FOURTH EDITION

CALCULUS

WITH ANALYTIC GEOMETRY

PROTTER AND PROTTER

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WITH ANALYTIC GEOMETRY

MURRAY H. PROTTER
UNIVERSITY OF CALIFORNIA, BERKELEY

PHILIP E. PROTTER
PURDUE UNIVERSITY

JONES AND BARTLETT PUBLISHERS
Boston • Portola Valley

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A TABLE OF INTEGRALS

Some Elementary Forms

1. $\int du = u + C$
2. $\int a \, du = au + C$
3. $\int [f(u) + g(u)]du = \int f(u)du + \int g(u)du$
4. $\int u^n \, du = \frac{u^{n+1}}{n + 1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$

Rational Forms Containing $a + bu$

6. $\int \frac{u \, du}{a + bu} = \frac{1}{b^2}[a + bu - a \ln|a + bu|] + C$
7. $\int \frac{u^2 \, du}{a + bu} = \frac{1}{b^3}\left[\frac{1}{2}(a + bu)^2 - 2a(a + bu) + a^2 \ln|a + bu|\right] + C$
8. $\int \frac{u \, du}{(a + bu)^2} = \frac{1}{b^2}\left[\frac{a}{a + bu} + \ln|a + bu|\right] + C$
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10. $\int \frac{u \, du}{(a + bu)^3} = \frac{1}{b^2}\left[\frac{a}{2(a + bu)^2} - \frac{1}{a + bu}\right] + C$
11. $\int \frac{du}{u(a + bu)} = \frac{1}{a} \ln\left|\frac{u}{a + bu}\right| + C$
12. $\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln\left|\frac{a + bu}{u}\right| + C$
13. $\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} + \frac{1}{a^2} \ln\left|\frac{u}{a + bu}\right| + C$

Forms Containing $\sqrt{a + bu}$

14. $\int u\sqrt{a + bu} \, du = \frac{2}{15b^3}(3bu - 2a)(a + bu)^{3/2} + C$
15. $\int u^2\sqrt{a + bu} \, du = \frac{2}{105b^3}(15b^2u^2 - 12abu + 8a^2)(a + bu)^{3/2} + C$
16. $\int u^n\sqrt{a + bu} \, du = \frac{2u^n(a + bu)^{3/2}}{b(2n + 3)} - \frac{2an}{b(2n + 3)} \int u^{n-1}\sqrt{a + bu} \, du$
17. $\int \frac{u \, du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + C$

18. $$\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2 - 4abu + 8a^2)\sqrt{a+bu} + C$$

19. $$\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$$

20. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C & \text{if } a > 0 \\ \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}} + C & \text{if } a < 0 \end{cases}$$

21. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

22. $$\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

23. $$\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bu}\,du}{u^{n-1}}$$

Forms Containing $a^2 \pm u^2$

24. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

25. $$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C = \begin{cases} \dfrac{1}{a}\tanh^{-1}\dfrac{u}{a} + C & \text{if } |u| < a \\ \dfrac{1}{a}\coth^{-1}\dfrac{u}{a} + C & \text{if } |u| > a \end{cases}$$

26. $$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C = \begin{cases} -\dfrac{1}{a}\tanh^{-1}\dfrac{u}{a} + C & \text{if } |u| < a \\ -\dfrac{1}{a}\coth^{-1}\dfrac{u}{a} + C & \text{if } |u| > a \end{cases}$$

Forms Containing $\sqrt{u^2 \pm a^2}$

In formulas 27 through 38, we may replace

$\ln(u + \sqrt{u^2+a^2})$ by $\sinh^{-1}\dfrac{u}{a}$

$\ln|u + \sqrt{u^2-a^2}|$ by $\cosh^{-1}\dfrac{u}{a}$

$\ln\left|\dfrac{a+\sqrt{u^2+a^2}}{u}\right|$ by $\sinh^{-1}\dfrac{a}{u}$

27. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + C$$

28. $$\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

29. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{8}(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

30. $$\int \frac{\sqrt{u^2+a^2}\,du}{u} = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$$

31. $$\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\frac{u}{a} + C$$

32. $$\int \frac{\sqrt{u^2 \pm a^2}\,du}{u^2} = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln|u + \sqrt{u^2 \pm a^2}| + C$$

33. $$\int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} - \frac{\pm a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

34. $$\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$$

35. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

36. $$\int \frac{du}{u^2\sqrt{u^2 \pm a^2}} = -\frac{\sqrt{u^2 \pm a^2}}{\pm a^2 u} + C$$

37. $$\int (u^2 \pm a^2)^{3/2}\,du = \frac{u}{8}(2u^2 \pm 5a^2)\sqrt{u^2 \pm a^2} + \frac{3a^4}{8}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

38. $$\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{\pm a^2\sqrt{u^2 \pm a^2}} + C$$

Forms Containing $\sqrt{a^2 - u^2}$

39. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

40. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

41. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

42. $$\int \frac{\sqrt{a^2-u^2}\,du}{u} = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C = \sqrt{a^2-u^2} - a\cosh^{-1}\frac{a}{u} + C$$

43. $$\int \frac{\sqrt{a^2-u^2}\,du}{u^2} = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

(This table is continued on the back endpapers.)

44. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

45. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C = -\frac{1}{a}\cosh^{-1}\frac{a}{u} + C$$

46. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

47. $$\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

48. $$\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$$

Forms Containing $2au - u^2$

49. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

50. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

51. $$\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2} + a\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

52. $$\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u} - \cos^{-1}\left(1-\frac{u}{a}\right) + C$$

53. $$\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(1-\frac{u}{a}\right) + C$$

54. $$\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

55. $$\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

56. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

57. $$\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}} + C$$

58. $$\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}} + C$$

Forms Containing Trigonometric Functions

59. $$\int \sin u\,du = -\cos u + C$$

60. $$\int \cos u\,du = \sin u + C$$

61. $$\int \tan u\,du = \ln|\sec u| + C$$

62. $$\int \cot u\,du = \ln|\sin u| + C$$

63. $$\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$$

64. $$\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln\left|\tan\tfrac{1}{2}u\right| + C$$

65. $$\int \sec^2 u\,du = \tan u + C$$

66. $$\int \csc^2 u\,du = -\cot u + C$$

67. $$\int \sec u\tan u\,du = \sec u + C$$

68. $$\int \csc u\cot u\,du = -\csc u + C$$

69. $$\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$$

70. $$\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$$

71. $$\int \tan^2 u\,du = \tan u - u + C$$

72. $$\int \cot^2 u\,du = -\cot u - u + C$$

73. $$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$$

74. $$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$$

75. $$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$$

76. $$\int \cot^n u\,du = -\frac{1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$$

77. $$\int \sec^n u\,du = \frac{1}{n-1}\sec^{n-2}u\tan u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$$

78. $$\int \csc^n u\,du = -\frac{1}{n-1}\csc^{n-2}u\cot u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$$

79. $$\int \sin mu\sin nu\,du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$$

80. $$\int \cos mu\cos nu\,du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$$

81. $$\int \sin mu\cos nu\,du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$$

82. $$\int u\sin u\,du = \sin u - u\cos u + C$$

83. $$\int u\cos u\,du = \cos u + u\sin u + C$$

84. $$\int u^2\sin u\,du = 2u\sin u + (2-u^2)\cos u + C$$

85. $\int u^2 \cos u \, du = 2u \cos u + (u^2 - 2) \sin u + C$

86. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$

87. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$

88. $\int \sin^m u \cos^n u \, du = -\dfrac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \dfrac{m-1}{m+n} \int \sin^{m-2} u \cos^n u \, du$

$= \dfrac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \dfrac{n-1}{m+n} \int \sin^m u \cos^{n-2} u \, du$

Forms Containing Inverse Trigonometric Functions

89. $\int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1 - u^2} + C$

90. $\int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1 - u^2} + C$

91. $\int \tan^{-1} u \, du = u \tan^{-1} u - \ln \sqrt{1 + u^2} + C$

92. $\int \cot^{-1} u \, du = u \cot^{-1} u + \ln \sqrt{1 + u^2} + C$

93. $\int \sec^{-1} u \, du = u \sec^{-1} u - \ln\left|u + \sqrt{u^2 - 1}\right| + C$

$= u \sec^{-1} u - \cosh^{-1} u + C$

94. $\int \csc^{-1} u \, du = u \csc^{-1} u + \ln\left|u + \sqrt{u^2 - 1}\right| + C$

$= u \csc^{-1} u + \cosh^{-1} u + C$

Forms Containing Exponential and Logarithmic Functions

95. $\int e^u \, du = e^u + C$

96. $\int a^u \, du = \dfrac{a^u}{\ln a} + C$

97. $\int ue^u \, du = e^u(u - 1) + C$

98. $\int u^n e^u \, du = u^n e^u - n \int u^{n-1} e^u \, du$

99. $\int u^n a^u \, du = \dfrac{u^n a^u}{\ln a} - \dfrac{n}{\ln a} \int u^{n-1} a^u \, du + C$

100. $\int \dfrac{e^u \, du}{u^n} = -\dfrac{e^u}{(n-1)u^{n-1}} + \dfrac{1}{n-1} \int \dfrac{e^u \, du}{u^{n-1}}$

101. $\int \dfrac{a^u \, du}{u^n} = -\dfrac{a^u}{(n-1)u^{n-1}} + \dfrac{\ln a}{n-1} \int \dfrac{a^u \, du}{u^{n-1}}$

102. $\int \ln u \, du = u \ln u - u + C$

103. $\int u^n \ln u \, du = \dfrac{u^{n+1}}{(n+1)^2}[(n+1) \ln u - 1] + C$

104. $\int \dfrac{du}{u \ln u} = \ln|\ln u| + C$

105. $\int e^{au} \sin nu \, du = \dfrac{e^{au}}{a^2 + n^2}(a \sin nu - n \cos nu) + C$

106. $\int e^{au} \cos nu \, du = \dfrac{e^{au}}{a^2 + n^2}(a \cos nu + n \sin nu) + C$

Forms Containing Hyperbolic Functions

107. $\int \sinh u \, du = \cosh u + C$

108. $\int \cosh u \, du = \sinh u + C$

109. $\int \tanh u \, du = \ln|\cosh u| + C$

110. $\int \coth u \, du = \ln|\sinh u| + C$

111. $\int \operatorname{sech} u \, du = \tan^{-1}(\sinh u) + C$

112. $\int \operatorname{csch} u \, du = \ln\left|\tanh \tfrac{1}{2} u\right| + C$

113. $\int \operatorname{sech}^2 u \, du = \tanh u + C$

114. $\int \operatorname{csch}^2 u \, du = -\coth u + C$

115. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

116. $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

117. $\int \sinh^2 u \, du = \tfrac{1}{4} \sinh 2u - \tfrac{1}{2} u + C$

118. $\int \cosh^2 u \, du = \tfrac{1}{4} \sinh 2u + \tfrac{1}{2} u + C$

119. $\int \tanh^2 u \, du = u - \tanh u + C$

120. $\int \coth^2 u \, du = u - \coth u + C$

121. $\int u \sinh u \, du = u \cosh u - \sinh u + C$

122. $\int u \cosh u \, du = u \sinh u - \cosh u + C$

123. $\int e^{au} \sinh nu \, du = \dfrac{e^{au}}{a^2 - n^2}(a \sinh nu - n \cosh nu) + C$

124. $\int e^{au} \cosh nu \, du = \dfrac{e^{au}}{a^2 - n^2}(a \cosh nu - n \sinh nu) + C$

FORMULAS FROM TRIGONOMETRY

THE EIGHT FUNDAMENTAL TRIGONOMETRIC IDENTITIES

$$\sin x \csc x = 1 \qquad \cos x \sec x = 1 \qquad \tan x \cot x = 1$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

SUM AND DIFFERENCE IDENTITIES

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad \tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

MULTIPLE-MEASURE IDENTITIES

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$\cos 2u = 1 - 2 \sin^2 u \qquad \cos 2u = 2 \cos^2 u - 1$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

$$\sin^2 \tfrac{1}{2}t = \frac{1 - \cos t}{2} \qquad \cos^2 \tfrac{1}{2}t = \frac{1 + \cos t}{2}$$

$$\tan \tfrac{1}{2}t = \frac{1 - \cos t}{\sin t} \qquad \tan \tfrac{1}{2}t = \frac{\sin t}{1 + \cos t}$$

IDENTITIES FOR THE PRODUCT, SUM, AND DIFFERENCE OF SINE AND COSINE

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\sin s + \sin t = 2 \sin\left(\frac{s + t}{2}\right) \cos\left(\frac{s - t}{2}\right)$$

$$\sin s - \sin t = 2 \cos\left(\frac{s + t}{2}\right) \sin\left(\frac{s - t}{2}\right)$$

$$\cos s + \cos t = 2 \cos\left(\frac{s + t}{2}\right) \cos\left(\frac{s - t}{2}\right)$$

$$\cos s - \cos t = -2 \sin\left(\frac{s + t}{2}\right) \sin\left(\frac{s - t}{2}\right)$$

SOME REDUCTION FORMULAS

$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$

$\sin(\frac{1}{2}\pi - x) = \cos x \qquad \cos(\frac{1}{2}\pi - x) = \sin x$

$\tan(\frac{1}{2}\pi - x) = \cot x$

$\sin(\frac{1}{2}\pi + x) = \cos x \qquad \cos(\frac{1}{2}\pi + x) = -\sin x$

$\tan(\frac{1}{2}\pi + x) = -\cot x$

$\sin(\pi - x) = \sin x \qquad \cos(\pi - x) = -\cos x$

$\tan(\pi - x) = -\tan x$

LAW OF SINES AND LAW OF COSINES

a, b, and c represent the measures of the sides of a triangle: α, β, and γ represent the measures of the angles opposite the sides of measures a, b, and c, respectively.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \qquad c^2 = a^2 + b^2 - 2ab \cos \gamma$$

FORMULAS FROM GEOMETRY

The following symbols are used for the measure:

r: radius h: altitude b: base a: base C: circumference
A: area S: surface area B: area of base V: volume

Circle: $A = \pi r^2$; $C = 2\pi r$

Triangle: $A = \frac{1}{2}bh$

Rectangle and parallelogram: $A = bh$

Trapezoid: $A = \frac{1}{2}(a + b)h$

Right circular cylinder: $V = \pi r^2 h$; $S = 2\pi rh$

Right circular cone: $V = \frac{1}{3}\pi r^2 h$; $S = \pi r\sqrt{r^2 + h^2}$

Sphere: $V = \frac{4}{3}\pi r^3$; $S = 4\pi r^2$

Prism (with parallel bases): $V = Bh$

Pyramid: $V = \frac{1}{3}Bh$

The Greek alphabet

α	alpha	ι	iota	ρ	rho
β	beta	κ	kappa	σ	sigma
γ	gamma	λ	lambda	τ	tau
δ	delta	μ	mu	υ	upsilon
ϵ	epsilon	ν	nu	φ	phi
ζ	zeta	ξ	xi	χ	chi
η	eta	o	omicron	ψ	psi
θ	theta	π	pi	ω	omega

EXAMPLE 3 Making these substitutions, we obtain the following simplification of an integral.

$$\int \frac{d\theta}{5-4\cos\theta} = \int \frac{(2\,dt)/(1+t^2)}{5-4[(1-t^2)/(1+t^2)]} = \int \frac{2\,dt}{1+9t^2}.$$ □

A SHORT TABLE OF INTEGRALS

The constant of integration is omitted.

Elementary formulas

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1}, \quad n \neq -1$
2. $\int \dfrac{du}{u} = \ln|u|$
3. $\int e^u\,du = e^u$
4. $\int a^u\,du = \dfrac{a^u}{\ln a}, \quad a>0,\ a\neq 1$
5. $\int \sin u\,du = -\cos u$
6. $\int \cos u\,du = \sin u$
7. $\int \sec^2 u\,du = \tan u$
8. $\int \csc^2 u\,du = -\cot u$
9. $\int \sec u\tan u\,du = \sec u$
10. $\int \csc u\cot u\,du = -\csc u$
11. $\int \sinh u\,du = \cosh u$
12. $\int \cosh u\,du = \sinh u$
13. $\int \operatorname{sech}^2 u\,du = \tanh u$
14. $\int \operatorname{csch}^2 u\,du = -\coth u$
15. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u$
16. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u$
17. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \arcsin\dfrac{u}{a}, \quad a>|u|$
18. $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\arctan\dfrac{u}{a}, \quad a\neq 0$
19. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{u}{a}, \quad |u|>a$
20. $\int \dfrac{du}{\sqrt{a^2+u^2}} = \begin{cases} \operatorname{argsinh}\dfrac{u}{a} \\ \ln\left(u+\sqrt{a^2+u^2}\right) \end{cases}$
21. $\int \dfrac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\operatorname{argtanh}\dfrac{u}{a}, & |u|<a \quad \text{or} \quad \dfrac{1}{2a}\ln\dfrac{a+u}{a-u}, \quad |u|<a \\ \dfrac{1}{a}\operatorname{argcoth}\dfrac{u}{a}, & |u|>a \quad \text{or} \quad \dfrac{1}{2a}\ln\dfrac{u+a}{u-a}, \quad |u|>a \end{cases}$
22. $\int \dfrac{du}{u\sqrt{a^2-u^2}} = -\dfrac{1}{a}\operatorname{argsech}\dfrac{u}{a}, \quad \text{or} \quad -\dfrac{1}{a}\ln\left(\dfrac{a+\sqrt{a^2-u^2}}{u}\right), \quad 0<u<a$

23. $\displaystyle\int \frac{du}{|u|\sqrt{u^2+a^2}} = -\frac{1}{a}\operatorname{argcsch}\frac{u}{a}, \quad u \neq 0 \quad \text{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{u^2+a^2}}{u}\right)$

24. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \begin{cases} \operatorname{argcosh}\dfrac{u}{a}, \\ \ln|u+\sqrt{u^2-a^2}|, \end{cases} \qquad |u| > a > 0$

Algebraic forms

25. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{u}{b} - \frac{a}{b^2}\ln(a+bu)$

26. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|$

27. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2}\left(\frac{1}{a+bu} + \frac{1}{a}\ln|a+bu|\right)$

28. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} + \frac{1}{a^2}\ln\left|\frac{u}{a+bu}\right|$

29. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|$

30. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2(3bu-2a)\sqrt{(a+bu)^3}}{15b^2}$

31. $\displaystyle\int \frac{\sqrt{a+bu}}{u}du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

32. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2(bu-2a)}{3b^2}\sqrt{a+bu}$

33. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}, \quad |u| < a$

34. $\displaystyle\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u+\sqrt{u^2 \pm a^2}|$

35. $\displaystyle\int \frac{\sqrt{a^2 \pm u^2}}{u}du = \sqrt{a^2 \pm u^2} - a\ln\left|\frac{a+\sqrt{a^2 \pm u^2}}{u}\right|$

36. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}du = \sqrt{u^2-a^2} - a\arccos\frac{a}{u}, \quad 0 < a < |u|$

Trigonometric forms

37. $\displaystyle\int \tan u\,du = -\ln|\cos u|$

38. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u|$

39. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u$

40. $\displaystyle\int \sin^n u\,du = -\frac{\sin^{n-1} u\cos u}{n} + \frac{n-1}{n}\int \sin^{n-2} u\,du$

41. $\int \cos^n u\, du = \dfrac{\cos^{n-1} u \sin u}{n} + \dfrac{n-1}{n} \int \cos^{n-2} u\, du$

42. $\int \dfrac{du}{\sin^n u} = -\dfrac{\cos u}{(n-1)\sin^{n-1} u} + \dfrac{n-2}{n-1} \int \dfrac{du}{\sin^{n-2} u}, \quad n \neq 1$

43. $\int \sin mu \sin nu\, du = \dfrac{\sin(m-n)u}{2(m-n)} - \dfrac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$

44. $\int \cos mu \cos nu\, du = \dfrac{\sin(m-n)u}{2(m-n)} + \dfrac{\sin(m+n)u}{2(m+n)}, \quad m \neq \pm n$

45. $\int \sin mu \cos nu\, du = -\dfrac{\cos(m-n)u}{2(m-n)} - \dfrac{\cos(m+n)u}{2(m+n)}, \quad m \neq \pm n$

46. $\int u^n \sin u\, du = -u^n \cos u + n \int u^{n-1} \cos u\, du$

47. $\int u^n \cos u\, du = u^n \sin u - n \int u^{n-1} \sin u\, du$

48. $\int \arcsin u\, du = u \arcsin u + \sqrt{1-u^2}$

49. $\int \arccos u\, du = u \arccos u - \sqrt{1-u^2}$

50. $\int u \arcsin u\, du = \frac{1}{4}[(2u^2-1)\arcsin u + u\sqrt{1-u^2}]$

Logarithmic and exponential forms

51. $\int \ln|u|\, du = u(\ln|u| - 1)$

52. $\int (\ln|u|)^2\, du = u(\ln|u|)^2 - 2u \ln|u| + 2u$

53. $\int u^n \ln|u|\, du = \dfrac{u^{n+1}}{n+1} \ln|u| - \dfrac{u^{n+1}}{(n+1)^2}, \quad n \neq -1$

54. $\int u^n e^u\, du = u^n e^u - n \int u^{n-1} e^u\, du$

55. $\int e^{au} \sin bu\, du = \dfrac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$

56. $\int e^{au} \cos bu\, du = \dfrac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$

Miscellaneous forms

57. $\int (a+bu)^n\, du = \dfrac{(a+bu)^{n+1}}{b(n+1)}, \quad n \neq -1$

58. $\displaystyle\int \frac{du}{u^2(a+bu)} = -(au)^{-1} + ba^{-2}[\log(a+bu) - \log u]$

59. $\displaystyle\int \frac{du}{(a+bu)^{1/2}(c+du)^{3/2}} = \frac{2}{bc-ad}\left(\frac{a+bu}{c+du}\right)^{1/2}, \quad bc-ad \neq 0$

60. $\displaystyle\int \frac{du}{u(a+bu^n)} = (an)^{-1}[\log(u^n) - \log(a+bu^n)], \quad n \neq 0$

61. $\displaystyle\int \frac{u\,du}{(a+bu)^{1/2}} = \frac{2(bu-2a)}{3b^2}(a+bu)^{1/2}$

62. $\displaystyle\int \frac{du}{u^2(u^2+a^2)^{1/2}} = -(a^2u)^{-1}(u^2+a^2)^{1/2}$

63. $\displaystyle\int \frac{(a^2-u^2)^{1/2}}{u^2}\,du = -u^{-1}(a^2-u^2)^{1/2} - \arcsin\frac{u}{a}$

64. $\displaystyle\int \sin^4 u\,du = \frac{3u}{8} - \frac{\sin 2u}{4} + \frac{\sin 4u}{32}$

65. $\displaystyle\int \cot^3 u\,du = -\tfrac{1}{2}\cot^2 u - \log \sin u$

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25. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{u}{b} - \frac{a}{b^2}\ln(a+bu)$

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The constant of integration is omitted.

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