EM 111 - DIFFERENTIAL CALCULUS

1 Functions

1.1 Definition, Domain, Range

The term **function** was first used by Leibniz in 1673 to denote the dependence of one quantity on another. In general, if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y, then we say that y is a "*function*" of x.

A **function** is a rule that assigns to each element in a nonempty set A one and only one element in set B. (A is the domain of the function, while B is the range of the function). We use a symbol such as *f*(x), which is read "*f* of x", to denote a function of x.

The **dependent variable** is a variable whose value always depends and determined by using the other variable called an independent variable. The dependent variable is also called the outcome variable. The result is being evaluated from the mathematical expression using an independent variable is called a dependent variable.

Independent variables are the inputs to the functions that define the quantity which is being manipulated in an experiment.

Let us consider an example y= 3x. Here, x is known as the independent variable and y is known as the dependent variable as the value of y is completely dependent on the value of x.

Example, given *f*(x) = $-x^2 + 6x - 11$, find (a) *f* (2), (b) *f* (-10), (c) *f* (t), and (d) *f* (t-3)

(a) *f* (2) = $-(2)^2 + 6(2) - 11 = \mathbf{-3}$
(b) *f* (-10) = $-(-10)^2 + 6(-10) - 11 = \mathbf{-171}$
(c) *f* (t) = $-(t)^2 + 6(t) - 11 = \boldsymbol{-t^2 + 6t - 11}$
(d) *f* (t-3) = $-(t-3)^2 + 6(t-3) - 11 = -(t^2 - 6t + 9) + 6(t-3) - 11$
$= -t^2 + 6t - 9 + 6t - 18 - 11$
$= \boldsymbol{-t^2 + 12t - 38}$

Domain is the set in which the independent variable is restricted to lie. Restrictions on the independent variable that affect the domain of the function generally are due to: physical or geometric considerations, natural restrictions that result from a formula used to define the function. and artificial restrictions imposed by a problem solver.

Range is the set of all images of points in the domain (*f* (x), x∈A). The range of a function is simply the set of all possible values that a function can take. **∈ means "is an element of"*

Example, if *f* (x) = 3x be a function, the domain values or the input values are {1, 2, 3} then the range of a function is given as:

f (1) = 3(1) = 3

f (2) = 3(2) = 6

f (3) = 3(3) = 9

Therefore, the range of the function will be **{3, 6, 9}**.

1.2 Intervals

An interval is defined as the range of numbers that are present between the two given numbers. The domains of the variables in many applications of calculus are interval like as shown in the figures below.

- The **open interval** *a b* or (*a*, *b*) ; $a < x < b$
- The **closed interval** *a b* or [*a*, *b*] ; $a \leq x \leq b$
- The interval *a* less than or equal to *x* less than *b* or [*a*, *b*) ; $a \leq x < b$
- The interval *a* less than *x* less than or equal to *b* or (*a*, *b*] ; $a < x \leq b$

1.3 Operations on Functions

Functions can also be constructed from previously defined functions by using the algebraic properties of the real number system. Hence, let *f* and *g* be functions.

i. Their sum, denoted by $f + g$, is a function defined by
$$(f + g)(x) = f(x) + g(x)$$
ii. Their difference, denoted by $f - g$, is a function defined by
$$(f - g)(x) = f(x) - g(x)$$
iii. Their product, denoted by fg, is a function defined by
$$(fg)(x) = f(x) * g(x)$$
iv. Their quotient, denoted by $\frac{f}{g}$, is a function defined by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (g(x) \neq 0)$$

Each of these functions has its domain the set of all values of x which are both in the domain of f and in the domain of g with an additional requirement for (iv) where values of x for which $g(x) = 0$ are excluded.

Example, let $f(x) = x^2$ and $g(x) = 3x + 2$. Find (i) f+g, (ii) f-g, (ii) f*g, and (iv) f/g.

i. $(f + g)(x) = \boldsymbol{x^2 + 3x + 2}$
ii. $(f - g)(x) = \boldsymbol{x^2 - 3x + 2}$
iii. $(f * g)(x) = x^2(3x + 2) = \boldsymbol{3x^3 + 2x^2}$
iv. $(f/g)(x) = \boldsymbol{\frac{x^2}{3x+2}}$

1.4 Composition of Functions

Let f and g be functions. The composite function of g and f, denoted by $g \circ f$, is defined by,
$$(g \circ f)(x) = g(f(x))$$
The domain of $g \circ f$ is the set of all x in the domain of f for which $f(x)$ is in the domain of g.

Example 1, Let $f(x) = x^2 + 1$ and $g(x) = 3x + 2$. Find (i) $f \circ g$, and(ii) $g \circ f$.

i. $(f \circ g)(x) = f(g(x))$
$= f(3x + 2)$
$= (3x + 2)^2 + 1$
$= 9x^2 + 12x + 4 + 1$
$= \boldsymbol{9x^2 + 12x + 5}$
ii. $(g \circ f)(x) = g(f(x))$
$= g(x^2 + 1)$
$= 3(x^2 + 1) + 2$
$= 3x^2 + 3 + 2$
$= \boldsymbol{3x^2 + 5}$

Example 2, Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$. Find (i) $(f \circ g)(5)$, and (ii) $(g \circ g)(x)$.

i. $(f \circ g)(5) = f(g(5))$
$= f(1 - 20(5))$
$= f(-99)$
$= 3(-99)^2 - (-99) + 10$
$= \boldsymbol{29512}$
ii. $(g \circ g)(x) = g(g(x))$
$= g(1 - 20x)$
$= 1 - 20(1 - 20x)$
$= 1 - 20 + 400x$
$= \boldsymbol{400x - 19}$

1.5 Classification of Functions

- A function that has only one element in its range is called a **constant function**.

$$f(x) = C$$

- A real-valued function may be classified as **algebraic** or **transcendental**.
- An **identity function** is a function that assigns to each element in the domain, the element itself.
- Any function formed by performing a finite number of algebraic operations on constant and identity functions is classified as an **algebraic function** in general.

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

  Where each a_i is a constant and n is any nonnegative integer is called a **polynomial** function of degree n.
- An algebraic function is said to be **rational** if it is a quotient of two polynomial functions.

$$f(x) = \frac{a_0 + a_1x_1 + \cdots + a_{n-1}x^{n-1} + a_nx_n}{b_0 + b_1x_1 + \cdots + b_{n-1}x^{n-1} + b_nx_n}$$

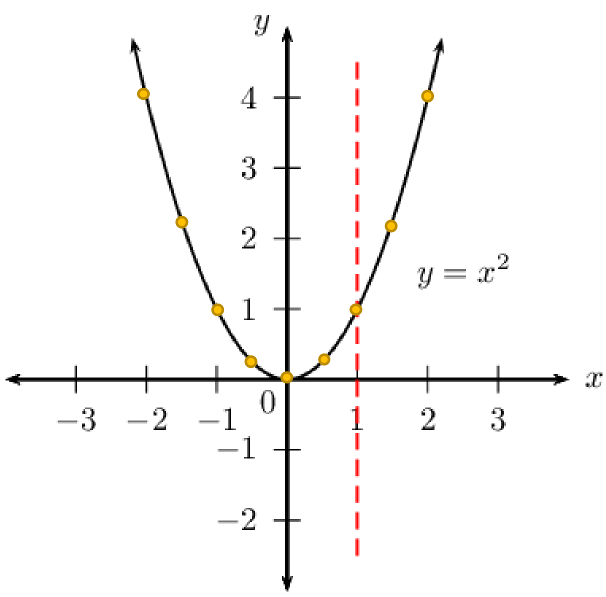
1.6 Graph of a Function

The graph of a function f defined by $y = f(x)$ is the set of all points (x, y) in a plane.

Example: Graph the function $y = x^2, -2 \leq x \leq 2$.

Sol'n: To graph a function, we have to make a table of pairs from the function.

x $-2 \leq x \leq 2$	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0
y $y = x^2$	4.0	2.25	1	0.25	0	0.25	1	2.25	4.0



Vertical Line Test

Given the graph of a relation, there is a simple test for whether or not the relation is a function. This test is called the vertical line test. If it is possible to draw any vertical line (a line of constant x) which crosses the graph of the relation more than once, then the relation is not a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x (one-to-many).

- If any vertical line cuts the graph only once, then the relation is a function (one-to-one or many-to-one).
- The red vertical line cuts the circle twice and therefore the circle is not a function.
- The red vertical line only cuts the parabola once and therefore the parabola is a function.

Hence, the graph above is a function since it cuts the graph only once.

1.7 Inverse of a Function

Given the function $f(x) = -\frac{1}{3}x + 1$, we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with y.
$$y = -\frac{1}{3}x + 1$$
2. Replace every x with a y and replace every y with an x.
$$x = -\frac{1}{3}y + 1$$
3. Solve for y.
$$x = -\frac{1}{3}y + 1$$
$$\frac{1}{3}y = -x + 1$$
$$\left[\frac{1}{3}y = -x + 1\right](3)$$
$$y = -3x + 3$$
4. Replace y with $f^{-1}(x)$.
$$\boldsymbol{f^{-1}(x) = -3x + 3}$$
5. You may verify by checking that $(f \circ f^{-1})(x) = x$ or $(f^{-1} \circ f)(x) = x$

Example: Given $g(x) = \sqrt{x-3}$, find $g^{-1}(x)$.

$g(x) = \sqrt{x-3}$

$y = \sqrt{x-3}$

$x = \sqrt{y-3}$

$x^2 = y - 3$

$y = x^2 + 3$

$\boldsymbol{g^{-1}(x) = x^2 + 3}$

Verify by using $(g^{-1} \circ g)(x) = x$

$$\begin{aligned}(g^{-1} \circ g)(x) &= g^{-1}(g(x)) \\ &= g^{-1}(\sqrt{x-3}) \\ &= (\sqrt{x-3})^2 + 3 \\ &= x - 3 + 3 \\ &= \boldsymbol{x}\end{aligned}$$

2 Limit of a Function and Continuity

Let f be a function that is defined at every number in the interval which contains the number a, except possibly at a itself. The number L is said to be the **limit of $f(x)$ as x approaches a**, or simply the **limit of f of a**, if $f(x)$ can be made very close to L as maybe desire by making x close enough to a and we write,

$$\lim_{x \to a} f(x) = L$$

The limit of a function describes the behavior of a function in the vicinity of a number as oppose to the value of the function at the given number. It tells us the approximate value assumed by the function in the vicinity of the given number. The definition of the limit above refers to the so called two sided limit since there are two ways or directions to approach the number a on the real line: from the right or from the left.

2.1 One-sided Limits

When the approach of x to a is confined only to one side of a, it is indicated by writing $x \to a^+$ or $x \to a^-$ instead of $x \to a$, depending upon whether x is approaching a from the right or from the left. The symbol $\lim_{x\to a^+} f(x)$ refers to the limit of $f(x)$ as x approaches a through values greater than a, and the symbol $\lim_{x\to a^-} f(x)$ refers to the limit of $f(x)$ as x approaches a through values less than a. If $f(x)$ approaches a definite number L as x becomes arbitrary large numerically, i.e. , x is increasing or decreasing without bound, we indicate it by writing $\lim_{x\to\infty} f(x) = L$. When $f(x)$ becomes arbitrary large numerically as x approaches the number a, the symbol $\lim_{x\to a} f(x) = \infty$ is used.

2.2 Theorems on Limits

Theorem 1: $\lim_{x\to a} c = c$, c = any constant

Theorem 2: $\lim_{x\to a} x = a$, a = any real number

Theorem 3: $\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$

Theorem 4: $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Theorem 5: $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

Theorem 6: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Theorem 7: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$,n = any positive integer and $f(x) \geq 0$ if n is even.

Theorem 8: $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

Example 1:

$$\lim_{x\to 2}(x^2 + 3x + 4) = \lim_{x\to 2} x^2 + \lim_{x\to 2} 3x + \lim_{x\to 2} 4 \qquad \textit{Theorem 4}$$

$$= \left[\lim_{x\to 2} x\right]^2 + 3\lim_{x\to 2} x + 4 \qquad \textit{Theorem 8,3,1}$$

$$= [2]^2 + 3(2) + 4 \qquad \textit{Theorem 2}$$

$$= \mathbf{14}$$

Example 2:

$$\lim_{x\to 2}(x + 4)\sqrt{2x + 5} = \lim_{x\to 2}(x + 4) \cdot \lim_{x\to 2} \sqrt{2x + 5} \qquad \textit{Theorem 5}$$

$$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 4\right) \cdot \sqrt{\lim_{x\to 2}(2x + 5)} \qquad \textit{Theorem 4,7}$$

$$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 4\right) \cdot \sqrt{\lim_{x\to 2} 2x + \lim_{x\to 2} 5} \qquad \textit{Theorem 4}$$

$$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 4\right) \cdot \sqrt{2\lim_{x\to 2} x + \lim_{x\to 2} 5} \qquad \textit{Theorem 3}$$

$$= (2 + 4) \cdot \sqrt{2(2) + 5} \qquad \textit{Theorem 2,1}$$

$$= \mathbf{18}$$

Example 3:

$$\lim_{x\to 3}(3x + 4)^2 = \left[\lim_{x\to 3}(3x + 4)\right]^2 \qquad \textit{Theorem 8}$$

$$= \left[\lim_{x\to 3}(3x) + \lim_{x\to 3}(4)\right]^2 \qquad \textit{Theorem 4}$$

$$= \left[3\lim_{x\to 3}(x) + \lim_{x\to 3}(4)\right]^2$$ *Theorem 3*

$$= [3(3) + 4]^2$$ *Theorem 2,1*

$$= \mathbf{169}$$

*Note that the limits of the functions in the above examples can be obtained by *straight substitution*.

$$\lim_{x\to 2}(x^2 + 3x + 4) = [(2)^2 + 3(2) + 4]$$

$$= \mathbf{14}$$

2.3 Indeterminate Forms

Evaluate $\lim_{x\to 4}\frac{x^2-5x+4}{x-4} = \frac{(4)^2-5(4)+4}{(4)-4} = \frac{0}{0}$ ∵ Indeterminate

Obtaining this form by straight substitution does not necessarily mean that $f(x)$ has no limit.

L'hopital's Rule – provides a technique to evaluate limits of indeterminate forms. Can be solved through derivative rule of factoring.

✓ Factoring Rule

$$\lim_{x\to 4}\frac{x^2 - 5x + 4}{x - 4} = \frac{(x-4)(x-1)}{(x-4)}$$

$$= \lim_{x\to 4}(x-1)$$

$$= 4 - 1$$

$$= \mathbf{3}$$

✓ Derivative Rule

$$\lim_{x\to 4}\frac{x^2 - 5x + 4}{x - 4} = \lim_{x\to 4}\frac{2x-5}{1}$$

$$= \frac{2(4)-5}{1}$$

$$= \mathbf{3}$$

2.4 Infinity

Theorem 9: $\lim_{x\to 0^+}\frac{1}{x} = +\infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

Theorem 10: $\lim_{x\to +\infty} k = \lim_{x\to -\infty} k = k$

Theorem 11: $\lim_{x\to +\infty}\frac{1}{x} = 0$, $\lim_{x\to -\infty}\frac{1}{x} = 0$

Example 1:

$$\lim_{x\to\infty}\frac{1}{x^3} = \lim_{x\to\infty}\left(\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}\right)\cdot\lim_{x\to\infty}\left(\frac{1}{x}\right)\cdot\lim_{x\to\infty}\left(\frac{1}{x}\right)$$ *Theorem 5*

$$= \mathbf{0}$$

Example 2:

$$\lim_{x\to\infty}\frac{1}{x^{\frac{1}{4}}} = \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{4}}$$

$$= \left[\lim_{x\to\infty}\frac{1}{x}\right]^{\frac{1}{4}}$$ *Theorem 8*

$$= \mathbf{0}$$ *Theorem 11*

Example 3:

$$\lim_{x\to\infty}\frac{8x-5}{2x+3} = \lim_{x\to\infty}\frac{\frac{8x}{x}-\frac{5}{x}}{\frac{2x}{x}+\frac{3}{x}}$$

Sol'n: Divide the numerator and the denominator by the **HIGHEST POWER OF X** occurring in either the numerator or denominator.

$$= \lim_{x\to\infty}\frac{8-\frac{5}{x}}{2+\frac{3}{x}}$$

$$= \frac{\lim_{x\to\infty} 8 - \lim_{x\to\infty} 5 \cdot \lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} 3 \cdot \lim_{x\to\infty}\frac{1}{x}}$$

$$= \frac{8-5(0)}{2+3(0)}$$ *Theorem 10,11*

$$= \mathbf{4}$$

Example 4:

$$\lim_{x\to\infty}\frac{4x^3+3x^2-6}{2x^3+5x+3} = \lim_{x\to\infty}\frac{\frac{4x^3}{x^3}+\frac{3x^2}{x^3}-\frac{6}{x^3}}{\frac{2x^3}{x^3}+\frac{5x}{x^3}+\frac{3}{x^3}}$$

$$= \lim_{x\to\infty}\frac{4+\frac{3}{x}-\frac{6}{x^3}}{2+\frac{5}{x^2}+\frac{3}{x^3}}$$

$$= \frac{\lim_{x\to\infty} 4 + \lim_{x\to\infty}\frac{3}{x} - \lim_{x\to\infty}\frac{6}{x^3}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{5}{x^2} + \lim_{x\to\infty}\frac{3}{x^3}}$$

$$= \frac{\lim_{x\to\infty} 4 + \lim_{x\to\infty} 3 \cdot \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty} 6 \cdot \lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} 5 \cdot \lim_{x\to\infty}\frac{1}{x^2} + \lim_{x\to\infty} 3 \cdot \lim_{x\to\infty}\frac{1}{x^3}}$$

$$= \frac{4+3(0)-6(0)}{2+5(0)+3(0)}$$

$$= \mathbf{2}$$

2.5 Continuity

A function f is said to be continuous at a number a provided,

i. a is in the domain of f, this is, $f(a)$ exists.
ii. $\lim_{x\to a} f(x)$ exists
iii. $\lim_{x\to a} f(x) = f(a)$

If any of these three conditions fails, f is not continuous at a and we say that f is discontinuous at a or that f has a discontinuity at a. A function is said to be continuous on an interval if it is continuous at every number in that interval.