
FOURTH EDITION

CALCULUS

WITH ANALYTIC GEOMETRY



PROTTER AND PROTTER



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WITH ANALYTIC GEOMETRY

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CONTENTS

1	INEQUALITIES. FUNCTIONS. THE LINE.	1
1	INEQUALITIES.	1
2	ABSOLUTE VALUE.	7
3	ABSOLUTE VALUE AND INEQUALITIES.	10
4	SETS. SET NOTATION. GRAPHS.	14
5	FUNCTIONS. FUNCTIONAL NOTATION.	19
6	RELATIONS. INTERCEPTS. ASYMPTOTES.	25
7	DISTANCE FORMULA. MIDPOINT FORMULA.	31
8	SLOPE OF A LINE. PARALLEL AND PERPENDICULAR LINES.	35
9	THE STRAIGHT LINE:	41
	REVIEW PROBLEMS.	46
2	LIMITS. THE DERIVATIVE. APPLICATIONS.	49
1	LIMITS (INTUITIVE).	49
2	COMPUTATION OF LIMITS.	55
3	THE DERIVATIVE.	58
4	GEOMETRIC INTERPRETATION OF THE DERIVATIVE.	62
5	INSTANTANEOUS VELOCITY AND SPEED: ACCELERATION.	66
6	DEFINITION OF LIMIT (OPTIONAL)	71
7	THEOREMS ON LIMITS.	75
8	CONTINUITY. ONE-SIDED LIMITS.	83
9	LIMITS AT INFINITY; INFINITE LIMITS (OPTIONAL).	89
	REVIEW PROBLEMS.	94
3	DIFFERENTIATION.	97
1	THEOREMS ON DIFFERENTIATION.	97
2	THE CHAIN RULE. APPLICATIONS. THE POWER FUNCTION.	104
3	IMPLICIT DIFFERENTIATION.	111
4	TRIGONOMETRIC LIMITS.	115
5	THE DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS.	118
	REVIEW PROBLEMS.	122
4	APPLICATIONS OF THE DERIVATIVE.	125
1	TOOLS FOR APPLICATIONS OF THE DERIVATIVE.	125

CONTENTS

2	FURTHER TOOLS: ROLLE'S THEOREM: MEAN VALUE THEOREM.	130
3	APPLICATIONS TO GRAPHS OF FUNCTIONS.	135
4	APPLICATIONS USING THE SECOND DERIVATIVE.	140
5	THE MAXIMUM AND MINIMUM VALUES OF A FUNCTION ON AN INTERVAL.	147
6	APPLICATIONS OF MAXIMA AND MINIMA.	149
7	THE DIFFERENTIAL: APPROXIMATION.	156
8	DIFFERENTIAL NOTATION.	161
9	NEWTON'S METHOD. (OPTIONAL).	165
10	RELATED RATES.	169
11	ANTIDERIVATIVES.	174
	REVIEW PROBLEMS.	178
5	THE DEFINITE INTEGRAL.	181
1	AREA.	181
2	THE DEFINITE INTEGRAL.	187
3	THE DEFINITE INTEGRAL AND AREA.	193
4	PROPERTIES OF THE DEFINITE INTEGRAL.	196
5	EVALUATION OF DEFINITE INTEGRALS. FUNDAMENTAL THEOREM OF CALCULUS.	200
6	MEAN VALUE THEOREM FOR INTEGRALS.	206
7	THE INDEFINITE INTEGRAL. INTEGRATION BY SUBSTITUTION.	213
	REVIEW PROBLEMS.	218
6	APPLICATIONS OF THE INTEGRAL.	223
1	AREA BETWEEN CURVES.	223
2	VOLUME OF A SOLID OF REVOLUTION. DISK METHOD.	228
3	VOLUME OF A SOLID OF REVOLUTION. SHELL METHOD.	233
4	ARC LENGTH.	236
5	AREA OF A SURFACE OF REVOLUTION.	242
	REVIEW PROBLEMS.	246
7	LOGARITHMS. THE EXPONENTIAL FUNCTION. INVERSE FUNCTIONS.	249
1	THE LOGARITHM FUNCTION.	249
2	RELATIONS AND INVERSE FUNCTIONS.	256
3	THE EXPONENTIAL FUNCTION. THE NUMBER e .	262
4	DIFFERENTIATION OF EXPONENTIAL FUNCTIONS. LOGARITHMIC DIFFERENTIATION.	269
5	THE INVERSE TRIGONOMETRIC FUNCTIONS.	273
6	INTEGRATIONS YIELDING INVERSE TRIGONOMETRIC FUNCTIONS.	281
7	APPLICATIONS.	283
8	THE HYPERBOLIC FUNCTIONS.	288
	REVIEW PROBLEMS.	288

CONTENTS

8	TECHNIQUES OF INTEGRATION.	291
1	INTEGRATION BY SUBSTITUTION.	291
2	INTEGRATION BY PARTS.	299
3	INTEGRATION OF TRIGONOMETRIC EXPRESSIONS.	305
4	TRIGONOMETRIC SUBSTITUTION.	310
5	INTEGRANDS INVOLVING QUADRATIC FUNCTIONS.	313
6	INTEGRATION OF RATIONAL FUNCTIONS; METHOD OF PARTIAL FRACTIONS.	315
7	TWO RATIONALIZING SUBSTITUTIONS (OPTIONAL).	323
	REVIEW PROBLEMS.	326
9	FURTHER APPLICATIONS OF INTEGRATION.	329
1	CENTER OF MASS.	329
2	CENTER OF MASS OF A PLANE REGION.	333
3	CENTER OF MASS OF A SOLID OF REVOLUTION.	339
4	CENTERS OF MASS OF WIRES AND SURFACES.	341
5	THEOREMS OF PAPPUS (OPTIONAL).	345
6	IMPROPER INTEGRALS.	347
7	WORK (OPTIONAL).	353
8	FLUID PRESSURE (OPTIONAL).	358
9	NUMERICAL INTEGRATION.	362
10	APPLICATIONS TO PROBABILITY THEORY (OPTIONAL).	369
	REVIEW PROBLEMS.	376
10	INFINITE SERIES.	379
1	INDETERMINATE FORMS.	379
2	SEQUENCES.	386
3	CONVERGENT AND DIVERGENT SERIES.	392
4	SERIES OF POSITIVE TERMS. COMPARISON AND LIMIT COMPARISON TESTS. THE INTEGRAL TEST.	397
5	SERIES OF POSITIVE AND NEGATIVE TERMS. THE ALTERNATING SERIES TEST. THE RATIO TEST. THE ROOT TEST.	407
6	POWER SERIES.	416
7	TAYLOR'S SERIES.	421
8	TAYLOR'S THEOREM WITH REMAINDER.	426
9	DIFFERENTIATION AND INTEGRATION OF SERIES.	433
10	NUMERICAL COMPUTATION WITH SERIES.	440
	REVIEW PROBLEMS.	445
11	ANALYTIC GEOMETRY IN THE PLANE.	449
1	POINTS, LINES, AND ANGLES.	449
2	THE PARABOLA.	455
3	THE CIRCLE AND THE ELLIPSE.	461
4	THE HYPERBOLA.	469
5	TRANSLATION OF AXES.	476
6	ROTATION OF AXES. THE GENERAL EQUATION OF THE SECOND DEGREE.	480
	REVIEW PROBLEMS.	487

12	PARAMETRIC EQUATIONS. ARC LENGTH. POLAR COORDINATES.	489
1	PARAMETRIC EQUATIONS.	489
2	DERIVATIVES AND PARAMETRIC EQUATIONS. ARC LENGTH.	496
3	CURVATURE.	503
4	POLAR COORDINATES. GRAPHS IN POLAR COORDINATES.	507
5	EQUATIONS AND CONIC SECTIONS IN POLAR COORDINATES.	514
6	DERIVATIVES IN POLAR COORDINATES.	521
7	AREA IN POLAR COORDINATES.	526
	REVIEW PROBLEMS.	529
13	ANALYTIC GEOMETRY IN THREE DIMENSIONS.	533
1	THE NUMBER SPACE R^3 . COORDINATES. THE DISTANCE FORMULA.	533
2	DIRECTION COSINES AND NUMBERS.	537
3	EQUATIONS OF LINES AND PLANES IN R^3 .	542
4	ANGLES. DISTANCE FROM A POINT TO A PLANE.	548
5	THE SPHERE. CYLINDERS.	553
6	CYLINDRICAL AND SPHERICAL COORDINATES.	556
	REVIEW PROBLEMS.	559
14	VECTORS IN THE PLANE AND IN THREE-SPACE.	563
1	VECTORS IN THE PLANE.	563
2	OPERATIONS WITH PLANE VECTORS. THE SCALAR PRODUCT.	570
3	VECTORS IN THREE DIMENSIONS.	575
4	THE SCALAR (INNER OR DOT) PRODUCT.	580
5	THE VECTOR OR CROSS PRODUCT.	584
6	PRODUCTS OF THREE VECTORS.	590
7	LINEAR DEPENDENCE AND INDEPENDENCE (OPTIONAL).	594
8	VECTOR FUNCTIONS IN THE PLANE AND THEIR DERIVATIVES.	598
9	APPLICATION OF VECTORS TO MOTION IN THE PLANE.	602
10	NEWTON'S LAW OF MOTION. DIFFERENTIAL EQUATIONS.	604
11	VECTOR FUNCTIONS IN SPACE. DERIVATIVES. TANGENTS. ARC LENGTH.	610
12	TANGENTIAL AND NORMAL COMPONENTS. THE MOVING TRIHEDRAL (OPTIONAL).	614
	REVIEW PROBLEMS.	619
15	PARTIAL DERIVATIVES. APPLICATIONS.	623
1	LIMITS AND CONTINUITY. PARTIAL DERIVATIVES.	623
2	IMPLICIT DIFFERENTIATION.	630
3	THE CHAIN RULE.	633
4	APPLICATIONS OF THE CHAIN RULE.	638
5	DIRECTIONAL DERIVATIVES. GRADIENT.	641

6	GEOMETRIC INTERPRETATIONS OF PARTIAL DERIVATIVES. TANGENT PLANES.	647
7	THE TOTAL DIFFERENTIAL. APPROXIMATION.	654
8	APPLICATIONS OF THE TOTAL DIFFERENTIAL.	659
9	SECOND AND HIGHER DERIVATIVES.	665
10	TAYLOR'S THEOREM WITH REMAINDER (OPTIONAL).	671
11	MAXIMA AND MINIMA.	676
12	MAXIMA AND MINIMA; LAGRANGE MULTIPLIERS (OPTIONAL).	686
13	EXACT DIFFERENTIALS.	692
	REVIEW PROBLEMS.	698
16	MULTIPLE INTEGRATION.	703
1	DEFINITION OF THE DOUBLE INTEGRAL.	703
2	PROPERTIES OF THE DOUBLE INTEGRAL.	707
3	EVALUATION OF DOUBLE INTEGRALS. ITERATED INTEGRALS.	709
4	AREA, DENSITY, AND MASS.	718
5	EVALUATION OF DOUBLE INTEGRALS BY POLAR COORDINATES.	720
6	MOMENT OF INERTIA AND CENTER OF MASS (OPTIONAL).	727
7	SURFACE AREA	734
8	VOLUMES OF SOLIDS OF REVOLUTION (OPTIONAL).	739
9	THE TRIPLE INTEGRAL.	741
10	MASS OF A SOLID. TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES.	747
11	MOMENT OF INERTIA. CENTER OF MASS (OPTIONAL).	753
	REVIEW PROBLEMS.	755
17	VECTOR FIELD THEORY.	759
1	DEFINITION OF A LINE INTEGRAL.	759
2	CALCULATION OF LINE INTEGRALS.	762
3	PATH-INDEPENDENT LINE INTEGRALS.	767
4	GREEN'S THEOREM.	774
5	SURFACES. SURFACE INTEGRALS (OPTIONAL).	781
6	ORIENTABLE SURFACES. STOKES' THEOREM.	790
7	THE DIVERGENCE THEOREM.	798
	REVIEW PROBLEMS.	806
18	DIFFERENTIAL EQUATIONS.	809
1	INTRODUCTION AND EXAMPLES.	809
2	SEPARABLE DIFFERENTIAL EQUATIONS.	813
3	EXACT DIFFERENTIALS. INTEGRATING FACTORS.	818
4	FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS.	822
5	APPLICATIONS.	824
6	SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.	830
7	NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.	836
8	SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS.	842
	REVIEW PROBLEMS.	846

APPENDIX 1 TRIGONOMETRY REVIEW. APP 1

- 1 BASIC FORMULAS. APP 1
- 2 GRAPHS OF TRIGONOMETRIC FUNCTIONS. APP 6

APPENDIX 2 CONIC SECTIONS. QUADRIC SURFACES. APP 10

- 1 THE CIRCLE (CONTINUED FROM CHAPTER 11, SEC. 3) APP 10
- 2 THE PARABOLA (CONTINUED FROM CHAPTER 11, SEC. 2) APP 13
- 3 THE ELLIPSE (CONTINUED FROM CHAPTER 11, SEC. 3) APP 17
- 4 THE HYPERBOLA (CONTINUED FROM CHAPTER 11, SEC. 4) APP 20
- 5 QUADRIC SURFACES. APP 24

APPENDIX 3 PROPERTIES OF HYPERBOLIC FUNCTIONS. APP 30

- 1 THE HYPERBOLIC FUNCTIONS. APP 30
- 2 THE INVERSE HYPERBOLIC FUNCTIONS. APP 33

APPENDIX 4 THEOREMS ON DETERMINANTS. APP 38

APPENDIX 5 PROOFS OF THEOREMS 8, 17, 18, AND 23 OF CHAPTER 14. APP 48

APPENDIX 6 INTRODUCTION TO THE USE OF A TABLE OF INTEGRALS. APP 53

- 1 SUBSTITUTION IN A TABLE OF INTEGRALS. APP 53
- 2 CERTAIN TRIGONOMETRIC AND HYPERBOLIC INTEGRALS. APP 53
- 3 TRIGONOMETRIC AND HYPERBOLIC SUBSTITUTIONS. APP 55
- 4 INTEGRALS INVOLVING QUADRATIC FUNCTIONS. APP 56
- 5 INTEGRATION BY PARTS. APP 56
- 6 INTEGRATION OF RATIONAL FUNCTIONS (QUOTIENTS OF POLYNOMIALS) APP 56
- 7 THREE RATIONALIZING SUBSTITUTIONS. APP 58

ANSWERS TO ODD-NUMBERED PROBLEMS. ANS 1

INDEX. INDEX 1

▶ A TABLE OF DERIVATIVES

- $D_x(u^n) = nu^{n-1} D_x u$
- $D_x(u + v) = D_x u + D_x v$
- $D_x(uv) = uD_x v + v D_x u$
- $D_x\left(\frac{u}{v}\right) = \frac{v D_x u - u D_x v}{v^2}$
- $D_x(e^u) = e^u D_x u$
- $D_x(a^u) = a^u \ln a D_x u$
- $D_x(\ln u) = \frac{1}{u} D_x u$
- $D_x(\sin u) = \cos u D_x u$
- $D_x(\cos u) = -\sin u D_x u$
- $D_x(\tan u) = \sec^2 u D_x u$
- $D_x(\cot u) = -\csc^2 u D_x u$
- $D_x(\sec u) = \sec u \tan u D_x u$
- $D_x(\csc u) = -\csc u \cot u D_x u$
- $D_x(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} D_x u$

- $D_x(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} D_x u$
- $D_x(\tan^{-1} u) = \frac{1}{1+u^2} D_x u$
- $D_x(\cot^{-1} u) = \frac{-1}{1+u^2} D_x u$
- $D_x(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}} D_x u$
- $D_x(\csc^{-1} u) = \frac{-1}{u\sqrt{u^2-1}} D_x u$
- $D_x(\sinh u) = \cosh u D_x u$
- $D_x(\cosh u) = \sinh u D_x u$
- $D_x(\tanh u) = \text{sech}^2 u D_x u$
- $D_x(\coth u) = -\text{csch}^2 u D_x u$
- $D_x(\text{sech } u) = -\text{sech } u \tanh u D_x u$
- $D_x(\text{csch } u) = -\text{csch } u \coth u D_x u$

▶ A TABLE OF INTEGRALS

Some Elementary Forms

- $\int du = u + C$
- $\int a du = au + C$
- $\int [f(u) + g(u)] du = \int f(u) du + \int g(u) du$
- $\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int \frac{du}{u} = \ln|u| + C$

Rational Forms Containing $a + bu$

- $\int \frac{u du}{a + bu} = \frac{1}{b^2} [a + bu - a \ln|a + bu|] + C$
- $\int \frac{u^2 du}{a + bu} = \frac{1}{b^3} \left[\frac{1}{2}(a + bu)^2 - 2a(a + bu) + a^2 \ln|a + bu| \right] + C$
- $\int \frac{u du}{(a + bu)^2} = \frac{1}{b^2} \left[\frac{a}{a + bu} + \ln|a + bu| \right] + C$
- $\int \frac{u^2 du}{(a + bu)^2} = \frac{1}{b^3} \left[a + bu - \frac{a^2}{a + bu} - 2a \ln|a + bu| \right] + C$
- $\int \frac{u du}{(a + bu)^3} = \frac{1}{b^2} \left[\frac{a}{2(a + bu)^2} - \frac{1}{a + bu} \right] + C$
- $\int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right| + C$
- $\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$
- $\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} + \frac{1}{a^2} \ln \left| \frac{u}{a + bu} \right| + C$

Forms Containing $\sqrt{a + bu}$

- $\int u\sqrt{a + bu} du = \frac{2}{15b^3} (3bu - 2a)(a + bu)^{3/2} + C$
- $\int u^n \sqrt{a + bu} du = \frac{2u^n(a + bu)^{3/2}}{b(2n + 3)} - \frac{2an}{b(2n + 3)} \int u^{n-1} \sqrt{a + bu} du$
- $\int u^2 \sqrt{a + bu} du = \frac{2}{105b^3} (15b^2 u^2 - 12abu + 8a^2)(a + bu)^{3/2} + C$
- $\int \frac{u du}{\sqrt{a + bu}} = \frac{2}{3b^2} (bu - 2a) \sqrt{a + bu} + C$

$$18. \int \frac{u^2 du}{\sqrt{a+bu}} = \frac{2}{15b^3} (3b^2u^2 - 4abu + 8a^2) \sqrt{a+bu} + C$$

$$19. \int \frac{u^n du}{\sqrt{a+bu}} = \frac{2u^n \sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)} \int \frac{u^{n-1} du}{\sqrt{a+bu}}$$

$$20. \int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}} \right| + C & \text{if } a > 0 \\ \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}} + C & \text{if } a < 0 \end{cases}$$

$$21. \int \frac{du}{u^n \sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a+bu}}$$

$$22. \int \frac{\sqrt{a+bu} du}{u} = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}$$

$$23. \int \frac{\sqrt{a+bu} du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)} \int \frac{\sqrt{a+bu} du}{u^{n-1}}$$

Forms Containing $a^2 \pm u^2$

$$24. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$26. \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C = \begin{cases} -\frac{1}{a} \tanh^{-1} \frac{u}{a} + C & \text{if } |u| < a \\ -\frac{1}{a} \coth^{-1} \frac{u}{a} + C & \text{if } |u| > a \end{cases}$$

$$25. \int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + C = \begin{cases} \frac{1}{a} \tanh^{-1} \frac{u}{a} + C & \text{if } |u| < a \\ \frac{1}{a} \coth^{-1} \frac{u}{a} + C & \text{if } |u| > a \end{cases}$$

Forms Containing $\sqrt{u^2 \pm a^2}$

In formulas 27 through 38, we may replace

$$\ln(u + \sqrt{u^2 + a^2}) \quad \text{by} \quad \sinh^{-1} \frac{u}{a}$$

$$\ln|u + \sqrt{u^2 - a^2}| \quad \text{by} \quad \cosh^{-1} \frac{u}{a}$$

$$\ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| \quad \text{by} \quad \sinh^{-1} \frac{a}{u}$$

$$27. \int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$28. \int \sqrt{u^2 \pm a^2} du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$29. \int u^2 \sqrt{u^2 \pm a^2} du = \frac{u}{8} (2u^2 \pm a^2) \sqrt{u^2 \pm a^2} - \frac{a^4}{8} \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$30. \int \frac{\sqrt{u^2 + a^2} du}{u} = \sqrt{u^2 + a^2} - a \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$$

$$31. \int \frac{\sqrt{u^2 - a^2} du}{u} = \sqrt{u^2 - a^2} - a \sec^{-1} \frac{u}{a} + C$$

$$32. \int \frac{\sqrt{u^2 \pm a^2} du}{u^2} = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$33. \int \frac{u^2 du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2} \sqrt{u^2 \pm a^2} - \frac{\pm a^2}{2} \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$34. \int \frac{du}{u\sqrt{u^2 + a^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$$

$$35. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

$$36. \int \frac{du}{u^2 \sqrt{u^2 \pm a^2}} = -\frac{\sqrt{u^2 \pm a^2}}{\pm a^2 u} + C$$

$$37. \int (u^2 \pm a^2)^{3/2} du = \frac{u}{8} (2u^2 \pm 5a^2) \sqrt{u^2 \pm a^2} + \frac{3a^4}{8} \ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$38. \int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{\pm a^2 \sqrt{u^2 \pm a^2}} + C$$

Forms Containing $\sqrt{a^2 - u^2}$

$$39. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$40. \int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$41. \int u^2 \sqrt{a^2 - u^2} du = \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$42. \int \frac{\sqrt{a^2 - u^2} du}{u} = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C = \sqrt{a^2 - u^2} - a \cosh^{-1} \frac{a}{u} + C$$

$$43. \int \frac{\sqrt{a^2 - u^2} du}{u^2} = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1} \frac{u}{a} + C$$

(This table is continued on the back endpapers.)

$$44. \int \frac{u^2 du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$45. \int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$= -\frac{1}{a} \cosh^{-1} \frac{a}{u} + C$$

$$46. \int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C$$

$$47. \int (a^2 - u^2)^{3/2} du = -\frac{u}{8} (2u^2 - 5a^2) \sqrt{a^2 - u^2}$$

$$+ \frac{3a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$48. \int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 - u^2}} + C$$

Forms Containing $2au - u^2$

$$49. \int \sqrt{2au - u^2} du = \frac{u-a}{2} \sqrt{2au - u^2} + \frac{a^2}{2} \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$50. \int u \sqrt{2au - u^2} du = \frac{2u^2 - au - 3a^2}{6} \sqrt{2au - u^2}$$

$$+ \frac{a^3}{2} \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$51. \int \frac{\sqrt{2au - u^2} du}{u} = \sqrt{2au - u^2} + a \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$52. \int \frac{\sqrt{2au - u^2} du}{u^2} = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$53. \int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$54. \int \frac{u du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$55. \int \frac{u^2 du}{\sqrt{2au - u^2}} = -\frac{(u+3a)}{2} \sqrt{2au - u^2}$$

$$+ \frac{3a^2}{2} \cos^{-1} \left(1 - \frac{u}{a} \right) + C$$

$$56. \int \frac{du}{u \sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$$

$$57. \int \frac{du}{(2au - u^2)^{3/2}} = \frac{u-a}{a^2 \sqrt{2au - u^2}} + C$$

$$58. \int \frac{u du}{(2au - u^2)^{3/2}} = \frac{u}{a \sqrt{2au - u^2}} + C$$

Forms Containing Trigonometric Functions

$$59. \int \sin u du = -\cos u + C$$

$$60. \int \cos u du = \sin u + C$$

$$61. \int \tan u du = \ln |\sec u| + C$$

$$62. \int \cot u du = \ln |\sin u| + C$$

$$63. \int \sec u du = \ln |\sec u + \tan u| + C = \ln \left| \tan \left(\frac{1}{2} \pi + \frac{1}{2} u \right) \right| + C$$

$$64. \int \csc u du = \ln |\csc u - \cot u| + C = \ln \left| \tan \frac{1}{2} u \right| + C$$

$$65. \int \sec^2 u du = \tan u + C$$

$$66. \int \csc^2 u du = -\cot u + C$$

$$67. \int \sec u \tan u du = \sec u + C$$

$$68. \int \csc u \cot u du = -\csc u + C$$

$$69. \int \sin^2 u du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C$$

$$70. \int \cos^2 u du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$$

$$71. \int \tan^2 u du = \tan u - u + C$$

$$72. \int \cot^2 u du = -\cot u - u + C$$

$$73. \int \sin^n u du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u du$$

$$74. \int \cos^n u du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u du$$

$$75. \int \tan^n u du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u du$$

$$76. \int \cot^n u du = -\frac{1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u du$$

$$77. \int \sec^n u du = \frac{1}{n-1} \sec^{n-2} u \tan u + \frac{n-2}{n-1} \int \sec^{n-2} u du$$

$$78. \int \csc^n u du = -\frac{1}{n-1} \csc^{n-2} u \cot u + \frac{n-2}{n-1} \int \csc^{n-2} u du$$

$$79. \int \sin mu \sin nu du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$$

$$80. \int \cos mu \cos nu du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$$

$$81. \int \sin mu \cos nu du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$$

$$82. \int u \sin u du = \sin u - u \cos u + C$$

$$83. \int u \cos u du = \cos u + u \sin u + C$$

$$84. \int u^2 \sin u du = 2u \sin u + (2-u^2) \cos u + C$$

$$85. \int u^2 \cos u \, du = 2u \cos u + (u^2 - 2) \sin u + C$$

$$86. \int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$$

$$87. \int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$$

$$88. \int \sin^m u \cos^n u \, du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} u \cos^n u \, du \\ = \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n} \int \sin^m u \cos^{n-2} u \, du$$

Forms Containing Inverse Trigonometric Functions

$$89. \int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$

$$90. \int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1-u^2} + C$$

$$91. \int \tan^{-1} u \, du = u \tan^{-1} u - \ln \sqrt{1+u^2} + C$$

$$92. \int \cot^{-1} u \, du = u \cot^{-1} u + \ln \sqrt{1+u^2} + C$$

$$93. \int \sec^{-1} u \, du = u \sec^{-1} u - \ln|u + \sqrt{u^2-1}| + C \\ = u \sec^{-1} u - \cosh^{-1} u + C$$

$$94. \int \csc^{-1} u \, du = u \csc^{-1} u + \ln|u + \sqrt{u^2-1}| + C \\ = u \csc^{-1} u + \cosh^{-1} u + C$$

Forms Containing Exponential and Logarithmic Functions

$$95. \int e^u \, du = e^u + C$$

$$96. \int a^u \, du = \frac{a^u}{\ln a} + C$$

$$97. \int u e^u \, du = e^u(u-1) + C$$

$$98. \int u^n e^u \, du = u^n e^u - n \int u^{n-1} e^u \, du$$

$$99. \int u^n a^u \, du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a} \int u^{n-1} a^u \, du + C$$

$$100. \int \frac{e^u \, du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1} \int \frac{e^u \, du}{u^{n-1}}$$

$$101. \int \frac{a^u \, du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1} \int \frac{a^u \, du}{u^{n-1}}$$

$$102. \int \ln u \, du = u \ln u - u + C$$

$$103. \int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2} [(n+1) \ln u - 1] + C$$

$$104. \int \frac{du}{u \ln u} = \ln|\ln u| + C$$

$$105. \int e^{au} \sin nu \, du = \frac{e^{au}}{a^2 + n^2} (a \sin nu - n \cos nu) + C$$

$$106. \int e^{au} \cos nu \, du = \frac{e^{au}}{a^2 + n^2} (a \cos nu + n \sin nu) + C$$

Forms Containing Hyperbolic Functions

$$107. \int \sinh u \, du = \cosh u + C$$

$$108. \int \cosh u \, du = \sinh u + C$$

$$109. \int \tanh u \, du = \ln|\cosh u| + C$$

$$110. \int \coth u \, du = \ln|\sinh u| + C$$

$$111. \int \operatorname{sech} u \, du = \tan^{-1}(\sinh u) + C$$

$$112. \int \operatorname{csch} u \, du = \ln|\tanh \frac{1}{2} u| + C$$

$$113. \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$114. \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$115. \int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$116. \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

$$117. \int \sinh^2 u \, du = \frac{1}{4} \sinh 2u - \frac{1}{2} u + C$$

$$118. \int \cosh^2 u \, du = \frac{1}{4} \sinh 2u + \frac{1}{2} u + C$$

$$119. \int \tanh^2 u \, du = u - \tanh u + C$$

$$120. \int \coth^2 u \, du = u - \coth u + C$$

$$121. \int u \sinh u \, du = u \cosh u - \sinh u + C$$

$$122. \int u \cosh u \, du = u \sinh u - \cosh u + C$$

$$123. \int e^{au} \sinh nu \, du = \frac{e^{au}}{a^2 - n^2} (a \sinh nu - n \cosh nu) + C$$

$$124. \int e^{au} \cosh nu \, du = \frac{e^{au}}{a^2 - n^2} (a \cosh nu - n \sinh nu) + C$$

FORMULAS FROM TRIGONOMETRY

THE EIGHT FUNDAMENTAL TRIGONOMETRIC IDENTITIES

$$\sin x \csc x = 1 \quad \cos x \sec x = 1 \quad \tan x \cot x = 1$$

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

SUM AND DIFFERENCE IDENTITIES

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \quad \tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

MULTIPLE-MEASURE IDENTITIES

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$\cos 2u = 1 - 2 \sin^2 u \quad \cos 2u = 2 \cos^2 u - 1$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\sin^2 u = \frac{1 - \cos 2u}{2} \quad \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

$$\sin^2 \frac{1}{2}t = \frac{1 - \cos t}{2} \quad \cos^2 \frac{1}{2}t = \frac{1 + \cos t}{2}$$

$$\tan^2 \frac{1}{2}t = \frac{1 - \cos t}{1 + \cos t} \quad \tan \frac{1}{2}t = \frac{\sin t}{1 + \cos t}$$

IDENTITIES FOR THE PRODUCT, SUM, AND DIFFERENCE OF SINE AND COSINE

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\sin s + \sin t = 2 \sin\left(\frac{s + t}{2}\right) \cos\left(\frac{s - t}{2}\right)$$

$$\sin s - \sin t = 2 \cos\left(\frac{s + t}{2}\right) \sin\left(\frac{s - t}{2}\right)$$

$$\cos s + \cos t = 2 \cos\left(\frac{s + t}{2}\right) \cos\left(\frac{s - t}{2}\right)$$

$$\cos s - \cos t = -2 \sin\left(\frac{s + t}{2}\right) \sin\left(\frac{s - t}{2}\right)$$

SOME REDUCTION FORMULAS

$$\begin{aligned}\sin(-x) &= -\sin x & \cos(-x) &= \cos x & \tan(-x) &= -\tan x \\ \sin(\tfrac{1}{2}\pi - x) &= \cos x & \cos(\tfrac{1}{2}\pi - x) &= \sin x \\ \tan(\tfrac{1}{2}\pi - x) &= \cot x \\ \sin(\tfrac{1}{2}\pi + x) &= \cos x & \cos(\tfrac{1}{2}\pi + x) &= -\sin x \\ \tan(\tfrac{1}{2}\pi + x) &= -\cot x \\ \sin(\pi - x) &= \sin x & \cos(\pi - x) &= -\cos x \\ \tan(\pi - x) &= -\tan x\end{aligned}$$

LAW OF SINES AND LAW OF COSINES

a , b , and c represent the measures of the sides of a triangle: α , β , and γ represent the measures of the angles opposite the sides of measures a , b , and c , respectively.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \quad c^2 = a^2 + b^2 - 2ab \cos \gamma$$

FORMULAS FROM GEOMETRY

The following symbols are used for the measure:

r : radius h : altitude b : base a : base C : circumference
 A : area S : surface area B : area of base V : volume

Circle: $A = \pi r^2$; $C = 2\pi r$

Triangle: $A = \frac{1}{2}bh$

Rectangle and parallelogram: $A = bh$

Trapezoid: $A = \frac{1}{2}(a + b)h$

Right circular cylinder: $V = \pi r^2 h$; $S = 2\pi r h$

Right circular cone: $V = \frac{1}{3}\pi r^2 h$; $S = \pi r \sqrt{r^2 + h^2}$

Sphere: $V = \frac{4}{3}\pi r^3$; $S = 4\pi r^2$

Prism (with parallel bases): $V = Bh$

Pyramid: $V = \frac{1}{3}Bh$

The Greek alphabet

α	alpha	ι	iota	ρ	rho
β	beta	κ	kappa	σ	sigma
γ	gamma	λ	lambda	τ	tau
δ	delta	μ	mu	υ	upsilon
ϵ	epsilon	ν	nu	ϕ	phi
ζ	zeta	ξ	xi	χ	chi
η	eta	$\omicron$	omicron	ψ	psi
θ	theta	π	pi	ω	omega

EXAMPLE 3 Making these substitutions, we obtain the following simplification of an integral.

$$\int \frac{d\theta}{5 - 4 \cos \theta} = \int \frac{(2 dt)/(1 + t^2)}{5 - 4[(1 - t^2)/(1 + t^2)]} = \int \frac{2 dt}{1 + 9t^2}. \quad \square$$

A SHORT TABLE OF INTEGRALS

The constant-of integration is omitted.

Elementary formulas

1. $\int u^n du = \frac{u^{n+1}}{n+1}, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln |u|$
3. $\int e^u du = e^u$
4. $\int a^u du = \frac{a^u}{\ln a}, \quad a > 0, a \neq 1$
5. $\int \sin u du = -\cos u$
6. $\int \cos u du = \sin u$
7. $\int \sec^2 u du = \tan u$
8. $\int \csc^2 u du = -\cot u$
9. $\int \sec u \tan u du = \sec u$
10. $\int \csc u \cot u du = -\csc u$
11. $\int \sinh u du = \cosh u$
12. $\int \cosh u du = \sinh u$
13. $\int \operatorname{sech}^2 u du = \tanh u$
14. $\int \operatorname{csch}^2 u du = -\coth u$
15. $\int \operatorname{sech} u \tanh u du = -\operatorname{sech} u$
16. $\int \operatorname{csch} u \coth u du = -\operatorname{csch} u$
17. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a}, \quad a > |u|$
18. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a}, \quad a \neq 0$
19. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a}, \quad |u| > a$
20. $\int \frac{du}{\sqrt{a^2 + u^2}} = \begin{cases} \operatorname{argsinh} \frac{u}{a} \\ \ln(u + \sqrt{a^2 + u^2}) \end{cases}$
21. $\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a} \operatorname{argtanh} \frac{u}{a}, & |u| < a \\ \frac{1}{2a} \ln \frac{a+u}{a-u}, & |u| < a \end{cases}$
 $\begin{cases} \frac{1}{a} \operatorname{argcoth} \frac{u}{a}, & |u| > a \\ \frac{1}{2a} \ln \frac{u+a}{u-a}, & |u| > a \end{cases}$
22. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \operatorname{argsech} \frac{u}{a}, \quad \text{or} \quad -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right), \quad 0 < u < a$

$$23. \int \frac{du}{|u|\sqrt{u^2 + a^2}} = -\frac{1}{a} \operatorname{arcsch} \frac{u}{a}, \quad u \neq 0 \quad \text{or} \quad -\frac{1}{a} \ln \left(\frac{a + \sqrt{u^2 + a^2}}{u} \right)$$

$$24. \int \frac{du}{\sqrt{u^2 - a^2}} = \begin{cases} \operatorname{arcosh} \frac{u}{a}, \\ \ln |u + \sqrt{u^2 - a^2}|, \end{cases} \quad |u| > a > 0$$

Algebraic forms

$$25. \int \frac{u \, du}{a + bu} = \frac{u}{b} - \frac{a}{b^2} \ln(a + bu) \qquad 26. \int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right|$$

$$27. \int \frac{u \, du}{(a + bu)^2} = \frac{a}{b^2} \left(\frac{1}{a + bu} + \frac{1}{a} \ln |a + bu| \right)$$

$$28. \int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} + \frac{1}{a^2} \ln \left| \frac{u}{a + bu} \right|$$

$$29. \int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right|$$

$$30. \int u\sqrt{a + bu} \, du = \frac{2(3bu - 2a)\sqrt{(a + bu)^3}}{15b^2}$$

$$31. \int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$

$$32. \int \frac{u \, du}{\sqrt{a + bu}} = \frac{2(bu - 2a)}{3b^2} \sqrt{a + bu}$$

$$33. \int \sqrt{a^2 - u^2} \, du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a}, \quad |u| < a$$

$$34. \int \sqrt{u^2 \pm a^2} \, du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln |u + \sqrt{u^2 \pm a^2}|$$

$$35. \int \frac{\sqrt{a^2 \pm u^2}}{u} \, du = \sqrt{a^2 \pm u^2} - a \ln \left| \frac{a + \sqrt{a^2 \pm u^2}}{u} \right|$$

$$36. \int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a \arccos \frac{a}{u}, \quad 0 < a < |u|$$

Trigonometric forms

$$37. \int \tan u \, du = -\ln |\cos u| \qquad 38. \int \sec u \, du = \ln |\sec u + \tan u|$$

$$39. \int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u$$

$$40. \int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du$$

$$41. \int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du$$

$$42. \int \frac{du}{\sin^n u} = -\frac{\cos u}{(n-1)\sin^{n-1} u} + \frac{n-2}{n-1} \int \frac{du}{\sin^{n-2} u}, \quad n \neq 1$$

$$43. \int \sin mu \sin nu \, du = \frac{\sin(m-n)u}{2(m-n)} - \frac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$$

$$44. \int \cos mu \cos nu \, du = \frac{\sin(m-n)u}{2(m-n)} + \frac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$$

$$45. \int \sin mu \cos nu \, du = -\frac{\cos(m-n)u}{2(m-n)} - \frac{\cos(m+n)u}{2(m+n)}, \quad m \neq \pm n$$

$$46. \int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$$

$$47. \int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$$

$$48. \int \arcsin u \, du = u \arcsin u + \sqrt{1-u^2}$$

$$49. \int \arccos u \, du = u \arccos u - \sqrt{1-u^2}$$

$$50. \int u \arcsin u \, du = \frac{1}{4}[(2u^2-1)\arcsin u + u\sqrt{1-u^2}]$$

Logarithmic and exponential forms

$$51. \int \ln |u| \, du = u(\ln |u| - 1)$$

$$52. \int (\ln |u|)^2 \, du = u(\ln |u|)^2 - 2u \ln |u| + 2u$$

$$53. \int u^n \ln |u| \, du = \frac{u^{n+1}}{n+1} \ln |u| - \frac{u^{n+1}}{(n+1)^2}, \quad n \neq -1$$

$$54. \int u^n e^u \, du = u^n e^u - n \int u^{n-1} e^u \, du$$

$$55. \int e^{au} \sin bu \, du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$$

$$56. \int e^{au} \cos bu \, du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$$

Miscellaneous forms

$$57. \int (a+bu)^n \, du = \frac{(a+bu)^{n+1}}{b(n+1)}, \quad n \neq -1$$

$$58. \int \frac{du}{u^2(a+bu)} = -(au)^{-1} + ba^{-2}[\log(a+bu) - \log u]$$

$$59. \int \frac{du}{(a+bu)^{1/2}(c+du)^{3/2}} = \frac{2}{bc-ad} \left(\frac{a+bu}{c+du} \right)^{1/2}, \quad bc-ad \neq 0$$

$$60. \int \frac{du}{u(a+bu^n)} = (an)^{-1} [\log(u^n) - \log(a+bu^n)], \quad n \neq 0$$

$$61. \int \frac{u \, du}{(a+bu)^{1/2}} = \frac{2(bu-2a)}{3b^2} (a+bu)^{1/2}$$

$$62. \int \frac{du}{u^2(u^2+a^2)^{1/2}} = -(a^2u)^{-1}(u^2+a^2)^{1/2}$$

$$63. \int \frac{(a^2-u^2)^{1/2}}{u^2} du = -u^{-1}(a^2-u^2)^{1/2} - \arcsin \frac{u}{a}$$

$$64. \int \sin^4 u \, du = \frac{3u}{8} - \frac{\sin 2u}{4} + \frac{\sin 4u}{32}$$

$$65. \int \cot^3 u \, du = -\frac{1}{2}\cot^2 u - \log \sin u$$

Algebraic forms

- $$25. \int \frac{u \, du}{a + bu} = \frac{u}{b} - \frac{a}{b^2} \ln(a + bu) \qquad 26. \int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right|$$
- $$27. \int \frac{u \, du}{(a + bu)^2} = \frac{a}{b^2} \left(\frac{1}{a + bu} + \frac{1}{a} \ln |a + bu| \right)$$
- $$28. \int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} + \frac{1}{a^2} \ln \left| \frac{u}{a + bu} \right|$$
- $$29. \int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right|$$
- $$30. \int u\sqrt{a + bu} \, du = \frac{2(3bu - 2a)\sqrt{(a + bu)^3}}{15b^2}$$
- $$31. \int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$
- $$32. \int \frac{u \, du}{\sqrt{a + bu}} = \frac{2(bu - 2a)}{3b^2} \sqrt{a + bu}$$
- $$33. \int \sqrt{a^2 - u^2} \, du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a}, \quad |u| < a$$
- $$34. \int \sqrt{u^2 \pm a^2} \, du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln |u + \sqrt{u^2 \pm a^2}|$$
- $$35. \int \frac{\sqrt{a^2 \pm u^2}}{u} \, du = \sqrt{a^2 \pm u^2} - a \ln \left| \frac{a + \sqrt{a^2 \pm u^2}}{u} \right|$$
- $$36. \int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a \arccos \frac{a}{u}, \quad 0 < a < |u|$$

Trigonometric forms

- $$37. \int \tan u \, du = -\ln |\cos u| \qquad 38. \int \sec u \, du = \ln |\sec u + \tan u|$$
- $$39. \int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u$$
- $$40. \int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du$$
- $$41. \int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du$$
- $$42. \int \frac{du}{\sin^n u} = -\frac{\cos u}{(n-1)\sin^{n-1} u} + \frac{n-2}{n-1} \int \frac{du}{\sin^{n-2} u}, \quad n \neq 1$$
- $$43. \int \sin mu \sin nu \, du = \frac{\sin(m-n)u}{2(m-n)} - \frac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$$
- $$44. \int \cos mu \cos nu \, du = \frac{\sin(m-n)u}{2(m-n)} + \frac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$$

(Continued inside back cover)

A SHORT TABLE OF INTEGRALS

The constant of integration is omitted.

Trigonometric forms

45. $\int \sin mu \cos nu \, du = -\frac{\cos(m-n)u}{2(m-n)} - \frac{\cos(m+n)u}{2(m+n)}, \quad m \neq \pm n$
46. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$
47. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$
48. $\int \arcsin u \, du = u \arcsin u + \sqrt{1-u^2}$
49. $\int \arccos u \, du = u \arccos u - \sqrt{1-u^2}$
50. $\int u \arcsin u \, du = \frac{1}{4}[(2u^2-1)\arcsin u + u\sqrt{1-u^2}]$

Logarithmic and exponential forms

51. $\int \ln |u| \, du = u(\ln |u| - 1)$
52. $\int (\ln |u|)^2 \, du = u(\ln |u|)^2 - 2u \ln |u| + 2u$
53. $\int u^n \ln |u| \, du = \frac{u^{n+1}}{n+1} \ln |u| - \frac{u^{n+1}}{(n+1)^2}, \quad n \neq -1$
54. $\int u^n e^u \, du = u^n e^u - n \int u^{n-1} e^u \, du$
55. $\int e^{au} \sin bu \, du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$
56. $\int e^{au} \cos bu \, du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$

Miscellaneous forms

57. $\int (a+bu)^n \, du = \frac{(a+bu)^{n+1}}{b(n+1)}, \quad n \neq -1$
58. $\int \frac{du}{u^2(a+bu)} = -(au)^{-1} + ba^{-2}[\log(a+bu) - \log u]$
59. $\int \frac{du}{(a+bu)^{1/2}(c+du)^{3/2}} = \frac{2}{bc-ad} \left(\frac{a+bu}{c+du} \right)^{1/2}, \quad bc-ad \neq 0$
60. $\int \frac{du}{u(a+bu^n)} = (an)^{-1}[\log(u^n) - \log(a+bu^n)], \quad n \neq 0$