inverse Laplace transforms, we get

$$y = L^{-1} \left(\frac{1}{s-1} \right) - 3 \cdot L^{-1} \left(\frac{1}{s+1} \right) + 2L^{-1} \left(\frac{1}{s+2} \right)$$

or

$$y = e^t - 3e^{-t} + 2e^{-2t}$$

which is the required solution of the given equation.

- 70. Solve :** $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$.

Using Laplace transform method.

Sol. Taking the Laplace transform of both sides, we get

$$[s^2\bar{x} - s.x(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\Rightarrow \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

$$\therefore \bar{x} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3} \text{ on resolving into partial fractions.}$$

On taking inverse Laplace transforms of both sides, we get

$$x = 2L^{-1} \left(\frac{1}{s-1} \right) - 3L^{-1} \left(\frac{1}{(s-1)^2} \right) + L^{-1} \left\{ \frac{1}{(s-1)^3} \right\}$$

$$\Rightarrow x = 2e^t - 3e^t \cdot \frac{t}{1!} + e^t \cdot \frac{t^2}{2!} \quad \text{or} \quad x = 2e^t - 3te^t + \frac{1}{2} t^2 \cdot e^t.$$

which is the desired solution of the given equation.

- 71. Solve** $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$ given that $y(0) = 1, y'(0) = 0, y''(0) = -2$.

Sol. Taking Laplace transforms of both sides, we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\text{Since,} \quad L(e^{at} \cdot t^n) = \frac{n!}{(s-a)^{n+1}}.$$

Using the given conditions, it reduces to

$$(s^3\bar{y} - s^2 + 2) - 3(s^2\bar{y} - s) + 3(s\bar{y} - 1) - \bar{y} = \frac{2}{(s-1)^3}$$

$$(s^3 - 3s^2 + 3s - 1)\bar{y} - s^2 + 2 + 3s - 3 = \frac{2}{(s-1)^3}$$

$$(s^3 - 3s^2 + 3s - 1)\bar{y} = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$(s-1)^3 \bar{y} = (s^2 - 3s + 1) + \frac{2}{(s-1)^3}$$

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

or
$$\bar{y} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Taking Laplace transform inversion, we obtain

$$y = L^{-1} \left(\frac{1}{s-1} \right) - L^{-1} \left(\frac{1}{(s-1)^2} \right) - L^{-1} \left(\frac{1}{(s-1)^3} \right) + 2L^{-1} \frac{1}{(s-1)^6}$$

or
$$y = e^t \left(1 - t - \frac{1}{2} t^2 + \frac{1}{60} t^5 \right)$$

which is the required solution of the given equation

72. Solve the differential equation $ty'' + 2y' + ty = \cos t$ given that $y(0) = 1$.

Sol. Taking Laplace transform of both sides of the equation and noting that

$$L\{tf(t)\} = -\frac{d}{ds} [L\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds} [s^2 \bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds} (\bar{y}) = \frac{s}{s^2 + 1}$$

or
$$-\left(s^2 \frac{d\bar{y}}{ds} + 2s\bar{y} \right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds} (\bar{y}) = \frac{s}{s^2 + 1}$$

or
$$(s^2 + 1) \frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2 + 1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2 + 1} - \frac{s}{(s^2 + 1)^2}$$

On taking inverse Laplace transforms and noting that $L^{-1}\{\bar{f}'(s)\} = -tf(t)$, we get

$$-ty = -\sin t - \frac{1}{2} t \sin t \quad \text{or} \quad y = \frac{1}{2} \left(1 + \frac{2}{t} \right) \sin t \quad \text{which is the solution.}$$

73. Solve $\frac{d^2 x}{dt^2} + 9x = \cos 2t$, given that $x(0) = -1$ and $x'(0) = 2$ using Laplace transform.

Sol.
$$\frac{d^2 x}{dt^2} + 9x = \cos 2t$$

Taking Laplace transform of both sides of the given equation, we have

$$L(x'') + 9.L(x) = L(\cos 2t)$$

$$s^2 L(x) - sx(0) - x'(0) + 9.L(x) = \frac{s}{s^2 + 4}$$

$$(s^2 + 9) L(x) - s(-1) - 2 = \frac{s}{s^2 + 4}$$

or
$$(s^2 + 9) L(x) = \frac{s}{s^2 + 4} - s + 2$$

or
$$\begin{aligned} L(x) &= \frac{s}{(s^2 + 4)(s^2 + 9)} - \frac{s}{(s^2 + 9)} + \frac{2}{(s^2 + 9)} \\ &= \frac{s}{5(s^2 + 4)} - \frac{s}{5(s^2 + 9)} - \frac{s}{(s^2 + 9)} + \frac{2}{(s^2 + 9)} \end{aligned}$$

$$= \frac{s}{5(s^2 + 4)} - \frac{6s}{5(s^2 + 9)} + \frac{2}{s^2 + 9}$$

$$\therefore x = \frac{1}{5} L^{-1} \left(\frac{s}{s^2 + 4} \right) - \frac{6}{5} L^{-1} \left(\frac{s}{s^2 + 9} \right) + 2 \cdot L^{-1} \left(\frac{1}{s^2 + 9} \right)$$

$$\text{or } x = \frac{1}{5} \cos 2t - \frac{6}{5} \cos 3t + \frac{2}{3} \sin 3t$$

which is the desired solution of the given equation.

74. Solve : $y'' + y' - 2y = t$, given that $y(0) = 1$ and $y'(0) = 0$, using Laplace transform.

Sol. Given equation is $\frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = t$

Taking Laplace transform of both sides

$$L(y'') + L(y') - 2L(y) = L(t)$$

$$s^2 L(y) - sy(0) - y'(0) + sL(y) - y(0) - 2L(y) = \frac{1}{s^2}$$

$$s^2 L(y) - s - 0 + sL(y) - 1 - 2L(y) = \frac{1}{s^2} \quad \text{or} \quad (s^2 + s - 2) L(y) = \frac{1}{s^2} + s + 1$$

$$L(y) = \frac{1}{s^2(s^2 + s - 2)} + \frac{s + 1}{s^2 + s - 2}$$

Taking inverse Laplace transform, we have

$$y = L^{-1} \left[\frac{1}{s^2(s^2 + s - 2)} \right] + L^{-1} \left[\frac{s + 1}{s^2 + s - 2} \right]$$

$$\text{or } y = L^{-1} \left[\frac{1}{s - 1} + \frac{1}{4(s + 2)} - \frac{1}{2s^2} - \frac{1}{4s} \right] \quad (\text{After resolving into partial fractions})$$

$$\text{or } y = e^t + \frac{1}{4} e^{-2t} - \frac{t}{2} - \frac{1}{4}$$

is the desired solution of the given equation.

75. Solve : $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$ using Laplace transform.

Sol. Taking Laplace transform on both sides, we get

$$L(y'') + L(4y') + 3L(y) = L(e^{-t})$$

$$[s^2 \bar{y} - sy(0) - y'(0)] + 4(s\bar{y} - y(0)) + 3\bar{y} = \frac{1}{s + 1}$$

$$(s^2 + 4s + 3)\bar{y} - s - 1 - 4 = \frac{1}{s + 1}$$

$$(s^2 + 4s + 3)\bar{y} = \frac{1}{s + 1} + s + 5$$

$$\bar{y} = \frac{s^2 + 6s + 6}{(s + 1)(s^2 + 4s + 3)} = \frac{s^2 + 6s + 6}{(s + 1)(s + 1)(s + 3)} = \frac{s^2 + 6s + 6}{(s + 1)^2(s + 3)}$$

$$\text{Let } \frac{s^2 + 6s + 6}{(s + 1)^2(s + 3)} = \frac{A}{s + 1} + \frac{B}{(s + 1)^2} + \frac{C}{s + 3}$$

$$\Rightarrow s^2 + 6s + 6 = A(s + 1)(s + 3) + B(s + 3) + C(s + 1)^2$$

$$\text{Put } s = -1, \quad 2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}$$

$$\text{Put } s = -3, \quad 4C = -3 \quad \Rightarrow \quad C = \frac{-3}{4}$$

Comparing constant terms on both sides, we have

$$3A + 3B + C = 6 \quad \text{or} \quad 3A + \frac{3}{2} - \frac{3}{4} = 6 \quad \Rightarrow \quad A = \frac{7}{4}$$

$$\therefore \quad \bar{y} = \frac{7}{4} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{(s+1)^2} \right) - \frac{3}{4} \cdot \left(\frac{1}{s+3} \right)$$

Taking inverse Laplace transform, we have

$$y = \frac{7}{4} e^{-t} + \frac{1}{2} \cdot t e^{-t} - \frac{3}{4} e^{-3t}$$

which is the desired solution of the given equation.

- 76.** A voltage Ee^{-at} is applied at $t = 0$, to a circuit of inductance L and resistance R . Show that the current at time t is $\frac{E}{R-aL} (e^{-at} - e^{-Rt/L})$.

Sol. Let I be the current in the circuit at any time t , then by Kirchhoff's law, we have

$$L \frac{dI}{dt} + R.I = E.e^{-at}, \text{ where } I(0) = 0$$

Taking Laplace transform of both sides, we have

$$L \cdot [s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a}$$

Using the given condition, it reduces to

$$(Ls + R)\bar{I} = \frac{E}{s+a} \quad \text{or} \quad \bar{I} = \frac{E}{(s+a)(Ls+R)} = \frac{E}{R-aL} \left(\frac{1}{s+a} - \frac{L}{Ls+R} \right)$$

Taking the inverse Laplace transform of both sides, we get

$$I = \frac{E}{R-aL} \cdot L^{-1} \left\{ \frac{1}{s+a} - \frac{1}{s+\frac{R}{L}} \right\} = \frac{E}{R-aL} \cdot [e^{-at} - e^{-Rt/L}].$$

- 77.** Solve the simultaneous equations $(D^2 - 3)x - 4y = 0$, $x + (D^2 + 1)y = 0$ for $t > 0$, given that $x = y = \frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$.

Sol. Taking the Laplace transforms of the given equations, we get

$$s^2 \cdot \bar{x} - sx(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0 \quad \text{or} \quad (s^2 - 3)\bar{x} - 4\bar{y} = 2 \quad \dots(1)$$

$$\text{and} \quad \bar{x} + s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 0$$

$$\text{or} \quad \bar{x} + (s^2 + 1)\bar{y} = 0 \quad \dots(2)$$

Solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{2(s^2 + 1)}{(s^2 - 1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

and

$$\bar{y} = -\frac{2}{(s^2 - 1)^2} = -\frac{1}{2} \left[\frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2} \right]$$

Taking the inverse Laplace transforms of both sides, we get

$$x = L^{-1} \left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2} \right] = te^t + te^{-t} = 2t \left(\frac{e^t + e^{-t}}{2} \right) \quad \text{or} \quad x = 2t \cosh t$$

and

$$y = -\frac{1}{2} L^{-1} \left[\frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2} \right] = -\frac{1}{2} \cdot (e^{-t} - e^t - te^{-t} + te^t)$$

$$= (1-t) \left\{ \frac{e^t - e^{-t}}{2} \right\} = (1-t) \sinh t$$

Hence the solution is given by $x = 2t \cosh t$ and $y = (1-t) \sinh t$.

78. Solve the simultaneous equations : $\frac{dx}{dt} - y = e^t$; $\frac{dy}{dt} + x = \sin t$, given that $x(0) = 1$, $y(0) = 0$.

Sol. Taking the Laplace transforms of the given equations, we get

$$\{s\bar{x} - x(0)\} - \bar{y} = \frac{1}{s-1} \quad \text{i.e.,} \quad s\bar{x} - 1 - \bar{y} = \frac{1}{s-1}$$

$$s\bar{x} - \bar{y} = \frac{1}{s-1} + 1 = \frac{s}{s-1} \quad \dots(1)$$

and

$$\{s\bar{y} - y(0)\} + \bar{x} = \frac{1}{s^2+1} \quad \text{or} \quad \bar{x} + s\bar{y} = \frac{1}{s^2+1} \quad \dots(2)$$

Solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} = \frac{1}{2} \left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1} \right] + \frac{1}{(s^2+1)^2}$$

and

$$\bar{y} = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)} = \frac{s}{(s^2+1)^2} - \frac{1}{2} \left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1} \right]$$

Taking the inverse Laplace transforms of both sides, we get

$$x = \frac{1}{2} L^{-1} \left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1} \right] + L^{-1} \left(\frac{1}{(s^2+1)^2} \right)$$

$$= \frac{1}{2} \cdot \{e^t + \cos t + \sin t\} + \frac{1}{2} (\sin t - t \cos t)$$

$$\therefore L^{-1} \left[\frac{1}{(s^2+a^2)^2} \right] = \frac{1}{2a^2} (\sin at - at \cos at)$$

$$x = \frac{1}{2} [e^t + \cos t + 2 \sin t - t \cos t]$$

$$y = L^{-1} \left[\frac{s}{(s^2+1)^2} \right] - \frac{1}{2} L^{-1} \left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1} \right]$$

$$= \frac{1}{2} t \sin t - \frac{1}{2} \{e^t - \cos t + \sin t\} \left[\because L^{-1} \left\{ \frac{s}{(s^2+a^2)^2} \right\} = \frac{1}{2a} t \sin at \right]$$

$$= \frac{1}{2} \{t \sin t - e^t + \cos t - \sin t\}$$

Hence
$$x = \frac{1}{2} (e^t + \cos t + 2 \sin t - t \cos t); \quad y = \frac{1}{2} (t \sin t - e^t + \cos t - \sin t)$$
 is the required solution of the given equation.

79. Solve the simultaneous equations: $\frac{d^2x}{dt^2} + 5 \frac{dy}{dt} - x = t$, $2 \frac{dx}{dt} - \frac{d^2y}{dt^2} + 4y = 2$, given that when $t = 0$,

$$x = 0, y = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = 0.$$

Sol. Let $L\{x(t)\} = \bar{x}(s)$, $L\{y(t)\} = \bar{y}(s)$

Taking Laplace transform of the given equations, we get

$$\{s^2\bar{x} - x(0) - x'(0)\} + 5\{s\bar{y} - y(0)\} - \bar{x} = \frac{1}{s^2}$$

$$\text{and} \quad 2\{s\bar{x} - x(0)\} - \{s^2\bar{y} - y(0) - y'(0)\} + 4\bar{y} = \frac{2}{s}$$

Using the given initial conditions, these equations reduce to

$$(s^2 - 1)\bar{x} + 5s\bar{y} = \frac{1}{s^2} \quad \dots(1) \quad \text{and} \quad 2s\bar{x} - (s^2 - 4)\bar{y} = \frac{2}{s} \quad \dots(2)$$

Eliminating $\bar{y}$ between (1) and (2), we have

$$\{(s^2 - 1)(s^2 - 4) + 10s^2\}\bar{x} = \frac{s^2 - 4}{s^2} + 10$$

$$\therefore \quad \bar{x} = \frac{11s^2 - 4}{s^2(s^2 + 1)(s^2 + 4)} = -\frac{1}{s^2} + \frac{5}{s^2 + 1} - \frac{4}{s^2 + 4}$$

Taking inverse Laplace transform, we get

$$x = -t + 5 \sin t - 2 \sin 2t \quad \dots(3)$$

Again eliminating $\bar{x}$ between (1) and (2), we have

$$\{10s^2 + (s^2 - 1)(s^2 - 4)\}\bar{y} = \frac{2}{s} - \frac{2(s^2 - 1)}{s}$$

$$\therefore \quad \bar{y} = \frac{4 - 2s^2}{s(s^2 + 1)(s^2 + 4)} \quad \text{or} \quad \bar{y} = \frac{1}{s} - \frac{2s}{s^2 + 1} + \frac{s}{s^2 + 4}$$

Taking inverse Laplace transform, we get

$$y = 1 - 2 \cos t + \cos 2t$$

Hence the desired solution of the given simultaneous equations is

$$\left. \begin{aligned} x &= -t + 5 \sin t - 2 \sin 2t \\ y &= 1 - 2 \cos t + \cos 2t \end{aligned} \right\}$$

80. Use Laplace transform to solve the simultaneous equations $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2, y = 0$ at $t = 0$.

Sol. Taking Laplace transform of the given equations, we get

$$s\bar{x} - x(0) + \bar{y} = \frac{1}{s^2 + 1} \quad \Rightarrow \quad s\bar{x} + \bar{y} = \frac{1}{s^2 + 1} + 2 \quad \dots(1)$$

$$\text{and} \quad s\bar{y} - y(0) + \bar{x} = \frac{s}{s^2 + 1} \quad \Rightarrow \quad \bar{x} + s\bar{y} = \frac{s}{s^2 + 1} \quad \dots(2)$$

Solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{2s}{s^2 - 1}, \bar{y} = \frac{1}{1 + s^2} + \frac{2}{1 - s^2}$$

$$\Rightarrow \bar{x} = \frac{1}{s + 1} + \frac{1}{s - 1}, \bar{y} = \frac{1}{1 + s^2} + \frac{1}{s + 1} - \frac{1}{s - 1}$$

Taking Laplace transform on both sides, we get

$$x = e^{-t} + e^t, y = \sin t + e^{-t} - e^t \quad \text{which is the desired solution.}$$

81. The co-ordinates (x, y) of a particle moving along a plane curve at any time t are given by

$$\frac{dy}{dt} + 2x = \sin 2t, \quad \frac{dx}{dt} - 2y = \cos 2t; (t > 0).$$

It is given that at $t = 0$, $x = 1$ and $y = 0$.

Show using Laplace transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Sol. The given equations are

$$\frac{dy}{dt} + 2x = \sin 2t \quad \dots(1) \quad \frac{dx}{dt} - 2y = \cos 2t \quad \dots(2)$$

Above equations may be rewritten as

$$2x + Dy = \sin 2t \quad \text{or} \quad Dx - 2y = \cos 2t, \quad \text{where } D = \frac{d}{dt}$$

Taking Laplace transform of equation (1) on both sides, we get

$$2\bar{x} + s\bar{y} - y(0) = \frac{2}{s^2 + 4}, \quad \text{where } \bar{x} = L(x), \bar{y} = L(y)$$

$$\Rightarrow 2\bar{x} + s\bar{y} = \frac{2}{s^2 + 4} \quad \dots(3)$$

Again taking Laplace transform of equation (2) on both sides, we get

$$s\bar{x} - x(0) - 2\bar{y} = \frac{s}{s^2 + 4} \quad \Rightarrow \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2 + 4} + 1 \quad \dots(4)$$

Multiplying equation (3) by 2 and equation (4) by s and then adding, we get

$$4\bar{x} + s^2\bar{x} = \frac{4}{s^2 + 4} + \frac{s^2}{s^2 + 4} + s = 1 + s \quad \text{or} \quad \bar{x} = \frac{1 + s}{4 + s^2} = \frac{1}{4 + s^2} + \frac{s}{4 + s^2}$$

Taking inverse Laplace transform, we get

$$x = \frac{1}{2} \sin 2t + \cos 2t \quad \dots(5)$$

Similarly, we solve equations (3) and (4) to obtain y as follows :

Multiplying equation (3) by s and equation (4) by 2 and then subtracting equation (4) from equation (3), we get

$$s^2\bar{y} + 4\bar{y} = \frac{2s}{s^2 + 4} - \frac{2s}{s^2 + 4} - 2 \quad \text{or} \quad \bar{y} = \frac{-2}{s^2 + 4}$$

Taking inverse Laplace transform, we get

$$y = -\sin 2t \quad \dots(6)$$

Now,

$$4x^2 = 4 \left[\frac{1}{4} \sin^2 2t + \cos^2 2t + \sin 2t \cos 2t \right]$$

$$5y^2 = 5 \sin^2 2t$$

$$4xy = 4 \left[\left(\frac{1}{2} \sin 2t + \cos 2t \right) (-\sin 2t) \right]$$

or

$$4xy = -(2 \sin^2 2t + 4 \sin 2t \cos 2t)$$

$$\therefore 4x^2 + 5y^2 + 4xy = 4 \sin^2 2t + 4 \cos^2 2t \quad \text{or} \quad 4x^2 + 5y^2 + 4xy = 4.$$

This shows the particle moves along this curve since the co-ordinates (x, y) given by equations (5) and (6) satisfy the curve.

82. Solve $y'' + ty' - y(t) = 0$ if $y(0) = 0$ and $y'(0) = 1$.

Sol. Taking the Laplace transform of both sides of the given equation, we get

$$L(y'') + L(ty') - L(y) = L(0)$$

$$\text{or} \quad \{s^2 \bar{y} - sy(0) - y'(0)\} - \frac{d}{ds} L(y') - y' = 0 \quad \Rightarrow \quad s^2 \bar{y} - 1 - \frac{d}{ds} \{s \bar{y} - y(0)\} - \bar{y} = 0$$

$$\Rightarrow \quad s^2 \bar{y} - 1 - \frac{d}{ds} (s \bar{y}) - \bar{y} = 0 \quad \Rightarrow \quad -s \frac{d\bar{y}}{ds} + (s^2 - 2) \bar{y} = 1$$

$$\frac{d\bar{y}}{ds} + \left(\frac{2}{s} - s \right) \bar{y} = -\frac{1}{s}$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x (or s) alone.

$$\text{I.F.} = e^{\int \left(\frac{2}{s} - s \right) ds} = s^2 \cdot e^{-s^2/2}$$

Solution of the differential equation is

$$\begin{aligned} \bar{y} s^2 e^{-s^2/2} &= \int \left(-\frac{1}{s} \right) s^2 e^{-s^2/2} + C = - \int s e^{-s^2/2} ds + C \\ &= C + e^{-s^2/2}, \quad \text{where } C \text{ is a constant.} \end{aligned}$$

C must vanish if $\bar{y}$ is a transform since $\bar{y} \rightarrow 0$ as $s \rightarrow \infty$.

$$\therefore \quad \bar{y} = \frac{1}{s^2} \quad \text{or} \quad y = L^{-1} \left(\frac{1}{s^2} \right) = t$$

Hence $y = t$ is the desired solution.

83. Solve the simultaneous equations $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, given that $x = y = 0$, when $t = 0$, by the Laplace transform method.

Sol. Taking the Laplace transform of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = \frac{1}{s^2} \quad \text{i.e.,} \quad (s+5)\bar{x} - 2\bar{y} = \frac{1}{s^2} \quad \dots(1)$$

$$\text{and} \quad s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad \text{i.e.,} \quad 2\bar{x} + (s+1)\bar{y} = 0 \quad \dots(2)$$

Solving (1) and (2) for $\bar{x}$,

$$\bar{x} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in (2), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} \quad \text{or} \quad \bar{y} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On taking inverse Laplace transform, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}, \quad y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

which is the desired solution of the given simultaneous differential equations.

- 84.** Solve the equations: $\frac{dx}{dt} - 6x + 3y = 8e^t$, $\frac{dy}{dt} - 2x - y = 4e^t$ given that $x(0) = -1$, $y(0) = 0$, by Laplace transformation. (M.D.U. May, 2009)

Sol. $\frac{dx}{dt} - 6x + 3y = 8e^t$... (1)

$\frac{dy}{dt} - 2x - y = 4e^t$... (2)

$x(0) = -1, \quad y(0) = 0.$

$$\left. \begin{aligned} (D-6)x + 3y &= 8e^t \\ (D-1)y - 2x &= 4e^t \end{aligned} \right\}$$
 ... (3)

Eliminating y in (3),

$$\begin{aligned} (D-6)(D-1)x + 3(D-1)y &= 8(D-1)e^t & \text{or} & \quad 3(D-1)y - 6x = 12e^t \\ \Rightarrow (D-6)(D-1)x + 6x &= 8(D-1)e^t - 12e^t & \text{or} & \quad [(D-6)(D-1) + 6]x = 8(D-1)e^t - 12e^t \\ (D^2 - 7D + 12)x &= -12e^t \end{aligned}$$

Its A.E. is $D^2 - 7D + 12 = 0 \Rightarrow D = 4, 3$

C.F. = $C_1e^{-4t} + C_2e^{-3t}$

P.I. = $\frac{-12e^t}{D^2 - 7D + 12} = \frac{-12e^t}{1 - 7 + 12} = -2e^t$

$\therefore x = C_1e^{-4t} + C_2e^{-3t} - 2e^t$

$\frac{dx}{dt} = -4C_1e^{-4t} - 3C_2e^{-3t} - 2e^t$

Putting the values of x and $\frac{dx}{dt}$ in equation (1)

$$(-4C_1e^{-4t} - 3C_2e^{-3t} - 2e^t) - 6(C_1e^{-4t} + C_2e^{-3t} - 2e^t) + 3y = 8e^t$$

or
$$-10C_1e^{-4t} - 9C_2e^{-3t} + 10e^t - 8e^t = -3y$$

or
$$y = \frac{10}{3}C_1e^{-4t} + 3C_2e^{-3t} - \frac{2}{3}e^t.$$

- 85.** Solve by transform method, the simultaneous differential equations $D^2x + 3x - 2y = 0$, $D^2y - 3x + 5y = 0$ Given $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$.

Sol. Taking Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0$$

i.e., $(s^2 + 3)\bar{x} - 2\bar{y} = 3$... (1)

and $[s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0$

i.e., $(s^2 - 3)\bar{x} + (s^2 + 5)\bar{y} = 5$... (2)

Solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{3s^2 + 25}{(s^2 + 1)(s^2 + 9)} = \frac{11}{4} \cdot \frac{1}{s^2 + 1} + \frac{1}{4} \cdot \frac{1}{s^2 + 9} \quad \text{and} \quad \bar{y} = \frac{2s^2 + 24}{(s^2 + 1)(s^2 + 9)} = \frac{11}{4} \cdot \frac{1}{s^2 + 1} + \frac{3}{4} \cdot \frac{1}{s^2 + 9}$$

On taking inverse Laplace transform, we get

$$x = \frac{11}{4} \sin t + \frac{1}{12} \sin 3t \quad \text{and} \quad y = \frac{11}{4} \sin t - \frac{1}{4} \sin 3t$$

which is the desired solution.

- 86.** Solve the equation : $t \frac{d^2 y}{dt^2} + \frac{dy}{dt} + 4ty = 0$ given that $y = 3$ and $\frac{dy}{dt} = 0$ when $t = 0$.

Sol. Taking Laplace transform on both sides of the given equation, we have

$$L(ty'') + L(y') + 4L(ty) = L(0)$$

$$\Rightarrow -\frac{d}{ds} \cdot L(y'') + L(y') - 4 \cdot \frac{d}{ds} L(y) = 0$$

$$\Rightarrow -\frac{d}{ds} \{s^2 \bar{y} - sy(0) - y'(0)\} + \{s\bar{y} - y(0)\} - 4 \frac{dy}{ds} = 0$$

$$\Rightarrow (s^2 + 4) \frac{dy}{ds} + s\bar{y} = 0$$

Separating the variables, we have $\frac{dy}{y} + \frac{sds}{s^2 + 4} = 0$

On integration, we get $\log \bar{y} + \frac{1}{2} \log (s^2 + 4) = \log C \Rightarrow \bar{y} = \frac{C}{\sqrt{s^2 + 4}}$

Taking inverse Laplace transform

$$y = C.L^{-1} \left\{ \frac{1}{\sqrt{s^2 + 4}} \right\} \quad \text{or} \quad L^{-1} \left\{ \frac{1}{\sqrt{s^2 + 1}} \right\} = J_0(t),$$

called Bessel function of order zero.

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$\therefore y = C J_0(2t)$$

Since $y(0) = 3$, from above $y(0) = C \cdot J_0(0) = C$, $\therefore C = 3$

Hence the required solution is $y = 3 \cdot J_0(2t)$.

- 87.** Using Laplace transform, solve the differential equation $y'' + 2ty' - y = t$, when $y(0) = 0$ and $y'(0) = 1$.

Sol. Taking Laplace transform on both sides, we get

$$L(y'') + 2L(ty') - L(y) = L(t) \quad \dots(1)$$

$$\Rightarrow s^2 \bar{y} - sy(0) - y'(0) + 2 \left[-\frac{d}{ds} L(y') \right] - \bar{y} = \frac{1}{s^2}$$

$$\Rightarrow s^2 \bar{y} - 1 - 2 \cdot \frac{d}{ds} \{s\bar{y} - y(0)\} - \bar{y} = \frac{1}{s^2}$$

$$\Rightarrow s^2 \bar{y} - 1 - 2 \cdot \left(s \frac{d\bar{y}}{ds} + \bar{y} \right) - \bar{y} = \frac{1}{s^2}$$

$$- 2s \frac{d\bar{y}}{ds} + \bar{y} (s^2 - 3) = \frac{1}{s^2} + 1$$

$$\Rightarrow \frac{d\bar{y}}{ds} + \left(\frac{3 - s^2}{2s} \right) \bar{y} = -\frac{1}{2s^3} - \frac{1}{2s} \quad \dots(2)$$

This is a linear differential equation of order 1 in $\bar{y}$ and s .

$$\text{I.F.} = e^{\frac{1}{2} \int \left(\frac{3}{s} - s \right) ds} = e^{\frac{1}{2} \left(3 \log s - \frac{s^2}{2} \right)} = e^{-(s^2/4)} \cdot s^{3/2}$$

Solution to equation (2) is

$$\bar{y} \cdot e^{-s^2/4} \cdot s^{3/2} = -\frac{1}{2} \int \left(\frac{1}{s^3} + \frac{1}{s} \right) s^{3/2} \cdot e^{-s^2/4} ds = -\frac{1}{2} \int \left(\sqrt{s} + \frac{1}{s^{3/2}} \right) e^{-s^2/4} ds$$

$$\text{Put } s^2 = 4t \Rightarrow s = 2\sqrt{t}, ds = \frac{dt}{\sqrt{t}}$$

$$\begin{aligned} \text{R.H.S. integral } & \frac{-1}{2} \int \left(\sqrt{s} + \frac{1}{s^{3/2}} \right) e^{-s^2/4} \cdot ds \\ &= \frac{-1}{2} \int \left(\sqrt{2t}^{1/4} + \frac{1}{2\sqrt{2}} t^{-3/4} \right) e^{-t} \cdot \frac{dt}{\sqrt{t}} = -\frac{1}{\sqrt{2}} \int \left(t^{-1/4} + \frac{1}{4} t^{-5/4} \right) e^{-t} \cdot dt \\ &= -\frac{1}{\sqrt{2}} \left[t^{-1/4} \cdot \frac{e^{-t}}{(-1)} + \int \frac{-1}{4} \cdot t^{-5/4} \cdot e^{-t} dt + \frac{1}{4} \int t^{-5/4} \cdot e^{-t} dt \right] \\ &= \frac{1}{\sqrt{2}} e^{-t} \cdot t^{-1/4} = \frac{1}{\sqrt{2}} e^{-s^2/4} \left(\frac{s^2}{4} \right)^{-\frac{1}{4}} = \frac{1}{\sqrt{s}} e^{-(s^2/4)} \end{aligned}$$

$$\Rightarrow \bar{y} \cdot e^{-s^2/4} \cdot s^{3/2} = \frac{1}{\sqrt{s}} e^{-(s^2/4)} \Rightarrow \bar{y} = \frac{1}{s^2} + C, \text{ where } C \text{ is a constant.}$$

C must vanish if $\bar{y}$ is a transform since $\bar{y} \rightarrow 0$ as $s \rightarrow \infty$.

$$\therefore \bar{y} = \frac{1}{s^2}$$

Taking inverse Laplace transform, we get $y = L^{-1} \left(\frac{1}{s^2} \right) = t$, which is the required solution.

88. Solve the simultaneous equations $\frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t$, $\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0$. Given that

$x = 0, y = 0, \frac{dx}{dt} = 0$ when $t = 0$, by using Laplace transforms. (M.D.U., 2005)

$$\text{Sol.} \quad \frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t \quad \dots(1)$$

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0 \quad \dots(2)$$

Taking Laplace transform on both sides of equation (1), we get

$$(s\bar{x} - x(0)) - [s\bar{y} - y(0)] - 2\bar{x} + 2\bar{y} = \frac{1}{s} - \frac{2}{s^2}$$

$$s\bar{x} - s\bar{y} - 2\bar{x} + 2\bar{y} = \frac{s-2}{s^2}$$

$$(s-2)\bar{x} - \bar{y}(s-2) = \frac{s-2}{s^2} \Rightarrow \bar{x} - \bar{y} = \frac{1}{s^2} \quad \text{or} \quad \bar{y} = \bar{x} - \frac{1}{s^2}$$

Taking Laplace transform of both sides of equation (2), we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 2[s\bar{y} - y(0)] + \bar{x} = 0$$

$$\Rightarrow s^2\bar{x} + 2s\bar{y} + \bar{x} = 0 \quad \text{or} \quad (s^2 + 1)\bar{x} + 2s\bar{y} = 0$$

Putting the value of $\bar{y}$, $\bar{y} = \bar{x} - \frac{1}{s^2}$, we get

$$(s^2 + 1)\bar{x} + 2s\left(\bar{x} - \frac{1}{s^2}\right) = 0 \quad \text{or} \quad (s^2 + 2s + 1)\bar{x} = \frac{2}{s} \quad \text{or} \quad \bar{x} = \frac{2}{s(s+1)^2}$$

Taking inverse Laplace transform

$$x = L^{-1}\left[\frac{2}{s(s+1)^2}\right]$$

$$x = L^{-1}\left[\frac{2}{s} - \frac{2}{s+1} + \frac{2}{(s+1)^2}\right] \quad \text{or} \quad x = 2 - 2e^{-t} - 2t \cdot e^{-t}$$

$$\Rightarrow x = 2(1 - e^{-t} - te^{-t})$$

$$\text{Now} \quad \bar{y} = \bar{x} - \frac{1}{s^2} \quad \text{or} \quad \bar{y} = \frac{2}{s(s+1)^2} - \frac{1}{s^2}$$

Taking inverse Laplace transform of this equation

$$y = L^{-1}\left[\frac{2}{s(s+1)^2}\right] - L^{-1}\left[\frac{1}{s^2}\right] = 2(1 - e^{-t} - te^{-t}) - t$$

$$\text{or} \quad y = 2(1 - e^{-t}) - t(1 + 2e^{-t})$$

Hence the solution of the given simultaneous equations is

$$x = 2(1 - e^{-t} - te^{-t})$$

$$y = 2(1 - e^{-t}) - t(1 + 2e^{-t}).$$

- 89.** By using Laplace transform, solve the equation $\frac{d^2x}{dt^2} + y = -5 \cos 2t$, $\frac{d^2y}{dt^2} + x = 5 \cos 2t$, where $x(0) = x'(0) = y'(0) = 1$, $y(0) = -1$. (M.D.U., 2005)

Sol. Taking Laplace transform of both sides of the equation $\frac{d^2y}{dt^2} + x = 5 \cos 2t$, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{x} = 5 \cdot \frac{s}{s^2 + 4}$$

$$s^2\bar{y} + s - 1 + \bar{x} = \frac{5s}{s^2 + 4} \quad \dots(1)$$

$$\text{Now,} \quad \frac{d^2x}{dt^2} + y = -5 \cos 2t$$

Taking Laplace transform on both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + \bar{y} = \frac{-5s}{s^2 + 4}$$

$$\text{or} \quad s^2\bar{x} - s - 1 + \bar{y} = -5 \cdot \frac{s}{s^2 + 4} \quad \dots(2)$$

To eliminate $\bar{y}$ from (1) and (2), we multiply equation (2) by s^2 and subtract from equation (1) and we have

$$s^2\bar{y} + s - 1 + \bar{x} - s^4\bar{x} + s^2(s+1) - s^2\bar{y} = \frac{5s + 5s^3}{s^2 + 4}$$

$$(1 - s^4)\bar{x} + s - 1 + s^2 + s^3 = \frac{5s + 5s^3}{s^2 + 4}$$

$$(1 - s^4)\bar{x} = \frac{5s + 5s^3}{s^2 + 4} - s + 1 - s^2 - s^3$$

or
$$(1 - s^4)\bar{x} = \frac{5s(1 + s^2)}{(s^2 + 4)} + (1 - s^2) - s(1 + s^2)$$

$$\Rightarrow \bar{x} = \frac{5s(1 + s^2)}{(1 - s^4)(s^2 + 4)} + \frac{1 - s^2}{1 - s^4} - \frac{s(1 + s^2)}{1 - s^4} = \frac{5s}{(1 - s^2)(s^2 + 4)} + \frac{1}{1 + s^2} - \frac{s}{1 - s^2}$$

$$= \frac{5s}{(1 - s^2)(s^2 + 4)} - \frac{s}{1 - s^2} + \frac{1}{1 + s^2} = \frac{5s - s(s^2 + 4)}{(1 - s^2)(s^2 + 4)} + \frac{1}{1 + s^2}$$

$$\Rightarrow \bar{x} = \frac{s}{s^2 + 4} + \frac{1}{1 + s^2}$$

Taking inverse Laplace transform, we get

$$x = L^{-1}\left(\frac{s}{s^2 + 4}\right) + L^{-1}\left(\frac{1}{1 + s^2}\right) \quad \text{or} \quad x = \cos 2t + \sin t$$

To eliminate $\bar{x}$ from equations (1) and (2), we multiply equation (1) with s^2 and subtract equation (2) from the same. Thus, we get

$$(s^4 - 1)\bar{y} = \frac{5s(1 + s^2)}{s^2 + 4} + s^2 - 1 - s(1 + s^2)$$

or
$$\bar{y} = \frac{5s}{(s^2 - 1)(s^2 + 4)} + \frac{1}{s^2 + 1} - \frac{s}{s^2 - 1} \Rightarrow \bar{y} = -\frac{s}{s^2 + 4} + \frac{1}{s^2 + 1}$$

Taking inverse Laplace transform, we have

$$y = -\cos 2t + \sin t.$$

Hence the solution of the given simultaneous differential equations is given by

$$x = \sin t + \cos 2t$$

$$y = \sin t - \cos 2t.$$

90. Discuss and describe some other useful functions and how to find their Laplace transforms.

(M.D.U. May, 2009)

Sol. 1. Unit Step Function or Heaviside's Unit Function

Definition : The unit step function $u(t - a)$ is defined as

$$u(t - a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}, \text{ where } a \geq 0$$

As a particular case, $u(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$

The product, $f(t) \cdot u(t - a) = \begin{cases} 0, & \text{for } t < 0 \\ f(t), & \text{for } t \geq 0 \end{cases}$

$$L\{u(t - a)\} = \frac{1}{s} e^{-as}$$

In particular, $L\{u(t)\} = \frac{1}{s}$

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(t-a) \cdot u(t-a)\} = e^{-as} \cdot \bar{f}(s)$

If $a = 0$, $L\{f(t) \cdot u(t)\} = \bar{f}(s) = L\{f(t)\}$.

Example. Find the Laplace transform of $(t-1)^2 \cdot u(t-1)$.

Sol. Comparing with $f(t-a) \cdot u(t-a)$, we have $a = 1$ and $f(t) = t^2$

$$\therefore \bar{f}(s) = L\{f(t)\} = L\{t^2\} = \frac{2}{s^3}$$

$$\therefore L\{(t-1)^2 u(t-1)\} = e^{-s} \cdot \bar{f}(s) = \frac{2e^{-s}}{s^3}.$$

2. Unit Impulse Function or Dirac-delta Function

$$\delta \varepsilon (t-a) = \begin{cases} 0, & \text{for } t < a \\ \frac{1}{\varepsilon}, & \text{for } a \leq t \leq a + \varepsilon \\ 0, & \text{for } t > a \end{cases}$$

As $\varepsilon \rightarrow 0$, the function $\delta \varepsilon (t-a)$ tends to be infinite at $x = a$ and zero elsewhere, with the characteristic property that its integral across $t = a$ is unity. If $\delta \varepsilon (t-a)$ represents a force acting for a short duration ε at time $t = a$, then the integral.

$\lim_{\varepsilon \rightarrow 0} \int_a^{a+\varepsilon} \delta \varepsilon (t-a) dt = 1$, represents unit impulse at $t = a$. Hence the limiting form of $\delta \varepsilon (t-a)$ as $\varepsilon \rightarrow 0$ is expressed as unit impulse function denoted by $\delta(t-a)$.

Thus, the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \begin{cases} \infty, & \text{for } t = a \\ 0, & \text{for } t \neq a \end{cases}$$

Such that, $\int_0^{\infty} \delta(t-a) dt = 1$.

Laplace transform of unit Impulse function

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st} \delta(t-a) dt = e^{-sa}.$$

$$L(\delta(t)) = 1.$$

3. Periodic Functions

If $f(t)$ is a periodic function with period

$$T, \text{ i.e., } f(t+T) = f(t), \text{ then } L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

4. The Error Function

The error function denoted by $\text{erf}\sqrt{t}$ is defined as $\text{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-\eta^2} d\eta$

Note. $\lim_{x \rightarrow 0} \text{erf } x = 0$ and $\lim_{x \rightarrow \infty} \text{erf } x = 1$.

$$L(\text{erf}\sqrt{t}) = L\left[\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-\eta^2} d\eta\right] = L\left[\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - \eta^2 + \frac{\eta^4}{2!} - \dots\right) d\eta\right]$$

$$= L\left[\frac{2}{\sqrt{\pi}} \left\{\sqrt{t} - \frac{t^{3/2}}{3} + \frac{1}{5.2!} \cdot t^{5/2} - \dots\right\}\right] = \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3s^{5/2}} + \dots\right]$$

$$\begin{aligned}
&= \frac{1}{s^{3/2}} - \frac{1}{2} \cdot \frac{1}{s^{5/2}} + \frac{1.3}{2.4} \cdot \frac{1}{s^{7/2}} - \dots \\
&= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1.3}{2.4} \cdot \frac{1}{s^2} - \dots \right] = \frac{1}{s^{3/2}} \left(1 + \frac{1}{s} \right)^{-1/2} = \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

Also,
$$L(\operatorname{erf} 2\sqrt{t}) = \frac{1}{4} \cdot \frac{1}{\frac{s}{4} \sqrt{\frac{s}{4} + 1}} = \frac{2}{s\sqrt{s+4}}$$

and
$$L(t \operatorname{erf} 2\sqrt{t}) = -\frac{d}{ds} \left[\frac{2}{s\sqrt{s+4}} \right] = \frac{3s+8}{s^2(s+4)^{3/2}}$$

5. The Bessel Function

(M.D.U., 2005; U.P., June, 2006, 2009; A.U.U.P., 2008)

Bessel function of order n is given by

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} \right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(n+r+1)} \left(\frac{x}{2} \right)^{n+2r}$$

Bessel function of order zero is given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$L\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2} \cdot \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{4!}{s^5} - \dots$$

$$= \frac{1}{s} \left[1 - \frac{1}{2} \cdot \left(\frac{1}{s^2} \right) + \frac{1.3}{2.4} \left(\frac{1}{s^4} \right) - \frac{1.3.5}{2.4.6} \left(\frac{1}{s^6} \right) + \dots \right] = \frac{1}{s} \left[\left(1 + \frac{1}{s^2} \right)^{-1/2} \right] = \frac{1}{\sqrt{s^2+1}}$$

By the recurrence relation, $J_0'(t) = -J_1(t)$, where $J_1(t)$ is the Bessel function of order one.

6. The Exponential Integral Function

The exponential integral is defined as

$$E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u} \cdot du \quad \text{or} \quad L\{E_i(t)\} = \frac{1}{s} \log(s+1)$$

7. Laguerre Polynomial

This polynomial is defined as
$$L_n(x) = \frac{e^x}{n!} \cdot \frac{d^n}{dx^n} (e^{-x} \cdot x^n)$$

Laplace transform of $L_n(t)$ i.e., $L\{L_n(t)\} = \frac{(s-1)^n}{s^{n+1}}$; $s > 1$ and n is a positive integer.

8. The Sine Integral

$$S_i(t) = \int_0^t \frac{\sin y}{y} dy$$

9. The Cosine Integral

$$C_i(t) = \int_0^{\infty} \frac{\cos u}{u} du, \quad L\{C_i(t)\} = \frac{1}{2s} \log(s^2+1)$$

10. The Complementary Error Function

This is defined as $\operatorname{erf}_c(t) = 1 - \operatorname{erf} t = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-u^2} \cdot du$

$$L(\operatorname{erf}_c \sqrt{t}) = L[1 - \operatorname{erf} \sqrt{t}] = L(1) - L\{\operatorname{erf} \sqrt{t}\} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}.$$

91. Express the following function in terms of unit step function and find its Laplace transform.

(M.D.U., May, 2009; U.P.T.U., 2007)

Sol. Here,
$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$$

or

$$f(t) = (t-1)\{u(t-1) - u(t-2)\} + u(t-2) = (t-1)u(t-1) - (t-2)u(t-2)$$

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(t-a)u(t-a)\} = e^{-as} \bar{f}(s)$

Here,
$$L\{f(t)\} = L(t) = \frac{1}{s^2}$$

$$\therefore L\{(t-1)u(t-1)\} = e^{-s} \cdot \frac{1}{s^2} \text{ and } L\{(t-2)u(t-2)\} = e^{-2s} \cdot \frac{1}{s^2}$$

Hence,
$$L\{(t-1)u(t-1) - (t-2)u(t-2)\} = \frac{e^{-s} - e^{-2s}}{s^2}.$$

92. Find the Laplace transform of the function $e^{-t}\{1 - u(t-2)\}$.

Sol. Expressing e^{-t} as a function of $t-2$, we have

$$e^{-t} = e^{-(t-2+2)} = e^{-2} \cdot e^{-(t-2)}$$

$$\therefore e^{-t} \cdot u(t-2) = e^{-2} \cdot e^{-(t-2)} \cdot u(t-2)$$

Comparing $e^{-(t-2)}u(t-2)$ with $F(t-a)u(t-a)$, we get $a = 2$ and $F(t) = e^{-t}$

$$\therefore f(s) = \frac{1}{s+1} = L\{F(t)\}$$

$$\therefore L[e^{-t} \cdot u(t-2)] = e^{-2} \cdot e^{-2s} \cdot \frac{1}{s+1}$$

$$\therefore L[e^{-t}\{1 - u(t-2)\}] = \frac{1}{s+1} - e^{-2} \cdot e^{-2s} \cdot \frac{1}{s+1} = \frac{1 - e^{-2(1+s)}}{(s+1)}.$$

93. Find Laplace transform of the function $F(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$ (U.P.T.U., 2009)

Sol.
$$F(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$$

$$= (t-3)u(t-3) - 2(t-2)u(t-2) + (t-1)u(t-1)$$

Hence,
$$L\{F(t)\} = L\{(t-3)u(t-3)\} - 2L\{(t-2)u(t-2)\} + L\{(t-1)u(t-1)\}$$

$$= \frac{e^{-3s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} = \frac{e^{-s}(1 - e^{-s})^2}{s^2}.$$

94. Express the function $f(t) = \begin{cases} t^2, & 0 < t \leq 2 \\ 4t, & t > 2 \end{cases}$ in terms of unit step function and obtain the Laplace transform.

Sol. In terms of unit-step function,

$$f(t) = t^2\{u(t) - u(t-2)\} + 4t \cdot u(t-2) = t^2 u(t) + (4t - t^2) u(t-2)$$

$$\begin{aligned}\therefore \quad L\{f(t)\} &= L\{t^2 u(t)\} + L\{[4 - (t-2)^2] u(t-2)\} \\ &= -\frac{2}{s^3} + e^{-2s} L(4 - t^2) = \frac{2}{s^3} + e^{-2s} \cdot \left(\frac{4}{s} - \frac{2}{s^3}\right) = \frac{2(1 - e^{-2s})}{s^3} + \frac{4e^{-2s}}{s}.\end{aligned}$$

95. Find the Laplace transform of the following periodic functions :

(i) $f(t) = \frac{t}{T}$, for $0 < t < T$ (saw-tooth wave of period T) (ii) $f(t) = \sin\left(\frac{\pi t}{a}\right)$ for $0 < t < a$ (Rectified sine wave of period a).

Sol. (i) Here, $L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$

$$\begin{aligned}&= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} \cdot \frac{t}{T} dt = \frac{1}{T(1 - e^{-sT})} \left[\left(\frac{te^{-st}}{-s} \right)_0^T - \int_0^T 1 \cdot \frac{e^{-st}}{-s} dt \right] \\ &= \frac{1}{1 - e^{-sT}} \left[-\frac{e^{-sT}}{s} + \frac{1 - e^{-sT}}{s^2 \cdot T} \right] = \frac{1}{s^2 T} - \frac{e^{-sT}}{s(1 - e^{-sT})}.\end{aligned}$$

(ii) $L\{f(t)\} = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} \sin\left(\frac{\pi t}{a}\right) dt$

Let $I = \int_0^a e^{-st} \cdot \sin\left(\frac{\pi t}{a}\right) dt = \left[\frac{e^{-st}}{s^2 + \frac{\pi^2}{a^2}} \left(-s \sin \frac{\pi t}{a} - \frac{\pi}{a} \cos \frac{\pi t}{a} \right) \right]_0^a$

$$= \left[\frac{e^{-as}}{s^2 + \frac{\pi^2}{a^2}} \left(\frac{\pi}{a} \right) \right] - \left[\frac{1}{s^2 + \frac{\pi^2}{a^2}} \left(-\frac{\pi}{a} \right) \right] = \frac{(1 + e^{-as}) a \pi}{a^2 s^2 + \pi^2}$$

$$\begin{aligned}\therefore L\{f(t)\} &= \frac{(1 + e^{-as}) a \pi}{(1 - e^{-as})(a^2 s^2 + \pi^2)} = \frac{(e^{as/2} + e^{-as/2}) \cdot (a \pi)}{(e^{as/2} - e^{-as/2})(a^2 s^2 + \pi^2)} \\ &= \left(\frac{a \pi}{a^2 s^2 + \pi^2} \right) \coth \frac{as}{2}.\end{aligned}$$

96. Find the Laplace transform of the triangular wave function of period $2c$ given by

$$f(t) = \begin{cases} t & , \quad 0 < t \leq c \\ 2c - t & , \quad c < t < 2c \end{cases}$$

(M.D.U. May 2009, Uttarakhand, June 2007)

Sol. $L\{f(t)\} = \frac{1}{1 - e^{-2cs}} \int_0^{2c} e^{-st} f(t) dt = \frac{1}{1 - e^{-2cs}} \left[\int_0^c e^{-st} \cdot t dt + \int_c^{2c} e^{-st} \cdot (2c - t) dt \right]$

$$\begin{aligned}&= \frac{1}{1 - e^{-2cs}} \left[\left\{ -\frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} + \frac{1}{s^2} \right\} + \left\{ \frac{e^{-2cs}}{s^2} + \frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} \right\} \right] \\ &= \frac{1}{1 - e^{-2cs}} \left(\frac{1 - 2e^{-cs} + e^{-2cs}}{s^2} \right) = \frac{1}{s^2} \left\{ \frac{(1 - e^{-cs})^2}{(1 + e^{-cs})(1 - e^{-cs})} \right\} = \frac{1}{s^2} \cdot \frac{1 - e^{-cs}}{1 + e^{-cs}} \\ &= \frac{1}{s^2} \tanh\left(\frac{cs}{2}\right).\end{aligned}$$

97. Find the Laplace transform of the rectified semi-wave function defined by

$$f(t) = \begin{cases} \sin wt, & 0 < t < \frac{\pi}{w} \\ 0, & \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$$

Sol. Here, $f(t)$ is a periodic function with period $\frac{2\pi}{w}$

$$\begin{aligned} \therefore L\{f(t)\} &= \frac{1}{1 - e^{-\frac{2\pi s}{w}}} \int_0^{2\pi/w} e^{-st} \cdot f(t) dt = \frac{1}{1 - e^{-\frac{2\pi s}{w}}} \left[\int_0^{\pi/w} e^{-st} \sin wt dt + \int_{\pi/w}^{2\pi/w} e^{-st} \cdot 0 \cdot dt \right] \\ &= \frac{1}{1 - e^{-\frac{2\pi s}{w}}} \left[\frac{e^{-st}(-s \sin wt - w \cos wt)}{s^2 + w^2} \right]_0^{\pi/w} \\ &= \frac{we^{-\frac{\pi s}{w}} + w}{\left(1 - e^{-\frac{\pi s}{w}}\right) \left(1 + e^{-\frac{\pi s}{w}}\right) (s^2 + w^2)} = \frac{w}{\left(1 - e^{-\frac{\pi s}{w}}\right) (s^2 + w^2)} \end{aligned}$$

98. Find the Laplace transform of the square-wave function of period a defined as :

$$f(t) = \begin{cases} 1, & 0 \leq t \leq \frac{a}{2} \\ -1, & \frac{a}{2} < t < a \end{cases}$$

$$\begin{aligned} \text{Sol. } L\{f(t)\} &= \frac{1}{1 - e^{-as}} \int_0^a e^{-st} \cdot f(t) dt = \frac{1}{1 - e^{-as}} \left[\int_0^{a/2} e^{-st} \cdot 1 \cdot dt + \int_{a/2}^a e^{-st} (-1) dt \right] \\ &= \frac{1}{1 - e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} + \left(\frac{e^{-st}}{s} \right)_{a/2}^a \right] = \frac{1}{s(1 - e^{-as})} [1 - e^{-as/2} + e^{-as} - e^{-as/2}] \\ &= \frac{1}{s(1 - e^{-as})} [1 - e^{-as/2}]^2 = \frac{1}{s} \left(\frac{1 - e^{-as/2}}{1 + e^{-as/2}} \right) \\ &= \frac{1}{s} \cdot \left(\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}} \right) = \frac{1}{s} \tanh \left(\frac{as}{4} \right) \end{aligned}$$

99. A condenser of capacity C is charged to potential E and discharged at $t = 0$ through an inductive resistance L , R . Show that the charge q at time t is given by $q = \frac{CE}{n} e^{-\mu t} (\mu \sin nt + n \cos nt)$, where

$$\mu = \frac{R}{2L} \text{ and } n^2 = \left(\frac{1}{LC} - \frac{R^2}{4L^2} \right) > 0.$$

Sol. Let q be the charge and i be the current in the circuit at time t then the potential difference across the inductance L , resistance R and capacitance C are respectively $L \frac{di}{dt}$, Ri and $\frac{q}{C}$.

When discharging, the applied potential is zero,

$$\therefore \quad L \frac{di}{dt} + Ri + \frac{q}{C} = 0 \quad \Rightarrow \quad L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \quad \dots(1) \quad \left(\because i = \frac{dq}{dt} \right)$$

The initial conditions are $q = EC$ and $i = 0$ when $t = 0$.

Hence Laplace transform of (1) is

$$\left(Ls^2 + Rs + \frac{1}{C} \right) \bar{q} = (Ls + R) EC \quad \Rightarrow \quad \bar{q} = \frac{EC \left(s + \frac{R}{L} \right)}{s^2 + \left(\frac{R}{L} \right) s + \frac{1}{LC}} = \frac{EC (s + 2\mu)}{s^2 + 2\mu s + \mu^2 + n^2}$$

Now, taking inverse Laplace transform, we get

$$q = EC \left[\frac{s + \mu}{(s + \mu)^2 + n^2} + \frac{\mu}{(s + \mu)^2 + n^2} \right]$$

$$\text{or} \quad q = EC \cdot \frac{e^{-\mu t}}{n} (n \cos nt + \mu \sin nt).$$

- 100.** In an electrical circuit with e.m.f. $E(t)$, resistance R and inductance L , the current i builds up at the rate given by $L \frac{di}{dt} + Ri = E(t)$. If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i at any instant. [U.P.T.U., (C.O.) 2005]

Sol. Here, $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E, & \text{for } 0 < t < a \\ 0, & \text{for } t > a \end{cases}$

Taking Laplace transform of both sides of the given equation, we have

$$L [s\bar{i} - i(0)] + R\bar{i} = \int_0^\infty e^{-st} E(t) dt$$

$$\text{or} \quad (Ls + R)\bar{i} = \int_0^a e^{-st} E dt = \frac{E}{s} (1 - e^{-as}) \quad \Rightarrow \quad \bar{i} = \frac{E}{s(Ls + R)} - \frac{E \cdot e^{-as}}{s(Ls + R)}$$

Taking inverse Laplace transform of both sides, we have

$$i = L^{-1} \left\{ \frac{E}{s(Ls + R)} \right\} - L^{-1} \left\{ \frac{E \cdot e^{-as}}{s(Ls + R)} \right\} \quad \dots(1)$$

$$\begin{aligned} \text{Now} \quad L^{-1} \left\{ \frac{E}{s(Ls + R)} \right\} &= L^{-1} \left\{ E \cdot \frac{1}{R} \left(\frac{1}{s} - \frac{L}{Ls + R} \right) \right\} \\ &= \frac{E}{R} \left[L^{-1} \left(\frac{1}{s} \right) - L^{-1} \left\{ \left(\frac{1}{s + R/L} \right) \right\} \right] = \frac{E}{R} (1 - e^{-Rt/L}) \end{aligned}$$

$$\text{and} \quad L^{-1} \left\{ \frac{E \cdot e^{-as}}{s(Ls + R)} \right\} = \frac{E}{R} [1 - e^{-R(t-a)/L}] u(t - a)$$

$$\therefore \text{ From (1), we have } i = \frac{E}{R} (1 - e^{-Rt/L}) - \frac{E}{R} [1 - e^{-R(t-a)/L}] u(t - a)$$

$$\text{Hence, for } 0 < t < a, \quad i = \frac{E}{R} (1 - e^{-Rt/L}) \quad \text{and for } t > a, \quad i = \frac{E}{R} e^{-Rt/L} \cdot (e^{Ra/L} - 1).$$

- 101.** An impulsive voltage $E \delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If I be the current at any subsequent time t , find the limit of I as $t \rightarrow 0$ and justify your result.

Sol. The equation for the circuit is $L \frac{dI}{dt} + RI + \frac{q}{C} = E \delta(t)$; $\frac{dq}{dt} = I$, with zero initial conditions.

Laplace transform of these equations are $(Ls + R)\bar{I} + \frac{q}{C} = E$, $qs = \bar{I}$

Multiplying by s , $\left(Ls^2 + Rs + \frac{1}{C}\right)\bar{I} = Es$ or $\left(s^2 + \frac{Rs}{L} + \frac{1}{LC}\right)\bar{I} = \frac{E}{L} \cdot s$

$$(s^2 + 2ms + m^2 + n^2)\bar{I} = \frac{E}{L} s, \text{ where } \frac{R}{L} = 2m, \frac{1}{LC} = m^2 + n^2$$

This gives, $\bar{I} = \frac{E}{L} \left[\frac{(s+m)-m}{(s+m)^2 + n^2} \right]$ so that, $I = \frac{E}{L} \cdot \left(\cos nt - \frac{m}{n} \sin nt \right) e^{-mt}$

We see that $I \rightarrow \frac{E}{L}$, where $t \rightarrow 0$.

Even though the initial current is zero, a large current will develop instantaneously in the circuit due to the impulsive voltage applied at $t = 0$. It will be the limit of this current which we have found above.

- 102.** A particle of mass m can perform small oscillations about a position of equilibrium under a restoring force mn^2 times the displacement. It is started from rest by a constant force F which acts for a time t and then ceases. Show that the amplitude of subsequent oscillation is $\frac{2F}{mn^2} \sin \frac{nT}{2}$.

Sol. The force F acting for $0 < t < T$, and ceasing afterwards can be represented by $F \{1 - u(t - T)\}$. Hence the equation of motion of mass m is

$$m \frac{d^2x}{dt^2} = -mn^2x + F\{1 - u(t - T)\} \quad \text{or} \quad \frac{d^2x}{dt^2} + n^2x = \frac{F}{m} \{1 - u(t - T)\}$$

Its Laplace transform is $(s^2 + n^2)\bar{x} = \frac{F}{ms} (1 - e^{-sT})$

$\therefore$ The initial displacement and velocity are zero.

$$\Rightarrow \bar{x} = \frac{F(1 - e^{-sT})}{ms(s^2 + n^2)} \Rightarrow \bar{x} = \frac{F}{m} (1 - e^{-sT}) \cdot \frac{1}{n^2} \left(\frac{1}{s} - \frac{s}{s^2 + n^2} \right)$$

Taking inverse Laplace transform, we get

$$\begin{aligned} x &= \frac{F}{mn^2} [(1 - \cos nt) - \{1 - \cos n(t - T)\} u(t - T)] \\ &= \frac{F}{mn^2} (1 - \cos nt), \text{ for } 0 < t < T \end{aligned}$$

$$\text{and for } t > T, \quad x = \frac{F}{mn^2} [(1 - \cos nt) - \{1 - \cos n(t - T)\}]$$

$$= \frac{F}{mn^2} [\cos n(t - T) - \cos nt] = \frac{F}{mn^2} \cdot 2 \sin \frac{nT}{2} \cdot \sin n \left(t - \frac{T}{2} \right)$$

$\therefore$ Amplitude of oscillation is $\left(\frac{2F}{mn^2} \right) \sin \frac{nT}{2}$.

- 103.** A body falls from rest in a liquid whose density is one fourth that of the body. If the liquid offers resistance proportional to the velocity and the velocity approaches a limiting value of 9 metres/sec., find the distance fallen in 5 seconds.

Sol. The equation of motion of the body is

$$m \frac{dv}{dt} = mg - \frac{1}{4} mg - mkv \Rightarrow \frac{dv}{dt} + kv = \frac{3}{4} g \quad \dots(1)$$

The initial conditions are $v = 0$ when $t = 0$

Taking Laplace transform of equation (1) on both sides, we get

$$s\bar{v} + k\bar{v} = \frac{3}{4} \cdot \frac{g}{s} \quad \text{or} \quad \bar{v} = \frac{3g}{4k} \left(\frac{1}{s} - \frac{1}{s+k} \right)$$

Taking inverse Laplace transform,

$$\therefore v = \frac{3g}{4k} (1 - e^{-kt}) \quad \dots(2)$$

When $s \rightarrow \infty$, $v \rightarrow \frac{3g}{4k}$

But limiting velocity is given to be = 9 m/sec

$$\therefore \frac{3g}{4k} = 9 \quad \text{or} \quad k = \frac{g}{12}$$

$\therefore$ Equation (2) becomes

$$v \text{ or } \frac{dx}{dt} = 9(1 - e^{-kt}) \quad \dots(3) \text{ and the initial condition is } x = 0 \text{ when } t = 0$$

Taking Laplace transform of (3), we get

$$s\bar{x} = 9 \left[\frac{1}{s} - \frac{1}{s+k} \right]$$

$$\text{i.e.,} \quad \bar{x} = 9 \left[\frac{1}{s^2} - \frac{1}{s(s+k)} \right] \Rightarrow \bar{x} = 9 \left[\frac{1}{s^2} - \frac{1}{k} \left(\frac{1}{s} - \frac{1}{s+k} \right) \right]$$

Taking inverse Laplace transform, we get

$$x = 9 \left[t - \left(\frac{1 - e^{-kt}}{k} \right) \right] \Rightarrow x = 9 \left[t - \frac{12(1 - e^{-gt/12})}{g} \right]$$

Putting $t = 5$ sec, $g = 9.8$ m/sec², the distance travelled in 5 sec is given by

$$x = 9 \left[5 - \frac{12(1 - e^{-49/12})}{9.8} \right] \Rightarrow x = 34.17 \text{ metres.}$$

- 104.** Discuss application of Laplace transforms to deflection of beams. Assuming that the deflection of a beam loaded by a weight $w(x)$ per unit length is given by $EI \frac{d^4 y}{dx^4} = w(x)$, obtain the deflection of a beam, clamped horizontally at one end and free at the other, carrying a uniform load P per unit length.

Sol. Suppose a beam is kept along x -axis and its ends are $x = 0$, $x = l$. Suppose the beam suffers a transverse deflection $y(x)$ which is produced by applying a vertical load $w(x)$ per unit length. The

deflection is given by $\frac{d^4 y}{dx^4} = \frac{w(x)}{EI}$, $0 < x < l$, where E is Young's modulus of elasticity for the beam and I is the moment of inertia of a cross-section of the beam about x -axis.

Following are the boundary conditions

- (i) If beam is hinged or simply supported ends, then $y = y'' = 0$
 (ii) If beam is clamped or built in or fixed end, then $y = y' = 0$
 (iii) If the beam has free end, then $y'' = y''' = 0$

In the given problem, the equation is $\frac{d^4 y}{dx^4} = \frac{P}{EI}$... (1)

Taking Laplace transform, we get

$$s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{P}{EI} \cdot \frac{1}{s}$$

$$\text{or } s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{a}{s} \quad \dots (2)$$

where, $a = \frac{P}{EI}$ (say).

$$\text{Boundary conditions are } y(0) = 0 = y'(0) \quad \dots (3)$$

$$y''(l) = 0 = y'''(l) \quad \dots (4)$$

From equation (2), in view of (3),

$$s^4 \bar{y} - s y''(0) - y'''(0) = \frac{a}{s} \Rightarrow s^4 \bar{y} = bs + c + \frac{a}{s}, \text{ where } y'''(0) = c(\text{say}), y''(0) = b(\text{say})$$

$$\Rightarrow \bar{y} = \frac{b}{s^3} + \frac{c}{s^4} + \frac{a}{s^5}$$

Taking inverse Laplace transform, we get

$$y = \frac{bx^2}{2} + \frac{cx^3}{6} + \frac{ax^4}{24} \quad \dots (5)$$

$$\therefore y'' = \frac{ax^2}{2} + b + cx \quad \dots (6) \quad \text{and} \quad y''' = ax + c \quad \dots (7)$$

Subjecting this to (4), we obtain

$$y''(l) = \frac{al^2}{2} + b + cl = 0$$

$$y'''(l) = 0 = al + c$$

On solving, we get $c = -al$ and $b = \frac{al^2}{2}$

$$\therefore y = \frac{ax^4}{24} + \frac{al^2}{4} x^2 - \frac{al}{6} x^3 = a \left(\frac{x^4}{24} + \frac{x^2 l^2}{4} - \frac{lx^3}{6} \right) = \frac{P}{24EI} (x^4 - 4lx^3 + 6x^2 l^2)$$

$$\Rightarrow y = \frac{P}{24EI} \cdot x^2 (x^2 - 4lx + 6l^2)$$

This gives the deflection of the beam.

105. A beam which is hinged at its ends $x = 0$ and $x = l$ carries a uniform load w_0 per unit length. Show that the deflection at any point is $y(x) = \frac{w_0 x}{24EI} (l - x)(l^2 + lx - x^2)$.

Sol. The deflection $y(x)$ is given by $\frac{d^4 y}{dx^4} = \frac{w_0}{EI} = a(\text{say})$... (1)

Boundary conditions are, $y(0) = 0 = y''(0)$... (2)

$y(l) = 0 = y''(l)$... (3)

Taking Laplace transform on both sides of equation (1)

$$s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{a}{s} \Rightarrow s^4 \bar{y} - s^2 y'(0) - y'''(0) = \frac{a}{s} \quad \dots(4)$$

Let $y'(0) = b$ and $y'''(0) = c$, then from (4),

$$s^4 \bar{y} = bs^2 + c + \frac{a}{s} \Rightarrow \bar{y} = \frac{b}{s^2} + \frac{c}{s^4} + \frac{a}{s^5}$$

Taking inverse Laplace transform,

$$y = bx + \frac{cx^3}{3!} + \frac{ax^4}{4!} = bx + \frac{cx^3}{6} + \frac{ax^4}{24} \quad \dots(5)$$

$$\text{Subjecting to (3), } y(l) = 0 = bl + \frac{cl^2}{6} + \frac{al^4}{24} \quad \dots(6)$$

$$y''(l) = 0 = cl + \frac{cl^2}{2} \quad \dots(7)$$

Solving equations (6) and (7), we get $c = -\frac{al}{2}$, $b = \frac{al^3}{24}$

Then equation (5) becomes

$$y = \frac{al^3 x}{24} - \frac{alx^3}{12} + \frac{ax^4}{24} \Rightarrow y = \frac{ax}{24} (l^3 - 2lx^2 + x^3) \Rightarrow y = \frac{ax}{24} (l-x)(l^2 + lx - x^2)$$

$$\text{or } y = \frac{w_0 x}{24EI} (l-x)(l^2 + lx - x^2).$$

Hence shown.

106. A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = \frac{l}{4}$. Find the resulting deflection of the beam at any point.

Sol. The equation for the deflection is $\frac{d^4 y}{dx^4} = \frac{w}{EI} \cdot \delta \left(x - \frac{l}{4} \right)$

Taking Laplace transform, $s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{w}{EI} e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI} \cdot \frac{e^{-ls/4}}{s^4}$$

On taking inverse Laplace transform, it gives

$$y = c_1 x + c_2 \cdot \frac{x^3}{3!} + \frac{w}{EI} \cdot \frac{(x - l/4)^3}{3!} \cdot u(x - l/4)$$

$$\text{i.e., } y = c_1 x + \frac{1}{6} c_2 x^3, \text{ for } 0 < x < l/4 = c_1 x + \frac{1}{6} c_2 x^3 + \frac{w}{6EI} (x - l/4)^3, \text{ for } \frac{l}{4} < x < l.$$

Using the conditions $y(l) = 0$, $y'(l) = 0$, we get

$$0 = c_1 l + \frac{1}{6} c_2 l^3 + \frac{9wl^3}{128EI} \quad \text{and} \quad 0 = c_1 + \frac{1}{2} c_2 l^2 + \frac{9wl^2}{32EI}$$

$$\Rightarrow c_1 = 9wl^2/256EI \quad \text{or} \quad c_2 = -81w/128EI$$

Substituting the values of c_1 and c_2 , we get

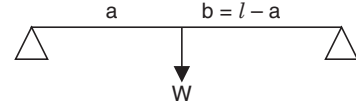
$$y = \frac{45wl^2}{128EI} \cdot x - \frac{27wx^3}{256EI} + \frac{w}{6EI} \left(x - \frac{l}{4} \right)^3 u \left(x - \frac{l}{4} \right).$$

107. A weight less beam of length l is freely supported at ends when a concentrated load W acts $\frac{-13}{692}$ at

$x = a$. Obtain the deflection of the beam. Equation for deflection is $EI \frac{d^4 y}{dx^4} = W \cdot \delta(x - a)$.

Sol. Taking Laplace transform of the equation, we get

$$s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{W}{EI} \cdot e^{-as}$$



Using the conditions $y(0) = y''(0) = 0$ and taking $y'(0) = c_1, y'''(0) = c_2$, we get

$$s^4 \bar{y} - c_1 s^2 - c_2 = \frac{W}{EI} \cdot e^{-as} \quad \text{or} \quad \bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{W}{EI} \cdot \frac{e^{-as}}{s^4}$$

Taking inverse Laplace transform, we have

$$y = c_1 x + c_2 \cdot \frac{x^3}{3!} + \frac{W}{EI} \frac{(x-a)^3}{3!} \cdot u(x-a) \quad \dots(1)$$

$$\therefore y' = c_1 + \frac{1}{2} \cdot c_2 x^2 + \frac{W}{2EI} (x-a)^2 u(x-a)$$

$$y'' = c_2 x + \frac{W}{EI} (x-a) u(x-a)$$

Using the conditions $y(l) = 0, y''(l) = 0$, we get

$$0 = c_1 l + c_2 \frac{l^3}{6} + \frac{W}{EI} \cdot \frac{b^3}{6}$$

$$0 = c_2 l + \frac{W}{EI} \cdot b \quad [\text{Since } u(x-a) = 1, \text{ for } x \geq a \text{ and } l-a = b]$$

$$\begin{aligned} \therefore c_2 &= -\frac{W}{EI} \cdot \frac{b}{l}, c_1 = -\frac{1}{l} \left[c_2 \frac{l^3}{6} + \frac{W}{EI} \cdot \frac{b^3}{6} \right] \\ &= -\frac{1}{l} \left[-\frac{W}{EI} \cdot \frac{bl^2}{6} + \frac{W}{EI} \cdot \frac{b^3}{6} \right] = \frac{Wab}{6EI} (l+b) \end{aligned}$$

$\therefore$ From (1), the solution is

$$y = \frac{Wab}{6EI} (l+b)x - \frac{W}{6EI} \cdot \frac{b}{l} x^3 + \frac{W}{EI} \cdot \frac{(x-a)^3}{3!} \cdot u(x-a)$$

When $0 < x < a, u(x-a) = 0$

$$\therefore y = \frac{W}{6EI} \left[\frac{ab(l+b)x}{l} - \frac{b}{l} x^3 \right] \quad \dots(2)$$

When $a < x < l, u(x-a) = 1$

$$\therefore y = \frac{W}{6EI} \left[\frac{ab(l+b)}{l} x - \frac{b}{l} x^3 + (x-a)^3 \right] \quad \dots(3)$$

From (2) and (3), we have

$$\begin{aligned} y(a) &= \frac{W}{6EI} [a^2 b(l+b) - ba^3] = \frac{Wa^2 b}{6EI} [l+b-a] = \frac{Wa^2 b}{6EI} (2b) \\ &= \frac{1}{3} \cdot \frac{W}{EI} \cdot \frac{a^2 b^2}{l} \end{aligned}$$

□ □ □

Partial Differential Equations

IMPORTANT FORMULAE AND DEFINITIONS

DEF : 1

A differential equation which involves partial derivatives is called **Partial Differential Equation**.

DEF : 2

The Order of a Partial Differential Equation is the order of the highest partial derivative occurring in the equation.

DEF : 3

The Degree of a Partial Differential Equation is the degree of the highest order partial derivative occurring in the equation.

Formation of Partial Differential Equation

(a) By the elimination of arbitrary constants

(b) By the elimination of arbitrary functions.

Method 1. (a) (i) If the number of arbitrary constants to be eliminated is equal to the number of independent variables, then the partial differential equation obtained will be of first order.

(ii) If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equations obtained will be of second or higher order.

(b) If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation, in general, equals to the number of arbitrary functions eliminated.

Notations. $\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial t^2} = t.$

Solutions to Partial Differential Equation

DEF : 4

Complete Solution or Complete Integral of a first order partial differential equation. It is a solution which contains two arbitrary constants.

DEF : 5

General Solution or General Integral consider the solution $f(x, y, z, a, b) = 0$ of a first order partial differential equation. If $b = \phi(a)$, and find the envelope of the family of surfaces $f(x, y, z, a, \phi(a)) = 0$.

We get a solution involving an arbitrary function ϕ . This solution is called general solution or general integral.

Equations Solvable by Direct Integration

Method 2. The equations containing *only one* partial derivative can be solved by direct integration. For constants of integration, we must use arbitrary functions of the variable kept constant.

DEF : 6

Linear Partial Differential Equation of First Order. A differential equation involving first order partial derivative p and q only, is called a partial differential equation of first order. If p and q both occur in first degree only and are not multiplied together, then it is called a linear partial differential equation of first order.

DEF : 7

Lagrange's Linear Equation. The partial differential equation of the form $P.p + Q.q = R$, where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

Solution of Partial Differential Equations

Method 3. To solve Lagrange's linear equation.

(i) Consider $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) Find two independent solutions $u(x, y) = c_1$ and $v(x, y) = c_2$

(iii) Then the complete solution or general solution is given by $\phi(u, v) = 0$ or $u = f(v)$

Very Important Note. If the functions P, Q, R occur in a cyclic order, we can apply either

$$(1) \frac{a}{d} = \frac{b}{e} = \frac{c}{f} \Rightarrow \frac{a-b}{d-e} = \frac{b-c}{e-f} = \frac{c-a}{f-d}$$

or (2) Choose the multipliers which make denominator of the desired fraction zero.

Non-Linear Partial Differential Equations of the First Order**DEF : 8**

A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the product of p and q is called a non-linear partial differential equation of the first order.

Remark. The complete solution of a non-linear partial differential equation involves only two arbitrary constants (*i.e.*, equal to the number of independent variables).

DEF : 9.**Some Standard Forms**

(a) **Equation of the form $f(p, q) = 0$**

i.e., the equations involving only p and q and no x, y, z . Then the complete solution is given by

$$z = ax + by + c, \text{ where } f(a, b) = 0$$

(b) **Equation of the form $z = px + qy + f(p, q)$.** The complete solution is given by

$$z = ax + by + f(a, b)$$

Note. The above form is called Clairaut's form.

(c) **Equation of the form $f(z, p, q) = 0$**

Step I. Take $u = x + ay$ so that

$$p = \frac{dz}{du}, q = a \frac{dz}{du}$$

Step II. Put these values of p and q in the given equation.

Step III. Solve the resulting equation obtained in step-II, in terms of z and u .

Step IV. Replace u by $x + ay$, we get the required solution.

(d) **Equations of the form $f_1(x, p) = f_2(y, q)$**

i.e., the equations in which z is absent and the terms involving x and p can be separated from those involving y and q .

Step I. Find p and q from the given equation

Step II. Use $dz = p dx + q dy$ and on integrating, find z in terms of x and y , as the required solutions.

To Solve Non-Linear Partial Differential Equation of the First Order (By Charpit's Method)

Step I. Write the given equation in the form $f(x, y, z, p, q) = 0$.

Step II. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial p}$, and $\frac{\partial f}{\partial q}$.

Step III. Consider Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-\left(p \frac{\partial f}{\partial p} + q \frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Step IV. Find p (or q) from **Step (III)** and use the given equation to find q (or p).

Step V. Use $dz = p dx + q dy$ and integrate to find z .

Homogeneous Linear Partial Differential Equations With Constant Co-efficients

DEF : 10

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad \dots(A)$$

where $a_1, a_2, \dots, a_n$ are constants, is called **homogeneous linear partial differential equation of order n** with constant co-efficients. It is called **homogeneous** because all the terms contain derivatives of the same order.

Method 4. Homogeneous linear partial differential equation of order n with constant co-efficients.

Step. Write D for $\frac{\partial}{\partial x}$, D' for $\frac{\partial}{\partial y}$, we get from (A)

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D^n) z = F(x, y)$$

$$\text{or } \phi(D, D')z = F(x, y), \text{ where } \phi(D, D') = D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D^n \quad \dots(B)$$

Then the complete solution of (B) is given by $z = \text{C.F.} + \text{P.I.}$

where C.F. = Complementary function
P.I. = Particular integral.

DEF : 11

Rules for finding Complementary Functions (C.F.) for a homogeneous linear partial differential equation of any order with constant co-efficients.

Consider $\phi(D, D')z = F(x, y)$. Then

Step. Put the A.E. $\phi(D, D') = 0$ and find the roots by putting $D = m$ and $D' = 1$. Let $m_1, m_2, m_3, \dots$ are the roots.

Case I. If the roots $m_1, m_2, m_3, \dots$ of the A.E. are all distinct then

$$C.F. = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \dots$$

Case II. If two roots of the A.E. are equal, then

$$C.F. = f_1(y + m_1x) + x f_2(y + m_1x) + f_3(y + m_3x) + \dots$$

Case III. If three roots of the A.E. are equal, then

$$C.F. = f_1(y + m_1x) + x f_2(y + m_1x) + x^2 f_3(y + m_1x) + f_4(y + m_4x) + \dots$$

and so on.

DEF : 12.

Rules For Finding Particular Integral

(Short method) for a homogeneous linear partial differential equation of order with constant co-efficient.

Consider $\phi(D, D')z = F(x, y)$

Then $P.I. = \frac{1}{\phi(D, D')} F(x, y)$

Case I. When $F(x, y) = e^{ax+by}$,

Then $P.I. = \frac{1}{\phi(D, D')} e^{ax+by} = \frac{1}{\phi(a, b)} e^{ax+by}$, Provided $\phi(a, b) \neq 0$

| i.e., Put $D = a, D' = b$

Note. If $\phi(a, b) = 0$, then it is called a case of failure. We then apply **General method**, to be discussed next.

Case II. When $F(x, y) = \sin(ax + by)$ [or $\cos(ax + by)$]

Then $P.I. = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax + by)$ [or $\cos(ax + by)$]
 $= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax + by)$ [or $\cos(ax + by)$],

Provided $\phi(-a^2, -ab, -b^2) \neq 0$

i.e., Put $D^2 = -a^2, DD' = -ab, D'^2 = -b^2$

Note. If $\phi(-a^2, -ab, -b^2) = 0$, then it is a case of failure. We then apply **General method**, to be discussed next.

Case III. When $F(x, y) = x^p y^q$, where p, q are +ve integers.

Then $P.I. = \frac{1}{\phi(D, D')} x^p y^q$

Take out common, the lowest degree terms in $\phi(D, D')$. Expand it by **Binomial theorem**. If the highest power of x is p . Then expand $\phi(D, D')$ upto D^p . (since $D^{p+1}(x^p) = 0, D^{p+2}(x^p) = 0$ etc.) If the highest power of y is q , then expand $\phi(D, D')$ upto D^q (since $D^{q+1}(y^q) = 0, D^{q+2}(y^q) = 0$ etc.

Remarks. (i) $\frac{1}{D} F(x, y) = \int_{x \text{ constant}} F(x, y) dx$

(ii) $\frac{1}{D'} F(x, y) = \int_{y \text{ constant}} F(x, y) dy$

(iii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$$\begin{aligned}
 (1+x)^{-1} &= 1 - x + x^2 - x^3 + \dots \\
 (1-x)^{-2} &= 1 + 2x + 3x^2 + 4x^3 + \dots \\
 (1+x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + \dots
 \end{aligned}$$

DEF : 13.

Very-Very Important Remarks

(1) When $F(x, y)$ (D) is not of the form e^{ax+by} or $\sin(ax+by)$ or $\cos(ax+by)$ or $x^p y^q, p > 0, q > 0$.

or

(2) When there are cases of failures in the short methods, we have a **General Method** to find the Particular Integral (P.I.)

General Method of Finding Particular Integral

$$\begin{aligned}
 \text{Write P.I.} &= \frac{1}{\phi(D, D')} F(x, y) \\
 &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_n D')} F(x, y) \\
 &\quad | \text{ i.e., make the linear factor of } \phi(D, D')
 \end{aligned}$$

$$\text{Consider } \frac{1}{(D - mD')} F(x, y) = \int F(x, (c - mx)) dx$$

where $y + mx = c$. Integrate and replace c by $y + mx$.

DEF : 14.

Non-Homogeneous Linear Partial Differential Equation with Constant Coefficients

In the equation $\phi(D, D')z = F(x, y)$, ... (1)

if the polynomial $\phi(D, D')$ is not homogeneous (when all the terms do not involve derivatives of same order). Then (1) is called a non-homogeneous linear partial differential equation with constant coefficients.

Method 5. To solve (1), non-homogeneous linear partial differential equation with constant co-efficients.

The complete solution is $z = \text{C.F.} + \text{P.I.}$, where

C.F. = complementary function

P.I. = particular integral

Note. The methods to find P.I. are exactly same as those for homogeneous linear equations.

DEF : 15.

General Method

Method 6. To find Complementary Functions (C.F.) for a Non-Homogeneous Linear Partial Differential equation with constant co-efficients.

Step. Consider $f(D, D')z = 0$, Resolve $\phi(D, D')$ into linear factors of the form $D - mD' - a$.

Case I. If all the factors are different

$$\text{i.e., } (D - m_1 D' - a_1)(D - m_2 D' - a_2) \dots (D - m_n D' - a_n)z = 0$$

$$\text{Then C.F.} = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \dots + e^{a_n x} f_n(y + m_n x)$$

Case II. If we have repeated factors, like $(D - mD' - a)^3 z = 0$, then

$$\text{C.F.} = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx) \text{ and so on}$$

Remark. When $\phi(D, D')$ cannot be resolved into the linear factors of the form $(D - mD' - a)$, then the above method fails. We adopt the following method.

Consider the equation $\phi(D, D')z = F(x, y)$. Then the complete solution is given by

$$z = \text{C.F.} + \text{P.I. where } \text{C.F.} = \Sigma A e^{hx + ky}, \text{ and } f(h, k) = 0.$$

SOLVED PROBLEMS

1. Form partial differential equations from the following equations by eliminating the arbitrary constants : (A.U.U.P., 2009)

$$(i) z = (x^2 + a)(y^2 + b) \quad (ii) 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (iii) \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$(iv) z = ax + by + ab \quad (v) z = (x + a)(y + b).$$

Sol. (i) Differentiating z partially w.r.t. x and y

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \quad \dots(1) \quad q = \frac{\partial z}{\partial y} = 2y(x^2 + a) \quad \dots(2)$$

Multiplying (1) and (2), we have

$$pq = 4xy(x^2 + a)(y^2 + b) = 4xyz$$

which is a partial differential equation of the first order.

$$(ii) \text{ Here, } 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Differentiating partially w.r.t. x and y ,

$$2 \frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{p}{x} \quad \text{and} \quad 2 \frac{\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{q}{y}$$

Substituting for $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in the given equation, we get

$$2z = px + qy \text{ which is a partial differential equation of the first order.}$$

- (iii) Here the number of arbitrary constants a, b, c is greater than the number of independent variables x, y . So, the partial differential equation may be second or higher order.

Differentiating partially w.r.t. x and y , we have

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2 x + a^2 z \cdot \frac{\partial z}{\partial x} = 0 \quad \dots(1)$$

$$\text{and} \quad \frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \cdot \frac{\partial z}{\partial y} = 0 \quad \dots(2)$$

Again differentiating (1) partially w.r.t. x , we have

$$c^2 + a^2 \left(\frac{\partial z}{\partial x} \right)^2 + a^2 \cdot z \cdot \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x} \right)^2 + z \cdot \frac{\partial^2 z}{\partial x^2} = 0$$

Substituting $\frac{c^2}{a^2} = -\frac{z}{x} \cdot \frac{\partial z}{\partial x}$ from equation (1), we have

$$-\frac{z}{x} \cdot \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x} \right)^2 + z \cdot \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad xz \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x} \right)^2 - z \frac{\partial z}{\partial x} = 0$$

which is a partial differential equation of the second order.

Instead of differentiating equation (1) partially w.r.t. x , if we differentiate (2) partially w.r.t. y

$$\text{and substitute for } \frac{c^2}{b^2} \text{ from (2), we shall get } yz \frac{\partial^2 z}{\partial y^2} + y \left(\frac{\partial z}{\partial y} \right)^2 - z \left(\frac{\partial z}{\partial y} \right) = 0$$

$$(iv) z = ax + by + ab$$

Differentiating z partially w.r.t. x and y , $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq \quad \text{which is a partial differential equation of the first order.}$$

$$(v) z = (x + a)(y + b)$$

Differentiating z partially w.r.t. x , we get $\frac{\partial z}{\partial x} = p = y + b$... (1)

Differentiating z partially w.r.t. y , we get $\frac{\partial z}{\partial y} = q = x + a$... (2)

Multiplying equations (1) and (2), we get

$$pq = (y + b)(x + a) = z$$

$\therefore pq = z$ This is a partial differential equation.

2. Form the partial differential equation by eliminating the function f from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

Sol. Differentiating z partially w.r.t. x and y ,

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right) \quad \text{or} \quad -px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \dots(1)$$

$$\text{and} \quad q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right) \quad \text{or} \quad qy = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right) \quad \dots(2)$$

From (1) and (2), we have $-px^2 = qy - 2y^2$ or $x^2p + yq = 2y^2$

which is a partial differential equation of the first order.

3. Form the partial differential equation by eliminating the arbitrary functions from

$$(i) z = f(x + it) + g(x - it), \text{ where } i = \sqrt{-1} \quad (ii) z = x f_1(x + t) + f_2(x + t)$$

$$(iii) f(x + y + z, x^2 + y^2 + z^2) = 0.$$

Sol. (i) Given $z = f(x + it) + g(x - it)$

Differentiating z twice partially w.r.t. x and t , we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'(x + it) + g'(x - it) \\ \frac{\partial^2 z}{\partial x^2} &= f''(x + it) + g''(x - it) \end{aligned} \quad \dots(1)$$

$$\frac{\partial z}{\partial t} = i f'(x + it) - i g'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x + it) + i^2 g''(x - it)$$

$$\text{or} \quad \frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \quad \dots(2)$$

$$\text{Adding (1) and (2), we have} \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is a partial differential equation of the second order.

$$(ii) \text{ Given, } z = x f_1(x + t) + f_2(x + t)$$

$$\therefore \frac{\partial z}{\partial x} = f_1(x + t) + x f_1'(x + t) + f_2'(x + t)$$

$$\frac{\partial^2 z}{\partial x^2} = f_1'(x+t) + f_1'(x+t) + xf_1''(x+t) + f_2''(x+t)$$

$$\text{or} \quad \frac{\partial^2 z}{\partial x^2} = 2f_1'(x+t) + xf_1''(x+t) + f_2''(x+t) \quad \dots(1)$$

$$\frac{\partial z}{\partial t} = xf_1'(x+t) + f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial t^2} = xf_1''(x+t) + f_2''(x+t) \quad \dots(2)$$

Subtracting (2) from (1),

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 2f_1'(x+t) \quad \dots(3)$$

$$\text{Also,} \quad \frac{\partial^2 z}{\partial x \partial t} = f_1'(x+t) + xf_1''(x+t) + f_2''(x+t) = \frac{1}{2} \left[\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} \right] + \frac{\partial^2 z}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} - 2 \cdot \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is a partial differential equation of the second order.

(iii) Let $x + y + z = u$, $x^2 + y^2 + z^2 = v$, then $f(u, v) = 0$

Differentiating partially w.r.t. x and y , we have

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot p \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot p \right) = 0$$

$$\text{or} \quad \frac{\partial f}{\partial u} (1+p) + \frac{\partial f}{\partial v} (2x+2zp) = 0 \quad \dots(1)$$

$$\text{and} \quad \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot q \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \cdot q \right) = 0$$

$$\text{or} \quad \frac{\partial f}{\partial u} (1+q) + \frac{\partial f}{\partial v} (2y+2zq) = 0 \quad \dots(2)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$(1+p)(2y+2zq) = (1+q)(2x+2zp) \quad \text{or} \quad (y-z)p + (z-x)q = x-y$$

which is a partial differential equation of the first order.

4. Form the Partial differential equation by eliminating the arbitrary functions from

$$z = x f(x+t) + g(x+t). \quad (\text{M.D.U., May 2007})$$

Sol. Given

$$z = x f(x+t) + g(x+t)$$

$$\text{Partial derivation w.r.t. } x \text{ gives} \quad \frac{\partial z}{\partial x} = x f'(x+t) + f(x+t) + g'(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = x f''(x+t) + 2f'(x+t) + g''(x+t) \quad \dots(1)$$

Partially differentiating w.r.t. t ,

$$\frac{\partial z}{\partial x} = x f'(x+t) + g'(x+t), \quad \frac{\partial^2 z}{\partial x^2} = x f''(x+t) + g''(x+t) \quad \dots(2)$$

$$\text{Now,} \quad \frac{\partial^2 z}{\partial x \partial t} = x f''(x+t) + f'(x+t) + g''(x+t)$$

$$f'(x+t) = \frac{\partial^2 z}{\partial x \partial t} - x f''(x+t) - g''(x+t) = \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} \quad \dots(3)$$

Putting the value of $f'(x+t)$ in (1), we get

$$\frac{\partial^2 z}{\partial x^2} = 2 \left(\frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} \right) + \frac{\partial^2 z}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} = 2 \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$$

5. Form the partial differential equation by eliminating arbitrary functions from the following
 $x + y + z = f(x^2 + y^2 + z^2)$ (M.D.U., Dec., 2009)

Sol. Let $x + y + z = u$, $x^2 + y^2 + z^2 = v$, then $u = f(v)$

Differentiating partially w.r.t. x and y , we have

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot p \right) = \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot p \right)$$

$$\frac{\partial f}{\partial u} (1 + p) = \frac{\partial f}{\partial v} (2x + 2zp) \quad \dots(1)$$

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot q \right) = \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \cdot q \right)$$

$$\frac{\partial f}{\partial u} (1 + q) = \frac{\partial f}{\partial v} (2y + 2zq) \quad \dots(2)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (1) and (2), we get

$$(1 + p)(1 + q) = (2x + 2zp)(2y + 2zq)$$

which is a partial differential equation of the first order.

6. Form the partial differential equation by eliminating the arbitrary function(s) from the following :
 (i) $z = f(x^2 - y^2)$ (ii) $z = \phi(x) \cdot \psi(y)$ (iii) $z = x + y + f(x, y)$.

Sol. (i) Differentiating z partially w.r.t. x , we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) \cdot 2x \quad \dots(1)$$

Differentiating z partially w.r.t. y , we get

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2) \cdot (-2y) \quad \dots(2)$$

Dividing equation (1) by equation (2), we get

$$\frac{p}{q} = -\frac{x}{y} \quad \text{or} \quad py + qx = 0, \text{ which is a partial differential equation.}$$

(ii) Differentiating z partially w.r.t. x , we have

$$\frac{\partial z}{\partial x} = p = \phi'(x) \cdot \psi(y) \quad \dots(1)$$

Differentiating z w.r.t. y partially, we get

$$\frac{\partial z}{\partial y} = q = \phi(x) \cdot \psi'(y) \quad \dots(2)$$

Differentiating equation (1) partially w.r.t. y , we get

$$\frac{\partial^2 z}{\partial y \partial x} = \phi'(x) \cdot \psi'(y) = s \quad \dots(3)$$

Multiplying equations (1) and (2), we have

$$pq = \phi(x) \cdot \psi(y) \cdot \phi'(x) \cdot \psi'(y) = zs \quad [\text{Using eqn. (3)}]$$

$$\Rightarrow pq - zs = 0$$

which is the required partial differential equation.

$$(iii) \quad z = x + y + f(x, y)$$

Differentiating z w.r.t. x, y we get

$$\frac{\partial z}{\partial x} = p = 1 + f'(x, y) \cdot y \Rightarrow p - 1 = y f'(x, y) \quad \dots(1)$$

$$\frac{\partial z}{\partial y} = q = 1 + f'(x, y) \cdot x \Rightarrow q - 1 = x f'(x, y) \quad \dots(2)$$

Dividing equation (1) by equation (2), we get

$$\frac{p-1}{q-1} = \frac{y}{x} \Rightarrow px - qy = x - y$$

which is the required partial differential equation.

7. Form partial differential equation by eliminating the arbitrary function from

$$z = f(ax + by) + g(ax + by) \quad (\text{M.D.U., May 2009})$$

Sol.

$$p = \frac{\partial z}{\partial x} = af'(ax + by) + ag'(ax + by)$$

$$q = \frac{\partial z}{\partial y} = bf'(ax + by) + bg'(ax + by)$$

$$\text{Now, } b \frac{\partial z}{\partial x} - a \frac{\partial z}{\partial y} = 0 \text{ or } bp - aq = 0 \text{ which is the required differential equation.}$$

8. Show that if u_1 and u_2 are two solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = A \cdot \frac{\partial^2 u}{\partial t^2} + B \cdot \frac{\partial u}{\partial t}, \text{ then } c_1 u_1 + c_2 u_2 \text{ is also a solution.}$$

(U.P.T.U., 2009)

Sol. Since u_1 and u_2 are solutions of the given equation, we have

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = A \frac{\partial^2 u_1}{\partial t^2} + B \frac{\partial u_1}{\partial t} \quad \dots(1)$$

$$\text{and } \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} = A \cdot \frac{\partial^2 u_2}{\partial t^2} + B \cdot \frac{\partial u_2}{\partial t} \quad \dots(2)$$

$$\begin{aligned} \text{Now, } \frac{\partial^2}{\partial x^2} (c_1 u_1 + c_2 u_2) + \frac{\partial^2}{\partial y^2} (c_1 u_1 + c_2 u_2) + \frac{\partial^2}{\partial z^2} (c_1 u_1 + c_2 u_2) \\ = c_1 \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) + c_2 \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} \right) \\ = c_1 \left(A \cdot \frac{\partial^2 u_1}{\partial t^2} + B \cdot \frac{\partial u_1}{\partial t} \right) + c_2 \left(A \cdot \frac{\partial^2 u_2}{\partial t^2} + B \cdot \frac{\partial u_2}{\partial t} \right) \\ = A \cdot \frac{\partial^2}{\partial t^2} (c_1 u_1 + c_2 u_2) + B \cdot \frac{\partial}{\partial t} (c_1 u_1 + c_2 u_2) \end{aligned}$$

$$\Rightarrow c_1 u_1 + c_2 u_2 \text{ is also a solution of the given equation.}$$

9. Verify that $e^{-n^2 t} \sin nx$ is a solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Hence show that $\sum_{n=1}^N c_n e^{-n^2 t}$

$\sin nx$, where $c_1, c_2, \dots, c_N$ are arbitrary constants, is a solution of this equation satisfying the boundary conditions $u(0, t) = 0$ and $u(\pi, t) = 0$.

Sol. Let $u = e^{-n^2 t} \sin nx$ then $\frac{\partial u}{\partial t} = -n^2 e^{-n^2 t} \sin nx$

$$\frac{\partial u}{\partial x} = n e^{-n^2 t} \cos nx \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} = -n^2 e^{-n^2 t} \sin nx$$

so that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$\therefore u = e^{-n^2 t} \sin nx$ is a solution of the given equation.

For $n = 1, 2, \dots, N$, we get N different solutions.

Their linear combination $u = c_1 e^{-1^2 t} \sin x + c_2 e^{-2^2 t} \sin 2x + \dots + c_N e^{-N^2 t} \sin Nx$

or $u(x, t) = \sum_{n=1}^N c_n e^{-n^2 t} \sin nx$ is also a solution.

Clearly, $u(0, t) = 0$ and $u(\pi, t) = 0$, since $n\pi = 0$, where n is an integer.

10. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\frac{\pi}{2}$.

Sol. Given $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x , keeping y as constant, we have

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \dots(1)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$

$$\therefore -2 \sin y = -\sin y + f(y) \quad \text{or} \quad f(y) = -\sin y$$

$$\therefore \text{From equation (1), } \frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Integrating w.r.t. y , keeping x as constant, we have

$$z = \cos x \cos y + \cos y + \phi(x) \quad \dots(2)$$

When y is an odd multiple of $\frac{\pi}{2}$, $z = 0$

$$\therefore 0 = 0 + 0 + \phi(x), \text{ since } \cos(2n+1)\frac{\pi}{2} = 0 \quad \text{or} \quad \phi(x) = 0$$

$$\therefore \text{From (2), } z = (1 + \cos x) \cos y$$

which is the required particular equation.

11. Solve : $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$. (M.D.U., Dec. 2008)

Sol. If z were a function of x alone, the solution would have been

$$z = c_1 \cos x + c_2 \sin x, \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

But here z is a function of x and y , therefore, c_1 and c_2 are arbitrary functions of y , the independent variable kept constant.

$\therefore$ The solution of the given equation is

$$z = f(y) \cos x + \phi(y) \sin x \quad \text{and} \quad \frac{\partial z}{\partial x} = -f(y) \sin x + \phi(y) \cos x$$

When $x = 0$, $z = e^y$, $\therefore e^y = f(y)$

When $x = 0$, $\frac{\partial z}{\partial x} = 1$, $\therefore \phi(y) = 1$.

$\therefore$ The required particular solution is $z = e^y \cdot \cos x + \sin x$.

12. Solve : $\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y$. (M.D.U., Dec., 2007)

Sol. $\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y$ or $\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$

Integrating w.r.t. $x + y$, we get

$$z = e^{x+y} + c.$$

13. Solve : $t = \sin(xy)$.

Sol. The given equation is $\frac{\partial^2 z}{\partial y^2} = t = \sin(xy)$

Integrating w.r.t. y , we get $\frac{\partial z}{\partial y} = -\frac{1}{x} \cos(xy) + f(x)$

The constant is being taken as a function of x .

Again integrating w.r.t. y , we get $z = -\frac{1}{x^2} \sin(xy) + yf(x) + \phi(x)$.

14. Find the surface satisfying $\frac{\partial^2 z}{\partial y^2} = t = 6x^2y$, *containing the two lines* $y = 0 = z$, $y = 2 = z$.

Sol. $\frac{\partial^2 z}{\partial y^2} = 6x^2y$...(1)

Integrating equation (1) w.r.t. y , we get

$$\frac{\partial z}{\partial y} = 6x^2 \left(\frac{y^2}{2} \right) + f(x) \quad \text{...(2)}$$

The constant is being taken as a function of x .

Again integrating w.r.t. y , we get

$$z = (3x^2) \left(\frac{y^3}{3} \right) + y \cdot f(x) + \phi(x) \quad \text{...(3)}$$

Since (3) contains $y = 0 = z$, we have

$$0 = 0 + 0 + \phi(x) \Rightarrow \phi(x) = 0.$$

$\therefore$ (3) reduces to $z = x^2y^3 + yf(x)$

Since the solution also contains the line $y = 2 = z$, we have

$$2 = 8x^2 + 2f(x) \Rightarrow f(x) = 1 - 4x^2$$

Hence the surface is given by the solution, $z = x^2y^3 + y(1 - 4x^2)$.

15. Solve the following differential equations :

(i) $\frac{y^2 z}{x} p + xzq = y^2$ (M.D.U., Dec., 2008)

(ii) $pz - qz = z^2 + (x + y)^2$ (U.P.T.U., 2009)

(iii) $(x^2 - yz) p + (y^2 - zx) q = z^2 - xy$.

Sol. (i) The given equation can be written as

$$y^2zp + x^2zq = xy^2$$

Comparing with $Pp + Qq = R$ (Lagrange's linear equation), we have

$$P = y^2z, Q = x^2z, R = xy^2$$

∴ The auxiliary equations are $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$

Taking the first 2 members, we have $x^2dx = y^2dy$

which on integration gives $x^3 - y^3 = a$... (1)

Again taking the first and third members,

We have $xdx = zdz$

which on integration gives $x^2 - z^2 = b$... (2)

From (1) and (2), the general solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$

(ii) Given equation is $pz - qz = z^2 + (x + y)^2$

Comparing with $Pp + Qq = R$, we have $P = z, Q = -z, R = z^2 + (x + y)^2$

The auxiliary equations are $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2}$

Taking the first 2 members, we have $dx + dy = 0$ which on integration gives

$$x + y = a$$
 ... (1)

Again taking the first and third members, we have

$$dx = \frac{zdz}{z^2 + (x + y)^2} \quad \text{or} \quad 2 dx = \frac{2zdz}{z^2 + (x + y)^2}$$

which on integration gives $\log(z^2 + a^2) = 2x + b$ or $\log\{z^2 + (x + y)^2\} - 2x = b$... (2)

From (1) and (2), the general solution is $\phi\{x + y, \log(x^2 + y^2 + z^2 + 2xy) - 2x\} = 0$.

(iii) Here $P = x^2 - yz, Q = y^2 - zx, R = z^2 - xy$

∴ Auxiliary equations are $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$... (1)

$$\begin{aligned} \therefore \frac{dx - dy}{x^2 - y^2 - yz + zx} &= \frac{dy - dz}{y^2 - z^2 - zx + xy} \\ \frac{dx - dy}{(x - y)(x + y + z)} &= \frac{dy - dz}{(y - z)(x + y + z)} \quad \text{or} \quad \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} \end{aligned}$$

which on integration gives $\log(x - y) = \log(y - z) + \log a$

$$\text{or} \quad \log\left(\frac{x - y}{y - z}\right) = \log a \quad \text{or} \quad \frac{x - y}{y - z} = a$$
 ... (2)

Using x, y, z as multipliers, each fraction of (1)

$$\begin{aligned} &= \frac{xdx + ydy + zdz}{x(x^2 - yz) + y(y^2 - zx) + z(z^2 - xy)} = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} \\ &= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)} \end{aligned}$$
 ... (3)

$$\text{Also each fraction of (1)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$
 ... (4)

$$\text{From (3) and (4),} \quad \frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

$$\Rightarrow xdx + ydy + zdz = (x + y + z) d(x + y + z)$$

$$\text{Integrating it gives,} \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x + y + z)^2}{2} + c$$

$$\begin{aligned}
 \text{or} & \quad x^2 + y^2 + z^2 = (x + y + z)^2 + 2c \\
 \text{or} & \quad 2xy + 2yz + 2zx + 2c = 0 \\
 \text{or} & \quad xy + yz + zx = b, \text{ where } b = -c \quad \dots(5)
 \end{aligned}$$

Combining equations (2) and (5), the general solution is $\phi\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0$.

16. Solve the following differential equations :

$$(i) (mz - ny)p + (nx - lz)q = ly - mx \quad (ii) x^2(y - z)p + y^2(z - x)q = z^2(x - y).$$

(M.D.U., May 2008)

Sol. (i) Here, $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$

The auxiliary equations are $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

Using x, y, z as multipliers, we get

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$$\text{which on integration gives } x^2 + y^2 + z^2 = a \quad \dots(1)$$

Again using l, m, n as multipliers, we get

$$\text{Each fraction} = \frac{ldx + mdy + ndz}{0}$$

$$\therefore ldx + mdy + ndz = 0$$

$$\text{which on integration gives } lx + my + nz = b \quad \dots(2)$$

From (1) and (2), the general solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz).$$

$$(ii) \text{ Here } P = x^2(y - z), Q = y^2(z - x), R = z^2(x - y)$$

$$\therefore \text{ The auxiliary equations are } \frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, we get

$$\text{Each fraction} = \frac{\frac{1}{x^2} \cdot dx + \frac{1}{y^2} \cdot dy + \frac{1}{z^2} \cdot dz}{0}$$

$$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

$$\text{which on integration gives } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \quad \dots(1)$$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{Each fraction} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

which on integration gives $\log x + \log y + \log z = \log b$ or $\log (xyz) = \log b \Rightarrow xyz = b$... (2)

From (1) and (2), the general solution is $\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$.

17. Solve the differential equation $x^2p + y^2q = (x + y)z$. (M.D.U., Dec. 2007)

Sol.

$$x^2p + y^2q = (x + y)z$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z} \quad \text{or} \quad \frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x + y)z} \quad \text{or} \quad \frac{dx - dy}{x - y} = \frac{dz}{z}$$

$$\log (x - y) = \log z$$

Integrating both sides, we have $x - y = z + c \Rightarrow \frac{x - y}{z} = c$... (1)

Also, $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating both sides, we get $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

Let,

$$x^2 = t, \quad y^2 = T$$

$$2x dx = dt, \quad 2y dy = dT$$

$$dx = \frac{dt}{2\sqrt{t}}, \quad dy = \frac{dT}{2\sqrt{T}}$$

$$\int \frac{dt}{2t^{3/2}} = \int \frac{dT}{2T^{3/2}} \Rightarrow \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{T}} \quad \text{or} \quad \frac{1}{x} = \frac{1}{y}$$

$\therefore$ From (1) and (2) $x - y = 0$... (2)

$\therefore$ Complete solution is $\phi\left(x - y, \frac{x - y}{z}\right) = 0$.

18. Solve the following differential equations :

$$(i) (x^2 - y^2 - z^2)p + 2xyq = 2xz \quad (ii) (z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$$

(M.T.U., 2010)

Sol. (i) Here $P = x^2 - y^2 - z^2$, $Q = 2xy$, $R = 2xz$

$\therefore$ The auxiliary equations are $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

Taking the last 2 members, we have $\frac{dy}{y} = \frac{dz}{z}$

which on integration gives $\log y = \log z + \log a \Rightarrow \frac{y}{z} = a$... (1)

Using x, y, z as multipliers, we get

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\therefore \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz} \quad \text{or} \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

which on integration gives $\log (x^2 + y^2 + z^2) = \log z + \log b$ or $\frac{x^2 + y^2 + z^2}{z} = b$... (2)

∴ From (1) and (2), the general solution is

$$\frac{x^2 + y^2 + z^2}{z} = f\left(\frac{y}{z}\right) \quad \text{or} \quad x^2 + y^2 + z^2 = z f\left(\frac{y}{z}\right)$$

(ii) The auxiliary equations are $\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$

Taking x, y, z as multipliers, we have

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$$\text{which on integration gives } x^2 + y^2 + z^2 = a \quad \dots(1)$$

Again, taking the last 2 members, we have

$$\frac{dy}{y + z} = \frac{dz}{y - z}$$

$$\text{or } (y - z) dy = (y + z) dz \quad \text{or } ydy - (zdy + ydz) - zdz = 0$$

$$\text{or } ydy - d(yz) - zdz = 0$$

$$\text{which on integration gives } y^2 - 2yz - z^2 = b \quad \dots(2)$$

From (1) and (2), the general solution is $\phi(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$.

19. Solve the following equations :

$$(i) p \cos(x + y) + q \sin(x + y) = z \quad (ii) px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3).$$

Sol. (i) Here the auxiliary equations are

$$\frac{dx}{\cos(x + y)} = \frac{dy}{\sin(x + y)} = \frac{dz}{z} \quad \dots(1)$$

Using 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx + dy}{\cos(x + y) + \sin(x + y)} \quad \dots(2)$$

Using 1, -1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{\cos(x + y) - \sin(x + y)} \quad \dots(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x + y) + \sin(x + y)} = \frac{dx - dy}{\cos(x + y) - \sin(x + y)} \quad \dots(4)$$

Taking the first 2 fractions of (4), we have

$$\frac{dz}{z} = \frac{d(x + y)}{\cos(x + y) + \sin(x + y)} \quad \text{or} \quad \frac{dz}{z} = \frac{dt}{\cos t + \sin t}, \text{ where } t = x + y$$

$$\text{or } \frac{dz}{z} = \frac{dt}{\sqrt{2} \left(\frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t \right)}$$

$$= \frac{dt}{\sqrt{2} \left(\sin \frac{\pi}{4} \cdot \cos t + \cos \frac{\pi}{4} \cdot \sin t \right)} = \frac{dt}{\sqrt{2} \cdot \sin \left(t + \frac{\pi}{4} \right)}$$

$$\text{or } \sqrt{2} \cdot \frac{dz}{z} = \operatorname{cosec} \left(t + \frac{\pi}{4} \right) dt$$

Integrating $\sqrt{2} \log z = \log \tan \left(\frac{t}{2} + \frac{\pi}{8} \right) + \log c_1$

$\therefore z^{\sqrt{2}} = c_1 \tan \left(\frac{t}{2} + \frac{\pi}{8} \right) \quad \text{or} \quad z^{\sqrt{2}} \cot \left(\frac{x+y}{2} + \frac{\pi}{8} \right) = c_1 \quad \dots(5)$

Taking the last 2 fractions of equation (4), we have

$$dx - dy = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} \cdot d(x+y)$$

or $dx - dy = \frac{\cos t - \sin t}{\cos t + \sin t} \cdot dt, \quad \text{where } t = x + y$

Integrating, we get $x - y = \log(\cos t + \sin t) - \log c_2$

or $\frac{\sin t + \cos t}{c_2} = e^{x-y} \quad \text{or} \quad e^{y-x} [\sin(x+y) + \cos(x+y)] = c_2 \quad \dots(6)$

Combining equations (5) and (6), the general solution is

$$\phi \left[z^{\sqrt{2}} \cot \left(\frac{x+y}{2} + \frac{\pi}{8} \right), e^{y-x} \{ \sin(x+y) + \cos(x+y) \} \right] = 0$$

(ii) The given equation is $px(z - 2y^2) + qy(z - y^2 - 2x^3) = z(z - y^2 - 2x^3)$

The auxiliary equations are $\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)} \quad \dots(1)$

Taking the last two fractions of equation (1), we have

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, $\log y = \log z + \log a \Rightarrow \frac{y}{z} = a \quad \dots(2)$

Taking the first and third fractions of (1), we have

$$\frac{dx}{x(z - 2y^2)} = \frac{dz}{z(z - y^2 - 2x^3)}$$

Using (2) i.e., $y = az$, we have

$$\frac{dx}{x(z - 2a^2z^2)} = \frac{dz}{z(z - a^2z^2 - 2x^3)} \quad \text{or} \quad \frac{dx}{x(1 - 2a^2z)} = \frac{dz}{z - a^2z^2 - 2x^3}$$

or $zdx - a^2z^2dx - 2x^3dx = xdz - 2a^2xzdz \quad \text{or} \quad (xdz - zdx) - a^2(2xzdz - z^2dx) + 2x^3dx = 0$

Dividing by x^2 , we have

$$\frac{xdz - zdx}{x^2} - a^2 \left\{ \frac{2xzd - z^2dx}{x^2} \right\} + 2xdx = 0 \quad \text{or} \quad d \left(\frac{z}{x} \right) - a^2 d \left(\frac{z^2}{x} \right) + d(x^2) = 0$$

Integrating, we get $\frac{z}{x} - a^2 \cdot \frac{z^2}{x} + x^2 = b \quad \text{or} \quad \frac{z}{x} - \frac{y^2}{x} + x^2 = b \quad \dots(3)$

Combining (2) and (3), the general solution is

$$\phi \left(\frac{y}{z}, \frac{z}{x} - \frac{y^2}{x} + x^2 \right) = 0.$$

20. Solve : $(z - y)p + (x - z)q = y - x$.

Sol. Comparing with $Pp + Qq = R$, $P = z - y$, $Q = x - z$, $R = y - x$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x}$$

Using 1, 1, 1 as multipliers, we get

$$\text{Each fraction} = \frac{dx + dy + dz}{0}$$

$\therefore dx + dy + dz = 0$ which on integration gives

$$x + y + z = a \quad \dots(1)$$

Again, using x, y, z as multipliers, we get

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0 \text{ which on integration gives } \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b \quad \dots(2)$$

From (1) and (2), general solution is $x^2 + y^2 + z^2 = f(x + y + z)$.

21. Solve the differential equation : $x(y - z)p + y(z - x)q = z(x - y)$

Sol. The auxiliary equations are $\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$

$$\text{Each fraction} = \frac{dx + dy + dz}{0} \Rightarrow dx + dy + dz = 0$$

$$\text{which on integration gives } x + y + z = a \quad \dots(1)$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as the multipliers, we have

$$\text{Each fraction} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0} \Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\text{which on integration gives } \log x + \log y + \log z = \log b \text{ or } xyz = b \quad \dots(2)$$

From (1) and (2), the general solution is given by $\phi(x + y + z, xyz) = 0$.

22. Solve the differential equation : $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$. (U.P.T.U., 2007)

Sol. Auxiliary equations are $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad \dots(1)$

$$\text{Using } \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \text{ as multipliers, each fraction of (1)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \text{ or } \log xyz = \log a \text{ or } xyz = a \quad \dots(2)$$

Using x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0 \text{ which on integration gives } x^2 + y^2 + z^2 = b \quad \dots(3)$$

From (2) and (3), we have the general solution given by $\phi(x^2 + y^2 + z^2, xyz) = 0$.

23. Solve : $\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{y} - \frac{1}{x}$.

Sol. The auxiliary equations are $\frac{dx}{\frac{1}{z} - \frac{1}{y}} = \frac{dy}{\frac{1}{x} - \frac{1}{z}} = \frac{dz}{\frac{1}{y} - \frac{1}{x}}$... (1)

Each fraction of (1) is $= \frac{dx + dy + dz}{0}$

$\therefore dx + dy + dz = 0$ which on integration gives $x + y + z = a$... (2)

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\frac{1}{x}\left(\frac{1}{z} - \frac{1}{y}\right) + \frac{1}{y}\left(\frac{1}{x} - \frac{1}{z}\right) + \frac{1}{z}\left(\frac{1}{y} - \frac{1}{x}\right)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$, which on integration gives

$\log x + \log y + \log z = \log b \Rightarrow xyz = b$... (3)

Combining (2) and (3), the general solution is $\phi(x + y + z, xyz) = 0$.

24. Solve : $y^2 p - xyq = x(z - 2y)$.

Sol. The auxiliary equations are $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$... (1)

Taking first and second members of (1), we have

$$\frac{dx}{y} = -\frac{dy}{x} \quad \text{or} \quad xdx + ydy = 0$$

which on integration gives $x^2 + y^2 = a$... (2)

Taking second and third members of (1), we have

$$\frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \quad \text{or} \quad (z - 2y) dy + ydz = 0$$

$$zdy + ydz - 2ydy = 0$$

which on integration gives $(yz) - y^2 = b$... (3)

Combining (2) and (3), the general solution is $\phi(x^2 + y^2, yz - y^2) = 0$.

25. Solve : (i) $\sqrt{p} + \sqrt{q} = 1$ (ii) $pq = p + q$. (U.P.T.U., 2007, 2009)

Sol. (i) The equation is of the form $f(p, q) = 0$

The complete solution is $z = ax + by + c$... (1)

where, $\sqrt{a} + \sqrt{b} = 1$ or $b = (1 - \sqrt{a})^2$

$\therefore$ From (1), the complete solution is $z = ax + (1 - \sqrt{a})^2 y + c$

(ii) The equation is of the form $f(p, q) = 0$

The complete solution is $z = ax + by + c$... (1)

where, $ab = a + b$ or $b = \frac{a}{a - 1}$

$\therefore$ From (1), the complete solution is $z = ax + \left(\frac{a}{a - 1}\right)y + c$.

26. Solve : $x^2 p^2 + y^2 q^2 = z^2$.

Sol. The given equation can be written as

$$\frac{x^2}{z^2} \cdot p^2 + \frac{y^2}{z^2} \cdot q^2 = 1 \quad \text{or} \quad \left(\frac{x}{z} \cdot \frac{\partial z}{\partial x} \right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y} \right)^2 = 1 \quad \dots(1)$$

Putting $\frac{dx}{x} = dX$, $\frac{dy}{y} = dY$, $\frac{dz}{z} = dZ$

so that $X = \log x$, $Y = \log y$, $Z = \log z$.

$$\frac{\partial Z}{\partial X} = \frac{x}{z} \cdot \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial Z}{\partial Y} = \frac{y}{z} \cdot \frac{\partial z}{\partial y}$$

$\therefore$ Equation (1) can be written as

$$\left(\frac{\partial Z}{\partial X} \right)^2 + \left(\frac{\partial Z}{\partial Y} \right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \text{ where } P = \frac{\partial Z}{\partial X}, Q = \frac{\partial Z}{\partial Y}$$

It is of the form $f(P, Q) = 0$

Its complete solution is $Z = aX + bY + c$...(2)

where, $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$

$\therefore$ From (2), the complete solution is $\log z = a \log x + \sqrt{1 - a^2} \log y + c$.

27. Solve the partial differential equation : $(y^2 + z^2)p - xyq + zx = 0$. (M.D.U., May 2009)

Sol.

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

$$\frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - z^2x} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore \quad \begin{aligned} xdx + ydy + zdz &= 0 \\ x^2 + y^2 + z^2 &= c_1 \end{aligned} \quad \dots(1)$$

Now taking second and third member,

$$\begin{aligned} \frac{dy}{-xy} &= \frac{dz}{-zx} \Rightarrow \frac{dy}{y} = \frac{dz}{z} \\ \log y &= \log z + \log c_2 \\ \frac{y}{z} &= c_2 \end{aligned} \quad \dots(2)$$

Hence from (1) and (2), solution is given by $f\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$.

28. Solve the partial differential equation $(bz - cy)p + (cx - az)q = ay - bx$ (M.D.U., Dec. 2009)

Sol. $(bz - cy)p + (cx - az)q = ay - bx$

Here the auxiliary equations are $\frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx}$

Using x, y, z as multipliers, we get

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0} \quad \text{or} \quad xdx + ydy + zdz = 0$$

which on integration gives $x^2 + y^2 + z^2 = c_1$...(1)

Again using a, b, c as multipliers, we get

$$\text{Each fraction} = \frac{adx + bdy + cdz}{0}$$

$\therefore adx + bdy + cdz = 0$ which on integration gives

$$ax + by + cz = c_2 \quad \dots(2)$$

From equations (1) and (2), the general solution is $x^2 + y^2 + z^2 = f(ax + by + cz)$.

29. Solve : $(y - x)(qy - px) = (p - q)^2$

Sol. Let $x + y = X, xy = Y$

So that
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x} \quad \text{or} \quad p = \frac{\partial z}{\partial X} + y \frac{\partial z}{\partial Y} \quad \text{and} \quad q = \frac{\partial z}{\partial X} + x \frac{\partial z}{\partial Y}$$

$$\therefore qy - px = (y - x) \frac{\partial z}{\partial X} \quad \text{and} \quad p - q = (y - x) \frac{\partial z}{\partial Y}$$

$\therefore$ Given equation can be written as

$$\frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y} \right)^2 \quad \text{or} \quad P = Q^2, \text{ where } P = \frac{\partial z}{\partial X} \text{ and } Q = \frac{\partial z}{\partial Y}$$

It is of the form $f(P, Q) = 0$

Its complete solution is $z = aX + bY + c \quad \dots(1)$

where, $a = b^2 \quad \text{or} \quad b = \sqrt{a}$.

$\therefore$ From (1), the complete solution is $z = a(x + y) + \sqrt{a} \cdot xy + c$.

30. Solve : $p^2 + q^2 = 1$.

Sol. The equation is of the form $f(p, q) = 0$.

The complete solution is $z = ax + by + c$

where $a^2 + b^2 = 1 \quad \text{or} \quad b^2 = 1 - a^2 \quad \text{i.e.,} \quad b = \sqrt{1 - a^2}$

$\therefore$ The complete solution is $z = ax + \sqrt{1 - a^2} \cdot y + c$.

31. Solve : $z = px + qy + \sqrt{1 + p^2 + q^2}$. (A.U.U.P., 2009)

Sol. The given equation is of the form

$$z = px + qy + f(p, q) = 0$$

Its solution is $z = ax + by + \sqrt{1 + a^2 + b^2}$.

32. Solve : $4xyz = pq + 2px^2y + 2qxy^2$ (U.P.T.U., 2006; M.T.U., 2010)

Sol. Let $x^2 = X$ and $y^2 = Y$

so that
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x \frac{\partial z}{\partial X} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y \frac{\partial z}{\partial Y}$$

$\therefore$ The given equation becomes

$$4xyz = 4xy \frac{\partial z}{\partial X} \frac{\partial z}{\partial Y} + 4x^3y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$$

or
$$z = x^2 \frac{\partial z}{\partial X} + y^2 \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

$$z = X \frac{\partial z}{\partial X} + Y \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} \quad \text{or} \quad z = PX + QY + PQ$$

where, $P = \frac{\partial z}{\partial X}, Q = \frac{\partial z}{\partial Y}$

It is of the form $z = PX + QY + f(P, Q)$

Its complete solution is $z = aX + bY + ab$ or $z = ax^2 + by^2 + ab$.

- 33.** Solve : $z^2(p^2 + q^2 + 1) = a^2$. (M.D.U., 2006)

Sol. The given equation is of the form $f(z, p, q) = 0$

Let $u = x + by$, so that $p = \frac{dz}{du}$, $q = b \frac{dz}{du}$

Substituting these values of p and q in the given equation, we get

$$z^2 \left[\left(\frac{dz}{du} \right)^2 + b^2 \left(\frac{dz}{du} \right)^2 + 1 \right] = a^2 \quad \text{or} \quad z^2 \left[(1 + b^2) \left(\frac{dz}{du} \right)^2 \right] = a^2 - z^2$$

$$\text{or} \quad z \sqrt{1 + b^2} \cdot \frac{dz}{du} = \pm \sqrt{a^2 - z^2} \quad \text{or} \quad \pm \sqrt{1 + b^2} \cdot \frac{z}{\sqrt{a^2 - z^2}} dz = du$$

Integrating, we have $\pm \sqrt{1 + b^2} \cdot \sqrt{a^2 - z^2} = u + c$ or $(1 + b^2)(a^2 - z^2) = (x + by + c)^2$

which is the required complete solution.

- 34.** Solve : $z^2(p^2x^2 + q^2) = 1$. (M.D.U., May 2007)

Sol. The given equation can be written as

$$z^2 \left[\left(x \frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1 \quad \dots(1)$$

Put $\frac{dx}{x} = dX$ so that $X = \log x$ and $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{1}{x} \cdot \frac{\partial z}{\partial X} \Rightarrow x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}$

$$\therefore \text{Equation (1) reduces to } z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1 \text{ or } z^2[P^2 + q^2] = 1, \text{ where } P = \frac{\partial z}{\partial X} \quad \dots(2)$$

Let $u = X + ay$ so that $P = \frac{dz}{du}$, $q = a \frac{dz}{du}$

$$\therefore \text{From (2), we have } z^2 \left[\left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 \right] = 1$$

$$\text{or} \quad (1 + a^2) z^2 \left(\frac{dz}{du} \right)^2 = 1 \quad \text{or} \quad \sqrt{1 + a^2} \cdot z \cdot dz = \pm du$$

which on integration gives $\sqrt{1 + a^2} \cdot \frac{z^2}{2} = \pm u + b$ or $\sqrt{1 + a^2} \cdot z^2 = \pm 2(X + ay) + 2b$

or $\sqrt{1 + a^2} \cdot z^2 = \pm 2(\log x + ay) + c$ which is the required complete solution.

- 35.** Solve : $p^2 - q^2 = x - y$. (M.D.U., Dec. 2009)

Sol. The given equation is $p^2 - x = q^2 - y$ which is of the form $f_1(x, p) = f_2(y, q)$

Let $p^2 - x = q^2 - y = a$ then $p^2 = x + a$, $q^2 = y + a$ i.e., $p = \sqrt{x + a}$, $q = \sqrt{y + a}$

Substituting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \sqrt{x + a} \cdot dx + \sqrt{y + a} \cdot dy$$

$$\text{On integrating, } z = \frac{2}{3} (x + a)^{3/2} + \frac{2}{3} (y + a)^{3/2} + b$$

which is the required complete solution.

36. Solve : $yp = 2yx + \log q$.

Sol. The given equation can be written as

$$p = 2x + \frac{1}{y} \log q \quad \text{or} \quad p - 2x = \frac{1}{y} \log q, \text{ which is of the form } f_1(x, p) = f_2(y, q)$$

$$\text{Let } p - 2x = \frac{1}{y} \log q = a \quad \text{then } p = 2x + a, q = e^{ay}$$

Substituting these values of p and q in $dz = p dx + q dy$, we get

$$dz = (2x + a) dx + e^{ay} dy$$

On integrating we obtain $z = x^2 + ax + \frac{1}{a} e^{ay} + b$, which is the required complete solution.

37. Solve : $z^2(p^2 + q^2) = x^2 + y^2$ (A.U.U.P., 2007, 2009; M.T.U., 2010)

Sol. The given equation can be written as

$$\left(z \frac{\partial z}{\partial x} \right)^2 + \left(z \frac{\partial z}{\partial y} \right)^2 = x^2 + y^2 \quad \dots(1)$$

$$\text{Let } z dz = dZ \text{ so that } Z = \frac{1}{2} z^2$$

$$\text{Now,} \quad \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = Z \frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = Z \cdot \frac{\partial z}{\partial y}$$

$$\therefore \text{ Equation (1) becomes } \left(\frac{\partial Z}{\partial x} \right)^2 + \left(\frac{\partial Z}{\partial y} \right)^2 = x^2 + y^2$$

$$\text{or} \quad P^2 + Q^2 = x^2 + y^2, \text{ where } P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial y}$$

$$\text{or} \quad P^2 - x^2 = y^2 - Q^2 \text{ which is of the form } f_1(x, P) = f_2(y, Q)$$

$$\text{Let} \quad P^2 - x^2 = y^2 - Q^2 = a, \text{ then } P = \sqrt{x^2 + a}, Q = \sqrt{y^2 - a}$$

Substituting these values of P and Q in $dZ = P dx + Q dy$, we get

$$dZ = \sqrt{x^2 + a} \cdot dx + \sqrt{y^2 - a} \cdot dy$$

On integrating, we get

$$Z = \frac{1}{2} x \sqrt{x^2 + a} + \frac{a}{2} \log(x + \sqrt{x^2 + a}) + \frac{1}{2} y \sqrt{y^2 - a} - \frac{a}{2} \log(y + \sqrt{y^2 - a}) + b$$

$$\text{or} \quad z^2 = x \sqrt{x^2 + a} + y \sqrt{y^2 - a} + a \log \frac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}} + c,$$

(where $c = 2b$), is the required complete solution.

38. Solve : $p^2 + q^2 = x + y$

Sol. Let $p^2 - x = y - q^2 = a$

$$\text{Then} \quad p^2 = x + a, q^2 = y - a \quad \text{i.e., } p = \sqrt{x + a}, q = \sqrt{y - a}$$

Substituting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \sqrt{x + a} \cdot dx + \sqrt{y - a} \cdot dy$$

$$\text{Integrating,} \quad z = \frac{2}{3} (x + a)^{3/2} + \frac{2}{3} (y - a)^{3/2} + b$$

which is the required complete solution.

39. Solve : $p(1+q) = qz$

Sol. Let $u = x + ay, \quad p = \frac{dz}{du}, \quad q = \frac{a dz}{du}$

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du} \left(1 + \frac{a dz}{du} \right) = \frac{a dz}{du} \cdot z \quad \text{or} \quad \frac{a dz}{du} = az - 1 \quad \text{or} \quad \frac{a dz}{az - 1} = du$$

This $\Rightarrow$ on integration

$$\log(az - 1) = u + b \quad \text{or} \quad \log(az - 1) = x + ay + b$$

which is the required complete solution.

40. Solve the following partial differential equations :

(i) $yp + xq + pq = 0$

(ii) $(p-1)(z - px - qy) = 1$ (M.D.U., May 2009)

Sol. (i) $yp + xq + pq = 0$ or $p(y+q) = -xq$ or $\frac{y+q}{q} = -\frac{x}{p} = \alpha$ (say)

$$f(y, q) = f(x, p)$$

$$\Rightarrow p = -\frac{x}{\alpha}, \quad \frac{y+q}{q} = \alpha \quad \text{or} \quad y+q = q\alpha \quad \text{or} \quad y = q(\alpha-1) \quad \text{or} \quad \frac{y}{\alpha-1} = q$$

Using

$$dz = p dx + q dy$$

$$dz = -\frac{x}{\alpha} \cdot dx + \frac{y}{\alpha-1} \cdot dy \quad \text{or} \quad z = -\frac{1}{\alpha} \frac{x^2}{2} + \frac{1}{\alpha-1} \cdot \frac{y^2}{2} + c$$

$$2z = -\frac{x^2}{\alpha} + \frac{y^2}{\alpha-1} + c.$$

(ii) $(p-1)(z - px - qy) = 1$ or $z - px - qy = \frac{1}{p-1}$ or $z = px + qy + \frac{1}{p-1}$

which is of the form $z = px + qy + f(p, z)$

Put $p = a, q = b,$ $z = ax + by + \frac{1}{a-1}.$

41. Solve : $q^2 = z^2 p^2 (1 - p^2).$

(M.T.U. 2010)

Sol. Let $u = x + ay, \quad p = \frac{dz}{du}, \quad q = a \frac{dz}{du}$

$$a^2 \left(\frac{dz}{du} \right)^2 = z^2 \left(\frac{dz}{du} \right)^2 \left[1 - \left(\frac{dz}{du} \right)^2 \right] \quad \text{or} \quad a^2 = z^2 \left[1 - \left(\frac{dz}{du} \right)^2 \right]$$

$$a^2 = z^2 - z^2 \left(\frac{dz}{du} \right)^2 \quad \text{or} \quad z^2 \left(\frac{dz}{du} \right)^2 = z^2 - a^2$$

$$\left(\frac{dz}{du} \right) = \frac{\sqrt{z^2 - a^2}}{z^2} = \frac{\sqrt{z^2 - a^2}}{z} \quad \text{or} \quad \frac{z dz}{\sqrt{z^2 - a^2}} = du$$

$$(z^2 - a^2)^{1/2} = u + b = (x + ay + b)$$

$$z^2 - a^2 = (x + ay + b)^2 \text{ which is the required complete solution.}$$

42. Solve : $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

Sol. This equation can be reduced to the form $f_1(x, p) = f_2(y, q)$ i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q .

Putting $u = x + y$, $v = x - y$ and $z = z(u, v)$

Then,
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \text{or} \quad p = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = P + Q$$

$$q = \frac{\partial z}{\partial y} = P - Q, \text{ where } P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2u + 4Q^2v = 1$$

$$4P^2u = 1 - 4Q^2v = a \text{ (say)}$$

$$\therefore P = \pm \frac{1}{2} \sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2} \sqrt{\frac{1-a}{v}}$$

$$\therefore dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv = \pm \frac{\sqrt{a}}{2} \cdot \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1-a}}{2} \cdot \frac{dv}{\sqrt{v}}$$

Integrating, we have
$$z = \pm \sqrt{a} \cdot \sqrt{u} \pm \sqrt{1-a} \cdot \sqrt{v} + b$$

or
$$z = \pm \sqrt{a(x+y)} \pm \sqrt{(1-a)(x-y)} + b$$

which is the required complete solution.

43. Solve by Charpit's method non-linear partial differential equation of the first order :

$$2zx - px^2 - 2qxy + pq = 0. \quad (\text{M.D.U., May 2008, Dec. 2009})$$

Sol. Here,
$$f = 2zx - px^2 - 2qxy + pq = 0 \quad \dots(1)$$

$$\therefore \frac{\partial f}{\partial x} = 2z - 2px - 2qy, \quad \frac{\partial f}{\partial y} = -2qx,$$

$$\frac{\partial f}{\partial z} = 2x, \quad \frac{\partial f}{\partial p} = -x^2 + q, \quad \frac{\partial f}{\partial q} = -2xy + p.$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

or
$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$$

$$\therefore dq = 0 \quad \text{or} \quad q = a$$

Putting $q = a$ in equation (1), we get

$$p = \frac{2x(z - ay)}{x^2 - a}$$

$$\therefore dz = p dx + q dy = \frac{2x(z - ay)}{x^2 - a} \cdot dx + a dy \quad \text{or} \quad \frac{dz - a dy}{z - ay} = \frac{2x}{x^2 - a} \cdot dx$$

Integrating, we get

$$\log(z - ay) = \log(x^2 - a) + \log b$$

or
$$z - ay = b(x^2 - a),$$

$$\therefore z = ay + b(x^2 - a) \text{ which is the required complete solution.}$$

44. Solve : $(p^2 + q^2)y = qz$ by Charpit's method.

Sol. Here,
$$f = (p^2 + q^2)y - qz = 0 \quad \dots(1)$$

$$\therefore \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = p^2 + q^2, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial p} = 2py, \quad \frac{\partial f}{\partial q} = 2qy - z$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

or
$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z} = \frac{dF}{0}$$

From the first two members, we have $pdp + qdq = 0$

Integrating, $p^2 + q^2 = a^2$...(2)

Putting in equation (1), $q = \frac{a^2 y}{z}$

$\therefore$ From (2), $p = \sqrt{a^2 - q^2} = \frac{a}{z} \sqrt{z^2 - a^2 y^2}$

$\therefore$ $dz = p dx + q dy = \frac{a}{z} \sqrt{z^2 - a^2 y^2} dx + \frac{a^2 y}{z} dy$

or $z dz - a^2 y dy = a \sqrt{z^2 - a^2 y^2} dx$ or $\frac{1}{2} \frac{d(z^2 - a^2 y^2)}{\sqrt{z^2 - a^2 y^2}} = a dx$

Integrating, we get

$$\sqrt{z^2 - a^2 y^2} = ax + b \quad \text{or} \quad z^2 - a^2 y^2 = (ax + b)^2 \quad \text{or} \quad z^2 = (ax + b)^2 + a^2 y^2$$

which is the required complete solution.

45. Solve : $2z + p^2 + qy + 2y^2 = 0$ by Charpit's method.

Sol. Here, $f = 2z + p^2 + qy + 2y^2 = 0$ (A.U.U.P., 2006, 2008)

The auxiliary equations are $\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$

From first and fourth members, $dp = -dx$ or $p = -x + a$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y} [-2z - 2y^2 - (a - x)^2]$$

$\therefore$ $dz = p dx + q dy$ or $dz = (a - x) dx - \frac{1}{y} [2z + 2y^2 + (a - x)^2] dy$

Multiplying both sides by $2y^2$

$$2y^2 dz + 4yz dy = 2y^2 (a - x) dx - 4y^3 dy - 2y(a - x)^2 dy$$

Integrating $2zy^2 = -[y^2(a - x)^2 + y^4] + b$ or $y^2 [(a - x)^2 + y^2 + 2z] = b$

which is the desired solution.

46. Solve by Charpit's method the partial differential equation :

$(p^2 + q^2)x = pz$ (M.D.U., Dec. 2007)

Sol. Here, $f = (p^2 + q^2)x - pz = 0$...(1)

$\therefore$ $\frac{\partial f}{\partial x} = p^2 + q^2, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = -p, \frac{\partial f}{\partial p} = 2px - z, \frac{\partial f}{\partial q} = 2qx$

Charpit's auxiliary equations are

$$\frac{dp}{q^2} = \frac{dq}{-pq} = \frac{dz}{-p(2px - z) - 2q^2 x} = \frac{dx}{-2px + z} = \frac{dy}{-2qx} = \frac{dF}{0}$$

From the first two members, we have

$$pdp + qdq = 0 \quad \text{which on integration gives} \quad p^2 + q^2 = a^2 \quad \dots(2)$$

Putting in equation (1), we get

$$p = \frac{a^2}{z} x, \quad q = \sqrt{a^2 - p^2} = \frac{a}{z} \sqrt{z^2 - a^2 x^2}$$

$$\therefore dz = pdx + qdy = \frac{a^2}{z} x dx + \frac{a}{z} \sqrt{z^2 - a^2 x^2} \cdot dy \quad \text{or} \quad z dz - a^2 x dx = a \sqrt{z^2 - a^2 x^2} \cdot dy$$

$$\frac{1}{2} d(z^2 - a^2 x^2) = a \sqrt{z^2 - a^2 x^2} \cdot dy \quad \text{or} \quad \frac{\frac{1}{2} d(z^2 - a^2 x^2)}{\sqrt{z^2 - a^2 x^2}} = a \cdot dy$$

which on integration gives $\sqrt{z^2 - a^2 x^2} = ay + b$ or $z^2 - a^2 x^2 = (ay + b)^2$
 $\Rightarrow z^2 = a^2 x^2 + (ay + b)^2$ which is the required complete solution.

47. Solve by Charpit's method : $px + qy = pq$

Sol. Here, $f = px + qy - pq = 0 \quad \dots(1)$

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p$$

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x - q) - q(y - p)} = \frac{dx}{-x + q} = \frac{dy}{-y + p} = \frac{dF}{0}$$

From first two members $\frac{dp}{p} = \frac{dq}{q}$

$$\log p = \log q + \log a \Rightarrow p = qa$$

Putting in equation (1) $px + \frac{py}{a} = \frac{p^2}{a} \Rightarrow p = 0, ax + y$

$$\therefore q = \frac{ax + y}{a}$$

$$dz = p \cdot dx + q \cdot dy$$

$$dz = (ax + y)dx \quad \text{or} \quad adz = a(ax + y)dx \Rightarrow az = \frac{1}{2} (ax + y)^2 + b.$$

48. Solve by Charpit's method : $pxy + pq + qy = yz$

Sol. Here, $f = pxy + pq + qy - yz = 0 \quad \dots(1)$

$$\therefore \frac{\partial f}{\partial x} = py, \quad \frac{\partial f}{\partial y} = px + q - z, \quad \frac{\partial f}{\partial z} = -y, \quad \frac{\partial f}{\partial p} = xy + q, \quad \frac{\partial f}{\partial q} = p + y$$

Then Charpit's auxiliary equations are :

$$\begin{aligned} \frac{dp}{0} &= \frac{dq}{px + q - z - qy} = \frac{dz}{-p(xy + q) - q(p + y)} = \frac{dx}{-(xy + q)} \\ &= \frac{dy}{-(p + y)} = \frac{dF}{0} \end{aligned}$$

$$\therefore dp = 0 \quad \text{or} \quad p = a$$

Putting $p = a$ in (1), we get

$$q = \frac{y(z - ax)}{a + y}$$

$$\therefore dz = pdx + qdy = a \cdot dx + \frac{y(z - ax)}{a + y} \cdot dy$$

$$\therefore \frac{dz - a \cdot dx}{z - ax} = \frac{y}{a + y} \cdot dy \quad \text{or} \quad \frac{dz - a \cdot dx}{z - ax} = \left(1 - \frac{a}{a + y}\right) dy$$

Integrating both sides, we get

$$\log(z - ax) = y - a \log(a + y) + b \quad \text{which is the required complete solution.}$$

49. Solve the partial differential equation $z = p^2x + q^2y$ by Charpit's method.

(M.D.U., 2005; M.T.U. 2010)

Sol.

$$z = p^2x + q^2y$$

Here,

$$f = z - p^2x - q^2y = 0$$

$$\therefore \frac{\partial f}{\partial x} = -p^2, \frac{\partial f}{\partial y} = -q^2, \frac{\partial f}{\partial z} = 1, \frac{\partial f}{\partial p} = -2px, \frac{\partial f}{\partial q} = -2qy$$

The auxiliary equations are

$$\begin{aligned} \frac{dx}{2px} &= \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{dp}{-p^2 + p} = \frac{dq}{-q^2 + q} = \frac{dF}{0} \\ \frac{\frac{dx}{2x} - \frac{dy}{2y}}{p - q} &= \frac{-\frac{dp}{p} + \frac{dq}{q}}{(p - 1) - (q - 1)} = \frac{-\frac{dp}{p} + \frac{dq}{q}}{p - q} \Rightarrow \frac{dx}{2x} - \frac{dy}{2y} = -\frac{dp}{p} + \frac{dq}{q} \end{aligned}$$

which on integrating gives

$$\frac{1}{2} \log x - \frac{1}{2} \log y = -\log p + \log q + \log a$$

$$\text{or} \quad \frac{1}{2} \log \frac{x}{y} = \log \frac{q \cdot a}{p} \quad \text{or} \quad \sqrt{\frac{x}{y}} = \frac{q}{p} \cdot a \quad \text{or} \quad q = \frac{p}{a} \sqrt{\frac{x}{y}}$$

Substituting the value of q in the given equation, we have

$$z = p^2x + \frac{p^2}{a^2} \cdot \frac{x}{y} \cdot y = p^2 \left(1 + \frac{1}{a^2}\right) x$$

$$\therefore p^2 = \frac{z \cdot a^2}{(a^2 + 1)x} \Rightarrow p = \frac{\sqrt{z} \cdot a}{\sqrt{a^2 + 1} \cdot \sqrt{x}}$$

$$\therefore q = \frac{\sqrt{z}}{\sqrt{a^2 + 1} \cdot \sqrt{y}}$$

Substituting the values of p and q in $dz = p dx + q dy$, we have

$$dz = \frac{\sqrt{z}}{\sqrt{a^2 + 1}} a \cdot x^{-1/2} \cdot dx + \frac{\sqrt{z}}{\sqrt{a^2 + 1}} y^{-1/2} \cdot dy$$

$$\frac{dz}{\sqrt{z}} = \frac{a}{\sqrt{a^2 + 1}} x^{-1/2} \cdot dx + \frac{1}{\sqrt{a^2 + 1}} y^{-1/2} \cdot dy$$

On integration, this gives

$$\frac{\sqrt{z}}{\frac{1}{2}} = \frac{a}{\sqrt{a^2 + 1}} \cdot \frac{\sqrt{x}}{\frac{1}{2}} + \frac{1}{\sqrt{a^2 + 1}} \cdot \frac{\sqrt{y}}{\frac{1}{2}} + b$$

$$\sqrt{a^2 + 1} \cdot \sqrt{z} = a\sqrt{x} + \sqrt{y} + b \Rightarrow z(a^2 + 1) = (a\sqrt{x} + \sqrt{y} + b)^2$$

$$\text{or} \quad z = \frac{(a\sqrt{x} + \sqrt{y} + b)^2}{(a^2 + 1)}, \quad \text{which is the required complete solution.}$$

50. Solve by using the Charpit's method : $p^2 + qy = z$ (M.D.U., 2006)

Sol. Let $f = p^2 + qy - z = 0$... (1)

$$\therefore \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = -1, \frac{\partial f}{\partial p} = 2p, \frac{\partial f}{\partial q} = y$$

The auxiliary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{-p} = \frac{dq}{0} \quad \text{or} \quad \frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}$$

$$dq = 0 \Rightarrow q = a$$
 ... (2)

$$\therefore \text{ From (1), } p^2 + ay = z \quad \text{or} \quad p = \sqrt{z - ay}$$
 ... (3)

Substituting the values of p and q in $dz = p dx + q dy$

$$dz = \sqrt{z - ay} \cdot dx + a \cdot dy$$

We have, $dz - a dy = \sqrt{z - ay} \cdot dx$ or $\frac{dz - a dy}{\sqrt{z - ay}} = dx$

which on integration gives

$$\frac{\sqrt{z - ay}}{\frac{1}{2}} = x + b \quad \text{or} \quad 4(z - ay) = (x + b)^2 \Rightarrow z = ay + \left(\frac{x + b}{2}\right)^2$$

which is the required complete solution.

51. Solve the partial differential equation using Charpit's method : $2(z + xp + yq) = yp^2$ (M.D.U., Dec. 2008)

Sol. $f(x, y) = 2z + 2xp + 2yq - yp^2 = 0$

$$\frac{\partial f}{\partial x} = 2p, \frac{\partial f}{\partial y} = 2q - p^2, \frac{\partial f}{\partial z} = 2, \frac{\partial f}{\partial p} = 2x - 2yp, \frac{\partial f}{\partial q} = 2y$$

Charpit's auxiliary equations are

$$\frac{\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}}}{\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}}} = \frac{\frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}}{\frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}} = \frac{\frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}}}{\frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}}} = \frac{\frac{dx}{-\frac{\partial f}{\partial p}}}{\frac{dx}{-\frac{\partial f}{\partial p}}} = \frac{\frac{dy}{-\frac{\partial f}{\partial q}}}{\frac{dy}{-\frac{\partial f}{\partial q}}}$$

$$\Rightarrow \frac{dp}{4p} = \frac{dq}{4q - p^2} = \frac{dz}{-2xp + 2p^2y - 2yq} = \frac{dx}{-2x + 2yp} = \frac{dy}{-2y}$$

Hence its solution is $z = \frac{ax}{y^2} - \frac{a^2}{4y^3} + \frac{b}{y}$.

52. Solve by Charpit's method $z^2 = pqxy$. (M.D.U., May 2008)

Sol. Here, $f = z^2 - pqxy = 0$... (1)

$$\frac{\partial f}{\partial x} = -pqy, \frac{\partial f}{\partial y} = -pqx, \frac{\partial f}{\partial z} = 2z, \frac{\partial f}{\partial p} = -qxy, \frac{\partial f}{\partial q} = -pxy$$

Auxiliary equations are

$$\frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy}$$

$$\frac{dp}{p(2z - qy)} = \frac{dq}{q(2z - px)} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy}$$

From first and second members,

$$\frac{\frac{dp}{p}}{2z - qy} = \frac{\frac{dq}{q}}{2z - px} \quad \text{or} \quad \frac{-\frac{dq}{q} + \frac{dp}{p}}{px - qy} = \frac{-\frac{dx}{x} + \frac{dy}{y}}{px - qy} \Rightarrow \frac{dp}{p} - \frac{dq}{q} = -\frac{dx}{x} + \frac{dy}{y}$$

Integrating, we get

$$\log p - \log q = -\log x + \log y + \log b^2$$

$$\text{or} \quad \log p + \log x - \log q - \log y = \log b^2 \quad \text{or} \quad \log \left(\frac{px}{qy} \right) = \log (b^2) \Rightarrow \frac{px}{qy} = b^2$$

$$px = b^2 qy \quad \text{or} \quad p = \frac{b^2 qy}{x} \quad \dots(1)$$

$$\text{Since,} \quad z^2 = pqxy \quad \dots(2)$$

$$\text{or} \quad z^2 = \frac{b^2 qy}{x} \cdot qxy \quad [\text{Using equation (2)}]$$

$$\Rightarrow \quad z^2 = b^2 q^2 y^2$$

$$\Rightarrow \quad z^2 = bqy \quad \text{or} \quad q = \frac{z}{by}$$

$$\therefore \quad p = \frac{b^2 y}{x} \cdot \frac{z}{by} = \frac{bz}{x} \quad \dots(3)$$

Substituting the values of p and q in

$$dz = p dx + q dy$$

$$\text{We have,} \quad dz = bz \frac{dx}{x} + \frac{z}{by} \cdot dy \quad \text{or} \quad \frac{dz}{z} = b \frac{dx}{x} + \frac{1}{b} \cdot \frac{dy}{y}$$

On integrating, we get

$$\log z = b \log x + \frac{1}{b} \log y + \log a = \log x^b + \log y^{1/b} + \log a$$

$$\Rightarrow \quad \log z = \log (ax^b \cdot y^{1/b}) \quad \text{or} \quad z = a \cdot x^b \cdot y^{1/b} \text{ is the solution.}$$

53. Solve :

$$(i) \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = 0 \quad (ii) (D + 2D')(D - 3D')^2 \cdot z = 0 \quad (iii) 4r - 12s + 9t = 0.$$

Sol. (i) The given equation is $(D^2 - DD' - 6D'^2)z = 0$ where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

Auxiliary equation is $m^2 - m - 6 = 0 \Rightarrow (m - 3)(m + 2) = 0 \Rightarrow m = 3, -2$

$$\therefore \quad \text{C.F.} = f_1(y + 3x) + f_2(y - 2x)$$

$$\text{P.I.} = 0$$

where f_1 and f_2 are arbitrary functions. Hence the complete solution is $z = f_1(y + 3x) + f_2(y - 2x)$.

(ii) The given equation is $(D + 2D')(D - 3D')^2 z = 0$

Auxiliary equation is $(m + 2)(m - 3)^2 = 0 \Rightarrow m = -2, 3, 3$

$$\text{C.F.} = f_1(y - 2x) + f_2(y + 3x) + x f_3(y + 3x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y - 2x) + f_2(y + 3x) + x f_3(y + 3x)$

where f_1, f_2 and f_3 are arbitrary functions.

(iii) The given equation is $4 \frac{\partial^2 z}{\partial x^2} - 12 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow (4D^2 - 12DD' + 9D'^2)z = 0$

Auxiliary equation is $4m^2 - 12m + 9 = 0$ or $(2m - 3)^2 = 0 \Rightarrow m = \frac{3}{2}, \frac{3}{2}$

$\therefore$ C.F. = $f_1 \left(y + \frac{3}{2}x \right) + xf_2 \left(y + \frac{3}{2}x \right)$, P.I. = 0

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1 \left(y + \frac{3}{2}x \right) + xf_2 \left(y + \frac{3}{2}x \right)$.

54. Solve :

(i) $(D^3 - 3D^2D' + 2DD'^2)z = 0$ (ii) $r = a^2t$

(iii) $(D^3D'^2 + D^2D'^3)z = 0$ (iv) $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$ (A.U.U.P., 2009; M.T.U., 2010)

Sol. (i) The given equation is $(D^3 - 3D^2D' + 2DD'^2)z = 0$

The auxiliary equation is $m^3 - 3m^2 + 2m = 0$

$\Rightarrow m(m^2 - 3m + 2) = 0$ or $m(m - 1)(m - 2) = 0$ or $m = 0, 1, 2$

$\therefore$ C.F. = $f_1(y) + f_2(y + x) + f_3(y + 2x)$

P.I. = 0

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} \Rightarrow z = f_1(y) + f_2(y + x) + f_3(y + 2x)$

where f_1, f_2, f_3 are arbitrary functions.

(ii) The given equation is $r - a^2t = 0$ or $(D^2 - a^2D'^2)z = 0$

The auxiliary equation is $m^2 - a^2 = 0 \Rightarrow m = \pm a$

$\therefore$ C.F. = $f_1(y + ax) + f_2(y - ax)$

P.I. = 0

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y + ax) + f_2(y - ax)$

where f_1, f_2 are arbitrary functions.

(iii) The given equation is $D^2D'^2(D + D')z = 0$

$\therefore$ C.F. = $f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x)$

P.I. = 0

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x)$

where f_1, f_2, f_3, f_4 and f_5 are arbitrary functions.

(iv) The given equation is $(D^4 - D'^4)z = 0$

The auxiliary equation is $m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = \pm 1, \pm i$

$\therefore$ C.F. = $f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

P.I. = 0

$\therefore$ The complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

where f_1, f_2, f_3, f_4 are arbitrary functions.

55. Solve the linear partial differential equation : $\frac{\partial^4 z}{\partial x^4} - 2 \frac{\partial^4 z}{\partial x^3 \partial y} + 2 \frac{\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0$.

(U.P.T.U., 2008)

Sol. The given equation is $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$, where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

Auxiliary equation is $m^4 - 2m^3 + 2m - 1 = 0$

or $m^4 - m^3 - m^3 + m^2 - m^2 + m + m - 1 = 0$

$$\begin{aligned}
 \text{or} \quad & m^3(m-1) - m^2(m-1) - m(m-1) + 1(m-1) = 0 \\
 \Rightarrow \quad & (m-1)[m^3 - m^2 - (m+1)] = 0 \\
 & (m-1)[m^2(m-1) - 1(m-1)] = 0 \\
 & (m-1)^2(m^2-1) = 0 \\
 & (m-1)^3(m+1) = 0 \Rightarrow m = -1, 1, 1, 1
 \end{aligned}$$

$$\therefore \text{C.F.} = f_1(y-x) + f_2(y+x) + xf_3(y+x) + x^2f_4(y+x)$$

$$\text{Also, P.I.} = 0$$

$$\text{Hence the complete solution is } z = f_1(y-x) + f_2(y+x) + xf_3(y+x) + x^2f_4(y+x)$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

56. Solve : $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$. (A.U.U.P., 2006, 2008, 2009)

Sol. The given equation is $(D^4 + D'^4)z = 0$

Auxiliary equation is $m^4 + 1 = 0 \Rightarrow m^4 + 1 + 2m^2 = 2m^2$ or $(m^2 + 1)^2 = 2m^2$

$$(m^2 + 1)^2 - (\sqrt{2}m)^2 = 0 \Rightarrow (m^2 + 1 + \sqrt{2}m)(m^2 + 1 - \sqrt{2}m) = 0$$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0 \Rightarrow m = \frac{-1 \pm i}{\sqrt{2}}, \frac{1 \pm i}{\sqrt{2}}$

Let $z_1 = \frac{-1+i}{\sqrt{2}}, z_2 = \frac{1+i}{\sqrt{2}}$ then $m = z_1, \bar{z}_1, z_2, \bar{z}_2$

Here $\bar{z}_1$ and $\bar{z}_2$ denote complex conjugate of z_1 and z_2 respectively.

$$\therefore \text{C.F.} = f_1(y + z_1x) + f_2(y + \bar{z}_1x) + f_3(y + z_2x) + f_4(y + \bar{z}_2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + z_1x) + f_2(y + \bar{z}_1x) + f_3(y + z_2x) + f_4(y + \bar{z}_2x)$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

57. Solve : $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$.

Sol. Auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$

$$\Rightarrow (m-1)(m-2)(m-3) = 0 \Rightarrow m = 1, 2, 3$$

$$\therefore \text{C.F.} = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

where f_1, f_2 and f_3 are arbitrary functions.

58. Solve : $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$.

Sol. Auxiliary equation is $m^3 - 6m^2 + 12m - 8 = 0$

By hit and trial method $m = 2$ satisfies it.

$$\therefore (m-2)(m^2 + 4 - 4m) = 0 \text{ or } (m-2)(m-2)^2 = 0$$

$$\therefore (m-2)^3 = 0 \text{ or } m = 2, 2, 2$$

$$\therefore \text{C.F.} = f_1(y+2x) + xf_2(y+2x) + x^2f_3(y+2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y+2x) + xf_2(y+2x) + x^2f_3(y+2x)$

where f_1, f_2 and f_3 are arbitrary functions.

59. Solve : $r - 4s + 4t = 0$

Sol. Given equation is $(D^2 - 4DD' + 4D'^2)z = 0$

Auxiliary equation is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$$\therefore \quad \begin{aligned} \text{C.F.} &= f_1(y + 2x) + xf_2(y + 2x) \\ \text{P.I.} &= 0 \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y + 2x) + xf_2(y + 2x)$ where f_1 and f_2 are arbitrary functions.

60. Solve : $\frac{\partial^2 z}{\partial x^2} + 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = x + y.$ (A.U.U.P., 2007)

Sol. The given equation is $(D^2 + 3DD' + 2D'^2)z = x + y$

The auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0$ or $m = -1, -2$

$$\therefore \quad \text{C.F.} = f_1(y - x) + f_2(y - 2x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 3DD' + 2D'^2} \cdot (x + y) = \frac{1}{(1)^2 + 3(1)(1) + 2(1)^2} \iint u \, du \, du, \text{ where } x + y = u. \\ &= \frac{1}{6} \cdot \frac{u^3}{6} = \frac{u^3}{36} = \frac{(x + y)^3}{36} \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y - x) + f_2(y - 2x) + \frac{(x + y)^3}{36}$

where f_1 and f_2 are arbitrary functions.

61. Solve : $(D^2 + 2DD' + D'^2)z = e^{3x+2y}.$ (U.P.T.U., June, 2007)

Sol. The auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0$ or $m = -1, -1$

$$\therefore \quad \text{C.F.} = f_1(y - x) + xf_2(y - x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 2DD' + D'^2} \cdot e^{3x+2y} = \frac{1}{(3)^2 + 2(3)(2) + (2)^2} \iint e^u \, du \, du, \text{ where } 3x + 2y = u \\ &= \frac{1}{25} \cdot e^u = \frac{1}{25} e^{3x+2y} \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$z = f_1(y - x) + xf_2(y - x) + \frac{1}{25} e^{3x+2y}$$

where f_1 and f_2 are arbitrary functions.

62. Solve the linear partial differential equation $\frac{\partial^2 z}{\partial x^2} - 2 \cdot \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y).$

Sol. The given equation is $\frac{\partial^2 z}{\partial x^2} - 2 \cdot \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y)$

$$\Rightarrow (D^2 - 2DD' + D'^2)z = \sin(2x + 3y) \text{ where } D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$$

The auxiliary equation is $m^2 - 2m + 1 = 0 \Rightarrow (m - 1)^2 = 0$ or $m = 1, 1$

$$\therefore \quad \text{C.F.} = f_1(y + x) + xf_2(y + x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2DD' + D'^2} \sin(2x + 3y) \\ &= \frac{1}{(D - D')^2} \sin(2x + 3y) = \frac{1}{(2 - 3)^2} \iint \sin u \, du \, du = -\sin u = -\sin(2x + 3y) \end{aligned}$$

Hence the complete solution is $z = f_1(y + x) + xf_2(y + x) - \sin(2x + 3y)$

where f_1, f_2 are arbitrary functions.

63. Solve : $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

Sol. The given equation is $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$

The auxiliary equation is $2m^2 - 5m + 2 = 0$ or $(2m - 1)(m - 2) = 0 \Rightarrow m = \frac{1}{2}, 2$

Hence, C.F. = $f_1\left(y + \frac{1}{2}x\right) + f_2(y + 2x) = \phi_1(2y + x) + f_2(y + 2x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2D^2 - 5DD' + 2D'^2} \cdot 24(y - x) && | \text{ Here } a = -1, b = 1 \\ &= \frac{24}{2(-1)^2 - 5(-1)(1) + 2(1)^2} \iint u \, du \, du, \text{ where } y - x = u \\ &= \frac{4}{9} u^3 = \frac{4}{9} (y - x)^3 \end{aligned}$$

$\therefore$ Complete solution is $z = \text{C.F.} + \text{P.I.} = \phi_1(2y + x) + f_2(y + 2x) + \frac{4}{9}(y - x)^3$.

64. Solve : $r + s - 2t = \sqrt{2x + y}$ (U.P.T.U., 2006, 2008)

Sol. The given equation is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial^2 z}{\partial y^2} = (2x + y)^{1/2}$ or $(D^2 + DD' - 2D'^2)z = (2x + y)^{1/2}$

Auxiliary equation is $m^2 + m - 2 = 0 \Rightarrow (m - 1)(m + 2) = 0 \Rightarrow m = 1, -2$

$\therefore$ C.F. = $f_1(y + x) + f_2(y - 2x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + DD' - 2D'^2} (2x + y)^{1/2} && | \text{ Here } a = 2, b = 1 \\ &= \frac{1}{(2)^2 + (2)(1) - 2(1)^2} \iint \sqrt{u} \cdot du \, du, \text{ where } u = 2x + y \\ &= \frac{1}{4} \cdot \frac{4}{15} \cdot u^{5/2} = \frac{1}{15} (2x + y)^{5/2} \end{aligned}$$

Hence complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y + x) + f_2(y - 2x) + \frac{1}{15}(2x + y)^{5/2}$.

65. Solve the linear partial differential equation : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x + y)$.

(A.U.U.P., 2009)

Sol. The given equation is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x + y)$

$\Rightarrow (D^2 + D'^2)z = \cos mx \cos ny + 30(2x + y)$

Hence auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F. = $f_1(y + ix) + f_2(y - ix)$

$$\text{P.I.} = \frac{1}{D^2 + D'^2} (\cos mx \cdot \cos ny) + \frac{1}{D^2 + D'^2} \cdot 30(2x + y) = P_1 + P_2$$

where, $P_1 = \frac{1}{D^2 + D'^2} \cos mx \cos ny$

$$= \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} [\cos(mx + ny) + \cos(mx - ny)]$$

$$\begin{aligned}
&= \frac{1}{2} \left[\frac{1}{D^2 + D'^2} \cos(mx + ny) + \frac{1}{D^2 + D'^2} \cos(mx - ny) \right] \\
\text{or } P_1 &= \frac{1}{2} \left[\frac{1}{m^2 + n^2} \iint \cos u \, du \, dv + \frac{1}{m^2 + n^2} \iint \cos v \, dv \, du \right] \\
&= \frac{1}{2} \left[-\frac{1}{m^2 + n^2} \cos u - \frac{1}{m^2 + n^2} \cos v \right], \text{ where } mx + ny = u, \, mx - ny = v \\
&= -\frac{1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)] \\
\text{and } P_2 &= \frac{1}{D^2 + D'^2} \cdot 30(2x + y) = \frac{30}{(2)^2 + (1)^2} \iint w \, dw \, dw, \text{ where } 2x + y = w \\
&= 6 \cdot \frac{w^3}{6} = w^3 = (2x + y)^3 \\
\therefore \text{ P.I.} &= -\frac{1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)] + (2x + y)^3 \\
&= -\frac{1}{m^2 + n^2} \cos mx \cos ny + (2x + y)^3
\end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$\Rightarrow z = f_1(y + ix) + f_2(y - ix) - \frac{1}{m^2 + n^2} \cos mx \cos ny + (2x + y)^3$$

where f_1 and f_2 are arbitrary functions.

66. Solve : $\frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x-2y)$. (U.P.T.U. 2006, 2009)

Sol. The given equation is $(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + \sin(x-2y)$

Auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\therefore$ C.F. = $f_1(y+x) + f_2(y+2x)$

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2} \cdot e^{2x+3y} + \frac{1}{D^2 - 3DD' + 2D'^2} \cdot \sin(x-2y) = P_1 + P_2$$

$$\text{where, } P_1 = \frac{1}{D^2 - 3DD' + 2D'^2} \cdot e^{2x+3y}$$

$$= \frac{1}{(2)^2 - 3(2)(3) + 2(3)^2} \iint e^u \, du \, dv, \text{ where } 2x + 3y = u$$

$$= \frac{1}{4} e^u = \frac{1}{4} e^{2x+3y}$$

$$P_2 = \frac{1}{D^2 - 3DD' + 2D'^2} \sin(x-2y) = \frac{1}{(1)^2 - 3(1)(-2) + 2(-2)^2} \iint \sin v \, dv \, dv$$

$$= \frac{1}{15} (-\sin v) = -\frac{1}{15} \sin(x-2y), \text{ where } x-2y = v$$

$$\therefore \text{ P.I.} = P_1 + P_2 = \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x-2y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+x) + f_2(y+2x) + \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x-2y)$$

where f_1 and f_2 are arbitrary functions.

67. Solve : $\frac{\partial^2 z}{\partial x^2} - 2 \cdot \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$. (U.P.T.U., 2007, 2009)

Sol. The given equation is $(D^2 - 2DD' + D'^2)z = \sin x$

Auxiliary equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$\therefore$ C.F. = $f_1(y+x) + x \cdot f_2(y+x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-D')^2} \sin(x+0 \cdot y) = \frac{1}{(1-0)^2} \iint \sin u \, du \, du, \text{ where } x = u \\ &= -\sin u = -\sin x \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y+x) + xf_2(y+x) - \sin x$

where f_1 and f_2 are arbitrary functions.

68. Solve the linear partial differential equation :

$$\frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 4 \sin(2x+y). \quad (\text{A.U.U.P. 2010})$$

Sol. The given equation is $(D^3 - 4D^2D' + 4DD'^2)z = 4 \sin(2x+y)$

The auxiliary equation is $m^3 - 4m^2 + 4m = 0$ or $m(m^2 - 4m + 4) = 0 \Rightarrow m = 0, 2, 2$

$\therefore$ C.F. = $f_1(y) + f_2(y+2x) + xf_3(y+2x)$

$$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 4 \sin(2x+y)$$

$$\begin{aligned} \text{or } \text{P.I.} &= \frac{4}{D(D^2 - 4DD' + 4D'^2)} \sin(2x+y) = \frac{4}{D} \left[\frac{1}{(D-2D')^2} \sin(2x+y) \right] \\ &= x \cdot \frac{4}{D} \left[\frac{1}{2(D-2D')} \sin(2x+y) \right] = 4x^2 \cdot \frac{1}{D} \left[\frac{1}{2} \sin(2x+y) \right] \\ &= 2x^2 \cdot \frac{1}{D} \sin(2x+y) = -2x^2 \cdot \frac{\cos(2x+y)}{2} = -x^2 \cos(2x+y) \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$\Rightarrow z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - x^2 \cos(2x+y)$$

where f_1, f_2 and f_3 are arbitrary functions.

69. Solve the linear partial differential equation :

$$2 \frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 5 \sin(2x+y). \quad (\text{M.D.U., 2010})$$

Sol. The given equation is $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y)$

The auxiliary equation is $2m^2 - 5m + 2 = 0$ or $(2m-1)(m-2) = 0 \Rightarrow m = \frac{1}{2}, 2$

$\therefore$ C.F. = $f_1\left(y + \frac{1}{2}x\right) + f_2(y+2x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2D^2 - 5DD' + 2D'^2} \cdot 5 \sin(2x+y) = 5x \cdot \frac{1}{4D - 5D'} \sin(2x+y) \\ &= 5x \cdot \frac{1}{4(2) - 5(1)} \int \sin u \, du, \text{ where } 2x+y = u \\ &= \frac{5}{3} \cdot x \cdot (-\cos u) = -\frac{5x}{3} \cos(2x+y) \end{aligned}$$

Hence complete solution is $z = f_1\left(y + \frac{1}{2}x\right) + f_2(y+2x) - \frac{5x}{3} \cos(2x+y)$

70. Solve : $(D^2 + 5DD' + 6D'^2)z = \frac{1}{y-2x}$. (M.T.U., 2010)

Sol. Auxiliary equation is $m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3$

$\therefore$ C.F. = $f_1(y-2x) + f_2(y-3x)$

$$\text{P.I.} = \frac{1}{D^2 + 5DD' + 6D'^2} \cdot \left(\frac{1}{y-2x} \right) = x \cdot \frac{1}{2D + 5D'} \left(\frac{1}{y-2x} \right)$$

$$= x \cdot \frac{1}{2(-2) + 5(1)} \int \frac{1}{u} \cdot du, \text{ where } y-2x = u$$

$$= x \log u = x \log(y-2x)$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y-2x) + f_2(y-3x) + x \log(y-2x)$

where f_1 and f_2 are arbitrary, functions.

71. Solve the linear partial differential equation : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$ (U.P.T.U. 2008, 2010)

Sol. The given equation is $(D^2 - DD')z = \sin x \cos y$

Auxiliary equation is $m^2 - m = 0$ or $m(m-1) = 0 \Rightarrow m = 0, 1$

$\therefore$ C.F. = $f_1(y) + f_2(y+x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \sin x \cdot \cos y = \frac{1}{2(D^2 - DD')} [\sin(x+y) + \sin(x-y)]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - DD'} \sin(x+y) + \frac{1}{D^2 - DD'} \sin(x-y) \right] = \frac{1}{2} (P_1 + P_2)$$

where, $P_1 = \frac{1}{D^2 - DD'} \sin(x+y) = x \cdot \frac{1}{2D - D'} \sin(x+y)$

or $P_1 = x \cdot \frac{1}{2(1) - (1)} \int \sin u \, du, \text{ where } x+y = u$
 $= x \cdot (-\cos u) = -x \cos(x+y)$

and $P_2 = \frac{1}{D^2 - DD'} \sin(x-y) = \frac{1}{(1)^2 - (1)(-1)} \iint \sin w \, dw \, dw \text{ where } x-y = w$
 $= \frac{1}{2} (-\sin w) = -\frac{1}{2} \sin(x-y)$

$\therefore$ $\text{P.I.} = \frac{1}{2} \left[-x \cos(x+y) - \frac{1}{2} \sin(x-y) \right]$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y+x) - \frac{x}{2} \cos(x+y) - \frac{1}{4} \sin(x-y)$$

where, f_1 and f_2 are arbitrary functions.

72. Solve : $4r - 4s + t = 16 \log(x+2y)$.

Sol. The given equation is $(4D^2 - 4DD' + D'^2)z = 16 \log(x+2y)$

Auxiliary equation is $4m^2 - 4m + 1 = 0$ or $(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$

$\therefore$ C.F. = $f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{1}{2}x\right)$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x+2y) = 16x \cdot \frac{1}{2(2D - D') \cdot 2} \log(x+2y)$$

$$= 4x \cdot \frac{1}{2D - D'} \log(x + 2y) = 4x^2 \cdot \frac{1}{2} \log(x + 2y) = 2x^2 \log(x + 2y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{1}{2}x\right) + 2x^2 \log(x + 2y)$$

where f_1 and f_2 are arbitrary functions.

73. Solve : $r + 2s + t = 2(y - x) + \sin(x - y)$.

(A.U.U.P., 2009)

Sol. Given equation is $(D^2 + 2DD' + D'^2)z = 2(y - x) + \sin(x - y)$

Auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

$\therefore$ C.F. = $f_1(y - x) + xf_2(y - x)$

$$\text{P.I.} = \frac{1}{(D + D')^2} \cdot 2(y - x) + \frac{1}{(D + D')^2} \cdot \sin(x - y) = P_1 + P_2$$

$$\text{where, } P_1 = \frac{1}{(D + D')^2} \cdot 2(y - x) = 2x \cdot \frac{1}{2(D + D')} (y - x) = x \cdot \frac{1}{D + D'} (y - x) = x^2 \cdot (y - x)$$

$$P_2 = \frac{1}{(D + D')^2} \sin(x - y) = x \cdot \frac{1}{2(D + D')} \sin(x - y) = x^2 \cdot \frac{1}{2} \cdot \sin(x - y)$$

$$\therefore \text{P.I.} = x^2(y - x) + \frac{x^2}{2} \sin(x - y)$$

Hence complete solution is $z = f_1(y - x) + xf_2(y - x) + x^2(y - x) + \frac{x^2}{2} \sin(x - y)$.

74. Solve : $2r - s - 3t = 5 \cdot \frac{e^x}{e^y}$.

(M.D.U., 2010)

Sol. The given equation is $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$

Auxiliary equation is $2m^2 - m - 3 = 0$ or $2m^2 - 3m + 2m - 3 = 0$

$$(m + 1)(2m - 3) = 0 \Rightarrow m = -1, \frac{3}{2}$$

$$\therefore \text{C.F.} = f_1(y - x) + f_2\left(y + \frac{3}{2}x\right)$$

$$\text{P.I.} = \frac{1}{2D^2 - DD' - 3D'^2} (5e^{x-y}) = \frac{5x \cdot e^{x-y}}{4D - D'} = 5x \cdot \frac{1}{4(1) - (-1)} \int e^u du = xe^{x-y}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2\left(y + \frac{3}{2}x\right) + xe^{x-y}$

where f_1 and f_2 are arbitrary functions.

75. Solve : $(D - D')^2 z = x + \phi(x + y)$.

Sol. The auxiliary equation is $(m - 1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$ C.F. = $f_1(y + x) + xf_2(y + x)$

$$\text{P.I.} = \frac{1}{(D - D')^2} \{x + \phi(x + y)\} = \frac{1}{(D - D')^2} \cdot x + \frac{1}{(D - D')^2} \phi(x + y) = P_1 + P_2$$

$$\text{where, } P_1 = \frac{1}{(D - D')^2} (x + 0 \cdot y) = \frac{1}{(1 - 0)^2} \iint u du du, \text{ where } x = u$$

$$= \frac{u^3}{6} = \frac{x^3}{6}$$

$$P_2 = \frac{1}{(D - D')^2} \cdot \phi(x + y) = x \cdot \frac{1}{2(D - D')} \cdot \phi(x + y) = \frac{x^2}{2} \phi(x + y)$$

$$\therefore \text{P.I.} = \frac{x^3}{6} + \frac{x^2}{2} \cdot \phi(x + y)$$

$$\text{Hence complete solution is } z = f_1(y + x) + xf_2(y + x) + \frac{x^3}{6} + \frac{x^2}{2} \phi(x + y).$$

$$76. \text{ Solve : } \frac{\partial^3 z}{\partial x^2 \partial y} - 2 \frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}. \quad (\text{U.P.T.U., 2007; M.T.U., 2010})$$

$$\text{Sol. The given equation is } (D^2 D' - 2DD'^2 + D'^3)z = \frac{1}{x^2} \Rightarrow D'(D^2 - 2DD' + D'^2)z = \frac{1}{x^2}$$

Part of C.F. Corresponding to D' is $f_1(x)$

Auxiliary equation of the remaining factor is

$$m^2 - 2m + 1 = 0 \text{ or } (m - 1)^2 = 0 \Rightarrow m = 1, 1$$

$$\therefore \text{ Another part of C.F.} = f_2(y + x) + xf_3(y + x)$$

$$\therefore \text{ Complete C.F.} = f_1(x) + f_2(y + x) + xf_3(y + x)$$

$$\text{P.I.} = \frac{1}{D'^3 + D^2 D' - 2DD'^2} \cdot \left(\frac{1}{x^2} \right) \quad | \quad x^{-2} = (x + 0 \cdot y)^{-2}$$

$$= y \cdot \frac{1}{3D'^2 + D^2 - 4DD'} \cdot \left(\frac{1}{x^2} \right)$$

$$= y \cdot \frac{1}{3(0)^2 + (1)^2 - 4(1)(0)} \iint \frac{1}{u^2} du du, \text{ where } x = u.$$

$$\text{P.I.} = y \cdot \int \left(-\frac{1}{u} \right) du = y(-\log u) = -y \log x$$

Hence the complete solution is $z = f_1(x) + f_2(y + x) + xf_3(y + x) - y \log x$

where f_1, f_2 and f_3 are arbitrary functions.

$$77. \text{ Solve : } (D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y) + e^{3x+y}.$$

(Uttarakhand, June 2009, A.U.U.P., 2010)

$$\text{Sol. Auxiliary equation is } m^3 - 7m - 6 = 0 \text{ or } m^3 + m^2 - m^2 - m - 6m - 6 = 0$$

$$m^2(m + 1) - m(m + 1) - 6(m + 1) = 0 \text{ or } (m^2 - m - 6)(m + 1) = 0$$

$$\Rightarrow (m + 2)(m - 3)(m + 1) = 0 \Rightarrow m = -1, -2, 3$$

$$\therefore \text{ C.F.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3} [\sin(x + 2y) + e^{3x+y}]$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x + 2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{3x+y}$$

P.I. Corresponding to :

$$\begin{aligned} \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cdot \sin(x + 2y) &= \frac{1}{(1)^3 - 7(1)(2)^2 - 6(2)^3} \iiint \sin u du du du, \text{ where } x + 2y = u \\ &= -\frac{1}{75} \cos u = -\frac{1}{75} \cos(x + 2y) \end{aligned}$$

$$\text{P.I. Corresponding to } e^{3x+y} = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cdot (e^{3x+y})$$

Here,

$$F(D, D') = D^3 - 7DD'^2 - 6D'^3$$

$$F(a, b) = F(3, 1) = (3)^3 - 7(3)(1) - 6(1)^3 = 0$$

∴ It is a case of failure.

$$\begin{aligned}\therefore \text{ P.I. Corresponding to } e^{3x+y} &= x \cdot \frac{1}{\frac{\partial}{\partial D} (D^3 - 7DD'^2 - 6D'^3)} \cdot e^{3x+y} \\ &= x \cdot \frac{1}{3D^2 - 7D'^2} \cdot e^{3x+y} = x \cdot \frac{1}{20} \cdot e^{3x+y}\end{aligned}$$

$$\therefore \text{ Required P.I.} = -\frac{1}{75} \cos(x+2y) + \frac{x}{20} e^{3x+y}$$

∴ Complete solution is $z = \text{C.F.} + \text{P.I.}$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75} \cos(x+2y) + \frac{x}{20} e^{3x+y}$$

where f_1, f_2 and f_3 are arbitrary functions.

78. Solve : $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$.

Sol. The given equation is $(D^3 - D'^3)z = x^3 y^3$

Auxiliary equation is $m^3 - 1 = 0 \Rightarrow m = 1, w, w^2$, where w is one of the cube roots of unity.

$$\therefore \text{ C.F.} = f_1(y+x) + f_2(y+wx) + f_3(y+w^2x)$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^3 - D'^3} (x^3 y^3) = \frac{1}{D^3 \left(1 - \frac{D'^3}{D^3}\right)} (x^3 y^3) \\ &= \frac{1}{D^3} \left(1 - \frac{D'^3}{D^3}\right)^{-1} (x^3 y^3) = \frac{1}{D^3} \left(1 + \frac{D'^3}{D^3}\right) (x^3 y^3) \\ &= \frac{1}{D^3} \left[x^3 y^3 + \frac{1}{D^3} D'^3 (x^3 y^3)\right] = \frac{1}{D^3} \left[x^3 y^3 + \frac{1}{D^3} (6x^3)\right] = \frac{1}{D^3} (x^3 y^3) + \frac{1}{D^6} (6x^3) \\ &= \left(\frac{x^6}{6 \cdot 5 \cdot 4}\right) (y^3) + \frac{6x^9}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = \frac{x^6 y^3}{120} + \frac{x^9}{10080}\end{aligned}$$

$$\text{Hence complete solution is } z = f_1(y+x) + f_2(y+wx) + f_3(y+w^2x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

where f_1, f_2 and f_3 are arbitrary functions.

79. Solve : $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

(M.T.U., 2010)

Sol. The auxiliary equation is $m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$

$$\therefore \text{ C.F.} = f_1(y+3x) + x f_2(y+3x)$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - 6DD' + 9D'^2} \cdot 12x^2 + \frac{1}{D^2 - 6DD' + 9D'^2} \cdot (36xy) \\ &= P_1 + P_2\end{aligned}$$

$$\begin{aligned}\text{where, } P_1 &= \frac{1}{D^2 - 6DD' + 9D'^2} (12x^2) = \frac{12}{(1)^2 - 6(1)(0) + 9(0)^2} \iint u^2 du du \text{ where } x = u \\ &= 12 \cdot \frac{u^4}{12} = u^4 = x^4\end{aligned}$$

$$\begin{aligned}\text{and } P_2 &= \frac{1}{D^2 - 6DD' + 9D'^2} (36xy) = \frac{36}{(D - 3D')^2} (xy) = \frac{36}{D^2} \left(1 - \frac{3D'}{D}\right)^{-2} (xy) \\ &= \frac{36}{D^2} \left(1 + \frac{6D'}{D}\right) (xy) \quad (\text{Ignoring higher power terms})\end{aligned}$$

$$= \frac{36}{D^2} \left[xy + \frac{6}{D}(x) \right] = \frac{36}{D^2} [xy + 3x^2] = 36 \left[\frac{x^3}{6} \cdot y + \frac{x^4}{4} \right] = 6x^3y + 9x^4$$

$$\therefore \text{P.I.} = 6x^3y + 10x^4$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y + 3x) + xf_2(y + 3x) + 6x^3y + 10x^4$ where f_1 and f_2 are arbitrary functions.

80. Solve : $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$. (A.U.U.P., 2010)

Sol. Auxiliary equation is $m^2 - 2m + 1 = 0$ or $(m - 1)^2 = 0$ or $m = 1, 1$

$$\therefore \text{C.F.} = f_1(y + x) + xf_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2DD' + D'^2} \cdot e^{x+2y} + \frac{1}{(D - D')^2} \cdot x^3 = P_1 + P_2$$

$$\text{where, } P_1 = \frac{1}{D^2 - 2DD' + D'^2} \cdot e^{x+2y} = \frac{1}{(1)^2 - 2(1)(2) + (2)^2} \iint e^u du du, \text{ where } x + 2y = u$$

$$\text{or } P_1 = e^{x+2y}$$

$$P_2 = \frac{1}{(D - D')^2} \cdot x^3 \text{ or } P_2 = \frac{1}{(1 - 0)^2} \iint u^3 du du, \text{ where } u = x$$

$$= \frac{u^5}{20} = \frac{x^5}{20}$$

$$\text{Hence complete solution is } z = \text{C.F.} + \text{P.I.} = \text{C.F.} + P_1 + P_2$$

$$= f_1(y + x) + xf_2(y + x) + e^{x+2y} + \frac{x^5}{20}$$

81. Solve the linear partial differential equation: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$.

(M.D.U., May 2010; R.G.P.V., Bhopal, June, 2009, Feb., 2008, June, 2006; U.P.T.U., 2009)

Sol. The given equation is $(D^2 + DD' - 6D'^2)z = y \cos x$

The auxiliary equation is $m^2 + m - 6 = 0 \Rightarrow m = 2, -3$

$$\therefore \text{C.F.} = f_1(y + 2x) + f_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} \cdot y \cos x = \frac{1}{(D - 2D')(D + 3D')} \cdot y \cos x$$

$$= \frac{1}{D - 2D'} \int (c + 3x) \cos x dx, \text{ where } y = c + 3x$$

$$= \frac{1}{D - 2D'} \left\{ c \sin x + 3 \int x \cos x dx \right\}$$

$$= \frac{1}{D - 2D'} \left[c \sin x + 3 \left\{ x \sin x - \int 1 \cdot \sin x dx \right\} \right]$$

$$= \frac{1}{D - 2D'} [c \sin x + 3x \sin x + 3 \cos x] = \frac{1}{D - 2D'} [(c + 3x) \sin x + 3 \cos x]$$

$$= \frac{1}{D - 2D'} [y \sin x + 3 \cos x] = \int (b - 2x) \sin x dx + 3 \sin x, \text{ where } y = b - 2x$$

$$= -b \cos x - 2 \left\{ x(-\cos x) - \int 1 \cdot (-\cos x) dx \right\} + 3 \sin x$$

$$= -b \cos x + 2x \cos x - 2 \sin x + 3 \sin x$$

$$= -(b - 2x) \cos x + \sin x = -y \cos x + \sin x.$$

$$\text{Hence complete solution is } z = f_1(y + 2x) + f_2(y - 3x) - y \cos x + \sin x.$$

82. Solve : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial^2 z}{\partial y^2} = (y-1)e^x$. (U.P.T.U., 2008)

Sol. The given equation is $(D^2 + DD' - 2D'^2)z = (y-1)e^x$

The auxiliary equation is $m^2 + m - 2 = 0$ or $(m-1)(m+2) = 0 \Rightarrow m = 1, -2$

$\therefore$ C.F. = $f_1(y+x) + f_2(y-2x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + DD' - 2D'^2} \cdot (y-1)e^x = \frac{1}{(D-D')(D+2D')} \cdot (y-1)e^x \\ &= \frac{1}{D-D'} \int (c+2x-1)e^x dx, \text{ where } y = c+2x \\ &= \frac{1}{D-D'} [(c-1)e^x + 2(x-1)e^x] = \frac{1}{D-D'} [(c+2x)e^x - 3e^x] = \frac{1}{D-D'} (ye^x - 3e^x) \\ &= \int (b-x)e^x dx - 3e^x, \text{ where } y = b-x \\ &= be^x - (x-1)e^x - 3e^x = (b-x-2)e^x = (y-2)e^x \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y+x) + f_2(y-2x) + (y-2)e^x$
where f_1 and f_2 are arbitrary functions.

83. Solve : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \tan x \tan y (\sec^2 x - \sec^2 y)$,

(M.D.U., 2010; M.T.U., 2010; A.U.U.P., 2007, 2009)

Sol. Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ C.F. = $f_1(y+x) + f_2(y-x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D'^2} \tan x \tan y (\sec^2 x - \sec^2 y) \\ &= \frac{1}{D+D'} \left[\frac{1}{D-D'} (\tan x \sec^2 x \tan y - \tan x \tan y \sec^2 y) \right] \\ &= \frac{1}{D+D'} \left[\int \tan x \sec^2 x \tan(c-x) dx - \int \tan x \tan(c-x) \sec^2(c-x) dx \right], \end{aligned}$$

where $y = c-x$.

$$\begin{aligned} &= \frac{1}{D+D'} \left[\tan(c-x) \frac{\tan^2 x}{2} + \int \sec^2(c-x) \frac{\tan^2 x}{2} dx + \tan x \frac{\tan^2(c-x)}{2} \right. \\ &\quad \left. - \int \sec^2 x \cdot \frac{\tan^2(c-x)}{2} dx \right] \\ &= \frac{1}{2(D+D')} [\tan(c-x) \tan^2 x + \tan x \tan^2(c-x) + \int \sec^2(c-x) (\sec^2 x - 1) dx \\ &\quad - \int \sec^2 x \{ \sec^2(c-x) - 1 \} dx] \\ &= \frac{1}{2(D+D')} \left[\tan(c-x) \tan^2 x + \tan x \tan^2(c-x) + \int \{ \sec^2 x - \sec^2(c-x) \} dx \right] \\ &= \frac{1}{2(D+D')} [\tan(c-x) \tan^2 x + \tan x \tan^2(c-x) + \tan x + \tan(c-x)] \\ &= \frac{1}{2(D+D')} [\tan x \sec^2(c-x) + \tan(c-x) \sec^2 x] \end{aligned}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{2(D+D')} [\tan x \sec^2 y + \tan y \sec^2 x] \\
&= \frac{1}{2} \left[\int \tan x \sec^2 (b+x) dx + \int \tan (b+x) \sec^2 x dx \right], \text{ where } y = b+x \\
&= \frac{1}{2} \left[\tan x \cdot \tan (b+x) - \int \sec^2 x \cdot \tan (b+x) dx + \int \tan (b+x) \sec^2 x dx \right] \\
&= \frac{1}{2} \tan x \cdot \tan (b+x) = \frac{1}{2} \tan x \tan y, \text{ where } b = y-x
\end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y+x) + f_2(y-x) + \frac{1}{2} \tan x \tan y$

where f_1 and f_2 are arbitrary functions.

84. Solve : $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy$.

Sol. Auxiliary equation is $m^2 - m - 2 = 0$ or $(m+1)(m-2) = 0 \Rightarrow m = -1, 2$

$\therefore$ C.F. = $f_1(y-x) + f_2(y+2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D+D')(D-2D')} [(2x-y)(x+y) \sin xy - \cos xy] \\
&= \frac{1}{(D+D')} \int [(4x-c)(c-x) \sin (cx-2x^2) - \cos (cx-2x^2)] dx, \text{ where } y = c-2x \\
&= \frac{1}{(D+D')} \int [(x-c)(c-4x) \sin (cx-2x^2) - \cos (cx-2x^2)] dx \\
&= \frac{1}{(D+D')} \left[(x-c) \left\{ -\cos (cx-2x^2) \right\} + \int \cos (cx-2x^2) dx - \int \cos (cx-2x^2) dx \right] \\
&= \frac{1}{(D+D')} (c-x) \cos \{x(c-2x)\} = \frac{1}{(D+D')} (y+x) \cos xy, \text{ where } c = y+2x \\
&= \int (b+2x) \cos (bx+x^2) dx, \text{ where } y = b+x \\
&= \sin (bx+x^2) = \sin xy
\end{aligned}$$

$\therefore$ Complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y-x) + f_2(y+2x) + \sin xy$

where f_1 and f_2 are arbitrary functions.

85. Solve : $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$.

(A.U.U.P. 2009)

Sol. The given equation is $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$

Auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$ or $m = -1, -1$

$\therefore$ C.F. = $f_1(y-x) + xf_2(y-x)$

$$\text{P.I.} = \frac{1}{(D+D')^2} 2 \cos y - \frac{1}{(D+D')^2} \cdot (x \sin y) = P_1 - P_2$$

$$\begin{aligned}
\text{where, } P_1 &= \frac{1}{(D+D')^2} \cdot 2 \cos y = \frac{2}{(0+1)^2} \iint \cos u du du, \text{ where } y = u \\
&= 2(-\cos y) = -2 \cos y
\end{aligned}$$

$$\begin{aligned}
P_2 &= \frac{1}{(D+D')^2} (x \sin y) = \frac{1}{D+D'} \int x \sin (c+x) dx, \text{ where } y = c+x \\
&= \frac{1}{D+D'} \left[x \{-\cos (c+x)\} - \int 1 \cdot \{-\cos (c+x)\} dx \right] \\
&= \frac{1}{D+D'} [-x \cos (c+x) + \sin (c+x)] = \frac{1}{D+D'} (-x \cos y + \sin y)
\end{aligned}$$

$$\begin{aligned}
&= \int -x \cos(b+x) dx + \int \sin(b+x) dx, \text{ where } y = b+x \\
&= (-x) \sin(b+x) - \int (-1) \cdot \sin(b+x) dx + \int \sin(b+x) dx \\
&= -x \sin(b+x) - 2 \cos(b+x) = -x \sin y - 2 \cos y
\end{aligned}$$

$\therefore$ P.I. = $x \sin y$

Hence complete solution is $z = f_1(y-x) + x f_2(y-x) + x \sin y$ where f_1, f_2 are arbitrary functions.

- 86.** Solve the partial differential equation : $(D - D' - 1)(D - D' - 3)z = 0$.

Sol. The given equation is $(D - D' - 1)(D - D' - 3)z = 0$

Its C.F. = $e^x f_1(y+x) + e^{3x} f_2(y+x)$ and P.I. = 0

Hence complete solution is $z = e^x f_1(y+x) + e^{3x} f_2(y+x)$ where f_1 and f_2 are arbitrary functions.

- 87.** Solve the linear partial differential equation : $(D + D' - 1)(D + 2D' - 2)z = 0$.

Sol. Given equation is $(D + D' - 1)(D + 2D' - 2)z = 0$

C.F. = $e^x f_1(y-x) + e^{2x} f_2(y-2x)$

P.I. = 0

Hence complete solution is $z = e^x f_1(y-x) + e^{2x} f_2(y-2x)$ where f_1 and f_2 are arbitrary functions.

- 88.** Solve : $DD'(D + 2D' + 1)z = 0$.

Sol. The given equation is $DD'(D + 2D' + 1)z = 0$

Corresponding to the factor D, Part of C.F. = $f_1(y)$

Corresponding to the factor D', Part of C.F. = $f_2(x)$

Corresponding to the factor $(D + 2D' + 1)$, Part of C.F. = $e^{-x} f_3(y-2x)$

Hence combined C.F. = $f_1(y) + f_2(x) + e^{-x} f_3(y-2x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(x) + e^{-x} f_3(y-2x)$

where f_1, f_2 and f_3 are arbitrary functions.

- 89.** Solve : $r + 2s + t + 2p + 2q + z = 0$.

Sol. The given equation is $[D^2 + 2DD' + D'^2 + 2D + 2D' + 1]z = 0$

$\Rightarrow [(D + D')^2 + 2(D + D') + 1]z = 0 \Rightarrow (D + D' + 1)^2 z = 0$

Its C.F. = $e^{-x} f_1(y-x) + x e^{-x} f_2(y-x)$ where f_1 and f_2 are arbitrary functions.

- 90.** Solve : $r - t + p - q = 0$.

Sol. Given equation is $(D^2 - D'^2 + D - D')z = 0$

$\Rightarrow (D - D')(D + D' + 1)z = 0$

Its C.F. = $f_1(y+x) + e^{-x} f_2(y-x)$

P.I. = 0

Hence complete solution is $z = f_1(y+x) + e^{-x} f_2(y-x)$ where f_1 and f_2 are arbitrary functions.

- 91.** Solve : $(D + 4D' + 5)^2 z = 0$. (M.T.U., 2009; A.U.U.P., 2008)

Sol. Given equation is $(D + 4D' + 5)^2 z = 0$

Its C.F. = $e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.} = e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$

where f_1 and f_2 are arbitrary functions.

- 92.** Solve : $(D^2 - DD' - 2D)z = 0$

Sol. The given equation is $(D^2 - DD' - 2D)z = 0 \Rightarrow D(D - D' - 2)z = 0$

Its C.F. = $f_1(y) + e^{2x} f_2(y+x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(y) + e^{2x} f_2(y+x)$
 where f_1 and f_2 are arbitrary functions.

93. Solve : $(D+1)(D+D'-1)z=0$ (A.U.U.P., 2010)

Sol. The given equation is $(D+1)(D+D'-1)z=0$

Its C.F. = $e^{-x} \cdot f_1(y) + e^x f_2(y-x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.}$ or $z = e^{-x} f_1(y) + e^x f_2(y-x)$

94. Solve : $(D^2 - D'^2 + D + 3D' - 2)z = 0$

Sol. The given equation is $(D^2 - D'^2 + D + 3D' - 2)z = 0$

$$\Rightarrow (D^2 - D'^2 + 2D - D + 2D' + D' - 2 + DD' - DD')z = 0$$

$$\Rightarrow [D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2]z = 0$$

$$[D(D - D' + 2) + D'(D - D' + 2) - 1(D - D' + 2)]z = 0$$

$$(D - D' + 2)(D + D' - 1)z = 0$$

Its C.F. = $e^{-2x} f_1(y+x) + e^x \cdot f_2(y-x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.} \Rightarrow z = e^{-2x} f_1(y+x) + e^x f_2(y-x)$

where f_1 and f_2 are arbitrary functions.

95. Solve : $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$.

Sol. The given equation is $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$

$$\Rightarrow [D^2 + DD' - 2DD' - 2D'^2 + 2(D + D')]z = 0$$

$$\Rightarrow [D(D + D') - 2D'(D + D') + 2(D + D')]z = 0$$

$$(D + D')(D - 2D' + 2)z = 0$$

Its C.F. = $f_1(y-x) + e^{-2x} f_2(y+2x)$

P.I. = 0

Hence complete solution is $z = \text{C.F.} + \text{P.I.} \Rightarrow z = f_1(y-x) + e^{-2x} \cdot f_2(y+2x)$

where f_1 and f_2 are arbitrary functions.

96. Solve : $(D^2 + D'^2 - p^2)z = 0$. (U.P.T.U., 2008, 2009; M.T.U., 2010)

Sol. Here $D^2 + D'^2 - p^2$ can not be resolved into linear factors in D and D' .

Let $z = Ae^{hx+ky}$

$$\therefore D^2 z = Ah^2 e^{hx+ky}$$

$$D'^2 z = Ak^2 e^{hx+ky}$$

$$\therefore (D^2 + D'^2 - p^2)z = A(h^2 + k^2 - p^2) e^{hx+ky}$$

$$\text{Then, } (D^2 + D'^2 - p^2)z = 0 \Rightarrow A(h^2 + k^2 - p^2) e^{hx+ky} = 0$$

$$\Rightarrow h^2 + k^2 - p^2 = 0 \text{ or } h^2 + k^2 = p^2$$

$$\therefore \text{C.F.} = \Sigma Ae^{hx+ky}, \text{ where } h^2 + k^2 - p^2 = 0$$

$$\text{P.I.} = 0$$

$$\therefore z = \Sigma Ae^{hx+ky}, \text{ where } h^2 + k^2 - p^2 = 0$$

Now h may be taken as $p \cos \alpha$ and k may be taken as $p \sin \alpha$

The complete solution is $z = \Sigma Ae^{p(x \cos \alpha + y \sin \alpha)}$ where A is arbitrary constant and Σ denotes that any number of terms may be taken.

97. Solve : $s + ap + bq + abz = e^{mx+ny}$.

Sol. The given equation is

$$(DD' + aD + bD' + ab)z = e^{mx+ny} \Rightarrow [D(D' + a) + b(D' + a)]z = e^{mx+ny}$$

$$\Rightarrow (D' + a)(D + b)z = e^{mx+ny}$$

Its

$$\text{C.F.} = e^{-bx} f_1(y) + e^{-ay} \cdot f_2(x)$$

$$\text{P.I.} = \frac{1}{(D+b)(D'+a)} \cdot e^{mx+ny} = \frac{e^{mx+ny}}{(m+b)(n+a)}$$

$$\text{Hence complete solution is } z = \text{C.F.} + \text{P.I.} = e^{-bx} f_1(y) + e^{-ay} \cdot f_2(x) + \frac{e^{mx+ny}}{(m+b)(n+a)}$$

where f_1 and f_2 are arbitrary functions.

98. Solve : $(D^3 - 3DD' + D' + 4)z = e^{2x+y}$.

Sol. Here $D^3 - 3DD' + D' + 4$ cannot be resolved into linear factors in D and D' .

Let

$$z = Ae^{hx+ky}$$

$$\therefore (D^3 - 3DD' + D' + 4)z = A(h^3 - 3hk + k + 4)e^{hx+ky}$$

$$\text{Then } (D^3 - 3DD' + D' + 4)z = 0 \text{ iff } h^3 - 3hk + k + 4 = 0$$

$$\therefore \text{C.F.} = \Sigma Ae^{hx+ky}, \text{ where } h^3 - 3hk + k + 4 = 0$$

$$\text{P.I.} = \frac{1}{D^3 - 3DD' + D' + 4} e^{2x+y} = \frac{e^{2x+y}}{2^3 - 3(2)(1) + 1 + 4} = \frac{1}{7} e^{2x+y}$$

$$\text{Hence complete solution is } z = \Sigma Ae^{hx+ky} + \frac{1}{7} e^{2x+y}, \text{ where } h^3 - 3hk + k + 4 = 0.$$

99. Solve : $D(D - 2D' - 3)z = e^{x+2y}$.

Sol.

$$\text{C.F.} = f_1(y) + e^{3x} f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{D(D - 2D' - 3)} \cdot e^{x+2y} = \frac{e^{x+2y}}{1(1 - 4 - 3)} = -\frac{1}{6} e^{x+2y}$$

$$\text{Hence the complete solution is } z = \text{C.F.} + \text{P.I.} \text{ or } z = f_1(y) + e^{3x} f_2(y + 2x) - \frac{1}{6} e^{x+2y}$$

where f_1 and f_2 are arbitrary functions.

100. Solve : $(D^2 - DD' + D' - 1)z = \sin(x + 2y)$. (U.P.T.U., 2011)

Sol. The given equation is

$$(D^2 - DD' + D' - 1)z = \sin(x + 2y)$$

$$\Rightarrow \{(D + 1)(D - 1) - D'(D - 1)\}z = \sin(x + 2y) \Rightarrow (D - 1)(D - D' + 1)z = \sin(x + 2y)$$

$$\therefore \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y + x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - 1)(D - D' + 1)} \sin(x + 2y) = \frac{1}{D^2 - DD' + D' - 1} \sin(x + 2y) \\ &= \frac{1}{-1 + 2 + D' - 1} \sin(x + 2y) = \frac{1}{D'} \sin(x + 2y) = -\frac{1}{2} \cos(x + 2y) \end{aligned}$$

$$\text{Hence complete solution is } z = \text{C.F.} + \text{P.I.} \text{ or } z = e^x f_1(y) + e^{-x} f_2(y + x) - \frac{1}{2} \cos(x + 2y)$$

where f_1 and f_2 are arbitrary functions.

101. Solve : $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. (A.U.U.P., 2009)

Sol. The given equation is $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$

$$\text{C.F.} = e^x f_1(y + x) + e^{2x} f_2(y + x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - D' - 1)(D - D' - 2)} \sin(2x + 3y) \\ &= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \cdot \sin(2x + 3y) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{-4 + 12 - 9 - 3D + 3D' + 2} \sin(2x + 3y) = \frac{1}{-3D + 3D' + 1} \sin(2x + 3y) \\
&= - \left[\frac{(3D - 3D') + 1}{\{(3D - 3D') + 1\} \{3D - 3D' - 1\}} \sin(2x + 3y) \right] \\
&= - \left[\frac{(3D - 3D') + 1}{9D^2 + 9D'^2 - 18DD' - 1} \sin(2x + 3y) \right] = - \left[\frac{3D - 3D' + 1}{-36 - 81 + 108 - 1} \sin(2x + 3y) \right] \\
&= \frac{1}{10} (3D - 3D' + 1) \sin(2x + 3y) \\
&= \frac{1}{10} [6 \cos(2x + 3y) - 9 \cos(2x + 3y) + \sin(2x + 3y)] \\
&= \frac{1}{10} [\sin(2x + 3y) - 3 \cos(2x + 3y)]
\end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{10} [\sin(2x + 3y) - 3 \cos(2x + 3y)]$$

where f_1 and f_2 are arbitrary functions.

102. Find particular integral of: $2s + t - 3q = 5 \cos(3x - 2y)$.

Sol. The given equation is $(2DD' + D'^2 - 3D')z = 5 \cos(3x - 2y)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{2DD' + D'^2 - 3D'} \cdot 5 \cos(3x - 2y) = \frac{1}{2(6) + (-4) - 3D'} 5 \cos(3x - 2y) \\
&= \frac{1}{8 - 3D'} \cdot 5 \cos(3x - 2y) = 5 \left[\frac{8 + 3D'}{64 - 9D'^2} \cos(3x - 2y) \right] \\
&= 5 \left[\frac{8 + 3D'}{64 - 9(-4)} \cos(3x - 2y) \right] = \frac{1}{20} [(8 + 3D') \cos(3x - 2y)] \\
&= \frac{1}{20} [8 \cos(3x - 2y) + 6 \sin(3x - 2y)]
\end{aligned}$$

$$\text{or P.I.} = \frac{1}{10} [4 \cos(3x - 2y) + 3 \sin(3x - 2y)]$$

103. Solve the linear partial differential equation :

(M.T.U., 2009)

$$(D - D' - 1)(D - D' - 2)z = e^{3x-y} + x.$$

Sol. The given equation is $(D - D' - 1)(D - D' - 2)z = e^{3x-y} + x$

$$\text{C.F.} = e^x f_1(y + x) + e^{2x} f_2(y + x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot (e^{3x-y} + x) \\
&= \frac{1}{(D - D' - 1)(D - D' - 2)} e^{3x-y} + \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot (x) = P_1 + P_2
\end{aligned}$$

$$\text{where } P_1 = \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot e^{3x-y} = \frac{1}{(3 + 1 - 1)(3 + 1 - 2)} \cdot e^{3x-y} = \frac{1}{6} \cdot e^{3x-y}$$

$$\begin{aligned}
P_2 &= \frac{1}{(D - D' - 1)(D - D' - 2)} (x) = \frac{1}{(1 - D + D')(2 - D + D')} (x) \\
&= \frac{1}{\{1 - (D - D')\} 2 \left\{ 1 - \left(\frac{D - D'}{2} \right) \right\}} (x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\{1 - (D - D')\}^{-1} \left\{ 1 - \left(\frac{D - D'}{2} \right) \right\}^{-1} (x) = \frac{1}{2} \left[(1 + D - D') \left(1 + \frac{D - D'}{2} \right) \right] (x) \right. \\
&= \frac{1}{2} \left[1 + \frac{D}{2} - \frac{D'}{2} + D + \frac{D^2}{2} - D' - \frac{DD'}{2} + \frac{D'^2}{2} \right] (x) \\
&= \frac{1}{2} \left[x + \frac{1}{2} - 0 + 1 + 0 \right] = \frac{1}{2} \left(x + \frac{3}{2} \right)
\end{aligned}$$

$$\therefore \text{P.I.} = \frac{1}{6} e^{3x-y} + \frac{1}{2} \left(x + \frac{3}{2} \right)$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6} e^{3x-y} + \frac{1}{2} \left(x + \frac{3}{2} \right)$

where f_1 and f_2 are arbitrary functions.

104. Solve : $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$.

Sol. The given equation is $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$

$$\text{C.F.} = e^x f_1(y-x) + e^{3x} f_2(y-2x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D + D' - 1)(D + 2D' - 3)} (4 + 3x + 6y) \\
&= \frac{1}{(1 - D - D')(3 - D - 2D')} (4 + 3x + 6y) \\
&= \frac{1}{\{1 - (D + D')\} 3 \left\{ 1 - \frac{D + 2D'}{3} \right\}} (4 + 3x + 6y) \\
&= \frac{1}{3} \left[\{1 - (D + D')\}^{-1} \left\{ 1 - \frac{D + 2D'}{3} \right\}^{-1} \right] (4 + 3x + 6y) \\
&= \frac{1}{3} \left[(1 + D + D') \left(1 + \frac{D + 2D'}{3} \right) \right] (4 + 3x + 6y) \\
&= \frac{1}{3} \left[1 + \frac{D}{3} + \frac{2D'}{3} + D + \frac{D^2}{3} + \frac{2DD'}{3} + D' + \frac{D'D}{3} + \frac{2D'^2}{3} \right] (4 + 3x + 6y) \\
&= \frac{1}{3} \left[1 + \frac{4D}{3} + \frac{5D'}{3} + \frac{D^2}{3} + DD' + \frac{2D'^2}{3} \right] (4 + 3x + 6y) \\
&= \frac{1}{3} \left[(4 + 3x + 6y) + \frac{4}{3}(3) + \frac{5}{3}(6) + 0 + 0 + 0 \right] = \frac{1}{3} [3x + 6y + 18] = x + 2y + 6
\end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = e^x f_1(y-x) + e^{3x} f_2(y-2x) + x + 2y + 6$

where f_1 and f_2 are arbitrary functions.

105. Solve : $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$. [U.P.T.U., 2007, and A.U.U.P, 2010]

Sol. The given equation is $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

$$\Rightarrow (D - D')(D + D' - 3)z = xy + e^{x+2y}$$

$$\therefore \text{C.F.} = f_1(y+x) + e^{3x} f_2(y-x)$$

$$\begin{aligned}
\text{P.I. Corresponding to } xy &= \frac{1}{(D - D')(D + D' - 3)} (xy) = -\frac{1}{3D} \left(1 - \frac{D'}{D} \right)^{-1} \left(1 - \frac{D}{3} - \frac{D'}{3} \right)^{-1} (xy) \\
&= -\frac{1}{3D} \left(1 + \frac{D'}{D} \right) \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \dots \right) (xy)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{3D} \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \dots \right) (xy) \\
&= -\frac{1}{3D} \left(xy + \frac{y}{3} + \frac{2}{3}x + \frac{x^2}{2} + \frac{2}{9} \right) = -\frac{1}{3} \left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x \right] \\
&\quad \text{(after integrating each term w.r.t. } x)
\end{aligned}$$

P.I. Corresponding to e^{x+2y}

$$\begin{aligned}
&= \frac{1}{(D - D')(D + D' - 3)} e^{x+2y} = \frac{1}{(1-2)(D + D' - 3)} \cdot e^{x+2y} \\
&= -e^{x+2y} \frac{1}{D + 1 + D' + 2 - 3} \quad (1) \\
&= -e^{x+2y} \cdot \frac{1}{D + D'} \quad (1) = -e^{-x+2y} \cdot \frac{1}{D'} \left(1 + \frac{D}{D'} \right)^{-1} = -e^{-x+2y} \cdot \frac{1}{D'} \quad (1) = -ye^{-x+2y}
\end{aligned}$$

Hence P.I. = $-\frac{1}{3} \left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x \right] - ye^{-x+2y}$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$= f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{3} \left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x \right] - ye^{-x+2y}.$$

106. Solve : $s + p - q = z + xy$.

Sol. The given equation is $(DD' + D - D' - 1)z = xy \Rightarrow (D - 1)(D' + 1)z = xy$

$\therefore$ C.F. = $e^x f_1(y) + e^{-y} f_2(x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D - 1)(D' + 1)} (xy) = -\{(1 - D)^{-1}(1 + D')^{-1}\} (xy) \\
&= -[(1 + D + D^2 + \dots)(1 - D' + \dots)] (xy) \\
&= -(1 + D - D' - DD') xy \quad (\text{leaving higher powers}) \\
&= -(xy + y - x - 1) = x + 1 - xy - y
\end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = e^x f_1(y) + e^{-y} \cdot f_2(x) + x + 1 - xy - y$ where f_1 and f_2 are arbitrary functions.

107. Solve : $r - s + p = 1$.

Sol. The given equation is $(D^2 - DD' + D)z = 1 \Rightarrow D(D - D' + 1)z = 1$

$\therefore$ C.F. = $f_1(y) + e^{-x} \cdot f_2(y+x)$

$$\text{P.I.} = \frac{1}{D(D - D' + 1)} (1) = \frac{1}{D} [1 + (D - D')]^{-1} (1) = \frac{1}{D} (1 - D + D') (1) = \frac{1}{D} (1) = x$$

$\therefore$ Complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y) + e^{-x} f_2(y+x) + x$

where f_1 and f_2 are arbitrary functions.

108. Solve : $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

Sol. C.F. = $f_1(y) + e^x \cdot f_2(y-x) + e^{2x} f_3(y-3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D(D + D' - 1)(D + 3D' - 2)} (x^2 - 4xy + 2y^2) \\
&= \frac{1}{2D} \{1 - (D + D')\}^{-1} \left\{ 1 - \frac{D + 3D'}{2} \right\}^{-1} (x^2 - 4xy + 2y^2) \\
\text{P.I.} &= \frac{1}{2D} \{1 + D + D' + (D + D')^2 + \dots\} \left\{ 1 + \frac{D + 3D'}{2} + \left(\frac{D + 3D'}{2} \right)^2 + \dots \right\} (x^2 - 4xy + 2y^2)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2D} \left[1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2} + \dots \right] (x^2 - 4xy + 2y^2) \\
&= \frac{1}{2D} \left[x^2 - 4xy + 2y^2 + 3(x - 2y) + 5(2y - 2x) + \frac{7}{2} + 19 - 22 \right] \\
&= \frac{1}{2D} \left[x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2} \right] = \frac{1}{2} \left[\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4yx + \frac{x}{2} \right]
\end{aligned}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y) + e^x f_2(y - x) + e^{2x} f_3(y - 3x) + \frac{1}{6} x^3 - x^2y + xy^2 - \frac{7}{4} x^2 + 2xy + \frac{x}{4}$$

where f_1, f_2 and f_3 are arbitrary functions.

109. Solve : $(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x + y)$.

Sol. Here,

$$\text{C.F.} = e^{2x} f_1(y + 3x) + x e^{2x} f_2(y + 3x) + x^2 e^{2x} f_3(y + 3x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D - 3D' - 2)^3} 6e^{2x} \sin(3x + y) = 6e^{2x} \cdot \frac{1}{(D + 2 - 3D' - 2)^3} \sin(3x + y) \\
&= 6e^{2x} \cdot \frac{1}{(D - 3D')^3} \sin(3x + y) = 6e^{2x} \cdot \frac{x^3}{6} \sin(3x + y) = x^3 \cdot e^{2x} \sin(3x + y)
\end{aligned}$$

Hence complete solution is

$$z = e^{2x} \cdot f_1(y + 3x) + x e^{2x} f_2(y + 3x) + x^2 e^{2x} f_3(y + 3x) + x^3 e^{2x} \sin(3x + y)$$

where f_1, f_2 and f_3 are arbitrary functions.

110. Solve : $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$.

Sol. C.F. = $e^{2x} f_1(y + 3x) + x e^{2x} f_2(y + 3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D - 3D' - 2)^2} \cdot 2e^{2x} \tan(y + 3x) = 2e^{2x} \cdot \frac{1}{(D + 2 - 3D' - 2)^2} \tan(y + 3x) \\
&= 2e^{2x} \left[\frac{1}{(D - 3D')^2} \tan(y + 3x) \right] = 2e^{2x} \left[x \cdot \frac{1}{2(D - 3D')} \tan(y + 3x) \right] \\
&= 2e^{2x} \cdot x^2 \cdot \frac{1}{2} \tan(y + 3x) = x^2 e^{2x} \tan(y + 3x)
\end{aligned}$$

Hence complete solution is $z = \text{C.F.} + \text{P.I.}$

$$= e^{2x} f_1(y + 3x) + x e^{2x} f_2(y + 3x) + x^2 e^{2x} \tan(y + 3x)$$

where f_1, f_2 are arbitrary functions.

111. Solve the linear partial differential equation :

$$(D^2 - DD' + D' - 1) z = \cos(x + 2y) + e^y.$$

(U.P.T.U., 2006, 2009)

Sol. The given equation is $(D^2 - DD' + D' - 1) z = \cos(x + 2y) + e^y$

or

$$(D^2 - 1 - DD' + D') z = \cos(x + 2y) + e^y$$

$\Rightarrow$

$$\{(D + 1)(D - 1) - D'(D - 1)\} z = \cos(x + 2y) + e^y$$

$\Rightarrow$

$$(D - 1)(D + 1 - D') z = \cos(x + 2y) + e^y$$

$$\text{C.F.} = e^x f_1(y) + e^{-x} f_2(y + x)$$

$$\text{P.I.} = \frac{1}{(D - D' + 1)(D - 1)} \{\cos(x + 2y) + e^y\} = P_1 + P_2$$

$$\text{where } P_1 = \frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y) = \frac{1}{-(1)^2 - (-2) + D' - 1} \cos(x + 2y)$$

$$P_1 = \frac{1}{D'} \cos(x + 2y) = \frac{\sin(x + 2y)}{2}$$

$$\begin{aligned} \text{and } P_2 &= \frac{1}{D - D' + 1} \left(\frac{1}{D - 1} e^y \right) = \frac{1}{D - D' + 1} \left(\frac{1}{0 - 1} e^y \right) \\ &= \frac{1}{D - D' + 1} (-e^y) = -e^y \cdot \frac{1}{(D + 0) - (D' + 1) + 1} \quad (1) \\ &= -e^y \cdot \frac{1}{D - D'} \quad (1) = -e^y \cdot \frac{1}{D} \left(1 - \frac{D'}{D} \right)^{-1} \quad (1) \\ &= -e^y \cdot \frac{1}{D} \left(1 + \frac{D'}{D} \right) \quad (1) = -e^y \cdot \frac{1}{D} \quad (1) = -xe^y \end{aligned}$$

$$\therefore \text{P.I.} = \frac{1}{2} \sin(x + 2y) - xe^y$$

Hence the complete solution is $z = e^x f_1(y) + e^{-x} f_2(y + x) + \frac{1}{2} \sin(x + 2y) - xe^y$

where f_1 and f_2 are arbitrary functions.

112. Solve : $r - 4s + 4t + p - 2q = e^{x+y}$.

Sol. The given equation is $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$

$$\Rightarrow \{(D - 2D')^2 + (D - 2D')\}z = e^{x+y}$$

$$\Rightarrow (D - 2D')(D - 2D' + 1)z = e^{x+y}$$

$$\therefore \text{C.F.} = f_1(y + 2x) + e^{-x} f_2(y + 2x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - 2D')(D - 2D' + 1)} e^{x+y} = \frac{1}{D - 2D' + 1} \left[\frac{1}{D - 2D'} e^{x+y} \right] \\ &= \frac{1}{D - 2D' + 1} \left[\frac{1}{1 - 2} \int e^u du \right] \quad \text{where } x + y = u \\ &= -\frac{1}{D - 2D' + 1} \cdot e^{x+y} = -e^{x+y} \cdot \frac{1}{D + 1 - 2(D' + 1) + 1} \\ &= -e^{x+y} \frac{1}{D - 2D'} \quad (1) = -e^{x+y} \cdot \frac{1}{D} \left(1 - \frac{2D'}{D} \right)^{-1} = -xe^{x+y} \end{aligned}$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y + 2x) + e^{-x} f_2(y + 2x) - xe^{x+y}$
where f_1 and f_2 are arbitrary functions.

113. Find the particular integral of $(D^2 - D')z = xe^{ax+a^2y}$.

Sol. The given equation is $(D^2 - D')z = xe^{ax+a^2y}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D'} (xe^{ax+a^2y}) = e^{ax+a^2y} \cdot \frac{1}{(D+a)^2 - (D'+a^2)} (x) \\ &= e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD - D'} (x) = e^{ax+a^2y} \cdot \frac{1}{2aD} \left(1 + \frac{D}{2a} - \frac{D'}{2aD} \right)^{-1} (x) \\ &= e^{ax+a^2y} \cdot \frac{1}{2aD} \left\{ 1 - \frac{D}{2a} + \frac{D'}{2aD} + \dots \right\} (x) = e^{ax+a^2y} \cdot \frac{1}{2aD} \left(x - \frac{1}{2a} \right) \\ &= e^{ax+a^2y} \cdot \frac{1}{2a} \cdot \frac{1}{D} \left(x - \frac{1}{2a} \right) = e^{ax+a^2y} \cdot \frac{1}{2a} \left(\frac{x^2}{2} - \frac{x}{2a} \right) = e^{ax+a^2y} \left(\frac{x^2}{4a} - \frac{x}{4a^2} \right) \end{aligned}$$

114. Solve the linear partial differential equation : $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4$.

(A.U.U.P., 2007, 2009)

Sol. In the given equation, coefficients of derivative are multiples of the variables. It can be reduce to partial differential equation with constant coefficients by putting $x = e^X$, $y = e^Y$

$$X = \log x, Y = \log y$$

The given equation becomes $[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{3X+4Y}$

Here note that the substitution $x = e^X$, $y = e^Y$ gives

$$\begin{aligned} x^2 \frac{\partial^2 z}{\partial x^2} &= (D-1)Dz, & y^2 \frac{\partial^2 z}{\partial y^2} &= (D'-1)D'z \\ y \frac{\partial z}{\partial y} &= D'z, & xy \frac{\partial^2 z}{\partial x \partial y} &= DD'z \end{aligned}$$

The given equation reduces to $\{(D^2 - 4DD' + 4D'^2) - (D - 2D')\}z = e^{3X+4Y}$
 $\Rightarrow (D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$

Its C.F. = $f_1(Y + 2X) + e^X f_2(Y + 2X)$

$$= f_1(\log y + 2 \log x) + x f_2(\log y + 2 \log x)$$

$$= f_1(\log yx^2) + x f_2(\log yx^2) = g_1(yx^2) + x g_2(yx^2)$$

$$\text{P.I.} = \frac{1}{D - 2D' - 1} \left[\frac{1}{D - 2D'} e^{3X+4Y} \right] = \frac{1}{D - 2D' - 1} \left[\frac{1}{3-8} \int e^u du \right] \text{ where } 3X + 4Y = u$$

$$= \frac{1}{D - 2D' - 1} \left[-\frac{1}{5} e^{3X+4Y} \right] = -\frac{1}{5} \left[\frac{1}{D - 2D' - 1} \cdot e^{3X+4Y} \right]$$

$$= -\frac{1}{5} \left[\frac{1}{3-8-1} e^{3X+4Y} \right] = -\frac{1}{5} \left[-\frac{1}{6} \cdot e^{3X+4Y} \right] = \frac{1}{30} \cdot e^{3X+4Y} = \frac{1}{30} x^3 y^4$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = g_1(yx^2) + x g_2(yx^2) + \frac{1}{30} x^3 y^4$

where g_1 and g_2 are arbitrary functions.

115. Solve : $x^2 r - y^2 t + px - qy = \log x$.

Sol. Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$

Given equation reduces to $[D(D-1) - D'(D'-1) + D - D']z = X \Rightarrow (D^2 - D'^2)z = X$

which is a homogeneous linear *Partial differential equation* with constant co-efficients.

$\therefore$ C.F. = $\phi_1(Y + X) + \phi_2(Y - X)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D'^2} (X) = \frac{1}{(1)^2 - (0)^2} \iint u du du, \text{ where } X = u \\ &= \int \frac{u^2}{2} du = \frac{u^3}{6} = \frac{X^3}{6} \end{aligned}$$

Hence solution is $z = \phi_1(Y + X) + \phi_2(Y - X) + \frac{X^3}{6}$

$$= \phi_1(\log y + \log x) + \phi_2(\log y - \log x) + \frac{(\log x)^3}{6}$$

$$= \phi_1(\log xy) + \phi_2\left(\log \frac{y}{x}\right) + \frac{1}{6} (\log x)^3$$

or
$$z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6} (\log x)^3$$

where f_1 and f_2 are arbitrary functions.

116. Solve : $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = xy$.

Sol. Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$

Given equation reduces to $[D(D-1) - D'(D'-1)] z = e^{X+Y}$

$$(D^2 - D'^2 - D + D') z = e^{X+Y}$$

$$\{(D - D')(D + D') - 1(D - D')\} z = e^{X+Y}$$

$$(D - D')(D + D' - 1) z = e^{X+Y}$$

$$\text{C.F.} = f_1(Y + X) + e^X f_2(Y - X)$$

$$= f_1(\log xy) + x f_2\left(\log \frac{y}{x}\right) = g_1(xy) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D - D')(D + D' - 1)} e^{X+Y} = \frac{1}{D - D'} \left[\frac{1}{1 + 1 - 1} \cdot e^{X+Y} \right]$$

$$\text{P.I.} = \frac{1}{D - D'} e^{X+Y} = X \cdot e^{X+Y} = xy \log x$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = g_1(xy) + x g_2\left(\frac{y}{x}\right) + xy \log x$

where g_1 and g_2 are arbitrary functions.

117. Solve : $(x^2 D^2 + 2xy DD' + y^2 D'^2) z = x^m \cdot y^n$.

Sol. Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$

Then the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)] z = e^{mX+nY}$$

$$\Rightarrow [D^2 + 2DD' + D'^2 - D - D'] z = e^{mX+nY}$$

$$[(D + D')^2 - 1(D + D')] z = e^{mX+nY}$$

$$(D + D')(D + D' - 1) z = e^{mX+nY}$$

$$\therefore \text{C.F.} = f_1(Y - X) + e^X f_2(Y - X) = f_1\left(\log \frac{y}{x}\right) + x f_2\left(\log \frac{y}{x}\right)$$

$$\text{C.F.} = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D + D')(D + D' - 1)} \cdot e^{mX+nY} = \frac{1}{(m+n)(m+n-1)} \cdot e^{mX+nY} = \frac{x^m \cdot y^n}{(m+n)(m+n-1)}$$

Hence complete solution is $z = \text{C.F.} + \text{P.I.} = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right) + \frac{x^m y^n}{(m+n)(m+n-1)}$

where g_1 and g_2 are arbitrary functions.

118. Solve : $x^2 r + 2y^2 t + px - 3xys + 2qy = x + 2y$.

Sol. Let $x = e^X$, $y = e^Y$, $X = \log x$, $Y = \log y$

Given equation reduces to

$$[D(D-1) + 2D'(D'-1) + D - 3DD' + 2D'] z = e^X + 2e^Y$$

$$\Rightarrow [D^2 - D + 2D'^2 - 2D' + D - 3DD' + 2D'] z = e^X + 2e^Y$$

$$\Rightarrow (D^2 - 3DD' + 2D'^2) z = e^X + 2e^Y$$

$$\Rightarrow (D - D')(D - 2D') z = e^X + 2e^Y$$

$$\therefore \text{C.F.} = f_1(Y + X) + f_2(Y + 2X)$$

$$= f_1(\log xy) + f_2(\log x^2 y) = g_1(xy) + g_2(x^2 y)$$

$$\text{P.I.} = \frac{1}{(D - D')(D - 2D')} \cdot e^X + \frac{1}{(D - D')(D - 2D')} \cdot 2e^Y = P_1 + P_2$$

$$\text{where, } P_1 = \frac{1}{(D - D')(D - 2D')} \cdot e^X = \frac{1}{(1 - 0)(1 - 0)} \iint e^u du du, \text{ where } X = u$$

$$= e^u = e^X = x$$

$$P_2 = \frac{1}{(D - D')(D - 2D')} (2e^Y) = \frac{2}{(0 - 1)(0 - 2)} \iint e^v dv dv, \text{ where } Y = v$$

$$= e^Y = y$$

$$\therefore \text{ P.I.} = x + y$$

Hence complete solution is $z = g_1(xy) + g_2(x^2y) + x + y$ where g_1 and g_2 are arbitrary functions.

119. Solve : $\frac{1}{x^2} \frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3} \cdot \frac{\partial z}{\partial x} = \frac{1}{y^2} \frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3} \cdot \frac{\partial z}{\partial y}$

Sol. Let $\frac{x^2}{2} = X$ and $\frac{y^2}{2} = Y$ so that $\frac{dX}{dx} = x, \frac{dY}{dy} = y$

$$\therefore \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{1}{x} \frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial X^2} = \frac{\partial}{\partial X} \left(\frac{1}{x} \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{1}{x} \frac{\partial z}{\partial x} \right) \frac{dx}{dX} = -\frac{1}{x^3} \frac{\partial z}{\partial x} + \frac{1}{x^2} \frac{\partial^2 z}{\partial x^2} \quad \dots(1)$$

Similarly, $\frac{\partial^2 z}{\partial Y^2} = -\frac{1}{y^3} \frac{\partial z}{\partial y} + \frac{1}{y^2} \frac{\partial^2 z}{\partial y^2} \quad \dots(2)$

Hence by (1) and (2), the given equation reduces

$$\frac{\partial^2 z}{\partial X^2} - \frac{\partial^2 z}{\partial Y^2} = 0 \quad \text{or} \quad (D^2 - D'^2)z = 0 \quad \text{or} \quad (D + D')(D - D')z = 0,$$

$$\therefore \text{ C.F.} = f_1(Y + X) + f_2(Y - X) = f_1\left(\frac{y^2}{2} + \frac{x^2}{2}\right) + f_2\left(\frac{y^2}{2} - \frac{x^2}{2}\right) = \phi_1(x^2 + y^2) + \phi_2(y^2 - x^2)$$

$$\text{P.I.} = 0$$

$$\therefore \text{ Complete solution is } z = \text{C.F.} + \text{P.I.} = \phi_1(x^2 + y^2) + \phi_2(y^2 - x^2).$$

120. How do you classify linear partial differential equation of second order

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

where A, B, C are constants or continuous function of x and y possessing continuous partial derivative and A is positive. (U.P., IInd Semester, 2007, 2009)

Sol. Given equation is

(i) Elliptic if $B^2 - 4AC < 0$

(ii) Hyperbolic if $B^2 - 4AC > 0$

(iii) Parabolic if $B^2 - 4AC = 0$

Note that if A, B, C are constants, then nature of the given equation will be the same in whole region i.e., for all values of x and y . If A, B, C are functions of x and y , then nature of the given equation will not be the same in the whole region i.e., for all values of x and y .

121. Classify the following operators :

(i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

(ii) $4 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

(iii) $5 \frac{\partial^2 u}{\partial x^2} - 9 \frac{\partial^2 u}{\partial x \partial t} + 4 \frac{\partial^2 u}{\partial t^2}$

Sol. (i) Here, $A = B = C = 1 \quad \therefore B^2 - 4AC = -3 < 0$

$\therefore$ Given operator is elliptic.

(ii) Here $A = 4, B = 4, C = 1 \quad \therefore B^2 - 4AC = 0$

$\therefore$ Given operator is parabolic.

(iii) Here $A = 5, B = -9, C = 4, \quad \therefore B^2 - 4AC = 1 > 0$

$\therefore$ Given operator is hyperbolic.

122. Classify the following operators :

$$(i) x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u \quad (ii) t \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \quad (iii) x \frac{\partial^2 u}{\partial x^2} + t \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}.$$

Sol. (i) Here, $A = x^2, B = 0, C = -1$

$$B^2 - 4AC = 0 - 4(x^2)(-1) = 4x^2$$

$\therefore$ Operator is hyperbolic if $4x^2 > 0$ i.e., $x > 0$

Parabolic if $4x^2 = 0$ i.e., $x = 0$

Since $4x^2$ is a square, cannot be negative, hence operator cannot be elliptic.

(ii) Here, $A = t, B = 2, C = x$

$$B^2 - 4AC = 4 - 4tx$$

$\therefore$ Operator is hyperbolic if $4 - 4tx > 0$ i.e., $tx < 1$

Elliptic if $4 - 4tx < 0$ i.e., $tx > 1$ and parabolic if $4 - 4tx = 0$ i.e., $tx = 1$

(iii) Here, $A = x, B = t, C = 1 \quad \therefore B^2 - 4AC = t^2 - 4x$

$\therefore$ The operator is hyperbolic if $t^2 - 4x > 0$ i.e., $t^2 > 4x$, elliptic if $t^2 < 4x$, parabolic if $t^2 = 4x$.

123. Classify the equation as to type in the second quadrant of xy -plane :

$$\sqrt{y^2 + x^2} \cdot u_{xx} + 2(x - y) u_{xy} + \sqrt{y^2 + x^2} \cdot u_{yy} = 0.$$

Sol. Here, $A = \sqrt{y^2 + x^2}, B = 2(x - y), C = \sqrt{y^2 + x^2}$

$$\text{Now, } B^2 - 4AC = 4(x - y)^2 - 4(y^2 + x^2) = -8xy$$

In the second quadrant, y is positive and x is -ve

$$\therefore B^2 - 4AC = +ve > 0$$

$\therefore$ Differential equation is hyperbolic.

124. Show that the equation $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

(U.P.T.U., 2009; A.U.U.P., 2010; M.T.U., 2010)

Sol. The given differential equation is $\frac{\partial^2 u}{\partial t^2} - C^2 \frac{\partial^2 u}{\partial x^2} = 0$

Here $A = 1, B = 0, C = -C^2$

Now $B^2 - 4AC = 4C^2$ which is always greater than zero.

Hence, the given equation is hyperbolic.

125. Classify the partial differential equation : $2 \frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial t} + 3 \frac{\partial^2 u}{\partial x^2} = 0.$

Sol. Here, $A = 2, B = 4, C = 3$

$$\therefore B^2 - 4AC = 16 - 24 = -8 < 0$$

Hence, the given equation is **elliptic**.

126. Classify the partial differential equation : $x^2 \frac{\partial^2 u}{\partial t^2} + 3 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + 17 \frac{\partial u}{\partial t} = 100u.$

Sol. Here, $A = x^2$, $B = 3$, $C = x$

$$B^2 - 4AC = (3)^2 - 4(x^2)(x) = 9 - 4x^3$$

The equation is elliptic if $9 - 4x^3 < 0$, parabolic if $9 - 4x^3 = 0$, hyperbolic if $9 - 4x^3 > 0$.

- 127.** Classify the Partial differential equation $\frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 6u = 0$

Sol. Here, $A = 1$, $B = t$, $C = x$

Now, $B^2 - 4AC = t^2 - 4x$

The equation is elliptic, parabolic, hyperbolic according as $t^2 - 4x < 0$, $= 0$ and > 0 respectively.

- 128.** Show that the equation : $u_{xx} + xu_{yy} + u_y = 0$ is elliptic for $x > 0$ and hyperbolic for $x < 0$.

Sol. The given equation is $u_{xx} + xu_{yy} + u_y = 0$

Here, $A = 1$, $B = 0$, $C = x$

($\because u_{xy}$ term is missing $\Rightarrow B = 0$)

$$B^2 - 4AC = -4x$$

Equation is elliptic if $B^2 - 4AC < 0$ i.e., if $-4x < 0$ or if $x > 0$ and equation is hyperbolic if $B^2 - 4AC > 0$ i.e., if $x < 0$.

- 129.** Show that the equation : $z_{xx} + 2xz_{xy} + (1 - y^2)z_{yy} = 0$ is elliptic for values of x and y in the region $x^2 + y^2 < 1$, parabolic on the boundary and hyperbolic outside this region.

Sol. Here $A = 1$, $B = 2x$, $C = 1 - y^2$ $\therefore B^2 - 4AC = 4x^2 - 4(1 - y^2) = 4(x^2 + y^2 - 1)$

Equation is elliptic if $B^2 - 4AC < 0 \Rightarrow x^2 + y^2 < 1$

Parabolic if $B^2 - 4AC = 0 \Rightarrow x^2 + y^2 = 1$ (on the boundary) and Hyperbolic if $x^2 + y^2 > 1$.

Hence the results.

- 130.** Classify the equation : $u_{xx} - 2u_{xy} + u_{yy} = 0$.

Sol. Here, $A = 1$, $B = -2$, $C = 1$

$$B^2 - 4AC = 4 - 4.1.1 = 0$$

$\therefore$ The given equation is parabolic.

- 131.** Classify the following equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$

Sol. Here the equation is a linear partial differential equation of II order in 3 independent variables.

Here, $a_{11} = 1$, $a_{22} = 1$, $a_{33} = 1$, $a_{44} = 0$; $a_{13} = a_{23} = a_{31} = a_{24} = a_{34} = 0$, etc.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Hence the given differential equation is parabolic.

- 132.** Classify the differential equation : $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$

Sol. Here $a_{11} = 1$, $a_{12} = -2$, $a_{13} = 0$, $a_{21} = 0$, $a_{22} = 2$, $a_{23} = -2$, $a_{31} = 0$, $a_{32} = 0$, $a_{33} = 1$

$$\therefore \Delta = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

$$\therefore \phi = \delta_1^2 - 2\delta_1\delta_2 + 2\delta_2^2 - 2\delta_2\delta_3 + \delta_3^2 = (\delta_1 - \delta_2)^2 + (\delta_2 - \delta_3)^2$$

which is positive definitely for all real values of $\delta_1, \delta_2, \delta_3$.

Hence the given equation is elliptic.

Note. $\phi = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \delta_i \delta_j$ for all real values of δ_i positive or negative. The differential equation is elliptic if

ϕ is positive for all real values of δ_i and it reduces to zero only when all δ_i 's are zero. The equation is hyperbolic if ϕ can be both positive or negative. In this question, ϕ being positive the given differential equation is therefore elliptic.

□ □ □

Applications of Partial Differential Equations

IMPORTANT DEFINITIONS AND FORMULAE

Introduction

In applied mathematics, the partial differential equations generally arise from the mathematical formulation of physical and engineering problems. Besides these, partial differential equations also play an important role in the theory of Elasticity, Hydraulics, Transmission Lines of Electricity and telegraph/radio, Flow of Heat by Conduction in a uniform bar etc. Since the general solution of a p.d.e. in a region R contains arbitrary constants or functions, the unique solution of a p.d.e. corresponding to a physical problem will satisfy certain other conditions at the boundary of the Region R. These are known as *boundary conditions*. When these conditions are specified for the time $t = 0$, they are known as *initial conditions*. Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables.

In this method, right from the beginning, we try to find the particular solutions of the p.d.e. which satisfy all or some of the boundary conditions and then adjust them till the remaining conditions are also satisfied. A combination of these particular solutions gives the solution of the problem.

Method of Separation of Variables

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

Method 1. Method of Separation of Variables

Step 1. Consider the partial differential equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

Here z is called dependent variable and x, y are called independent variables. Assume $Z = XY$, where X is a function of x only and Y is a function of y only.

Step 2. Find $\frac{\partial z}{\partial x} = X'Y$, $\frac{\partial z}{\partial y} = XY'$, $\frac{\partial^2 z}{\partial x^2} = X''Y$, $\frac{\partial^2 z}{\partial y^2} = XY''$ etc., and put in the given partial differential equation.

Step 3. Separate the variables and by solving the resulting equations, find X and Y .

Step 4. $Z = XY$ will be the required solution.

1. DEF.

One Dimensional Wave Equation or Vibrations of a Stretched String

$$\text{The equation } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, y = y(x, t) \quad \dots(1)$$

$$\text{where, } \left. \begin{array}{l} y(0, t) = 0 \\ y(l, t) = 0 \end{array} \right\} \text{ Boundary conditions} \quad \dots(2)$$

$$\left. \begin{array}{l} y(x, 0) = f(x) \\ \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \end{array} \right\} \text{ Initial conditions,} \quad \dots(3)$$

is called one dimensional wave equation or vibrations of a stretched string.

Remark 1. Eqn. (1) is called one dimensional wave equation and $y(x, t)$ is called Displacement function.

Remark 2. Solution of one dimensional wave equation

The solution of one dimensional wave equation (1) satisfying the boundary conditions (2) is given by

$$y = y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Remark 3. The solution of one dimensional wave equation (1) satisfying the boundary conditions (2) and the initial conditions (3) is given by

$$y = y(x, t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \text{ and } y(x, 0) = f(x)$$

DEF : 2. Two Dimensional Heat Flow

Its equation is given by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

Remark 4. In the steady state, u is independent of t , so that $\frac{\partial u}{\partial t} = 0$ and the equation (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is known as **Laplace's equation in two dimensions.**

DEF : 3. Three Dimensional Heat Flow

The equation of heat flow in a solid (Three dimensional heat flow) is given by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \dots(2)$$

Remark 5. In the steady state, u is independent of t , so that (2) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

which is known as **Laplace's equation in three dimensions.**

Remark 6. Solution of Laplace's equation in two dimensions

DEF : 4. Vibrating Membrane—Two Dimensional Wave Equation

The wave equation in two dimensions is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

where the membrane is rectangular and is stretched between the lines $x = 0, x = a, y = 0, y = b$.

$$\left. \begin{aligned} \text{The boundary conditions are } u(0, y, t) = u(a, y, t) = 0 \\ u(x, 0, t) = u(x, b, t) = 0 \end{aligned} \right\} \quad \dots(2)$$

$$\left. \begin{aligned} \text{and the initial conditions are } u(x, y, 0) = f(x, y) \\ \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \end{aligned} \right\} \quad \dots(3)$$

DEF : 5. Solution of Two Dimensional Wave Equation

(a) The solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$, satisfying the boundary conditions

$$\left. \begin{aligned} u(0, y, t) = u(a, y, t) = 0 \\ u(x, 0, t) = u(x, b, t) = 0 \end{aligned} \right\}$$

is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos pt + B_{mn} \sin pt] \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

(b) The solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$, satisfying the boundary conditions

$$\left. \begin{aligned} u(0, y, t) = u(a, y, t) = 0 \\ u(x, 0, t) = u(x, b, t) = 0 \end{aligned} \right\}$$

as well as the initial conditions

$$\left. \begin{aligned} u(x, y, 0) = f(x, y) \\ \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \end{aligned} \right\} \text{ is given by}$$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \cos pt$$

where,

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dy \, dx$$

and

$$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Note. Proof is beyond the scope of the book.

DEF : 6. Transmission Line Equations

Consider the flow of electricity in an insulated cable. Let V be the potential and I the current at time t at a point P of the cable at a distance x from a given point.

Let R = Resistance, L = Inductance, C = Capacitance, G = Leakance, to the ground per unit length of the cable, each assumed to be constant.

Then the equations

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} + (RC + LG) \frac{\partial V}{\partial t} + RGV \quad \dots(1)$$

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + (RC + LG) \frac{\partial I}{\partial t} + RGI \quad \dots(2)$$

are called **Telephone Equations**.

DEF : 7. Telegraph Equations

If $L = 0$, $G = 0$, the equations (1) and (2) reduce to

$$\frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial x} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RC \frac{\partial I}{\partial t},$$

which are known as **Telegraph Equations**.

DEF : 8. Radio Equations

If $R = 0$, $G = 0$, then the equations (1) and (2) reduce to

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2},$$

which are known as **Radio Equations**.

Remark 7. Telegraph equations are similar to one dimension heat flow.

Remark 8. Radio equations are similar to one dimension wave equation.

DEF : 9

Laplace Equation in Two Dimension Cartesian Form is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and its various possible solutions are given by

$$(i) \ u(x, y) = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py)$$

$$(ii) \ u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py})$$

$$(iii) \ u(x, y) = (c_1 x + c_2) (c_3 y + c_4).$$

DEF : 10

Laplace Equation in Three Dimensional Cartesian Form is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \text{ and its various possible solutions are}$$

$$(i) \ u(x, y, z) = (c_1 e^{kx} + c_2 e^{-kx}) (c_3 e^{ly} + c_4 e^{-ly}) (c_5 \cos \sqrt{k^2 + l^2} z + c_6 \sin \sqrt{k^2 + l^2} z)$$

$$(ii) \ u(x, y, z) = (c_1 \cos kx + c_2 \sin kx) (c_3 \cos ly + c_4 \sin ly) (c_5 e^{\sqrt{k^2 + l^2} z} + c_6 e^{-\sqrt{k^2 + l^2} z})$$

DEF : 11

Laplace Equation in Polar Co-ordinates (r, θ) is given by

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \text{ and its various possible solutions are}$$

- (i) $u(r, \theta) = (c_1 r^p + c_2 r^{-p}) (c_3 \cos p\theta + c_4 \sin p\theta)$
 (ii) $u(r, \theta) = (c_1 \cos(p \log r) + c_2 \sin(p \log r)) (c_3 e^{p\theta} + c_4 e^{-p\theta})$
 (iii) $u(r, \theta) = (c_1 \log r + c_2) (c_3 \theta + c_4)$.

DEF : 12

Laplace Equation in Cylindrical Co-ordinates (r, θ, z) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ and its solution is given by}$$

$$u(r, \theta, z) = (c_1 J_n(kr) + c_2 Y_n(kr)) (c_3 \cos n\theta + c_4 \sin n\theta) (c_5 e^{kz} + c_6 e^{-kz}).$$

DEF : 13

Laplace Equation in Spherical Co-ordinates (r, θ, ϕ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

and its solution is given by

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [c_1 P_n^m(\cos \theta) + c_2 Q_n^m(\cos \theta)] \times [c_3 \cos m\phi + c_4 \sin m\phi] [c_5 r^n + c_6 r^{-n-1}]$$

Note. Proofs of the above are beyond the scope of the book.

DEF : 14. One Dimensional Heat Equation

The solution of *one dimensional heat equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given by

$$u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}, \text{ where } u(x, t) \text{ is called } \textit{Temperature Function}.$$

SOLVED PROBLEMS

1. Solve the equation using the method of separation of variables

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u, \text{ given that } u(x, 0) = 6e^{-3x}. \quad (\text{U.P.T.U., 2006})$$

Sol. The given equation is $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$... (1)

Here, u is a function of x and t .

Let $u = XT$... (2)

where X is a function of x only and T is a function of t only.

$$\frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad X'T = (2T' + T)X \Rightarrow \frac{X'}{X} = \frac{2T' + T}{T} \quad \dots(3)$$

Since x and t are independent variables, as t varies x remains constant, so that the L.H.S. and hence the R.H.S. is constant. Therefore, equation (3) can hold only when each side is equal to some constant, say k .

$$\therefore \quad \frac{X'}{X} = k \quad \text{i.e.,} \quad \log X = kx + \log c_1$$

or

$$\log \frac{X}{c_1} = kx \quad \text{or} \quad X = c_1 e^{kx} \quad \dots(4)$$

and

$$\frac{2T' + T}{T} = k \quad \text{i.e.,} \quad \frac{T'}{T} = \frac{1}{2}(k - 1)$$

$$\Rightarrow \quad \log T = \frac{1}{2}(k - 1)t + \log c_2 \quad \text{or} \quad T = c_2 e^{\frac{1}{2}(k-1)t} \quad \dots(5)$$

From (2), (4) and (5), we have $u = u(x, t) = c_1 e^{kx} \cdot c_2 e^{\frac{1}{2}(k-1)t}$

Since $u(x, 0) = 6e^{-3x} \quad \therefore \quad c_1 c_2 = 6, k = -3$

$\therefore$ The unique solution of the given equation is $u = 6e^{-3x} \cdot e^{-2t}$ i.e., $u = 6e^{-(3x+2t)}$.

2. Use the method of separation of variables to solve the equation : $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$.

(U.P.T.U., 2005, 2009)

Sol. Here u is a function of x and y . Let $u = XY$... (1) where X is a function of x on y and Y is a function of y only.

Then,

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y$$

Substituting in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \Rightarrow (X'' - 2X')Y + XY' = 0$$

Separating the variables, we have

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} \quad \dots(2)$$

Since x and y are independent variables, equation (2) can hold only when each side is equal to some constant, say k .

$$\therefore \quad \frac{X'' - 2X'}{X} = k \quad \text{or} \quad X'' - 2X' - kX = 0 \quad \dots(3)$$

and

$$\frac{Y'}{Y} = k \quad \text{or} \quad Y' + kY = 0 \quad \dots(4)$$

These are ordinary differential equations.

For equation (3), the auxiliary equation is $m^2 - 2m - k = 0 \Rightarrow m = 1 \pm \sqrt{1+k}$

$\therefore$ Solution of equation (3) is $X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$

Also the solution of (4) is $Y = c_3 e^{-ky}$

Substituting these values of X and Y in equation (1), we get

$$u = \left\{ c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x} \right\} c_3 e^{-ky} \quad \text{or} \quad u = [Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}] \cdot e^{-ky}$$

where, $A = c_1 c_3$ and $B = c_2 c_3$.

3. Use the method of separation of variables to solve $4 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u$, $u = 3e^{-x} - e^{-5x}$, when $t = 0$.

Sol. Let $u = XT$... (1)

where, X is a function of x only and T is a function of t only.

$$\therefore \frac{\partial u}{\partial t} = XT', \quad \frac{\partial u}{\partial x} = X'T$$

$\therefore$ Given equation becomes

$$4XT' + X'T = 3XT \Rightarrow \frac{4T'}{T} + \frac{X'}{X} = 3 \quad \text{or} \quad \frac{4T'}{T} - 3 = -\frac{X'}{X} = p^2 \text{ (say)} \quad \dots (2)$$

$$(i) \quad \frac{4T'}{T} = p^2 + 3$$

$$\text{Integration yields, } \log T = \left(\frac{p^2 + 3}{4} \right) t + \log c_1 \Rightarrow T = c_1 e^{\left(\frac{p^2 + 3}{4} \right) t} \quad \dots (3)$$

$$(ii) \quad \frac{X'}{X} = -p^2 \Rightarrow X = c_2 e^{-p^2 x} \quad \dots (4)$$

$$\begin{aligned} \text{From (1), we get} \quad u = XT \quad \text{or} \quad u &= c_2 e^{-p^2 x} \cdot c_1 e^{\left(\frac{p^2 + 3}{4} \right) t} \\ &= c_1 c_2 e^{-p^2 x + \left(\frac{p^2 + 3}{4} \right) t} \quad \text{or} \quad u(x, t) = b_n e^{-p^2 x + \left(\frac{p^2 + 3}{4} \right) t} \end{aligned}$$

Most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \cdot e^{-p^2 x + \left(\frac{3+p^2}{4} \right) t} \quad \dots (5)$$

$$\text{When } t = 0, \quad u(x, 0) = 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n \cdot e^{-p^2 x}$$

Comparing, when $p^2 = 1$, $b_1 = 3$ and when $p^2 = 5$, $b_2 = -1$

Hence from equation (5), general solution is $u(x, t) = 3e^{-x+t} - e^{-5x+2t}$ which is the required solution.

4. Solve : $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$. (U.P. II Semester 2009, 2010)

Sol. Let $u = XY$... (1)

where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X'' \cdot Y$$

Substituting these values in the given equation, we get

$$\begin{aligned} X''Y &= XY' + 2XY \\ \frac{X''}{X} &= \frac{Y'}{Y} + 2 = -p^2 \text{ (say)} \end{aligned}$$

$$(i) \quad \frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

Auxiliary equation is $m^2 + p^2 = 0$

$$m = \pm pi$$

$$\therefore \text{C.F.} = c_1 \cos px + c_2 \sin px$$

$$\text{P. I.} = 0$$

$$\therefore X = c_1 \cos px + c_2 \sin px \quad \dots(2)$$

$$(ii) \quad \frac{Y'}{Y} + 2 = -p^2 \quad \text{or} \quad \frac{Y'}{Y} = -(p^2 + 2).$$

$$\text{Integration yields, } \log Y = -(p^2 + 2)y + \log c_3 \Rightarrow Y = c_3 e^{-(p^2 + 2)y} \quad \dots(3)$$

$$\therefore u(x, y) = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-(p^2 + 2)y}$$

is the required solution of the given equation.

5. Use the method of separation of variables to solve the equation: $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ given that $v = 0$ when $t \rightarrow \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.

$$\text{Sol. Let } v = XT, \frac{\partial v}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = X''T \quad \dots(1)$$

From the given equation

$$X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -p^2 \text{ (say)} \quad \dots(2)$$

$$(i) \quad \frac{T'}{T} = -p^2 \Rightarrow \log T = -p^2 t + \log c_1 \Rightarrow T = c_1 e^{-p^2 t} \quad \dots(3)$$

$$(ii) \quad \frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

Auxiliary equation is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$\text{C.F.} = c_2 \cos px + c_3 \sin px$$

$$\text{P.I.} = 0$$

$$\therefore X = c_2 \cos px + c_3 \sin px \quad \dots(4)$$

$$\text{Hence, } v = XT = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px) \quad \dots(5)$$

Putting $x = 0, v = 0$ in (5), we get

$$0 = c_1 e^{-p^2 t} \cdot c_2 \quad \therefore c_2 = 0 \quad (\because c_1 \neq 0)$$

$$\therefore \text{From (5), } v = c_1 c_3 e^{-p^2 t} \sin px \quad \dots(6)$$

At $x = l, v = 0$

$$\therefore \text{From (6), } 0 = c_1 c_3 e^{-p^2 t} \sin pl \Rightarrow \sin pl = 0 = \sin n\pi \quad (\because c_3 \neq 0)$$

$$\Rightarrow p = \frac{n\pi}{l}, n \in \mathbb{I}$$

$$\therefore \text{From (6), } v = c_1 c_3 e^{-\frac{n^2 \pi^2}{l^2} t} \cdot \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l} \cdot e^{-\left(\frac{n^2 \pi^2}{l^2}\right) t} \quad \dots(7)$$

The above equation (7) satisfies the given conditions for all values of n .

Hence the most general solution is $v = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n^2 \pi^2}{l^2}\right)t} \sin \frac{n\pi x}{l}$.

6. Solve by the method of separation of variables : $\frac{4}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} = 3u$; $u(0, y) = 4e^{-y} - e^{-5y}$.

Sol. Let $u = XY$... (1)

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY'$$

$\therefore$ Given equation becomes

$$4X'Y + XY' = 3XY$$

$$\frac{4X'}{X} + \frac{Y'}{Y} = 3 \quad \text{or} \quad \frac{4X'}{X} - 3 = -\frac{Y'}{Y} = p^2 \text{ (say)} \quad \dots (2)$$

$$(i) \quad \frac{4X'}{X} = p^2 + 3, \quad \frac{X'}{X} = \frac{p^2 + 3}{4}$$

$$\text{Integration yields} \quad \log X = \left(\frac{p^2 + 3}{4}\right)x + \log c_1 \quad \text{or} \quad X = c_1 e^{\left(\frac{p^2 + 3}{4}\right)x} \quad \dots (3)$$

$$(ii) \quad \frac{-Y'}{Y} = p^2 \Rightarrow \frac{Y'}{Y} = -p^2$$

$$\text{Integration yields} \quad \log Y = -p^2 y + \log c_2 \quad \text{or} \quad Y = c_2 e^{-p^2 y} \quad \dots (4)$$

From (1), we get $u = XY$

$$u = c_1 c_2 e^{-p^2 y} \cdot e^{\left(\frac{p^2 + 3}{4}\right)x} = c_1 c_2 e^{-p^2 y + \left(\frac{p^2 + 3}{4}\right)x}$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-p^2 y + \left(\frac{p^2 + 3}{4}\right)x} \quad \dots (5)$$

$$\text{When } x = 0, \quad u(0, y) = 4e^{-y} - e^{-5y} = \sum b_n e^{-p^2 y}$$

$$\text{Comparing when } p^2 = 1, \quad b_1 = 4$$

$$\text{when } p^2 = 5, \quad b_2 = -1$$

Hence from (5), general solution is $u(x, y) = 4e^{x-y} - e^{2x-5y}$.

7. Solve by the method of separation of variables $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$; $u(x, 0) = 10e^{-x} - 6e^{-4x}$.

$$\text{Sol. Given equation is } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u \quad \dots (1)$$

$$\text{Let } u = XT \quad \dots (2)$$

$$\frac{\partial u}{\partial t} = XT', \quad \frac{\partial u}{\partial x} = X'T \quad \therefore XT' = X'T - 2XT$$

$$\frac{T'}{T} = \frac{X'}{X} - 2 = -p^2 \text{ (say)}$$

$$(i) \quad \frac{T'}{T} = -p^2 \Rightarrow \log T = -p^2 t + \log c_1 \Rightarrow T = c_1 e^{-p^2 t} \quad \dots(3)$$

$$(ii) \quad \frac{X'}{X} - 2 = -p^2 \quad \text{or} \quad \frac{X'}{X} = -p^2 + 2$$

$$\log X = (2 - p^2)x + \log c_2$$

$$X = c_2 e^{(2-p^2)x} \quad \dots(4)$$

$$\therefore u = c_1 c_2 e^{-p^2 t + (2-p^2)x}$$

$$\text{When } t = 0, \quad u(x, 0) = 10e^{-x} - 6e^{-4x} = \sum_{n=1}^{\infty} b_n e^{(2-p^2)x}$$

$$\text{Comparing, when } \begin{matrix} p^2 = 3, & b_1 = 10 \\ p^2 = 6, & b_2 = -6 \end{matrix}$$

$$\text{Hence general solution is } u(x, t) = 10e^{-3t-x} - 6e^{-6t-4x} = 10e^{-(x+3t)} - 6e^{-2(3t+2x)}.$$

8. What is one-dimensional wave equation giving the transverse vibrations of a stretched string? What are the boundary conditions? What are the initial conditions? What is the solution of the wave equation?

Sol. The equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is called the one dimensional wave equation. The boundary conditions, which the wave equation has to satisfy are :

$$(i) \quad y = 0, \quad \text{when } x = 0 \quad (ii) \quad y = 0, \quad \text{when } x = l.$$

These should be satisfied for every value of t . If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the initial conditions are :

$$(i) \quad y = f(x), \quad \text{when } t = 0 \quad (ii) \quad \frac{\partial y}{\partial t} = 0, \quad \text{when } t = 0.$$

Solution of the wave equation is

$$y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{where, } a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

when $f(x)$ i.e., $y(x, 0)$ is known.

9. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any

$$\text{point at a distance } x \text{ from one end at time } t \text{ is given by } y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}.$$

(U.P., II Sem. 2009)

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$ and the initial conditions are

$$y(x, 0) = a \sin \frac{\pi x}{l} \quad \text{and} \quad \frac{\partial y}{\partial t} = 0, \quad \text{when } t = 0.$$

Solution of the wave equation (as stated in the previous question) is

$$y(x, t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

where

$$a_n = \frac{2}{l} \int_0^l y(x, 0) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^l a \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{2a}{l} \int_0^l \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx$$

which vanishes for all values of n except $n = 1$.

$$\therefore a_1 = \frac{2a}{l} \int_0^l \sin^2 \frac{\pi x}{l} dx = \frac{a}{l} \int_0^l \left(1 - \cos \frac{2\pi x}{l}\right) dx = \frac{a}{l} \left[x - \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right]_0^l = a$$

Hence the required solution is $y(x, t) = a \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l}$.

- 10.** A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t .

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$. (U.P., III Semester, 2008)

We have,
$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \dots(1)$$

Since the string was at rest initially, $y(x, 0) = 0$

$$\therefore \text{ From (1), } 0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \Rightarrow a_n = 0$$

$$\therefore y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(2)$$

and
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But
$$\frac{\partial y}{\partial t} = \lambda x(l - x) \quad \text{when } t = 0$$

$$\therefore \lambda x(l - x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

$$\begin{aligned} \Rightarrow \frac{\pi c}{l} n b_n &= \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} \cdot dx \\ &= \frac{2\lambda}{l} \left[x(l - x) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l - 2x) \left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2) \left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right]_0^l \\ &= \frac{4\lambda l^2}{n^3\pi^3} (1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n] \\ &= 0, \text{ when } n \text{ is even} \\ &= \frac{8\lambda l^2}{n^3\pi^3}, \text{ when } n \text{ is odd.} \end{aligned}$$

i.e.,
$$\frac{8\lambda l^2}{\pi^3(2m-1)^3}, \text{ taking } n = 2m - 1$$

$$\Rightarrow b_n = \frac{8\lambda l^3}{c\pi^4(2m-1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi ct}{l} \cdot \sin \frac{(2m-1)\pi x}{l}.$$

11. Show how the wave equation $c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ can be solved by the method of separation of variables.

If the initial displacement and velocity of a string stretched between $x = 0$ and $x = l$ are given by $y = f(x)$ and $\frac{\partial y}{\partial t} = g(x)$, determine the constants in the series solution. (U.P.T.U., 2005)

Sol. The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (1)

Let $y = XT$... (2)

where, X is a function of x only and T is a function of t only.

$$\frac{\partial^2 y}{\partial t^2} = XT'', \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in equation (1), we get

$$XT'' = c^2 X''T \Rightarrow \frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} \quad \dots (3)$$

Case I. When $\frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} = p^2$ (say)

(i) $\frac{1}{c^2} \frac{T''}{T} = p^2 \Rightarrow \frac{d^2 T}{dt^2} - p^2 c^2 T = 0$

Auxiliary equation is

$$m^2 - p^2 c^2 = 0 \Rightarrow m = \pm pc$$

$$\text{C.F.} = c_1 e^{pct} + c_2 \cdot e^{-pct}$$

$$\text{P.I.} = 0$$

$\therefore T = \text{C.F.} + \text{P.I.} = c_1 e^{pct} + c_2 \cdot e^{-pct}$

(ii) $\frac{X''}{X} = p^2 \Rightarrow \frac{d^2 X}{dx^2} - p^2 X = 0$

Auxiliary equation is $m^2 - p^2 = 0$ or $m = \pm p$

$\therefore X = c_3 e^{px} + c_4 e^{-px}$

Hence the solution is $y = XT = (c_1 e^{pct} + c_2 e^{-pct}) (c_3 e^{px} + c_4 e^{-px})$... (4)

Case II. When $\frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} = -p^2$ (say)

(i) $\frac{1}{c^2} \cdot \frac{T''}{T} = -p^2 \Rightarrow \frac{d^2 T}{dt^2} + p^2 c^2 T = 0$

Auxiliary equation is $m^2 + p^2 c^2 = 0 \Rightarrow m = \pm pci$

$\therefore T = c_5 \cos pct + c_6 \sin pct$

(ii) $\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2 X}{dx^2} + p^2 X = 0$

Auxiliary equation is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$\therefore X = c_7 \cos px + c_8 \sin px$

Hence the solution is $y = XT = (c_5 \cos pct + c_6 \sin pct) (c_7 \cos px + c_8 \sin px)$... (5)

Case III. When $\frac{1}{c^2} \cdot \frac{T''}{T} = \frac{X''}{X} = 0$

$$(i) \frac{1}{c^2} \cdot \frac{T''}{T} = 0 \Rightarrow \frac{d^2 T}{dt^2} = 0 \text{ or } T = c_9 + c_{10} t$$

$$(ii) \frac{X''}{X} = 0 \Rightarrow \frac{d^2 X}{dx^2} = 0 \text{ or } X = c_{11} + c_{12} \cdot x$$

$$\text{Hence the solution is } y(x, t) = (c_9 + c_{10} t) (c_{11} + c_{12} x) \quad \dots(6)$$

of the above three solutions given by (4), (5) and (6), we have to choose the solution which is consistent with the physical nature of the problem. Since we are dealing with a problem on vibrations, y must be a periodic function of x and t . Therefore the solution must have trigonometric terms. Hence solution (5) is the required solution.

Boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$

$$y = f(x) \text{ when } t = 0$$

$$\frac{\partial y}{\partial t} = g(x) \text{ when } t = 0$$

$$\text{From equation (5), } y(0, t) = (c_5 \cos cpt + c_6 \sin cpt) c_7$$

$$0 = (c_5 \cos cpt + c_6 \sin cpt) c_7$$

$$\Rightarrow c_7 = 0$$

$$\text{Hence from (5), } y(x, t) = (c_5 \cos cpt + c_6 \sin cpt) c_8 \sin px \quad \dots(7)$$

$$y(l, t) = 0 = (c_5 \cos cpt + c_6 \sin cpt) c_8 \sin pl$$

$$\Rightarrow \sin pl = 0 = \sin n\pi \quad (n \in \mathbb{I}) \Rightarrow p = \frac{n\pi}{l}$$

$$\therefore \text{ From (7), } y(x, t) = \left(c_5 \cos \frac{n\pi ct}{l} + c_6 \sin \frac{n\pi ct}{l} \right) c_8 \sin \frac{n\pi x}{l} \quad \dots(8)$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$$\text{where } c_5 c_8 = a_n \text{ and } c_6 c_8 = b_n.$$

$$\text{The general solution is } y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \dots(9)$$

$$y(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx \quad \dots(10)$$

$$\text{From (9), } \frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_1^{\infty} \left(-na_n \sin \frac{n\pi ct}{l} + nb_n \cos \frac{n\pi ct}{l} \right) \cdot \sin \frac{n\pi x}{l}$$

$$\text{At } t = 0, \left(\frac{\partial y}{\partial t} \right)_{t=0} = g(x) = \frac{\pi c}{l} \sum_1^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\text{where } \frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \cdot dx.$$

$$\Rightarrow b_n = \frac{2}{n\pi c} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \cdot dx \quad \dots(11)$$

Hence the required solution is

$$y(x, t) = \sum_1^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

where

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx \quad \text{and} \quad b_n = \frac{2}{n\pi c} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \cdot dx.$$

- 12.** A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

(U.P.T.U., 2006)

Sol. The equation of the string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (1)

The solution of equation (1) is $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px)$... (2)

Boundary conditions are $y(0, t) = 0$... (3)

$y(l, t) = 0$... (4)

$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$... (5)

$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l}$... (6)

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_3 \Rightarrow c_3 = 0$$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px$... (7)

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$

$\Rightarrow \sin pl = 0 = \sin n\pi \quad (n \in \mathbb{I})$

$\therefore p = \frac{n\pi}{l}$

Hence from (7), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l} \right) c_4 \sin \frac{n\pi x}{l}$... (8)

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l} \left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l} \right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 = \frac{n\pi c}{l} \cdot c_2 c_4 \sin \frac{n\pi x}{l} \Rightarrow c_2 = 0$

$\therefore$ From (8), $y(x, t) = c_1 c_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$ or $y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$

Most general solution is $y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$... (9)

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$\Rightarrow y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \dots$

Comparing, we get $b_1 = \frac{3y_0}{4}$, $b_2 = 0$, $b_3 = -\frac{y_0}{4}$, $b_4 = b_5 = 0$

Hence, from (9), $y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$

which is the required displacement.

13. A string is stretched between two fixed points $(0, 0)$ and $(l, 0)$ and released at rest from the initial

$$\text{deflection given by } f(x) = \begin{cases} \left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

Find the deflection of the string at any time.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

Solution of equation (1) is $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px)$...(2)

Boundary conditions are $y(0, t) = 0$

$$y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0$$

$$y(x, 0) = \begin{cases} \left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

From (2),

$$y(0, t) = c_3 (c_1 \cos cpt + c_2 \sin cpt)$$

$$0 = c_3 (c_1 \cos cpt + c_2 \sin cpt)$$

$\Rightarrow$

$$c_3 = 0$$

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px \quad \dots(3)$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$$

$\Rightarrow$

$$\sin pl = 0 = \sin n\pi; n \in \mathbb{I}$$

$$p = \frac{n\pi}{l}$$

$\therefore$ From (3),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l} \right) c_4 \sin \frac{n\pi x}{l} \quad \dots(4)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l} \left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l} \right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 = \frac{n\pi c}{l} \left[c_2 c_4 \sin \frac{n\pi x}{l} \right] \Rightarrow c_2 = 0$$

$\therefore$ From (4),

$$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(5)$$

where $c_1 c_4 = b_n$.

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(6)$$

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l} \int_0^l y(x, 0) \sin \frac{n\pi x}{l} \cdot dx$$

$$= \frac{2}{l} \left[\int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l} dx + \int_{l/2}^l \frac{2k}{l} (l-x) \sin \frac{n\pi x}{l} \cdot dx \right]$$

$$= \frac{4k}{l^2} \left[\int_0^{l/2} x \sin \frac{n\pi x}{l} dx + \int_{l/2}^l (l-x) \sin \frac{n\pi x}{l} \cdot dx \right]$$

$$= \frac{4k}{l^2} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_0^{l/2} - \int_0^{l/2} 1 \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx$$

$$+ \left\{ (l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_{l/2}^l - \int_{l/2}^l (-1) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$= \frac{4k}{l^2} \left[-\frac{l}{n\pi} \cdot \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_0^{l/2} + \frac{l}{2} \cdot \frac{l}{n\pi} \cos \frac{n\pi}{2} - \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_{l/2}^l$$

$$= \frac{4k}{l^2} \left[\frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{l^2}{n^2 \pi^2} \left(\sin n\pi - \sin \frac{n\pi}{2} \right) \right]$$

$$= \frac{4k}{l^2} \left[\frac{2l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right] = \frac{8k}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

$\therefore$ From (6),
$$y(x, t) = \frac{8k}{\pi^2} \sum \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

14. A tightly stretched violin string of length l fixed at both ends, is plucked at $x = \frac{l}{3}$ and assumes initially the shape of a triangle of height a . Find the displacement y at any distance x and any time t after the string is released from rest.

Sol. One dimensional wave equation is

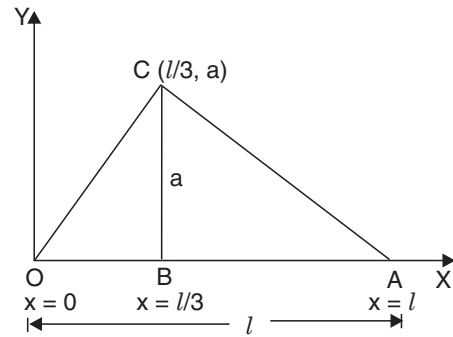
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

The solution of equation (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px) \quad \dots(2)$$

Equation of line OC is

$$y - 0 = \frac{a - 0}{\frac{l}{3} - 0} (x - 0) \quad \text{or} \quad y = \frac{3a}{l} x \quad \dots(3)$$



Equation of line CA is
$$y - a = \frac{0 - a}{l - \frac{l}{3}} \left(x - \frac{l}{3} \right) \quad \text{or} \quad y - a = -\frac{3a}{2l} \left(x - \frac{l}{3} \right)$$

or
$$y - a = -\frac{3ax}{2l} + \frac{a}{2} \quad \text{or} \quad y = -\frac{3ax}{2l} + \frac{3a}{2} \quad \text{or} \quad y = \frac{3a}{2} \left(1 - \frac{x}{l} \right) \quad \dots(4)$$

Hence the boundary conditions are

$$y(0, t) = 0 \quad \dots(5)$$

$$y(l, t) = 0 \quad \dots(6)$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \quad \dots(7)$$

and

$$y(x, 0) = \begin{cases} \frac{3ax}{l}, & 0 < x < \frac{l}{3} \\ \frac{3a}{2} \left(1 - \frac{x}{l}\right), & \frac{l}{3} < x < l \end{cases} \quad \dots(8)$$

From (2), $y(0, t) = 0 = c_3 (c_1 \cos cpt + c_2 \sin cpt) \Rightarrow c_3 = 0$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px \quad \dots(9)$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$$

$\Rightarrow \sin pl = 0 = \sin n\pi \ (n \in \mathbb{I}) \quad \Rightarrow \quad p = \frac{n\pi}{l}$

$\therefore y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l} \right) c_4 \sin \frac{n\pi x}{l} \quad \dots(10)$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l} \left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l} \right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $\frac{\partial y}{\partial t} = 0 = \frac{n\pi c}{l} \left[c_2 c_4 \sin \frac{n\pi x}{l} \right] \Rightarrow c_2 = 0$

$\therefore y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

The most general solution is

$$y(x, t) = \sum_1^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(11)$$

From (11),

$$\begin{aligned} y(x, 0) &= \sum_1^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where} \\ b_n &= \frac{2}{l} \int_0^l y(x, 0) \sin \frac{n\pi x}{l} \cdot dx \\ &= \frac{2}{l} \left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l} \cdot dx + \int_{l/3}^l \frac{3a}{2} \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} \cdot dx \right] \\ &= \frac{2}{l} \left[\frac{3a}{l} \int_0^{l/3} x \sin \frac{n\pi x}{l} \cdot dx + \frac{3a}{2} \int_{l/3}^l \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} \cdot dx \right] \\ &= \frac{6a}{l^2} \left[\left(x \left\{ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right\} \right) \Big|_0^{l/3} - \int_0^{l/3} -\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \cdot dx \right] \\ &\quad + \frac{3a}{l} \left[\left(\left(1 - \frac{x}{l}\right) \left\{ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right\} \right) \Big|_{l/3}^l - \int_{l/3}^l \left(-\frac{1}{l}\right) \left\{ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right\} dx \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{6a}{l^2} \left[\frac{-l}{n\pi} \cdot \frac{l}{3} \cos \frac{n\pi}{3} + \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_{l/3}^{l/3} \right] + \frac{3a}{l} \left[\frac{l}{n\pi} \cdot \frac{2}{3} \cos \frac{n\pi}{3} - \frac{1}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_{l/3}^l \right] \\
&= \frac{6a}{l^2} \left[\frac{-l^2}{3n\pi} \cos \frac{n\pi}{3} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{3} \right] + \frac{3a}{l} \left[\frac{2l}{3n\pi} \cos \frac{n\pi}{3} + \frac{l}{n^2\pi^2} \cdot \sin \frac{n\pi}{3} \right] \\
&= \frac{6a}{n\pi} \left[-\frac{1}{3} \cos \frac{n\pi}{3} + \frac{1}{n\pi} \sin \frac{n\pi}{3} \right] + \frac{6a}{n\pi} \left(\frac{1}{3} \cos \frac{n\pi}{3} \right) + \frac{3a}{n^2\pi^2} \sin \frac{n\pi}{3} \\
&= \frac{6a}{n^2\pi^2} \sin \frac{n\pi}{3} + \frac{3a}{n^2\pi^2} \sin \frac{n\pi}{3} \quad \Rightarrow \quad b_n = \frac{9a}{n^2\pi^2} \sin \frac{n\pi}{3}
\end{aligned}$$

$$\therefore \text{ From (11), } y(x, t) = \frac{9a}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \cdot \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

which is the required expression for displacement at any distance x and any time t .

15. The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

Sol. The equation for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

The solution of equation (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px) \quad \dots(2)$$

Let l be the length of string.

$$\text{Equation of OB is } y = \frac{h-0}{\frac{l}{3}-0} (x-0)$$

$$\Rightarrow y = \frac{3h}{l} \cdot x \quad \dots(3)$$

$$\text{Equation of BC is } y-h = \frac{-2h}{\left(\frac{l}{3}\right)} \left(x-\frac{l}{3}\right)$$

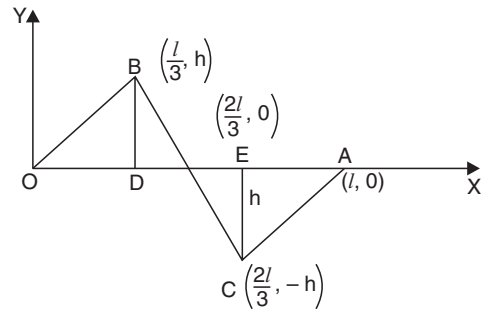
$$\text{or } y-h = -\frac{6h}{l} \left(x-\frac{l}{3}\right)$$

$$\Rightarrow y = 3h \left(1 - \frac{2x}{l}\right) \quad \dots(4)$$

$$\text{Equation of CA is } y+h = \frac{h}{l-\frac{2l}{3}} \left(x-\frac{2l}{3}\right) = \frac{3h}{l} \left(x-\frac{2l}{3}\right) = \frac{3hx}{l} - 2h \Rightarrow y = 3h \left(\frac{x}{l} - 1\right) \quad \dots(5)$$

Hence boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$, $\frac{\partial y}{\partial t} = 0$ when $t = 0$

$$\text{and } y(x, 0) = \begin{cases} \frac{3h}{l} x, & 0 \leq x \leq \frac{l}{3} \\ \frac{3h}{l} (l-2x), & \frac{l}{3} \leq x \leq \frac{2l}{3} \\ \frac{3h}{l} (x-l), & \frac{2l}{3} \leq x \leq l \end{cases}$$



$$\begin{aligned}
\text{From (2),} \quad & y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_3 \Rightarrow c_3 = 0 \\
\therefore \text{ From (2),} \quad & y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px \quad \dots(6) \\
& y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl \\
\Rightarrow \quad & \sin pl = 0 = \sin n\pi \quad (n \in \mathbb{I})
\end{aligned}$$

$$\begin{aligned}
\therefore \quad & p = \frac{n\pi}{l} \\
\therefore \text{ From (6),} \quad & y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l} \right) c_4 \sin \frac{n\pi x}{l} \quad \dots(7)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial y}{\partial t} = \frac{n\pi c}{l} \left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l} \right) c_4 \sin \frac{n\pi x}{l} \\
\text{At } t = 0, \quad & \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l} \Rightarrow c_2 = 0
\end{aligned}$$

$$\therefore \text{ From (7),} \quad y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(8)$$

$$y(x, 0) = \sum b_n \sin \frac{n\pi x}{l},$$

$$\begin{aligned}
\text{where } b_n &= \frac{2}{l} \int_0^l y(x, 0) \sin \frac{n\pi x}{l} \cdot dx \\
&= \frac{2}{l} \left[\int_0^{l/3} \frac{3h}{l} x \sin \frac{n\pi x}{l} dx + \int_{l/3}^{2l/3} \frac{3h}{l} (l-2x) \sin \frac{n\pi x}{l} \cdot dx + \int_{2l/3}^l \frac{3h}{l} (x-l) \sin \frac{n\pi x}{l} dx \right] \\
&= \frac{2}{l} \cdot \frac{3h}{l} \int_0^{l/3} x \sin \frac{n\pi x}{l} dx + \frac{2}{l} \cdot \frac{3h}{l} \int_{l/3}^{2l/3} (l-2x) \sin \frac{n\pi x}{l} dx \\
&\quad + \frac{2}{l} \cdot \frac{3h}{l} \int_{2l/3}^l (x-l) \sin \frac{n\pi x}{l} dx \\
&= -\frac{2h}{n\pi} \cos \frac{n\pi}{3} + \frac{6h}{n^2\pi^2} \sin \frac{n\pi}{3} + \frac{2h}{n\pi} \cos \frac{2n\pi}{3} + \frac{2h}{n\pi} \cos \frac{n\pi}{3} \\
&\quad - \frac{12h}{n^2\pi^2} \left(\sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right) - \frac{2h}{n\pi} \cos \frac{2n\pi}{3} + \frac{6h}{n^2\pi^2} \left(0 - \sin \frac{2n\pi}{3} \right) \\
&= \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} - \frac{18h}{n^2\pi^2} \sin \frac{2n\pi}{3} = \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} - \frac{18h}{n^2\pi^2} \sin \left(n\pi - \frac{n\pi}{3} \right) \\
&= \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} + \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} \cos n\pi = \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} (1 + \cos n\pi) \\
&= \begin{cases} \frac{36h}{n^2\pi^2} \sin \frac{n\pi}{3}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ From (8),} \quad & y(x, t) = \frac{36h}{\pi^2} \sum_{n=2,4,\dots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \\
&= \frac{9h}{\pi^2} \sum_{m=1,2,\dots}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cos \frac{2m\pi ct}{l} \cdot \sin \frac{2m\pi x}{l} \quad \dots(9)
\end{aligned}$$

(where $n = 2m$) which is the required expression for $y(x, t)$.

Putting $x = \frac{l}{2}$ in equation (6), we get

$$y\left(\frac{l}{2}, t\right) = \frac{9h}{\pi^2} \sum_{m=1}^{\infty} \sin \frac{2m\pi}{3} \cdot \frac{1}{m^2} \cos \frac{2m\pi ct}{l} \sin m\pi = 0$$

Hence mid point of the string is always at rest.

- 16.** If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, find the displacement $y(x, t)$.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

The solution of equation (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px) \quad \dots(2)$$

Boundary conditions are,

$$y(0, t) = 0 \quad \dots(3) \quad y(l, t) = 0 \quad \dots(4) \quad y(x, 0) = 0 \quad \dots(5)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l} \quad \text{at } t = 0 \quad \dots(6)$$

From equation (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$$

$$\Rightarrow \sin pl = 0 = \sin n\pi \quad (n \in \mathbb{I}) \quad \therefore p = \frac{n\pi}{l}$$

$$\therefore \text{ From (7), } y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \dots(8)$$

$$y(x, 0) = 0 = c_1 c_4 \sin \frac{n\pi x}{l} \quad \Rightarrow c_1 = 0$$

$$\therefore \text{ From (8), } y(x, t) = c_2 c_4 \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ where } c_2 c_4 = b_n$$

The general solution is

$$y(x, t) = \sum_1^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \dots(9)$$

$$\frac{\partial y}{\partial t} = \sum b_n \cdot \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$\text{At } t = 0, \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum b_n \cdot \frac{n\pi c}{l} \cdot \sin \frac{n\pi x}{l}$$

$$b \sin^3 \frac{\pi x}{l} = \sum b_n \cdot \frac{n\pi c}{l} \cdot \sin \frac{n\pi x}{l}$$

$$\frac{b}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2b_2 \pi c}{l} \sin \frac{2\pi x}{l} + 3b_3 \cdot \frac{\pi c}{l} \sin \frac{3\pi x}{l} + \dots$$

$$\Rightarrow \frac{b_1 \pi c}{l} = \frac{3b}{4} \Rightarrow b_1 = \frac{3bl}{4\pi c}, b_2 = 0$$

$$b_3 = \frac{-bl}{12\pi c}, b_4 = 0 = b_5 = \dots \text{ etc.}$$

$$\begin{aligned} \text{Hence from (9), } y(x, t) &= \frac{3bl}{4\pi c} \sin \frac{\pi ct}{l} \cdot \sin \frac{\pi x}{l} - \frac{bl}{12\pi c} \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l} \\ &= \frac{bl}{12\pi c} \left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l} \right]. \end{aligned}$$

17. Transform the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ to its normal form using the transformation $u = x + ct$,

$v = x - ct$ and hence solve it. Show that the solution may be put in the form $y = \frac{1}{2} [f(x + ct)$

$+ f(x - ct)]$. Assume initial conditions $y = f(x)$ and $\left(\frac{\partial y}{\partial t}\right) = 0$ at $t = 0$.

Sol. One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

Let us introduce two new independent variables

$$u = x + ct \quad \dots(2)$$

$$v = x - ct \quad \dots(3)$$

So that y becomes a function of u and v .

$$\text{Then, } \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \quad \dots(4) \quad [\text{Using (2) and (3)}]$$

$$\text{Also, } \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \dots(5)$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad \dots(6)$$

$$\text{Also, } \frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial t} = c \cdot \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} (-c) = c \left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right) \quad \dots(7)$$

$$\Rightarrow \frac{\partial}{\partial t} \equiv c \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \quad \dots(8)$$

$$\begin{aligned} \therefore \frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) = c \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) c \left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right) \\ &= c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \quad \dots(9) \end{aligned}$$

From (1), (6) and (9), we have

$$\begin{aligned} c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) &= c^2 \left(\frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \\ \Rightarrow -4c^2 \frac{\partial^2 y}{\partial u \partial v} &= 0 \quad \Rightarrow \frac{\partial^2 y}{\partial u \partial v} = 0 \quad \dots(10) \quad (\because c^2 \neq 0) \end{aligned}$$

Integrating equation (10) partially, w.r.t. u , we get

$$\frac{\partial y}{\partial v} = f_1(v)$$

Integrating again w.r.t. v partially, we get

$$y = \int f_1(v) \partial v + \psi(u) = \phi(v) + \psi(u)$$

$$\Rightarrow y(x, t) = \phi(x - ct) + \psi(x + ct) \quad \dots(11)$$

which is D' Alembert's solution of wave equation.

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$ in (11), we get

$$f(x) = \phi(x) + \psi(x) \quad \text{and} \quad 0 = -\phi'(x) + \psi'(x)$$

Hence, $\phi(x) = \psi(x) = \frac{1}{2} f(x) \quad \therefore y = \frac{1}{2} [f(x + ct) + f(x - ct)]$

- 18.** A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$ then find the displacement $y(x, t)$ at any point of string at any time t .

Sol. The equation for the vibrations of a string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

Its solution is $y(x, t) = (c_1 \cos pct + c_2 \sin pct) (c_3 \cos px + c_4 \sin px) \quad \dots(2)$

Boundary conditions are $y(0, t) = 0 = y(\pi, t)$, $y(x, 0) = 0$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$

From (2), $y(0, t) = 0 = (c_1 \cos pct + c_2 \sin pct) c_3 \Rightarrow c_3 = 0$

From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px \quad \dots(3)$

$y(\pi, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin p\pi$

$\Rightarrow \sin p\pi = 0 = \sin n\pi \quad (n \in \mathbb{I}) \Rightarrow p = n$

From (3), $y(x, t) = (c_1 \cos nct + c_2 \sin nct) c_4 \sin nx \quad \dots(4)$

$y(x, 0) = 0 = c_1 c_4 \sin nx \Rightarrow c_1 = 0$

$\therefore$ From (4), $y(x, t) = c_2 c_4 \sin nct \sin nx$ or $y(x, t) = b_n \sin nct \sin nx \quad \dots(5)$

where, $c_2 c_4 = b_n$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx \quad \dots(6)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} nc b_n \cos nct \sin nx$$

At $t = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} nc b_n \sin nx$

$0.03 \sin x - 0.04 \sin 3x = cb_1 \sin x + 2cb_2 \sin 2x + 3cb_3 \sin 3x + \dots$

$\Rightarrow cb_1 = 0.03 \Rightarrow b_1 = \frac{0.03}{c}, b_2 = 0, 3cb_3 = -0.04 \Rightarrow b_3 = \frac{-0.0133}{c}$

$\therefore$ From (6), $y(x, t) = \frac{0.03}{c} \sin ct \sin x - \frac{0.0133}{c} \sin 3ct \sin 3x$

$$= \frac{1}{c} [0.03 \sin ct \sin x - 0.0133 \sin 3x \sin 3ct]$$

which is the required expression.

19. Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

Sol. By D' Alembert's method, the solution is

$$y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] \quad [\text{See question 17}]$$

$$\begin{aligned} y(x, t) &= \frac{1}{2} [k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}] \\ &= k [\sin x \cos ct - \sin 2x \cos 2ct] \end{aligned}$$

Also $y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

20. What is a two-dimensional wave equation? Give its solution.

Sol. The wave equation in two dimensions is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

Its solution is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \quad \dots(2)$$

where
$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx \quad \dots(3)$$

and
$$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad \dots(4), m \text{ and } n \text{ are integers}$$

Equation (1) is the equation for the vibrations of a tightly stretched membrane (which is rectangular and is stretched between the lines $x = 0, x = a, y = 0$ and $y = b$).

21. Find the deflection $u(x, y, t)$ of the square membrane with $a = b = c = 1$, if the initial velocity is zero and the initial deflection $f(x, y) = A \sin \pi x \sin 2\pi y$.

Sol. The vibrations of the square membrane are governed by two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

Here the boundary conditions are $u(0, y, t) = 0, u(1, y, t) = 0, u(x, 0, t) = 0, u(x, 1, t) = 0$

and the initial conditions are $u(x, y, 0) = f(x, y) = A \sin \pi x \sin 2\pi y$ and $\left(\frac{\partial u}{\partial t} \right)_{t=0} = 0$

To solve equation (1),

Let
$$u = XYT \quad \dots(2)$$

where, X is a function of x only, Y is a function of y only,

and T is a function of t only.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2} (XYT) = XYT''$$

$$\frac{\partial^2 u}{\partial x^2} = X''YT, \quad \frac{\partial^2 u}{\partial y^2} = XY''T$$

Substituting these values in equation (1), we have

$$XYT'' = c^2 (X''YT + XY''T) \quad \Rightarrow \quad \frac{1}{c^2} \cdot \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

This will be true only when each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0, \quad \frac{d^2T}{dt^2} + (k^2 + l^2) c^2T = 0$$

The solutions of these equations are respectively,

$$X = c_1 \cos kx + c_2 \sin kx, \quad Y = c_3 \cos ly + c_4 \sin ly$$

$$T = c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct.$$

$$\therefore u(x, y, t) = (c_1 \cos kx + c_2 \sin kx) (c_3 \cos ly + c_4 \sin ly) (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \quad \dots(3)$$

$$\text{From (3), } u(0, y, t) = 0 = c_1 (c_3 \cos ly + c_4 \sin ly) (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \\ \Rightarrow c_1 = 0$$

$\therefore$ From (3),

$$u(x, y, t) = c_2 \sin kx (c_3 \cos ly + c_4 \sin ly) (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \quad \dots(4)$$

$$u(1, y, t) = 0 = c_2 \sin k (c_3 \cos ly + c_4 \sin ly) (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \\ \Rightarrow \sin k = 0 = \sin m\pi \quad (m \in \mathbb{I}) \quad \text{or} \quad k = m\pi$$

$$\therefore u(x, y, t) = c_2 \sin m\pi x (c_3 \cos ly + c_4 \sin ly) (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \quad \dots(5)$$

$$u(x, 0, t) = 0 = c_2 \sin m\pi x \cdot c_3 (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \Rightarrow c_3 = 0$$

$$\therefore \text{From (5), } u(x, y, t) = c_2 c_4 \sin m\pi x \sin ly (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \quad \dots(6)$$

$$u(x, 1, t) = 0 = c_2 c_4 \sin m\pi x \sin l (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \\ \Rightarrow \sin l = 0 = \sin n\pi \quad (n \in \mathbb{I}) \quad \Rightarrow l = n\pi$$

$$\therefore \text{From (6), } u(x, y, t) = c_2 c_4 \sin m\pi x \sin n\pi y (c_5 \cos \sqrt{k^2 + l^2} \cdot ct + c_6 \sin \sqrt{k^2 + l^2} \cdot ct) \\ u(x, y, t) = \sin m\pi x \sin n\pi y (A_{mn} \cos pt + B_{mn} \sin pt) \quad \dots(7)$$

$$\text{where } p = \pi c \sqrt{m^2 + n^2}$$

The most general solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin m\pi x \sin n\pi y (A_{mn} \cos pt + B_{mn} \sin pt) \quad \dots(8)$$

$$\text{Applying the condition } \frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \text{ we get } B_{mn} = 0 \quad \dots(9)$$

Also, using the condition $u = f(x, y) = A \sin \pi x \sin 2\pi y$

$$\text{When } t = 0, \text{ we get } A \sin \pi x \sin 2\pi y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y$$

This is a double Fourier Series.

$$A_{mn} = \frac{2}{1} \cdot \frac{2}{1} \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dx \, dy$$

$$\text{Obviously } A_{m_1} = A_{m_3} = A_{m_4} = \dots = 0$$

$$\begin{aligned}
\text{But,} \quad A_{m_2} &= 4A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin^2 2\pi y \cdot dx dy \\
&= 2A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x (1 - \cos 4\pi y) dx dy \\
&= 2A \int_0^1 \sin \pi x \sin m\pi x \left(y - \frac{\sin 4\pi y}{4\pi} \right)_0^1 \cdot dx \\
&= 2A \int_0^1 \sin \pi x \sin m\pi x dx
\end{aligned}$$

$$\text{Again obviously,} \quad A_{22} = A_{32} = A_{42} = \dots = 0$$

$$\begin{aligned}
\text{But,} \quad A_{12} &= 2A \int_0^1 \sin^2 \pi x dx = A \int_0^1 (1 - \cos 2\pi x) dx \\
&= A \left(x - \frac{\sin 2\pi x}{2\pi} \right)_0^1 = A
\end{aligned} \tag{10}$$

$\therefore$ From (8), (9) and (10), we get

$$u(x, y, t) = A \sin \pi x \sin 2\pi y \cos pt \tag{11}$$

$$\text{where,} \quad p = \pi c \sqrt{m^2 + n^2} = \pi(1) \sqrt{1 + 4} = \pi\sqrt{5}$$

$\therefore$ From (11), $u(x, y, t) = A \cos \pi\sqrt{5} t \sin \pi x \sin 2\pi y$ which is the required deflection.

22. Find the deflection $u(x, y, t)$ of a rectangular membrane ($0 \leq x \leq a$, $0 \leq y \leq b$) whose boundary is fixed ; given that it starts from rest and $u(x, y, 0) = xy(a - x)(b - y)$. Show that the deflection u is

$$\text{given by} \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\text{where} \quad A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6} (1 - \cos m\pi) (1 - \cos n\pi) \text{ and } k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

Sol. As given in question 20, we have

$$u(x, y, t) = \sum \sum A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

$$\text{where, } p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{1}$$

$$\text{From (1),} \quad u(x, y, 0) = \sum \sum A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$xy(a - x)(b - y) = \sum \sum A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\text{where, } A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b xy(a - x)(b - y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx$$

$$= \frac{4}{ab} \int_0^a x(a - x) \sin \frac{n\pi x}{a} dx \cdot \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy$$

$$\begin{aligned}
&= \frac{4}{ab} \left[x(a-x) \left(\frac{-\cos \frac{m\pi x}{a}}{\frac{m\pi}{a}} \right) - (a-2x) \left(\frac{-\sin \frac{m\pi x}{a}}{\frac{m^2\pi^2}{a^2}} \right) + (-2) \left(\frac{\cos \frac{m\pi x}{a}}{\frac{m^3\pi^3}{a^3}} \right) \right]_0^a \\
&\quad \times \left[y(b-y) \left(\frac{-\cos \frac{n\pi y}{b}}{\frac{n\pi}{b}} \right) - (b-2y) \left(\frac{-\sin \frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}} \right) - 2 \left(\frac{\cos \frac{n\pi y}{b}}{\frac{n^3\pi^3}{b^3}} \right) \right]_0^b \\
&= \frac{4}{ab} \left[\frac{-2a^3}{m^3\pi^3} \cos m\pi + \frac{2a^3}{m^3\pi^3} \right] \left[\frac{-2b^3}{n^3\pi^3} \cos n\pi + \frac{2b^3}{n^3\pi^3} \right] \\
&= \frac{4}{ab} \cdot \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} [1 - (-1)^n] [1 - (-1)^m] \\
A_{mn} &= \frac{16a^2b^2}{m^3n^3\pi^6} [1 - (-1)^n] [1 - (-1)^m]
\end{aligned}$$

Hence from (1),

$$u(x, y, t) = \sum \sum A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\text{where, } A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6} (1 - \cos m\pi) (1 - \cos n\pi) \quad \text{and} \quad k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

- 23.** A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos (\sqrt{5} \pi ct)$.

Sol. Here we have to solve the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

with boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$

and the initial conditions $u(x, y, 0) = f(x, y) = k \sin 2\pi x \sin \pi y$, $\frac{\partial u}{\partial t} = 0$ when $t = 0$

Proceeding as in question 21, we have

$$u(x, y, t) = \sum \sum A_{mn} \sin m\pi x \sin n\pi y \cos pt \quad \dots(1)$$

Since $a = b = 1$, where $p = \pi c \sqrt{m^2 + n^2}$

$$\begin{aligned}
\text{and} \quad A_{mn} &= \frac{4}{1 \times 1} \int_0^1 \int_0^1 k \sin 2\pi x \sin \pi y \sin m\pi x \sin n\pi y \, dy \, dx \\
&= 4k \int_0^1 \sin m\pi x \sin 2\pi x \, dx \int_0^1 \sin n\pi y \sin \pi y \, dy \\
&= 0 \text{ for } m \neq 2 \text{ or } n \neq 1.
\end{aligned}$$

$$\therefore A_{21} = 4k \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = k \quad \text{and} \quad p = \pi c \sqrt{(2)^2 + (1)^2} = \sqrt{5} \pi c$$

Hence solution (1) reduces to $u(x, y, t) = k \sin 2\pi x \sin \pi y \cos (\sqrt{5} \pi ct)$.

24. What is one-dimensional heat flow equation ? Give its solution.

Sol. The heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$... (1)

which shows the flow of heat by conduction in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible.

Solution of equation (1) is given by $u = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$... (2)

25. Determine the solution of one-dimensional heat equation : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $u(0, t) = 0 = u(l, t)$, $t > 0$ and $u(x, 0) = x$, l being the length of the bar. (M.D.U., May 2007; Dec., 2009)

Sol. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$... (1)

Let $u = XT$ where X is a function of x only and T is a function of t only, be a solution of equation (1)

Then, $\frac{\partial u}{\partial t} = XT'$, $\frac{\partial^2 u}{\partial x^2} = X''T$

Then, $XT' = c^2 X''T$

Separating the variables, $\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}$

Now LHS is a function of x only and RHS is a function of t only.

Since x and t are independent variables, this equation can hold only when both sides reduce to a constant k .

The above equation leads to the partial differential equation :

$$\frac{d^2 X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{dT}{dt} - kc^2 T = 0$$

On solving, we get

(i) When k is +ve and $= p^2$ (say)

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2 p^2 t}$$

(ii) When k is -ve and $= -p^2$ (say)

$$X = c_1 \cos px + c_2 \sin px, \quad T = c_3 e^{-c^2 p^2 t}$$

(iii) When k is 0, $X = c_1 x + c_2$, $T = c_3$

Thus the various solutions of the Heat equations are :

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t}$$

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$$

$$u = (c_1 x + c_2) c_3$$

Of these 3 solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$$

Now, $u(0, t) = 0 = u(l, t)$, $t > 0$ (Given)

$$\therefore 0 = c_1 e^{-c^2 p^2 t} \Rightarrow c_1 = 0$$

As $u(l, t) = 0$, $0 = c_2 \sin pl e^{-c^2 p^2 t}$

$$\sin pl = 0 \Rightarrow pl = n\pi \text{ or } p = \frac{n\pi}{l}, n \text{ being an integer}$$

$$u(x, t) = b_n \cdot \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} \cdot t} \quad (\text{On replacing } c_2 \text{ by } b_n)$$

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, \dots$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} t}$$

Since $u(x, 0) = x$, we have

$$x = \sum b_n \sin \frac{n\pi x}{l} \text{ which is half range series for } x.$$

$$b_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} = \frac{-2l}{\pi} \cos \frac{n\pi}{l}$$

$$\therefore \text{ Complete solution is, } u(x, t) = \frac{-2l}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{l} \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} t}.$$

- 26.** A rod of length l with insulated sides is initially at a uniform temperature u_0 . Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x, t)$.

Sol. The temperature function $u(x, t)$ satisfies the differential equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

As given in previous question (Q. No. 24), we have

$$u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t} \quad \dots(1)$$

Since, the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary conditions are $u(0, t) = u(l, t) = 0$ for all t .

Also $u(x, 0) = u_0$ is the initial condition.

Since $u(0, t) = 0$, we have from (1)

$$0 = c_1 e^{-c^2 p^2 t} \Rightarrow c_1 = 0$$

$$\Rightarrow \text{From (1), } u(x, t) = c_2 \sin px \cdot e^{-c^2 p^2 t} \quad \dots(2)$$

Since $u(l, t) = 0$, we have from (2), $0 = c_2 \sin pl \cdot e^{-c^2 p^2 t} \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi$

$$\therefore p = \frac{n\pi}{l}, n \text{ being an integer.}$$

Solution (2) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t}$ on replacing c_2 by b_n .

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, \dots$

$$\therefore u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t} \quad \dots(3)$$

Since $u(x, 0) = u_0$, we have

$$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ which is half-range sine series for } u_0.$$

$$\therefore b_n = \frac{2}{l} \int_0^l u_0 \sin \frac{n\pi x}{l} \cdot dx = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Hence the temperature function

$$\begin{aligned} u(x, t) &= \frac{4u_0}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} \sin \frac{n\pi x}{l} \cdot e^{\frac{-c^2 n^2 \pi^2 t}{l^2}} \\ &= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} \cdot e^{\frac{-c^2 (2n-1)^2 \pi^2 t}{l^2}} \end{aligned}$$

- 27.** An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at a distance x from A at time t .

Sol. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

Prior to the temperature change at the end B , when $t = 0$, the heat flow was independent of time (steady state condition). When the temperature function u depends only upon x and not on t , (1)

$$\text{reduces to } \frac{\partial^2 u}{\partial x^2} = 0. \text{ Its general solution is } u = ax + b \quad \dots(2)$$

where a, b are arbitrary constants.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, we get

$$\text{From (2),} \quad b = 0 \quad \text{and} \quad a = \frac{100}{l}$$

$$\therefore \text{ The initial condition is expressed by } u(x, 0) = \frac{100}{l} x.$$

Also the boundary conditions for the subsequent flow are $u(0, t) = 0$ and $u(l, t) = 0$ for all values of t .

The most general solution of (1) satisfying the boundary conditions, is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-c^2 n^2 \pi^2 t}{l^2}} \quad \dots(3)$$

$$\text{Since } u(x, 0) = \frac{100}{l} x, \text{ we have } \frac{100}{l} x = \sum b_n \sin \frac{n\pi x}{l}$$

which is half-range sine series for $\frac{100}{l} x$.

$$\therefore b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} \cdot dx = \frac{200}{l^2} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l} \right)^2} \right) \right]_0^l$$

$$= \frac{200}{l^2} \left[-\frac{l^2}{n\pi} \cos n\pi \right] = -\frac{200}{n\pi} (-1)^n = \frac{200}{n\pi} (-1)^{n+1}$$

$$\text{Hence the temperature function } u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}.$$

28. In the previous problem find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

Sol. Initial condition remains the same i.e., $u(x, 0) = \frac{100}{l} x$

Boundary conditions are $u(0, t) = 20$, $u(l, t) = 80$

The final temperature distribution is $u_2 = 20 + \left(\frac{80 - 20}{l} \right) x = 20 + \frac{60}{l} \cdot x$

Then, $u = u_2(x) + u_1(x, t)$

$$u = 20 + \frac{60}{l} x + \sum (a_n \cos px + b_n \sin px) e^{-c^2 p^2 t}$$

$$u(0, t) = 20 = 20 + \sum_{n=1}^{\infty} a_n e^{-c^2 p^2 t} \Rightarrow a_n = 0$$

$$\therefore u = 20 + \frac{60}{l} x + \sum_{n=1}^{\infty} b_n \sin px \cdot e^{-c^2 p^2 t} \quad \dots(A)$$

$$u(l, t) = 80 = 20 + \frac{60}{l} \cdot l + \sum b_n \sin pl \cdot e^{-c^2 p^2 t} \Rightarrow 0 = \sum b_n \sin pl e^{-c^2 p^2 t}$$

$$\sin pl = 0 = \sin n\pi, n \in \mathbb{I}, \therefore p = \frac{n\pi}{l} \quad \dots(B)$$

$$\text{From (A) and (B), } u = 20 + \frac{60}{l} x + \sum b_n \sin \frac{n\pi x}{l} e^{-\left(\frac{n\pi c}{l}\right)^2 t} \quad \dots(C)$$

Using initial condition,

$$u(x, 0) = \frac{100}{l} \cdot x = 20 + \frac{60}{l} x + \sum b_n \sin \frac{n\pi x}{l} \Rightarrow \frac{40}{l} x - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where, } b_n = \frac{2}{l} \int_0^l \left(\frac{40}{l} x - 20 \right) \sin \frac{n\pi x}{l} \cdot dx$$

$$\begin{aligned} &= \frac{2}{l} \left[\left(\frac{40}{l} x - 20 \right) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_0^l - \int_0^l \frac{40}{l} \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \\ &= \frac{2}{l} \left[\frac{-20l}{n\pi} \cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right]_0^l = \frac{-40}{n\pi} (1 + \cos n\pi) \end{aligned}$$

$$= \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

Hence equation (C) becomes

$$\begin{aligned} u(x, t) &= 20 + \frac{60}{l}x - \frac{80}{\pi} \sum_{n=2,4,\dots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \cdot e^{-\left(\frac{n\pi c}{l}\right)^2 t} \\ &= 20 + \frac{60}{l}x - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-\frac{4c^2 m^2 \pi^2 t}{l^2}} \quad (\text{Taking } n = 2m) \end{aligned}$$

which is the required solution.

- 29.** The ends A and B of a rod of length 20 cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function $u(x, t)$. The specific heat, density and the thermal conductivity of the material of the rod are such that the combination $\frac{k}{\rho\sigma} = c^2 = 1$.

Sol. Initial temperature distribution in the rod is $u_1 = 30 + \left(\frac{80-30}{20}\right)x = 30 + \frac{5}{2}x$

Final temperature distribution (i.e., in steady state) is $u_2 = 40 + \left(\frac{60-40}{20}\right)x = 40 + x$

To get u in the intermediate period $u = u_1(x, t) + u_2(x)$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to 0 as t increases.

$\therefore u_1(x, t)$ satisfies one-dimensional heat flow equation

$$\therefore u = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} \quad \dots(1)$$

$$u(0, t) = 40 \quad \dots(2)$$

$$\text{In steady state, } u(20, t) = 60 \quad \dots(3)$$

$$\text{From (1), } u(0, t) = 40 = 40 + \sum a_n e^{-p^2 t} \Rightarrow a_n = 0 \quad \dots(4)$$

$$\therefore \text{ From (1) and (4), } u = 40 + x + \sum b_n \sin px e^{-p^2 t}$$

$$\text{Again, } u(20, t) = 60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20p e^{-p^2 t} \Rightarrow 0 = \sum_{n=1}^{\infty} b_n \sin 20p e^{-p^2 t}$$

$$\sin 20p = 0 = \sin n\pi (n \in \mathbb{I}) \Rightarrow p = \frac{n\pi}{20}$$

$$\therefore u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t} \quad \dots(5)$$

Using initial condition, $u(x, 0) = 30 + \frac{5}{2}x$ in equation (5), we get

$$30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \Rightarrow \frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where
$$b_n = \frac{2}{20} \int_0^{20} \left(\frac{3}{2}x - 10 \right) \sin \frac{n\pi x}{20} dx = -\frac{20}{n\pi} [2(-1)^n + 1]$$

From (5),
$$u = 40 + x - \frac{20}{\pi} \sum \left\{ \frac{2(-1)^n + 1}{n} \right\} \sin \frac{n\pi x}{20} \cdot e^{-\left(\frac{n\pi}{20}\right)^2 t}$$

which is the required solution.

- 30.** The temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and

$x = \pi$ is given by the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Assuming the initial temperature distribution as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time.

Sol.
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

Its solution is
$$u(x, t) = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px) \quad \dots(2)$$

Since ends of bar are insulated, no heat can pass from either sides and boundary conditions are

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \quad \dots(3) \quad \text{and} \quad \frac{\partial u}{\partial x} = 0 \text{ at } x = \pi \quad \dots(4)$$

From (2),
$$\frac{\partial u}{\partial x} = c_1 \cdot e^{-p^2 t} (-pc_2 \sin px + pc_3 \cos px)$$

At $x = 0$,
$$0 = c_1 e^{-p^2 t} pc_3 \Rightarrow c_3 = 0$$

$\therefore$ From (2),
$$u(x, t) = c_1 c_2 e^{-p^2 t} \cos px \quad \dots(5)$$

Again
$$\frac{\partial u}{\partial x} = -pc_1 c_2 e^{-p^2 t} \sin px$$

At $x = \pi$,
$$0 = -pc_1 c_2 e^{-p^2 t} \sin p\pi \Rightarrow \sin p\pi = 0 = \sin n\pi \quad (n \in \mathbb{I})$$

 $p\pi = n\pi \Rightarrow p = n$

$\therefore$ From (5),
$$u(x, t) = b_n e^{-n^2 t} \cos nx \text{ where } c_1 c_2 = b_n$$

Most general solution is
$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \cos nx \quad \dots(6)$$

$$u(x, 0) = \cos 2x = \sum_{n=1}^{\infty} b_n \cos nx$$

Comparing, we get $b_2 = 1$ and $n = 2$

All other b_i 's are zero.

$\therefore$ From (6),
$$u(x, t) = e^{-4t} \cdot \cos 2x \text{ which is the required solution.}$$

- 31.** Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary condition $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(l, t) = 0$ where $0 < x < l$.

Sol. The solution to the equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

is given by
$$u(x, t) = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px) \quad \dots(2)$$

From (2), $u(0, t) = c_1 c_2 e^{-p^2 t} \Rightarrow 0 = c_1 c_2 e^{-p^2 t} \Rightarrow c_2 = 0$

$\therefore$ From (2), $u(x, t) = c_1 c_3 e^{-p^2 t} \sin px \quad \dots(3)$

$$u(l, t) = 0 = c_1 c_3 e^{-p^2 t} \cdot \sin pl \Rightarrow \sin pl = 0 = \sin n\pi \quad (n \in \mathbb{I})$$

$\therefore \quad p = \frac{n\pi}{l}$

From (3), $u(x, t) = c_1 c_3 e^{\frac{-n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l} = b_n e^{\frac{-n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$

The most general solution is

$$u(x, t) = \sum b_n e^{\left(\frac{-n^2 \pi^2 t}{l^2}\right)} \sin \frac{n\pi x}{l} \quad \dots(4)$$

$\therefore$ From (4), $u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \Rightarrow 3 \sin n\pi x = \sum b_n \sin \frac{n\pi x}{l}$

Comparison gives $b_n = 3, l = 1$.

Hence from (4), the required solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \sin n\pi x.$$

- 32.** The initial temperature of an insulated infinite rod is given by $u(x, 0) = (-1)^n U$ between $x = nc$ and $x = (n+1)c$, where $n \in \mathbb{I}$. Show that for $t > 0$,

$$u(x, t) = \frac{4U}{\pi} \sum_{p=0}^{\infty} \frac{1}{2p+1} \sin \left\{ (2p+1) \frac{\pi x}{c} \right\} \cdot e^{\frac{-(2p+1)^2 d^2 \pi^2 t}{c^2}}.$$

Sol. Since the initial temperature is alternatively U and $-U$ over equal distances on the infinite rod, the final temperature as $t \rightarrow \infty$ will be the average of these, namely 0.

The temperature $u(x, t)$ at any time will be an odd periodic function of distance with period $2c$. It will satisfy the conditions :

(i) $u = 0$ at $x = 0$ and (ii) $u = 0$ at $x = c$.

The equation satisfied by $u(x, t)$ is $\frac{\partial u}{\partial t} = d^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$

($\because$ c has been used in the question, we are therefore using here d instead of c).

Most general solution of (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \cdot e^{\frac{-n^2 \pi^2 d^2}{c^2} \cdot t} \quad \dots(2)$$

where, $b_n = \frac{2}{c} \int_0^c u(x, 0) \sin \frac{n\pi x}{c} \cdot dx \quad \dots(3)$

Since $u = U$ (for $0 < x < c$) when $t = 0$

$\therefore$ From (3), $b_n = \frac{2}{c} \int_0^c U \sin \frac{n\pi x}{c} \cdot dx = \frac{2U}{c} \left(-\frac{c}{n\pi} \cos \frac{n\pi x}{c} \right)_0^c = \frac{2U}{n\pi} (1 - \cos n\pi)$

$$= \begin{cases} \frac{4U}{n\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Putting $n = 2p + 1$ to get odd n , (2) gives

$$u(x, t) = \frac{4U}{\pi} \sum_{p=0}^{\infty} \frac{\sin(2p+1) \frac{\pi x}{c}}{2p+1} \cdot e^{\frac{-(2p+1)^2 d^2 \pi^2 t}{c^2}}$$

33. A bar with insulated sides is initially at a temperature 0°C throughout. The end $x = 0$ is kept at 0°C , and heat is suddenly applied at the end $x = l$ so that $\frac{\partial u}{\partial x} = A$ for $x = l$, where A is a constant. Find the temperature function $u(x, t)$.

Sol. One dimensional heat flow equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$... (1)

Its solution is $u(x, t) = c_1 e^{-p^2 c^2 t} (c_2 \cos px + c_3 \sin px)$

or $u(x, t) = (A_1 \cos px + B \sin px) e^{-p^2 c^2 t}$... (2)

Applying the zero end conditions as,

$$u(0, t) = 0 = A_1 e^{-p^2 c^2 t} \Rightarrow A_1 = 0$$

$\therefore$ From (2), $u(x, t) = B \sin px \cdot e^{-p^2 c^2 t}$... (3)

From (3), $\frac{\partial u}{\partial x} = p B \cos px \cdot e^{-p^2 c^2 t}$

At $x = l$, $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 = p B \cos pl \cdot e^{-p^2 c^2 t} \Rightarrow \cos pl = 0 = \cos \left(n\pi - \frac{\pi}{2}\right); n \in \mathbb{I}$

$$pl = (2n - 1) \frac{\pi}{2} \Rightarrow p = (2n - 1) \frac{\pi}{2l}$$

$\therefore$ From (3), $u(x, t) = B \sin px \cdot e^{-p^2 c^2 t}$... (4)

where $p = (2n - 1) \frac{\pi}{2l}$

The most general solution is

$$u(x, t) = \sum B_n \sin px \cdot e^{-p^2 c^2 t} \quad \dots (5)$$

where $p = (2n - 1) \frac{\pi}{2l}$

The end conditions given for this problem are

$$(i) u = 0 \text{ at } x = 0 \quad (ii) \frac{\partial u}{\partial x} = A \text{ at } x = l \quad \dots (6)$$

These conditions are different from the zero end conditions.

So we add to (5) the solution $u = A_1 x + B$

Choosing A_1 and B so that (6) is satisfied.

This gives $B = 0$ and $A_1 = A$

$$\therefore u(x, t) = Ax + \sum_{n=1}^{\infty} B_n \sin px \cdot e^{-p^2 c^2 t} \quad \dots (7)$$

where $p = (2n - 1) \pi / 2l$.

Applying the condition that $u = 0$ at $t = 0$, we have

$$0 = Ax + \sum_{n=1}^{\infty} B_n \sin px \quad \text{or} \quad -Ax = \sum_{n=1}^{\infty} B_n \sin px$$

where

$$\begin{aligned} B_n &= \frac{2}{l} \int_0^l (-Ax) \sin px \cdot dx \quad \text{where } p = (2n-1) \frac{\pi}{2l} \\ &= -\frac{2A}{l} \left[\left\{ x \left(\frac{-\cos px}{p} \right) \right\}_0^l - \int_0^l 1 \cdot \left(\frac{-\cos px}{p} \right) dx \right] \\ &= -\frac{2A}{l} \left[\frac{-l \cos pl}{p} + \frac{1}{p} \left(\frac{\sin px}{p} \right)_0^l \right] = -\frac{2A}{l} \left[\frac{-l \cos pl}{p} + \frac{1}{p^2} \sin pl \right] \\ &= -\frac{2A(2l)^2}{l(2n-1)^2 \pi^2} \sin(2n-1) \frac{\pi}{2} \quad (\because \cos pl = 0) \\ &= \frac{-8Al}{\pi^2(2n-1)^2} \sin \left(n\pi - \frac{\pi}{2} \right) = \frac{8Al}{\pi^2(2n-1)^2} \cdot (-1)^n \quad \dots(8) \end{aligned}$$

$$\therefore \text{ From (7), } u(x, t) = Ax + \frac{8Al}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin(2n-1) \frac{\pi x}{2l} \cdot e^{-\left[\frac{(2n-1)^2 \pi^2 c^2 t}{4l^2} \right]}$$

which is the required temperature function.

34. Solve : $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ under the conditions

- (i) $u \neq \infty$ if $t \rightarrow \infty$ (ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$
 (iii) $u = l x - x^2$ for $t = 0$ between $x = 0$ and $x = l$.

$$\text{Sol. Solution to } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \text{ is } u(x, t) = c_1 e^{-c^2 kt} (c_2 \cos cx + c_3 \sin cx) \quad \dots(1)$$

Equation (1) satisfies the condition $u \neq \infty$ if $t \rightarrow \infty$.

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (1), we get

$$c_3 = 0 \quad \text{and} \quad c = \frac{n\pi}{l}, \quad n \in \mathbb{I}$$

$$\therefore u = c_1 c_2 e^{-\left(\frac{n^2 \pi^2 kt}{l^2} \right)} \cos \frac{n\pi x}{l} = a_n \cos \frac{n\pi x}{l} \cdot e^{-\left(\frac{n^2 \pi^2 kt}{l^2} \right)} \quad \dots(2)$$

Again, the second possible solution is

$$u = c_1 (c_2 x + c_3) \quad \dots(3)$$

| if $c^2 = 0$

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (3), we get $c_2 = 0$

$$\therefore u = c_1 c_3 = \frac{a_0}{2} \text{ (say)} \quad \dots(4)$$

The general solution is the sum of solutions (2) and (4) for various n .

$$\therefore u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \cdot e^{-\left(\frac{n^2 \pi^2 kt}{l^2}\right)} \quad \dots(5)$$

Now applying $u = lx - x^2$ for $t = 0$ to equation (5), we get

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

Here,

$$a_0 = \frac{2}{l} \int_0^l (lx - x^2) dx = \frac{l^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l} dx = \begin{cases} -\frac{4l^2}{n^2 \pi^2}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd.} \end{cases}$$

$$\therefore u = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum_{n=2,4,\dots} \frac{1}{n^2} \cdot \cos \frac{n\pi x}{l} \cdot e^{-\left(\frac{n^2 \pi^2 kt}{l^2}\right)}$$

Put $n = 2m$, we get

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \frac{2m\pi x}{l} \cdot e^{-\left(\frac{4m^2 \pi^2 kt}{l^2}\right)}$$

which is the required solution.

35. What is the two dimensional heat flow equation ? Give its solution.

Sol. Two dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

In steady state, u is independent of t so that $\frac{\partial u}{\partial t} = 0$ and (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(2)$$

which is known as **Laplace's Equation in two dimensions.**

The equation of heat flow in a solid (3-dimensional heat flow) can be written as

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

In the steady state, it reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

This is known as **Laplace's Equation in three dimensions.**

Solution of Laplace's Equation in Two Dimensions is given by

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 \cdot e^{-py}).$$

36. Use separation of variables method to solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary

conditions $u(0, y) = u(l, y) = u(x, 0) = 0$, $u(x, a) = \sin \frac{n\pi x}{l}$.

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$... (1)

The 3 possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py) \quad \dots (2)$$

$$u = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots (3)$$

$$u = (c_1 x + c_2) (c_3 y + c_4) \quad \dots (4)$$

Keeping in view the given boundary conditions, the only possible solution is (3).

$$\therefore u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots (5)$$

Since $u(0, y) = 0$, $\therefore 0 = c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0$

Equation (5) reduces to

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \dots (6)$$

Since $u(l, y) = 0$, $\therefore 0 = c_2 \sin pl (c_3 e^{py} + c_4 e^{-py})$

$$\Rightarrow \sin pl = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{l}, n \text{ being an integer.}$$

Also $u(x, 0) = 0 \therefore 0 = c_2 \sin \frac{n\pi x}{l} (c_3 + c_4)$

$$\Rightarrow c_3 + c_4 = 0 \therefore c_4 = -c_3$$

$\therefore$ Equation (6) becomes

$$u(x, y) = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}} \right)$$

Replacing $c_2 c_3$ by b_n , we have

$$u(x, y) = b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right) = 2b_n \sin \frac{n\pi x}{l} \cdot \sinh \frac{n\pi y}{l} \quad \dots (7)$$

Putting $y = a$, we have $u(x, a) = \sin \frac{n\pi x}{l} = 2b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi a}{l}$

$$\Rightarrow 2b_n \sinh \frac{n\pi a}{l} = 1 \text{ or } b_n = \frac{1}{2 \sinh \frac{n\pi a}{l}}$$

Hence (7) reduces to $u(x, y) = \sin \frac{n\pi x}{l} \left[\frac{\sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}} \right]$, which is the required solution of (1).

37. Solve the following partial differential equation by the method of separation of variables :

$$u_{xx} = u_y + 2u \text{ with } u(0, y) = 0, \left(\frac{\partial u}{\partial x} \right)_{0, y} = 1 + e^{-3y}$$

(M.D.U., May 2009, U.P.T.U., 2009, 2010)

Sol. $u_{xx} = u_y + 2u$... (1)

Let $u = XY$ where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y$$

Putting these values in equation (1), we get

$$X''Y = XY' + 2XY \text{ or } X''Y = X(Y' + 2Y) \Rightarrow \frac{X''}{X} = \frac{Y' + 2Y}{Y} = \frac{Y'}{Y} + 2 = k \text{ (say)}$$

$$(a) \quad \frac{X''}{X} = k$$

$$(D^2 - k)X = 0 \Rightarrow D = \pm \sqrt{k} \quad X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$$

$$(b) \quad \frac{Y'}{Y} + 2 = k \quad \text{or} \quad \frac{Y'}{Y} = k - 2$$

Integrating it, we get

$$\log Y = (k - 2)y + \log c_3 \quad \text{or} \quad Y = c_3 e^{(k-2)y}$$

$$\therefore U = XY \quad \text{or} \quad u = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x} \right) c_3 e^{(k-2)y}$$

$$\text{Given } u(0, y) = 0 \quad \text{when } x = 0 \quad \dots(2)$$

$$\therefore 0 = (c_1 + c_2)c_3 e^{(k-2)y} \quad \dots(3)$$

$$\text{and} \quad \left(\frac{\partial u}{\partial x} \right)_{0,y} = 1 + e^{-3y} \quad (\text{given})$$

$$\begin{aligned} \text{Now,} \quad \frac{\partial u}{\partial x} &= c_3 e^{(k-2)y} \left(\sqrt{k} c_1 e^{\sqrt{k}x} - \sqrt{k} c_2 e^{-\sqrt{k}x} \right) \\ \left(\frac{\partial u}{\partial x} \right)_{0,y} &= 1 + e^{-3y} = c_3 e^{(k-2)y} \cdot \sqrt{k} (c_1 - c_2) \quad \dots(4) \end{aligned}$$

From (3) and (4), we get the values of c_1 and c_2 . Then we put in equation (2) and simplify.

$$\text{Hence,} \quad u = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x + e^{-3y} \cdot \sin x.$$

38. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by $u = 20x$ for $0 \leq x \leq 5$ and $u = 20(10 - x)$ for $5 \leq x \leq 10$ and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C , prove that the temperature u at any point (x, y) is given by $u = \frac{800}{\pi^2} \sum \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-\frac{(2n-1)\pi y}{10}}$.

$$\text{Sol. The temperature } u(x, y) \text{ at any point } P(x, y) \text{ satisfies the equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

The boundary conditions are

$$u(0, y) = 0 \quad \text{for all values of } y \geq 0 \quad \dots(2)$$

$$u(10, y) = 0 \quad \text{for all values of } y \geq 0 \quad \dots(3)$$

$$u(x, \infty) = 0 \quad \text{in } 0 \leq x \leq 10 \quad \dots(4)$$

$$u(x, 0) = \begin{cases} 20x & , \quad 0 \leq x \leq 5 \\ 20(10 - x) & , \quad 5 \leq x \leq 10 \end{cases} \quad \dots(5)$$

Now the 3 possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py) \quad \dots(6)$$

$$u = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots(7)$$

$$u = (c_1 x + c_2) (c_3 y + c_4) \quad \dots(8)$$

The solution (6) can't satisfy the condition (2) since we get $u \neq 0$ for $x = 0$, for all values of y .

The solution (8) can't satisfy the condition (4).

Thus the only possible solution is (7).

$$\therefore u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots(9)$$

Since $u(0, y) = 0, \therefore 0 = c_1(c_3e^{py} + c_4e^{-py}) \Rightarrow c_1 = 0$
 $\therefore$ Equation (9) reduces to $u(x, y) = c_2 \sin px (c_3e^{py} + c_4e^{-py})$... (10)

Since, $u(10, y) = 0, 0 = c_2 \sin 10p (c_3e^{py} + c_4e^{-py})$
 $\sin 10p = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{10}, n$ being an integer.

Also, $u(x, \infty) = 0, \therefore c_3 = 0$

Hence from (10) a solution satisfying (2), (3) and (4) is $u(x, y) = c_2c_4 \sin \frac{n\pi x}{10} \cdot e^{-\frac{n\pi y}{10}}$.

Replacing c_2c_4 by b_n , the most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \cdot e^{-\frac{n\pi y}{10}} \quad \dots (11)$$

Putting $y = 0$, we have

$$\begin{aligned} u(x, 0) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \\ \therefore b_n &= \frac{2}{10} \int_0^{10} u(x, 0) \sin \frac{n\pi x}{10} \cdot dx = \frac{1}{5} \left[\int_0^5 20x \sin \frac{n\pi x}{10} \cdot dx + \int_5^{10} 20(10-x) \sin \frac{n\pi x}{10} \cdot dx \right] \\ &= 4 \left[x \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10} \right)^2} \right) \right]_0^5 + 4 \left[(10-x) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) + 1 \left(\frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10} \right)^2} \right) \right]_5^{10} \\ &= 4 \left[\frac{-50}{n\pi} \cos \frac{n\pi}{2} + \left(\frac{10}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right] + 4 \left[\frac{50}{n\pi} \cos \frac{n\pi}{2} + \left(\frac{10}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right] \\ &= 4 \cdot 2 \cdot \frac{100}{n^2\pi^2} \cdot \sin \frac{n\pi}{2} = \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2} \\ &= \begin{cases} \frac{800}{n^2\pi^2} (-1)^{\frac{n-1}{2}}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases} \end{aligned}$$

Hence from (11) the required solution is

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1,3,5} \frac{1}{n^2} (-1)^{\frac{n-1}{2}} \sin \frac{n\pi x}{10} \cdot e^{-\frac{n\pi y}{10}}$$

Replacing n by $2n-1$,

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} (-1)^{n-1} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-\frac{(2n-1)\pi y}{10}}$$

or

$$u = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-\frac{(2n-1)\pi y}{10}}$$

Since, $(-1)^{n+1} = (-1)^{n-1} \cdot (-1)^2 = (-1)^{n-1}$.

39. A rectangular plate with insulated surfaces is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}$, $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C , show that the steady state temperature at any point of the plate is given by $u(x, y) = 100 e^{-\frac{\pi y}{8}} \cdot \sin \frac{\pi x}{8}$.

Sol. Let $u(x, y)$ be the temperature at any point of the plate.

Two-dimensional heat flow equation in steady state is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

$$\text{Its solution is } u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots(2)$$

Boundary conditions are

$$u(0, y) = 0, u(8, y) = 0, \lim_{y \rightarrow \infty} u(x, y) = 0$$

$$u(x, 0) = 100 \sin \frac{\pi x}{8}, 0 < x < 8$$

$$\begin{aligned} \text{From (2), } u(0, y) = 0 &= c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0 \\ \therefore \text{ From (2), } u(x, y) &= c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \end{aligned} \quad \dots(3)$$

$$u(8, y) = 0 = c_2 \sin 8p (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \sin 8p = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{8}, n \in \mathbb{I}$$

$$\therefore \text{ From (3), } u(x, y) = c_2 \sin \frac{n\pi x}{8} \left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}} \right) \quad \dots(4)$$

$$\lim_{y \rightarrow \infty} u(x, y) = 0 = c_2 \cdot \sin \frac{n\pi x}{8} \lim_{y \rightarrow \infty} \left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}} \right)$$

which is satisfied only when $c_3 = 0$

$$\therefore \text{ From (4), } u(x, y) = c_2 c_4 \sin \frac{n\pi x}{8} \cdot e^{-\frac{n\pi y}{8}} = b_n \sin \frac{n\pi x}{8} \cdot e^{-\frac{n\pi y}{8}} \quad \dots(5)$$

$$\text{From (5), } u(x, 0) = b_n \sin \frac{n\pi x}{8}$$

$$100 \sin \frac{\pi x}{8} = b_n \sin \frac{n\pi x}{8} \Rightarrow b_n = 100, n = 1$$

$$\therefore \text{ From (5), } u(x, y) = 100 \sin \frac{\pi x}{8} \cdot e^{-\frac{\pi y}{8}}$$

which is the required steady state temperature at any point of the plate.

40. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π . This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

Boundary conditions are $u(0, y) = 0 = u(\pi, y)$

$$\lim_{y \rightarrow \infty} u(x, y) = 0 \quad (0 < x < \pi) \quad \text{and} \quad u(x, 0) = u_0 \quad (0 < x < \pi)$$

Solution to equation (1) is

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots(2)$$

$$\text{From (2), } u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0$$

$$\begin{aligned} \text{From (2), } u(x, y) &= c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \dots(3) \\ u(\pi, y) &= 0 = c_2 \sin p\pi (c_3 e^{py} + c_4 e^{-py}) \Rightarrow \sin p\pi = 0 = \sin n\pi \quad (n \in \mathbb{I}) \end{aligned}$$

$$\therefore p = n$$

$$\therefore \text{ From (3), } u(x, y) = c_2 \sin nx (c_3 e^{ny} + c_4 e^{-ny}) \quad \dots(4)$$

$$\lim_{y \rightarrow \infty} u(x, y) = 0 = c_2 \sin nx \cdot \lim_{y \rightarrow \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

which is satisfied only when $c_3 = 0$.

$$\therefore \text{ From (4), } u(x, y) = c_2 c_4 \sin nx \cdot e^{-ny} = b_n e^{-ny} \sin nx, \text{ where } c_2 c_4 = b_n$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \cdot \sin nx \quad \dots(5)$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} \text{where, } b_n &= \frac{2}{\pi} \int_0^{\pi} u(x, 0) \sin nx \, dx = \frac{2u_0}{\pi} \left(\frac{-\cos nx}{n} \right)_0^{\pi} \\ &= \frac{2u_0}{n\pi} \{1 - (-1)^n\} = \begin{cases} \frac{4u_0}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

$$\therefore \text{ From (5), } u(x, y) = \frac{4u_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin nx}{n} e^{-ny} \quad (n \text{ is odd})$$

$$\text{or } u(x, y) = \frac{4u_0}{\pi} \sum \frac{1}{(2n-1)} \sin (2n-1)x \cdot e^{-(2n-1)y}$$

which is the required solution.

$$41. \text{ Solve : } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi \text{ which satisfies the conditions}$$

$$u(0, y) = u(\pi, y) = u(x, \pi) = 0 \text{ and } u(x, 0) = \sin^2 x.$$

$$\text{Sol. The given equation is } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

Its solution consistent with boundary conditions is

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \quad \dots(2)$$

$$\text{From (2), } u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0$$

$$\therefore \text{ From (2), } u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \dots(3)$$

$$u(\pi, y) = 0 = c_2 \sin p\pi (c_3 e^{py} + c_4 e^{-py}) \Rightarrow \sin p\pi = 0 = \sin n\pi, \therefore p = n$$

$$\text{Hence from (3), } u(x, y) = c_2 \sin nx (c_3 e^{ny} + c_4 e^{-ny})$$

$$u(x, y) = \sin nx \cdot (Ae^{ny} + Be^{-ny}) \quad \dots(4)$$

$$\text{where, } A = c_2 c_3 \text{ and } B = c_2 c_4$$

$$\text{From (4), } u(x, \pi) = \sin nx (Ae^{n\pi} + Be^{-n\pi})$$

$$0 = \sin nx (Ae^{n\pi} + Be^{-n\pi})$$

$$\Rightarrow 0 = Ae^{n\pi} + B.e^{-n\pi} \Rightarrow Ae^{n\pi} = -B.e^{-n\pi} = -\frac{1}{2} B_n \text{ (say)}$$

Equation (4) becomes

$$u(x, y) = \sin nx \left[-\frac{1}{2} B_n e^{-n\pi} \cdot e^{ny} + \frac{1}{2} B_n e^{n\pi} \cdot e^{-ny} \right]$$

$$\text{Then (4) becomes } u(x, y) = \frac{1}{2} B_n [e^{n(\pi-y)} - e^{-n(\pi-y)}] \sin nx = B_n \sinh n(\pi - y) \cdot \sin nx \quad \dots(5)$$

$$u(x, 0) = \sin^2 x = \sum B_n \sinh n\pi \sin nx$$

$$\text{where, } B_n \sinh n\pi = \frac{2}{\pi} \int_0^\pi \sin^2 x \cdot \sin nx \, dx = \frac{1}{\pi} \int_0^\pi (1 - \cos 2x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^\pi \left[\sin nx - \frac{1}{2} \{ \sin (n+2)x + \sin (n-2)x \} \right] dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos nx}{n} + \frac{\cos (n+2)x}{2(n+2)} + \frac{\cos (n-2)x}{2(n-2)} \right]_0^\pi$$

$$= \frac{1}{2\pi} \left[\left(\frac{1}{n+2} + \frac{1}{n-2} - \frac{2}{n} \right) \{ (-1)^n - 1 \} \right], \text{ when } n \neq 2$$

$$B_n \sinh n\pi = \begin{cases} \frac{-8}{\pi n (n^2 - 4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even and } \neq 2. \end{cases}$$

$$\text{When } n = 2, \quad B_2 \sinh 2\pi = \frac{2}{\pi} \int_0^\pi \sin^2 x \sin 2x \, dx$$

$$= \frac{1}{\pi} \int_0^\pi (1 - \cos 2x) \sin 2x \, dx = \frac{1}{\pi} \int_0^\pi \left(\sin 2x - \frac{1}{2} \sin 4x \right) dx$$

$$= \frac{1}{\pi} \left(-\frac{\cos 2x}{2} + \frac{1}{8} \cos 4x \right)_0^\pi = \frac{1}{\pi} \left[-\frac{1}{2} (1-1) + \frac{1}{8} (1-1) \right] = 0$$

$$\therefore B_2 = 0$$

Hence the solution (5) becomes,

$$u(x, y) = \frac{-8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin nx \cdot \sinh n(\pi - y)}{n(n^2 - 4) \sinh n\pi}$$

$$\therefore u(x, y) = \frac{-8}{\pi} \sum_{m=1,2,3,\dots}^{\infty} \frac{\sin (2m-1)x \sinh (2m-1)(\pi - y)}{(2m-1) \{ (2m-1)^2 - 4 \} \sinh (2m-1)\pi}.$$

42. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$; given that

$$\begin{aligned} u(x, b) &= u(0, y), \\ &= u(a, y) = 0 \text{ and } u(x, 0) = x(a - x). \end{aligned}$$

Sol. The equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$... (1)

Its solution is $u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py})$... (2)

$$u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0$$

$\therefore$ From (2), $u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py})$... (3)

$$u(a, y) = 0 = c_2 \sin ap (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \sin ap = 0 = \sin n\pi \quad (n \in \mathbb{I}) \quad \text{or} \quad p = \frac{n\pi}{a}$$

$$\begin{aligned} \text{From (3),} \quad u(x, y) &= c_2 \sin \frac{n\pi x}{a} \left(c_3 \cdot e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}} \right) \\ u(x, y) &= \sin \frac{n\pi x}{a} \cdot \left(A e^{\frac{n\pi y}{a}} + B e^{-\frac{n\pi y}{a}} \right) \end{aligned} \quad \dots (4)$$

where

$$\begin{aligned} A &= c_2 c_3, \quad B = c_2 c_4 \\ u(x, b) &= \sin \frac{n\pi x}{a} \cdot \left(A e^{\frac{n\pi b}{a}} + B \cdot e^{-\frac{n\pi b}{a}} \right) \\ 0 &= \sin \frac{n\pi x}{a} \cdot \left(A \cdot e^{\frac{n\pi b}{a}} + B \cdot e^{-\frac{n\pi b}{a}} \right) \Rightarrow A e^{\frac{n\pi b}{a}} + B \cdot e^{-\frac{n\pi b}{a}} = 0 \\ A e^{\frac{n\pi b}{a}} &= -B \cdot e^{-\frac{n\pi b}{a}} = -\frac{1}{2} \cdot B_n \text{ (say)} \end{aligned}$$

$$\begin{aligned} \text{Then (4) becomes} \quad u(x, y) &= \sin \frac{n\pi x}{a} \cdot \left(-\frac{1}{2} B_n e^{-\frac{n\pi y}{a}} \cdot e^{\frac{n\pi b}{a}} + \frac{1}{2} B_n \cdot e^{\frac{n\pi b}{a}} \cdot e^{-\frac{n\pi y}{a}} \right) \\ &= \frac{1}{2} B_n \cdot \sin \frac{n\pi x}{a} \left[e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)} \right] \\ &= \frac{1}{2} B_n \sin \frac{n\pi x}{a} \cdot 2 \sinh \frac{n\pi}{a} (b - y) = B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b - y) \end{aligned}$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \cdot \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi}{a} (b - y) \quad \dots (5)$$

Applying to this the condition $u(x, 0) = x(a - x)$, we get

$$\text{From (5),} \quad u(x, 0) = \sum B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \Rightarrow x(a - x) = \sum B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

$$\begin{aligned}
\text{where, } B_n \sinh \frac{n\pi b}{a} &= \frac{2}{a} \int_0^a x(a-x) \sin \frac{n\pi x}{a} \cdot dx \\
&= \frac{2}{a} \left[\left\{ (ax - x^2) \left(\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) \right\}_0^a - \int_0^a (a-2x) \left(\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) dx \right] \\
&= \frac{2}{a} \cdot \frac{a}{n\pi} \int_0^a (a-2x) \cdot \cos \frac{n\pi}{a} \cdot x \, dx \\
&= \frac{2}{n\pi} \left[\left\{ (a-2x) \frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right\}_0^a - \int_0^a (-2) \left(\frac{\sin \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) dx \right] \\
&= \frac{4}{n\pi} \cdot \frac{a}{n\pi} \left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right)_0^a = \frac{4a}{n^2\pi^2} \cdot \frac{a}{n\pi} (1 - \cos n\pi) \\
&= \frac{4a^2}{n^3\pi^3} (1 - \cos n\pi) = \begin{cases} \frac{8a^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}
\end{aligned}$$

$$\therefore B_n = \frac{8a^2}{\sinh \left(\frac{n\pi b}{a} \right) (n^3\pi^3)} \text{ for odd } n \text{ and } 0 \text{ for even } n.$$

$$\therefore \text{ From (5), } u(x, y) = \frac{8a^2}{\pi^3} \sum \frac{\sin \frac{n\pi x}{a}}{n^3 \sinh \frac{n\pi b}{a}} \cdot \sinh \frac{n\pi}{a} (b-y) \quad (n \text{ is odd})$$

$$\text{or } u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \cdot \sin (2n+1) \frac{\pi x}{a}.$$

$$\left(\frac{\sinh \frac{(2n+1)\pi}{a} (b-y)}{\sinh \frac{(2n+1)\pi}{a} \cdot b} \right) \text{ which is the required solution.}$$

43. A thin rectangular plate whose surface is impervious to heat flow has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. Its four edges $x = 0, x = a, y = 0, y = b$ are kept at zero temperature. Determine the temperature at a point of a plate as t increases. Discuss the problem when

$$f(x, y) = \beta \sin \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi y}{b} \right).$$

Sol. Two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t} \quad \dots(1)$$

Boundary conditions are $u(0, y, t) = 0, u(a, y, t) = 0, u(x, 0, t) = 0, u(x, b, t) = 0$

$u(x, y, t) = f(x, y)$ at $t = 0$.

Let the solution be $u = XYT$ where X is a function of x only, Y is a function of y only and T is a function of t only.

$$(1) \text{ is } \quad X''YT + XY''T = \frac{1}{c^2} (XYT') \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{c^2 \cdot T} \quad \dots(2)$$

There are three possibilities :

$$(i) \quad \frac{X''}{X} = 0, \quad \frac{Y''}{Y} = 0, \quad \frac{T'}{c^2 \cdot T} = 0 \quad (ii) \quad \frac{X''}{X} = K_1^2, \quad \frac{Y''}{Y} = K_2^2, \quad \frac{T'}{c^2 \cdot T} = K^2$$

$$(iii) \quad \frac{X''}{X} = -K_1^2, \quad \frac{Y''}{Y} = -K_2^2, \quad \frac{T'}{c^2 \cdot T} = -K^2$$

where $K^2 = K_1^2 + K_2^2$. Of these 3 solutions, we have to select the solution which is consistent with physical nature of the problem. The solution satisfying the given boundary conditions will be given by (iii).

Then,

$$X = c_1 \cos K_1 x + c_2 \sin K_1 x, \quad Y = c_3 \cos K_2 y + c_4 \sin K_2 y$$

$$T = c_5 e^{-c^2 K^2 t}$$

$$\therefore \quad u = XYT$$

$$\Rightarrow \quad u(x, y, t) = (c_1 \cos K_1 x + c_2 \sin K_1 x) (c_3 \cos K_2 y + c_4 \sin K_2 y) (c_5 e^{-c^2 K^2 t}) \quad \dots(3)$$

$$u(0, y, t) = 0 = c_1 (c_3 \cos K_2 y + c_4 \sin K_2 y) c_5 e^{-c^2 K^2 t} \Rightarrow c_1 = 0$$

$$\begin{aligned} \therefore \text{ From (3), } \quad u(x, y, t) &= c_2 \sin K_1 x (c_3 \cos K_2 y + c_4 \sin K_2 y) (c_5 e^{-c^2 K^2 t}) \\ &= c_6 \sin K_1 x (c_3 \cos K_2 y + c_4 \sin K_2 y) (e^{-c^2 K^2 t}) \end{aligned} \quad \dots(4)$$

where

$$c_6 = c_2 c_5.$$

$$\text{From (4), } \quad u(a, y, t) = 0 = c_6 \sin K_1 a (c_3 \cos K_2 y + c_4 \sin K_2 y) e^{-c^2 K^2 t}$$

$$\Rightarrow \quad \sin K_1 a = 0 = \sin n\pi \quad (n \in \mathbb{I}) \quad \therefore \quad K_1 = \frac{n\pi}{a}$$

$$\text{From (4), } \quad u(x, y, t) = c_6 \sin \frac{n\pi x}{a} (c_3 \cos K_2 y + c_4 \sin K_2 y) (e^{-c^2 K^2 t}) \quad \dots(5)$$

$$u(x, 0, t) = 0 = c_6 \sin \frac{n\pi x}{a} \cdot c_3 \cdot e^{-c^2 K^2 t} \Rightarrow c_3 = 0$$

$$\therefore \text{ From (5), } \quad u(x, y, t) = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 y \cdot e^{-c^2 K^2 t} \quad \dots(6)$$

$$u(x, b, t) = 0 = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 \cdot b \cdot e^{-c^2 K^2 t}$$

$$\Rightarrow \quad \sin K_2 b = 0 = \sin m\pi \quad (m \in \mathbb{I}) \Rightarrow K_2 b = m\pi \Rightarrow K_2 = \frac{m\pi}{b}$$

$$\begin{aligned} \therefore \text{ From (6), } \quad u(x, y, t) &= c_6 c_4 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cdot e^{-c^2 K^2 t} \\ &= A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cdot e^{-c^2 K^2 t} \end{aligned} \quad \dots(7) \quad (\text{where, } c_6 c_4 = A_{mn})$$

But

$$K^2 = K_1^2 + K_2^2 = \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right)$$

Equation (7) becomes

$$u(x, y, t) = \sum \sum A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cdot e^{-c^2 K^2 t} \quad \dots(8)$$

which is the most general solution.

$$u(x, y, 0) = f(x, y) = \sum \sum A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

which is the double Fourier half-range sine series for $f(x, y)$

where,
$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^a \int_{y=0}^b \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} f(x, y) dx dy$$

When $f(x, y) = \beta \sin \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi y}{b} \right)$

$$\begin{aligned} A_{mn} &= \frac{4\beta}{ab} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{\pi x}{a} dx \int_0^b \sin \frac{m\pi y}{b} \sin \frac{\pi y}{b} dy \\ &= \frac{\beta}{ab} \int_0^a \left[\cos(n-1) \frac{\pi x}{a} - \cos(n+1) \frac{\pi x}{a} \right] dx \cdot \int_0^b \left[\cos(m-1) \frac{\pi y}{b} - \cos(m+1) \frac{\pi y}{b} \right] dy \\ &= \frac{\beta}{ab} \left[\frac{\sin(n-1) \frac{\pi x}{a}}{(n-1) \frac{\pi}{a}} - \frac{\sin(n+1) \frac{\pi x}{a}}{(n+1) \frac{\pi}{a}} \right]_0^a \times \left[\frac{\sin(m-1) \frac{\pi y}{b}}{(m-1) \frac{\pi}{b}} - \frac{\sin(m+1) \frac{\pi y}{b}}{(m+1) \frac{\pi}{b}} \right]_0^b \\ &= \frac{\beta}{ab} \cdot 0 = 0 \quad (\because \sin n\pi = 0 \text{ and } \sin m\pi = 0) \end{aligned}$$

Hence from (8), $u(x, y, t) = 0$ when $f(x, y) = \beta \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$.

44. The edges of a thin plate of side π cm are kept at 0°C and faces are perfectly insulated. The initial temperature is $u(x, y, 0) = f(x, y) = xy(\pi - x)(\pi - y)$. Determine the temperature in the plate at time t .

Sol. Referring to solution of Q. 43 where $a = \pi$, $b = \pi$ and $f(x, y) = xy(\pi - x)(\pi - y)$.

and
$$K^2 = \pi^2 \left(\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2} \right) = m^2 + n^2$$

$$\therefore u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin nx \sin my \cdot e^{-c^2(m^2+n^2)t} \quad \dots(1)$$

(Pl. See Sol. of Previous Q. No. 43)

From (1),
$$u(x, y, 0) = \sum \sum A_{mn} \sin nx \sin my \quad \dots(2)$$

where
$$\begin{aligned} A_{mn} &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi xy(\pi - x)(\pi - y) \sin nx \sin my dx dy \\ &= \frac{4}{\pi^2} \int_0^\pi (\pi x - x^2) \sin nx dx \int_0^\pi (\pi y - y^2) \sin my dy \quad \dots(3) \end{aligned}$$

Now
$$\int_0^\pi (\pi x - x^2) \sin nx dx = \left[(\pi x - x^2) \left(\frac{-\cos nx}{n} \right) - (\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^\pi$$

$$= \frac{2}{n^3} [1 - (-1)^n]$$

$$\text{and } \int_0^\pi (\pi y - y^2) \sin my \, dy = \frac{2}{m^3} [1 - (-1)^m]$$

$$\begin{aligned} \therefore A_{mn} &= \frac{16}{\pi^2 m^3 n^3} [1 - (-1)^n] [1 - (-1)^m] \\ &= \begin{cases} 0, & \text{when } m = 2p \text{ and } n = 2q \\ \frac{64}{\pi^2 (2p-1)^3 (2q-1)^3}, & \text{when } m = 2p-1 \text{ and } n = 2q-1 \end{cases} \end{aligned}$$

Hence the required solution is

$$u(x, y, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \sin (2q-1)x \sin (2p-1)y \cdot e^{-\lambda^2 pq^2 t}$$

$$\text{where } \lambda_{pq}^2 = c^2 [(2p-1)^2 + (2q-1)^2]$$

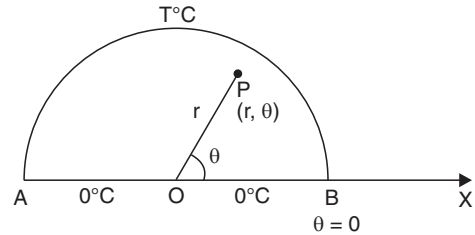
$$\text{and } A_{pq} = \frac{64}{\pi^2 (2p-1)^3 (2q-1)^3}.$$

45. The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is $T^\circ\text{C}$. Show that the steady state temperature in the plate is given by

$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\theta.$$

Sol. Take the centre of the circle as the pole and the bounding diameter as the initial line. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(1)$$



This is Laplace's equation in polar coordinates.

Let

$$u = RT$$

...(2)

where R is a function of r only and T is a function of θ only, be a solution of (1).

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r} (RT) = T \frac{dR}{dr} = TR'$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r} (TR') = T \cdot R''$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} (RT) = R \frac{d^2 T}{d\theta^2} = RT''$$

$$\therefore \text{ From (1), } r^2 T R'' + r T R' + RT'' = 0 \quad \text{or} \quad \frac{r^2 R'' + r R'}{R} = -\frac{T''}{T}$$

$$\text{Case I. When } \frac{r^2 R'' + r R'}{R} = -\frac{T''}{T} = p^2 \text{ (say)}$$

$$(i) \frac{r^2 R'' + r R'}{R} = p^2 \Rightarrow r^2 R'' + r R' - p^2 R = 0$$

Put $r = e^z$ so that $z = \log r$ and let $D = \frac{d}{dz}$

Then above equation reduces to

$$\{D(D-1) + D - p^2\} R = 0 \quad \text{or} \quad (D^2 - p^2) R = 0$$

Auxiliary equation is $m^2 - p^2 = 0$, $m = \pm p$

$$\text{C.F.} = c_1 e^{pz} + c_2 \cdot e^{-pz} = c_1 e^{p \log r} + c_2 \cdot e^{-p \log r} = c_1 \cdot r^p + c_2 \cdot r^{-p}$$

$$\text{P.I.} = 0$$

$$\therefore R = c_1 \cdot r^p + c_2 \cdot r^{-p}$$

$$(ii) \quad \frac{-T''}{T} = p^2, \quad \frac{d^2 T}{d\theta^2} + p^2 T = 0$$

Auxiliary equation is $m^2 + p^2 = 0$ or $m = \pm pi$

$$\text{C.F.} = c_3 \cos p\theta + c_4 \sin p\theta$$

$$\text{P.I.} = 0$$

$$\therefore T = c_3 \cos p\theta + c_4 \sin p\theta$$

$$\therefore u(r, \theta) = (c_1 r^p + c_2 r^{-p}) (c_3 \cos p\theta + c_4 \sin p\theta) \quad \dots(3)$$

Case II. When $\frac{r^2 R'' + rR'}{R} = -\frac{T''}{T} = -p^2$ (say)

$$(i) \quad \frac{r^2 R'' + rR'}{R} = -p^2 \Rightarrow r^2 R'' + rR' + p^2 \cdot R = 0$$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$ then above equation reduces to

$$[D(D-1) + D + p^2] R = 0 \quad \text{or} \quad (D^2 + p^2) R = 0$$

Auxiliary equation is $m^2 + p^2 = 0$ or $m = \pm pi$

$$\text{C.F.} = (c_5 \cos pz + c_6 \sin pz)$$

$$\text{P.I.} = 0$$

$$\therefore R = c_5 \cos pz + c_6 \sin pz \\ = c_5 \cos (p \log r) + c_6 \sin (p \log r).$$

$$(ii) \quad \frac{T''}{T} = -p^2 \Rightarrow \frac{d^2 T}{d\theta^2} - p^2 T = 0$$

Auxiliary equation is $m^2 - p^2 = 0$ or $m = \pm p$

$$\therefore \text{C.F.} = c_7 e^{p\theta} + c_8 \cdot e^{-p\theta}$$

$$\text{P.I.} = 0$$

$$\therefore T = c_7 e^{p\theta} + c_8 \cdot e^{-p\theta}$$

$$\text{Hence } u(r, \theta) = [c_5 \cos (p \log r) + c_6 \sin (p \log r)] (c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad \dots(4)$$

Case III. When $\frac{r^2 R'' + rR'}{R} = -\frac{T''}{T} = 0$ (say)

$$(i) \quad r^2 R'' + rR' = 0$$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$ then above equation reduces to

$$[D(D-1) + D] R = 0 \quad \text{or} \quad D^2 R = 0$$

Auxiliary equation is $m^2 = 0 \Rightarrow m = 0, 0$

$$\text{C.F.} = (c_9 + c_{10} z) e^{0 \cdot z} = c_9 + c_{10} \log r$$

$$R = c_9 + c_{10} \log r$$

$$(ii) - \frac{T''}{T} = 0 \Rightarrow T'' = 0 \Rightarrow T = c_{11} + c_{12} \cdot \theta$$

$$\therefore u(r, \theta) = (c_9 + c_{10} \log r) (c_{11} + c_{12} \theta) \quad \dots(5)$$

Of these three solutions (3), (4) and (5), we choose the solution consistent with the given boundary conditions.

Boundary conditions are

$$u(r, 0) = 0 \quad \dots(6) \quad u(r, \pi) = 0 \quad \dots(7) \quad u(a, \theta) = T \quad \dots(8)$$

$$\text{and} \quad u \rightarrow 0 \text{ as } r \rightarrow 0 \quad \dots(9)$$

Solutions (4) and (5) do not satisfy boundary condition (9).

Hence the consistent solution is

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p}) (c_3 \cos p\theta + c_4 \sin p\theta) \quad \dots(10)$$

$$\text{From (10), } u(r, 0) = 0 = (c_1 r^p + c_2 r^{-p}) c_3 \Rightarrow c_3 = 0$$

$$\text{From (10), } u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta \quad \dots(11)$$

$$u(r, \pi) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi$$

$$0 = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi$$

$$\Rightarrow \sin p\pi = 0 = \sin n\pi \quad (n \in \mathbb{I})$$

$$\therefore p = n$$

$$\therefore \text{ From (11), } u(r, \theta) = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta \quad \dots(12)$$

Condition $u \rightarrow 0$ as $r \rightarrow 0$ is satisfied only and only when $c_2 = 0$

Hence from (12), $u(r, \theta) = c_1 c_4 r^n \sin n\theta = b_n r^n \sin n\theta$

The most general solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \dots(13)$$

$$u(a, \theta) = T = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

$$\text{where, } b_n a^n = \frac{2}{\pi} \int_0^{\pi} \{T \sin n\theta d\theta\} = \frac{2T}{\pi} \left(\frac{-\cos n\theta}{n} \right)_0^{\pi}$$

$$= \frac{2T}{n\pi} (1 - \cos n\pi) = \begin{cases} \frac{4T}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\therefore b_n = \begin{cases} \frac{4T}{n\pi a^n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\text{From (13), } u(r, \theta) = \frac{4T}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \cdot \frac{r^n}{a^n} \sin n\theta$$

$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a} \right)^{2n-1} \sin (2n-1) \theta$$

which is the required steady state temperature in the plate.

46. Solve : $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0$ with boundary conditions :

(i) V is finite when $r \rightarrow 0$

(ii) $V = \sum C_n \cos n\theta$ on $r = a$.

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Sol. Solution to given differential equation is

$$V = \sum (A_n r^n + B_n r^{-n}) \cos (n\theta + \alpha)$$

When $r = a$, $V = \sum c_n \cos n\theta$

$$\therefore \sum c_n \cos n\theta = \sum (A_n a^n + B_n \cdot a^{-n}) \cos (n\theta + \alpha)$$

$$\therefore c_n = A_n a^n + B_n \cdot a^{-n}, \alpha = 0$$

When $r \rightarrow 0$, V is finite.

$$\therefore B_n = 0 \quad \therefore A_n = \frac{c_n}{a^n} \quad \therefore V = \sum c_n \cdot \left(\frac{r}{a}\right)^n \cos n\theta.$$

- 47.** The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.

Sol. Here we have to take the solution in polar coordinates.

The solution is $u = (c_1 \cos p\theta + c_2 \sin p\theta) (c_3 r^p + c_4 \cdot r^{-p})$... (1)

Since the temperature remains finite at $r = 0$

$$\therefore c_4 = 0$$

Also, if we increase θ by 2π , we arrive at the same point. So the solution (1) should be periodic with period 2π .

$$\therefore p = n, \text{ an integer.}$$

Hence we may write the general solution as

$$\begin{aligned} u &= \sum_{n=0}^{\infty} (c_1 \cos n\theta + c_2 \sin n\theta) c_3 r^n \\ &= \sum (A_n \cos n\theta + B_n \sin n\theta) r^n \mid \text{ where } c_1 c_3 = A_n, c_2 c_3 = B_n \text{ (say)} \end{aligned}$$

Applying to this, the condition

$$u = f(\theta) \text{ for } r = a, \text{ we get}$$

$$f(\theta) = \sum (A_n \cos n\theta + B_n \sin n\theta) a^n$$

$$\therefore a^n A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \cdot d\theta$$

$$a^n B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \cdot d\theta$$

Hence the result.

- 48.** Find the harmonic function ϕ in the semi-circle $r < a$, $0 < \theta < \pi$ which vanishes on $\theta = 0$ and takes the value c when $\theta = \pi$ and on the curved portion $r = a$.

Sol. A harmonic function is a function satisfying Laplace's equation.

Solution to $r^2 \frac{\partial^2 \phi}{\partial r^2} + r \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2} = 0$ is $\phi = (A \cos p\theta + B \sin p\theta) (C \cdot r^p + D \cdot r^{-p})$... (1)

Since $\phi \rightarrow 0$ as $r \rightarrow 0$, $\therefore D = 0$

$$\therefore \phi = (A \cos p\theta + B \sin p\theta) C r^p \quad \dots (2)$$

Imposing on (2), the zero boundary conditions $\phi(r, 0) = 0$, $\phi(r, \pi) = 0$, we get

$$A = 0 \quad \text{and} \quad p = n \quad (n \in \mathbb{I})$$

Putting in (2) and adding up various solutions for $n = 1, 2, 3, \dots$, we get

$$\phi = \sum_{n=1}^{\infty} B_n r^n \sin n\theta \quad \dots(3)$$

This solution will not satisfy both the given boundary conditions of the problem namely

$$(i) \phi = 0 \quad \text{when} \quad \theta = 0$$

$$\text{and} \quad (ii) \phi = c \quad \text{when} \quad \theta = \pi \quad \dots(4)$$

So we add to (3) the solution $\phi = A_0\theta + B_0$

which satisfies Laplace's equation.

We choose A_0, B_0 so that (4) is satisfied.

$$\text{Then,} \quad B_0 = 0, \quad c = A_0 \pi$$

$$\text{Hence,} \quad \phi = \frac{c}{\pi} \cdot \theta + \sum B_n r^n \sin n\theta \quad \dots(5)$$

Satisfies equation (4).

Applying to (5), the condition that $\phi = c$ when $r = a$, we get

$$c = \frac{c}{\pi} \theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta \Rightarrow c \left(1 - \frac{\theta}{\pi}\right) = \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$\therefore B_n \cdot a^n = \frac{2}{\pi} \int_0^{\pi} c \left(1 - \frac{\theta}{\pi}\right) \sin n\theta \, d\theta$$

$$B_n \cdot a^n = \frac{2c}{n\pi} \Rightarrow B_n = \frac{2c}{n\pi a^n} \quad \dots(6)$$

$$\text{From (5) and (6),} \quad \phi = \frac{c \cdot \theta}{\pi} + \frac{2c}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \sin n\theta.$$

49. What are transmission line equations. Discuss their similarities under different conditions.

Sol. Consider the flow of electricity in an insulated cable. Let V be the potential and I the current at time t at a point p of the cable at a distance x from a given point. Let R, L, C, G be resistance, inductance, capacitance and leakance to the ground per unit length of the cable. The following equations are known as telephone equations :

$$\frac{\partial^2 V}{\partial x^2} = LC \cdot \frac{\partial^2 V}{\partial t^2} + (RC + LG) \frac{\partial V}{\partial t} + RGV \quad \dots(1)$$

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + (RC + LG) \frac{\partial I}{\partial t} + RGI \quad \dots(2)$$

(a) If $L = G = 0$, equations become :

$$\frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial t} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RC \frac{\partial I}{\partial t}$$

which are known as the telegraph equations. These are similar to the equations in one-dimensional heat flow.

(b) If $R = G = 0$, the above equations become

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2}$$

which are known as the radio equations. They are similar to the wave equation.

(c) If R and G are negligible, the transmission lines become

$$\frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t} \quad \text{and} \quad \frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t}$$

(d) If $L = C = 0$, the above equations become

$$\frac{\partial^2 V}{\partial x^2} = RGV \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RGI$$

which are equations for sub-marine cable.

50. Find the current i and voltage e in a line of length l , t seconds after the ends are suddenly grounded,

given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin \frac{\pi x}{l}$. Also R and G are negligible.

Sol. Since R and G are negligible, the transmission line equations become

$$\frac{\partial e}{\partial x} = -L \frac{\partial i}{\partial t} \quad \dots(1)$$

$$\frac{\partial i}{\partial x} = -C \frac{\partial e}{\partial t} \quad \dots(2)$$

For elimination of i , differentiating (1) partially *w.r.t.* x and (2) partially *w.r.t.* t , we have

$$\begin{aligned} \frac{\partial^2 e}{\partial x^2} &= -L \cdot \frac{\partial^2 i}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 i}{\partial t \partial x} = -C \frac{\partial^2 e}{\partial t^2} \\ \frac{\partial^2 e}{\partial x^2} &= LC \frac{\partial^2 e}{\partial t^2} \end{aligned} \quad \dots(3)$$

$$\text{The initial conditions are } i(x, 0) = i_0 \text{ and } e(x, 0) = e_0 \sin \frac{\pi x}{l} \quad \dots(4)$$

Since, the ends are suddenly grounded, the boundary conditions are

$$e(0, t) = e(l, t) = 0 \quad \dots(5)$$

Also, $i = i_0$ (constant) when $t = 0$

$$\therefore \frac{\partial i}{\partial x} = 0 \text{ which gives } \frac{\partial e}{\partial t} = 0 \text{ when } t = 0 \quad \dots(6)$$

Now, let $e = XT$ be a solution of (3) where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 e}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial^2 e}{\partial t^2} = XT''$$

$\therefore$ From (3),

$$X''T = LCXT''$$

Separating the variables, $\frac{X''}{X} = LC \frac{T''}{T} = -p^2$ (say)

This leads to the ordinary differential equations

$$\frac{d^2 X}{dx^2} + p^2 X = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + \frac{p^2}{LC} \cdot T = 0$$

$$\therefore X = c_1 \cos px + c_2 \sin px$$

$$T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}$$

$$\Rightarrow e = XT = (c_1 \cos px + c_2 \sin px)$$

$$+ \left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad \dots(7)$$

Applying the boundary conditions (5), we have

$$c_1 = 0, p = \frac{n\pi}{l}, n \text{ being an integer.}$$

$$\therefore \text{ Equation (7) becomes } e = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi t}{l\sqrt{LC}} + c_4 \sin \frac{n\pi t}{l\sqrt{LC}} \right)$$

$$\text{or } e = \sin \frac{n\pi x}{l} \left(A \cos \frac{n\pi t}{l\sqrt{LC}} + B \sin \frac{n\pi t}{l\sqrt{LC}} \right) \quad \dots(8)$$

where

$$A = c_2 c_3 \text{ and } B = c_2 c_4$$

$$\frac{\partial e}{\partial t} = \sin \frac{n\pi x}{l} \left(-\frac{A n\pi}{l\sqrt{LC}} \sin \frac{n\pi t}{l\sqrt{LC}} + \frac{B n\pi}{l\sqrt{LC}} \cos \frac{n\pi t}{l\sqrt{LC}} \right)$$

Since $\frac{\partial e}{\partial t} = 0$ when $t = 0$, we get $B = 0$

$$\therefore \text{ From (8), } e = A \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

$$e = \sum A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \text{ is also a solution.}$$

But $e = e_0 \sin \frac{\pi x}{l}$ when $t = 0$

$$\therefore e_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \Rightarrow A_1 = e_0 \text{ and } A_2 = A_3 = \dots = 0.$$

$$\text{Hence, } e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

$$\text{Now } -L \frac{\partial i}{\partial t} = \frac{\partial e}{\partial x} \quad [\text{From (1)}]$$

$$\therefore \frac{\partial i}{\partial t} = -\frac{1}{L} \cdot e_0 \frac{\pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Integrating w.r.t. t , regarding x as a constant.

$$i = -\frac{e_0 \pi}{Ll} \cos \frac{\pi x}{l} \cdot \frac{l\sqrt{LC}}{\pi} \cdot \sin \frac{\pi t}{l\sqrt{LC}} + f(x) \quad \dots(9)$$

where $f(x)$ is an arbitrary constant function.

Since $i = i_0$ when $t = 0$, we have $i_0 = 0 + f(x)$ or $f(x) = i_0$

$$\therefore \text{ From (9) we have } i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}}.$$

- 51.** A transmission line 1000 km. long is initially under steady-state conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the L , G are negligible, find the potential $E(x, t)$.

Sol. Since L and G are negligible, we use the telegraph equation

$$\frac{\partial^2 E}{\partial x^2} = RC \frac{\partial E}{\partial t} \quad \text{or} \quad \frac{\partial E}{\partial t} = \frac{1}{RC} \frac{\partial^2 E}{\partial x^2} \quad \dots(1)$$

Here E_s = initial steady voltage satisfying $\frac{\partial^2 E}{\partial x^2} = 0$

$$= 1300 - \frac{1300 - 1200}{1000} \cdot x = 1300 - 0.1 x = E(x, 0) \quad \dots(2)$$

E_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached

$$= 1300 - \frac{1300 - 0}{1000} x = 1300 - 1.3 x$$

$\therefore E(x, t) = E_s' + E_r(x, t)$ where $E_r(x, t)$ is the transient part.

$$= 1300 - 1.3 x + \sum b_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2 \pi^2 t}{l^2 RC}} \quad \dots(3)$$

where $l = 1000$ km.

Putting $t = 0$ in (3), we have

$$E(x, 0) = 1300 - 1.3 x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{or} \quad 1300 - 0.1 x = 1300 - 1.3 x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad [\text{Using (2)}]$$

$$\Rightarrow \quad 1.2 x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where,} \quad b_n = \frac{2}{l} \int_0^l 1.2x \sin \frac{n\pi x}{l} \cdot dx$$

$$= \frac{2.4}{l} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l} \right)^2} \right) \right]_0^l = \frac{2.4}{l} \cdot \frac{l}{n\pi} [-l \cos n\pi]$$

$$= -\frac{2.4 \times 1000}{n\pi} (-1)^n = \frac{2400}{\pi} \cdot \frac{(-1)^{n+1}}{n}$$

$$\text{Hence,} \quad E(x, t) = 1300 - 1.3 x + \frac{2400}{\pi} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{\frac{-n^2 \pi^2 t}{l^2 RC}}$$

52. Find the e.m.f. $v(x, t)$ in a line of length l , t seconds after the ends were suddenly grounded, given

that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_2 \sin \frac{5\pi x}{l}$. Given that R and G are negligible.

Sol. Since R and G are negligible, we use the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \dots(1)$$

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, v(l, t) = 0 \quad \dots(2)$$

Also the initial conditions are

$$i(x, 0) = i_0$$

and

$$v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} \quad \dots(3)$$

$$\therefore \frac{\partial i}{\partial x} = -c \frac{\partial v}{\partial t} \text{ gives } \left(\frac{\partial v}{\partial t} \right)_{t=0} = 0 \quad \dots(4)$$

Let $v = XT$ be the solution of (1) where X is a function of x only and T is a function of t only.

$$\therefore \text{ From (1), } TX'' = LCXT'' \Rightarrow \frac{X''}{X} = LC \frac{T''}{T} = -p^2 \text{ (say)}$$

$$\therefore X'' + p^2 X = 0 \Rightarrow X = c_1 \cos px + c_2 \sin px \text{ and } T'' + \left(\frac{p^2}{LC} \right) T = 0$$

$$\Rightarrow T = c_3 \cos \frac{p}{\sqrt{LC}} \cdot t + c_4 \sin \frac{p}{\sqrt{LC}} \cdot t$$

$$\therefore v = XT = (c_1 \cos px + c_2 \sin px) \left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad \dots(5)$$

Using the boundary conditions (2), we get

$$c_1 = 0 \text{ and } p = \frac{n\pi}{l}, n \in \mathbb{I}$$

$$\therefore v = \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi t}{l\sqrt{LC}} + b_n \sin \frac{n\pi t}{l\sqrt{LC}} \right) \quad \dots(6)$$

Using initial condition (4), we get

$$b_n = 0$$

$$\therefore \text{ From (6), } v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Thus the most general solution of (1) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \quad \dots(7)$$

Finally by the initial condition (3), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\therefore a_1 = e_1, a_5 = e_5 \text{ while all other } a\text{'s are zero.}$$

$$\text{Hence, } v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.



ABOUT THE BOOK

The basic objective of this book is to provide solutions to as many problems as is possible together to the students of engineering colleges and technology institutions. The solutions are of the topics included in the latest syllabus of Engineering Mathematics for First Year Semester-I & II engineering students. This book has been prepared to help the students in solving problems which have been selected from different textbooks as well as questions from recent papers (2000 onwards) of various University examinations. Solutions of 1115 problems and other questions have been provided in this book in an easily understandable manner which is the most distinguishing feature of this book.

ABOUT THE AUTHOR

Dr. T. C. Gupta has an extraordinary calibre and significant academic achievements. He obtained first class Master's degree in Mathematics from Punjab University in 1968 and secured second position in the university in order of merit in this examination. He started his career as a Teacher and Researcher with defence Research and Development Organization, New Delhi. He has done his Ph.D. in Statistics. He has worked for many years as a Research Scientist with Indian Council of Medical Research, New Delhi and retired as Deputy Director-General. During the past decade he has visited various countries-USA, Canada, Italy, Hongkong and Singapore. He has attended many Scientific Conferences in India and abroad and presented his original research papers. He has to his credit several research publications in the national and international journals of great repute.

At present, Dr. T. C. Gupta is teaching at Manav Rachna Engineering College, Faridabad (Haryana) as Professor of Mathematics with the Department of Applied Sciences & Humanities.



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