CALCULUS 1

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Bachelor of Science in Mechanical Engineering
College of Engineering

VISION

Laguna University shall be a socially responsive educational institution of choice providing holistically developed individuals in the Asia-Pacific Region.

MISSION

Laguna University is committed to produce academically prepared and technically skilled individuals who are socially and morally upright citizens.

Department of Mechanical Engineering

MISSION

The Department of Mechanical Engineering of Laguna University is committed to produce academically prepared and technically skilled mechanical engineers who are socially and morally upright citizens.

VISION

The Department of Mechanical Engineering of Laguna University is envisioned to be the provincial college of choice producing well-equipped mechanical engineers who specializes on energy management.

Table of Contents

Course Code: **ENG'G 301**

Course Description: An introductory course covering the core concepts of limit, continuity and differentiability of functions involving one or more variables. This also includes the application of differential calculations in solving problems on optimization, rates of change, related rates, tangents and normal, and approximations; partial differentiation and transcendental curve tracing.

Course Intended Learning Outcomes (CILO):

At the end of the course, students should be able to:

1. Differentiate algebraic and transcendental functions
2. Apply the concept of differentiation in solving word problems
3. Analyze and trace transcendental curves

Course Requirements:

Assessment Tasks	**- 60%**
Major Exams	**- 40%**
Periodic Grade	**100%**

Computation of Grades:

PRELIM GRADE = 60% (Activity 1-4) + 40% (Prelim exam)

MIDTERM GRADE = 30 %(Prelim Grade) + 70 % [60% (Activity 5-7) + 40% (Midterm exam)]

FINAL GRADE = 30 %(Midterm Grade) + 70 % [60% (Activity 8-10) + 40 %(Final exam)]

MODULE 1
LIMITS AND CONTINUITY

Introduction

In mathematics the two main mathematical operations are differentiation and integration. These operations include derivative calculation and the definite integral, each based on the notion of limit (Leithold, 1990).

The main concepts we work with in calculus are functions. Functions are the foundation of real-world phenomenon engineering. We clearly demonstrate the connection between two (or more) quantities. When we have such a relationship, various questions naturally arise (Brown, 2013).

According to Leithold (1990) the principle of limiting a function was first offered a step-by - step incentive, which takes the topic up to a comprehensive epsilon-delta description by calculating the value of the function near a number, by an understandable analysis of the limiting process. Limit theorems are used to facilitate estimation of basic function limitations. The definition of limit is generalized to include different feature forms. Limits including infinity are viewed and used to describe feature graphs with vertical and horizontal asymptotes. For mathematics the most important type of functions are potentially continuous functions.

In mathematics, continuity is another specific term. A task may be either continuous or discontinuous. One simple way to check for a function's continuity is to see how a function's graph can be traced with a pen without taking the pen off the page. A logical interpretation of consistency like this one is possibly appropriate for the mathematics we do in precalculus and calculus, but a more rigorous interpretation is required for advanced math. Using limits we can also know a simpler and even more accurate way to describe continuity. You are prepared for the calculus with knowledge of the principles of limits and continuity (“Continuity and Limits”, 2005).

Learning Outcomes

At the end of this module, students should be able to:

1. Familiarize with the different terms and classifications of functions.
2. Learn about the theorems on limits.
3. Determine if a function is continuous.

Lesson 1. Review on Functions

All functions are classified as either algebraic or transcendental. The algebraic functions are rational integral functions, or polynomials; rational fractions, or quotients of polynomials; and irrational functions, of which the simplest are those formed from rational functions by the extraction of roots. The elementary transcendental functions are trigonometric and inverse trigonometric functions; exponential functions, in which the variable occurs as an exponent; and logarithms (Love & Rainville, 1981).

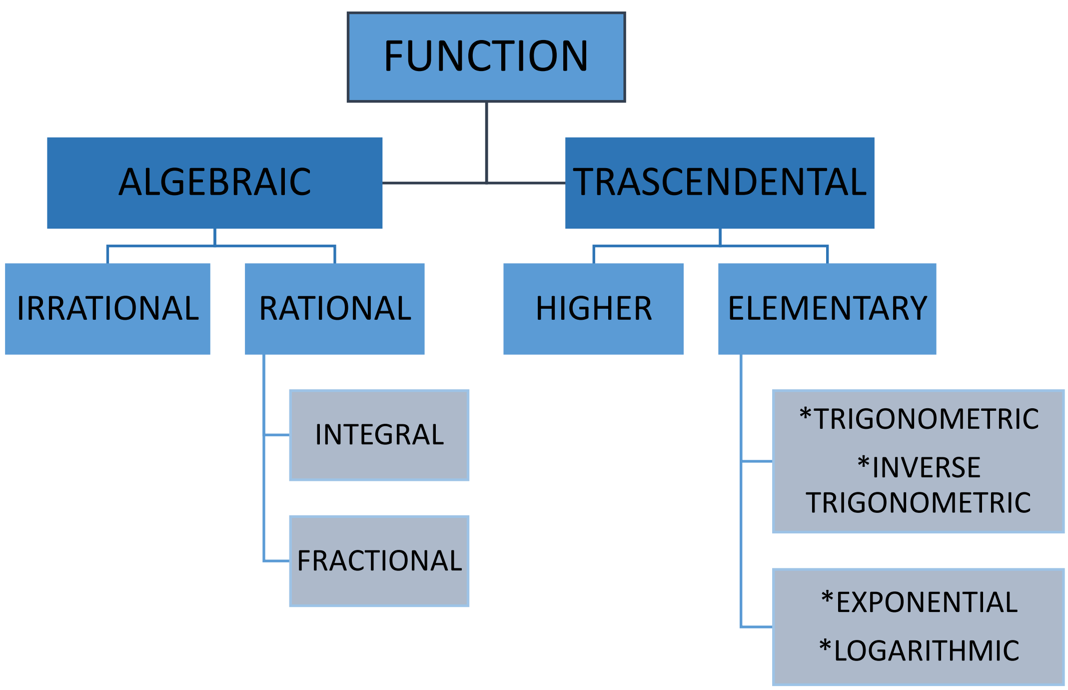

Figure 1.1 Classifications of Function

Lesson 2. Limits

According to Feliciano and Uy (1983) this deals with several theorems by means of which we shall be able to evaluate the limits of functions rapidly and efficiently. To evaluate or to find

$$\lim_{x \to a} f(x) = A$$

means that we are to find the number L that f(x) is near, whenever x is near a but not equal to a. Of course, when x = a, the value of the function is f(a). It may be that f(a0 is also the limit, i.e., L = F(a). Thus to evaluate

$$\lim_{x \to 1}(4 - x^2)$$

Means to find a number of which $4 - x^2$ is near whenever x is near the number 1. We know that

$$\lim_{x \to 1}(4 - x^2) = 3$$

since by choosing x sufficiently close to 1, $4 - x^2$ can be made to come as close to 3.

To obtain the limits of more complicated functions, we shall use the following theorems which we shall state symbolically without proof (Feliciano & Uy, 1983).

Lesson 3. Theorems on Limits

(1) $\lim_{x \to a} c = c$ c = any constant

(2) $\lim_{x \to a} x = a$ a = any real number

(3) $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$

(4) $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(5) $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(6) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(7) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, n = any positive integer and f(x) ≥ 0 if n is even.

(8) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

In stating above theorems, we assume that f(x) and g(x) are defined for all values of x in some interval containing, except possibly at a itself. These theorems may be stated briefly in words. For instance, (4) is sometimes stated as "the limit of a sum is the sum of the limits" (Feliciano & Uy, 1983).

Example 1.

$$\lim_{x \to 2}(x^2 + 3x + 4) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x + \lim_{x \to 2} 4 \qquad (4)$$

$$= \lim_{x \to 2}[x]^n + 3 \lim_{x \to 2} x + 4 \qquad (8), (3), (1)$$

$$= [2]^2 + 3(2) + 4 \qquad (2)$$

= 14

Example 2

$$\lim_{x \to 2}(x + 4)\sqrt{2x + 5} = \lim_{x \to 2}(x + 4) \lim_{x \to 2} \sqrt{2x + 5} \qquad (5)$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4) \sqrt{\lim_{x \to 2}(2x + 5)} \qquad (4), (7)$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4) \sqrt{\lim_{x \to 2} 2x + \lim_{x \to 2} 5} \qquad (4)$$

$$= (\lim_{x \to 2} x + \lim_{x \to 2} 4) \sqrt{2 \lim_{x \to 2} x + \lim_{x \to 2} 5} \qquad (3)$$

$$= (2 + 4)\sqrt{2(2) + 5} \qquad (2), (1)$$

= 18

Example 3

$$\lim_{x\to 3}(3x+4)^2 = [\lim_{x\to 3}(3x+4)]^2 \quad (8)$$

$$= [\lim_{x\to 3} 3x + \lim_{x\to 3} 4]^2 \quad (4)$$

$$= [3\lim_{x\to 3} x + \lim_{x\to 3} 4]^2 \quad (3)$$

$$= [3(3)+4]^2 \quad (2), (1)$$

$$= 169$$

Note that the limits of the functions in the above examples can be obtained by straight substitution. For instance, in example 5 we see that straight substitution of x = 2 gives the desired limit. Thus, the solution may simply be written as follows:

$$\lim_{x\to 2}(x+4)\sqrt{2x+5} = (2+4)\sqrt{2(2)+5}$$

$$= (6)\sqrt{9}$$

$$= 18$$

Lesson 4. Limits at Infinity

According to Feliciano and Uy (1983) bear in mind that ∞ is not a number which results from division by zero. Recall that in the real number system, division by zero is not permissible. In fact, it can be argued that

$$\lim_{x\to a} f(x) = \infty$$

is not an equation at all since ∞ does not represent a number. It is merely used as a symbol to imply that the value of f(x) increases numerically without bound as x approaches a.

A function f(x) may have a finite limit even when the independent variable x becomes infinite. This statement "x becomes infinite" is customarily expressed in symbolism by "$x \to \infty$".

Consider the function f(x)= 1/x. It can be shown (intuitively or formally) that 1/x approaches a finite limit (the number zero) as x increases without bound. That is,

$$\frac{1}{x} \to 0 \quad as \quad x \to \infty$$

We shall consider this fact as an additional theorem on limits and in symbol, we write

(9) $\lim_{x\to\infty} \frac{1}{x} = 0$ (Feliciano & Uy, 1983).

Example 4

$$\lim_{x\to\infty} \frac{1}{x^3} = \lim_{x\to\infty} (\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x})$$

$$= \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} \qquad (5)$$

$= 0$

Example 5

$$\lim_{x\to\infty} \frac{4}{x^2} = 4 \lim_{x\to\infty} \frac{1}{x^2}$$

$$= 4 \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{x}$$

$$= 4 \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} \qquad (5)$$

= 0 (9)

Example 6.

$$\lim_{x\to\infty} \frac{1}{x^{\frac{1}{4}}} = \lim_{x\to\infty} (\frac{1}{x})^{\frac{1}{4}}$$

$$= [\lim_{x\to\infty} \frac{1}{x}]^{\frac{1}{4}} \qquad (8)$$

= 0 (9)

From the example above we institutively feel that if n is any positive number, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

This is given as a theorem in some books. Note that when n = 1, we have (9).

A function $f(x) = \frac{N(x)}{D(x)}$ may assume an intermediate form $\frac{\infty}{\infty}$ when x is replaced by ∞. However, the limit of f(x) as x becomes infinite may be definite. To find the limits we first divide N(x) and D(x) by the highest power of x. Then we evaluate the limit by the use of theorem (9) (Feliciano & Uy, 1983).

Example 7.

Evaluate $\lim_{x\to\infty} \frac{4x^3+3x^2-6}{2x^{2+5x+3}}$

Solution: The functions assume the indeterminate form $\frac{\infty}{\infty}$ when x is replaced by ∞. Dividing the numerator and denominator by x^3, we get

$$\lim_{x\to\infty} \frac{4x^3+3x^2-6}{2x^{2+5x+3}} = \lim_{x\to\infty} \frac{4+\frac{3}{x}-\frac{6}{x^3}}{2+\frac{5}{x^2}+\frac{3}{x^3}}$$

$$= \frac{5+0-0}{2+0+0}$$

$$= 2$$

Lesson 5. Continuity

According to Leithold (1990) the function f is said to be continuous at the number if and only if the following three conditions are satisfied:

(i) *f(a)* exists;

(ii) $\lim_{x \to a} f(x)$ exists;

(iii) $\lim_{x \to a} f(x) = f(a)$.

If one or more of these conditions fails to hold a, the function is said to be discontinuous at a.

A function f(x) is said to be continuous in an interval if it is continuous for every value of x in the interval. The graph of this function is “unbroken” over that interval. That is, the graph of f(x) can be drawn without lifting the pencil from the paper, see figure in example 1 (Leithold, 1990).

Example 8.

The function f(x) = x^2 is continuous at x = 2 because $\lim_{x \to 2} x^2$ is continuous since the graph of this function is “unbroken.”

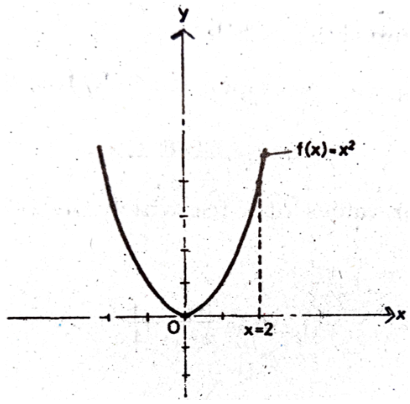

Figure 1.2 Graph of Continuous Function

Example 9.

Is the function $f(x) = \frac{4x}{x^2-4}$ continuous over the interval $0 \leq x \leq 5$?

Answer: No, since at x = 2, f(2) is undefined.

Assessment Task - 1

Evaluate the following.

1. $\lim_{x \to 2}(x^2 - 4x + 3)$
2. $\lim_{x \to \frac{\pi}{4}}(\tan x + \sin x)$
3. $\lim_{x \to 8}(2x + \sqrt[3]{x} - 4)$
4. $\lim_{x \to 2} \frac{\sqrt{3x}}{x\sqrt{x+1}}$
5. $\lim_{x \to \infty} \frac{6x^3+4x^2+5}{8x^3+7x-3}$
6. $\lim_{x \to \infty} \frac{4x+5}{x^2+1}$
7. $\lim_{x \to \infty} \frac{8x-5}{\sqrt{4x^2+3}}$
8. $\lim_{x \to \infty} \frac{(x+2)^3-(x-2)^3}{6x+1}$

Find the value of x for which the function for discontinuous.

1. $\frac{3x}{x-5}$
2. $\frac{1}{2^x-8}$

Summary

According to Leithold (1990) let f be a function specified in some open interval at any number containing a, except probably at the number a itself. The limit of f(x) as x approaches a is L, written as $\lim_{x \to a} f(x) = L$.

Leithold (1990) also explained that we let f be a function that is defined at every number in some open interval (a, +∞). The limit if f(x), as x increases without bound, is L, written as $\lim_{x \to +\infty} f(x) = L$.

Function f is said to be continuous at the number a if and only if the following three conditions are satisfied:

(i) *f(a)* exists; (ii) $\lim_{x \to a} f(x)$ exists; and (iii) $\lim_{x \to a} f(x) = f(a)$.

References

Brown, Peter. (2013). Limits and continuity – A guide for teachers (Years 11–12). Education Services Australia PO Box 177 Carlton South Vic 3053 Australia.

Continuity and Limits. 2005. SparkNotes Editors. Retrieved from http://origin-www.sparknotes.com/math/precalc/continuityandlimits/summary/

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Leithold, Louis, (1990).The Calculus with Analytic Geometry, Harper & Row Publishers, Inc., New York. 6th Ed.

Love, Clyde E. & Rainville,Earl D., Differential and Integral Calculus, Macmilan Publishing Co., Inc., Philippines, 1981.

MODULE 2
THE DERIVATIVE

Introduction

According to "Derivative" (2020) function derivative of a real variable tests the ability to adjust the function value (output value) in relation to an adjust in its statement (input value). Derivatives are an integral calculation tool. For illustration, the velocity of the object is the derivative of a moving object's location in relation to time: this calculates how rapidly the object's position changes as time progresses.

The derivative of a single variable function at a specified input value, where it occurs, is the slope of the tangent line to the function graph at that level. The tangent line is near that input value the best linear approximation of the function. For this purpose, the derivative is also defined as "instantaneous rate of change," the ratio of the instant change in the dependent variable to the independent variable (“Derivative”, 2020).

Learning Outcomes

At the end of this module, students should be able to:

1. Use the basic principles of algebra and limits to define the derivative
2. Use the rules for differentiation in finding the derivative of a function.

Lesson 1. The Derivative of a Function

Delta Notation

According to Ayres & Mendelson (2013) let f be a function and as usual we let x stand for any argument of f, and we let y be the corresponding value of f. Thus, y = f(x). Consider any number x_0 in the domain of f. Let Δx (read “delta x”) represent a small change in the value of x, from x_0 to $x_0 + \Delta x$, and then let Δy (read “delta y”) denote the corresponding change in the value of y. So, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then the ratio

$$\frac{\Delta y}{\Delta x} = \frac{change\ in\ y}{change\ in\ x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called the average rate of change of the function f on the interval between x_0 and $x_0 + \Delta x$.

Example :

Let $y = f(x) = x^2 + 2x$. Starting at $x_0 = 1$, change x to 1.5. Then the corresponding change in y is $y = f(1.5) - f(1) = 5.25 - 3 = 2.25$. Hence, the average rate of change of y on the interval between x = 1 and x = 1.5 is $\frac{\Delta y}{\Delta x} = \frac{2.25}{0.5} = 4.5$.

Derivative

If y = f(x) and x_0 is in the domain of f, then by the instantaneous rate of change of f at x_0 we mean the limit of the average rate of change between x_0 and $x_0 + \Delta x$ as Δx approaches 0:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided that this limit exists. This limit is also called the derivative of f at x_0.

Notation for Derivatives

Let us consider the derivative of f at an arbitrary point x in its domain:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The value of the derivative is a function of x, and will be denoted by any of the following expressions:

$$D_x y = \frac{dy}{dx} = y' = f'(x) = \frac{d}{dx} y = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The value f′(a) of the derivative of f at a particular point a is sometimes denoted by $\left.\frac{dy}{dx}\right|_{x=a}$ (Ayres & Mendelson, 2013).

Lesson 2. Rules for differentiation

According to Feliciano and Uy (1983) in the following formulas, it is assumed that *u, v*, and *w* are functions that are differentiable at *x*; *c* and *m* are assumed to be constants.

(1) $\frac{d}{dx}(c) = 0$ (The derivative of a constant function is zero.)

(2) $\frac{d}{dx}(x) = 1$ (The derivative of the identity function is 1.)

(3) $\frac{d}{dx}(cu) = c\frac{du}{dx}$

(4) $\frac{d}{dx}(u+v+\cdots)=\frac{du}{dx}+\frac{dv}{dx}+\cdots$ (Sum Rule)

(5) $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ (Product Rule)

(6) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ provided that $v\neq 0$ (Quotient Rule)

(7) $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$ (Power Rule)

(8) $\frac{d}{dx}(\sqrt{u})=\frac{\frac{du}{dx}}{2\sqrt{u}}$

(9) $\frac{d}{dx}\left(\frac{1}{u^n}\right)=\frac{-u}{u^{n+1}}\frac{du}{dx}$

Example 1.

Find $\frac{dy}{dx}$ if $y=x^3-4x^2+5.$

Solution:

$$\frac{dy}{dx}=\frac{d}{dx}(x^3-4x^2+5)$$

$$=\frac{d}{dx}(x^3)+\frac{d}{dx}(-4x^2)+\frac{d}{dx}(5) \quad (4)$$

$$=\frac{d}{dx}(x^3)-4\frac{d}{dx}(x^2)+\frac{d}{dx}(5) \quad (3)$$

$$=3x^2-4(2x)+0 \quad (7)\ (1)$$

$$=3x^2-8x$$

Example 2.

If $y=\sqrt{3x+2}$, find $\frac{dy}{dx}$.

1st solution: Transform the radical into the exponential form. Thus,

$$\frac{dy}{dx}=\frac{1}{2}(3x+2)-\frac{d}{dx}(3x+2) \quad (7)$$

$$=\frac{1}{2}(3x+2)-\frac{d}{dx}(3+0) \quad (4)\ (3)$$

$$=\frac{3}{2(3x+2)^{\frac{1}{2}}}$$

$$=\frac{3}{2\sqrt{3x+2)}}$$

2nd solution: $y = \sqrt{3x+2}$

$$= \frac{\frac{d}{dx}(3x+2)}{2\sqrt{3x+2)}} \qquad (8)$$

$$= \frac{3}{2\sqrt{3x+2)}}$$

Example 3.

Find $\frac{dy}{dx}$ if $y = \frac{4}{(2x+1)^3}$

1st solution: $y = \frac{4}{(2x+1)^3}$

$$\frac{dy}{dx} = \frac{(2x+1)^3\frac{d}{dx}(4) - 4\frac{d}{dx}(2x+1)^3}{[(2x+1)^3]^2} \qquad (6)$$

$$= \frac{(2x+1)^3(0) - 4(2x+1)^3(2)}{(2x+1)^6}$$

$$= \frac{-24(2x+1)^2}{(2x+1)^6}$$

$$= \frac{-24}{(2x+1)^4}$$

2nd solution: $y = \frac{4}{(2x+1)^3} = 4(2x+1)^{-3}$

$$\frac{dy}{dx} = 4\frac{d}{dx}(2x+1)^{-3} \qquad (3)$$

$$dx = 4(-3)(2x+1)^{-4}\frac{d}{dx}(2x+1) \qquad (7)$$

$$= -12(2x+1)^{-4}(2)$$

$$= -24(2x+1)^{-4}$$

$$= \frac{-24}{(2x+1)^4}$$

3rd solution: $y = \frac{4}{(2x+1)^3}$

$$\frac{dy}{dx} = \frac{4(-3)}{(2x+1)^4}\frac{d}{dx}(2x+1) \qquad (9)$$

$$= \frac{-12}{(2x+1)^4}(2)$$

$$= \frac{-24}{(2x+1)^4}$$

Example 4.

Find $\frac{dy}{dx}$ if $y = (2x+1)^3(4x-1)^2$

Solution:

$$\frac{dy}{dx} = (2x+1)^3\frac{d}{dx}(4x-1)^2 + (4x-1)^2\frac{d}{dx}(2x+1)^3 \qquad (5)$$

$$= (2x+1)^3 \cdot 2(4x-1)(4) + (4x-1)^2 \cdot 3(2x+1)^2(2) \qquad (7)$$

$$= 2(2x+1)^2(4x-1)[4(2x+1) + 3(4x-1)]$$

$$= 2(2x+1)^2(4x-1)(20x+1)$$

Assessment Task 2

Find $\frac{dy}{dx}$ of each of the following.

1. $y = \sqrt{5 - 6x}$
2. $y = (3x^2 - 4x + 1)^5$
3. $y = \frac{4x-5}{2x+1}$
4. $y = (2x + 5)\sqrt{4x - 1}$
5. $y = (\frac{2x-3}{5x+1})^4$
6. $y = \left(\frac{x-6}{3x+4}\right)^{\frac{1}{3}}$
7. $y = 4(\sqrt{x} + 1)^5$

Summary

According to "Derivative" (2020) differentiation is the calculating action of a derivative. The function y = f(x) derivative of a variable x is a measure of the rate at which the function's value y changes in relation to the variable x change. With respect to x, it is called the derivative of f. If x and y are real numbers, and if f is plotted against x, the derivative at each point is the slope of that graph.

References

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MODULE 3
CHAIN RULE AND INVERSE FUNCTION RULE

Introduction

According to Feliciano and Uy (1983) certain functions are formed out of simpler functions by a process of substitution. Functions which result in this manner are called composite functions.

For a general discussion of composite functions, consider the functions f and g given by $y = f(u)$ and $u = g(x)$ respectively. We have here a situation in which y depends on u and u in turn depends on x. to eliminate u, we simply substitute $u = g(x)$ in $y = f(u)$ and thereby obtain a new function h expressed symbolically in the form $y = h(x) = f[g(x)]$. Then $y = f[g(x)]$ is a composite function since y is a function of u and u in turn is a function of x. Note that $y = h(x) = f[g(x)]$ expresses directly as a function of x (Feliciano & Uy, 1983).

The chain rule states that the $f(g(x))$ derivative is $f'(g(x))$with $g'(x)$. It helps in the differentiation of composite functions, in other words ("The Chain Rule: Introduction", 2019).

Learning Outcomes

At the end of this module, students should be able to:

1. Understand the use of chain rule in differentiating a composite function.
2. Learn about the inverse function rule.

Lesson 1. The Chain Rule

Example 1:

Find $\frac{dy}{dx}$ if $y = 4u^3$ and $u = x^2 + 5x$

Solution: Substituting $u = x^2 + 5x$ in $y = 4u^3$, we get $y = 4(x^2 + 5x)^3$

By the rule 7 in derivatives, we have

$$\frac{dy}{dx} = 12(x^2 + 5x)^2(2x + 5)$$

Chain Rule: if y is a differentiable function of u given by y = f(u) and if u is a differentiable function of x given by u = g(x), then y is a differentiable function of x and

(10) $\frac{dy}{dx} = \frac{dy}{du} = \frac{du}{dx}$ (Feliciano & Uy, 1983).

Example 2:

Consider the functions given in example 1. Since $y = 4u^3$, then $\frac{dy}{du} = 12u^2$. Likewise, since $u = x^2 + 5x$, then $\frac{du}{dx} = 2x + 5$. Then by (10),

$$\frac{dy}{dx} = (12u^2)(2x + 5)$$

$$= 12(x^2 + 5x)^2(2x + 5)$$

Since $u = x^2 + 5x$

We now give the proof of (10). If $\Delta u \neq 0$, then we can write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta\Delta x}$$

Since u is a differentiable of x, then

$\Delta u \to 0$ as $\Delta x \to 0$

Applying the theorems on limits, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x})$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

And by definition, we have

$$\frac{dy}{dx} = \frac{dy}{du} = \frac{du}{dx}$$

Lesson 2. Differentiation of Inverse Function

According to Feliciano and Uy (1983) considering again the function defined by the equation y = f'(x), this equation may be solved for x, giving x = g(y) the functions f and g are said to be inverse functions. To distinguish between f and g, we shall call f the direct function and g the inverse function.

Let us now focus our attention to the problem of finding the derivative of y with respect to x or $\frac{dy}{dx}$ of the function written in the form x = g(y). This is accomplished by using the so called inverse function rule (Feliciano & Uy, 1983).

Inverse Function Rule: If y is a differentiable function of x defined by y = f(x), then its inverse function defined by x = g(y) is a differentiable function of y and

(11) $\frac{dy}{dx} = \frac{1}{dy/dx}$ (Feliciano & Uy, 1983).

Note that (11) clearly shows that the rate of change of y with respect to x (dy/dx) and the rate of change of x with respect to y (dx/dy) are reciprocals. It also says that the derivative of the inverse function is equal to the reciprocal of the derivative of the direct function.

Proof of (11): Let y = f(x) and x = g(y) be inverse functions. Then y is a function x and x is a function of y. By (10) in previous lesson,

$\frac{dy}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}$ and $1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$ or $\frac{dy}{dx} = \frac{1}{dy/dx}$ (Feliciano & Uy, 1983).

Example 3:

If $x = y^3 - 4y^2$, find $\frac{dy}{dx}$.

Solution: since $x = y^3 - 4y^2$, then $\frac{dy}{dx} = 3y^2 - 8y$ and by (11),

$$\frac{dy}{dx} = \frac{1}{3y^2 - 8y}$$

Assessment Task 3

Summary

Use chain rule to find $\frac{dy}{dx}$ and express the final answer in terms of x.

1. $y = u^2 + u,\ u = 2x + 1$
2. $y = (u - 4)^{\frac{3}{2}},\ u = x^2 + 4$
3. $y = \sqrt{u + 2},\ u = 4x - 2$
4. $y = \sqrt{u},\ u = \sqrt{x}$

Use inverse function rule to find $\frac{dy}{dx}$.

5. $x = \sqrt{y} + \sqrt[3]{y}$
6. $x = 2(4y + 1)^3$
7. $x = \sqrt{1 + \sqrt{1 + \sqrt{y}}}$

According to Feliciano and Uy (1983),

Chain Rule: if y is a differentiable function of u given by y = f(u) and if u is a differentiable function of x given by u = g(x), then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} = \frac{du}{dx}$$

Inverse Function Rule: If y is a differentiable function of x defined by y = f(x), then its inverse function defined by x = g(y) is a differentiable function of y and

$$\frac{dy}{dx} = \frac{1}{dy/dx}$$

References

Feliciano, Florentino T. & Uy, Fausto B., Differential & Integral Calculus, Merriam & Webster Bookstore, Inc., Manila, Philippines. 1983.

The Chain Rule: Introduction. (2019, August 1). Khan Academy. Retrieved from https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-3-1a/v/chain-rule-introduction

MODULE 4
RELATED RATES

Introduction

Based on Feliciano and Uy (1983) we recall that if y = f(x), then dy / dx is the rate of y varying from x. Therefore if y = f(t), then dy / dt is the rate of y change over t. If t represents the time, then dy / dt is generally referred to as the y change rate. Similarly, dx / dt is the time rate of change of x. These change rates are associated with the $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. For example, if $y = x^2 + 4x + 3$, then $\frac{dy}{dx} = 2x + 4$ and by equation $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2x + 4)\frac{dx}{dt}$ This equation is indeed the outcome of differentiating both sides of $y = x^2 + 4x + 3$ with respect to the time t. In practice, therefore, to find dy / dt of the equation y = f(x), we simply obtain the derivative of y as regards x and then multiply the result by dx/dt.

Many physical issues deal with quantity change rates regarding time. For example, when water is poured into a tank, the surface of the water is rising relative to time. This rate of water level change can be expressed in terms of the rate of the water depth change. If we denote this change by h, then dh / dt is the time rate of change of the depth. Likewise, if V represents volume, then dV/dt is the time rate of change of the volume. If V = f(h), then by the equation $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, we have $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ (Feliciano & Uy, 1983).

Learning Outcomes

At the end of this module, students should be able to:

1. Solve for the time rate of change in related rates.

Lesson 1. Time Rates

In solving "time rate" problems, it is important to observe that all quantities which change with respect to the time must be denoted by letters. Do not substitute the numerical values of such variable until after differentiation with respect to the time t is done (Feliciano & Uy, 1983).

Example 1:

Water is poured into a conical tank 6m across the top and 8m deep at the rate of $10m^3/min$. How fast is the water level rising when the water in the tank is 5m deep?

Solution:

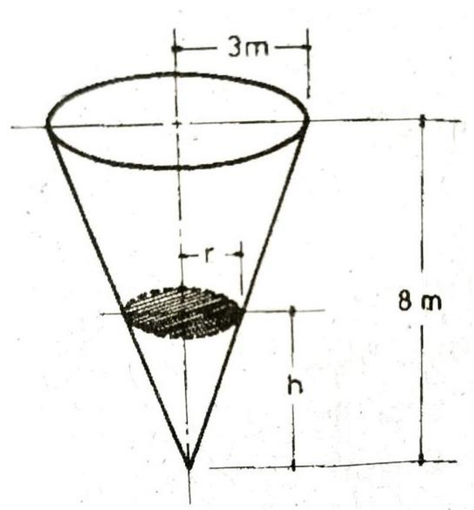

Figure 4.1 Conical Tank

At time t, let:

r = radius of the water surface

h = depth of the water

V = volume of the water

It is given that $\frac{dy}{dt} = 10m^3/min.$ and it is required to find $\frac{dh}{dt}$ at the instant when h = 5m.

The volume of the water in the tank at the time t is

$$V = \frac{1}{3}\pi r^2 h \qquad (1)$$

Since we are to find $\frac{dh}{dt}$, then we have to express V as a function of h. In the figure given and by similar triangles, we have

$$\frac{r}{3} = \frac{h}{8} \qquad (2)$$

Solving for r in (2), we get

$$r = \frac{3h}{8} \qquad (3)$$

Substituting (3) in (1) and simplifying, we obtain

$$V = \frac{3\pi h^3}{64} \qquad (4)$$

Differentiating (4) with respect to t

$$\frac{dV}{dt} = \frac{9\pi h^2}{64} \cdot \frac{dh}{dt} \qquad (5)$$

Substituting $\frac{dV}{dt} = 10$ and h = 5 in (5),

$$10 = \frac{225\pi}{64} \cdot \frac{dh}{dt} \qquad (6)$$

Solving for $\frac{dh}{dt}$ in (6), we obtain

$$\frac{dh}{dt} = \frac{128}{45\pi} m/min.$$

Example 2:

A ship A is 20 km west of another ship B. If A sails east at 10 km/hr and at the same time B sails north at 30 km/hr, find the rate of change of the distance between them at the end of ½ hr.

Solution:

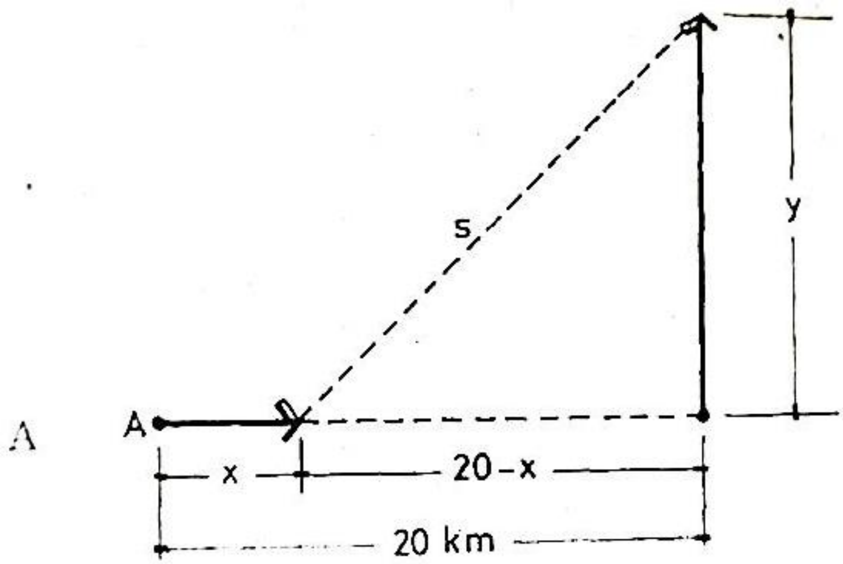

Figure 4.2 Position and Distance Traveled by Ship A & B

At time t, let:

s = distance between the ships

x = distance traveled by ship A

y = distance traveled by ship B

where x = 10t and y = 30t. Hence $\frac{dx}{dt} = 10$ and $\frac{dy}{dt} = 30$.

It is required that we find ds/dt when t= ½ hr. Using the right triangle in the figure, we get the relation

$$s^2 = (20 - x)^2 + y^2 \qquad (1)$$

Differentiating (1) with respect to t

$$\frac{ds}{dt} = \frac{-(20-x)\frac{dx}{dt}+y\frac{dy}{dt}}{s} \qquad (2)$$

When t = ½, we get

$$x = 10\left(\frac{1}{2}\right) = 5$$

$$y = 30\left(\frac{1}{2}\right) = 15$$

Solving for s in (1) and substituting these values of x and y, we have

$$s = \sqrt{(20 - x)^2 + y^2}$$

$$= \sqrt{(20 - 5)^2 + 15^2}$$

` $$= 15\sqrt{2}$$

Substituting the values of x, y, s, dx/dt and dy/dt in (2), we get

$$\frac{ds}{dt} = \frac{-(20-5)10+15(30)}{15\sqrt{2}}$$

$$10\sqrt{2} km/hr$$

Alternative Solution: Another approach is to express s in terms of t only. To obtain this, we substitute x = 10t and y = 30t in (1). Thus

$$s^2 = (20 - 10t)^2 + (30t)^2$$

or $$s = \sqrt{(20 - 10t)^2 + (30t)^2}$$

Differentiating

$$\frac{ds}{dt} = \frac{2(20-10t)(-10)+2(30t)(30)}{2\sqrt{(20-10t)^2+(30t)^2}}$$

Substituting t = ½, we get

$$\frac{ds}{dt} = 10\sqrt{2} km/hr$$

Assessment Task - 4

Solve for the following time rate problems.

1. Water is flowing into a conical cistern at the rate of $8m^3/min$. If the height of the inverted cone is 12m and the radius of its circular opening is 6m. How fast is the water rising when the water is 4m depth?
2. A through filled with water is 2m long and has a cross section in the shape of an isosceles trapezoid 30cm wide at the bottom, 60cm wide at the top, and a height of 50cm. If the through leaks water at the rate of $2000cm^3/min$, how fast is the water level falling when the water is 20cm deep?

Summary

According to Capote and Mandawe (2007) here are the suggested steps in solving time rates problems:

- Make an equation based on the given situation (construct figures if necessary).
- If the right side of the equation has two or more variables, and if the rate of change of the other variable are not given then converting it to one variable is very necessary.
- Differentiate it with respect to time.
- Substitute the boundary conditions and solve for the unknown.

References

Capote, Roger S. & Mandawe, Joel A., Mathematics & Basic Engineering Sciences, JAM Publisher, Philippines. 2007.

Feliciano, Florentino T. & Uy, Fausto B., Differential & Integral Calculus, Merriam & Webster Bookstore, Inc., Manila, Philippines. 1983.