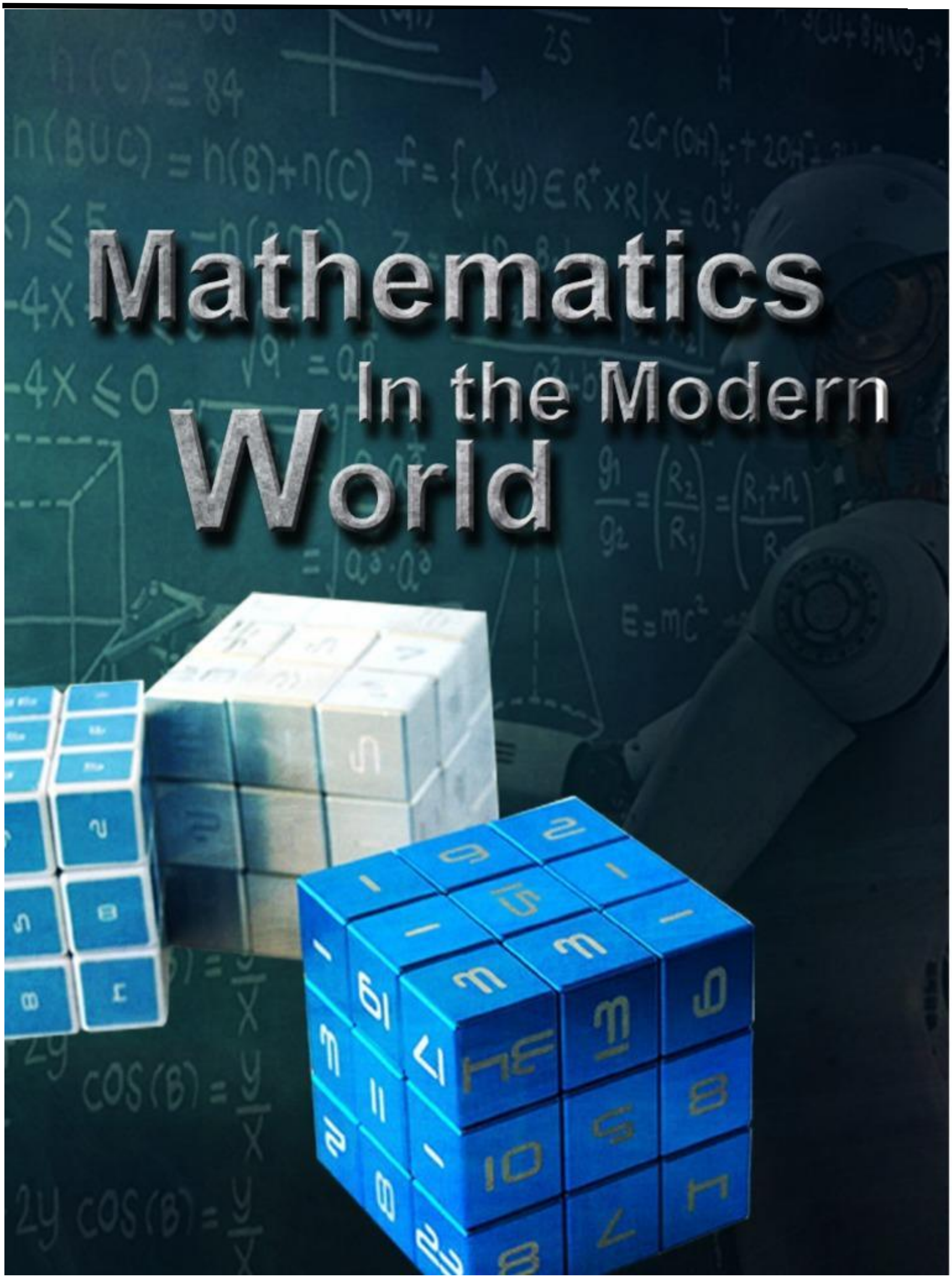
Mathematics
In the Modern
World

FOREWORD

Mathematics in the Modern World is a general education course that deals with the nature of mathematics. It is concerned with the appreciation of its practical, intellectual, aesthetic, and application of mathematical tools in daily life. Its primary objective is to educate and secondarily to train learners in this area of discipline. On educative side, it aims to equip students with necessary intelligence to become leaders and partakers in nation building. While on the training side, it provides them necessary and sufficient skills that they can harness in order to combat the challenges of daily living.

This learning material being guided by CHED CMO No. 20, series of 2013, is divided into several modules encompassing both the nature of mathematics and the utility of mathematics in the modern world. Each module is subdivided into a number of lessons designed to introduce each topic pedagogically in a management fashion intended for independent learning. Learning activities as well as Chapter Tests are provided in compliance with the learning plan suggested by the Commission on Higher Education and of the Institution.

Module 1 is concerned with the mathematics in our world. It provides a new way of looking at mathematics as a science of patterns. Basically, it encapsulates the entirety of the course by providing insights that mathematical structure is embedded in the structure of the natural world.

Module 2 is focused on mathematical language. It expounded the idea that like any language, mathematics has its own symbols, syntax and rules. It explained the conventions and usefulness of mathematics as a language.

Module 3 discussed about problem solving and reasoning. It asserted that mathematics is not just about numbers and much of it is problem solving and reasoning. It recalibrated the learner's problem solving skills, by providing new understanding on the relevance of the Polya's method in solving mathematical problems.

Module 4 introduced the mathematical system. It was a recreational mathematics that finally put into the limelight at the dawn of the modern world. Its utilization served as a backbone of commerce in the information age. It is an indispensable tool of the modern time.

Module 5 is entitled Data Management. It explained that statistical tools derived from mathematics are useful in processing and managing numerical data in order to describe a phenomenon and predict values. It is intended to go beyond the typical understanding as merely set of formulas but as tools that can decode nature's numbers.

Module 6 explored the very fabric that woven of mathematical landscape. It is the art and science of correct thinking and reasoning: Logic. It disciplined learner's understanding by exploring the application of formal logic to mathematics.

Module 7 is an innovative mathematical concept concerning network and connectivity. It basically concerned on how networks can be encoded and solved economically. It challenges the mind of the learner to metaphorically settle the Konigsberg Bridge Problem of the modern world.

Over-all the modules comprising this book are eclectically treated in such a way that both interpretative and applicative dimensions of learning become an integral part of the lesson presented, and the learning activities as well.

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MODULE ONE

THE NATURE OF MATHEMATICS

CORE IDEA

Module One is an introduction to the nature of mathematics as an exploration of patterns. It is a useful way to think about nature and our world.

Learning Outcome:

1. To identify patterns in nature and regularities in the world.
2. To articulate importance of mathematics in one's life.
3. To argue about the nature of mathematics, what it is, how it is expressed, represented, and used.
4. Express appreciation of mathematics as a human endeavor.

Unit Lessons:

Lesson 1.1 Mathematics of Our World
Lesson 1.2 Mathematics in Our World
Lesson 1.3 Mathematics of Sequence

Time Allotment: Four lecture hours

The Mathematics of Our World

Specific Objectives

1. To understand the mathematics of the modern world.
2. To revisit and appreciate the mathematical landscape.
3. To realize the importance of mathematics as a utility.
4. To gain awareness of the role of mathematics as well as our role in mathematics.

Lesson 1.1 does not only attempt to explain the essence of mathematics, it serves also as a hindsight of the entire course. The backbone of this lesson draws from the Stewart's ideas embodied in his book entitled Nature's Numbers. The lesson provides new perspective to understand the irregularity and chaos of our world as we move through the landscape of regularity and order. It poses some thought-provoking questions to draw one's innate mathematical intelligence by making one curious, not so much to seek answers, but to ask more right questions.

Discussions

The Nature of Mathematics

In the book of Stewart, Nature's Number, he that mathematics is a formal system of thought that was gradually developed in the human mind and evolved in the human culture. Thus, in the long course of human history, our ancestors at a certain point were endowed with insight to realize the existence of “form” in their surroundings. From their realization, a system of thought further advanced their knowledge into understanding measures. They were able to gradually develop the science of measures and gained the ability to count, gauge, assess, quantify, and size almost everything.

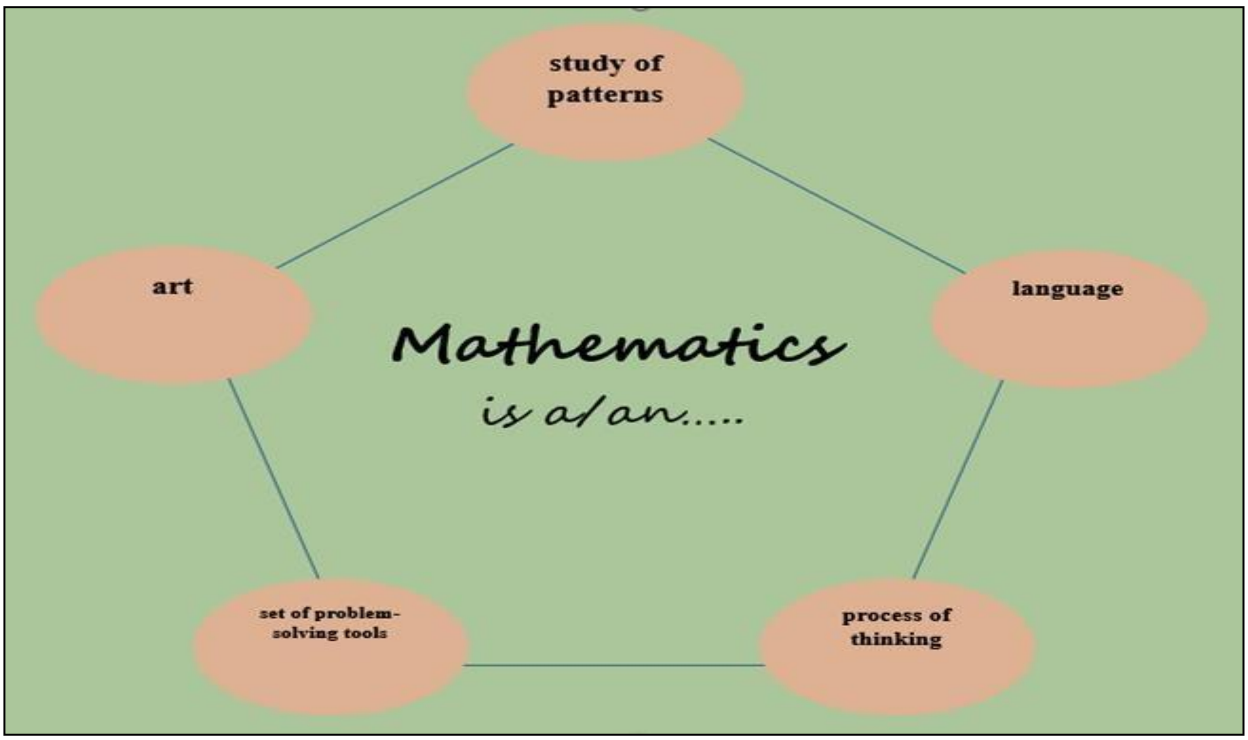
From our ancestor’s realization of measures, they were able to notice and recognize some rudiment hints about patterns. Thus, the concept of recognizing shapes made its course towards classifying contour and finally using those designs to build human culture: an important ingredient for a civilization to flourish. From then, man realized that the natural world is embedded in a magnanimously mathematical realm of patterns----and that natural order efficiently utilizes all mathematical patterns to its advantage. As a result, we made use of mathematics as a brilliant way to understand the nature by comprehending the structure of its underlying patterns and regularities.

Mathematics is present in everything we do; it is all around us and it is the building block of our daily activities. It has been at the forefront of each and every period of our development, and as our civilized societies advanced, our needs of mathematics pioneering arose on the frontier of our course as we prepare our human species to traverse the cosmic shore.

Mathematics is a Tool

Mathematics, as a tool, is immensely useful, practical, and powerful. It is not about crunching numbers, formulas, and symbols but rather, it is all about forming new ways to see problems so we can understand them by combining insights with imagination. It also allows us to perceive realities in different contexts that would otherwise be intangible to us. It can be likened to our sense of sight and touch. Mathematics is our sense to decipher patterns, relationships, and logical connections. It is our whole new way to see and understand the modern world.

Mathematics, being a broad and deep discipline, deals with the logic of shape, quantity, and arrangement. Once, it was perceived merely a collective thoughts dealing with counting numbers, but it is now being understood as a universal language dealing with symbols, arts, equations, geometric shapes and patterns. It is asserting that mathematics is a powerful tool in decision-making and it is a way of life.



The nature of mathematics
Figure 1.1

In the Figure 1.1 illustrated by Nocon and Nocon, it portrays the function of mathematics. As shown, it is stated that mathematics is a set of problem-solving tools. It provides answers to existing questions and presents solutions to occurring problems. It has the power to unveil the reasons behind occurrences and it offers explanations. Moreover, mathematics, as a study of patterns, allows people to observe, hypothesize, experiment, discover, and recreate. On the other hand, mathematics is an art and a process of thinking. For it involves reasoning, which can be inductive or deductive, and it applies methods of proof both in fashion that is conventional and unventional.

Mathematics is Everywhere

We use mathematics in their daily tasks and activities. It is our important tool in the field of sciences, humanities, literature, medicine, and even in music and arts; it is in the rhythm of our daily activities, operational in our communities, and a default system of our culture. There is mathematics wherever we go. It helps us cook delicious meals by exacting our ability to measure and moderately control of heat. It also helps us to shop wisely, read maps, use the computer, remodel a home with constrained budget with utmost economy.

Source: Space Telescope Science/NASA

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Even the cosmic perspective, the patterns in the firmament are always presented as a mystery waiting to be uncovered by us-the sentient being. In order to unearthed this mystery, we are challenged to investigate and deeply examine its structure and rules to the infinitesimal level. The intertwined governing powers of cosmic mystery can only be decoded by seriously observing and studying their regularities, and patiently waiting for the signature of some kind interference. It is only by observing the abundance of patterns scattered everywhere that this irregularities will beg to be noticed. Some of them are boldly exposed in a simple and obvious manner while others are hidden in ways that is impossible to perceive by easy to discern. While our ancestors were able to discover the presence of mathematics in everything, it took the descendants, us, a long time to gradually notice the impact of these patterns in the persistence of our species to rightfully exist.

The Essential Roles of Mathematics

Mathematics has countless hidden uses and applications. It is not only something that delights our mind but it also allows us to learn and understand the natural order of the world. This discipline was and is often studied as a pure science but it also finds its place in other areas of perpetuating knowledge. Perhaps, science would definitely agree that, when it comes to discovering and unveiling the truth behind the inherent secrets and occurrences of the universe, nothing visual, verbal, or aural come close to matching the accuracy, economy, power and elegance of mathematics. Mathematics helps us to take the complex processes that is naturally occuring in the world around us and it represents them by utilizing logic to make things more organized and more efficient.

Further, mathematics also facilitate not only to weather, but also to control the weather ---- be it social, natural, statistical, political, or medical. Applied mathematics, which once only used for solving problems in physics, and it is also becoming a useful tool in biological sciences: for instance, the spread of various diseases can now be predicted and controlled. Scientists and researchers use applied mathematics in doing or performing researches to solve social, scientific, medical, or even political crises.

It is a common fact that mathematics plays an important role in many sciences. It is and it provides tools for calculations. We use of calculations in other disciplines whenever we are underrating some kind of research or experiment. The use of mathematical calculations is indispensable method in scientifically approaching most of the problems. In a similar way, mathematics, provides new questions to think about. Indeed, in learning and doing mathematics, there will always be new questions to answer, new problems to solve, and new things to think about (Vistru-Yu PPT presentation).

The Mathematical Landscape

The human mind and culture developed a conceptual landscape for mathematical thoughts and ideas to flourish and propagate. There is a region in the human mind that is capable of constructing and discerning the deepest insights being perceived from the natural world. In this region, the mathematical landscape exists- wherein concepts of numbers, symbols, equations, operations calculations, abstractions, and proofs are the inhabitants as well as the constructs of the impenetrable vastness of its unchartered territories. In this landscape, a number is not simply a mathematical tree of counting. Also, infinite variables can be encapsulate to finite. Even those something that is hard to express in decimal form can be expressed in terms of fractions. Those things that seemed eternal $\mathbb{Z}$ can further be exploited using mathematical operations. This landscape claimed complex numbers as the firmament and even asserted that imaginary numbers also exist. To the low state negative numbers relentlessly enjoying recognition as existent beings. The wind in this landscape is unpredictable that the rate of change of the rate of change of weather is known as calculus. And beneath the surface of this mathematical landscape are firmly-woven proofs, theorems, definitions, and axioms which are intricately “fertilized” by reasoning, analytical, critical thinking and germicide by mathematical logic that made them precise, exact and powerful.

With this landscape, the mathematician's instinct and curiosity entice to explore further the vast tranquil lakes of functions and impassable crevasse of the unchartered territories of abstract algebra. For to claim ownership is to understand the ebb and flow of prime numbers. To predict the behavior of its Fibonacci weather, to be amazed with awe and wonder the patternless chaos of fractal clouds, and to rediscover that after all, the numbers in mathematics is not a "thing" but a process. Conventionally, we are just simply made ourselves comfortable on the “thingification” of those processes and we forgot that 1+1 is not a noun but a verb.

How Mathematics is Done

Math is a way of thinking, and it is undeniably important to see how that thinking is going to be developed rather than just merely see face value of the results. For some people, few math theorems can bring up as much remembered pain and anxiety. For others, this discipline is so complex and they have to understand the confusing symbols, the difficult procedures, and the dreaded graphs and charts. For most, mathematics is just nothing but something to survive, rather than to learn.

To the untrained eye, doing mathematics is quite difficult and challenging. It is ambiguous, for it follows a set of patterns, formulas, and sequences that make it more demanding to do and to learn. It is abstract and complex ---- and for these reasons, a lot of people adopt the belief that they are not math people.

Mathematics builds upon itself. More complex concepts are built upon simpler concepts, and if you do not have a strong grasp of the fundamental principles, then a more complex problem is more likely going to stump you. If you come across a mathematical problem that you cannot solve, the first thing to do is to identify the components or the operations that it wants you to carry out, and everything follows. Doing and performing mathematics is not that simple. It is done with curiosity, with a penchant for seeking patterns and generalities, with a desire to know the truth, with trial and error, and without fear of facing more questions and problems to solve. (Vistru-Yu)

Mathematics is for Everyone

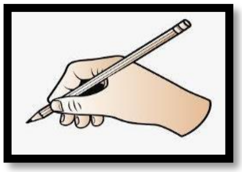
The relationship of the mathematical landscape in the human mind with the natural world is so strange that in the long run, the good math provides utilization and usefulness in the order of things. Perhaps, for most people, they simply need to know the basics of the mathematical operations in order to survive daily tasks; but for the human society to survive and for the human species to persistently exist, humanity needs, beyond rudiment of mathematics. To safeguard our existence, we already have delegated the functions of mathematics across all disciplines. There is mathematics we call pure and applied, as there are scientists we call social and natural. There is mathematics for engineers to build, mathematics for commerce and finance, mathematics for weather forecasting, mathematics that is related to health, and mathematics to harness energy for utilization. To simply put it, everyone uses mathematics in different degrees and levels. Everyone uses mathematics, whoever they are, wherever they are, and whenever they need to. From mathematicians to scientists, from professionals to ordinary people, they all use mathematics. For mathematics puts order amidst disorder. It helps us become better persons and helps make the world a better place to live in. (Vistru-Yu).

The Importance of Knowing and Learning Mathematics

Why do we want to observe and describe patterns and regularities? Why do we want to understand the physical phenomena governing our world? Why do we want to dig out rules and structures that lie behind patterns of the natural order? It is because those rules and structures explain what is going on. It is because they are beneficial in generating conclusions and in predicting events. It is because they provide clues. The clues that make us realize that interference in the motion of heavenly bodies can predict lunar eclipse, solar eclipse as well as comets' appearances. That the position of the sun and the moon relative to the earth can predict high tide and low tide events affecting human activities. And that human activities need clues for the human culture to meaningfully work.

Mathematical training is vital to decipher the clues provided by nature. But the role of mathematics goes clues and it goes beyond prediction. Once we understand how the system works, our goal is to control it to make it do what we want. We want to understand the mathematical pattern of a storm to avoid or prevent catastrophes. We want to know the mathematical concept behind the contagion of the virus to control its spread. We want to understand the unpredictability of cancer cells to combat it before it even exists. Finally, we want to understand the butterfly effect as much as we are so curious to know why the "die" of the physical world play god.

> *"Whatever the reasons, mathematics is a useful way to think about nature. What does it want to tell us about the patterns we observe? There are many answers. We want to understand how they happen; to understand why they happen, which is different; to organize the underlying patterns and regularities in the most satisfying way; to predict how nature will behave; to control nature for our own ends; to make practical use of what we have learned about our world. Mathematics helps us to do all these things, and often, it is indispensable." [Stewart]*



Learning Activity 1.1

Answer one of the following questions (15-20 lines) and submit your answer to your course facilitator.

1. What are the new things that you learned about the nature of mathematics?
2. What aspect of the lesson significant changed your view about mathematics?
3. What is the most important contribution of mathematics in humankind?

Observe the following this format:

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Lesson 1.2

The Mathematics in Our World

Specific Objective

:

1. To develop one's understanding about patterns;
2. To identify different patterns in nature;
3. To recognize different symmetries in nature; and
4. To explain the presence of Fibonacci numbers in nature

The mathematics in our world is rooted in patterns. Patterns are all around us. Finding and understanding patterns give us great power to play like god. With patterns, we can discover and understand new things; we learn to predict and ultimately control the future for our own advantage.

A pattern is a structure, form, or design that is regular, consistent, or recurring. Patterns can be found in nature, in human-made designs, or in abstract ideas. They occur in different contexts and various forms. Because patterns are repetitive and duplicative, their underlying structure regularities can be modelled mathematically. In general sense, any regularity that can be explained mathematically is a pattern. Thus, an investigation of nature's patterns is an investigation of nature's numbers. This means that the relationships can be observed, that logical connections can be established, that generalizations can be inferred, that future events can be predicted, and that control can possibly be possible.

Discussions

Different Kinds of Pattern

As we look at the world around us, we can sense the orchestrating great regularity and diversity of living and non-living things. The symphonies vary from tiny to gigantic, from simple to complex, and from dull to the bright. The kaleidoscope of patterns is everywhere and they make the nature look only fascinating but also intriguing. Paradoxically, it seemed that everything in the world follows a pattern of their own and tamed by the same time pattern of their own.

Patterns of Visuals. Visual patterns are often unpredictable, never quite repeatable, and often contain fractals. These patterns are can be seen from the seeds and pinecones to the branches and leaves. They are also visible in self-similar replication of trees, ferns, and plants throughout nature.

Patterns of Flow. The flow of liquids provides an inexhaustible supply of nature's patterns. Patterns of flow are usually found in the water, stone, and even in the growth of trees. There is also a flow pattern present in meandering rivers with the repetition of undulating lines.

Patterns of Movement. In the human walk, the feet strike the ground in a regular rhythm: the left-right-left-right-left rhythm. When a horse, a four-legged creature walks, there is more of a complex but equally rhythmic pattern. This prevalence of pattern in locomotion extends to the scuttling of insects, the flights of birds, the pulsations of jellyfish, and also the wave-like movements of fish, worms, and snakes.

Patterns of Rhythm. Rhythm is conceivably the most basic pattern in nature. Our hearts and lungs follow a regular repeated pattern of sounds or movement whose timing is adapted to our body's needs. Many of nature's rhythms are most likely similar to a heartbeat, while others are like breathing. The beating of the heart, as well as breathing, have a default pattern.

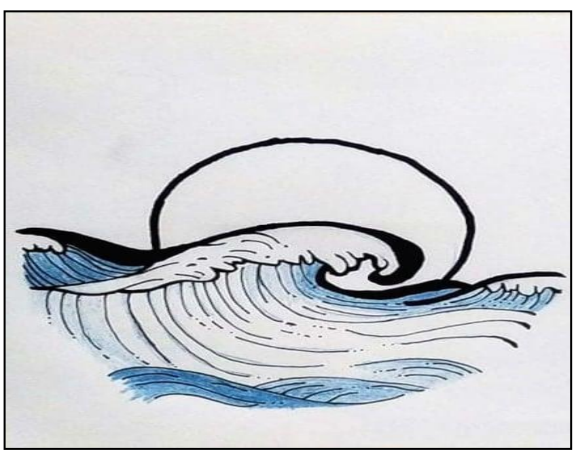
Patterns of Texture. A texture is a quality of a certain object that we sense through touch. It exists as a literal surface that we can feel, see, and imagine. Textures are of many kinds. It can be bristly, and rough, but it can also be smooth, cold, and hard.

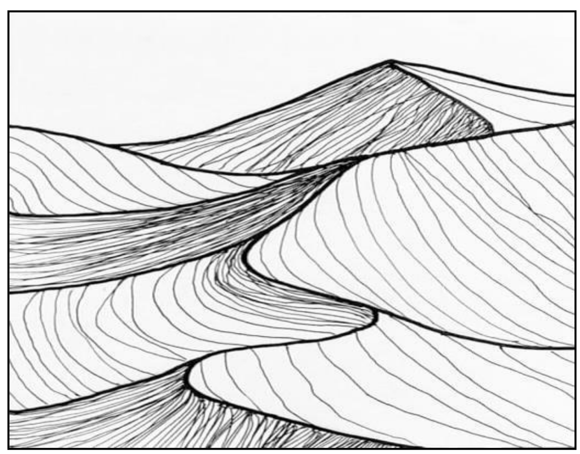
Geometric Patterns. A geometric pattern is a kind of pattern which consists of a series of shapes that are typically repeated. These are regularities in the natural world that are repeated in a predictable manner. Geometrical patterns are usually visible on cacti and succulents.

Patterns Found in Nature

Common patterns appear in nature, just like what we see when we look closely at plants, flowers, animals, and even at our bodies. These common patterns are all incorporated in many natural things.

Waves and Dunes





A wave is any form of disturbance that carries energy as it moves. Waves are of different kinds: mechanical waves which propagate through a medium ---- air or water, making it oscillate as waves pass by. Wind waves, on the other hand, are surface waves that create the chaotic patterns of the sea. Similarly, water waves are created by energy passing through water causing it to move in a circular motion. Likewise, ripple patterns and dunes are formed by sand wind as they pass over the sand.

Spots and Stripes

We can see patterns like spots on the skin of a giraffe. On the other hand, stripes are visible on the skin of a zebra. Patterns like spots and stripes that are commonly present in different organisms are results of a reaction-diffusion system (Turing, 1952). The size and the shape of the pattern depend on how fast the chemicals diffuse and how strongly they interact.

Spirals

Jean Beaufort has released this "Spira Galaxy" image under Public Domain license

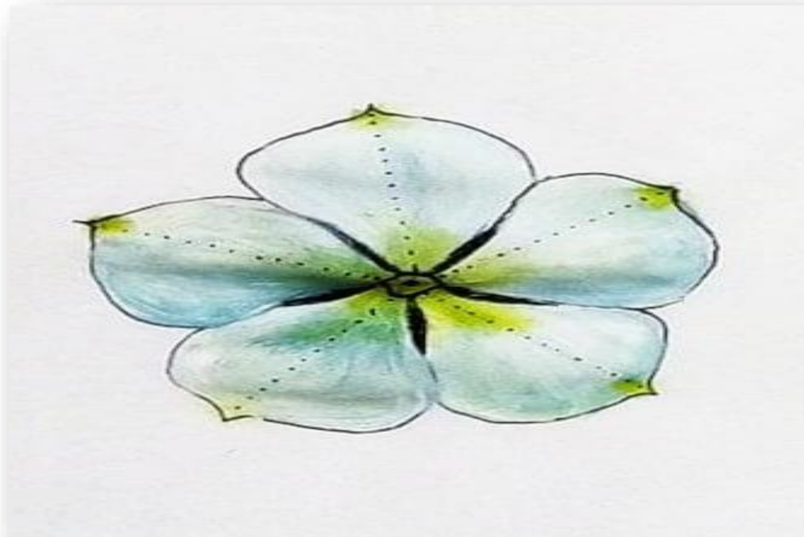
The spiral patterns exist on the scale of the cosmos to the minuscule forms of microscopic animals on earth. The Milky Way that contains our Solar System is a barred spiral galaxy with a band of bright stars emerging from the center running across the middle of it. Spiral patterns are also common and noticeable among plants and some animals. Spirals appear in many plants such as pinecones, pineapples, and sunflowers. On the other hand, animals like ram and kudu also have spiral patterns on their horns.

Symmetries

In mathematics, if a figure can be folded or divided into two with two halves which are the same, such figure is called a symmetric figure. Symmetry has a vital role in pattern formation. It is used to classify and organize information about patterns by classifying the motion or deformation of both pattern structures and processes. There are many kinds of symmetry, and the most important ones are reflections, rotations, and translations. These kinds of symmetries are less formally called flips, turns, and slides.

Reflection symmetry, sometimes called line symmetry or mirror symmetry, captures symmetries when the left half of a pattern is the same as the right half.

Rotations, also known as rotational symmetry, captures symmetries when it still looks the same after some rotation (of less than one full turn). The degree of rotational symmetry of an object is recognized by the number of distinct orientations in which it looks the same for each rotation.



Translations. This is another type of symmetry. Translational symmetry exists in patterns that we see in nature and in man-made objects. Translations acquire

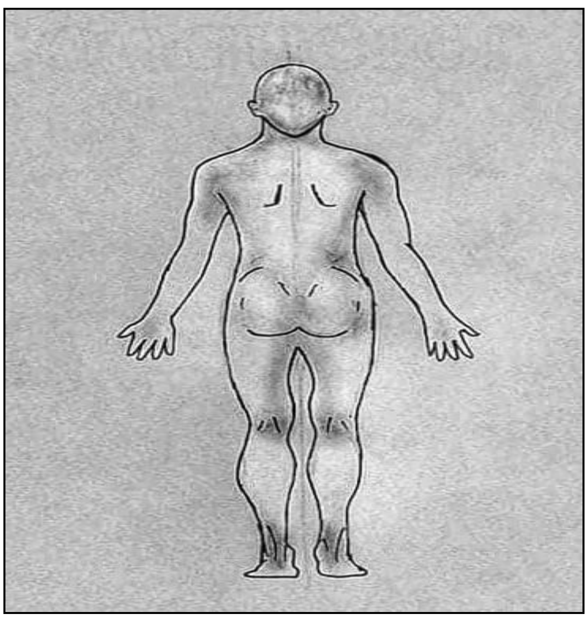
symmetries when units are repeated and turn out having identical figures, like the bees' honeycomb with hexagonal tiles.

Symmetries in Nature

From the structure of subatomic particles to that of the entire universe, symmetry is present. The presence of symmetries in nature does not only attract our visual sense, but also plays an integral and prominent role in the way our life works.

Human Body

The human body is one of the pieces of evidence that there is symmetry in nature. Our body exhibits bilateral symmetry. It can be divided into two identical halves.



Animal Movement

The symmetry of motion is present in animal movements. When animals move, we can see that their movements also exhibit symmetry.

Sunflower

One of the most interesting things about a sunflower is that it contains both radial and bilateral symmetry. What appears to be "petals" in the outer ring are actually small flowers also known as ray florets. These small flowers are bilaterally symmetrical. On the other hand, the dark inner ring of the sunflower is a cluster of radially symmetrical disk florets.

Snowflakes

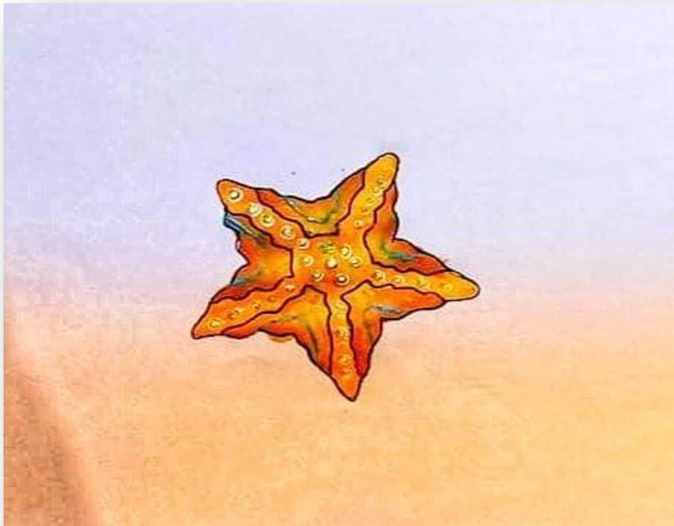
Snowflakes have six-fold radial symmetry. The ice crystals that make-up the snowflakes are symmetrical or patterned. The intricate shape of a single arm of a snowflake is very much similar to the other arms. This only proves that symmetry is present in a snowflake.

Honeycombs/Beehive

Honeycombs or beehives are examples of wallpaper symmetry. This kind of symmetry is created when a pattern is repeated until it covers a plane. Beehives are made of walls with each side having the same size enclosed with small hexagonal cells. Inside these cells, honey and pollen are stored and bees are raised.

Starfish

Starfish have a radial fivefold symmetry. Each arm portion of the starfish is identical to each of the other regions.



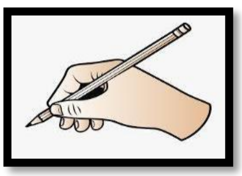
Fibonacci in Nature

By learning about nature, it becomes gradually evident that the nature is essentially mathematical, and this is one of the reasons why explaining nature is dependent on mathematics. Mathematics has the power to unveil the inherent beauty of the natural world.

In describing the amazing variety of phenomena in nature we stumble to discover the existence of Fibonacci numbers. It turns out that the Fibonacci numbers appear from the smallest up to the biggest objects in the natural world. This presence of Fibonacci numbers in nature, which was once existed realm mathematician's curiously, is considered as one of the biggest mysteries why the some patterns in nature is Fibonacci. But one thing is definitely made certain, and that what seemed solely mathematical is also natural.

For instance, many flowers display figures adorned with numbers of petals that are in the Fibonacci sequence. The classic five-petal flowers are said to be the most common among them. These include the buttercup, columbine, and hibiscus. Aside from those flowers with five petals, eight-petal flowers like clematis and delphinium also have the Fibonacci numbers, while ragwort and marigold have thirteen. These numbers are all Fibonacci numbers.

Apart from the counts of flower petals, the Fibonacci also occurs in nautilus shells with a logarithmic spiral growth. Multiple Fibonacci spirals are also present in pineapples and red cabbages. The patterns are all consistent and natural.



Learning Activity 1.2

Synthesis

I. Read the entire book entitled Nature's Numbers by Stewart.

II. Write synthesis about all the things that you learned about nature's numbers.

III. It is highly recommended that at the outset, an outline must be made.

IV. Please ensure that topic sentence can be clearly understood.

-Your topic sentence must be supported by at least three arguments.

V. Your synthesis must be around 1400-1500 words.

VI. Rules on referencing and citation must be strictly observed.

VII. You may use either MLA or APA system.

VIII. The last page must contain the references or bibliography.

IX. Please observe the following format:

Paper Substance (if printed)	Margin	Orientation	Paper Size	Font Type	Font Size	All Line Spacing	Page Number
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Please bear in mind the following criteria for grading your work.

1 Point : The student unable to elicit the ideas and concepts.

2 Point : The student is able to elicit the ideas and concepts but shows erroneous understanding of these.

3 Points: The student is able to elicit the ideas and concepts and shows correct understanding of these.

4 Points: The students not only elicits the correct ideas but also shows evidence of internalizing these.

5 Points: The student elicits the correct ideas, shows evidence of internalizing these, and consistently contributes additional thoughts to the Core Idea.

The Fibonacci Sequence

Specific Objectives

1. **To define sequence and its types**
2. **To differentiate Fibonacci sequence from other types of sequence**
3. **To discover golden ratio and golden rectangle; and**
4. **To learn how to compute for the nth term in the Fibonacci Sequence**

As we have discussed in the preceding lesson, human mind is capable of identifying and organizing patterns. We were also to realized that there are structures and patterns in nature that we don’t usually draw attention to. Likewise, we arrived at a position that in nature, some things follow mathematical sequences and one of them follow the Fibonacci sequence. We noticed that these sequences is observable in some flower petals, on the spirals of some shells and even on sunflower seeds. It is amazing to think that the Fibonacci sequence is dramatically present in nature and it opens the door to understand seriously the nature of sequence.

Discussion

Sequence

Sequence refers to an ordered list of numbers called **terms,** that may have repeated values. The arrangement of these terms is set by a definite rule. (*Mathematics in the Modern World,* 14^{th} Edition, *Aufmann, RN.* et al.). Cosider the given below example:

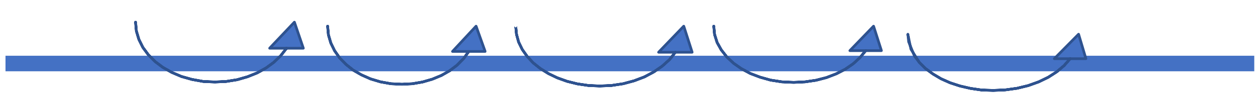
1,	**3,**	**5,**	**7, …**
(1^{st}term)	(2^{nd} term)	(3^{rd} term)	(4^{th} term)

As shown above, the elements in the sequence are called terms. It is called sequence because the list is ordered and it follows a certain kind of pattern that must be recognized in order to see the expanse. The three dots at the end of the visible patterns means that the sequence is infinite.

There are different types of sequence and the most common are the arithmetic sequence, geometric sequence, harmonic sequence, and Fibonacci sequence.

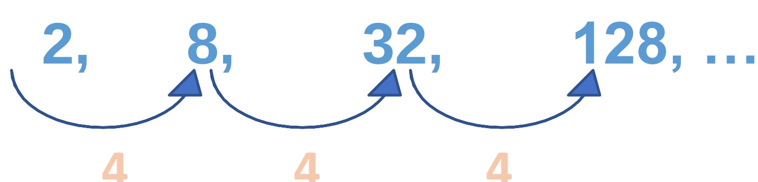
Arithmetic sequence. It is a sequence of numbers that follows a definite pattern. To determine if the series of numbers follow an arithmetic sequence, check the difference between two consecutive terms. If common difference is observed, then definitely arithmetic sequence governed the pattern. To clearly illustrate the arrangement, consider the example below:

2, 4, 6, 8, 10, 12 …



Notice in the given example above, the common difference between two consecutive terms in the sequence is two. The common difference is the clue that must be figure out in a pattern in order to recognize it as an arithmetic sequence.

Geometric sequence. If in the arithmetic sequence we need to check for the common difference, in geometric sequence we need to look for the common ratio. The illustrated in the example below, geometric sequence is not as obvious as the arithmetic sequence. All possibilities must be explored until some patterns of uniformity can intelligently be struck. At first it may seemed like pattern less but only by digging a little bit deeper that we can finally delve the constancy. That is $\frac{2}{8}, \frac{8}{32}, \frac{32}{128}, \ldots$ generate 4, 4, 4,…



Harmonic Sequence. In the sequence, the reciprocal of the terms behaved in a manner like arithmetic sequence. Consider the example below and notice an interesting pattern in the series. With this pattern, the reciprocal appears like arithmetic sequence. Only in recognizing the appearance that we can finally decode the sequencing the govern the series.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$$

Fibonacci Sequence. This specific sequence was named after an Italian mathematician Leonardo Pisano Bigollo (1170 - 1250). He discovered the sequence while he was studying rabbits. The Fibonacci sequence is a series of numbers governed by some unusual arithmetic rule. The sequence is organized in a way a number can be obtained by adding the two previous numbers.

1, 1, 2, 3, 5, 8, 13, 21, …

0 + 1 = 1 0, **1, 1**
1 + 1 = 2 0, **1, 1, 2**
1 + 2 = 3 0, **1, 1, 2, 3**
2 + 3 = 5 0, **1, 1, 2, 3, 5**

Notice that the number **2** is actually the sum of **1** and **1.** Also the 5th term which is number **5** is based on addition of the two previous terms **2,** and **3.** That is the kind of pattern being generated by the Fibonacci sequence. It is infinite in expanse and it was once purely maintained claim as a mathematical and mental exercise but later on the it was observed that the ownership of this pattern was also being claimed by some species of flowers, petals, pineapple, pine cone, cabbages and some shells.

1, 1, 2, 3, 5, 8, 13, 21, …

To explore a little bit more about the Fibonacci sequence, the location of the term was conventionally tagged as Fib(n). This means that Fib(1)=1, Fib(2)=1, Fib(3)=2 and Fib(4)=3. In this method, the Fib(n) is actually referring to the the nth term of the sequence. It is also possible to make some sort of addition in this sequence. For instance:

Fib (2) + Fib (6) = _?__

Fib(2) refers to the 2nd term in the sequence which is "**1**". And Fib(6) refers to the 6th term which is "**8**". So, the answer to that equation is simply "9"

Formula for computing for the nth term in the Fibonacci Sequence

$$x_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

Where:

X_n stands for the Fibonacci number we're looking for

N stands for the position of the number in the Fibonacci sequence

Φ stands for the value of the golden ratio

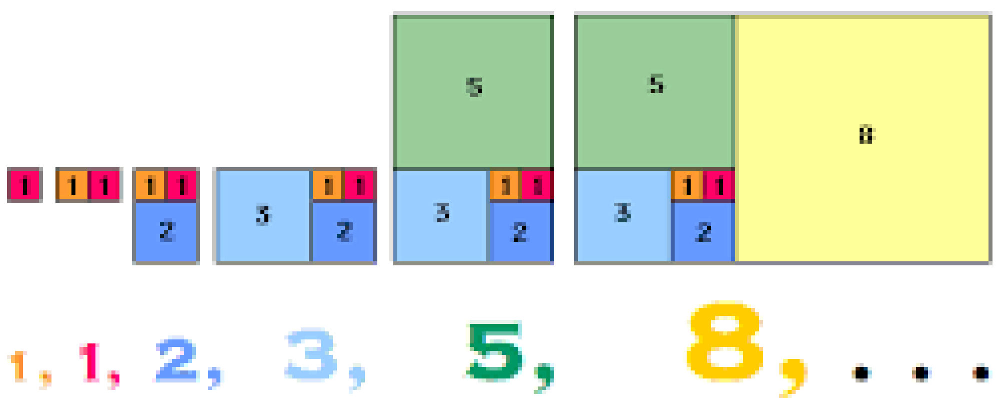
Let us try for example: What is the 5th Fibonacci number? By using the formula we'll get:

$$X_5 = \frac{(1.618)^5 - (1-1.618)^5}{\sqrt{5}}$$

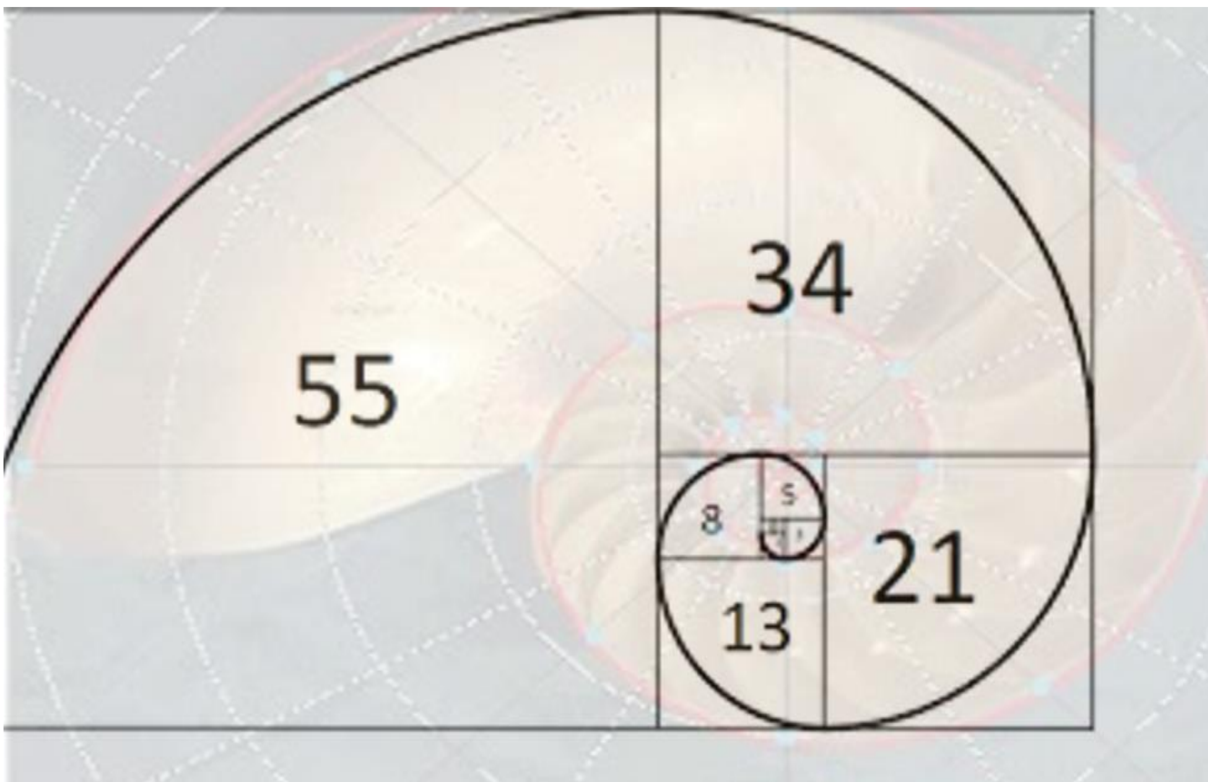
$$X_5 = 5$$

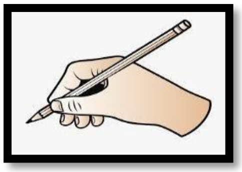
The amazing grandeur of Fibonacci sequence was also discovered in the structure of **Golden rectangle.** The golden rectangle is made up of squares whose sizes, surprisingly is also behaving similar to the Fibonacci sequence. Take a serious look at the figure:

The Golden Ratio



As we can see in the figure, there is no complexity in forming a spiral with the use of the golden rectangle starting from one of the sides of the first Fibonacci square going to the edges of each of the next squares. This golden rectangle shows that the Fibonacci sequence is not only about sequence of numbers of some sort but it is also a geometric sequence observing a rectangle ratio. The spiral line generated by the ratio is generously scattered around from infinite to infinitesimal.





Learning Activity 1.3

I. Identify what type of sequence is the one below and supply the sequence with the next two terms:

1. 1, 4, 7, 10, _, _? Type of Sequence:______________________
2. 80, 40, 20, _, _ ? Type of Sequence:______________________
3. 1, 1, 2, 3, 5, 8, _, _ ? Type of Sequence:______________________
4. 56, 46, 36, 26, _, _ ? Type of Sequence:______________________
5. 2, 20, 200, 2000, _, _ ? Type of Sequence:______________________

II. Compute for the following Fibonacci numbers and perform the given operation:

1. What if Fib (13) ?
2. What is Fib (20) ?
3. What is Fib (8) + Fib (9) ?
4. What is Fib (1) * Fib (7) + Fib (12) – Fib (6) ?
5. What is the sum of Fib (1) up to Fib (10) ?

Chapter Test 1

Multiple Choice. Choose the letter of the correct answer and write it on the blank provided at the of the test paper.

_____________ **1. What is said to be the most basic pattern in nature?**

A. Pattern of Flow

B. Pattern of Movement

C. Pattern of Rhythm

D. Pattern of Visuals

_____________**2. This kind of pattern is unpredictable and it often contains fractals.**

A. Geometric Patterns

B. Pattern of Forms

C. Pattern of Movement

D. Pattern of Visuals

_____________**3. What kind of pattern is a series of shapes that are repeating?**

A. Geometric Pattern

B. Pattern of Flows

C. Pattern of Texture

D. Pattern of Visuals

_____________**4. Among the following, what is not a type of symmetry?**

A. Reflection

B. Rotation

C. Transformation

D. Translation

_____________**5. All of the following statements are correct about Fibonacci except one:**

A. The logarithmic spiral growth of the Nautilus shell

B. The total number of family members correspond to a Fibonacci number.

C. Fibonacci numbers are the root of the discovery of the secret behind sunflower seeds.

D. The numbers of petals of almost all flowers in the world correspond to the Fibonacci numbers.

_____________**6. What type of sequence deals with common ratio?**

A. Arithmetic Sequence
B. Fibonacci Sequence
C. Geometric Sequence
D. Harmonic Sequence

____________**7. What is the sum of Fib (10) + Fib(5) ?**

A. 58
B. 59
C. 60
D. 61

____________**8. What is Fib (12) ?**

A. 144
B. 233
C. 377
D. 89

____________**9. What are the next two terms of the sequence, 8, 17, 26, 35?**

A. 49, 58
B. 39, 48
C. 44, 53
D. 54, 63

____________**10. What type of sequence is 5, 8, 13, 21, 34, 55, … ?**

A. Fibbonacci Sequence
B. Fibonaacii Sequence
C. Fibonacci Sequence
D. Fibonacii Sequence

ANSWER KEY

1.	**C**	**6.**	**C**
2.	**D**	**7.**	**C**
3.	**A**	**8.**	**B**
4.	**C**	**9.**	**C**
5.	**B**	**10.**	**C**

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MODULE TWO

MATHEMATICAL LANGUAGE AND SYMBOLS

CORE IDEA

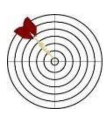
Like any language, mathematics has its own symbols, syntax and rules.

Learning Outcome:

1. Discuss the language, symbols, and conventions of mathematics.
2. Explain the nature of mathematics as a language
3. Perform operations on mathematical expressions correctly.
4. Acknowledge that mathematics is a useful language.

□ **Time Allotment:** Four (4) lecture hours

Characteristics and Conventions in the Mathematical Language



Specific Objective

At the end of this lesson, the student should be able to:

1. Understand what mathematical language is.
2. Name different characteristics of mathematics.
3. Compare and differentiate natural language into a mathematical language and expressions into sentences.
4. Familiarize and name common symbols use in mathematical expressions and sentences.
5. Translate a sentence into a mathematical symbol.

Introduction:

Have you read about one of the story in the bible known as "The Tower of Babel?" This story is about constructing a tower in able to reach its top to heaven; the Kingdom of God.

At first, the construction of a tower is smoothly being done since all of the workers have only one and only one language. But God disrupted the work of the people by making their language different from each other. There were a language barrier and the people were confused what the other people are talking about resulting the tower was never finished and the people were spread in all over and different places of the earth.

Based on the story, what was the most important thing that people should have in order to accomplish a certain task? Yes, a "language". Language is one of the most important thing among the people because it has an important role in communication. But the question is, what is language? Why is it so important? In this module, we will be discussing about mathematical relative on what you have learned in your English subject.

Discussion:

For sure you may be asked what the real meaning of a language is. Perhaps you could say that language is the one we use in able to communicate with each other or this is one of your lessons in English or in your Filipino subject. According to Cambridge English Dictionary, a language is a system of communication consisting of sounds, words and grammar, or the system of communication used by people in a particular country or type of work.

Did you know that mathematics is a language in itself? Since it is a language also, mathematics is very essential in communicating important ideas. But most mathematical language is in a form of symbols. When we say that "Five added by three is eight", we could translate this in symbol as "5 + 3 = 8." Here, the first statement is in a form of group of words while the translation is in a form of symbol which has the same meaning and if your will be reading this, for sure all of you have a common understanding with this. But let us take a look at this mathematical symbols:

$$f(x) = L$$

$$\forall \varepsilon > 0, \exists\ \delta > 0\ \rightarrow |x - a| < \delta,\ |f(x) - L| < \varepsilon,\ x \epsilon R$$

Did you understand what these symbols are? This mathematical sentence is a complex idea; yet, it is contained and tamed into a concise statement. It may sound or look Greek to some because without any knowledge of the language in which the ideas are expressed, the privilege to understand and appreciate its grandeur can never be attained. Mathematics, being a language in itself, may appear complex and difficult to understand simply because it uses a different kind of alphabet and grammar structure. It uses a kind of language that has been historically proven effective in communicating and transmitting mathematical realities. The language of mathematics, like any other languages, can be learned; once learned, it allows us to see fascinating things and provides us an advantage to comprehend and exploit the beauty of beneath and beyond. Hence, in able to understand better different topics in mathematics, it is very important that you must learn first on how to read and understand different symbols in mathematics which used in mathematical language.

A. Characteristics of Mathematical Language

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express.

It is:

1. precise (able to make very fine distinction)
2. concise (able to say things briefly); and
3. powerful (able to express complex thoughts with relative cases).

B. Vocabulary vs. Sentences

Every language has its vocabulary (the words), and its rules for combining these words into complete thoughts (the sentences). Mathematics is no exception. As a first step in discussing the mathematical language, we will make a very broad classification between the 'nouns' of mathematics (used to name mathematical objects of interest) and the 'sentences' of mathematics (which state complete mathematical thoughts)'

You must study the Mathematics Vocabulary!

- Student must learn on how to use correctly the language of Mathematics, when and where to use and figuring out the incorrect uses.
- Students must show the relationship or connections the mathematics language with the natural language.
- Students must look backward or study the history of Mathematics in order to understand more deeply why Mathematics is important in their daily lives.

Importance of Mathematical Language

- Major contributor to overall comprehension
- Vital for the development of Mathematics proficiency
- Enables both the teacher and the students to communicate mathematical knowledge with precision

C. Comparison of Natural Language into Mathematical Language

The table below is an illustration on the comparison of a natural language (expression or sentence) to a mathematical language.

	English	Mathematics
Expressions		
Name given to an object of interest.	Noun such as person, place and things and pronouns Example: a) Ernesto b) Batangas City c) Book d) He	2 3 – 2 3x 3x + 2 ax + by + c
Sentence		
It has a complete thought	Group of words that express a statement, question or command. Example: a) Ernesto is a boy. b) He lives in Batangas City. c) Allan loves to read book. d) Run! e) Do you love me?	3 + 2 = 5 a + b = c ax + by + c = 0 $(x + y)^2 = x^2 + 2xy + y^2$

D. Expressions versus Sentences

Ideas regarding sentences:

Ideas regarding sentences are explored. Just as English sentences have verbs, so do mathematical sentences. In the mathematical sentence;

$$3 + 4 = 7$$

the verb is =. If you read the sentence as 'three plus four is equal to seven, then it's easy to hear the verb. Indeed, the equal sign = is one of the most popular mathematical verb.

Example:

1. The capital of Philippines is Manila.
2. Rizal park is in Cebu.
3. 5 + 3 = 8
4. 5 + 3 = 9

Connectives

A question commonly encountered, when that; presenting the sentence example 1 + 2 = 3 is If = is the verb, then what is + ?

The answer is the symbol + is what we called a connective which is used to connect objects of a given type to get a 'compound' object of the same type. Here, the numbers 1 and 2 are connected to give the new number 1 + 2.

In English, this is the connector "and". Cat is a noun, dog is a noun, cat and dog is a 'compound' noun.

Mathematical Sentence

Mathematical sentence is the analogue of an English sentence; it is a correct arrangement of mathematical symbols that states a complete thought. It makes sense to as about the TRUTH of a sentence: Is it true? Is it false? Is it sometimes true/sometimes false?

Example:

1. The capital of Philippines is Manila.
2. Rizal park is in Cebu.
3. 5 + 3 = 8
4. 5 + 3 = 9

Truth of Sentences

Sentences can be true or false. The notion of "truth" (i.e., the property of being true or false) is a fundamental importance in the mathematical language; this will become apparent as you read the book.

Conventions in Languages

Languages have conventions. In English, for example, it is conventional to capitalize name (like Israel and Manila). This convention makes it easy for a reader to distinguish between a common noun (carol means Christmas song) and proper noun (Carol i.e. name of a person). Mathematics also has its convention, which help readers distinguish between different types of mathematical expression.

Expression

An expression is the mathematical analogue of an English noun; it is a correct arrangement of mathematical symbols used to represent a mathematical object of interest.

An expression does NOT state a complete thought; in particular, it does not make sense to ask if an expression is true or false.

E. Conventions in mathematics, some commonly used symbols, its meaning and an example

a) Sets and Logic

SYMBOL	NAME	MEANING	EXAMPLE
∪	**Union**	**Union of set A and set B**	**A ∪ B**
∩	**Intersection**	**Intersection of set A and set B**	**A ∩ B**
∈	**Element**	**x is an element of A**	**x ∈ A**
∉	**Not an element of**	**x is not an element of set A**	**x ∉ A**
{ }	**A set of..**	**A set of an element**	**{a, b, c}**
⊂	**Subset**	**A is a subset of B**	**A ⊂ B**
⊄	**Not a subset of**	**A is not a subset of B**	**A ⊄ B**
…	**Ellipses**	**There are still other items to follow**	**a, b, c, …** **a + b + c + ….**
∧	**Conjunction**	**A and B**	**A ∧ B**
∨	**Disjunction**	**A or B**	**A ∨ B**
~	**Negation**	**Not A**	**~A**
→	**Implies (If-then statement)**	**If A, then B**	**A → B**
↔	**If and only if**	**A if and only if B**	**A ↔ B**
∀	**For all**	**For all x**	**∀x**
∃	**There exist**	**There exist an x**	∃
∴	**Therefore**	**Therefore C**	∴C

|	Such that	x such that y	x | y
■	End of proof		
≡	Congruence / equivalent	A is equivalent to B a is congruent to b modulo n	A ≡B a ≡ b mod n
a, b, c, …, z (lower case)	Variables *First part of English Alphabet uses as fixed variable* *Middle part of English alphabet use as subscript and superscript variable* *Last part of an English alphabet uses as unknown variable*	$(ax_o)^p$	$(5x_2)^6$

b) Basic Operations and Relational Symbols

SYMBOL	NAME	MEANING	EXAMPLE
+	Addition; Plus sign	a plus b a added by b a increased by b	3 + 2
-	Subtraction; minus sign	a subtracted by b a minus b a diminished by b	3 - 2
· ( )	Multiplication sign *we do not use x as a symbol for multiplication in our discussion since its use as a variable*	a multiply by b a times b	4 · 3 (4)(3)
÷ or |	Division sign; divides	a ÷ b b | a	10 ÷ 5 5 | 10

o	Composition of function	f of g of x	f o g(x)
=	Equal sign	a = a a + b = b + a	5 = 5 3 + 2 = 2 + 3
≠	Not equal to	a ≠ b	3 ≠ 4
>	Greater than	a > b	10 > 5
<	Less than	b < a	5 < 10
≥	Greater than or equal to	a ≥ b	10 ≥ 5
≤	Less than or equal to	b ≤ a	5 ≤ 10
*	Binary operation	a * b	a * b = a + 17b

c) Set of Numbers

SYMBOL	NAME	MEANING	EXAMPLE
$\mathbb{N}_0$	natural numbers / whole numbers set (with zero)	$\mathbb{N}_0 = \{0,1,2,3,4,...\}$	$0 \in \mathbb{N}_0$
$\mathbb{N}_1$	natural numbers / whole numbers set (without zero)	$\mathbb{N}_1 = \{1,2,3,4,5,...\}$	$6 \in \mathbb{N}_1$
$\mathbb{Z}$	integer numbers set	$= \{...-3,-2,-1,0,1,2,3,...\}$	$-6 \in \mathbb{Z}$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x=a/b,\ a,b \in \mathbb{Z} \text{ and } b \neq 0\}$	$2/6 \in \mathbb{Q}$
$\mathbb{R}$	real numbers set	$= \{x \mid -\infty < x < \infty\}$	$6.343434 \in \mathbb{R}$
$\mathbb{C}$	complex numbers set	$\mathbb{C} = \{z \mid z=a+bi,\ -\infty<a<\infty,\ -\infty<b<\infty\}$	$6+2i \in \mathbb{C}$

F. Translating words into symbol

- Practical problems seldom, if ever, come in equation form. The job of the problem solver is to translate the problem from phrases and statements into mathematical expressions and equations, and then to solve the equations.

- As problem solvers, our job is made simpler if we are able to translate verbal phrases to mathematical expressions and if we follow step in solving applied problems. To help us translate from words to symbols, we can use the Mathematics Dictionary.

Examples:

Let x be a number. Translate each phrase or sentence into a mathematical expression or equation.

1. Twelve more than a number.

Ans.: $12+x$

2. Eight minus a number.

Ans.: $8-x$

3. An unknown quantity less fourteen. Ans.: $x-14$

4. Six times a number is fifty-four.

Ans.: $6x=54$

5. Two ninths of a number is eleven.

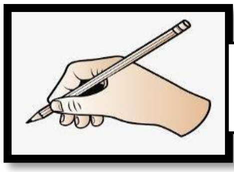
Ans.: $2/9x=11$

6. Three more than seven times a number is nine more than five times the number.

Ans.: $3+7x=9+5x$

7. Twice a number less eight is equal to one more than three times the number.

Ans.: $2x-8=3x+1$ or $2x-8=1+3x$



Self -Learning Activity

Directions: Do as indicated.

A. **Bingo Game:** Your teacher will be asking you to make your own bingo card (one card only) with different mathematical symbols like the one below.

MATHEMATICS IN THE MODERN				
W	**O**	**R**	**L**	**D**
o	∀	≤	-	↔
$\mathbb{C}$	$\mathbb{N}_1$	$\mathbb{Q}$	$\mathbb{N}_0$	*
|	$\mathbb{R}$	**FREE**	~	■
≥	∧	∪	≡	>
∃	( )	+	∨	$\mathbb{Z}$

No symbol/s must be repeated in a single card. Just like an ordinary bingo game, you will be playing a "Block-out Game" where your teacher would be the game master. Whoever student/s complete all the symbols in a card (block out game) won the game. The game master will be check if the symbols are all correct. **Note: He or she will be given an incentive points for this item.**

B. Translate each of the following mathematical phrase into a mathematical expression if possible. Let x and y be the numbers.

1. A number increased by five

   Ans.

2. Twice the square of a number

   Ans.

3. The square of the sum of two numbers

   Ans.

4. The sum of the squares of two numbers

   Ans.

5. A number less by three

   Ans.

6. Twice of a number added by four

   Ans.

7. The cube of a number less than five

   Ans.

8. The area of a rectangle whose length is seven more than its width

   Ans.

9. The difference of a square of two numbers

   Ans.

10. The quotient of the sum of two numbers by another number

    Ans

Lesson 3.2

Four Basic Concepts

Specific Objective

At the end of the lesson, the student should be able to:

1. Define what a set and its basic terminologies.
2. Differentiate two ways in describing sets.
3. Perform basic operations on set.
4. Define what a relation and a function is.
5. Translate relation and function into a diagram.
6. Name and apply the different properties of a relation and function.
7. Identify the domain and range in a relation and function.
8. Evaluate a function.
9. Define and perform a binary operation.

Introduction

In this module, it will be discussed the four basic concepts in mathematics such as sets and its basic operation, the functions, relations and the binary operations.

Discussion:

I. SETS AND SUBSETS

A. The Language of Sets

Use of the word "set" as a formal mathematical term was introduced in 1879 by Georg Cantor. For most mathematical purposes we can think of a set intuitively, as Cantor did, simply as a collection of elements.

So, by definition:

A set is a collection of well-defined objects.

Illustration:

A set of counting numbers from 1 to 10.

A set of an English alphabet from a to e.
A set of even numbers
A set of an integers

Note: A set is denoted with braces or curly brackets { } and label or name the set by a capital letter such as A, B, C,…etc.

a. A set of counting numbers from 1 to 5.
   A = { 1, 2, 3, 4, 5 }

b. A set of English alphabet from a to d.
   B = { a, b, c, d }

c. A set of all even positive integers.
   C = { 2, 4, 6, 8, … }

d. A set of an integers.
   D = { …, -3, -2, -1, 0, 1, 2, 3, …}

Now, if S is a set, the notation x ∈ S means that x is an element of S. The notation x ∉ S means that x is not an element of S.

So, what is an element of a set?

Element of a set

Each member of the set is called an element and the ∈ notation means that an item belongs to a set.

Illustration:

Say A = { 1, 2, 3, 4, 5 }
1 ∈ A; 3 ∈ A; 5 ∈ A

Is 6 is an element of set A? Since in a given set A above, we could not see six as an element of set A, thus we could say that;

6 is not an element of set A or
6 ∉ A

Note: Each element is a set should be separated by comma.

Terminologies of Sets

1. Unit Set

Unit set is a set that contains only one element.

Illustration:

A = { 1 }; B = { c }; C = { banana }

2. Empty set or Null set; ∅

Empty or null set is a set that has no element.

Illustration:

A = { }

A set of seven yellow carabaos

3. Finite set

A finite set is a set that the elements in a given set is countable.

Illustration:

A = { 1, 2, 3, 4, 5, 6 }

B = { a, b, c, d }

4. Infinite set

An infinite set is a set that elements in a given set has no end or not countable.

Illustration:

A set of counting numbers

A = { …-2, -1, 0, 1, 2, 3, 4, … }

5. Cardinal Number; n

Cardinal number are numbers that used to measure the number of elements in a given set. It is just similar in counting the total number of element in a set.

Illustration:

A = { 2, 4, 6, 8 } n = 4

B = { a, c, e } n = 3

6. Equal set

Two sets, say A and B, are said to be equal if and only if they have equal number of cardinality and the element/s are identical. There is a 1 -1 correspondence.

Illustration:

A = { 1, 2, 3, 4, 5}   B = { 3, 5, 2, 4, 1}

7. Equivalent set

Two sets, say A and B, are said to be equivalent if and only if they have the exact number of element. There is a 1 – 1 correspondence.

Illustration:

A = { 1, 2, 3, 4, 5 }   B = { a, b, c, d, e }

8. Universal set

The universal set U is the set of all elements under discussion.

Illustration:

A set of an English alphabet

U = {a, b, c, d, …, z}

9. Joint Sets

Two sets, say A and B, are said to be joint sets if and only if they have common element/s.

A = { 1, 2, 3}B = { 2, 4, 6 }

Here, sets A and B are joint set since they have common element such as 2.

10. Disjoint Sets

Two sets, say A and B, are said to be disjoint if and only if they are mutually exclusive or if they don't have common element/s.

A = { 1, 2, 3}B = { 4, 6, 8 }

B. Two ways of Describing a Set

1. Roster or Tabular Method

It is done by listing or tabulating the elements of the set.

2. Rule or Set-builder Method

It is done by stating or describing the common characteristics of the elements of the set. We use the notation A = { x / x … }

Illustration:

a. A = { 1, 2, 3, 4, 5 }
A = {x | x is a counting number from 1 to 5}
A = { x | x ∈ N, x < 6}

b. B = { a, b, c, d, …, z }
B = {x | x ∈ English alphabet}
B = { x | x is an English alphabet}

C. Subsets

A subset, A ⊆ B, means that every element of A is also an element of B.

If x ∈ A, then x ∈ B.
In particular, every set is a subset of itself, A ⊆ A.

A subset is called a proper subset, A is a proper subset of B, if A ⊂ B and there is at least one element of B that is not in A:

If x ⊂ A, then x ⊂ B and there is an element b such that b ∈ B and b ∉ A.

NOTE1: The empty set. or {} has no elements and is a subset of every set for every set A, A ⊂ A.

The number of subsets of a given set is given by 2^n, where n is the number of elements of the given set.

Illustration:

How many subsets are there in a set

A = {1, 2, 3 }? List down all the subsets of set A. Number of subsets = $2^n = 2^3$ = 8 subsets

With one element
{ 1 } ; { 2 } ; { 3 }

With two elements
{ 1, 2 } ; { 1, 3 }; { 2, 3 }

With three elements
{ 1, 2, 3 }

With no elements
{ }

D. Ordered Pair

Given elements a and b, the symbol (a, b) denotes the ordered pair consisting of a and b together with the specification that "a" is the first element of the pair and "b" is the second element. Two ordered pairs (a,b) and (c,d) are equal iff a = c and b = d. Symbolically;

(a, b) = (c, d) means that a = c and b = d

Illustration:

a) If (a, b) = (3, 2), what would be the value of a and b.

Here, by definition that two ordered

and b = d.

pairs (a,b) and (c,d) are equal iff a =

c Hence, a = 3 and b = 2.

b) Find x and y if $(4x + 3, y) = (3x + 5, -2)$.

Solution:

Since $(4x + 3, y) = (3x + 5, -2)$, so

$$4x + 3 = 3x + 5$$

Solving for x, we got $x = 2$ and obviously $y = -2$.

E. OPERATION ON SETS

Sets can be combined in a number of different ways to produce another set. Here are the basic operations on sets.

1. Union of Sets

The union of sets A and B, denoted by $A \cup B$, is the set defined as:

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

Example 1: If A = {1, 2, 3} and B = {4, 5} , then

$$A \cup B = \{1, 2, 3, 4, 5\} .$$

Example 2: If A = {1, 2, 3} and B = {1, 2, 4, 5} , then

$$A \cup B = \{1, 2, 3, 4, 5\} .$$

Note that elements are not repeated in a set.

2. Intersection of Sets

The intersection of sets A and B, denoted by $A \cap B$, is the set defined as :

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

Example 1: If A = {1, 2, 3} and B = {1, 2, 4, 5}
then $A \cap B = \{1, 2\}$.

Example 2: If A = {1, 2, 3} and B = {4, 5}
then $A \cap B = \emptyset$

3. Difference of Sets

The difference of sets A from B , denoted by A - B , is the set defined as

$$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$$

Example 1: If A = {1, 2, 3} and B = {1, 2, 4, 5}
then A - B = {3} .

Example 2: If A = {1, 2, 3} and B = {4, 5} ,
then A - B = {1, 2, 3} .

Example : 3 If A = {a, b, c, d } and B = {a, c, e } ,
then A - B = {b, d } .

Note that in general A - B ≠ B - A

4. Compliment of Set

For a set A, the difference U - A , where U is the universe, is called the complement of A and it is denoted by A^c . Thus A^c is the set of everything that is not in A.

Example: Let U = { a, e, i, o, u } and A = { a, e }

then A^c = { i, o u }

5. Cartesian Product

Given sets A and B, the Cartesian product of A and B, denoted by A x B and read as "A cross B", is the set of all ordered pair (a,b) where a is in A and b is in B. Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Note that A x B is not equal to B x A.

Illustration:

If A = { 1, 2} and B = {a, b}, what is A x B?

A x B = {(1,a), (1, b), (2, a), (2, b)}. How many elements in a A x B?

Example 1: Let A = {1, 2, 3} and B = {a, b}. Then

A x B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)} .

Example 2: For the same A and B as in Example 1,

B x A = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)} .

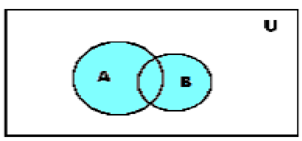
Venn Diagram

A Venn diagram is an illustration of the relationships between and among sets, groups of objects that share something in common. Usually, Venn diagrams are used to depict set intersections (denoted by an upside-down letter U). This type of diagram is used in scientific and engineering presentations, in theoretical mathematics, in computer applications, and in statistics.

Venn Diagram on Sets Operation

A. Union of Sets

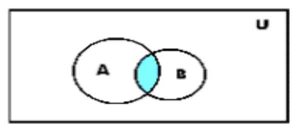
Fig. 1



A U B

B. Intersection of Sets

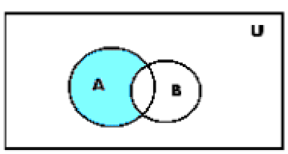
Fig. 2



A ∩ B

C. Difference of Sets

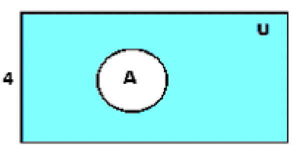
Fig. 3



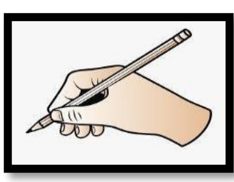
A - B

D. Complement

Fig. 4



Complement of A



Self- Learning Activity

Directions: Do as indicated.

1. Tell whether the following is true or false:
    1. Empty set is also called a unit set.
    2. { ∅ } is an empty set.

3. A set with two elements has 2 subsets.
4. Equivalent set is also an equal set.
5. Counting number is an example of a finite set.

2. List down all the subsets of a set

$$A = \{ a, b, c, d \}.$$

3. If A = {1,2,3} and B={i,o,u}, find A x B.

4. Find *x* and *y* if $(x - y, x + y) = (6, 10)$

5. Let U = {{0,1,2,3,4,5,6,7,8,9}; A = {1, 3, 5, 7}, B = {2, 4, 6, 8} and C = {3, 6, 9, 0}. Find the following:

a) $A \cup B$ b) $B \cap C$ c) $A' \cap B'$ d) $(A \cup B)'$

6. A group of students were asked whether they are like basketball, softball, or both. If 456 like basketball, 384 like softball, and 252 like both games, how many students were there?

7. A survey of 100 fourth year high school students revealed that 42 like mathematics, 62 like Filipino, 44 like History, 22 like both Math and History, 25 like both Math and Filipino, 17 like Filipino and History, and 10 like all the three subjects. How many like Math only? Filipino only? History only? How many did not like any of the three subjects?

II. FUNCTIONS AND RELATIONS

A. THE LANGUAGE OF RELATIONS AND FUNCTIONS

Sometimes, we asked ourselves that, "What is my relationship with other people, with the environment and most of all, with the God?"

How strong is your relationship with your parents, brother, sister, friends and even your teacher? Are we related by blood? Are we related through sharing a common ideas and ideology? Also, we talked about relationship between student and teacher, a manager and the subordinates or even people who share common religion, ethnic or culture.

How are we going to relate the word relation in Mathematics? The objects of mathematics may be related in various ways. A set "A" may be said to be related to a set "B" if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common. A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$.

To be able to understand better what a relation is all about more specifically if we talked about relation in mathematics, let us have a simple illustration.

Let A = {1,2,3} and B = {2, 3, 4} and let us say that an element x in A is related to an element y in B if and only if, x is less than y and let us use the notation x R y as translated mathematical term for the sentence "x is related to y. Then, it follows that:

1 R 2 since 1 < 2
1 R 3 since 1 < 3
1 R 4 since 1 < 4
2 R 3 since 2 < 3
2 R 4 since 2 < 4
3 R 4 since 3 < 4.

Now, can we say that 1 R 1? Is 3 R 2?

Recall the Cartesian product. What are the elements of A x B? It is clearly stated A x B = { (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}. Here, the elements of some ordered pairs in A x B are related, whereas the element of other ordered pairs are not.

What are the elements (ordered pair) in A x B, based on the given conditions, that are related? Perhaps your answer would be:

{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}

Observe that knowing which ordered pairs lie in this set is equivalent to knowing which elements are related to which. The relation can be therefore be thought of the totality of ordered pairs whose elements are related by the given condition. The formal mathematical definition of relation, based on this idea, was introduced by the American mathematicians and logician C.S. Peirce in the nineteenth century.

What is a relation?

1. A relation from set X to Y is the set of ordered pairs of real numbers (x, y) such that to each element x of the set X there corresponds at least one element of the set Y.

2. Let A and B sets. A relation R from A to B is a subset of A x B. Given an ordered pair (x, y) in A x B, x is related to y by R, written x R y, if and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co- domain.

Notation:

The notation for a relation R may be written symbolically as follows:

x R y meaning (x, y) ∈ R.

The notation x $\underset{\sim}{R}$ y means that x is not related to y by R;

x $\underset{\sim}{R}$ y meaning (x, y) ∉ R.

Example:

1. Given a set of an ordered pairs:

{(0, -5), (1, -4), (2, -3), (3, -2), (4, -1), (5, 0)}

The domain are x = {0, 1, 2, 3, 4, 5}

The co-domain are y = {-5, -4, -3, -2, -1, 0}

2. Let A = {1,2} and B = {1,2,3} and define a relation R from A to B as follows: Given any (x,y) ∈ A x B,

(x,y) ∈ R means that $\frac{x-y}{2}$ is an integer;

a. State explicitly which ordered pairs are in A x B and which are in R.
b. Is 1 R 3? Is 2 R 3? Is 2 R 2?
c. What are the domain and the co-domain of R?

Solution:

a. A x B = {(1,1), (1,2), (1,3), (2,1), (2,2),(2,3). To determine explicitly the composition of R, examine each ordered pair in A x B to see whether its element satisfy the defining condition for R.

(1,1) ∈R because $\frac{1-1}{2} = \frac{0}{2} = 0$, which is an integer.

(1,2) ∉ R because $\frac{1-2}{2} = \frac{-1}{2}$, which is not an integer.

(1,3) ∈R because $\frac{1-3}{2} = \frac{0-2}{2} = -1$, which is an integer.

(2,1) ∉R because $\frac{2-1}{2} = \frac{1}{2}$, which is not an integer.

2 2

(2,2) ∈R because $\frac{2-2}{2} = \frac{0}{2} = 0$, which is an integer.

(2,3) ∉R because $\frac{2-3}{2} = \frac{-1}{2}$, which is not an integer.

Thus, R = {(1,1), (1,3), (2,2)}

b. Yes! 1 R 3 because (1,3) ∈ R
   No! 2 $\underset{\sim}{R}$ 3 because (2,3) ∉ R.
   Yes! 2 R 2 because (2,2) ∈ R.

c. The domain of R is {1,2} and the co-domain is {1,2,3}

C. ARROW DIAGRAM OF A RELATION

Suppose R is a relation from a set A to a set B. The arrow diagram for R is obtained as follows:

1. Represent the elements of A as a points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y, and only if, x is related to y by R. Symbolically:

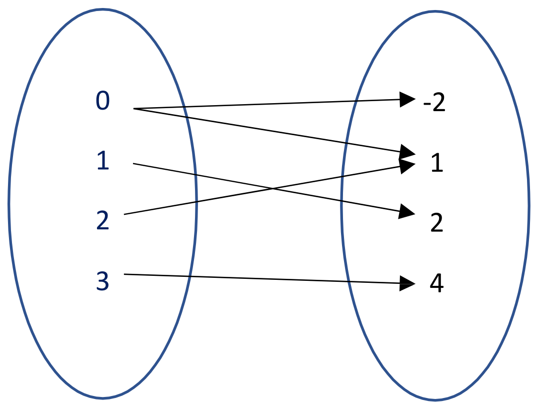
Draw an arrow from x to y

Example:

I
f
a
n
d
o
n
l
y

i
f
,
x
R
y
If and only if, $(x, y) \in R$.

1. Given a relation $\{(1, 2), (0, 1), (3, 4), (2, 1), (0, -2)\}$. Illustrate the given relation into an arrow diagram.



2. What is the domain and co-domain of an example 1?

The domain are as follows: {0, 1, 2, 3}
The co-domain are as follows: {-2, 1, 2, 4}

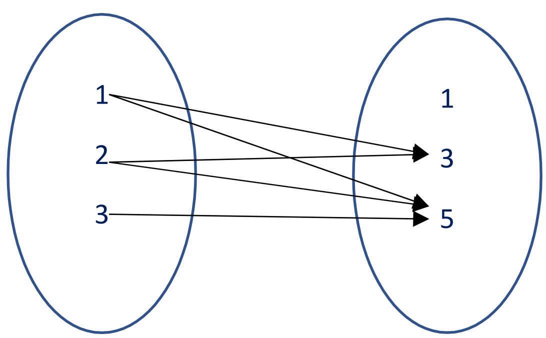
3. Let A = {1,2,3} and b = {1,3,5} and define relations S and T from A to B as follows:
   For all (x, y) ∈ A x B, (x,y) ∈S means that x < y, i.e., S is a "less than" relation.

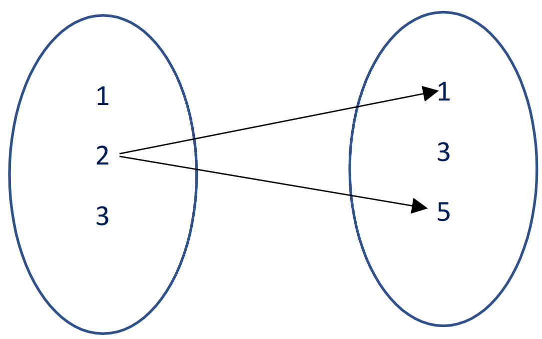
   T = {(2,1), (2,5)}

Draw arrow diagrams for S and T.

Solution:

A x B = {(1,1), (1,3), (1,5), (2,1), (2,3), (2,5), (3,1), (3,3), (3,5)}. It is given by x < y so S = {(1,3), (1,5), (2,3), (2,5), (3,5)}





D. PROPERTIES OF A RELATION

When a relation R is defined from a set A into the same set A, the three properties are very useful such as reflexive, symmetric and the transitive.

A. Reflexive

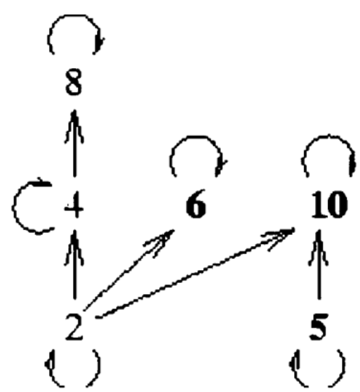
A relation R on A is said to be reflexive if every element of A is related to itself. In notation, a R a for all a ∈ A.

Examples of reflexive relations include:

_ "is equal to" (equality)
_ "is a subset of" (set inclusion)
_ "is less than or equal to" and "is greater than or equal to" (inequality)
_ "divides" (divisibility).

An example of a non reflexive relation is the relation "is the father of" on a set of people since no person is the father of themself.

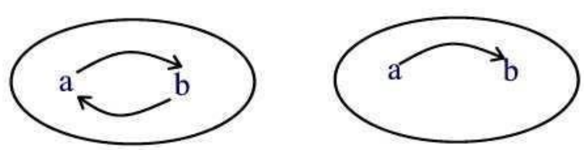
When looking at an arrow diagram, a relation is reflexive if every element of A has an arrow pointing to itself. For example, the relation in a given figure below is a reflexive relation.



B. Symmetric

A relation R on A is symmetric if given a R b then b R a.

For example, "is married to" is a symmetric relation, while, "is less than" is not. The relation "is the sister of" is not symmetric on a set that contains a brother and sister but would be symmetric on a set of females. The arrow diagram of a symmetric relation has the property that whenever there is a directed arrow from a to b, there is also a directed arrow from b to a.



symmetric nonsymmetric

C. Transitive

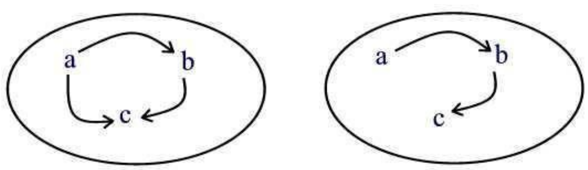
A relation R on A is transitive if given a R b and b R c then a R c.

Examples of reflexive relations include:

_ "is equal to" (equality)
_ "is a subset of" (set inclusion)
_ "is less than or equal to" and "is greater than or equal to" (inequality)
_ "divides" (divisibility).

On the other hand, "is the mother of" is not a transitive relation, because if Maria is the mother of Josefa, and Josefa is the mother of Juana, then Maria is not the mother of Juana.

The arrow diagram of a transitive relation has the property that whenever there are directed arrows from a to b and from b to c then there is also a directed arrow from a to c:



Transitive Nontransitive

A relation that is refexive, symmetric, and transitive is called an equivalence relation on A.

Examples of equivalence relations include:

_ The equality ("=") relation between real numbers or sets.
_ The relation "is similar to" on the set of all triangles.
_ The relation "has the same birthday as" on the set of all human beings.

On the other hand, the relation " ⊆ " is not an equivalence relation on the set of all subsets of a set A since this relation is not symmetric.

E. WHAT IS A FUNCTION?

A function is a relation in which every input is paired with exactly one output. A function from set X to Y is the set of ordered pairs of real numbers (x, y) in which no two distinct ordered pairs have the same first component. Similar to a relation, the values of x is called the domain of the function and the set of all resulting value of y is called the range or co-domain of the function.

A function F from a set A to a set B is a relation with domain and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that (x,y) ∈F.
2. For all elements x in A and y and z in B,

If (x,y) ∈ F and (x,z) ∈F, then y = z

These two properties; (1) and (2) can be stated less formally as follows:

1. Every element of A is the first element of an ordered pair of F.
2. No two distinct ordered pairs in F have the same first element.

- Is a function a relation? Focus on the ***x*-coordinates**, when given a relation.
- If the set of ordered pairs have **different *x*-coordinates**, it is a function.
- If the set of ordered pairs have **same *x*-coordinates**, it is **NOT** a function but it could be said a relation.

Note:

a) Y-coordinates have no bearing in determining functions

b) Function is a relation but relation could not be said as function.

Example 1: Determine if the following is a function or not a function.

1. {(0, -5), (1, -4), (2, -3), (3, -2), (4, -1), (5, 0)}

2. {(-1, -7), (1, 0), (2, -3), (0, -8), (0, 5), (-2, -1)}

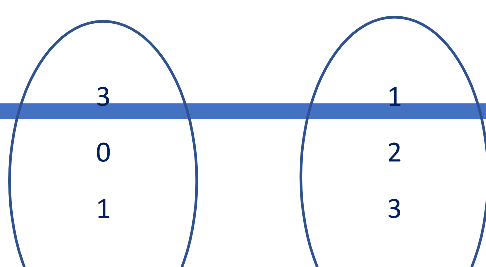
3. $2x + 3y – 1 = 0$

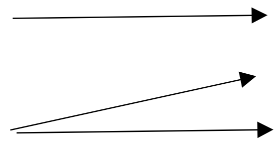
4. $x^2 + y^2 = 1$

5. $y^2 = x + 1$

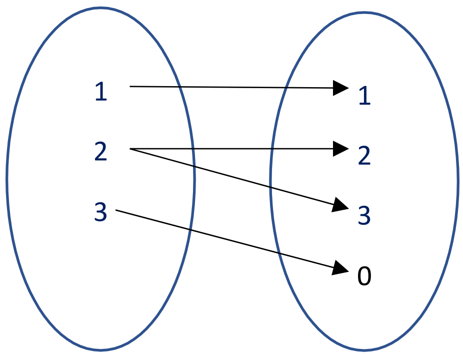
Example 2. Which of the following mapping represent a function?

1.





2.



Function Notations:

The symbol *f(x)* means function of *x* and it is read as "*f* of *x*." Thus, the equation y = 2x + 1 could be written in a form of f(x) = 2x + 1 meaning y = f(x). It can be stated that y is a function of x.

Let us say we have a function in a form of f(x) = 3x – 1. If we replace x = 1, this could be written as f(1) = 3(1) – 1. The notation f(1) only means that we substitute the value of x = 1 resulting the function value. Thus

$$f(x) = 3x - 1;\ \text{let } x = 1$$
$$f(1) = 3(1) - 1 = 3 - 1 = 2.$$

Another illustration is given a function $g(x) = x^2 - 3$ and let x = -2, then $g(-2) = (-2)^2 - 3 = 1$

Operations on Functions

The following are definitions on the operations on functions.

a. The sum or difference of f and g, denoted by f ± g is the function defined by (f ± g)(x) = f(x) ± g(x).

b. The product of f and g, denoted by f · g is the function defined by (f·g)(x) = f(x)·g(x).

c. The quotient of f and g denoted by f/g is the function defined by f(x)/g(x), where g(x) is not equal to zero.

d. The composite function of f and g denoted by f o g is the function defined by (f o g)(x) = f(g(x)). Similarly, the composite function of g by f, denoted by g o f, is the function defined by (g o f)(x) = g(f(x)).

Examples:

1. If f(x) = 2x + 1 and g(x) = 3x + 2, what is (f+g)(x)?

Solution:

(f+g)(x) = f(x) + g(x)
= (2x + 1) + (3x + 2)
= 2x + 3x + 1 + 2
= 5x + 3

2. What is (f • g)(x) if f(x) = 2x + 1 and g(x) = 3x + 2?

Solution:

(f • g)(x) = f(x) • g(x)
= (2x + 1)(3x + 2)
= $6x^2 + 7x + 2$

3. What is $\left(\frac{f}{g}\right)(x)$ if f(x) = 2a + 6b and g(x) = a + 3b?

Solution:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2a+6b}{a+3b} = \frac{2(a+3b)}{a+3b} = 2$$

4. If f(x) = 2x + 1 and g(x) = 3x + 2, what is (g o f)(x)?

Solution:

(g o f)(x) = g(f(x)) = g(2x + 1) = 3(2x + 1) + 2 = 6x + 3 + 2 = 6x + 5

III. BINARY OPERATION

R = {(x, y) | y = x +

6

$\mathbf{x}$; **where** $\mathbf{x}, \mathbf{y} \in \mathbf{N}, \mathbf{x} < \mathbf{6}\}$

–

A binary operation on a set G, then, is simply a method (or formula) by which the members of an ordered pair from G combine to yield a new member of G. This condition is called closure. **The most familiar binary operations are ordinary addition, subtraction, and multiplication of integers**. **Division of integers is not a binary operation on the integers because an integer divided by an integer need not be an integer.**

In mathematics, a binary operation on a set is a calculation that combines two elements of the set (called operands) to produce another element of the set.

Definition of Binary Operations

Let *G* be a non-empty set. An operation * on *G* is said to be a binary operation on *G* if for every pair of elements, a, b is in *G* that is a,b ∈ *G*; the product a * b ∈ *G*.

Note: For each (a,b) ∈G, we assign an element a * b of G.

Illustrative examples:

Tell whether the following is a binary operation or not.

1) ***G*** **∈ Z define a * b = a + b (usual addition on Z)**

Solution:

Let a ∈ Z and b ∈ Z. If we add a and b, then the sum a + b ∈ Z. Hence a * b = a + b is a binary operation.

2) ***G*** **∈ Z define a * b = ab (usual multiplication on Z)**

Solution:

Let a, b ∈ Z. Then the product of a and b, that is ab ∈ Z. Hence a * b = ab is a binary operation.

3) ***G*** **∈ R^+ defined by a * b = a + 17b**

Solution:

Let a, b $\in R^+$. If we take the sum of a + 17b $\in R^+$. Hence it is a binary operation.

4) **G $\in Z^+$, defined * by a * b = a – b for all set a, b $\in Z^+$.**

Solution:

If a > b = a – b > 0 $\in Z^+$

If a < b = a – b < 0 $\notin Z^+$

Therefore, * is NOT a binary operation

5) **G $\in$ R defined by a * b = a^b**

To be able to determine if the above statement is a binary operation or not, we need to have a counter example.

If a = 0 and b = 0, then a^b does not exist, hence it is not an element of R

If a = -4 and b = ½, then a^b would be an element of a complex number C, so $a^b \notin$ R.

6) **G $\in$ Z \{-1} , defined * by a * b = a + b + ab for all set a, b $\in$ Z.**

Solution

Let a, b $\in$ Z except -1, then a > -1 and b > -1 and a < -1 and b < -1. If **a * b = a + b + ab,** it follows that **a * b = a + b + ab** $\in$ Z since in both case such as a and b > -1 and a and b < -1, the result would be Z, hence ***** is a binary operation.

CLOSED

Definition: A set is "closed" under operation if the operation assigns to every ordered pair of elements from the set an element of the set.

Illustrative examples:

1) Is S = { ±1, ±3, ±5, ±7, …} is closed under usual addition?

Solution:

By giving a counter example, $S = \{ \pm1, \pm3, \pm5, \pm7, \ldots\}$ is NOT closed under usual addition. Why? Let us say we are going to a 1 and 3. The sum of 1 and 3 is 4 where 4 is not an element of S. Hence, it is not closed.

2) Let + and • be usual binary operations of addition and multiplication of Z and let $H = \{n^2 \mid n \in Z^+\}$. Is H closed under addition? Under multiplication?

Solution:

a. To be able to determine if H is closed under addition, we need to have a counter-example. Let us take two elements in Z, say 1 and 4. If we are going to add this two numbers, the result would be 5 and obviously, $5 \in n^2$ or 5 is not a perfect square. Hence, H is not closed under addition.
b. Let $r \in H$ and $s \in H$. Using $H \times H \in (r, s) = r \bullet s$. Since $r \in H$ and $s \in H$, that means there must be an integers n and $m \in Z^+$ such that $r = n^2$ and $s = m^2$. So;

$$(r, s) = r \bullet s = n^2 \bullet m^2 = (nm)^2$$

and $n, m \in Z^+$. It follows that $nm \in Z^+$, then $(nm)^2 \in H$. Hence, H is closed under multiplication.

Example. Consider the binary operation $*$ on R given by $x * y = x + y - 3$. $(x * y) * z = (x + y - 3) * z = (x + y - 3) + z - 3 = x + y + z - 6$, $x * (y * z) = x * (y + z - 3) = x + (y + z - 3) - 3 = x + y + z - 6$.

Therefore, $*$ is associative. Since $x * y = x + y - 3 = y + x - 3 = y * x$, $*$ is commutative.

Example. Consider the binary operation * on R given by a*b = ab/2. Show that a*b = b*c.

Solution:

Let a*b = ab/2. We need to show that a*b = b*a. In b*a = ba/2. But by commutative properties under multiplication, that is ab = ba, then it follows that b*a = ab/2. Hence a*b = b*a

Definition:

Let * be a binary operation of a set S. Then;

(a) * is associative if for all a, b, c ∈ S, (a*b)*c = a *(b*c)

(b) * is commutative if for all a, b ∈ S, a*b = b *a

(c) An element e ∈ S is called a left identity element if for all a ∈ S, we have e * a = a

(d) An element e ∈ S is called a right identity element if for all a ∈ S, we have a * e = a

(e) An element e ∈ S is called an identity element if for all a ∈ S, we have a * e = a and e * a = a.

(f) Let e be an identity element is S and a ∈ S, then b is called an inverse of the element "a" if a * b = e and b * a = e.

Note that a * b = b * a = e or $a * a^{-1} = a^{-1} * a = e$

If a ∈ S, then the inverse of "a" is denoted by a^{-1}. Here -1 is not an exponent of a.

Example: Let S = Z^+ as define * on S by a * b = a + b – ab. Show the associativity and the commutativity of S in a binary operation. Find also its identity and inverse if any.

(a) Associativity

Let a, b, c ∈ Z^+. Then; (a * b) * c = a * (b * c)

For (a * b) * c

(a * b) * c = (a + b – ab) * c

= (a + b – ab) + c - (a + b – ab)c

$= a + b + c - ab - ac - bc + abc$

For $a * (b * c)$

$a * (b * c) = a * (b + c - bc)$

$= a + (b + c - bc) - (a)(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

Hence * is associative on $S \in Z^+$.

(b) Commutative

$a * b = b * a$
$a + b - ab = b + a - ba$
$a + b - ab = a + b - ab$

Hence * is commutative on $S \in Z^+$.

(c) Identity

$a * e = a$	$e * a = a$
$a + e - ae = a$	$e + a - ea = a$
$e - ae = a - a$	$e - ea = a - a$
$e(1 - a) = 0$	$e(1 - a) = 0$
$e = 0$	$e = 0$

hence, the identity exist except when a = 1.

(d) Inverse

$a * a^{-1} = e$	$a^{-1} * a = e$

Example: Let $S = Z^+$ as define * on S by $a * b = a^2 + ab + b^2$. Is the operation * associative? Commutative? What is its identity? What is its inverse?

(a) Commutative

$$a * b = b * a$$

$$a^2 + ab + b^2 = b^2 + ba + a^2$$

$$a^2 + ab + b^2 = a^2 + ab + b^2$$

Hence, the operation * is commutative.

(b) Associative

$$(a * b) * c = a * (b * c)$$

$$(a^2 + ab + b^2) * c = a * (b^2 + bc + c^2)$$

$$(a^2 + ab + b^2)^2 + (a^2 + ab + b^2)(c) + c^2 \neq a^2 + (a)(b^2 + bc + c^2) + (b^2 + bc + c^2)^2$$

Hence, the operation * is not associative.

Cayley Tables

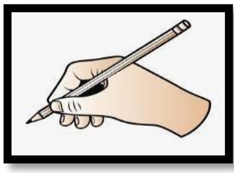
A (binary) operation on a finite set can be represented by a table. This is a square grid with one row and one column for each element in the set. The grid is filled in so that the element in the row belonging to x and the column belonging to y is x*y.

A binary operation on a finite set (a set with a limited number of elements) is often displayed in a table that demonstrates how the operation is performed.

Example: The table below is a table for a binary operation on the set {a, b, c, d}

*	a	b	C	d
a	a	b	C	d
b	b	c	D	a
c	c	d	A	b
d	d	a	B	c

a. Is the * commutative?
b. Is the * associative?
c. What is its identity?



Self - Learning Activity

Directions. Do as indicated.

A. Define a relation C from R to R as follows: For any (x,y) ∈R x R,

$$(x,y) \in C \text{ meaning that } x^2 + y^2 = 1.$$

a. Is (1,0) ∈ C? Is (0,0) ∈ C? Is -2 C 0? Is 0 C (-1)?

b. What are the domain and the co-domain of C?

B. If $f(x) = 2x^2$ and $g(x) = 3x + 1$, evaluate the following:

a. (f + g)(x) b) (f • g)(x) c) $(\frac{f}{g})(x)$ d) (g o f)(x)

C. Tell whether the following is a binary operation or not.

a) G ∈ Z, defined * by a * b = $a^2 - b^2$ for all set a, b ∈ Z.

Explanation:

b) G ∈ N, defined * by a * b = 2 + 3ab for all set a, b ∈ N.

Explanation:

c) **G ∈ Z^-, defined * by a * b = a + ab - b for all set a, b ∈ Z^-.**

Explanation:

d) **G ∈ R, defined * by a * b = $\sqrt{a - b}$ for all set a, b ∈ R.**

Explanation:

D. Let S = Z^+ as define * on S by a * b = a + b + 1. Show the associativity and the commutativity of S in a binary operation. Determine also the identity if there is.

E. Let A = Z – {0} and let S = {f_1, f_2, f_3, f_4, f_5, f_6} be the set of functions as A defined as follows:

$f_1(x) = x$ $\quad\quad$ $f_4(x) = \frac{1}{x}$

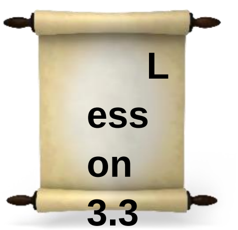
$f_2(x) = \frac{1}{1-x}$ $\quad\quad$ $f_5(x) = \frac{x}{x-1}$

$f_3(x) = \frac{x-1}{x}$ $\quad\quad$ $f_6 = 1 - x$

Show that the composition of mappings is a binary operation by completing the multiplication table for *

*	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2						
f_3						
f_4						
f_5						
f_6						

Present your solution below.



Logic and Formality

Specific Objective

At the end of this lesson, the student should be able to:

1. Define what logic is.
2. Tell whether the statement is formal or non-formal.
3. Show the relationship between grammar in English and logic in Mathematics.

Introduction

What comes first in your mind when we speak about logic? Do you have any idea what logic is all about? Could we say that if a person thinks correctly, then he has logic? Perhaps until now, there are some people arguing whether a logic is an art or it is a science. Now, whether it is an art or a science, studying logic could be very important not only in the field of mathematics but in other sciences such as natural science and social science. On this module, we will studying the fundamental concept of logic but basically logic as mathematical language.

Discussion

I. What is logic?

In this particular module, we are going to talk about logic as a mathematical language but a deeper discussion logic as a science as well as its application will be tackled in module 6. It is very essential to understand better what logic is as a language.

But first, let us have a definition in logic. In your social science courses, logic could define as the study of the principles of correct reasoning and it is not a psychology of reasoning. Based on the definition which is logic is the study of the principle of correct reasoning, one of the principles in logic that is very much important to study is on how to determine the validity of ones argument. Studying mathematics is also studying theorems. The proof of the theorem uses the principle of arguments in logic. So, in this case, we could say actually that the language of mathematics is logic.

In short, mathematical statement is also a grammar. In English, when we construct a sentence or sentences, we always check if it is grammatically correct but in Mathematics, we check mathematical statement or sentence in a logical structure. Wherever you go, we have a common language in mathematics. In order not to conflict with in an English word, we use appropriate symbols in mathematics so that there will no ambiguity on how to communicate as to the meaning of a mathematical expression or even in mathematical sentences

II. Formality

As stated by Heylighen F. and Dewaele J-M in the “Formality of Language: Definition and Measurement”, an expression is completely formal when it is context-independent and precise (i.e. non-fuzzy), that is, it represents a clear distinction which is invariant under changes of context. In mathematics, we are always dealing in a formal way.

Suppose that somebody asked you that the result of adding 5 to 3 is 8 or let us say that if a variable x is an even number then the square of this variable x would be also an even number, you would agree that both mathematical sentences or statements are true and there is no reason for you to doubt. Those two examples statements are precise and it is also an independent. These are the two characteristics in mathematics that the statement must have to say the mathematical sentence is in a formal manner. Speaking of statement, statement is the main component of logic in mathematics.

When we say mathematical logic, it is a statements about mathematical objects that are taken seriously as mathematical objects in their own right. More generally, in mathematical logic we formalize, that is, we formulate in a precise mathematical way its definition, theorem, lemma, conjecture, corollary, propositions and the methods of proof which will be discussed in our next lesson. These are the major part of formality in mathematics.

a) Definition

One of the major parts of formality in mathematics is the definition itself. When we say definition, it is a formal statement of the meaning of a word or group of words and it could stand alone.

Example of this is a definition of a right triangle. What is the exact or formal definition of this? A right triangle consists of two legs and a hypotenuse. The two legs meet at a 90° angle and the hypotenuse is the longest side of the right triangle and is the side opposite the right angle. Here, you will see the exactness and the precision of the definition of a right triangle.

Now suppose we are going to define "carabao". Can you give a definition for this? Maybe, some of you will define a carabao is a black and strong animal helps the farmer in plowing the rice field. But, have you noticed that this is not a formal definition? How about the cow and the horse? These are also an animal that could also help the farmers in plowing the field. How about the machine tractor? Are we not consider this machine that could possibly help our farmers in plowing the rice field? So, we cannot say that is a formal definition since it cannot stand alone.

b) Theorem

Another statement that could we consider as a formal statement is the theorem. You will encounter this word in all books of mathematics especially if it is pure mathematics. In your algebra subject during your high school days, have you studied different laws and principles in mathematics? These are just really theorems that proven true and justified using the concept of mathematical logic and all you need to do is to apply those laws and principles, isn't it? But what does theorem means? A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments. In general, a theorem is an embodiment of some general principle that makes it part of a larger theory. The process of showing a theorem to be correct is called a proof.

An example of a theorem that we all know is the Pythagorean Theorem. This is a very well-known theorem in mathematics. The theorem stated that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. If the hypotenuse (the side opposite the right angle, or the long side) is called c and the other two sides are a and b, then this theorem with the formula $a^2 + b^2 = c^2$. You will notice that the theorem is precise in a form of if-then statement. The if-then statement is one of the statements in logic.

So, a statement could not be considered theorem unless it was proven true using mathematical logic.

c) Proof

To be able to say that a theorem is true, it should be undergo on the process of proving. But what do we mean by proof or a mathematical proof. Proof is a rigorous mathematical argument which unequivocally demonstrates the truth of a given proposition. The different methods on proof are as follows:

1. Deductive
2. Inductive
3. Direct Proof
4. Indirect Proof
5. Proof by Counterexample
6. Proof by Contradiction

All of these methods of proof are written together with the correct mathematical logic and precise. Discussion and illustrative examples on these different methods of proof will be tackled in module 3.

d) Proposition

When we say proposition, it is a declarative statement that is true or false but not both. This statement is another major part of formality since all types of proposition are precise and concise. Different propositions that can be also said as logical connectives are as follows:

1. Negation

How does the statement translate into its negation. Say, given any statement P, another statement called the negation of P can be formed by writing “It is false that …” before P, or if possible, by inserting in P the word “not”.

For example, the given statement is “Roderick attends Mathematics class”. Translating this into its negation, the new statement would be “Roderick will not attend Mathematics class” or it could be “It is false that Roderick attends Mathematics class”.

2. Conjunction

Another logical connective is what we called conjunction. If two statements are combined by the word “and”, then the proposition is called conjunction. In other words, any two statements can be combined by the word “and” to form a composite statement which is called the conjunction of the original statements.

An example for this is, let us say the first statement is “Ernesto is handsome” and the second statement is “Ernesto is rich”. The new statement after connecting this two statements by the word “and”, this would become “Ernesto is handsome and Ernesto is rich market”.

3. Disjunction

Disjunction is another form of proposition. Any two statements can be combined by the word "or" to form a new statement which is called the disjunction of the original two statements.

Let us have an example for this kind of proposition. Say, the first statement is "Life is beautiful" while the second statement is "Life is challenging. Now, combining these two statements by the word "or" the new combined statement is "Life is beautiful or life is challenging."

4. Conditional

The fourth type of proposition is that what we called conditional. To be able to easily identify that the proposition is in a form of conditional statement, you will notice of the word "If-then". Most of mathematical definition is in a form of this statement. So, in other words, it is state that a true statement cannot imply a false statement. In this proposition, the first statement would be a premise and the second statement is the conclusion.

Let us have an example for this. Say the premise is "If x is positive, then its square is also positive." We can show the proposition is true with the use of one of the methods of proving.

5. Biconditional

The last type of proposition is the biconditional. Its uses a connector for two statements "if an only if". If your statement is in this form, then your statement is called biconditional.

Here is one of the examples of a biconditional statement. Let us say our first statement is "I will attend mass." The second statement is "Tomorrow is Sunday." So, the new statement using biconditional statement would be "I will attend mass if and only if tomorrow is Sunday."

Now, supposedly our statement goes like this. "Let's go!" Can we considered this as a precise formal statement? Perhaps you will be saying no since you may be asking; Who will be my companion?; What time are we going to go?; Where will we go?. This statement is not precise hence it is not formal.

All of these statements can be transformed into symbols. More details and specific lesson about this will be tackled in module 6.

e) Corollary

What is corollary? When we say corollary in mathematics, it is also a proposition that follows with little or no proof required from one already proven. An example of this is it is a theorem in geometry that the angles opposite two congruent sides of a triangle are also congruent. A corollary to that statement is that an equilateral triangle is also equiangular.

f) Lemma

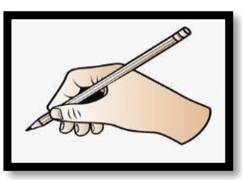
Another formal statement is a lemma and it can also be considered as a theorem. The only difference of a lemma into a theorem is that lemma is a short theorem used in proving a larger theorem. As we all know that a theorem is a precise statement since it was proved to be true with the use of mathematical logic. So, it is precise. If lemma is a shorter version of a larger theorem and theorem is a precise statement, we could say that a lemma is also a precise statement.

Let us have a concrete presentation for a lemma. Let us say the theorem stated that “Let f be a function whose derivative exists in every point, then f is a continuous function.” Then another theorem about Pythagorean and this theorem is about right triangles that can be summarized in an equation $x^2 + y^2 = z^2$. The consequence on the previous theorem is stated in a corollary which is “There is no right triangle whose sides measure 3cm, 4cm and 6cm. Now, we can more simplify our given theorem in a form of “Given two line segments whose lengths are a and b respectively, there is a real number r such that b = ra”.

g) Conjecture

A proposition which is consistent with known data, but has neither been verified nor shown to be false. It is synonymous or identical with hypothesis also known as educated guess. We can only disproved the truthfulness of a conjecture when after using a counterexample we found at least one that says the statement is false. Let us say we have 75 different balls in a bingo urn labelled as 1 – 75. What will be our conjecture? We could say that “All number in an urn is a counting number from 1 to 75.

Based on the previous discussion, you will observe that all of these statement follows the characteristics of mathematics and that is they are all precise and independent.



Self - Learning Activity

Direction: Do as indicated.

A. Tell whether the following statements is formal or non-formal. Write F if your answer is formal and NF if it is non-formal on the space provided before each item.

1. An acute triangle is a triangle that all included angles is less than 90 degrees.

2. The diagonal of a rhombus is perpendicular to each other.

3. A number is an even number if and only if n must be squared.

4. A number n is an odd number if and only if n = 2k + 1 where k is any integer.

5. An odd number raised to a third power is always an odd number.

B. Make a formal and non-formal definition for the following terms:

1. Table
2. Graph
3. Letters

C. Make exactly three formal statements in mathematics.

1. ____

2. ____

3. ____

Chapter Test 2

Test 1.

Directions: Read the following statement carefully. Encircle the letter of the best correct answer. If the correct answer is not on the choices, write N before the item.

1. Which of the following is the correct mathematical translation for an expression; "the square of the sum of the square of two numbers"?
   a. $(a + b)^2$ c. $a^2 + b^2$
   b. $(a^2 + b^2)^2$ d. $2(a + b)^2$
2. The correct mathematical symbol translation for "The product of the sum and difference of two numbers is the difference of the square of two numbers" is:
   a. $(a + b)(a – b) = (a – b)^2$ c. $(a^2 + b^2)(a^2-b^2) = a^2 – b^2$
   b. $(a + b)(a – b) = a^2 – b^2$ d. $(a + b)^2(a-b)^2 = a^2 – b^2$
3. The correct mathematical translation for "A number raised by the third power increased by one" is:
   a. $3x + 1$ c. $x^3 + 1$
   b. $(x + 1)^3$ d. $3(x + 1)$
4. Which of the following is the correct translation to an English expression for a mathematical expression; $x^3 + y^3$?
   a. The sum of two numbers
   b. The cube of the sum of two numbers
   c. The sum of the cube of two numbers
   d. Two numbers raised to the third power
5. The perimeter (P) of a rectangle is twice of its length (L) and twice of its width (W). If you are going to translate this into mathematical sentence, which of the following translation is correct?

   a. $P = 2L + 2W$ c. $P = LW$
   b. $P = L^2 + W^2$ d. $P = 2LW$

6. Which of the following is true?

   a. $A \subseteq \emptyset$ c. $A \times B = B \times A$
   b. $1 \notin N$ d. $A \subseteq A$

7. Which of the following is false?

   a. $4 \in \{1, 2, 3, 4\}$ c. $-5 \in N$

b. ½ ∉ Integer

d. The set of nice car is not a well-defined set

8. Let Z^+, B = {n∈Z, 10 ≤ n ≤ 100}, and C = {100, 200, 300, 400, 500}. Which of the following is false?

a. C is a proper subset of A
b. C and B have at least one element in common
c. C is a subset of C
d. B is a subset of A

9. Which of the following is false?

a. A ∩ B = B ∩ A
b. A ∪ B = B ∪ A
c. (A ∪ B) ∩ C = (A∩C)∪(B∩C)
d. A – B = B – A

10. Which of the following is NOT an empty set?

a. { }
b. ∅
c. A set of yellow carabao
d. { 0 }

11. Which of the following is false?

a. (0, 0) = (a, a) if a = 0
b. (a – b, 2x) = (b-a, 6x/3)
c. (ab, cd) = (ba, dc)
d. (x + a, y + b) = (a+x, b + y)

12. Which of the following is false?

a. ((6x2), (4x2)) = (12, 8)
b. $(\sqrt{9}, \frac{6}{2}) = (\pm 3, 3)$
c. (1/4, 1/5) = (4/16, 5/25)
d. (2/3, 3/2) = (3/2, 2/3)

13. It is the mathematical analogue of an English noun.

a. Equation
b. Expression
c. mathematical sentence
d. equal sign

14. Given two non-empty sets A and B, the set of all ordered pairs (*x*, *y*), where *x* ∈ A and *y* ∈ B is called:

a. Cartesian product of A and B
b. Intersection of A and B
c. Union of A and B
d. Difference of A and B

15. Which of the following sets is a null set ?

I. X = {x | x= 9, 2x = 4 } II. Y = {x | x= 2x, x ≠ 0 } III. Z = { x | x−8 = 4 }

a. I and II only c. I, II and III

b. I and III only

d. II and III only

16. If A is not equal to B, then which of the following is correct?

a. A x B = B x A
b. A x B > B x A
c. A x B < B x A
d. A x B ≠ B x A

17. In a mathematical sentence; 2ax + by = 6, what particular symbol is pertaining to a verb?

a. a and b
b. 2 and 6
c. x and y
d. =

18. In a mathematical sentence in item number 25, the + sign is called:

a. adverb
b. connective
c. noun
d. pronoun

19. What property of a relation does the set of ordered pairs (1,3), (2,4), (1,2),(3,1),(4,2), (2,1) belong?

a. Transitive
b. Equivalence
c. Symmetric
d. Reflexive

20. Which of the following is **NOT** a function?

a. $y = x^2 + 1$
b. (1,2), (2,1), (3,4), (4,3)
c. $16x^2 + 9y^2 = 16$
d. $y = x^3$

21. In a given set A = { a, b, c, d}, B = { a, c, e} and C = { b, c, f}, what is A - (B∩C)?

a. { a, b, c }
b. { a, b, f }
c. { a, b, d }
d. { a, b, e }

22. If U= { a, e, i, o, u} and A = { a, c, e } and B = { g, o, d }, what is $A^c – B^c$?

a. { a }
b. { i }
c. { e }
d. { o }

23. If A = {a, b}, what is A x A?

a. { a, b}
b. { (a,a), (b,b)}
c. { (a, b), (b, a), (a,a) }
d. { (a,a), (b,b), (a, b), (b,a)}

24. How many subsets are there in a given set A = { j, h, u, n }?

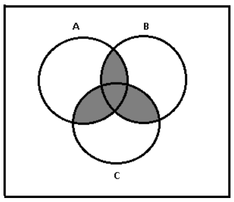
a. 16
c. 8

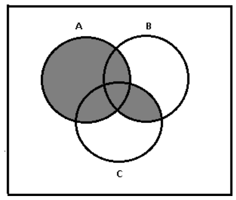
b. 4 d. 1

25. What is the intersection of a two sets A = {a, b, c} and B = {1, 2, 3}?

a. {∅} c. { }
b. 1 d. 0

26. Which of the following diagram represent of the set notation A ∪ (B ∩ C)?

a.


c.


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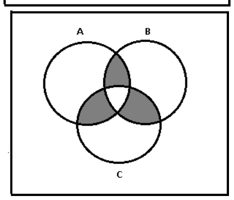
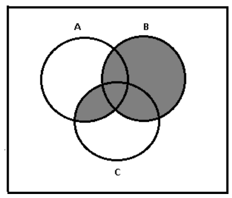


d.


27. If (x^2 – 1, 4) = (0, y – 1), what is the value of x and y?

a. x = 0 and y = 5 c. x = 1 and y = 3
b. x = 1; -1 and y = 5d. x = -1, 0 and y = 5

28. What is the value of x and y in a given two equal ordered pair below (2x + 1, 3y + 4) = (-4x - 2, 5 - 2y)?

a. x = -2 ; y = -5 c. x = 2; y = 5
b. x = 1/5; y = ½ d. x= -1/2; y = 1/5

29. A group of students were asked whether they are like Mathematics, English, or both. If 256 like Mathematics, 184 like English, and 120 like both subjects, how many students were there?

a. 520 c. 320
b. 420 d. 220

30. If A = {2,3} and B = {2,6} where a ∈ A and b ∈ B, what is the set if ordered pair such that “a” is a factor of “b” and a < b.

a. (2,2), (2,6) c. (2,6), (3,2), (3,6)
b. (3,2), (3,6) d. (2,6), (3.6)

31. If (2a + b, a – b) = (8, 3), what would be the value of a and b?

a. a=11/3 and b= 2/3 c. a = 2/3 and b = 11/3
b. a= 3/11 and b=3/2 d. a=3/2 and b=3/11

32. What is the domain of the relation R given by:

$$\mathbf{R} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} = \mathbf{x} + \frac{\mathbf{6}}{\mathbf{x}}; \text{ where } \mathbf{x}, \mathbf{y} \in \mathbf{N}, \mathbf{x} < \mathbf{6}\}$$

a. {1,2} c. {1,2,3}
b. {7,5} d. {5,6,7}

33. Given a function $C(t) = 2t^3 – 3t$ whose domain is {0, 2, 4,}, what is its range?

a. {0, 10, 116} c. {2, 12, 24}

b. {0, -10, -116} d. {-2, -12, -24}

34. If $f(x) = 2x^2 – 3$ and $g(x) = 3x + 2$, what is f(x) + g(x)?

a. $x^2 + 3x – 1$ c. $3x^2 + 3x – 1$

b. $2x^2 + 3x – 1$ d. $4x^2 + 3x – 1$

35. Which of the following could **NOT** be a binary operation?

a. Addition of integers c. Multiplication of integers
b. Subtraction of integers d. Division of integers

36. A set is______________under operation if the operation assigns to every ordered pair of elements from the set an element of the set.

a. Associative c. closed
b. Commutative d. binary

37. * is____________________if for all a, b ∈ S, a*b = b *a.

a. Associative c. closed
b. Commutative d. binary

38. A binary operation on a finite set can be represented by a table and this table is called:

a. Polya's Table c. Frequency Table
b. Cayley's Table d. Multiplication Table

39. Which of the following is NOT a binary operation?

a. G ∈ Z is defined as a * b = a – b.

b. G ∈ N is defined as a * b = 2a + 3b
c. G ∈ Z^+ is defined as a * b = a – b
d. G ∈ Q is defined as a * b = ab/2

40. Which of the following is NOT closed in an operation *?

a. N = {1,2,3,…} under usual addition
b. N = {1,2,3,…} under usual subtraction
c. Z = {…-3, -2, -1, 0, 1, 2, 3, …} under usual multiplication
d. Z^+ under usual multiplication

41. An operation * is defined on the set R by x * y = x + y + 6 where x, y are elements of R. What is -3*(5*2)?

a. 10
b. 16
c. 12
d. -2

42. The operation * is defined over the set of real numbers "R" by a * b = a + b + 1/2ab. What is 6 * (2 * 5)?

a. 28
b. 0
c. 13
d. 54

43. Supposing the binary operation * is defined on the set T = {1, 2, 3, 4, 5} by a * b = a + b + 2ab. Say 2, 3 ∈ T, so 2 * 3 is closed under operation *.

a. Yes
b. No
c. Maybe
d. Insufficient Information

44. The operation * is defined over the set R of real numbers by a * b = a + b + 2ab. What is the identity element under the operation *.

a. 0
b. 1
c. -1
d. 4ab

45 – 49. Refer to a table below:

∇	**a**	**b**	**c**	**d**
A	a	b	c	d
B	b	c	d	a
C	c	d	a	b
D	d	a	b	c

Answer the questions below.

45. What is (a ∇ b) ∇ c?

a. a
b. b
c. c
d. d

46. What is b ∇ (c ∇ d) ?

a. a
b. b
c. c
d. d

47. What is [(a ∇ b) ∇ d] ∇ [b ∇ (a ∇ c)]?

a. a
b. b
c. c
d. d

48. What is its identity?

a. a
b. b
c. c
d. d

49. What is the inverse of c?

a. a
b. b
c. c
d. d

Test II. APPLICATION

50 – 54. Problem 1. Given that $a * b = a + ab + b$ for all $a,b \in Z$. (a) Tell whether the operation * is a binary operation or not. (b) Show if it is commutative and (c) associative. (d) Find the identity and (e) the inverse if possible.

50. Write your explanation below.

51. Commutativity

52. Associativity

53. Identity

54. Inverse

55 – 60. Problem 2. A binary operation * is defined on the set S = {0, 1, 2, 3, 4} by a * b = a + ab + b. Complete the table and solve the following questions below. Write your solution on the space provided. (6 points)

*	0	1	2	3	4
0					
1					
2					
3					
4					

Do as indicated.

55. Is it closed in an operation *? Justify your answer.

56. Evaluate (1 * 3) * (2 * 4).

57. Is (2 * 3) = (3 * 2)? Justify your answer whether the answer is yes or no.

58. Is 2 * (3 * 4) equal to (2 * 3) * 4. Write your justification.

59. Look for the identity and the inverse if any.

B

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MODULE THREE

PROBLEM SOLVING AND REASONING

CORE IDEA

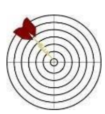
Module three is basically showing that mathematics is not just about numbers but much of it is problem solving and reasoning.

Learning Outcome:

1. State different types of reasoning to justify statements and arguments made about mathematics and mathematical concept.
2. Write clear and logical proofs.
3. Solve problems involving patterns and recreational problems following Polya's four steps.
4. Organize one's methods and approaches for proving and solving problems.

- **Time Allotment:** Six (6) lecture hours

Inductive and Deductive Reasoning



Specific Objectives

At the end of this lesson, the student should be able to:

1. Define inductive and deductive reasoning.
2. Differentiate inductive reasoning from deductive reasoning.
3. Demonstrate the correct way in using the two kinds of reasoning.
4. Apply the concept of patterns in mathematics to solve problems in inductive and deductive reasoning which

lead into correct conjecture by creating their own reasoning.

In mathematics, sometimes we need to use inductive and deductive reasoning to be able to solve some practical problems that we may encounter in our daily lives. During your senior high school, your teacher taught you on how to solve problems in a most scientific way and there are steps to be followed in order to solve problems in a particular math subject, specifically in Algebra. Some of these problems are the number problem, age problem, coin problem, work problem, mixture problem, etc.

In this module, we will be studying on how to solve problems in a different way. We will be using what we called an inductive and deductive reasoning way. But before we give an example on how to use this method, let us define first what inductive and deductive reasoning is.

A. Inductive Reasoning

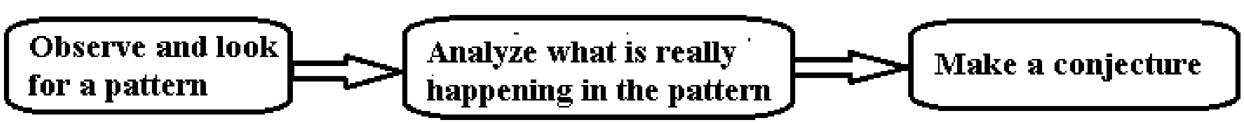
The type of reasoning that forms a conclusion based on the examination of specific examples is called ***inductive reasoning.*** The conclusion formed by using inductive reasoning is often called a ***conjecture***, since it may or may not be correct or in other words, it is a concluding statement that is reached using inductive reasoning.

Inductive reasoning uses a set of specific observations to reach an overarching conclusion or it is the process of recognizing or observing patterns and drawing a conclusion.

So in short, ***inductive reasoning*** is the process of reaching a general conclusion by examining specific examples.

Take note that inductive reasoning does not guarantee a true result, it only provides a means of making a conjecture.

Based on the given definition above, we could illustrate this by means of a diagram.



Also, in inductive reasoning, we use the "then" and "now" approach. The "then" idea is to use the data to find pattern and make a prediction and the "now" idea is to make a conjecture base on the inductive reasoning or find a counter-example. Definition for counter example will be discussed on the latter part of our lecture.

Let us have some examples on how to deal with this kind of reasoning.

Examples:

1. Use inductive reasoning to predict the next number in each of the following list:

   3, 6, 9, 12, 15, ?

   Explanation

   The given sequence of number is clearly seen that each successive number is three (3) larger than the preceding number, which is if the first number is increased by 3 the result is 6. Now, when this 6 is increased by 3 the next number would be 9. If we are going to continue the process, if 15 is increased by 3 then the next number would be 18. Hence the required number is 18.

2. Write a conjecture that describe the pattern 2, 4, 12, 48, 240. Then use the conjecture to find the next item in the sequence.

   Step 1. Look for a pattern

   2 4 12 48 240 … ?

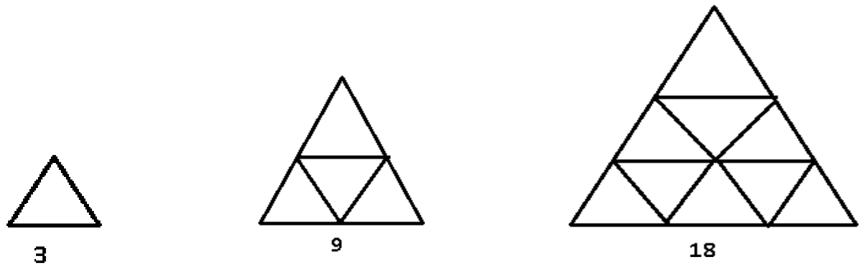
   Step 2. Analyze what is happening in the given pattern.

   The numbers are multiplied by 2, then 3, then 4, then 5. The next number will be the product of 240 times 6 or 1,140.

   Step 3: Make a conjecture

   Now, the answer is 1,140

3. Write a conjecture that describes the pattern shown below. How many segments could be formed on the fifth figure?



Step 1. Look for a pattern

3-segments 9-segments 18-segments

Step 2. Analyze what is happening in the given pattern.

3 9 18 30 ?

+6 +9 +12 +15

This could be written in a form of:

(3)(2) (3)(3) (3)(4) (3)(5)

The figure will increase by the next multiple of 3. If we add 15, the next or the fifth figure is made of 45 segments.

Step 3. Make a conjecture

Hence the fifth figure will have 45 segments.

Application of Inductive Reasoning (Using inductive reasoning to solve a problem)

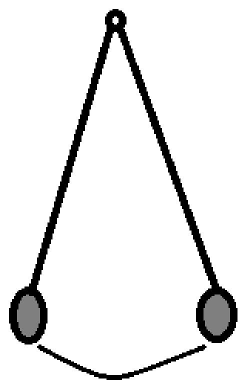
Inductive reasoning is very essential to solve some practical problems that you may encounter. With the use of inductive reasoning, we can easily predict a solution or an answer of a certain problem.

Here, we can see an illustrative examples on how to solve a certain problem using inductive reasoning.

Example 1.

Use the data below and with the use of inductive reasoning, answer each of the following questions:

1. If a pendulum has a length of 49 units, what is its period?
2. If the length of a pendulum is quadrupled, what happens to its period?



Note: The period of a pendulum is the time it takes for the pendulum to swing from left to right and back to its original position.

Length of Pendulum in Units	**Period of Pendulum in heartbeats**
1	1
4	2
9	3
16	4
25	5
36	6

Solution:

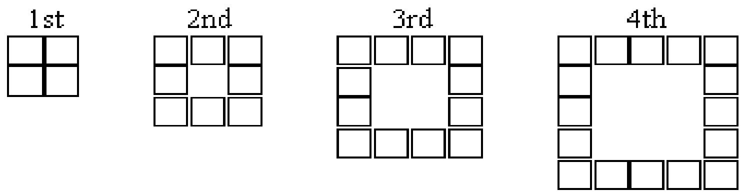
1. In the table, each pendulum has a period that is the square root of its length. Thus, we conjecture that a pendulum with a length of 49 units will have a

period of 7 heartbeats.

2. In the table, a pendulum with a length of 4 units has a period that is twice that of pendulum with a length of 1 unit. A pendulum with a length of 16 units has a period that is twice that of pendulum with a length of 4 units. It appears that quadrupling the length of a pendulum doubles its period.

Example 2.

The diagram below shows a series of squares formed by small square tiles. Complete the table below.



Let us make a table.

Figure	1st	2nd	3rd	4th	5th	6th	10th	15th
Number of Tiles	4	8	12	16				

Solution:

1. Based on the given figures from the first up to fourth, we need to observe and analyse what is really happening in the said figures.

2. Next, take a look if there is a pattern. Is there any pattern that you may observe? If so, what it is? For sure you could say that from the first figure, each subsequent square increases by four (4) tiles. How? Let us take a look at this.

Number of Tiles : 4 8 12 16

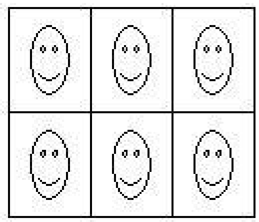
Patterm: +4 +4 +4

So, if each subsequent square increases by four, we could say that the 5^{th}, 6^{th}, 10^{th}, and the 15^{th} figure should have 20, 24, 40 and 60 squares respectively. Hence, the complete table would be;

Figure	1st	2nd	3rd	4th	5th	6th	10th	15th
Number of Tiles	4	8	12	16	20	24	40	60

Example 3.

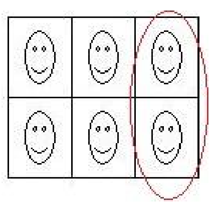
Two stamps are to be torn from the sheet shown below. The four stamps must be intact so that each stamp is joined to another stamp along at least one edge.



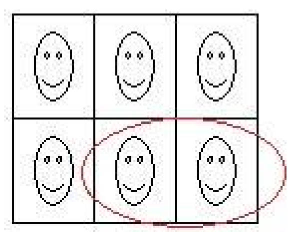
What would be the possible patterns for these four stamps after the two stamps were torn?

Solution:

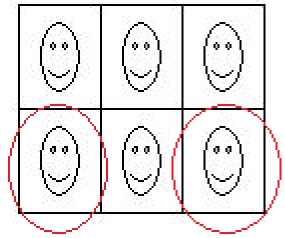
The first possible pattern is if we tear the two right most stamps as shown below.



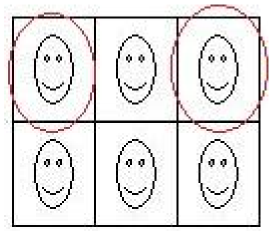
Next is if we tear the two stamps on the lower right portion as shown below.



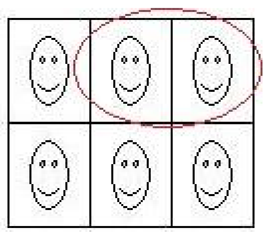
Then, the next possible pattern if we tear the lower rightmost and leftmost stamp as shown below.



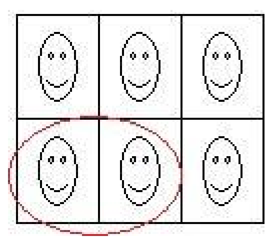
Also, if we tear the upper rightmost and upper leftmost stamp could be another possible pattern as shown below.



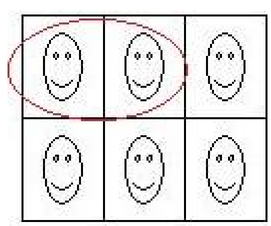
Next possible pattern is if we tear the two upper right most stamps as shown below.



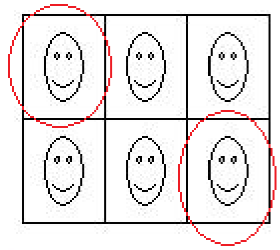
Then, it could be followed two stamps to be torn on the lower leftmost as shown below.



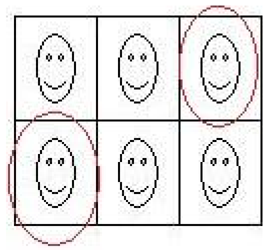
Next is the two stamps at the upper rightmost as shown below.



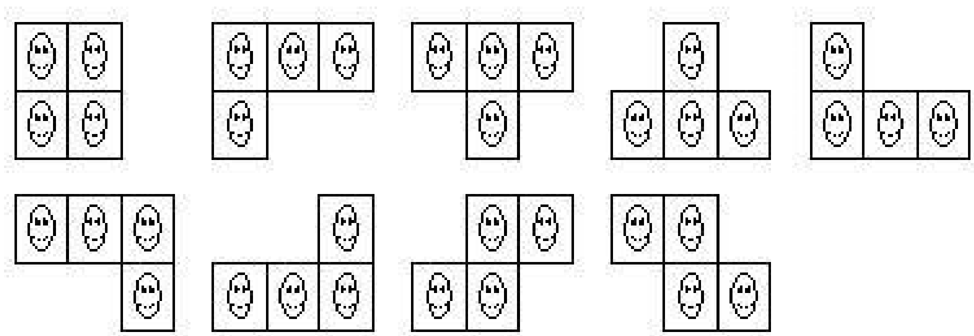
The eight possible pattern is if we tear one stamp at the upper leftmost and one stamp at the lower rightmost as shown.



Lastly, if we tear one stamp at the upper rightmost and another one stamp on the lower leftmost as shown.



Hence, below are the different possible pattern based on the given question above.



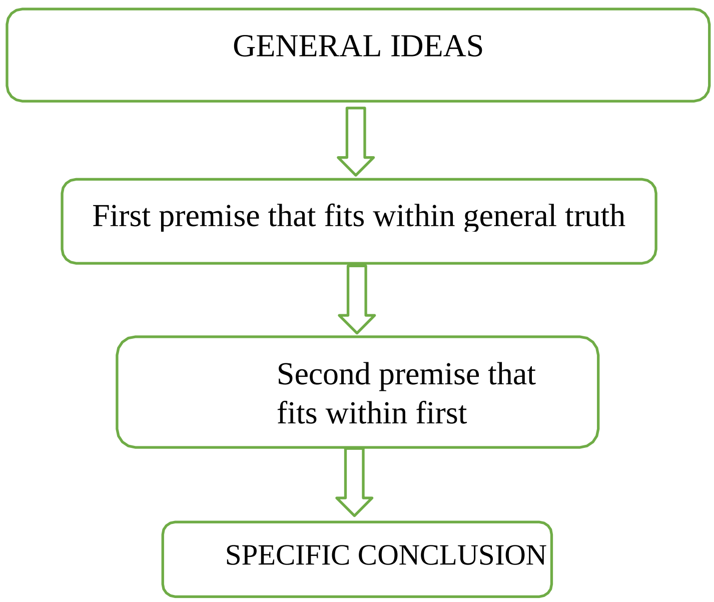
Note: The sequence of these pattern could be interchanged.

B. Deductive Reasoning

Another type of reasoning is called deductive reasoning. It is a basic form of valid reasoning starts out with a general statement, or hypothesis, and examines the possibilities to reach a specific, logical conclusion. So, we could say also that this kind of reasoning works from the more general to the more specific.

By definition, **deductive reasoning** is the process of reaching specific conclusion by applying general ideas or assumptions, procedure or principle or it is a process of reasoning logically from given statement to a conclusion.

The concept of deductive reasoning is often expressed visually using a funnel that narrows a general idea into a specific conclusion.



Example 1.

First Premise: All positive counting numbers whose unit digit is divisible by two are even numbers.

Second Premise: A positive counting number 1,236 has a unit digit of 6 which is divisible by two.

Conclusion: Therefore, 1,236 is an even number.

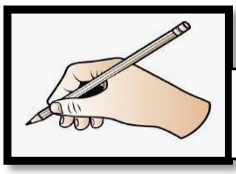
Example 2.

First Premise: If the Department of Education strictly observed health conditions of the students due to Covid 19, then there is no face-to-face teaching and learning activity in a classroom.

Second Premise: The Philippines is currently experiencing Covid 19 pandemic.

Conclusion: Therefore, there will be no face-to-face teaching and learning style in a classroom.

Note: Not all arguments are valid! Can you make an example of a deductive reasoning that could be considered as an invalid argument?



Self -Learning Activity

Directions: Do as indicated.

A. Identify the premise and conclusion in each of the following arguments. Tell whether also if the following arguments is an inductive or deductive reasoning.

a) The building of College of Informatics and Computing Sciences in BatStateU Alangilan is made out of cement. Both building of the College of Engineering, Architecture and Fine Arts and the College of Industrial Technology in BatStateU Alangilan are made out of cement. Therefore, all building of Batangas State University is made out of cement.

b) All birds has wings. Eagle is a bird. Therefore, eagle is a bird.

B. Use inductive reasoning to predict the next three numbers on the following series of numbers.

a) 3, 7, 11, 15, 19, 23,_____,______,______, …

b) 1, 2, 6, 15, 31,______,_______,_______, …

c) 1, 4, 9, 16, 25, 36, 49,_______,______,______, ….

C. Write the next possible equation on the following series of an equation.

37 x 3 = 111

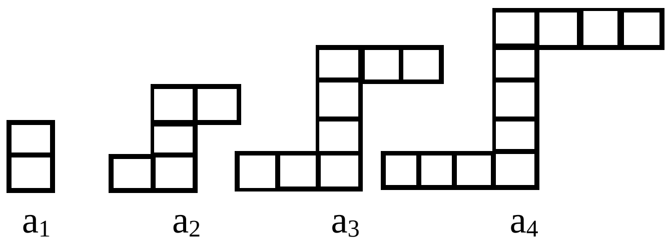
37 x 6 = 222

37 x 9 = 333

37 x 12 = 444

_____?_______

D. Assume that the figure below is made up of square tiles.



What would be the correct formula to determine the number of square tiles in the nth term of a sequence?

Lesson 3.2

Intuition, Proof and Certainty

Specific Objective

At the end of the lesson, the student should be able to:

1. Define and differentiate intuition, proof and certainty.
2. Make use of intuition to solve problem.
3. Name and prove some mathematical statement with the use of different kinds of proving.

Introduction

Sometimes, we tried to solve problem or problems in mathematics even without using any mathematical computation and we just simply observed, example a pattern to be able on how to deal with the problem and with this, we can come up with our decision with the use of our intuition. On the other hand, we use another method to solve problems in mathematics to come up with a correct conclusion or conjecture with the help of different types of proving where proofs is an example of certainty.

Discussion

A. Intuition

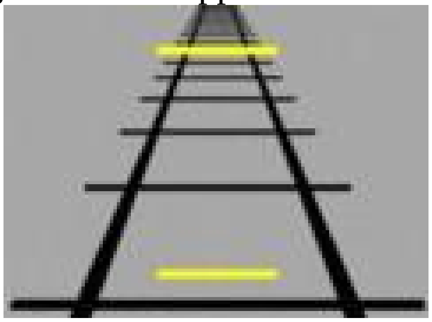
There are a lot of definition of an intuition and one of these is that it is an immediate understanding or knowing something without reasoning. It does not require a big picture or full understanding of the problem, as it uses a lot of small pieces of abstract information that you have in your memory to create a reasoning leading to your decision just from the limited information you have about the problem in hand. Intuition comes from noticing, thinking and questioning.

As a student, you can build and improve your intuition by doing the following:

a. Be observant and see things visually towards with your critical thinking.
b. Make your own manipulation on the things that you have noticed and observed.
c. Do the right thinking and make a connections with it before doing the solution.

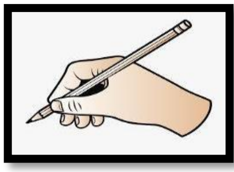
Illustration

1. Based on the given picture below, which among of the two yellow lines is longer? Is it the upper one or the lower one?



What are you going to do to be able to answer the question? Your own intuition could help you to answer the question correctly and come up with a correct conclusion. For sure, the first thing that you are going to do is to make a keen observation in the figure and you will be asking yourself (starting to process your critical thinking) which of these two yellow lines is longer compare to other line or is it really the yellow line above is more longer than the yellow line below? But what would be the correct explanation?

The figure above is called Ponzo illusion (1911). There are two identical yellow lines drawn horizontally in a railway track. If you will be observing these two yellow lines, your mind tells you that upper yellow line looks longer that the below yellow line. But in reality, the two lines has equal length. For sure, you will be using a ruler to be able to determine which of the two is longer than the other one. The exact reasoning could goes like this. The upper yellow line looks longer because of the converging sides of a railway. The farther the line, it seems look line longer that the other yellow line below. Now, have you tried to use a ruler? What have you noticed?



Self- Learning Activity

Now, let us test your intuition. We have here a set of problems. Make your own conclusion based on the given problem without solving it mathematically.

1. Which of the two have the largest value? Explain it accurately towards to correct conclusion.

10^3 ; 3^{10}

Write your explanation here.

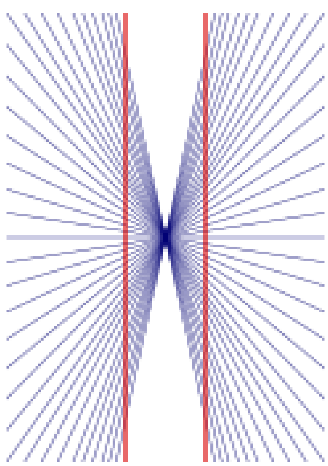
____
____
____

2. Which among of the following has a largest product?

34 x 12 = ; 21 x 43 = ; 54 x 31=

____
____
____

3. Look at the figure below. Are two lines a straight line? . What is your intuition?



____
____
____

B. Proof and Certainty

Another equally important lesson that the student should be learned is on how to deal with mathematical proof and certainty. By definition, a **proof** is an inferential argument for a mathematical statement while proofs are an example of mathematical logical **certainty**.

A mathematical proof is a list of statements in which every statement is one of the following:

(1) an axiom

(2) derived from previous statements by a rule of inference

(3) a previously derived theorem

There is a hierarchy of terminology that gives opinions about the importance of derived truths:

(1) A proposition is a theorem of lesser generality or of lesser importance.

(2) A lemma is a theorem whose importance is mainly as a key step in something deemed to be of greater significance.

(3) A corollary is a consequence of a theorem, usually one whose proof is much easier than that of the theorem itself.

METHODS OF PROOF

In methods of proof, basically we need or we have to prove an existing mathematical theorem to be able to determine if this theorem is true or false.

In addition, there is no need to prove any mathematical definition simply because we assumed that this is already true or this is basically true.

Usually, a theorem is in the form of if-then statement. So, in a certain theorem, it consists of hypothesis and conclusion.

Let us say P and Q are two propositions. In an if-then statement, proposition P would be the hypothesis while the proposition Q would be our conclusion denoted by:

$$P \rightarrow Q$$

Example:

If a triangle is a right triangle with sides a, b, and c as hypotenuse, then $a^2 + b^2 = c^2$.

There are two ways on how to present a proof. One is with the use of an outline form and the other one is in a paragraph form. Either of the two presentations could be used by the student.

TWO WAYS ON HOW TO PRESENT THE PROOF

a. Outline Form

Proposition: If P then Q.

1. Suppose/Assume P
2. Statement
3. Statement

.

.

. Statement

Therefore Q . €

b. Paragraph Form

Proposition: If P then Q.

Assume/Suppose P.______________._____________.________________________. ______________ . . . _______________. _________________. ___________________.
Therefore Q. €

Let us have a very simple and basic example on how to prove a certain mathematical statement.

Illustration 1: Prove (in outline form) that "If x is a number with 5x + 3 = 33, then x = 6"

Proof:

1. Assume that x is a number with 5x + 3 = 33.

2. Adding -3 both sides of an equation will not affect the equality of the two members on an equation, thus 5x + 3 – 3 = 33 – 3

3. Simplifying both sides, we got 5x = 30.

4. Now, dividing both member of the equation by 5 will not be affected the equality so $\frac{5x}{5} = \frac{30}{5}$.

5. Working the equation algebraically, it shows that x =

6. Therefore, if 5x + 3 = 33, then x = 6. €

Let us have a very simple and basic example on how to prove a certain mathematical statement in paragraph form.

Illustration 2: Prove (in paragraph form) that "If x is a number with 5x + 3 = 33, then x = 6"

Proof:

If 5x + 3 = 33, then 5x + 3 − 3 = 33 − 3 since subtracting the same number from two equal quantities gives equal results. 5x + 3 − 3 = 5x because adding 3 to 5x and then subtracting 3 just leaves 5x, and also, 33 − 3 = 30. Hence 5x = 30. That is, x is a number which when multiplied by 5 equals 30. The only number with this property is 6.

Therefore, if 5x + 3 = 33 then x = 6. €

Note: It is up to the student which of the two forms would be their preferred presentation.

KINDS OF PROOF

1. **DIRECT PROOF**

DEFINITION. A direct proof is a mathematical argument that uses rules of inference to derive the conclusion from the premises.

In a direct proof, let us say we need to prove a given theorem in a form of $P \rightarrow Q$. The steps in taking a direct proof would be:

1. Assume P is true.

2. Conclusion is true.

Example 1: Prove that if x is an even integer, then $x^2 - 6x + 5$ is odd.

Proof: (by outline form)

1. Assume that x is an even integer.
2. By definition of an even integer, $x = 2a$ for some $a \in Z$.
3. So, $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 4a^2 - 12a + 4 + 1 = 2(2a^2 - 6a + 2) + 1$ where $2a^2 - 6a + 2 \in k$.
4. Therefore, $2(2a^2 - 6a + 2) + 1 = 2k + 1$, so $x^2 - 6x + 5$ is odd. €

Example 2: With the use of direct proving, prove the following in both form (outline and paragraph).

Prove: (in an outline form)

If a and b are both odd integers, then the sum of a and b is an even integer.

Proof:

1. Assume that a and b are both odd integers.
2. There exists an integer k_1 and k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$ (by definition of an odd number).
3. Now, $a + b = (2k_1 + 1) + (2k_2 + 1) = 2k_1 + 2k_2 + 2$. Factoring 2, it follows that $a + b = 2(k_1 + k_2 + 1)$.
4. So; $a + b = 2(k_1 + k_2 + 1)$. Let $k_1 + k_2 + 1 = k \in Z$, hence $a + b = 2k$.
5. Therefore, if a and b are both odd integer, then a + b is even. €

Prove: (in paragraph form)

Assume that a and b are both odd integers. By definition of an odd number, there exists an integer k_1 and k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Now, adding a and b, that is, $a + b = (2k_1 + 1) + (2k_2 + 1) = 2k_1 + 2k_2 + 2$. Factoring 2, it follows that $a + b = 2(k_1 + k_2 + 1)$. So; $a + b = 2(k_1 + k_2 + 1)$ and let $k_1 + k_2 + 1 = k$

$\in Z$, hence $a + b = 2k$. Therefore, if a and b are both odd integers, their sum is always and even integer. €

Example 3: With the use of direct proving, prove the following in paragraph form.

Prove:

If x and y are two odd integers, then the product of x and y is also an odd integer.

Proof:

Assume that x and y are two different odd integers. There exists k_1 and $k_2 \in Z$ such that $x = 2k_1 + 1$ and $y = 2k_2 + 1$ by definition of an odd number. Now, taking the product of x and y, we got $xy = (2k_1 + 1)(2k_2 + 1) = 4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$. Let $2k_1k_2 + k_1 + k_2 = k \in Z$. Hence $(2k_1+1)(2k_2+1) = 2k + 1$. Therefore, $xy = 2k + 1$ where the product of two odd integers is also an odd integer. €

Example 4. Prove the proposition (in outline form) that is " **If x is an positive integer, then x^2 is also an odd integer**".

Prove: (In outline form)

1. Suppose x is odd.
2. Then by definition of an odd integer, $x = 2a + 1$ for some $a \in Z$.
3. Thus $x^2 = (2a + 1)^2 = 4a^2 + 4a +1 = 2(2a^2 + a) + 1$.
4. So $x^2 = 2b+1$ where b is the integer $b = 2a^2 + 2a$.
5. Thus $x^2 = 2b + 1$ for an integer b.
6. Therefore x^2 is odd, by definition of an odd number. €

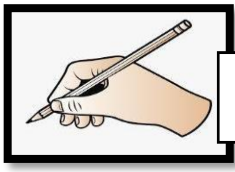
Example 5. Prove: Let a,b and c be integers. If a| b and b | c, then a | c.

Proof (in outline form)

1. Suppose a, b and c are integers and a | b and b | c.
2. We all know that if a | b, there is a certain integer say d which is b = ad.
3. Similarly, when b | c, there is an integer say e which is c = be.
4. Now, since b = ad, substitute the value of b in c = be, it follows that c = (ad)e = a(de).
5. So, c = a(de) = ax for x = de $\in Z$.

6. Therefore $a \mid c$. €

Now, it’s your turn to do some direct proving. You can use any of the two forms of presentation for proving.



Self- Learning Activity

Direction: Prove the following propositions with the use of direct proving. Show your answer on the space provided after each item. **(5 marks each)**

1. If a is an odd integer, then $a^2 + 3a + 5$ is odd.

2. Suppose $x, y \in Z$. If x^3 and y^3 are odd, then $(xy)^3$ is odd.

3. Suppose $x, y \in Z$. If x is even, then xy is even.

4. If $n - m$ is even, then $n^2 - m^2$ is also an even.

5. If x is odd positive integer then $x^2 - 1$ is divisible by 4.

6. If x is an odd integer, then 8 is a factor of $x^2 - 1$.

2. INDIRECT PROOF (CONTRAPOSITIVE PROOF) DEFINITION:

Indirect proof or contrapositive proof is a type of proof in which a statement to be proved is assumed false and if the assumption leads to an impossibility, then the statement assumed false has been proved to be true.

Recall that the proposition p →q is a conditional statement. This proposition is logically equivalent to ~q →~p. Now, the expression ~q →~p is the contrapositive form of the statement p →q.

In an indirect proof, let us say we need to prove a given theorem in a form of P → Q. The steps or outline in taking an indirect proof would be:

Assume/Suppose ~Q is true.

.

.

.

Therefore ~P is true. €

Example 1. Using indirect/contrapositive proof, prove that "**If x is divisible by 6, then x is divisible by 3**".

Here in example 1, we let that p : x is divisible by 6 and q : x is divisible by 3. So, this original statement to become a contrapositive could be transformed into "If x is not divisible by 3, then x is not divisible by 6".

Note that, we let p: x is divisible by 6 and q: x is divisible by 3. With the use of indirect proof, we assume that ~q is true and the conclusion ~p is also true.

So, the formal proof would be;

Proof:
1. Assume x is not divisible by 3.
2. Then $x \neq 3k$ for all $k \in Z$
3. It follows that $x \neq (2m)(3)$ for all $m \in Z$
4. So, $x \neq 6m$ for all $m \in Z$
5. Therefore, x is not divisible by 6. €

Example 2: Prove using indirect proof or contraposition.

Let x be an integer. Prove that, if x^2 is even, then x is even.

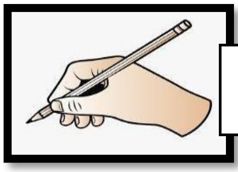
Note that, we let p: x^2 is even and q: x is even. With the use of indirect proof, we assume that ~q is true and the conclusion ~p is also true.

So, the original statement would become "If x is odd, then x^2 is odd".

Now, the formal proof would be;

Proof:
1. Assume x is odd.
2. Then $x = 2k + 1$ for some $k \in Z$
3. It follows that $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ where $q = 2k^2 + 2k$
4. So, $x^2 = 2q + 1$
5. Therefore, x^2 is odd. €

Again, it is your turn to prove the following propositions with the use of indirect or contrapositive proof.



Self- Learning Activity

Direction: Prove the following propositions with the use of indirect proving. Show your answer on the space provided after each item.

1. If a is an even integer and b is odd integer, then a + b is odd integer.

2. If n is an odd integer, then $n^3 + 2n^2$ is also an odd integer.

3. If n – m is even, then $n^2 - m^2$ is also an even.

4. If x is odd positive integer then $x^2 - 1$ is divisible by 4.

5. If x is an odd integer, then 8 is a factor of $x^2 - 1$.

6. Suppose $x, y \in Z$. If x is even, then xy is even.

3. Proof by Counter Example (Disproving Universal Statements)

A conjecture may be described as a statement that we hope is a theorem. As we know, many theorems (hence many conjectures) are universally quantified statements. Thus it seems reasonable to begin our discussion by investigating how to disprove a universally quantified statement such as

$$\forall x \in S, P(x).$$

To disprove this statement, we must prove its negation. Its negation is

$$\sim (\forall x \in S, P(x)) = \exists\, x \in S, \sim P(x).$$

Things are even simpler if we want to disprove a conditional statement $P(x) \Rightarrow Q(x)$. This statement asserts that for every x that makes P(x) true, Q(x) will also be true. The statement can only be false if there is an x that makes P(x) true and Q(x) false. This leads to our next outline for disproof.

The question is "How to disprove $P(x) \Rightarrow Q(x)$"? The answer is simple. Produce an example of an x that makes P(x) true and Q(x) false.

In both of the above outlines, the statement is disproved simply by exhibiting an example that shows the statement is not always true. (Think of it as an example that proves the statement is a promise that can be broken.) There is a special name for an example that disproves a statement: It is called a **counterexample**.

Example 1. Prove or disprove: All prime numbers are odd.
*Negation : Some prime numbers are even.

By counterexample: Let n = 2. By definition of a prime, $2 = (2)\cdot(1)$. But 2 is even where the only factor of 2 is 2 and 1 so we could say that 2 is a prime number.

Since we have found an even prime number so the original statement is not true. €

Example 2. Prove or disprove: For all integers x and y, if x + y is even, then both x and y are even.

*Negation : For some integers x and y,if x + y is odd, then x and y is odd.

Proof: $\forall(x)\forall(y)$: $x + y = 2k_1 + 2k_2$, for $k \in Z$

$$\exists x\, \exists y: x + y = 2k_1 + 1 + 2k_2 + 1$$
$$= 2k_1 + 2k_2 + 2$$
$$= 2(k_1 + k_2 + 1)$$

By giving a counterexample, if x = 1 and y = 1, then x + y = 2.
But x and y are both odd, therefore the theorem is false. €

Example 3. Prove that "For every $n \in Z$, the integer $f(n) = n^2 - n + 1$ is prime.

Note that the negation of this would be "For some $n \in Z$, the integer $f(n) = n^2 - n + 1$ is composite.

We all know that a prime number is a number whose factors are 1 and the number itself, thus if p is prime number then p = (p)(1) where $p \in Z$

To be able to resolve the truth or falsity of the above statement, let us construct a table for f(n) for some integers n. If we could find at least one number for f(n) which is not prime (composite), then we could conclude that the statement is false.

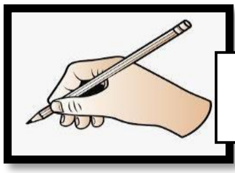
n	0	1	2	3	4	5	6	7	8	9	10	11
f(n)	11	11	13	17	23	31	41	53	67	83	101	?

In every case, f (n) is prime, so you may begin to suspect that the conjecture is true. Before attempting a proof, let's try one more n. Unfortunately, f (11) = $11^2 - 11 + 11 = 11^2$ is not prime. The conjecture is false because n = 11 is a counterexample.

We summarize our disproof as follows:

Disproof. The statement "For every $n \in Z$, the integer $f(n) = n^2 - n + 11$ is prime," is false. For a counterexample, note that for $n = 11$, the integer f

(11) = 121 = 11·11 is not prime. €



Self- Learning Activity

Direction: Prove the following propositions with the use of counter-example. Show your answer on the space provided after each item.

1. Prove: For all integer n which is a multiples of 3 are multiples of 6.

2. Prove: For all real numbers a and b, if $a^2 = b^2$, then a = b.

3. Prove: For all positive integers n, $n^2 - n + 41$ is prime.

4. Prove: For all positive integers n, $2^{2n} + 1$ is prime.

5. Prove: For all real number n, $n^2 + 4 < 5$.

4. Proof by Contradiction

Another method of proving is what we called "Proving by Contradiction". This method works by assuming your implication is not true, then deriving a contradiction.

Recall that if p is false then p → q is always true, thus the only way our implication can be false is if p is true and q is false.

So, if we let p → q be a theorem, a proof by contradiction is given by this way;

1. Assume p is true.
2. Suppose that ~q is also true.
3. Try to arrive at a contradiction.
4. Therefore q is true

So, in practice then, we assume our premise is true but our conclusion is false and use these assumptions to derive a contradiction.

This contradiction may be a violation of a law or a previously established result. Having derived the contradiction you can then conclude that your assumption (that p → q is false) was false and so the implication is true.

Be careful with this method: make sure that the contradiction arise because of your original assumptions, not because of a mistake in method. Also, if you end up proving ~p then you could have used proof by contraposition.

Example 1: Prove by contradiction that "If x + x = x, then x = 0.

Proof:

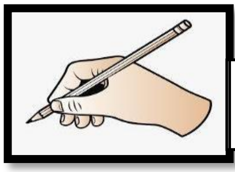
1. Assume that x + x = x.
2. Suppose that x ≠ 0.
3. Now, x + x = x, so 2x = x and since x ≠ 0, we could multiply both sides of the equation by the reciprocal of x, i.e., 1/x.
4. Multiplying by the reciprocal of x, it follows that 2 = 1 which is a contradiction.

5. Therefore, the original implication is proven to be true. €

Example 2: Prove by contradiction that "If x is even then x + 3 is odd.

Proof:

1. Assume x is even, so x = 2k.
2. Suppose x + 3 is even. Since x + 3 is even, there exist k ∈Z such that x + 3 = 2k.
3. It follows that x = 2k -3. We can rewrite this as x = 2k – 4 + 1. Now, x = 2(k – 2) + 1. Let k – 2 = q. So, x = 2q + 1. It is clearly seen that x is an odd number. This is a contradiction to the assumption.
4. Therefore, x + 3 is odd. €



Self- Learning Activity

Direction: Prove the following propositions with the use of contradiction. Show your answer on the space provided after each item.

1. There are no natural number solutions to the equation $x^2 - y^2 = 1$.

2. For all integers n, if $n^3 + 5$ is odd then n is even.

3. If x^2 is irrational then x is irrational.

Lesson 3.3

Polya's Four Steps in Problem Solving

Specific Objective

At the end of this lesson, the student should be able to:

1. Tell all the Polya's four steps in problem solving.
2. Select the appropriate strategy to solve the problem.
3. Solve problems with the use of Polya's four step.

<u>Introduction</u>

One of the major problems of a student in mathematics is on how to solve worded problems correctly and accurately. Sometimes, they have difficulty understanding in grasping the main idea of a problem on how to deal with it and to solve it. It is very important that there is always a clear understanding on how to solve problems most especially in a Mathematics as a course.

When you were in your senior high school, your teacher in mathematics especially in the course of Algebra taught you on how to solve problem using scientific method. Some of these problems are number problem, age problem, coin problem, work problem, mixture problem, etc. But not all problems in mathematics could be solve on what you have learned in your senior high school.

Here, in this "Polya's Four Steps in Solving Problem", we will be learning on how to solve mathematical problem in a different way.

<u>Discussion</u>

Maybe you would ask yourself that who is Polya? Why do we need to use his four steps in solving a mathematical problem? How are we going to use this to be able to solve problems? The answer for these question will be answered in this lesson.

George Polya is one of the foremost recent mathematicians to make a study of problem solving. He was born in Hungary and moved to the United States in 1940. He is also known as "The Father of Problem Solving".

He made fundamental contributions to combinatorics, number theory, numerical analysis and probability theory. He is also noted for his work in heuristics and mathematics education. Heuristic, a Greek word means that "find" or "discover" refers to experience-based techniques for problem solving, learning, and discovery that gives a solution which is not guaranteed to be optimal.

The George Polya's Problem-Solving Method are as follows:

Step 1. Understand the Problem.

This part of Polya's four-step strategy is often overlooked. You must have a clear understanding of the problem. To help you focus on understanding the problem, consider the following questions. These are some questions that you may be asked to yourself before you solve the problem.

a. Are all words in a problem really understand and clear by the reader?
b. Do the reader really know what is being asked in a problem on how to find the exact answer?
c. Can a reader rephrase the problem by their own without deviating to its meaning?
d. If necessary, do the reader can really visualize the real picture of the problem by drawing the diagram?
e. Are the information in the problem complete or is there any missing information in a problem that could impossible to solve the problem?

Step 2. Devise a Plan

Sometimes, it is necessary for us that to be able to solve a problem in mathematics, we need to devise a plan. Just like a Civil Engineer that before he construct a building, he needs to do a floor plan for a building that he wants to build. To be able to succeed to solve a problem, you could use different techniques or way in order to get a positive result. Here are some techniques that could be used. You could one of these or a combination to be able to solve the problem.

a. As much as possible, list down or identify all important information in the problem
b. Sometimes, to be able to solve problem easily, you need to draw figures or diagram and tables or charts.
c. Organized all information that are very essential to solve a problem.
d. You could work backwards so that you could get the main idea of the problem.
e. Look for a pattern and try to solve a similar but simpler problem.
f. Create a working equation that determines the given (constant) and variable.
g. You could use the experiment method and sometimes guessing is okay.

Step 3. Carry out the plan

After we devised a plan, the next question is "How are we going to carry out the plan?" Now, to be able to carry out the plan, the following suggestions could help us in order to solve a problem.

a. Carefully and accurately working on the problem.
b. There must be a clear and essential information or data in the problem.
c. If the first plan did not materialize, make another plan. Do not afraid to make mistakes if the first plan that you do would not materialize. There is a saying that "There is a second chance."

Step 4. Look back or Review the Solution

Just like on what you do in solving worded problems in Algebra, you should always check if your answer is correct or not. You need to review the solution that you have made. How will you check your solution? The following could be your guide.

a. Make it sure that your solution is very accurate and it jibed all important details of the problem.
b. Interpret the solution in the context of the problem.
c. Try to ask yourself that the solutions you've made could also be used in other problems.

As it was mentioned in this lesson, there are different strategies that you could employ or use to solve a problem. These strategies will help you to solve the problem easily. These are the following strategies that you could be used:

1. Draw a picture, diagram, table or charts. Label these with correct information or data that you could see in the problems. Sometimes, there are hidden information that is very much important also to solve the problem. So, be cautious.
2. Identify the known and unknown quantities. Choose appropriate variable in identifying unknown quantities. For example, the unknown quantity is height. You could use "h" as your variable.
3. You have to be systematic.
4. Just like on what we have in devising a plan, look for a pattern and try to solve a similar but simpler problem.
5. Sometimes, guessing is okay. There is no problem in guessing and it is not a bad idea to be able to begin in solving a problem. In guessing, you could examine how closed is your guess based on the given problem.

Illustrative examples will be solved with the use of Polya's four step method.

1. The sum of three consecutive positive integers is 165. What are these three numbers?

Step 1. Understand the problem

When we say consecutive numbers, these are like succeeding numbers. Say, 4, 5, 6 are three consecutive numbers for single-digit numbers. For the two-digit number, example of these three consecutive is 32, 33, and 34. Noticing that the second number added by 1 from the first number and the third number is increased by 2 from the first number.

Step 2. Devise a plan

From the previous discussion of this lesson, devising a plan is very essential to solve a problem. We could use an appropriate plan for this kind of problem and that is formulating a working equation. Since we do not know what are these three consecutive positive integers, we will be using a variable, say x to represent a particular number. This variable x could be the first number. Now, since it is consecutive, the second number will be increased by 1. So, the possible presentation would be x + 1. The third number was increased by 2 from the first number so the possible presentation would be x + 2. Since, based on the problem that the sum of these three consecutive positive integers is 165, the working equation is:

$$(x) + (x + 1) + (x + 2) = 165$$

where x be the first positive integer, x + 1 be the second number and x + 2 be the third number.

Step 3. Carry out the plan

We already know the working formula. To be able to determine the three positive consecutive integers, we will be using the concept of Algebra here in order to solve the problem. Manipulating algebraically the given equation;

$$x + x + 1 + x + 2 = 165$$

Combining similar terms;

$$3x + 3 = 165$$

Transposing 3 to the right side of the equation;

$$3x = 165 - 3$$

Simplifying;

$$3x = 162$$

Dividing both side by 3 to determine the value of x;

$$x = 54$$

and this would be the first number. Now, the second number is x + 1 and we already know the value of x = 54. So, the next number is 55. Then the third number would be x + 2 and again we know that x = 54 so the third number is 56. Hence, the three positive consecutive integers whose sum is 165 are 54, 55 and 56.

Step 4. Look back and review the solution

We need to review our solution to check if the answer is correct. How are we going to do that? Just simply add the identified three consecutive positive integers and the result should be 165. So, adding these three numbers, 54 + 55 + 56 will give us a sum of 165.

2. There are ten students in a room. If they give a handshake for his classmate once and only once, how many handshakes can be made?

Step 1. Understand the problem

Let us say that you are one of those ten students in a room. To know how many handshakes could be made with of those ten students, we need name those ten students as A, B, C. …, I, and J. If you would be the A student you can give a handshake to different nine students. Now, if you give your hand to B, i.e., A to B, it is can be said the giving handshake by B to A is just only one count. Obviously, you cannot make a handshake to yourself as well as B to himself and so on.

Step 2. Devising a plan

To determine the total number of handshakes, it is very easy if we are going to create a table. Say,

	A	B	C	D	E	F	G	H	I	J
A										
B										
C										
D										
E										
F										
G										
H										
I										
J										

The x symbol represent that you cannot make a handshake to yourself and √ symbol meaning that a handshake was made.

Step 3. Carry out the plan

	A	B	C	D	E	F	G	H	I	J
A	x	√	√	√	√	√	√	√	√	√
B	x	x	√	√	√	√	√	√	√	√
C	x	x	x	√	√	√	√	√	√	√
D	x	x	x	x	√	√	√	√	√	√
E	x	x	x	x	x	√	√	√	√	√
F	x	x	x	x	x	x	√	√	√	√
G	x	x	x	x	x	x	x	√	√	√
H	x	x	x	x	x	x	x	x	√	√
I	x	x	x	x	x	x	x	x	x	√
J	x	x	x	x	x	x	x	x	x	x
Total		1	2	3	4	5	6	7	8	9

So, adding this 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45. Hence, there are a total of 45 handshakes if these ten students give a handshake for his classmate once and only once.

Step 4. Look back and review the solution

Trying to double check the diagram, it is clearly seen that the total number of handshakes that could be made which is 45 is correct.

3. Five different points, say A, B, C, D, and E are on a plane where no three points are collinear. How many lines can be produced in these five points?

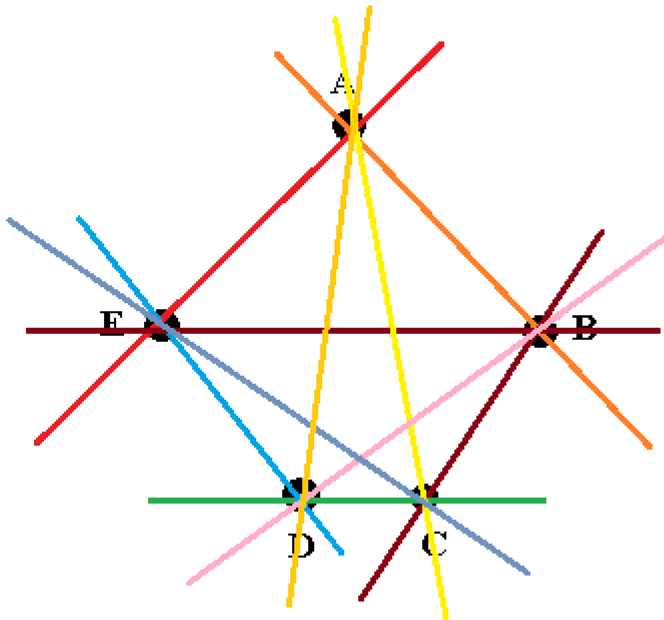
Step 1. Understand the problem

Based on the given problem, five points are on a plane where no three points are collinear. If you want to determine the number of lines from these five points, remember that the minimum number of points to produce a line is two. So, two points determine a line.

Step 2. Device a plan

The best way in order to determine the number of lines that can produce in a five different points where no three points are collinear is to plot those five points in a plane and label it as A, B, C, D, and E. Then list down all the possible lines. Take note that line AB is as the same as line BA.

Step 3. Carry out the plan

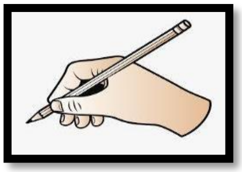


List:

AB	BC	CD	DE
AC	BD	CE	
AD	BE		
AE			

Step 4. Look back and review the solution

Counting all possible lines connected in five different points where no three points are collinear is still we could check that the total number of line could be produced of these five points are ten lines.



Learning Activity

Directions: Solve the following problems with the use of Polya's four-step problem solving procedures similar on the presentation on this topic.

1. The sum of three consecutive odd integers is 27. Find the three integers.
2. If the perimeter of a tennis court is 228 feet and the length is 6 feet longer than twice the width, then what are the length and the width?
3. There are 364 first-grade students in Park Elementary School. If there are 26 more girls than boys, how many girls are there?
4. If two ladders are placed end to end, their combined height is 31.5 feet. One ladder is 6.5 feet shorter than the other ladder. What are the heights of the two ladders?
5. A shirt and a tie together cost $50. The shirt costs $30 more than the tie. What is the cost of the shirt?

Mathematical Problems Involving Patterns

Specific Objective

At the end of this lesson, the student should be able to:

1. Demonstrate appreciation in solving problems involving patterns.
2. Show the appropriate strategies in solving problems which involve patterns.
3. Apply the Polya's 4-step rule method in solving problems with patterns.
4. Make a correct conclusion based on their final result.

Introduction

There are some problems that patterns may involve. One of the examples of problems that patterns are involve is an “abstract reasoning” where this kind of pattern is one of the type of exam that most of the Universities used in their entrance examination.

Discussion

Solving problems which involve pattern do not follow the steps on how to solve the problem on its traditional way. To be able to solve for this kind of problem, the following may be used as a guide:

(i) showing an understanding of the problem,

(ii) organising information systematically,

(iii) describing and explaining the methods used and the results obtained,

(iv) formulating a generalisation or rule, in words or algebraically.

The following sample of questions gives an indication of the variety likely to occur in the examination.

1. A group of businessmen were at a networking meeting. Each businessman exchanged his business card with every other businessman who was present.

a) If there were 16 businessmen, how many business cards were exchanged?

b) If there was a total of 380 business cards exchanged, how many businessmen were at the meeting?

Solution:

a) $15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 120$ exchanges $120 \times 2 = 240$ business cards. If there were 16 businessmen, 240 business cards were exchanged.

b) $380 \div 2 = 190$

$$190 = (19 \times 20) \div 2 = 19 + 18 + 17 + \ldots + 3 + 2 + 1$$

If there was a total of 380 business cards exchanged, there were 20 businessmen at the meeting.

2. Josie takes up jogging. On the first week she jogs for 10 minutes per day, on the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days each week, what will be her total jogging time on the sixth week?

Solution:

Understand

We know in the first week Josie jogs 10 minutes per day for six days.
We know in the second week Josie jogs 12 minutes per day for six days.
Each week, she increases her jogging time by 2 minutes per day and she jogs 6 days per week.
We want to find her total jogging time in week six.

Strategy

A good strategy is to list the data we have been given in a table and use the information we have been given to find new information.

We are told that Josie jogs 10 minutes per day for six days in the first week and 12 minutes per day for six days in the second week. We can enter this information in a table:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72

You are told that each week Josie increases her jogging time by 2 minutes per day and jogs 6 times per week. We can use this information to continue filling in the table until we get to week six.

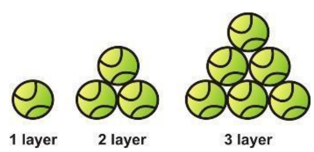
Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72
3	14	84
4	16	96
5	18	108
6	20	120

Apply strategy/solve

To get the answer we read the entry for week six.

Answer: In week six Josie jogs a total of 120 minutes.

3. You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 rows?



Solution

Understand

We know that we arrange tennis balls in triangles as shown.

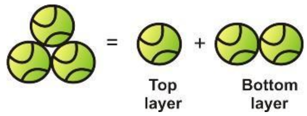
We want to know how many balls there are in a triangle that has 8 rows.

Strategy

A good strategy is to make a table and list how many balls are in triangles of different rows.

One row: It is simple to see that a triangle with one row has only one ball.

Two rows: For a triangle with two rows, we add the balls from the top row to the balls from the bottom row. It is useful to make a sketch of the separate rows in the triangle.



3=1+2

Three rows: We add the balls from the top triangle to the balls from the bottom row.

6=3+3

Now we can fill in the first three rows of a table.

Number of Rows	Number of Balls
1	1
2	3
3	6

We can see a pattern. To create the next triangle, we add a new bottom row to the existing triangle. The new bottom row has the same number of balls as there are rows. (For example, a triangle with 3 rows has 3 balls in the bottom row.) To get the total number of balls for the new triangle, we add the number of balls in the old triangle to the number of balls in the new bottom row.

Apply strategy/solve:

We can complete the table by following the pattern we discovered. Number of balls = number of balls in previous triangle + number of rows in the new triangle

Number of Rows	Number of Balls
1	1
2	3
3	6
4	6+4=10
5	10+5=15
6	15+6=21
7	21+7=28
8	28+8=36

Answer There are 36 balls in a triangle arrangement with 8 rows.

Check

Each row of the triangle has one more ball than the previous one. In a triangle with 8 rows, row 1 has 1 ball, row 2 has 2 balls, row 3 has 3 balls, row 4 has 4 balls, row 5 has 5 balls, row 6 has 6 balls, row 7 has 7 balls, row 8 has 8 balls.

When we add these we get: 1+2+3+4+5+6+7+8=36 balls

4. Andrew cashes a $180 check and wants the money in $10 and $20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?

Solution
Method 1: Making a Table
Understand
Andrew gives the bank teller a $180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table of the different ways Andrew can have twelve bills in tens and twenties.

Andrew could have twelve \$10 bills and zero \$20 bills, or eleven \$10 bills and one \$20 bill, and so on.

We can calculate the total amount of money for each case.

Apply strategy/solve

\$10 bills	\$ 20 bills	Total amount
12	0	\$10(12)+\$20(0)=\$120
11	1	\$10(11)+\$20(1)=\$130
10	2	\$10(10)+\$20(2)=\$140
9	3	\$10(9)+\$20(3)=\$150
8	4	\$10(8)+\$20(4)=\$160
7	5	\$10(7)+\$20(5)=\$170
6	6	\$10(6)+\$20(6)=\$180
5	7	\$10(5)+\$20(7)=\$190
4	8	\$10(4)+\$20(8)=\$200
3	9	\$10(3)+\$20(9)=\$210
2	10	\$10(2)+\$20(10)=\$220
1	11	\$10(1)+\$20(11)=\$230
0	12	\$10(0)+\$20(12)=\$240

In the table we listed all the possible ways you can get twelve \$10 bills and \$20 bills and the total amount of money for each possibility. The correct amount is given when Andrew has six \$10 bills and six \$20 bills.

Answer: Andrew gets six \$10 bills and six \$20 bills.

Check

Six \$10 bills and six \$20 bills →6(\$10)+6(\$20)=\$60+\$120=\$180
The answer checks out.
Let's solve the same problem using the method "Look for a Pattern."

Method 2: Looking for a Pattern

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table just as we did above. However, this time we will look for patterns in the table that can be used to find the solution.

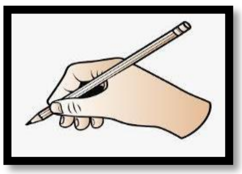
Apply strategy/solve

Let's fill in the rows of the table until we see a pattern.

\$10 bills	\$20 bills	Total amount
12	0	\$10(12)+\$20(0)=\$120
11	1	\$10(11)+\$20(1)=\$130
10	2	\$10(10)+\$20(2)=\$140

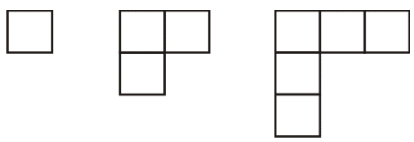
We see that every time we reduce the number of $10 bills by one and increase the number of $20 bills by one, the total amount increases by $10. The last entry in the table gives a total amount of $140, so we have $40 to go until we reach our goal. This means that we should reduce the number of $10 bills by four and increase the number of $20 bills by four. That would give us six $10 bills and six $20 bills.
6($10)+6($20)=$60+120=$180

Answer: Andrew gets six $10 bills and six $20 bills.



Learning Activity

1. A pattern of squares is put together as shown.



   How many squares are in the 12th diagram?

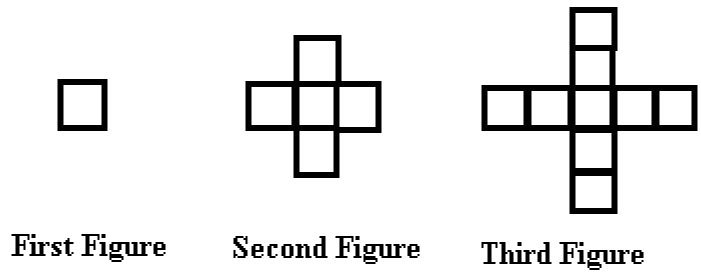
2. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, then cuts down to 21 cups the second week and 18 cups the third week, how many weeks will it take him to reach his goal?

3. A new theme park opens in Milford. On opening day, the park has 120 visitors; on each of the next three days, the park has 10 more visitors than the day before; and on each of the three days after that, the park has 20 more visitors than the day before.

   How many visitors does the park have on the seventh day?
   How many total visitors does the park have all week?

4. Mark is three years older than Janet, and the sum of their ages is 15. How old are Mark and Janet?

5. A pattern of squares is put together as shown.



How many squares are in the 10^{th} figure?

Specific Objective

At the end of this lesson, the student should be able to:

1. Demonstrate appreciation of recreational games using the concept of mathematics.
2. Show student's interest in a mathematical games by solving mathematical games.
3. Develop a sense of correct thinking to finish the game successfully.

Introduction

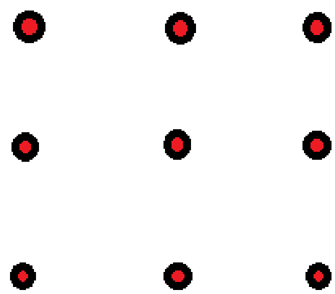
Puzzle, number games and mathematical riddles are some exciting games that we can solve or play. There are very essential most especially for the students in order to develop their critical thinking, enhance students' computational work, deepen understanding with numbers and use different strategies and style of techniques through recreational games. In this new modern day, there are a lot of games that you may encounter not only in social media but also in different internet site. Games are now easily downloaded and we can play this game on our palm hand. At the same time, person or persons that has high skills and knowledge in mathematics are those persons who could solve problems or games in mathematics easily.

Discussion

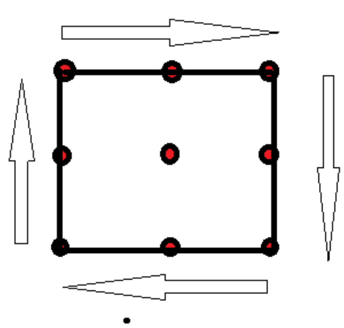
Now-a-days, there are a lot of recreational games that we could play whether it is an online or an offline games. Sometimes, we call this game as brain booster since our mind needs a lot of correct thinking on how to deal with the games and finish the game successfully.

In this lesson, there would be an illustrative example that the students may look into on how to deal with a games using mathematical concept.

1. With the use of pencil or pen, connect by means of a line the nine dots (see figure below) without lifting a pen and re-tracing the line.

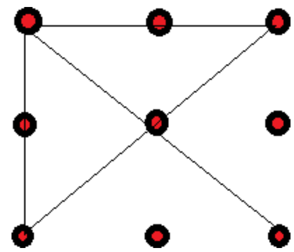


To begin, you may think that it is easy for you to connect these nine dots by means of a line with the use of your pen or pencil with lifting the pen and re-tracing the line. On your first attempt, it could be like this.

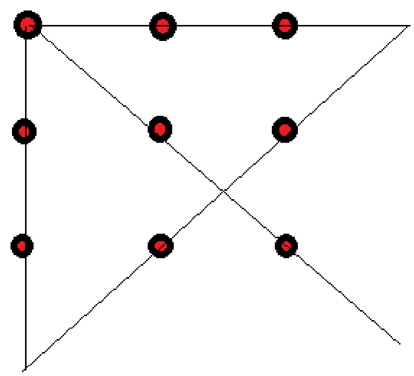


But as you can see, there is one dot left which is disconnected.

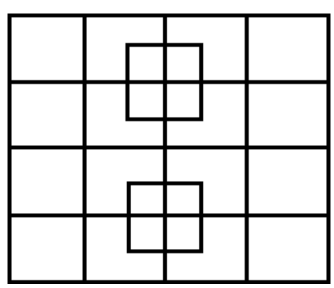
Trying for the second time, perhaps your presentation may looks like this.



But still, there are dots which are disconnected. Let us use the technique called "think outside the box strategy" and for sure we can solve the puzzle. With this, we could extend the line or lines to connect the dots just like this one below.

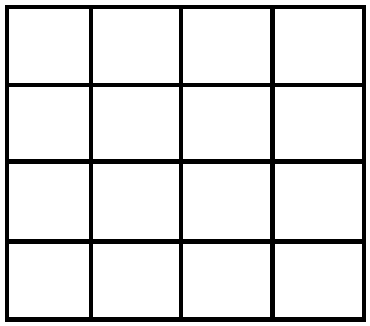


2. How many squares could you find at the picture below?



Some students would count manually the number of squares on the figure above. It is very tedious on the part of the student and it is prone to error. You cannot get the answer correctly at once if manually counting would be done. Not unless if you are very lucky to get the correct number of squares. But if we use the concept of mathematics here, you could be able to get accurately the total number of squares in the figure. Remember that a square has an equal sides. Let us say an "n by n" is a square.

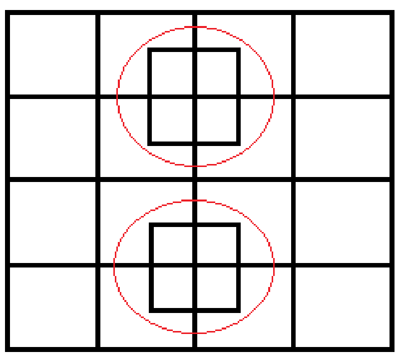
The question is, how many squares are really on the figure above? First, let us always think that the square has an equal side. Let us ignore first the squares within the big square.



On this figure, there are different "n x n" size of a square. The size of a big square is a 1 x 1 (1^2). But there is also a 2 x 2 (2^2) square on it as well as 3 x 3 and a 4 x 4 square. So, how many squares are there. If we look at the mathematical concept and we want to know the number of square, we need to add the different sizes of the squares such as;

$$1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30 \text{ squares}$$

Now, let us take a look on the squares within the big square.



If you want to know the number of squares are there, let us use the upper selected square. The number of squares on the upper part would be;

$$1^2 + 2^2 = 1 + 4 = 5 \text{ squares}$$

The same as the selected square below, that is;

$$1^2 + 2^2 = 1 + 4 = 5 \text{ squares}$$

Hence the total number of squares on the given figure is 30 + 5 + 5 = 40 squares.

3. A 3 x 3 grid table is given below. Filled out each cell of a digit from 1 – 9 except 5 since it is already given and without repetition where the sum of horizontal, vertical and diagonal are all equal to 15.

	5	

There are several ways to present the 3 x 3 grid table magic square number. First thing that you're going to do is just to add all digits from 1 to 9 giving a sum of 45. In a 3 x 3 square number, you have to add three numbers again and again hence it will give an average that the sum of three number is 15, i.e. 45/3 = 15. This number 15 is what we called the magic number of a 3 x 3 square number where when you add three numbers horizontally, vertically and diagonally will give us a sum of 15. To achieve this, the number 5 should be placed in the middle part of a 3 x 3 square number just like in the given figure above. You can also achieve 15, if you add the middle number 5 three times.

You can reduce 15 in a sum of three summands eight times:

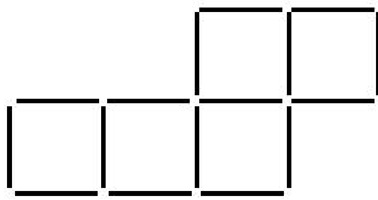
15=1+5+9	15=2+4+9	15=2+6+7	15=3+5+7
15=1+6+8	15=2+5+8	15=3+4+8	15=4+5+6

The odd numbers 1,3,7, and 9 occur twice in the reductions, the even numbers 2,4,6,8 three times and the number 5 once. Therefore you have to place number 5 in the middle of the magic 3x3 square. The remaining odd numbers have to be in the middles of a side and the even numbers at the corners. Under these circumstances there are eight possibilities building a square and two of these are presented below.

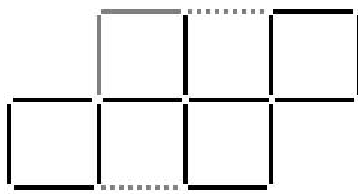
2	9	4
7	5	3
6	1	8

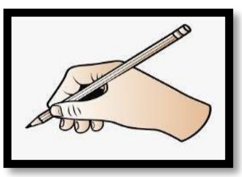
2	7	6
9	5	1
4	3	8

4. The figure below is arranged using 16 matchsticks to form 5 squares. Rearrange exactly 2 of the matchsticks to form 4 squares of the same size, without leaving any stray matchsticks.



Solution:





Learning Activity

Directions: Do as indicated.

1. Place 10 coins in five straight lines so that each line contains exactly four coins. Can you arrange four coins so that if you choose any three of them (i.e., no matter which three of the four you pick), the three coins you chose form the corners of an equilateral triangle?

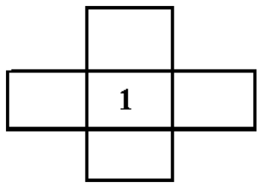
2. If you add the square of Tom's age to the age of Mary, the sum is 62; but if you add the square of Mary's age to the age of Tom, the result is 176. Can you say what the ages of Tom and Mary are?

3. A man and his wife had three children, John, Ben, and Mary, and the difference between their parents' ages was the same as between John and Ben and between Ben and Mary. The ages of John and Ben, multiplied together, equalled the age of the father, and the ages of Ben and Mary multiplied together equalled the age of the mother. The combined ages of the family amounted to ninety years. What was the age of each person?

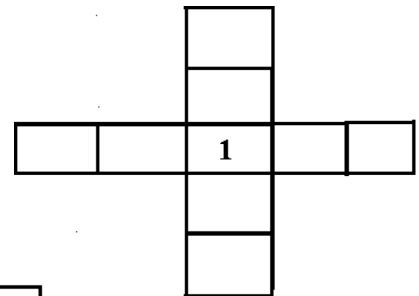
4. Refer to example number 3. Look for another remaining six (6) 3 x 3 magic square number.

5. Complete the grid table. Filled out the table (a) by digit from 1 to 4, (b) by digit from 1 to 9 and (c) by digit from 1 to 12; some digits are already identified whose sum is 8, 23 and 25 respectively.

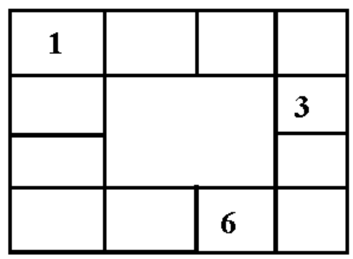
a.



b.



c.



Chapter Test 3

Test 1. TRUE OR FALSE

Directions: Read the following statement carefully. Write T if the statement is true, otherwise write F on the space provided before each item.

_______1. Deductive reasoning uses a set of specific observations to reach an overarching conclusion or it is the process of recognizing or observing patterns and drawing a conclusion.

_______2. The conclusion formed by using inductive reasoning is often called a conjecture.

_______3. Inductive reasoning is the process of reaching conclusion by applying general assumptions, procedure or principle or it is a process of reasoning logically from given statement to a conclusion.

_______4. Conjecture is a form of deductive reasoning where you arrive at a specific conclusion by examining two other premises or ideas.

_______5. In deductive reasoning, the two premises are the major and the minor premises and these are called an argument also known as syllogism.

______6. A categorical syllogisms follow the statement that "If A is part of C, then B is part of C".

_____7. Intuition is an immediate understanding or knowing something without reasoning.

_____8 A certainty is an inferential argument for a mathematical statement while proofs are an example of mathematical logical proof.

____9. An indirect proof is also known as contrapositive proof.

____10. Greg Polya is known as the “Father of Problem Solving”.

Test 2. **MULTIPLE CHOICE**

Directions: Read the following statement carefully. Choose only the letter of the best correct answer. Write your answer on the space provided before each item.

______11. It is the process of reaching a general conclusion by examining specific examples.

a. Deductive reasoning c. inductive reasoning
b. Conjecture d. syllogism

______12. It is the kind of reasoning in which, roughly, the truth of the premise logically guarantees the truth of the conclusion, provided that no mistake has been made in the reasoning.

a. Deductive reasoning c. inductive reasoning
b. Conjecture d. syllogism

______13. It is a form of deductive reasoning where you arrive at a specific conclusion by examining two other premises or ideas.

a. Deductive reasoning c. inductive reasoning
b. Conjecture d. syllogism

______14. Given a syllogism "All men are rational being. Anton is a man. Therefore, Anton is a rational being". Which of the following is the minor premise?

a. All men are rational being. c. Anton is a rational being.
b. Anton is a man. d. Cannot be determined

______15. How many rectangles can you find in a 2 x 2 grid of squares?

a. 4 c. 5
b. 9 d. 8

______16. Analyze the given question. How many squares can you find in a 5 x 5 grid of squares?

a. 55 c. 25
b. 52 d. 100

________17. Examine the given question. How many lines can you find in five points in a plane where no three points are collinear?

a. 5 c. 16
b. 8 d. 10

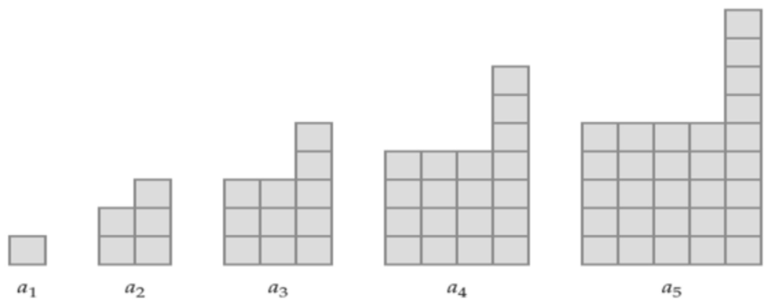
______18. Inspect the given sequence of number. What will be the next number on the sequence 2, 5, 10, 17, 26, ?

a. 37
b. 38
c. 39
d. 40

_______19. Through inspection, what would be the next number on the sequence 1, 14, 51, 124, 245, 426, …?

a. 576
b. 769
c. 679
d. 976

_______20. Assume that the pattern shown by the square tiles in the following figures continue.



How many tiles are in the tenth figure of the sequence?

a. 100
b. 910
c. 109
d. 901

_____21. What number would come next based on the given sequence 2, 4, 8, 10, 20, 22, … ?

a. 24
b. 34
c. 44
d. 54

____22. Given a number sequence 1, 7, 17, 31, 49, 71, …, what would be the next number after 71?

a. 97
b. 62
c. 26
d. 75

____23. What comes next based on the given name sequence?

ErnestO, OtsenrE, IsraeL, LearsI, LennarD, _______

a. DarnneL
b. DranneL
c. DranenL
d. DarennL

____24. What word comes next; are, era, was, saw, war,_____?

a. wars c. eras
b. wra d. raw

_____25. Using the nth-term formula below, $a_n = n/(n+1)$

$$a_n = \frac{n}{n+1},$$

what would be the first five terms of the sequence?

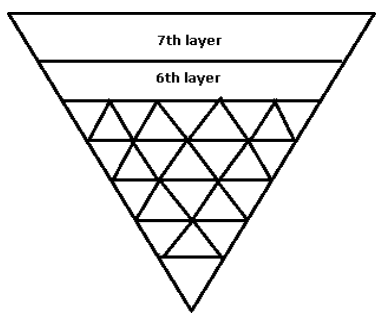
a. 1, 2, 3, 4, 5 c. -2, -1, 0, 1, 2

b. 1/2, 2/3, 3/4, 4/5, 5/6 d. 2, 3, 3, 4, 4

_____26. One cut of a stick of licorice produces two pieces. Two cuts produce three pieces. Three cuts produce four pieces. How many pieces produced by six cuts?

a. 6 c. 8
b. 7 d. 9

______27. Consider the figure below.



How many triangles are there on the 7th layer?
a. 11 c. 12
b. 14 d. 13

______28. Two different lines can intersect in at most one point, three different lines intersect at most three points, and four different lines can intersect in at most six points. How many points of intersection that six different lines can produced?

a. 21 c. 15
b. 28 d. 10

_____29. What would be the next number on the given sequence of numbers 0, 2, 6, 12, 20, 30, … ?

a. 40 c. 35

b. 42 d. 60

_____30. On the given sequence of numbers 1, 4, 27, 256, 3125, …, the next number after 3125 would be:

a. 46,656 c. 66,656
b. 56,656 d. 76,656

-END OF CHAPTER TEST-

B

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http://jnsilva.ludicum.org/HMR13_14/536.pdf

MODULE FOUR

MATHEMATICAL SYSTEM

CORE IDEA

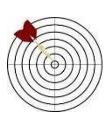
Module four is basically discussing about two equally important topics in Math in the Modern World and these are the mathematical system and fundamental concept on group theory. These two different topics are very essential most especially in dealing with encryption and decryption of a certain text in relation to identification, privacy and security purposes and to determine the validity of different codes and identification number such as the Universal Product Code, the ISBN, credit card, etc. Simple design for a wall paper, table cover, wall décor and other household decorations could also be done using mathematical system more specifically in dealing with modular arithmetic and operation on group.

Learning Outcome: At the end of this module, the student should be able to:

1. Compare modular arithmetic into group theory.
2. Illustrate accurately the modular arithmetic and its operation as well as operations on group theory.
3. Make use of modular arithmetic and group theory to apply in a real situation.

 Time Allotment: Ten (10) lecture hours

Modular Arithmetic



Specific Objective

At the end of this lesson, the student should be able to:

1. Define modular arithmetic.
2. Compare a clock arithmetic into modular arithmetic.

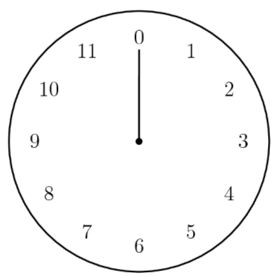
3. Explain comprehensively modular arithmetic.
4. Name different properties of modular arithmetic.
5. Tell whether the congruence is a congruence or not.
6. Solve and perform operations on clock and in modular arithmetic.
7. Construct a clock and a modulo n addition table.

Introduction

Special type of arithmetic which involves only integer (Z) is what we called "modular arithmetic". Usually, this topic is being discussed in number theory studying the integers and its properties and it is very essential for students like you who are taking up science and technology as their program to learn the concept of modular arithmetic since one the applications of this topic is on how to code and decode or encrypt and decrypt secret message for privacy and security purposes. Basically, the modular arithmetic emphasized the concept of remainder theorem when solving problems.

Discussion

Before we define modular arithmetic and study some of its application, let us use a 12-hour clock as an illustration to get an idea what a modular arithmetic is.



Most of our clock has a 12-hour design. This design designated whether the time is before noon or after noon with the use of an abbreviation A.M. and P.M. The abbreviation A.M. and P.M. came from the Latin word ante meridiem means before midday and post meridiem means after midday. Now, let us take a look the 12-hour clock. As what you have observed, after we reached the 12 o'clock, we begin again with 1. Right?

To understand better the "modular arithmetic", let us have a simple illustration with the use of the 12-hour clock below. Here, we use the numbers 0-11 instead of the numbers 1-12 to deal with modular arithmetic. The reason is that 0-11 are the remainders modulo 12. In general, when we work modulo n we replace all the numbers by their remainders modulo n. So, 12 here is replace by zero.

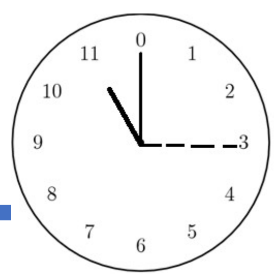


Figure 1

If we want to determine a time in the future or in the past, it is necessary to consider whether we have passed 12 o'clock. To determine time 8 hours after 3 o'clock, we add 3 and 8. Because we did not pass 12 o'clock, the time is 11 o'clock. See figure 1.

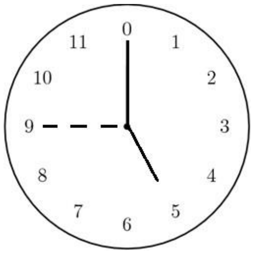
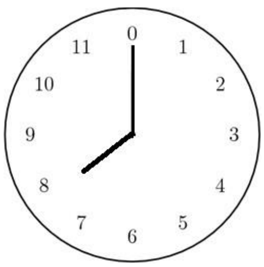


Figure 2

However, to determine the time 8 hours after 9 o'clock, we must take into consideration that once we have passed 12 o'clock, we begin again with 1. Hence, 8 hours after 9 o'clock is 5 o'clock as shown in figure 2. We will use symbol + to denote addition on a 12-hour clock and - to denote subtraction.

Now, let us use this notation for a 12-hour clock.

Let the clock would be:



Example1: Perform the + or - operator.

a) 8 + 3 = 11
b) 8 - 5 = 3
c) 8 + 16 = 0
d) 8 + 9 = 5
e) 8 - 10 = 10
f) 8 - 12 = 8

Example 2. If it is 11 o'clock and you have to finish your math homework in 18 hours, what hour will it be at that time?

Answer: 11 + 18 = 5. Hence the time that the homework could it be finished is 5 o'clock.

Example 3. If it is 12:00 now, what time is it in 12 hours? What is the remainder when you divide 12 by 12?

Answer: Using the 12-hour clock based on the given figure above, if it is now 12:00, the time after another 12 hour is also 12 o'clock and it has an equivalent number as zero (0). If we are going to divide 12 by 12, it is very obvious that the remainder is 0.

Example 4. If it is 12:00 now, what time is it in 18 hours? What is the remainder when you divide 18 by 12?

Answer: Similar on what we do to example 3, if it is now 12:00, adding 18 hours starting from 12 (0) would give us 6 o'clock. Dividing 18 by 12, the remainder would be 6. Hence, the time after 18 hours starting to 12 o'clock is 6 o'clock.

A similar example which involve the modular arithmetic is the day-of-the-week. If we assigned a number for each day of the week as shown below, then 6 days after Friday is Thursday and 16 days after Monday is Wednesday. We could write this as;

$$4 + 6 = 3 \text{ (Thursday)} \qquad \text{and} \qquad 0 + 16 = 2 \text{ (Wednesday)}$$

Monday = 0
Tuesday = 1
Wednesday = 2
Thursday = 3
Friday = 4
Saturday = 5
Sunday = 6

Another way to determine the day of the week is to note that when the sum 4 + 6 = 10 is divided by 7, the number of days in a week, the remainder is 3, which is assigned to Thursday. When 0 + 16 = 16 is divided by 7, the remainder 2 is the number assigned in Wednesday. This works because the days of the week repeat every 7 days. This could be done also to 12-hour clock arithmetic.

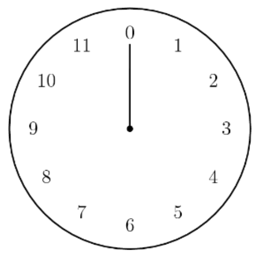
Example 5. Let us say today is Wednesday. What would be the day 11 days after Wednesday? What would be the remainder if we are going to divide 11 by 7?

Answer: Based on our given days in a week above, the equivalent number for Wednesday is 2. Adding 11 days after Wednesday, we got Sunday where this day has an equivalent number as 6. Adding 2 by 11, the result is 13 and if we are going to divide 13 by 7, the result is 6. Hence, 2 + 11 = 6.

Self -Learning Activity

Directions: Do as indicated.

A. Calculate using the 12-hour clock.



a) 7 + 12 = ?

b) 6 + 18 = ?

c) 9 + 21 = ?

d) 11- 13 = ?

e) 5 - 10 = ?

f) 8 - 12 = ?

B. Based on the assigned number for 7 days in a week, what would be the day on the after each of the following days:

Monday = 0
Tuesday = 1
Wednesday = 2
Thursday = 3
Friday = 4
Saturday = 5
Sunday = 6

a) 5 + 11 = ?
b) 3 + 9 = ?
c) 6 – 12 = ?
d) 5 + 8 = ?
e) 7 + 10 = ?

C. In exercises a to j, evaluate each expression, where + and – indicate addition and subtraction, respectively, using the 26 letters of an English alphabet where a =0, b =1, c = 2, …, y = 24 and z = 25. Determine the letter that corresponds to your answer.

a) 15 + 7 =
b) 10 + 5 =
c) 18 – 30 =
d) 20 + 9 =
e) 16 + 20 =
f) 8 – 16 =
g) 18 + 8 =
h) 13 – 16 =
i) 12 + 12 =
j) 20 – 26 =

As you can see, the situation like these that repeat in cycles are represented mathematically by using modular arithmetic or known as the arithmetic modulo n (mod n).

But the question is "What is modular arithmetic?"

Definition: Modular Arithmetic

i. Two integers a and b are said to be congruent modulo n, where n ∈ N, if $\frac{a-b}{n}$ is an integer. In this case, we write a ≡ b (mod n). The number n is called the modulus. The statement a ≡ b (mod n) is called a congruence.

This could be stated in this form.

ii. If a, b ∈ Z and n ∈ Z^+, then a ≡ b (mod n) if and only if n | a – b.

So, looking back in our clock arithmetic, instead of 13 = 1, in modular arithmetic we write 13 ≡ 1 (mod 12) and read it "13 is congruent to 1 modulo 12" or, to abbreviate, "13 is 1 modulo 12".

Let us take a look a comparison between the 12-hour clock addition table and the modulo 12 addition table.

Clock Addition Table

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

Modulo 12 Addition Table

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Based on the given table 10 + 7 = 5 or we could say that 17 ≡ 5 mod 12. Let us have some illustrative example using modulo 12 addition table.

Example 1. Based on the given table above, each expression follows the modular arithmetic under modulo 12.

a) $12 \equiv 0 \pmod{12}$

b) $21 \equiv 9 \pmod{12}$

c) $37 \equiv 1 \pmod{12}$

d) $17 \equiv 5 \pmod{12}$

Example 2. Write in the form $a \equiv b \pmod{n}$ the statement 3 | 6.

Answer:

3 | 6 could be written as 3 | (18 – 12); here n = 3, a = 18 and b = 12. So, we could write this as

$18 \equiv 12 \pmod{3}$ or $12 \equiv 18 \pmod{3}$

Note: There are more possible answers that you can give in our example 2. Can you think another possible answer?

Example 3. Is 53 ≡ 17 (mod 3)? How about 53 ≡ 14 (mod 3)? What about 53 ≡ 11(mod 3)?

Answer:

The 53 ≡ 17 (mod 3) can be written in a form of $\frac{a-b}{n}$. Now, let a = 53, b = 17 and n = 3. $\frac{53-17}{3} = \frac{36}{3} = 11 \in Z$. So the congruence is true. Now, let us take a look for 53 ≡ 14 (mod 3). Let a = 53, b=14 and n = 3. It is seen that $\frac{53-14}{3} = \frac{39}{3} = 13 \in Z$. So, it is congruence and so as 53 ≡ 11(mod 3).

Example 4. Another way to be able to write in a congruence modulo n is by dividing by n and take the remainder. Let us say n = 3. Then;

14(mod 3) ≡ 2; that is 14 = (3)(4) + 2. The remainder is 2.
9 (mod 3) ≡ 0; that is 9 = (3)(3) + 0. The remainder is 0.

2 (mod 3) ≡ 2; since 2 = (3)(0) + 2.

-1 (mod 3) ≡ 2; since -1 = (3)(-1) + 2
-5 (mod 3) ≡ 1; since -5 = (3)(-2) + 1. The remainder is 1.

Example 5. Tell whether the congruence is true or not.

29 ≡ 8 mod 3	This is a true congruence! Why?
7 ≡ 12 mod 5	This is a true congruence! Why?
15 ≡ 4 mod 6	This is not a true congruence. Why?

An alternative method to determine a true congruence in a ≡ b(mod n), where a and b are whole numbers, then when a and b is divided by n, they must have the same remainder.

Example 6. Let us say the given modulo is 53 ≡ 17 (mod 3). Now, if we divide 53 to 3, then;

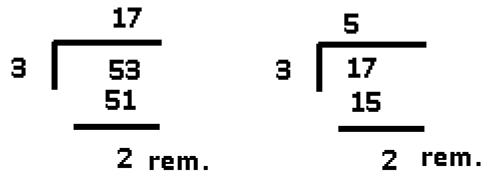
53 = (3)(17) + 2. The remainder is 2

and if we divide 17 to 3, we get;

17 = (3)(5) + 2. The remainder is also 2.

Hence, $53 \equiv 17 \pmod 3$ is a true congruence.

Or



So, the theorem states that:

For arbitrary integers a and b, $a \equiv b \pmod n$ if and only if a and b have the same remainder when divided by n.

Proof:

(⇒) Assume that $a \equiv b \pmod n$. Let $a = nq_1 + r_1$ and $b = nq_2 + r_2$ where $0 \le r_1 < n$ and $0 \le r_2 < n$. We need to show that $r_1 = r_2$. Since $a \equiv b \pmod n$, then $a - b = nk$ for some integer k. By substitution;

$$nq_1 + r_1 - nq_2 + r_2 = nk$$
$$n(q_1 - q_2) + r_1 - r_2 = nk$$
$$r_1 - r_2 = nk - n(q_1 - q_2)$$
$$r_1 - r_2 = n[k - (q_1 - q_2)] \text{ where } [k - (q_1 - q_2)] \in Z.$$

Now, $n \mid r_1 - r_2$. We claim that $r_1 - r_2 = 0$. Suppose $r_1 - r_2$ is not equal to zero, then $n < |r_1 - r_2|$. This is contradiction since $0 \le r_1 < n$ and $0 \le r_2 < n$. Therefore, $r_1 - r_2 = 0$ which implies that $r_1 = r_2$.

(⇐) Assume that a and b leaves the same remainder when divided by n. Then, $a = nq_1 + r$ and $b = nq_2 + r$. Now,

$$a - b = (nq_1 + r) - (nq_2 + r)$$
$$= nq_1 + r - nq_2 - r$$
$$= nq_1 - nq_2$$
$$= n(q_1 - q_2) \text{ where } (q_1 - q_2) \in Z$$

Then, $n \mid a - b$.
Therefore , $a \equiv b \pmod n$. €

Theorem: Properties on congruence

Let $n > 0$ be fixed and a, b, c and d are arbitrary integers. Then,

a) $a \equiv a \pmod{n}$
b) if $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$
c) if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$
d) if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c = b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$
e) if $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$ and $ac \equiv bc \pmod{n}$
f) If $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$

Proof:

a) Let $a \in Z$. If $a = a$, then $a – a = 0$. It follows that $n \mid 0$ which implies that $n \mid a – a$. Hence, $a \equiv a \pmod{n}$. €

Illustration:

1) $6 \equiv 6 \pmod{2}$ 2) $(x + 1) \equiv (x + 1) \pmod{3}$

b) Assume that $a \equiv b \pmod{n}$ if and only if $n \mid a – b$ or $a – b = nk$ for some integer k. From $a – b = nk$, it follows that $–[a – b] = -nk$, i.e. multiplying both sides by -1. Then $b – a = -nk$ or $b – a = n(-k)$ where $–k \in Z$. It only shows that $n \mid b – a$. Hence, $b \equiv a \pmod{n}$.€

Illustration:

1) If $13 \equiv 10 \pmod{3}$, then $10 \equiv 13 \pmod{3}$
2) $2 \equiv 12 \pmod{5}$ is as the same as $12 \equiv 2 \pmod{5}$

c) Assume that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. In $a \equiv b \pmod{n}$, it follows that $a – b = nk_1$(eq.1) and for $b \equiv c \pmod{n}$ we have $b – c = nk_2$ (eq. 2)for $k_1, k_2 \in Z$. In eq. 1, $b = a - nk_1$ and in eq.2 we have $b = nk_2 + c$. Then $a - nk_1 = nk_2 + c$. It follows that $a – c = nk_2 + nk_1 = n(k_2 + k_1)$ for $(k_2 + k_1) \in Z$. It only shows that $n \mid a – c$. Hence, $a \equiv c \pmod{n}$. €

Illustration:

1) If $2 \equiv 14 \pmod{3}$ and $14 \equiv 5 \pmod{3}$, then $2 \equiv 5 \pmod{3}$.

2) If $7 \equiv 19 \pmod{2}$ and $19 \equiv 15 \pmod{2}$, then $7 \equiv 15 \pmod{2}$

d) Assume that $a \equiv b(\bmod n)$ and $c \equiv d(\bmod n)$. Here, $n \mid a - b$ and $n \mid c - d$ respectively. In $n \mid a - b$ it follows that $a - b = nk_1$ and for $n \mid c - d$ we have $c - d = nk_2$ for $k_1, k_2 \in Z$. But $a - b + c - d = nk_1 + nk_2$. Using the commutative property on the left side, we have $(a + c) + (- b - d) = nk_1 + nk_2$. Now, $(a + c) - (b + d) = nk_1 + nk_2$. Then, $(a + c) - (b + d) = n (k_1 + k_2)$ for $k_1, k_2 \in Z$. It is clearly seen that $n \mid (a + c) - (b + d)$. Hence $(a + c) \equiv (b + d) (\bmod n)$.

Similarly, let $a = nq_1 + b$ and $c = nq_2 + d$ for any $q_1, q_2 \in Z$. Multiplying a and c, we got $ac = (nq_1 + b)(nq_2 + d) = n^2q_1q_2 + dnq_1 + bnq_2 + bd = bd + n(nq_1q_2 + bq_2 + dq_1)$ for $(nq_1q_2 + bq_2 + dq_1) \in Z$. It follows that $ac - bd = nq$. It only shows that $n \mid ac - bd$. Hence $ac \equiv bd(\bmod n)$.€

Illustration:

1) If $7 \equiv 19(\bmod 2)$ and $27 \equiv 11 (\bmod 2)$ then $(7 + 27) \equiv (19 + 11) (\bmod 2)$
2) If $7 \equiv 19(\bmod 2)$ and $27 \equiv 11 (\bmod 2)$ then $(7\cdot 27) \equiv (19 \cdot 11) (\bmod 2)$

e) Assume that $a \equiv b(\bmod n)$. We know that $c \equiv c (\bmod n)$. By property (d) $a + c \equiv (b + c) (\bmod n)$ and $ac \equiv bc (\bmod n)$.€

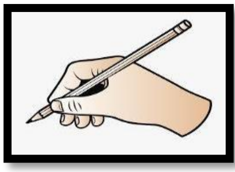
Illustration:

1) Given that $31 \equiv 13 (\bmod 2)$. Let $c = 3$, then $(31 + 3) \equiv (13 + 3)(\bmod 2)$ and $(31\cdot3) \equiv (13\cdot3)(\bmod 2)$

f) Assume that $a \equiv b(\bmod n)$. By the property of mathematical induction, if $k = 1$ then $a \equiv b(\bmod n)$ is true. Now, assume that it is true for $k = n$, i.e. $a^n \equiv b^n (\bmod n)$. We need to show that $a^{n+1} \equiv b^{n+1} (\bmod n)$. But $a \equiv b (\bmod n)$ and $a^n \equiv b^n (\bmod n)$. With the use of property (d), we can see that $(a)(a^n) \equiv (b)(b^n)(\bmod n)$. It could be written in a form of $a^{n+1} \equiv b^{n+1} (\bmod n)$ where $n + 1 = k$. Hence, $a^k \equiv b^k (\bmod n)$.€

Illustration:

1) Given that $25 \equiv 7 (\bmod 3)$ and $k = 2$, then $25^2 \equiv 7^2 (\bmod 3)$



Self -Learning Activity

Direction: Determine whether the following is congruence or not congruence. Write your answer on the right side of each item.

a) $5 \equiv 8 \pmod 3$

b) $5 \equiv 20 \pmod 4$

c) $21 \equiv 45 \pmod 6$

d) $88 \equiv 5 \pmod 9$

e) $100 \equiv 20 \pmod 8$

f) $11 \equiv 15 \pmod 4$

g) $7 \equiv 21 \pmod 3$

h) $18 \equiv 60 \pmod 7$

i) $72 \equiv 30 \pmod 5$

j) $25 \equiv 85 \pmod{12}$

Direction: Answer the following questions. Write your answer after each item.

a) What is 13mod1?
b) What is -4mod9?
c) What is -13mod1?
d) What is -14mod2?
e) What is 14mod2?

Directions: Which of the following integers are valid for solutions for x? Encircle all letters that could apply a correct answer.

1. Given $x \equiv 17 \bmod 4$

   a) -43 b) -17 c) 15 d) 25

2. Given $x \equiv 11 \bmod 8$

   a) -77 b) 77 c) 27 d) 25

Directions: Filled out the table of a modulo 4 addition table and transform each item in a form of $a \equiv b \bmod n$.

+	0	1	2	3
0				
1				

2				
3				

a) 3 + 3 =
b) 2 + 3 =
c) 1 + 2 =
d) 0 + 0 =
e) 2 + 2 =

Operations on Modular Arithmetic

Specific Objective

At the end of the lesson, the student should be able to:

1. Name and explain the different operations on modular arithmetic.
2. Perform and solve the operations on modulo n such as addition, subtraction, multiplication, the additive and multiplicative inverse.
3. Solve the congruence equation.

Introduction

During your senior high school, you have learned how to perform the four fundamental operations in mathematics in a higher level compare your lesson during elementary days. You encounter on how to deal with the addition, subtraction, multiplication and division of numbers as well as the different properties of real numbers.

For this lesson, you will be encountering another operation but this time it will be dealing in different operations in modulo n.

Discussion

In the previous theorem for a ≡ b(mod n), property (d) and (e) is a part of arithmetic operations modulo n. The different arithmetic operations modulo n are as follows: 1) addition modulo n, 2) subtraction modulo n, 3) multiplication modulo n and 4) the additive and multiplicative inverses.

In 29 ≡ 8 (mod 3), we could verify that its congruence is true since both 29 and 8 have remainder 2 when divided by 3 which is the modulus. There are many other numbers congruent to 8 modulo 3, but of all these, only one is a whole number less than the modulus. This number is the result when evaluating a modulo expression, and in this case we use an equal sign. Because 2 ≡ 8 mod 3 and 2 is less than the modulus, we can write 8 mod 3 = 2. In general, m mod n becomes the remainder when m is divided by n.

Arithmetic modulo n (where n is a natural number) requires us to evaluate a modular expression after using the standard rules of arithmetic. Thus, we perform the arithmetic operation and then divide by the modulus. The answer is the remainder. The result of an arithmetic operation mod n is always whole number less than n.

Illustration:

Let n = 3, then

14 (mod 3) ≡ 2 since 14 = 3·4 + 2 (2 is the remainder)

9(mod 3) ≡ 0 since 9 = 3·3 + 0 (0 is the remainder)

2(mod 3) ≡ 2 since 2 = 3·0 + 2 (2 is the remainder)

A. Addition Modulo n

To do the addition modulo n, let us have some example.

Evaluate (23 + 38) mod 12.

Solution:

Add 23 + 38 to produce 61. To evaluate 61 mod 12, divide 61 by modulus, 12. The answer is the remainder.

$$\begin{array}{r|l} & 5 \\ \hline 12 & 61 \\ & 60 \\ \hline & 1 \end{array}$$

So (23 + 38) mod 12 ≡ 1 since 61 = 12·5 + 1 where 1 is the remainder.

In modular arithmetic, adding the modulus to a number does not change the equivalent value of the number.

For instance; 13 ≡ 6 (mod 7)

20 ≡ 6 (mod 7) add 7 to 13

27 ≡ 6 (mod 7) add 7 to 20 and so on.

Another example is;

12 ≡ 7 (mod 5)

17 ≡ 7 (mod 5) add 5 to 12

22 ≡ 7 (mod 5) add 5 to 17 and so on.

This property of modular arithmetic is sometimes used in subtraction. It is possible to use negative numbers modulo n. For instance;

-2 ≡ 5 mod 7 is a true congruence. Why? Applying the definition; a ≡ b (mod n) where n | a – b, then

$$\frac{-2-5}{7} = \frac{-7}{7} = -1 \in Z .$$

B. Subtraction Modulo n

The following examples give you on how be able to perform subtraction.

1. Evaluate (33 – 16)mod 6.

Here, subtracting 16 from 33, we will be able to get a positive result, i.e. 17. Divide the difference by the modulus, 6 we get:

$$17 = 6{\cdot}2 + 5.$$

So, (33 – 16) mod 6 = 5

2. Evaluate (14 – 21) mod 5.

If we subtract 21 from 14, we will get a negative answer, i.e., -13. On that case, we must find x so that -13 ≡ x mod 5. Thus we must find x so that the value of $\frac{-13-x}{5} = \frac{-(13+x)}{5}$ is an integer. Trying the whole number values of x less than 5, the modulus, i.e. x = 0, 1, 2, 3, and 4, then;

$$\frac{-(13+0)}{5} \notin Z$$

$$\frac{-(13+1)}{5} \notin Z$$

$$\frac{-(13+2)}{5} \in Z$$

$$\frac{-(13+3)}{5} \notin Z$$

$$\frac{-(13+4)}{5} \notin Z$$

So, the only value for x is 2. Hence, (14 – 27) mod 5 = 2.

C. Multiplication Modulo n

Like in addition and subtraction, let us evaluate given example under multiplication modulo n to perform the operation multiplication.

Evaluate $(15 \cdot 23) \bmod 11$.

We need to find the product of 15 and 23. The product is 345. This product must be divided by the modulus, 11 to be able to find its remainder.

$345 = 11 \cdot 31 + 4$

Hence, $(15 \cdot 23) \bmod 11 = 4$

D. Additive and Multiplicative Inverses in Modular Arithmetic

a. Recall that if the sum of two numbers is zero (0),i.e. a + (-a) = 0, then the numbers are additive inverses of each other. For instance, 5 + (-5) = 0. So 5 is the additive invers of -5 and -5 is the additive inverse of 5.

The same concept applies in modular arithmetic. For example;

$$(3 + 5) \equiv 0 \bmod 8.$$

Thus, in mod 8 arithmetic, 3 is the additive inverse of 5, and 5 is the additive inverse of 3. Here, we consider only those whole number smaller that the modulus. Note that 3 + 5 = 8; that is, the sum of a number and its additive inverse equals the modulus. Using this fact, we can easily find the additive invers of a number for any modulus. For instance, in mod 11 arithmetic, the additive inverse of 5 is 6 because 5 + 6 = 11.

Let us have additional example.

Find the additive inverse of 7 in mod 16 arithmetic.

Solution:

In mod 16 arithmetic, 7 + 9 = 16. So, the additive inverse of 7 is 9.

b. If the product of two number is 1, then the numbers are multiplicative inverses of each other. This is one of the properties of real number, i.e., $(a)(\frac{1}{a}) = 1$. So, the multiplicative inverse of 2 is 1/2 and the multiplicative inverse of 1/2 is 2.

The same concept applies to modular arithmetic (although the multiplicative inverses will always be natural number). For example in mod 7 arithmetic, 5 is the multiplicative invers of 3 (and 3 is the multiplicative inverse of 5) because 5·3 ≡ 1 mod 7. Here, we will concern ourselves only with natural numbers less than the modulus. To find the multiplicative inverse of a mod m, solve the modular equation ax = 1 mod m for x.

Example:

In mod 7 arithmetic, find the multiplicative inverse of 2.

Solution:

To find the multiplicative inverse of 2, solve the equation 2x ≡ 1 mod 7 by trying different natural number values of x less than the modulus.

Here, x = 1,2,3,4,5, and 6.

2x ≡ 1 mod 7

2(1) ≡ 1 mod 7 (this is not a true congruence)
2(2) ≡ 1 mod 7 (this is not a true congruence)
2(3) ≡ 1 mod 7 (this is not a true congruence)
2(4) ≡ 1 mod 7 (this is a true congruence)
2(5) ≡ 1 mod 7 (this is not a true congruence)
2(6) ≡ 1 mod 7 (this is not a true congruence)

Hence, in mod 7 arithmetic, the multiplicative of 2 is 4.

SOLVING CONGRUENCE EQUATION

Solving a congruence equation means finding all whole numbers values of the variable for which the congruence is true.

Let us have an example on how to solve the congruence equation.

Example: Solve 3x + 5 ≡ 3 mod 4. Here, we need to search for whole number values of x for which the congruence is true.

Solution:

$$3x + 5 \equiv 3 \bmod 4$$

If x = 0	3(0) + 5 ≡ 3 mod 4	(this is not a true congruence)
If x = 1	3(1) + 5 ≡ 3 mod 4	(this is not a true congruence)
If x = 2	3(2) + 5 ≡ 3 mod 4	(this is a true congruence, so 2 is a solution)
If x = 3	3(3) + 5 ≡ 3 mod 4	(this is not a true congruence)
If x = 4	3(4) + 5 ≡ 3 mod 4	(this is not a true congruence)
If x = 5	3(5) + 5 ≡ 3 mod 4	(this is not a true congruence)
If x = 6	3(6) + 5 ≡ 3 mod 4	(this is a true congruence, so 6 is a solution).

Now, if we continue trying to find the other values to be a solution, we could find that 10 and 14 are also a solution. Note that 6, 10 and 14 are all congruent to 2 mod 4. Thus the solutions of 3x + 5 ≡ 3 mod 4 are 2, 6, 10, 14, 18, …

Example : Solve 3x + 4 = 2x + 8 (mod 9)
Solution:

3x + 4 = 2x + 8 (mod 9)

− 4 = − 4 (mod 9)

3x = 2x + 4 (mod 9)

− 2x = − 2x +

(mod 9) x =

4 (mod 9)

The solution is x = 4 (mod 9)

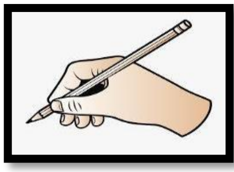
Or

3x + 4 = 2x + 8 (mod 9)

(3x – 2x) = (8 – 4)(mod 9)

x = 4(mod 9)

Hence, the solution is x = 4 (mod 9)



Self- Learning Activity

A. Evaluate the following arithmetic operations modulo n. Note: a ≤ n and a is positive.

1) (46 + 53)mod 8 =
2) (43 + 29)mod 10 =
3) (56 – 24) mod 17 =
4) (29 – 18) mod 3 =
5) (67 – 93) mod 9 =
6) (46 – 87)mod 5 =
7) (8)(13)mod 4 =
8) (16)(25) mod 18 =
9) (-23)(35)mod 29 =
10) (-24)(-32)mod 13 =

B. Find the additive and multiplicative inverse of the following if any. If there is no additive nor multiplicative inverse, explain why there is no such as inverses.

1) (x + 15) = 0 mod 29 x=
2) (23 + x) = 0 mod 27 x=
3) (x – 45) = 0 mod 89 x=
4) (46 – x) = 0 mod 16 x=
5) (2x – 16) = 0 mod 18 x=
6) 5x = 1mod 9 x=
7) 14x = 1mod 41 x=
8) 6x = 1mod 41 x=
9) 7x = 1 mod 13 x =
10) 11x = 1 mod 7 x=

C. Find the value of x if any. Show your complete solution at the back of this paper.

1) (13 + x) = 3mod 16 x =
2) (3x + 24) = (4 + x) mod 9 x =

D. Complete the following table.

Suppose that n = 7. Filled out the table on Z_7 = {0,1,2,3,4,5,6}

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

x	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Applications of Modular Arithmetic

Specific Objective

At the end of the lesson, the student should be able to:

1. To apply the concept of modular arithmetic to determine the validity of a certain serial number such the UPC, ISBN and the credit card.
2. To explain the validity and invalidity of the product code.
3. To differential cryptography to cryptography.
4. To make use of modular arithmetic to encrypt and decrypt the text.
5. To apply modular operations in making a modulo art.

Introduction

All of us are familiar with the number of hours in a day, number of days in a week, number of months in a year, etc. But what have you observed? Say, in a 12-hour clock. It goes on and on and on and after reaching in a 12 o'clock, it starts again to 1 o'clock and running all over again. This is the concept of modular arithmetic where numbers "wrap around" as what as stated previously.

The concept of modular arithmetic has an important role in different industry and some of these industries are banking and finance, information and technology, medicine and health, trade and industry, and in education sector. Perhaps you may be wondering why this modular arithmetic is so important. When we speak the determination of validity and invalidity of credit card, on how be able to manage book catalogue, how to have correct serial number of a particular product that the consumer could be bought, how to create an art and how to make a security code most especially in the banking sector, these are just some applications of a modular arithmetic.

Discussion

Remember in our previous lesson, in modular arithmetic, we select an integer, n, to be our “modulus". Then our system of numbers only includes the numbers 0, 1, 2, 3, ..., n-1. In order to have arithmetic make sense, we have the numbers “wrap around" once they reach n.

If we pick the modulus 5, then our solutions are required to be in the set Z_5 = {0,1, 2, 3, 4}. We have 2+1=3 and 2+2=4 as usual. Then 2+3=5, which is not in our set, so it wraps around giving 2+3=0. Then 2+4=6, which wraps around to be 1.

Z_5 = {1,2,3,4,0}

+	**1**	**2**	**3**	**4**	**0**
1	2	3	4	0	1
2	3	4	0	1	2
3	4	0	1	2	3
4	0	1	2	3	4
0	1	2	3	4	0

But how are we going to write again the illustration above in modular arithmetic? Some examples are written below.

2 + 1 = 3(mod 5) = 3
4 + 2 = 6(mod 5) = 1
2 + 4 = 6(mod 5) = 1

3 + 4 = 7(mod 5) = 2
4 + 4 = 8(mod 5) = 3
0 + 0 = 0(mod 5) = 0

Now, this could be done also in multiplication.

Z_5 = {1,2,3,4,0}

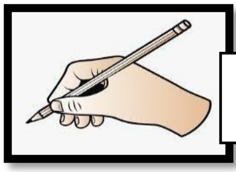
·	**1**	**2**	**3**	**4**	**0**
1	1	2	3	4	0
2	2	4	1	3	0
3	3	1	4	2	0
4	4	3	2	1	0
0	0	0	0	0	0

2 x 1 = 2(mod 2) = 2

3 x 4 = 12(mod 5) = 2

4 x 2 = 8(mod 5) = 3	4 x 4 = 16(mod 5) = 1
2 x 4 = 8(mod 5) = 3	0 x 0 = 0(mod 5) = 0

In a 12-hour clock, we go to 1 then 2 then 3, until we reach 12. This is an example of modulos12. We use {1,2,3,…12} but in modulus it could be presented as {1,2,3, …, 0}. These are the same because we consider 0 and 12 be the same in terms of wrapping around.



Self- Learning Activity

Direction: Do as what is indicated.

1. Complete the table below and evaluate the following modular arithmetic.

1.1) For Z_{12} = {0,1,2,3,4,5,6,7,8,9,10,11}

+	0	1	2	3	4	5	6	7	8	9	10	11
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												

1.2) Evaluate the following:

5 + 11 = 16(mod 12) =

10 11 = 21(mod 12) =

8 + 8 = 16(mod 12) =

10 + 9 = 19(mod 12) =

1.3) For Z_6 = {1,2,3,4,5,0}

·	1	2	3	4	5	0
1						
2						
3						

4						
5						
0						

1.4) Evaluate the following:

5 x 4 = 20(mod 6) = 4 x 2 = 8(mod 6) =
3 x 5 = 15(mod 6) = 5 x 5= 25(mod 6) =

APPLICATIONS OF MODULAR ARITHEMETIC

A. International Standard Book Number (ISBN)

One of the applications of modular arithmetic is on how to check or how to determine whether the ISBN (International Standard Book Number) is valid or not. Every book that is catalogued in the Library of Congress must have an ISBN. The ISBN consists of 13 digits and this was created to help to ensure that orders for books are filled accurately and that books are catalogued correctly.

The first digits of an ISBN are 978 (or 979), followed by 9 digits that are divided into three groups of various lengths. These indicate the country or region, the publisher, and the title of the book. The last digit (13^{th} digit) is called a **check digit**.

Illustration:

978 – 971 – 23 – 9357 – 0

978	**-**	**971**	**-**	**23**	**-**	**9357**	**-**	**0**
The first three digits	-	Country/region	-	publisher	-	Title of the book	-	Check digit

If we label the first digit of an ISBN as d_1, the second digit as d_2, and so on to the 13^{th} digit as d_3, then the check digit is given by the modular formula as:

$d_{13} = 10 – (d_1 + 3d_2 + d_3 + 3d_4 + d_5 + 3d_6 + d_7 + 3d_8 + d_9 + 3d_{10} + d_{11} + 3d_{12})$mod 10.

If $d_{13} = 10$, then the check digit is 0.

Example 1: The ISBN of Richard Aufmann's book entitled "Mathematics in the Modern World" published by Rex Bookstore in 2018 is:

978 – 971 – 23 – 9357 – 0

Is ISBN valid?

Solution:

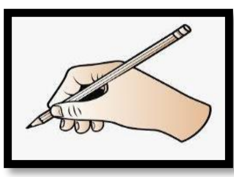
$d_{13} = 10 - (d_1 + 3d_2 + d_3 + 3d_4 + d_5 + 3d_6 + d_7 + 3d_8 + d_9 + 3d_{10} + d_{11} + 3d_{12}) \bmod 10.$

$d_{13} = 10 - [9 + 3(7) + 8 + 3(9) + 7 + 3(1) + 2 + 3(3) + 9 + 3(3) + 5 + 3(7)] \bmod 10.$

$d_{13} = 10 - [9 + 21 + 8 + 27 + 7 + 3 + 2 + 9 + 9 + 9 + 5 + 21] \bmod 10.$

$d_{13} = 10 - 130 \bmod 10 \qquad 130 = (10)(13) + 0$

$d_{13} = 10 - 0 = 10$. Since $d_{13} = 10$, then its check digit is 0. Therefore, it is a valid ISBN.



Self- Learning Activity

Directions: Solve the following problems.

1. The book entitled "The Equation that Couldn't be Solved" by Mario Livio has an ISBN 978-0-7432-5820-?. What is its check digit?

2. A book "The Mathematical Tourist" by Ivars Peterson has an ISBN 978-0-716-73250-5. Check whether if it is a valid or not valid number ISBN.

3. Is the ISBN: 978-971-9645-41-2 a valid number?Explain.

4. What is the check digit of this ISBN: 0-471-31055 – x?

B. Universal Product Code (UPC) and Credit Card

Another coding scheme that is closely related to the ISBN is the UPC (Universal Product Code). This number is placed on many items and is particularly useful in grocery stores. A check-out clerk passes the product by a scanner, which reads the number from a bar code and records the price on the cash register. If the price of an item changes for a promotional sale, the price is updated in the computer, thereby relieving a clerk of having to reprice each item. In addition to pricing items, the UPC gives the store manager accurate information about inventory and the buying habits of the store's customers.

The UPC is a 12-digit number that satisfies a modular equation that is similar to the one for ISBNs. The last digit is the check digit. The formula for the UPC check digit is:

$d_{12} = 10 - (3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11}) \bmod 10.$

If $d_{12} = 10$, then the check digit is 0.

Example. The staple wire has a bar code of 9-02870-766290. Is the UPC number of this product a valid number?

Solution:

$d_{12} = 10 - (3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11}) \bmod 10$

$d_{12} = 10 - [3(9) + 0 + 3(2) + 8 + 3(7) + 0 + 3(7) + 6 + 3(6) + 2 + 3(9)] \bmod 10$

$d_{12} = 10 - (27 + 0 + 6 + 8 + 21 + 0 + 21 + 6 + 18 + 2 + 27) \bmod 10$

$d_{12} = 10 - 136 \bmod 10$ but $136 = (10)(13) + 6$

$d_{12} = 10 - 6 = 4$

Since the computed check digit is not the last digit in a given code, then the bar code is not valid.

In terms of checking the credit card numbers if it is valid, this modular arithmetic can be used whether a credit card is valid or not. This is especially important in e-commerce, where credit card information is frequently sent over the internet. The primary coding method is based on the Luhn Algorithm, which uses mod 10 arithmetic. Credit card is usually or normally have 13 to 16 digits long. The first one to six digits are used to identify the card issuer. The table below shows some of the identification prefixes used by four popular card issuers.

Card Issuer	Prefix	Number of digits
Master Card	51 to 55	16
Visa	4	13 or 16
American Express	34 or 37	15
Discover	6011	16

The Lugh algorithm, used to determine whether a credit card is valid. It is calculated as follows:

1. Beginning with the next-to-last digit (the last digit is the check digit) and reading from right to left.
2. Double every other digit. If a digit becomes a two digit number after being doubled, treat the number as two individual digits.
3. Find the sum of the new list of digits. The final sum must be congruent to 0 mod 10.

Example:

Determine whether 5234-8213-3410-1298 is a valid card number.

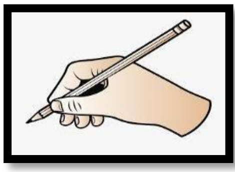
Solution:

5	2	3	4	8	2	1	3	3	4	1	0	1	2	9	8
x2		x2		x2		x2		x2		x2		x2		x2	
10	2	6	4	16	2	2	3	6	4	2	0	2	2	18	8

Then;

$(1+0) + 2 + 6 + 4 + (1 + 6) + 2 + 2 + 3 + 6 + 4 + 2 + 0 + 2 + 2 + (1 + 8) + 8 = 60$

Since $60 \equiv 0 \mod 10$, this is a valid card number.



Self- Learning Activity

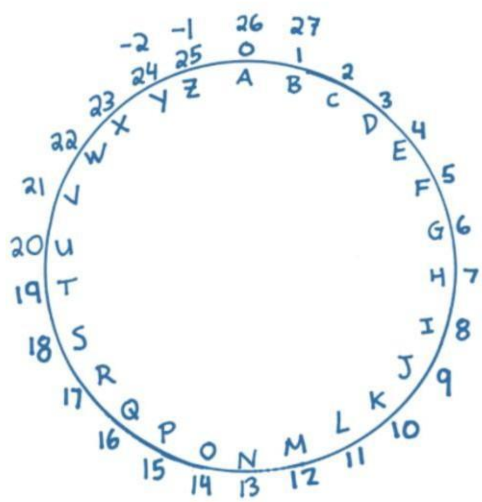
Direction: Answer the following questions.

1. A certain product has a bar code 4-804888-9027-5. Is the product code valid? Explain.
2. Tell whether the given Universal Product Code 300450180186 is a valid number. Explain.
3. Find the check number of a certain product whose code number is 0-332334-8272-x.
4. Is 6011-0123-9145-2317 a valid card number? Why?
5. How about a credit card whose number is 5155-0123-4356-0080? Is this a valid card number? Why?
6. Is the credit card number 4000001234567899 is a valid number? Explain.

C. Cryptology and Cryptography

Another usage of modular arithmetic is cryptography. But what is the difference between cryptology and cryptography? Based on dictionary.com, cryptology is the study of codes while cryptography is the art of writing and solving them.

Let us have simple activity to be able to understand what we mean by cryptography. Here is the activity to be done by the students. Try to decrypt the secret code.



In an English alphabet, line up with wheels so that “a" lines up with “R".

Riddle No. 1. They come out at night without being called, and are lost in the day without being stolen. What are they?

Answer: JKRI = ______________________

Riddle No. 2. What has a face and two hands but no arms and legs?

Answer: TCFTB = ____________________

Riddle No. 3: Why was the math book sad?

Answer: ZK YRU KFF DREP GIFSCVDJ = __________________________________

Now, cryptology is the study of making and breaking secret codes. It is very important to learn because there are industries or government agencies that need to transmit secret message or information that it cannot be understood by an unauthorized person most especially if this message is intercepted.

Different answer from our previous activity are encrypted words that need to decrypt to be able to break the code. The encrypted word is called the ciphertext while the decrypted word is called the plaintext. So, plaintext is a message before it is coded while the ciphertext is the message after it has been written in code. The method of changing from plaintext to ciphertext is called encryption.

But how the modular arithmetic can be used in encrypting and decrypting the code? If the encrypting code is to shift each letter of the plaintext message in "m" positions, then the corresponding letter in the ciphertext message is given by;

$$c \equiv (p + m) \bmod 26$$

where c is the encrypted code, p is the number corresponds to a letter in an English alphabet in a normal position, m is the shifted position and n = 26 is the modulus since there are 26 letters in an English alphabet.

Table
Cyclical English Alphabet

a	b	c	d	e	F	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0
-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	

Note that the letter Z is coded as zero because $26 \equiv 0 \bmod 26$.

So, with the use of modular arithmetic, how are we going to convert the plaintext "LOVE" into its ciphertext if each letter was shifted in 22 positions ($m = 22$)?

For L: $c \equiv (12 + 22) \bmod 26 = 34 \bmod 26 = 8$. So the coded letter for L would be H.
For O: $c \equiv (15 + 22) \bmod 26 = 37 \bmod 26 = 11$. So the coded letter for O would be K.
For V: $c \equiv (22 + 22) \bmod 26 = 44 \bmod 26 = 18$. So the coded letter for V would be R.
For E: $c \equiv (5 + 22) \bmod 26 = 27 \bmod 26 = 1$. So the coded letter for E would be A.

Hence, the encrypted code for the word LOVE is HKRA.

But how are we going to break the encrypted word. There must be a method by which the person who received encrypted message into its original message (plaintext). For the cyclical code, the congruence is;

$$p \equiv (c + n) \bmod 26$$

where p and c are defined as before and $n = 26 – m$. The letter H in ciphertext is decoded below using the congruence $p \equiv (c + n) \bmod 26$ such as;

Code: H: $n = 26 – 22 = 4$, then $p \equiv (8 + 4) \bmod 26 = 12 \bmod 26 = 12$

Hence, if you are going to decode H, it would be letter L.

The practicality of a cyclical alphabetic coding scheme is limited because it is relatively easy for a cryptologist to determine the coding scheme. A coding scheme that is a little more difficult to break is based on the congruence;

$$c \equiv (ap + m) \bmod 26$$

where a and 26 do not have a common factor. For example, "a" cannot be 14 because 14 and 26 have a common factor of 2. The reason why "a" and 26 cannot have a common factor is related to the procedure for determining the decoding congruence.

Example: Use the congruence $c \equiv (5p + 2) \bmod 26$ to encode the message LASER.

Solution:

The encrypting congruence is $c \equiv (5p + 2) \bmod 26$. We replace p by the numerical equivalent of each letter from the given previous table (Cyclical English Alphabet). So the coded for each letter in a word LASER are as follows:

L: $c \equiv [(5)(12) + 2] \bmod 26 = 62 \bmod 26 = 10$ since $62 = (26)(2) + 10$. So we could code L as J

A: $c \equiv [(5)(1) + 2] \bmod 26 = 7 \bmod 26 = 7$ since $7 = (26)(0) + 7$. So we could code A as G

S: $c \equiv [(5)(19) + 2] \bmod 26 = 97 \bmod 26 = 19$. So we could code S as S
E: $c \equiv [(5)(5) + 2] \bmod 26 = 27 \bmod 26 = 1$. So we could code E as A.

R: $c \equiv [(5)(18) + 2] \bmod 26 = 92 \bmod 26 = 14$. So we could code R as N.

So, the encrypted word for LASER would be JGSAN after using the congruence $c \equiv (5p + 2) \bmod 26$.

If we want to decode an encrypted word using the congruence $c \equiv (ap + m) \bmod n$, it require us to solve the congruence for p. To be able to solve it, we are going to use the method which relies on multiplicative inverses.

In previous congruence, i.e. $c \equiv (5p + 2) \bmod 26$, $c = 5p + 2$. Subtracting both sides by 2 we got $c – 2 = 5p$. Now, multiply each member of the equation by the multiplicative of 5 which is 21 since $(21)(5) = 1 \bmod 26$, it will become;

$$21(c – 2) = (21)(5p)$$
$$[21(c – 2)] \bmod 26 \equiv p.$$

Using this congruence, we can decode the ciphertext message JGSAN.

J: $p \equiv [21(10 – 2)] \bmod 26 = (21)(8) \bmod 26 = 168 \bmod 26 = 12$. So J will be decoded as L.

G: $p \equiv [21(7 – 2)] \bmod 26 = (21)(5) \bmod 26 = 105 \bmod 26 = 1$. So G will be decoded as A.

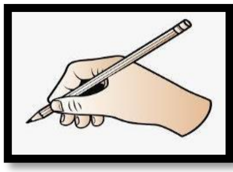
S: $p \equiv [21(19 – 2)] \bmod 26 = (21)(17) \bmod 26 = 357 \bmod 26 = 19$. So S will be decoded as S.

A: $p \equiv [21(1 – 2)] \bmod 26 = (21)(-1) \bmod 26 = (-21) \bmod 26 = 5$. So A will be decoded as E.

Note: To decode A, it is necessary to determine $(-21) \bmod 26$. Recall that this requires adding the modulus until a whole number less than 26 results. So, $(-21) + 26 = 5$, we have $(-21) \bmod 26 = 5$

N: $p \equiv [21(14 - 2)] \bmod 26 = (21)(12) \bmod 26 = 252 \bmod 26 = 18$. So N will be decoded as R.

Hence, the decoded word for "JGSAN" would be "LASER".



Self- Learning Activity

1. Encrypt the following words using the congruence $c \equiv (3p + 5) \bmod 26$ that corresponds to the previous table (cyclical English alphabet). Write the encrypted word opposite to the given plaintext.

Plaintext	Ciphertext
a) PATRIOTISM	____________________
b) INTEGRITY	____________________
c) EXCELLENCE	____________________
d) SERVICE	____________________
e) RESILIENCE	____________________
f) FAITH	____________________

2. Using the previous table (cyclical alphabetic),

   Code MATHEMATICS IN THE MODERN WORLD using $c \equiv (p + 3) \bmod 26$.

Answer:

____

3. Decode the message ACXUT CXRT, which was encrypted using the congruence $c \equiv (3p + 5) \bmod 26$ that corresponds to the previous table (cyclical English alphabet. Write the plaintext on the space provided below.

____

D. Modulo Operation in Designing Modulo Art

We could not deny the fact that all of us love to see the beauty of our nature and environment. But have you observed that all things that you can see with your naked eyes formed patterns? Straight, curve and circular patterns are just some design that you might see on the things that you will observe and mathematics has an important role to play on this.

According to Mr. Livin G. Rejuso, based on his slide share presentation that Mathematics can be considered as study of patterns. One of the ways in which we use number patterns is in the creation of unique and artistically pleasing design. This design is from number pattern in Modular Arithmetic formed modulo art.

Now, let us take a look once again our modulo 4 addition table.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

And let us take the encircled part.

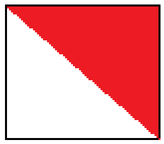
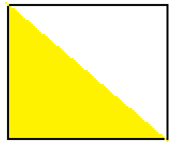
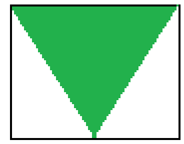
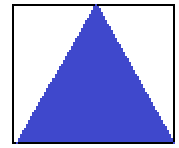
+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

We got;

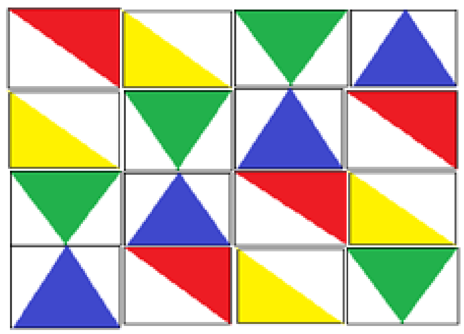
0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

Can you see a pattern?

Now, let us make a design based on the modulo 4 addition table. Our legend would be as follows:

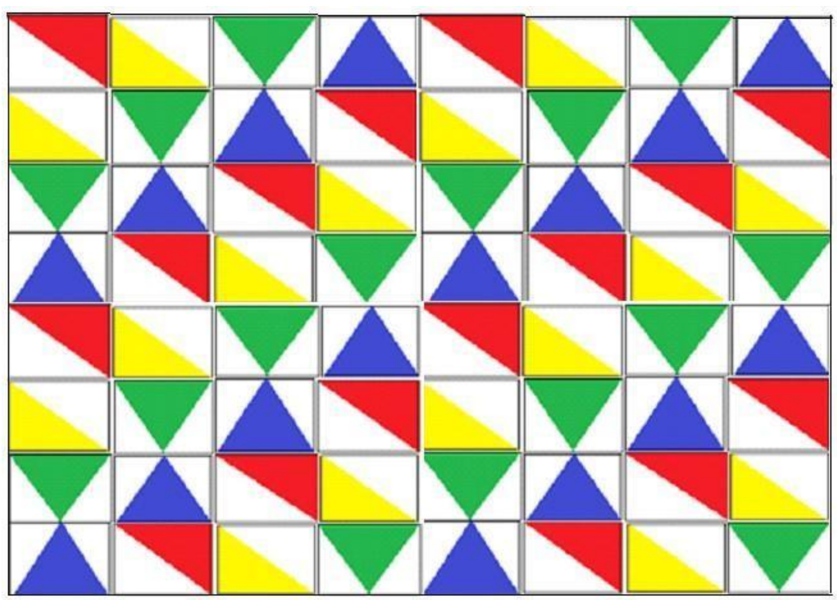
0  1  2  3 

Substituting each legend on the modulo 4 addition table, what we have got? After substituting the figure above based on the legend on the table, the pattern for this modulo 4 addition would be like this.

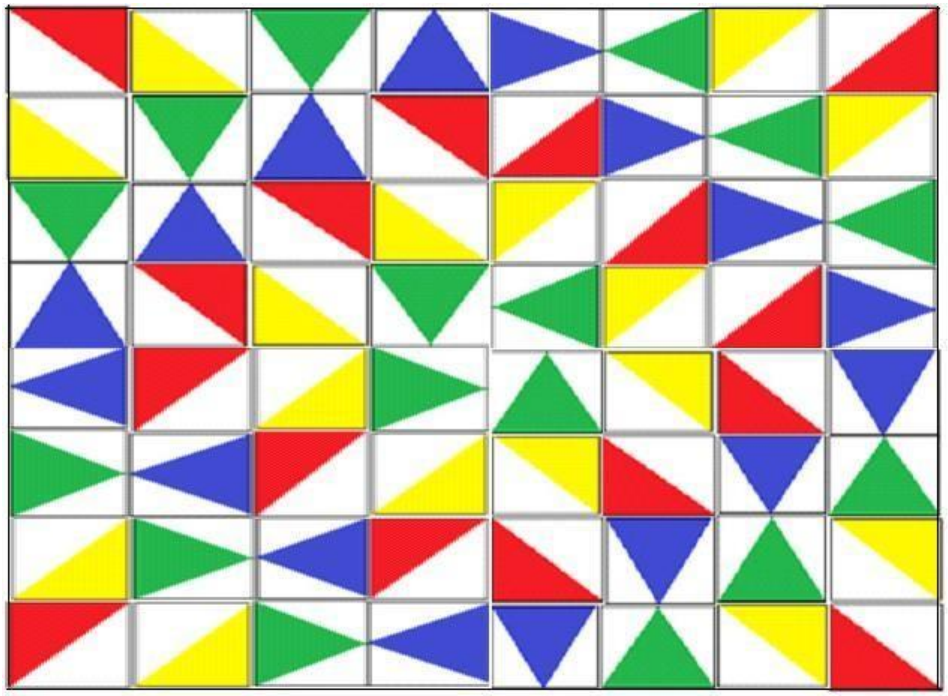


This design is also known as a Latin square design and the created pattern could be repeated, reflected or rotated.

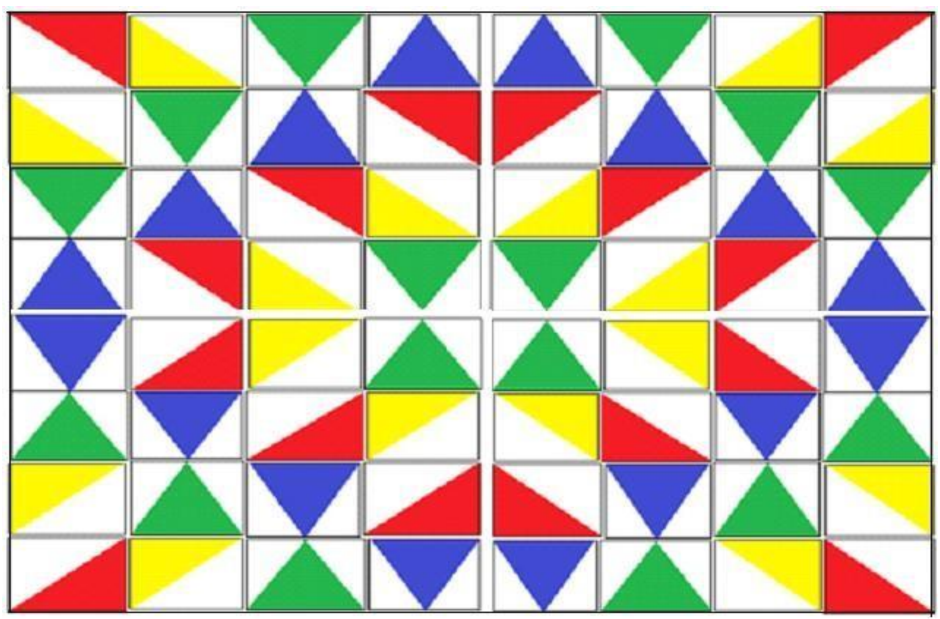
Repeated:



Rotated:



Reflected:



We may use also the converging pattern, standard kaleidoscope and the circular kaleidoscope. The illustration for these design are beyond on this lesson.

Self- Learning Activity

Directions: Do as what is indicated.

1. Make your own design with the use of modulo 4 multiplication.
2. Make a research about other patterns such as converging pattern, standard kaleidoscope and the circular kaleidoscope and do this on the created pattern that you made in item number 1.

Lesson 4.4

Group Theory

Specific Objective

At the end of this lesson, the student should be able to:

1. Relate clock arithmetic into the fundamental concept of group.
2. Explain the concept of group theory in permutation.
3. Define and differentiate group from an abelian group.
4. Name and list all the properties in order to says that the elements in a given set could be said a group.
5. Show that the integers under its operation form a group.
6. Define what an order of a group is.
7. Construct Cayley's table.
8. Explain what a symmetry of group and symbolic notation is.
9. Compare and evaluate symmetry of group and its symbolic notation.

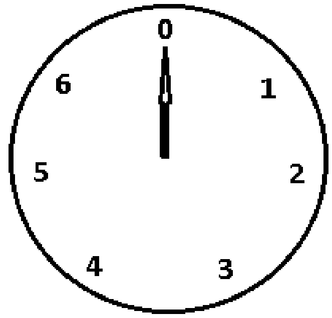
<u>Introduction</u>

Another important topics in the field of mathematics is what we called a group theory and of course, this could be important and very useful tool in teaching some topics in mathematics. Based on "The Evolution of Group Theory: A Brief", that the four major sources in the evolution of group theory together with the name of the creators and the date are as follows: a) Classical Algebra by J.L. Lagrange (1770), b) Number Theory by C.F. Gauss (1801), c) Geometry by F. Klein (1874) and d) Analysis by S. Lie (1874) and by H. Poincare and F. Klein (1876).

Discussion

A. Introduction to Group

Before we define group, let us recall the clock arithmetic!


0
1
2
3
4
5
6

Now, let's do the clock arithmetic.

$$3 + 5 =$$

$$4 + 3 =$$

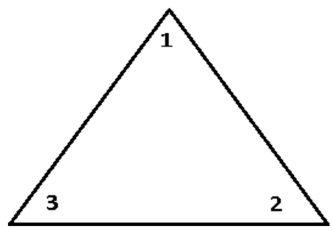
$$1 – 5 =$$

Here, we could say that clock arithmetic is also a modular arithmetic.

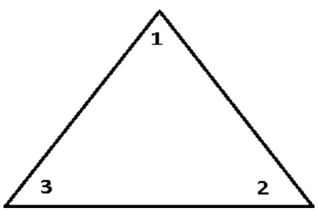
In 7 hours clock, this would be in an integer 7 or modulo 7.

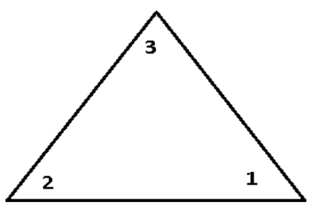
Thus, Z_7 = {1,2,3,4,5,6,,0}

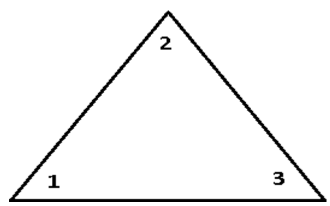
Let us answer the question, "How many ways can you rotate/flip the triangle so that it looks the same before and after"?

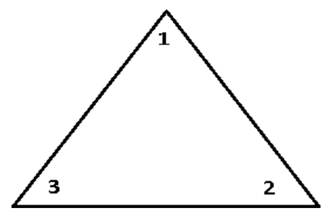


Original Position Rotate 120 degrees Rotate 120 degrees Rotate 120
degrees



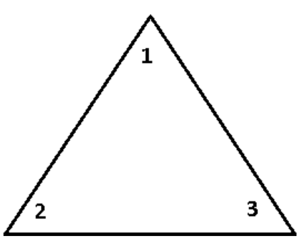


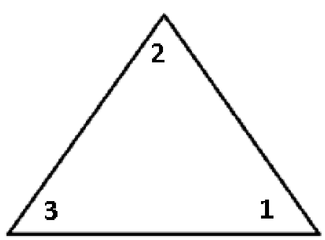


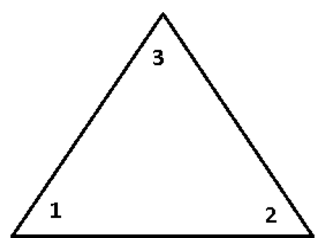


I r r^2 $r^3 = I$

Flip Flip and Rotate Flip and Rotate
Twice







f, but f^2 (flip twice = I) rf r^2f

So, G = {I,r, r^2,f, rf, r^2 f}

	Clock Arithmetic	Symmetry of Triangles	Set of an integer , Z
Element	{0,1,2,3,4,5,6}	{1, r, r^2, f, fr, fr^2}	{ …,-3, -2, -1, 0, 1, 2, 3, …}
Operation	addition	multiplication	Addition
Closure	yes	yes	yes
Identity	0	1	0
Inverse	3 (the inverse is -3) 3(the inverse is 4 since (3+4)=0mod7) x + (-x) = 0	r (the inverse is r^2 , since (r)(r^2) = r^3 = 1 $(x)(x^{-1}) = 1$	3 (the inverse is -3) -4 (the inverse is 4) x + (-x) = 0
Associative	(a+b)+c=a+(b+c)	(ab)c = a(bc)	(a+b)+c=a+(b+c)

In arithmetic, there are only two operations, the addition and the multiplication. The opposite of addition is subtraction and the opposite of multiplication is its reciprocal.

Operation	+	·
Opposition	negative	Reciprocal
Identity	0	1

Now, what is a GROUP?

In our previous lesson, we discussed operations modulo n. Now, let us consider the set of elements {0,1,2,3,4,5} and the operation is addition modulo 6. Here, we have only one operation.

+	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

The previous table is an example of an algebraic system called group. An algebraic system is a set of elements along with one or more operations for combining the elements. The real numbers with the operations of addition and multiplication are an example of an algebraic system. Mathematicians classify this particular system as a field.

Definition : Group

A group is a set of elements, with one operation and it must satisfy the following four properties:

P1: The set is close with respect to the operation. For all a, b ∈ G, then a * b ∈ G. Note that the operation would be + or · or in general * .

P2: The operation satisfies the associative law. For all a, b, c ∈ G, then (a * b) * c = a * (b * c)

P3: There must be an identity element. For every e ∈ G, such that e * a = a * e = a for all a ∈ G

P4: Each element has an inverse. For each a ∈ G then for every a^{-1} ∈ G, such that a * a^{-1} = a^{-1}* a = e.

Example 1: The binary operator addition mod 5, denoted by * is defined on the set Z = {0,1,2,3,4}. Complete the table and show that (Z, *) is a group.

*	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

P1: (Z, +) is closed since all members of the Cayley Table are in Z.

P2: In number theory, (a + b) + c mod n = a + (b + c) mod n. Hence, it is associative

P3: The identity element is 0, e = 0.

P4: Each element has an inverse. $0^{-1} = 0$; $1^{-1} = 4$; $2^{-1} = 3$; $3^{-1} = 2$; $4^{-1} = 1$

Since all the properties satisfied, hence it is a group.

So, For all $n \in$ N, the *integers mod n*, which we denote Zn, forms a group under addition. Then, the identity is 0, and the inverse of x is $-x$.

Example 2: Show that the integers with addition as the operation form a group.

Solution:

To be able to show that the integers with addition as the operation form a group, the four properties must be satisfied.

P1. Let a, b ∈ Z. Now, a + b ∈ Z and (–a) + (– b) ∈ Z. Hence it is closed under addition.

P2. Let a, b, and c are element of an integer. The associative property of addition holds true for the integers,i.e. (a + b) + c = a + (b + c).

P3. Let Z = {…-3,-2,-1,0,1,2,3,…}. The identity element of Z is 0 and 0 is an integer. Hence, there is an identity for addition.

P4. Let Z = {…-3,-2,-1,0,1,2,3,…}. Each element of Z has an inverse, i.e. if a ∈ Z, then –a is the inverse of a.

Because each of the four conditions of a group is satisfied, the integers with addition as the operation form a group or (Z, +) is a group.

Recall that the commutative property for an operation states that the order in which two elements are combined does not affect the result. For each of the groups discussed, the operation has satisfied the commutative property. For example {0,1,2,3,4,5} with addition modulo 6 satisfies the commutative property.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Groups in which the operation satisfies the commutative property are called commutative group or abelian group (came after the name of Neil Abel). If the group is a non-commutative property, it is called a non-abelian group.

Example 3: The binary operator multiplication mod 5, denoted by * is defined on the set Z = {0,1,2,3,4}. Complete the table and show that (Z, *) is a group.

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Condition one, $1 \leq m \leq 5$ and condition two, m and 5 must be relative prime, i.e. gcd(m,5) = 1

U(5) = {1,2,3,4} and we claim that U(5) is a group under multiplication. Note that U(n) is called units mod n.

·	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

For a given integer $n > 1$, let m be an integer s.t. $1 \leq m < n$ and gcd(m,n) = 1. Then the set of all such integers m forms a group, denoted by U(n), called the units modulo n.

B. Cayley Tables of Groups

In our previous lesson, we discuss about "Cayley's Table". Here, we are going to define first the meaning of an order of the group.

Definition: Order of the group is just the number of element in a group and it is denoted by |G|.

*	e
e	e

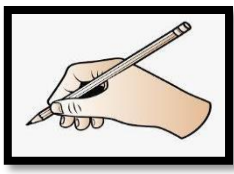
Illustration above is what we called a group of order 1. This group is called as a trivial group.

*	e	a
e	e	a
a	a	e

This group is called a group of order 2.

*	e	a	b
e	e	a	b
a	a	?	?
b	b	?	?

And this one is a group of order 3. Can you complete the table?



Self- Learning Activity

Direction: Do as indicated.

1. Show that Z = {0,1,2,3,4,5,6} under addition forms a group.

2. Show that Z = {0,1,2,3,4,5,6} under multiplication forms a group.

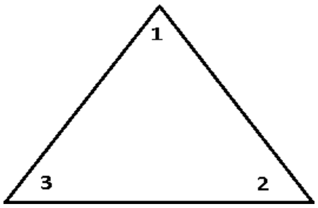
3. Below is a partially completed Cayley table of a group. Complete the table in the unique possible way (G,*) = {a,b,c,d,e} and find the inverse of the ff: a,b,c,d, and e.

*	A	b	c	d	e
a	B	c	?	?	?
b	?	?	e	a	?
c	?	?	?	b	?
d	E	?	?	?	?
e	A	b	c	d	e

C. Symmetry of Groups

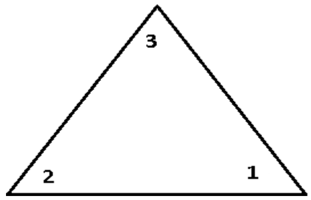
The concept of group is very general. The elements that make up a group do not have to be numbers, and the operation does not have to be addition or multiplication. Symmetry group is another type of group and it is based on regular polygon (polygon whose sides are on the same length and with the same angle measure).

Original Position



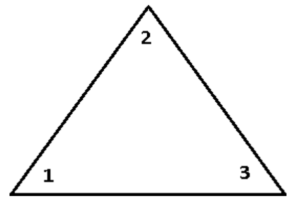
I

Rotate 120 degrees



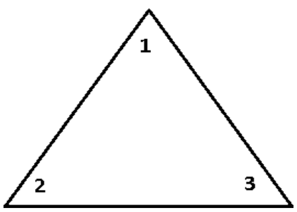
R120

Rotate 120 degrees



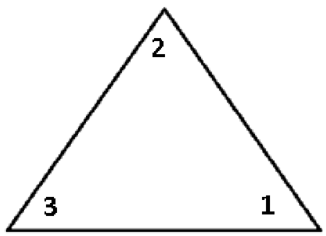
R240

Flip



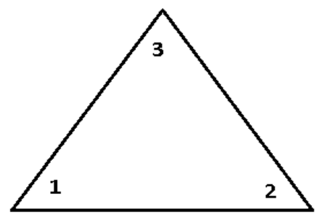
f

Flip and Rotate



rf

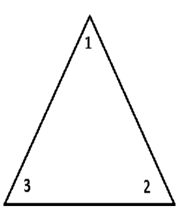
Flip and Rotate Twice



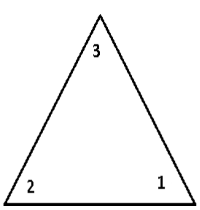
r^2f

A group must have an operation, a method by which two elements of the group can be combined to form a third element that must also be a member of the group. The operation we will use is called "followed by" and it symbolized by Δ.

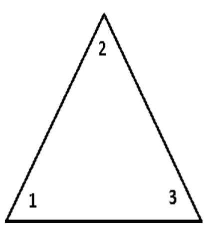
Original Position Rotate 120 degrees Rotate 120 degrees Flip Flip and Rotate Flip and Rotate Twice



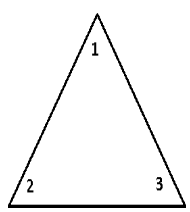
I



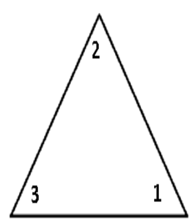
R120



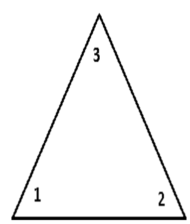
R240



f

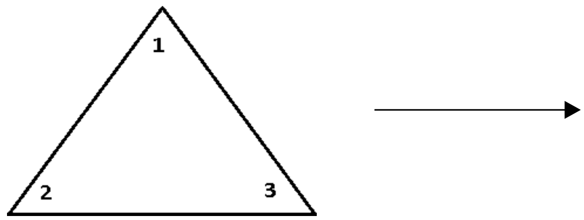


Rf



R2f

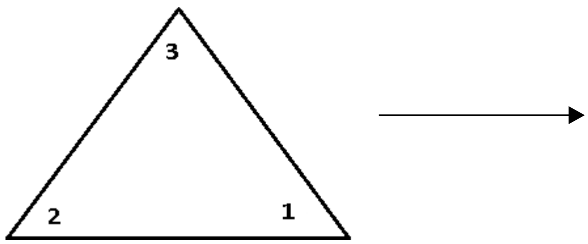
Example 1: Find f Δ R120



F
o
l
l
o
w
e
d
b
y
R
1
2
0

Hence, f Δ R120 = Rf

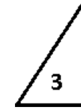
Example 2: Find R120 Δ R240



Hence, R120 Δ R240 = I

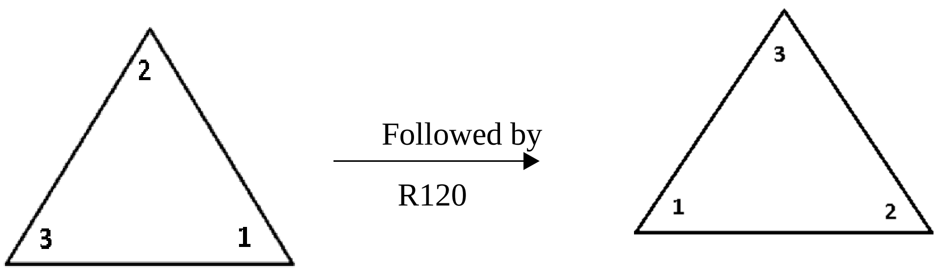
F
o
l
l
o
w
e

d	2
b	4
y	0
R	

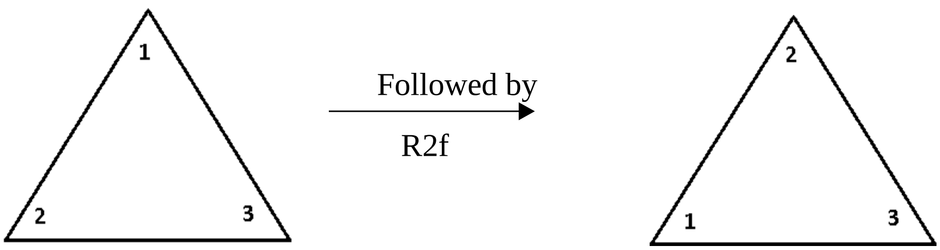


Example 3: Find f Δ (Rf Δ R120).

First, find Rf Δ R120

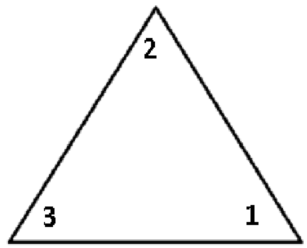


So, Rf Δ R120 = R2f Now, find f Δ R2f



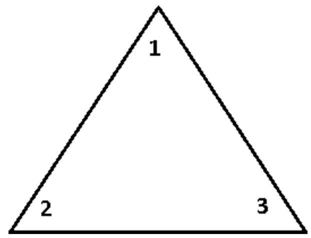
Hence, f Δ (Rf Δ R120) = R240

Example 4: Is Rf Δ R240 = R240 Δ Rf?

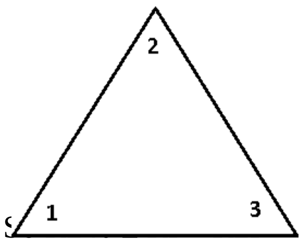


Followed by

R240

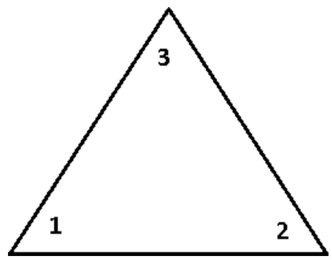


So Rf Δ R240 = f Now, find R240 Δ Rf.



Followed by

Rf



Hence, Rf Δ R240 and R240 Δ Rf is not equal so it is not commutative

D. Symbolic Notation

The operation notation for symmetry triangle can be presented into other notation called as symbolic notation and/or permutation.

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ A= R120= $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ B= R240= $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

C= f= $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ D= rf= $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ E = r2f= $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$A \,\Delta\, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Delta \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$

$B \,\Delta\, D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \Delta \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = E$

$$A \,\Delta\, C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Delta \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = E$$

$$D \,\Delta\, E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \Delta \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = A$$

Also, to find the inverse of any of these arrangement, say A, we need to follow the notation such that;

$$A \,\Delta A^{-1} = I.$$

or in other words that A followed by A^{-1} is equal to its identity.

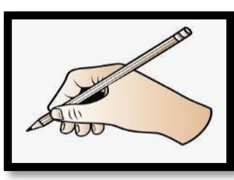
Now, what would be the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

To be able to find the inverse of A, remember that $A\Delta\, A^{-1} = I$. So;

$$A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Let us check:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Delta \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$



Self- Learning Activity

Given that:

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $R120 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ A= $R240 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & \end{pmatrix}$ B=

$C = f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $D = rf = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $E = r2f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

A. Direction: Evaluate the following:

a) R240 Δ R2f

b) R120 Δ f

c) r2f Δ (Rf Δ f)

d) rf Δ Rf

e) (r2f Δ Rf) Δ f

f) (R240 Δ f) Δ (r2f Δ f)

B. Based on the given, find the inverse of the following:

a) A b) B c) C d) D e) E

Chapter Test 4

Test 1. TRUE OR FALSE

A. True or False

Directions: Read the following statement carefully. Write the T if the given statement is true and F if it is false. Your answer must be written before each item.

1. In a modular arithmetic, if a, b ∈ Z and n ∈ Z^+, then a ≡ b (mod n) if and only if n | a – b.

2. The number n in a ≡ b (mod n) is called modulus.

3. The statement a ≡ b (mod n) is called a congruence.

4. 18 ≡ 12 (mod 3) is not a true congruence.

5. For arbitrary integers a and b, a ≡ b(mod n) if and only if a and b do not have the same remainder when divided by n.

6. Let n > 0 be fixed and a, b, c and d are arbitrary integers, then, if a ≡ b(mod n) and c ≡ d(mod n), then a + c = b + d (mod n) and ac ≡ bd (mod n).

7. The remainder in 14 (mod 3) is 3.

8. The remainder in (23 + 38) mod 12 is 2.

9. Statement 12 ≡ 7 (mod 5) is equivalent to 22 ≡ 7 (mod 5).

10. Statement (14 – 21) mod 5 is equal to 2.

11. In mod 7 arithmetic, the multiplicative inverse of 2 is 4.

12. In ISBN, the last digit or the 13^{th} digit is called verified digit.

13. Cryptology is the study of codes while cryptography is the art of writing and solving them.

14. A group of order 1 is called an abelian group.

15. A set of G = Z_5 is a group.

16. A set of integer is closed under addition and multiplication.
17. In a 12-hour clock, 8 + 13 = 21.

18. A clock arithmetic is also called as modular arithmetic.

19. The identity element of addition is 1.

20. The operation notation for symmetry triangle can be presented into another notation called symbolic notation or permutation.

Test 2. **MULTIPLE CHOICE**

Directions: Read the following statement carefully. Choose only the letter of the best correct answer. Write your answer before each item.

21. Which of the following is a false statement?

a. $-7 \equiv 3 \bmod 5$

b. $33 \equiv 5 \bmod 7$

c. $34 \equiv 4 \bmod 8$

d. $15 \equiv 1 \bmod 7$

22. The expression 13 (mod 6) is congruent to:

a. 1

b. 2

c. 3

d. 4

23. What is the remainder when the difference between 6002 and 601 is divided by 6.

a. 0

b. 1

c. 2

d. 3

24. In a 15-hour clock, what is the result if we add 47 to 7?

a. 5

b. 8

c. 7

d. 9

25. Ernesto’s family planned a vacation 25 months from July 2012. What year and month is their family vacation?

a. June, 2014

c. August, 2015

b. August, 2014 d. June, 2015

26. Today is Monday; what day will it be in 13 days?

a. Tuesday c. Sunday

b. Wednesday d. Friday

27. What are the values of d so that 15(mod d) = 3?

a. d = {4, 6, 12} c. d = {2, 3, 6}

b. d = {8, 12, 24} d. d = {5, 7, 13}

28. Let Z_5 = {0, 1, 2, 3, 4}. What would be the value of 4 + 3?

a. 7 c. 3

b. 2 d. 0

29. In item number 28, what is the additive inverse of 3?

a. -3 c. 2

b. 1 d. 0

30. The statement 96(mod 13) is equal to:

a. -5 c. 5

b. 1/5 d. 0

31. The statement (15·23)mod 11 is equal to:

a. 1 c. 3

b. 2 d. 4

32. The statement [(11 mod 8) x (15 mod 8)] mod 8 is equal to:

a. 5 c. 3

b. 4 d. 2

33. Which of the following is NOT a property of a group?

a. Close
b. Associative
c. Inverse
d. Reciprocal

34. Let A = addition, B = Subtraction, C= Multiplication and D = Division. Which of the following is/are the operations of arithmetic?

a. A only
b. B and D
c. A and C
d. C only

35. Groups in which the operation satisfies the commutative property are called:

a. Abelian group
b. Non-abelian group
c. Abel group
d. Non-abel group

36. Group of order 1 is called:

a. Unit group
b. One group
c. Sole group
d. Trivial group

37. Disregarding A.M. or P.M., if it is 5 o'clock now, what time was it 57 hours ago?

a. 7 o'clock
b. 8 o'clock
c. 9 o'clock
d. 10 o'clock

38. What is the additive inverse of 7 in mod 16 arithmetic?

a. 7
b. 8
c. 9
d. 10

39. A book has an ISBN of 978-0-7432-5820-?. What would be the check digit?

a. 3
b. 4
c. 5
d. 0

40. What is the check digit for the UPC whose number is 0-25192-21221-?.

a. 3 c. 5

b. 4 d. 0

41 – 43. Complete the table for Z_4 under addition and answer the given questions below.

+	0	1	2	3
0				
1				
2				
3				

41. What is 3 + 2 = 5(mod 4).

a. 0 c. 2

b. 1 d. 3

42. What is the inverse of 2?

a. 0 c. 2

b. 1 d. 3

43. What is the identity?

a. 0 c. 2

b. 1 d. 3

44. How many arrangement are there if you are going to rotate/flip a

square? a. 4 c. 12

b. 8 d. 24

45. For a given integer n > 1, let m be an integer such that $1 \leq m < n$ and gcd (m,n) = 1. Then the set of all such integers m forms a group, denoted by U(n) is called:

a. units modulo n c. unity of modulo

b. Group
d. union modulo n

46 – 48. Complete the table and answer the questions below.

*	e	a	b
e	e	a	b
a	a		
b	b		

46. What is a * a?

a. a
c. b
b. e
d. none

47. What is the inverse of a?

a. a
c. b
b. e
d. none

48. What is (a * b) * e?

a. a
c. b
b. e
d. none

49 – 51. Use the table below to encrypt the given words.

Cyclical English Alphabet

a	b	c	d	e	f	g	h	I	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

49. What is the correct equivalent code for letter H using the congruence $c \equiv (p + m) \bmod 26$, where $m = 22$?

a. a
c. c
b. b
d. d

50. What equivalent letter if we decode letter S using the congruence p ≡ (c + n) mod 26, where n = 26 – m based on the previous item?

a. w c. y

b. x d. z

51. What equivalent letter for A if we decode this letter using the congruence c ≡ (3p + 5)mod 26?

a. Y c. P

b. A d. E

52. Who is the card issuer if a card number has a prefix number of 37?

a. Master Card c. Visa

b. American Express d. Discover

53. It is a method on how the ciphertext message converted into its equivalent plaintext.

a. Encrypt c. decrypt

b. translation d. coding

54 – 60. Let

A = (**1 2 3**) **I** = (**1 2** **1 2 3**)
3) **B** =

2 3 1 **1 2 3** (**1 2**

E = (**1 2 3**) **D** = (**1 1 2 3**)
2 3) **C** =

2 1 3 **3 2 11** (**3 2**

54. What is A Δ B?

a. A c. I

b. B d. D

55. What is C Δ C?

a. A

b. B

c. I

d. D

56. Is A Δ B and C Δ C equal?

a. Yes
b. No
c. Maybe
d. Insufficient Information

57. What is C Δ E?

a. A
b. B
c. I
d. D

58. What is (B Δ E) Δ C?

a. B
b. C
c. A
d. E

59. Is (B Δ E) Δ C = B Δ (E Δ C)?

a. Yes
b. No
c. Maybe
d. Insufficient Information

60. What is the inverse of B?

a. B
b. C
c. A
d. E

-END OF CHAPTER TEST-

B

https://www.math.uci.edu/~mathcircle/materials/Modular_Arithmetic_and_Cryptography_Jan22_2015.pdf

http://www.acm.ciens.ucv.ve/main/entrenamiento/material/ModularArithmetic-Presentation.pdf

http://www.dehn.wustl.edu/~blake/courses/WU-Ed6021-2011-Summer/handouts/Clock%20Arithmetic.pdf

https://www.slideshare.net/nivilosujer/modulo-art-computerbased-design

https://www.khanacademy.org/computing/computer-science/cryptography/modarithmetic/e/modulo-operator

https://www.math.lsu.edu/~adkins/m7200/GroupHistory.pdf

MODULE FIVE

DATA MANAGEMENT

CORE IDEA

Statistical tools derived from mathematics are useful in processing and managing numerical data to describe a phenomenon and predict values.

Learning Outcome:

5. Use a variety of statistical tools to process and manage numerical data.
6. Use the methods of linear regression and correlations to predict the value of a variable given certain conditions.
7. Advocate the use of statistical data in making important decisions.

Unit Lessons:

Lesson 5.1 The Data
Lesson 5.2 Measures of Central Tendency
Lesson 5.3 Measures of Dispersion
Lesson 5.4 Measures of Relative Position
Lesson 5.5 Normal Distributions
Lesson 5.6 Linear
Correlation Lesson 5.7
Linear Regression

Time Allotment: Ten lecture hours

Lesson 5.1

The Data

Specific Objectives

1. To Understand the nature of statistics.
2. To gain deeper insights on the different levels. of measurements.
3. To clarify the meaning of some important key concepts.
4. To explore the strengths and limitations of graphical representation.

It is written in the Holy Book that "the truth shall set us free;" therefore, understanding statistics paves the way towards intellectual freedom. For without sufficient knowledge about it, we may be doomed to a life of half-truth. Statistics will provide deeper insights to critically evaluate information and to bring us to the well-lit arena of practicality.

Discussions

General Fields of Statistics: Descriptive Statistics and Inferential Statistics

Descriptive Statistics. If statistics, in general, basically deals with analysis of data,

then descriptive statistics part of the general field is about "describing" data in symbolic forms and abbreviated fashions. Sometimes we dealing with a large amount of data and that it is impossible to describe it as it is being a large amount of data but descriptive statistics will provide us certain tools to make the data manageable to handle and conveniently neat to describe.

To explore the characteristics of descriptive statistics, let us create a fictitious situation. What does it mean if someone tells you that majority of workers earn approximately P20,000.00 in a month? Were you able to dissect the idea behind the plain statement? Does it trigger your mind to question further?

This statement is a piece of information that described a particular trait or characteristic of a group of workers. Supplied with this singular information but armed with statistical inquisitiveness, descriptive statistics can further describe the given information to the extent of its depth and breadth.

Inferential Statistics. We could probably argue that descriptive statistics, with its characteristic to describe, is sufficient to depict any given information. While it is effective to describe a manageable size of data, it can hardly engulf a sizeable amount of data. Thus, for this kind of situation, inferential statistics is the alternative technique that can be used. Inferential statistics has the ability to “infer” and to generalize and it offers the right tool to predict values that are not really known.

Let us consider the fictitious situation we made under descriptive statistics, but this time instead of reporting the approximate monthly earning of some workers, we want to determine the estimated monthly earnings of all the workers in a certain region. By attempting to apply descriptive statistics, it would be impossible to ask all the workers in the entire region about their monthly income. But by using inferential statistics, we would instead practically decide to select just a small number of workers and ask them of their monthly income. From there, we can predict or approximate in a “more or less” fashion the monthly income of all workers in the entire region.

Of course, inference or generalization is a risky process that is why we need to ensure that the small group of workers we selected are the approximate representative of the workers in the entire region. But nevertheless, this inference or prediction is better than chance accuracy.

Measurement

It essentially means quantifying an observation according to a certain rule. For instance, the presence of fever can be quantified by using a thermometer. Body weight can be determined by using a weighing scale. Or the mental ability can be quantified by using written examination that can generate scores. The quantification sometimes can be done is simply counting. In quantifying an observation, there are two types of quantitative informations: variable and constant. A **variable** is something that can be measured and observed to vary. While a **constant** is something that does not vary, and it only maintains a single value.

Scales of Measurement

- Nominal Scale : Categorical Data
- Ordinal Scale : Ranked Data
- Interval/Ratio Scale : Measurement Data

To quantify an observation, it is necessary to identify its scale of measurement, it is known as level of measurement. Scale of measurement is the gateway to the fascinating world of statistics. Without sufficient knowledge of it, all our statistical learnings lead to nowhere.

Nominal Scale. It concerns with categorical data. It simply means using numbers to label categories. This is done by counting the occurrence of frequency within categories. One condition is that the categories must be independent or mutually exclusive. This implies that once something is identified under a certain category, then that something cannot be reassigned at the same time to another category.

An example for this, if we want to measure a group of people according to marital status. We can categorize marital status by simply assigning a number. For instance *"1"* for single and *"2"* for married.

Marital Status: Single (1) and Married (2)	
(1)	(2)
𝍸	𝍸 𝍸 𝍸

Obviously, those numbers only serve as labels and they do not contain any numerical weight. Thus, we cannot say that married people (having been labelled 2) have more marital status than single people (having been labelled *1*).

Ordinal Scale: It concerns with ranked data. There are instances wherein comparison is necessary and cannot be avoided. Ordinal scale provides ranking of the observation in order to generate information to the extent of "greater than" or "less than;". But the ranked data generated is limited also the extent of "greater than" or "less than;". It is not capable of telling information about how much greater or how much less.

Ordinal scale can be best illustrated in sports activities like fun run. Finding the order finish among the participants in a fun run always come up with a ranking. However, ranked data cannot provide information as to the difference in time between 1st placer and 2nd placer. Relative to this, reading reports with ordinal information is also tricky. For example, a TV commercial extol a certain brand for being the number one product in the country. This may seem acceptable, but if you learned that there is no other product then definitely the message of the commercial will be swallowed with an smirking face.

Interval Scale: It deals with measurement data. In the nominal scale, we use numbers to label categories while in the ordinal scale we use numbers to merely provide information regarding *greater than* or *less than.* However, in interval scale we assign numbers in such a way that there is meaning and weight on the value of points between intervals. This scale of measurement provides more information about the data. Consider the comparative illustration below:

Academic performance of five students in a certain class

	Student A	Student B	Student C	Student D	Student E
Interval Data	99	74	73	70	70
Ordinal Data	1st	2nd	3rd	4th	5th
Nominal Data	Passed	Failed	Failed	Failed	Failed

As you may have noticed, the interval scale provides substantial information about the grades of students. Student A earned a grade of 99, and so on and so forth. Now look at the information given by ordinal data. It is simply about ranking. With this of information, Student B can proudly and rightfully claim the 2nd place in the ranking. Ordinal scale is a trusted friend to keep a secret, that the grade of student B though claiming 2nd place is actually 74. Let us analyze the nominal data in our example. With this scale, it is also alright for the school sadly to announce that only one student passed and four students failed. Nominal data cannot provide more information specifically provide brighter limelight to student A. Audience may assume that Student A just got passing grade a little bit higher than the passing mark but student A grade of 99 will remain hidden forever.

Ratio Scale. This is an extension of an interval scale. It also pertains with measurement data but ratio's point of view is about absolute value. Because of this, we oftentimes cannot utilize ratio scale in the social sciences. We cannot justify an absolute value to gauge intelligence. We cannot say that our student A with a grade of 99 has an intelligence several points superior than student E who hardly but successfully achieved a grade of 70.

Key Concepts in Statistics

Population. A *population* can be defined as an entire group people, things, or events having at least one trait in common (Sprinthall, 1994). A common trait is the binding factor in order to group a cluster and call it a population. Merely having a clustering of people, things or events cannot be considered as a population. At least one common trait must be established to make a population. But, on the other hand, adding too many common traits can also limit the size of the population. In the illustration below, notice how a trait can severely reduce the size or membership in the population.

A group of students (this is a population, since the common trait is "students")
A group of male students.

A group of male students attending the Statistics class

A group of male students attending the Statistics class with iPhone

A group of male students attending the Statistics class with iPhone and Earphone

As we read the list, we can mentally visualize that the size of the population is dramatically becoming smaller and as we add more traits we may wonder if anyone still qualifies. The more common traits we add, the more we reduce the designated population.

Parameter. In gauging the entire population, any measure obtained is called a *parameter*. Situationally, if someone asks you as to what is the parameter of the study, then bear in mind that he is referring to the size of the entire population. In some situations where the actual size of the population is difficult to obtain; the parameters are in the form of estimate or inference.

Sample. The small number of observation taken from the total number making up a population is called a *sample*. As long as the observation or data is not the totality of the entire population, then it is always considered a *sample*. For instance, in a population of 100, then 1 is considered as a sample. 30 is clearly a sample. It may seem absurd but 99 taken from 100 is still considered a sample. Not until we include that last number (making it 100) could we claim that it is already a population and no longer a sample.

Statistic. In gauging the sample, any measure obtained from the sample is called a statistic. Whenever we describe the sample, then it is called statistics. Since a sample is easier to observe or gather than the population, then statistics are simpler to gather than the parameter.

Graphical representation

Graphs. It is another way to visually show the behavior of data. To create a graph, distribution of scores must be organized. For instance, in the scores provided below, presenting the scores in an unorganized manner can provide confusing or no information at all; Reporting raw can even hide some significant scores to be noticed.

120, 65, 110, 75, 105, 80, 105,

85, 100, 85, 100, 90, 95, 90, 90

But when we arrange the scores from highest to lowest, which is a form of **score distribution**, some pieces of information can gradually brought forth and exposed.

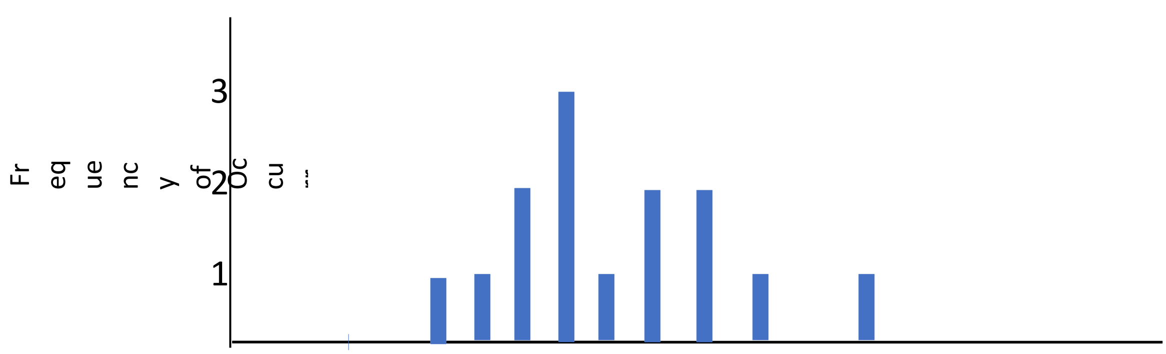
Distribution of Scores

120
110
105
105
100
100
95
90
90
90
85
85
80
75
65

The score distribution can still be organized in a form of a frequency distribution. Frequency distribution provides information about raw scores, and the frequency of occurrences. Frequency distribution provides clearer insights about the behavior of scores.

X (Raw score)	f (Frequency of Occurrence)
120	1
110	1
105	2
100	2
95	1
90	3
85	2
80	1
75	1
65	1

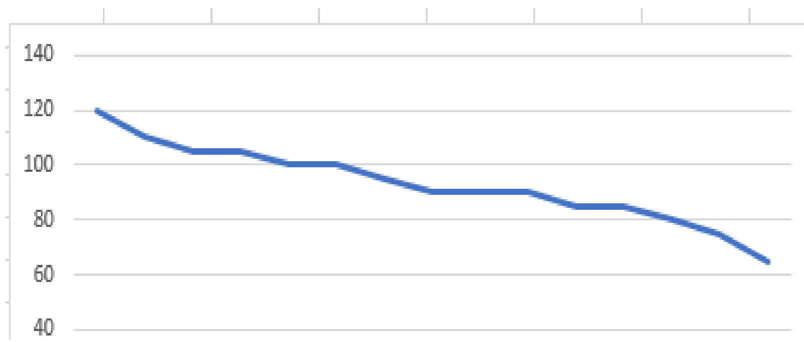
Another alternative way of presenting data in frequency distribution is to present them in a tabular form. A tabular form has the advantage of showing the visual representation of the data. This kind of presentation is more appealing to the general audience.

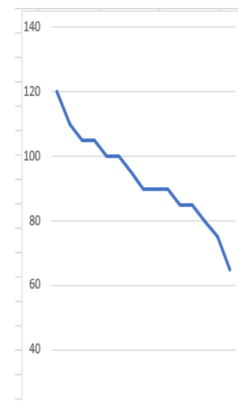


0 60 70 80 90 100 110 120 130

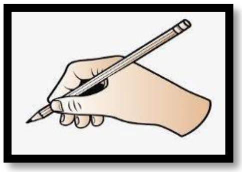
Raw scores

Another way of showing the data in graphical form is by using Microsoft Excel, as also illustrated in the graphs below. It is the frequency polygon of the scores in our cited example above.





Notice in the illustration of the frequency polygon, the two graphs may appear different but they are actually the same and they disclose the similar information. This illustration will allow you realize that unless you see things with a critical eye, a graph can create a false impression of what the data really reveal. This is an obvious situation showing how graphs can be used to distort reality if you are not equipped with a critical statistical mind. This type of deceitful cleverness in distorting graphs is common in some corporations devising the tinsel to camouflage and also to portray some gigantic leaps in sales in order to attract more clients or buyers.



Learning Activity 5.1

Indicate which scale of measurement- nominal ordinal or interval is being used.

1. Both Globe and Smart phone number prefix 0917 and 0923 served 1 million and 2.5 subscribers, respectively.

2. The Philippine Statistics Office announces that the average height of Filipino male is 156.41 cm tall.

3. Postal Office shows that 4,231 individuals have a zip code of 4231.

4. The Sportsfest committee posted the names of individuals with their order of finish for the first 50 runners to reach the finish line.

5. The University Admission Office posted the names and scores of student applicants who took the entrance examination.

Lesson 5.2

Measures of Central Tendency

Specific Objectives

:

1. To know the different measures of central tendency.
2. To comprehend the limitations of the three measures.
3. To realize the effect of the measures in the distribution.
4. To critically know how to select appropriate measure to describe a certain distribution.

Discussion

As we venture into the realm of descriptive statistics, let us now focus in describing the nature of a quantitative data. By using an appropriate descriptive technique, we can organize and neatly summarize small amounts and large amounts of data distribution. The procedure, utilizing measures of central tendency, allows us to precisely describe the centrality of data distribution.

Measures of central tendency are methods that can used to determine information regarding average, ranking, and category of any data distribution. Mean, median and mode are the three tools in obtaining the measures of central tendency. But only by knowing and using the appropriate tool that most accurate estimation of centrality can be achieved. The objective of the measures of central tendency is to describe the centrality of the distribution into a single numerical unit. This single numerical unit must provide clear description about the common trait being observed in the distribution of scores.

The Mean

The most widely used measure of the central tendency is the mean ($\bar{x}$ or $\bar{X}$). It is the arithmetic average of all the scores. The mean can be determined by adding all the scores together and then by dividing by the total number of scores. The basic formula for the mean is as follows:

The operational term "summation" indicating to add all measures of x

The raw scores

$$\bar{X} = \frac{\sum x}{N}$$

Mean

The entire number of observations being dealt with

In the example below concerning the annual income of 12 workers, the mean can be found by calculating the average score of the distribution.

X
Php 200,000.00
200,000.00
195,000.00
194,000.00
194,000.00
194,000.00
193,000.00
190,000.00
185,000.00
180,000.00
180,000.00
176,000.00

$\sum x$ = Php 2, 281,000.00

$$\overline{X} = \frac{\sum x}{N} = \frac{2,281,000.00}{12} = \text{Php } 190,083.00$$

In this example, the mean is an appropriate measure of central tendency because the distribution is fairly well-balanced. This means that there are no extremely high or extremely low scores in either direction that can unusually influence the average of the scores. Thus, the mean value of 190,083.00 represents the total picture of the distribution (i.e. annual incomes). This means that in a "more or less" or approximate fashion it describes the entire distribution.

Mean of Skewed Distribution. There are situations wherein the mean cannot be trusted to provide a measure of central tendency because it portrays an extremely distorted picture of the average value of a distribution of scores. For instance, let us still consider our example of annual incomes but this time with some adjustment. Let us introduce another score. The annual income of an affluent new neighbor who happened to move to this town just recently. This new neighbor has a frugal high annual income so extremely far above the others.

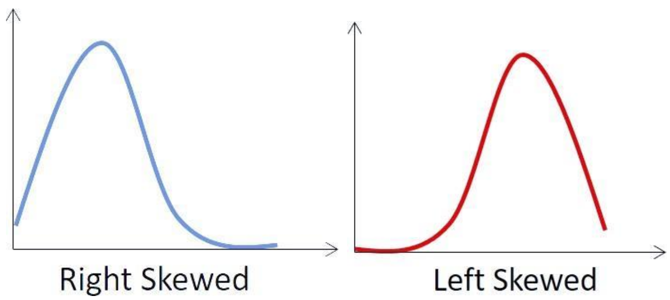
X	
Php 2, 500,000.00	New neighbor
200,000.00	
200,000.00	
195,000.00	
194,000.00	
194,000.00	
194,000.00	
193,000.00	
190,000.00	
185,000.00	
180,000.00	
180,000.00	
176,000.00	

$\sum x$ = Php 4, 481,000.00

$\overline{X} = \frac{\sum x}{N} = \frac{4,281,000.00}{13} =$ Php 367,769.00

As you may have noticed, the mean income of Php 367,769.00 this time provides a highly misleading picture of great prosperity for this neighborhood. The distribution was unbalanced by an extreme score of the new affluent neighbor. This is what we call an skewed distribution.

Here are some graphic illustration of a skewed distribution:



When the tail goes to the right, the curve is positively skewed; when it goes to the left, it is negatively skewed. The skew is in the direction of the tail-off of scores, not of the majority of scores. The mean is always pulled toward the extreme score in a skewed distribution. When the extreme score is at the low end, then the mean is too low to reflect centrality. When the extreme score is at the high end, the mean is too high.

The Median

The median is the point that separates the upper half from the lower half of the distribution. It is the middle point or midpoint of any distribution. If the distribution is made up of an even number of scores, the median can be found by determining the point that lies halfway between the two middlemost scores.

193,000.00
190,000.00
185,000.00
180,000.00

$$\text{Median} = \frac{(190,000+185,000)}{2}$$

Arranging scores to form a distribution means listing them sequentially either highest to lowest or lowest to highest. Unlike the mean, the median is not affected by skewed distribution. Whenever the mean cannot provide centrality because of extreme scores present, the median can be used to provide a more accurate representation.

Calculation of the Median

X

===========================

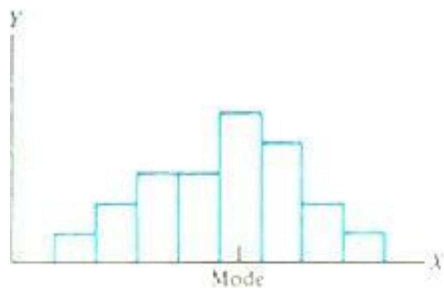
Php 2, 500,000.00
200,000.00
200,000.00
195,000.00
194,000.00
194,000.00 __
194,000.00 194,000.00 Median
193,000.00
190,000.00
185,000.00
180,000.00
180,000.00
176,000.00 __

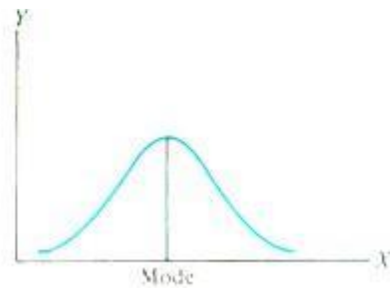
===========================

As you observed, even with the presence of extreme score at the high end of the distribution- the value of the median is still undisturbed.

The Mode

Another measure of central tendency is called the *mode*. It is the most frequently occurring score in a distribution. In a histogram, the mode is always located beneath the tallest bar.





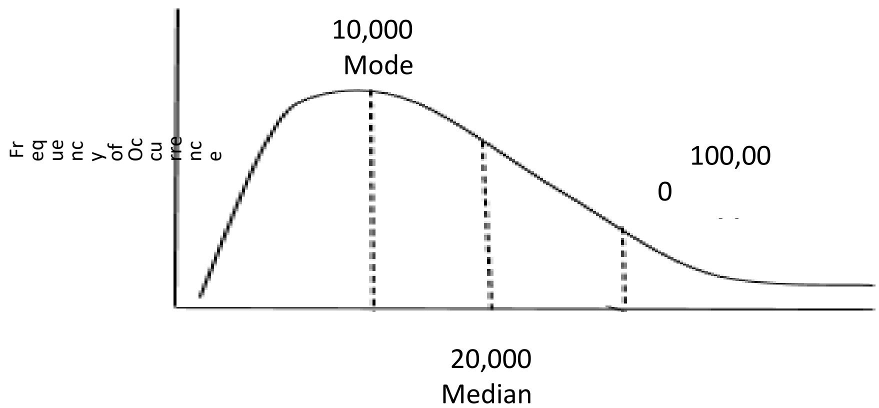
Finding the mode of a distribution of raw scores (Annual Income)

X	
Php 2, 500,000.00	
200,000.00	
200,000.00	
195,000.00	
194,000.00	**Mode**
194,000.00	**Mode**
194,000.00	**Mode**
193,000.00	
190,000.00	
185,000.00	
180,000.00	
180,000.00	
176,000.00	

The mode provides an extremely fast way of knowing the centrality of the distribution. You can immediately spot the mode by simply looking at the data and find the dominant constant. It is the frequently occurring scores.

Appropriate Use of the Mean, Median and Mode

The best way to illustrate the comparative applicability of the mean, median and mode is to look again at the skewed distribution.



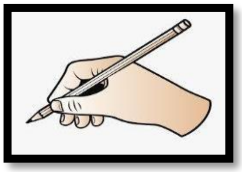
Distribution of monthly income per household in a certain municipality.

Most income is always skewed to the right because the low end has a fixed limit of zero while the high end has no limit. If we consider that the area of the curve is 100 percent, then the median is the exact midpoint of the distribution. The area below and above the median is both equal to 50 percent. Thus, if the median income is P20,000.00 this means that 50% of the households have an income below P20,000.00 and 50% of the households have an income above P20,000.00. On the other hand, the mean in our figure above indicates a high income of P 100,000. This makes the curve positively skewed. The value of the mean gives a distorted picture of reality. The value of the mean is being unduly influenced by few affluent income earners at the high end of the curve whose monthly income is almost around P 500,000.00. Looking at the modal income, which is P 10,000 per month, seemed also to distort the reality towards the low side. The *mode* is always the highest point of the curve. In this example, the mode represents the most frequently-earned income; it is far lower than the median income of P 20,000.00. Both the mean and the mode give a false portrait of distribution typicality and the truth lies somewhere in between.

Effects of the Scale of Measurement Used

The scale of measurement in which the data are based oftentimes dictates the measures of central tendency to be used. The interval data can entertain the calculations of all three measures of central tendency. The modal and ordinal data cannot be used to calculate for the mean. Ordinal mean can provide an extremely confusing wrong result. Since median is about ranking, a rank above the score falls and a rank below a score falls; the ordinal arrangement is necessary in finding the median. For the nominal data, however, neither the mean nor the median can be used. Nominal data are restricted by simply using a number as a label for a category and the only measure of central tendency permissible for nominal data is the mode.

In summary, if the interval data distribution is fairly well balanced, it is appropriate to use the mean to measure the central tendency. If the distribution of the interval data is skewed, you may either remove the outlier or adopt the median. If the interval data distribution manifests a significant clustering of scores, then consider to visually analyze the scores and find the presence of dominant constant which is the Mode.



Learning Activity 5.2

1. A class of 13 students takes a 20-item quiz on Science 101. Their scores were as follows: 11, 11, 13, 14, 15, 18, 19, 9, 6, 4, 1, 2, 2.

   a. Find the mean. b. Find the median c. Find the mode.

2. A day after, the of 13 students mentioned in problem 1 takes the same test a second time. This time their scores were: 10, 10, 10, 10, 11, 13, 19, 9, 9, 8, 1, 7, 8.

   a. Find the mean. b. Find the median c. Find the mode.

   d. Was there a difference in their performance when taking the test a second time?

3. For the set of scores: 1000, 50, 120, 170, 120, 90, 30, 120.

   a. Find the mean. b. Find the median c. Find the mode.

   d. Which measure of central tendency is the most appropriate, and why?

Lesson 5.3

Measures of Dispersion

Specific Objectives

1. To know the different measures of variability.
2. To comprehend the strengths of the three measures
3. To realize the effect of the measures in the distribution
4. To critically an select appropriate tool for a certain situation

The measures of central tendency only provide information about the similarity or typicality of scores. But to fully describe the distribution, we need to gain information about how scores differ or vary. The description of the distribution can only be complete if some information of its variability is known. To substantiate the information provided by the measures of centrality, some degree of dispersion must also be brought into the light.

Discussion

Measures of Variability

There are three measures of variability: the **range**, the **standard deviation** and the **variance**. These three measures give information about the spread of the scores in a distribution. Metaphorically, variability assert that a glass half-full is also half empty. Being half-full is about centrality and being half-empty is about variability.

The Range. The *range*, symbolized by R, describes the variability of scores by merely providing the width of the entire distribution. The range can be found by simply determining the difference between the highest score and the lowest score. This difference always has a single value answer.

The example below shows the calculation of the range from a distribution of annual incomes:

X	
Php 200,000.00	Highe
200,000.00	
195,000.00	
194,000.00	
194,000.00	HS-LS
194,000.00	=Range 200,000 –176,000
193,000.00	= 24,000
190,000.00	
185,000.00	
180,000.00	
180,000.00	
176,000.00	Lowe

The capability of the *range* is to give information about the scattering of the scores by merely using two extreme points. But one the hand, capability of range to report score deviation poses a severe limitation. If you add new scores within the distribution, the range can never report any changes in the deviation. Also, just by adding one extreme score amidst normal distribution can definitely increase or decrease in range even if there are no other deviations that transpired within the distribution. The range is not stable enough to indicate variability. But nevertheless it is still a method in finding the variability of any given distribution.

The Standard Deviation. The standard deviation (*SD*) is the life-blood of the variability concept. It provides measurement about how much all of the scores in the distribution normally differ from the mean of the distribution. Unlike the range, which utilizes only two extreme scores, SD employs every score in the distribution. It is computed with reference to the mean (not the median or the mode) and it requires that the scores must be in interval form.

A distribution with small standard deviation shows that the trait being measured is homogenous. While a distribution with a large standard deviation is indicative that the trait being measured is heterogeneous. A distribution with zero standard deviation implies that scores are all the same (i.e. 10, 10, 10, 10, 10). Although it may seem like stating the obvious, it is important to note that if all the scores are the same, there is no dispersion, no deviation, and no scattering of scores in the distribution --- so much so that there can never be less than zero variability.

In calculating the standard deviation, we can either use the computational method or the deviation method. Both methods provide the same answer. However, in this lesson, we will use the computational method because it is designed for electronic calculators.

The formula for computational method is provided below:

$$SD = \sqrt{\frac{\Sigma X^2}{N} - \bar{X}^2}$$

The raw score in a distribution is symbolized as X

The mean of a distribution is symbolized as $\bar{X}$

The number of scores in a distribution is symbolized as N

The formula simply states that the standard deviation (SD) is equal to the square root of the difference between the sum of raw score squared, which is divided by the number of cases, and the mean squared (Sprinthall, 1994). Below is an example on how to obtain the standard deviation using the computational method.

X	X^2
200,000	40,000,000,000
200,000	40,000,000,000
195,000	38,025,000,000
194,000	37,636,000,000
194,000	37,636,000,000
194,000	37,636,000,000
193,000	37,249,000,000
190,000	36,100,000,000
185,000	34,225,000,000
180,000	32,400,000,000
180,000	32,400,000,000
176,000	30,976,000,000

$N = 12$

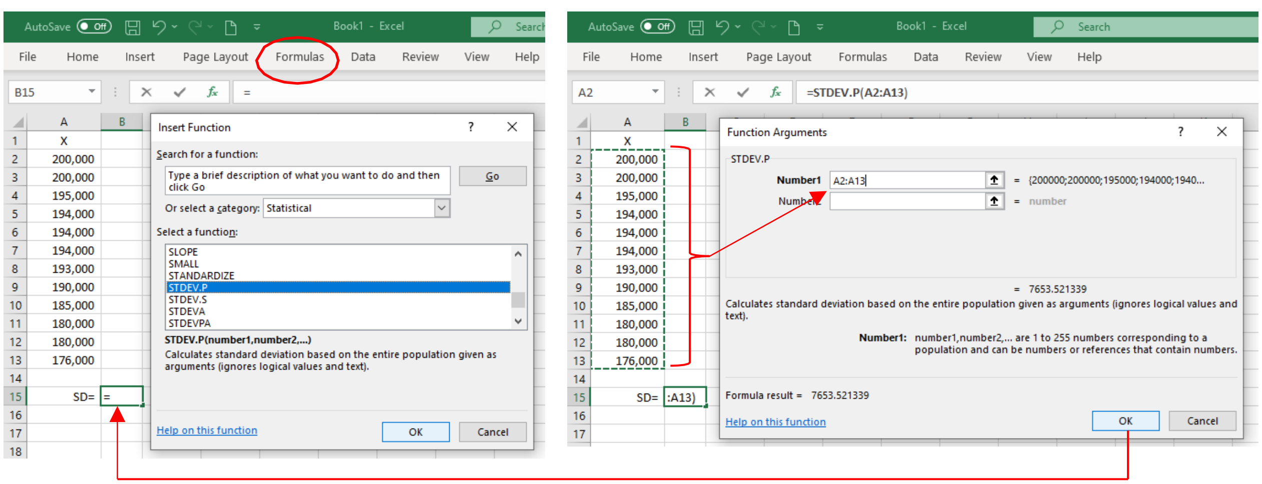
$\Sigma X = 2,281,000$ $\quad \Sigma X^2 = 434,283,000,000$

$\bar{X} = \frac{\Sigma X}{N} = \frac{2,281,000}{12} = 190,083$

$$SD = \sqrt{\frac{\Sigma X^2}{N} - \bar{X}^2}$$

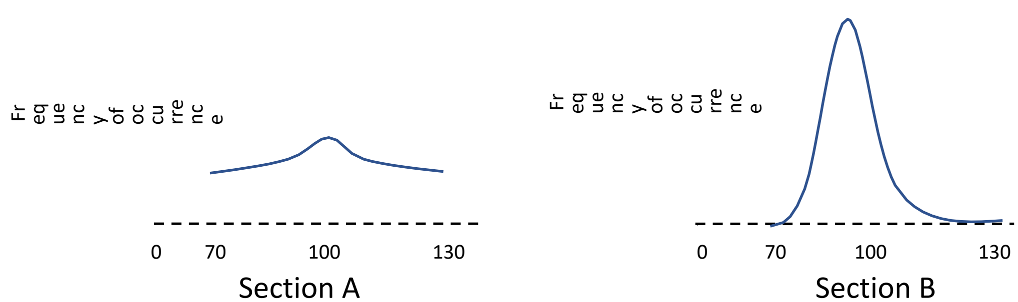
$$SD = 7653.521$$

Computer Note: Exploring MS Excel to find the value of SD



The concept of standard deviation can further be clarified by using an illustration of score distribution of students in Section A and in Section B, assuming that both distributions (Section A scores and Section B scores) have precisely the same measures of central tendency and the same range. The only unusual things about these two distributions is that they differ in terms of their standard deviations, Section A having a value that is greater than the value of Section B. The data are clearly shown in the figure below.

Math Quiz	Section A Scores	Section A Scores
Mean	100	100
Median	100	100
Mode	100	100
N	30	30
HS, LS, Range	130, 70, 60	130, 70, 60
SD	**10**	**2**



Two Frequency Distributions of Scores

As can be noticed in the figure above, there is just a slight bulge in the middle of the distribution of Section A. This means that it has many scores deviating widely from the mean (100) and this is the result of having a large standard deviation (10). However, Section B having a smaller standard deviation (2), most of the scores gathers closely around the mean (100) thereby creating a towering lump. These two distributions being compared reveals the disparity in the values of standard deviation between the two sections. The section A having a large standard deviation, is behaving in a heterogenous manner while the section B having small standard deviation acting in a homogenous way.

The Variance. *Variance* is another technique for assessing disparity in a distribution. In the simplest sense, variance is the square of the standard deviation. The formula is illustrated below:

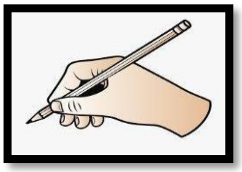
X is any raw score in the distribution

x it is the deviation score. It is equal to the raw score, X, minus the mean, $\bar{X}$: $x = X - \bar{X}$

$$V = SD^2 = \frac{\Sigma X^2}{N} - \bar{X}^2 = \frac{\Sigma x^2}{N}$$

Conceptually, *variance* is the same as standard deviation. If both *standard deviation* and *variance* manifest large values then it means heterogenous distribution and when they both manifest small values, they provide similar outcomes about the homogeneity of the distribution.

While standard deviation finds out how to spread out the distribution scores from the mean by exploring the square root of the variance, the variance, on the other hand, calculates the average degree by which each score differs from the mean - i.e. the average of all the scores in the distribution. It may appear to be unnecessary to study variance where, in fact, standard deviation seems complete. But there are situations wherein it is more efficient to work directly with variances than to frequently make courtesy appearances to the *standard deviation.* In fact, F Ratio takes full utilization of this special property of variability.



Learning Activity 5.3

1. At ABC University, a group of students was selected and asked how much of their weekly allowance they spent in buying mobile phone load. The following is the list of amounts spent: Php 120, 110, 100, 200, 10, 90, 100, 100. Calculate the mean, the range, and the standard deviation.

2. At XYZ University, another group of students was selected and asked how much of their weekly allowance they spent in buying mobile phone load. The following is the list of amounts spent: Php 200, 180, 30, 20, 10, 160, 150, 80. Calculate the mean, the range, and the standard deviation.

3. Consider the data in problems 1 and 2, in what way do the two distribution differ? Which group is more homogeneous?

Lesson 5.4

Measures of Relative Position

Specific Objectives

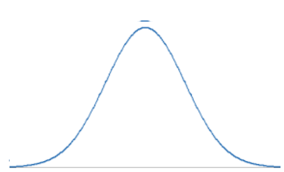
1. To gain deeper understanding about the Z-score
2. To realize the important role of percentile, and quartile in a distribution
3. To interpret the analysis reported by box-and-whisker plots

In the previous lesson, we have demonstrated two separate but related measures that can show the characteristics of the scores in a distribution. These are the measures of central tendency and the measures of variability. In this lesson, we can further explore all the possibilities that might occur in the relationship of centrality and variability (i.e., mean and standard deviation). Let us consider having two sets of distribution and different case scenarios that might occur in comparing their respective means and standard deviations.

Discussion

The z- Score

Case A

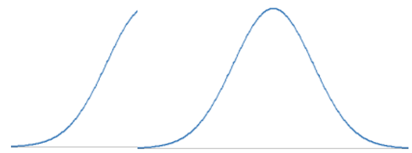


$\mu_1 = \mu_2$

$\sigma_1 = \sigma_2$

As shown in Case A, it is possible that two distributions can generate almost the same *means* (μ) and almost the same *standard deviations* (σ).

Case B

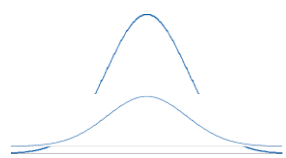


$\mu_1 \neq \mu_2$

$\sigma_1 = \sigma_2$

It is also possible that two distributions have different means (μ) but similar standard deviations (σ).

Case C

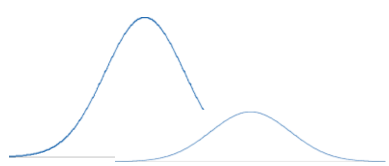


$\mu_1 = \mu_2$

$\sigma_1 \neq \sigma_2$

Here in Case C, the two distributions have the same *means* (μ) but they differ in *standard deviation* (σ).

Case D



$\mu_1 \neq \mu_2$

$\sigma_1 \neq \sigma_2$

In Case D, the distributions differ in terms of *means* (μ) and in terms of *standard deviations* (σ).

This preliminary discussion basically shows that comparing two distributions is complex. Case scenarios must be considered. Sometimes two distributions differ in terms of means and sometimes they differs in terms of standard deviations. The groups usually differ in terms of centrality as well as in terms of disparity. Thus, in order to compare two different groups, there must be a common scale that can reconcile both means and standard deviation in a single standard form. It is only when we convert scores from different distributions to common scores that direct comparison is possible. This common score being referred to is called the **z-score**. Below is the formula in finding the z-score.

$$z = \frac{X-\mu}{\sigma} \qquad z = \frac{X-\bar{x}}{s}$$

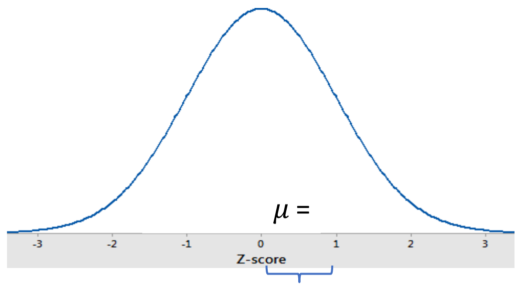
X refers to the raw scores from the population.
μ pertains to the mean of the population
σ population standard deviation

X refers to raw score from the sample
$\bar{x}$ pertains to the mean of the sample
s estimated standard deviation

Both formulas indicate the same relationship shared by the raw score, mean and standard deviation. The only distinction between the two formulas is that whether the distribution was generated from the population or from the sample. The formula in the left refers to the z-scores from the population while the formula in the right refers to the z-scores from the sample.

$$z = \frac{X-\mu}{\sigma}$$

The formula explains that values generated by the mean and standard deviation can be integrated to transform a raw score (X) into a standard score (z). The z-score equation, $z = \frac{X-\mu}{\sigma}$, can convert the raw score of any group into a common value and it enables comparison between scores coming from different group distributions. The below is an illustration of a standardized scale. As you may have noticed in this z-scaling, the mean is always zero and the standard deviation is always one unit.



To further clarify the concept of z-score, let us assume that you are taking physics and biology courses. In your final examinations, you earned a grade of 95 in physics and 85 in biology. Now the question is: **In which exam did you do better?**

It seems obvious based on the face value of the scores, that you did better in physics than in biology. But to come up with a serious comparison about your scores between the two tests, we must take into consideration the question about how well your classmates perform as a whole group. This requires additional information about the mean and standard deviation values of both physics and biology groups. But let us assume that we can right away get those needed information. As such:

	μ (population mean)	σ (population SD)
Physics	85	10
Biology	75	5

Now, let us substitute that information into the *z-score* formula and compute for the *z score* values

Physics	Biology
$Z_p = \frac{X_p - \mu_p}{\sigma_p}$	$Z_b = \frac{X_b - \mu_b}{\sigma_b}$
$Z_p = \frac{95-85}{10} = 1.0$	$Z_b = \frac{85-75}{5} = 2.0$

Finally, let us place these *z-score* values into a *z-scale* to clearly illustrate the measures.

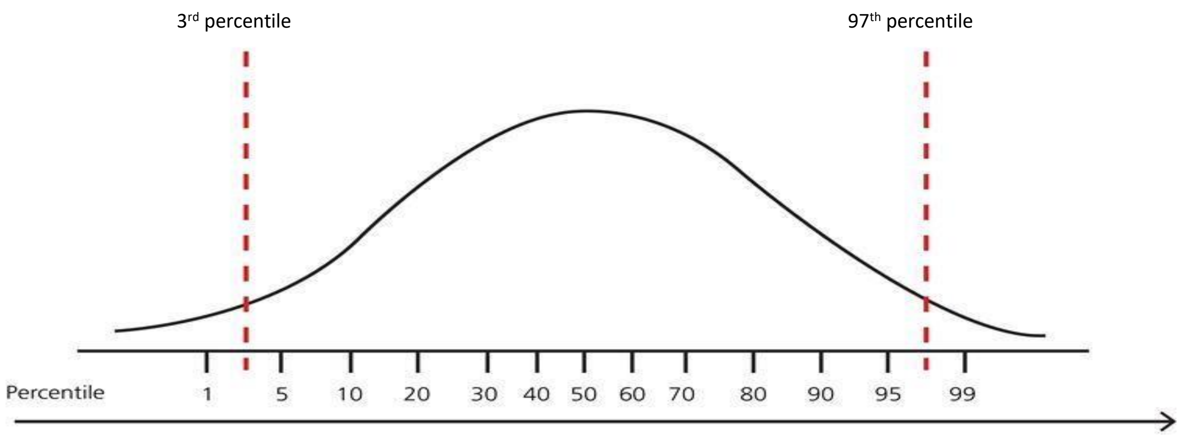
Z_p=1.0 Z_b=2.0

Z	**-4**	**-3**	**-2**	**-1**	**0**	**+1**	**+2**	**+3**	**+4**
Physics 45	45	55	65	75	85	95	105	115	125
Biology 55	55	60	65	70	75	80	85	90	95

Notice that in the illustration, we can clearly compare the relative position of scores in one standardized scale. Notice also that the means of both subjects reconcile to adopt a common mean of 0 (μ = 0). Likewise, both subjects agree to calibrate their standard deviations into a unit of one (σ = 1). Thus, comparison can now be made on your final examination scores. As displayed, your score of 95 in physics falls directly below 1.0 on the z-scale. Your score of 85 in biology falls directly below 2.0 on the z-scale. It is clear that you did much better in the biology exam (Z_b = 2.0) than what we previously thought that you did better in physics. This example is only a glimpse to show that standardized scores are the building blocks that provide the foundation to inferential statistics.

Percentile

To locate a specific point in any distribution, *percentiles*, *quartiles* and *deciles* are the tools that can be used. The relative position of the raw score can be described precisely by converting it into a percentile. A percentile refers to a point in the distribution below which a given percentage of scores fall.



Based on the figure above, a score at the 97th percentile (P_{97}) is at the very high end of the distribution because an enormous number (97%) of scores are below that point. A score at the 3rdpercentile (P_3), however, is an extremely low score because only 3% of the scores are below that point. The figure above also show that the 50th percentile divides the distribution exactly in half. The position of the 50th percentile is also the location of the median.

To provide a better understanding on the role of the percentile, let us assume that your College Admission Test Result reflected the 97th percentile score. This does not indicate that out of 100 items of questions, you just made around three mistakes. Instead, it means that 97% of those who took the exam did not perform better than you. However, a significant 3% did perform better than you.

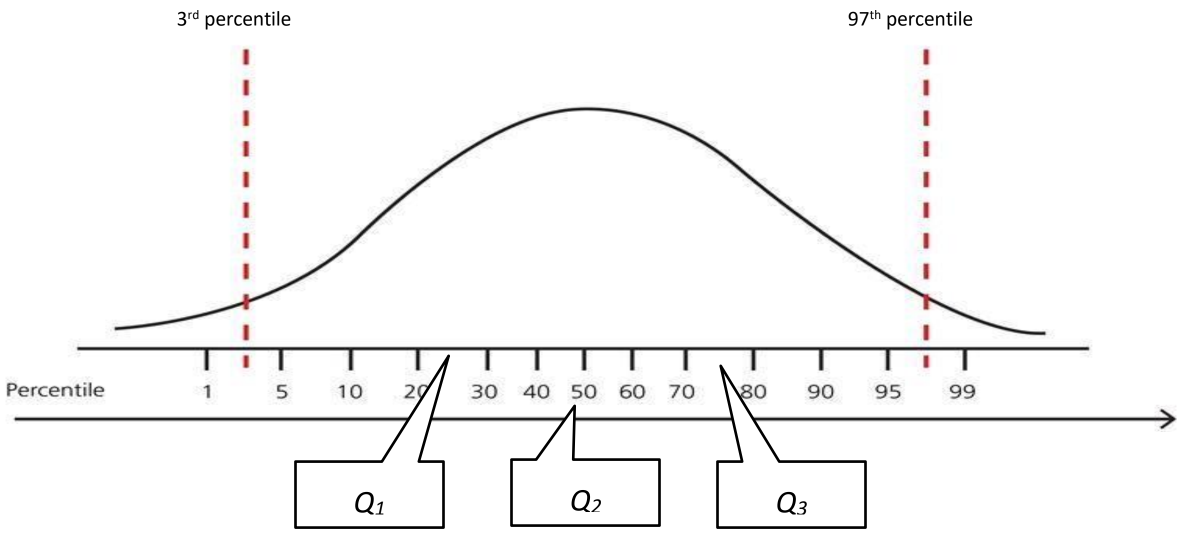
The percentile of any given data value score (x) can be determined by dividing the number of data values less than x with total number of data values, and then multiplying the obtained result by 100. For instance, consider a College Admission Test administered to 5000 students, and your score of 800 was higher than the scores of 4000 examinees. With this information , we can determine the percentile of your score by using the formula the:

$$\text{Percentile Score (x)} = \frac{\text{number of examinees who did worse than you (4000)}}{\text{Total number of examinees which is 5000}} \times 100$$

$= 80$

Your score of 800 places you at the 80th percentile.

Quartiles. As the name implies, *quartiles* divide the distribution into quarters.



The first quartile, Q_1, is actually on the 25thpercentile. The second quartile, Q_2, coincides with the median, which is on the 50th percentile. The 3rd Quartile, Q_3, is on the 75th percentile. The Q can be determined by using the following procedures:

For Q1. : The value of x is in the position .25 (n+1)
For Q_2: The value of x is in the position .50 (n +1)
For Q_3: The value of x is in the position .75 (n+1)

Let us consider this example and determine Q_1, Q_2, and Q_3.

X

============================

Php 200,000.00
200,000.00
195,000.00
194,000.00
193,000.00
192,000.00
191,000.00
190,000.00
185,000.00
181,000.00
180,000.00
176,000.00

============================

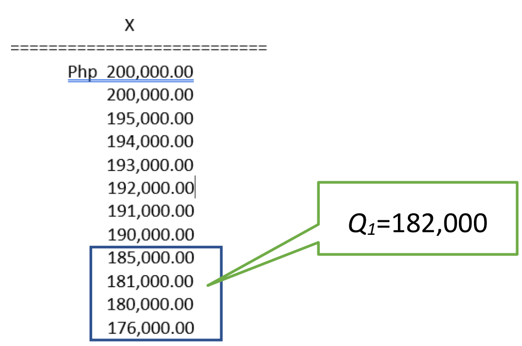
First, make sure that the scores are arranged from highest to lowest.

1. Calculating for the 1st quartile (Q_1) or the 25th percentile
   The *x* score is in the position of $Q_1 = .25\,(n+1)$

$Q_1 = .25\,(n+1)$

$Q_1 = .25\,(12+1)$

$Q_1 = 3.25$



The value of *x* corresponding to the position is 181,000 + .25 (185,000-181,000).
Thus, Q_1 = 182.000

2. Calculating for the 2nd quartile (Q_2) or the 50th percentile
   The x score is in the position of Q_2 =.50 (n+1)

x
Php 200,000.00
200,000.00
195,000.00
194,000.00
193,000.00
192,000.00
191,000.00
190,000.00
185,000.00
181,000.00
180,000.00
176,000.00

Q_2=191,500

Q_2 =.50 (n+1)

Q_2 =.50 (12+1)

Q_2 = 6.5

The value of x corresponding to the position is 191,000 + .50 (192,000-191,000). Thus, Q_2 =191,500

3. Calculating for the 3rd quartile (Q_3) or the 75th percentile
   The x score is in the position of Q_3 =.75 (n+1)

x
Php 200,000.00
200,000.00
195,000.00
194,000.00
193,000.00
192,000.00
191,000.00
190,000.00
185,000.00
181,000.00
180,000.00
176,000.00

Q_3=194,750

Q_3 =.75 (n+1)

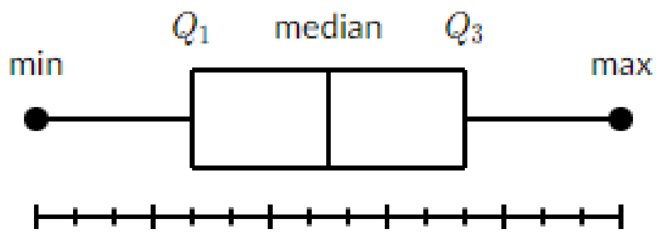
Q_3 =.75 (12+1)

Q_3 = 9.75

The value of x corresponding to the position is 194,000 + .75 (195,000-194,000). Thus, Q_3 = 194,750

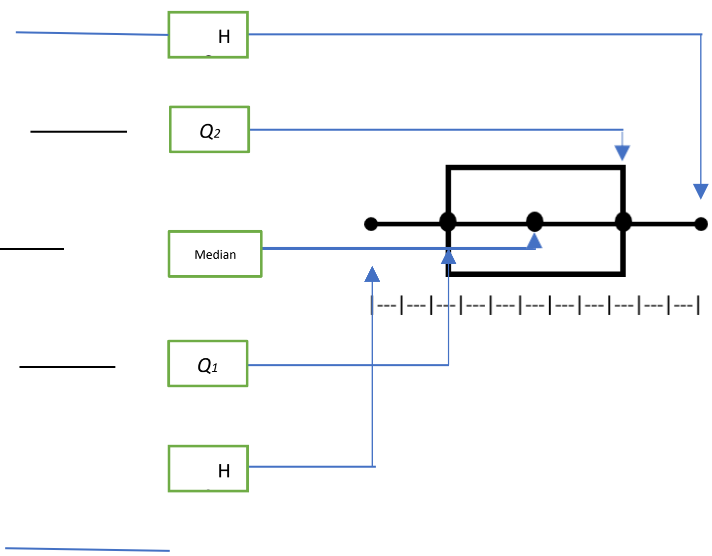
Box-and-Whisker Plots

A box and whisker plot displays a graphical summary of a set of data. It provides information about the minimum and the maximum scores in the distribution, the 1st Quartile and 3rdQuartile as well as the 2nd quartile or the median. Observe the figure below.

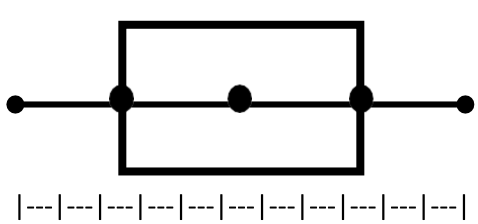


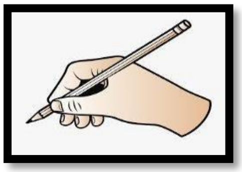
Now, let us find the five-point summary of our previous example.

X
===============
Php 200,000.00
200,000.00
195,000.00
194,000.00
193,000.00
192,000.00
191,000.00
190,000.00
185,000.00
181,000.00
180,000.00
176,000.00
================



Box-and-Whisker plots are easy to construct and they outrightly show important information about the distribution of scores in a simple diagram. Also, it is not necessary to label the final product.





Learning Activity 5.4

1. You have taken final exams. Your score in science 101 was 80. Your score in math 101 was 95

	n	Σx	Σx^2
Science 101	120	7120	2800
Math 101	75	2275	325

a. Compute for the means of both classes.
b. Compute for standard deviations of both classes
c. Convert the final score into z-scores
d. Plot the standard scores on a z-scale, include the appropriate raw score scale values for the two classes.
e. In which class did you do better? Explain how did you analyze it.

2. The score of all students at ABC school were obtained. The highest score was 140, and the lowest score was 110. The following scores were identified as to their percentile:

X	**Percentile**
112	**10th**
119	**25th**
123	**50th**
127	**75th**
134	**90th**

a. What is the range of the distribution?
b. What is the median?
c. What is the 1st quartile, 2nd quartile, 3rd quartile? Figure out---What is the interquartile range?
d. What is the interdecile range?

3. The data given are the calories per 200 milliliters of popular sodas.

   21,18,21,20,26,31,18,16,25,27,13,27,36,24,25

   a. Find the 25^{th} percentile.
   b. Find the median
   c. Find the 75^{th} percentile
   d. Construct a box plot for these data.

Lesson 5.5

The Normal Distributions

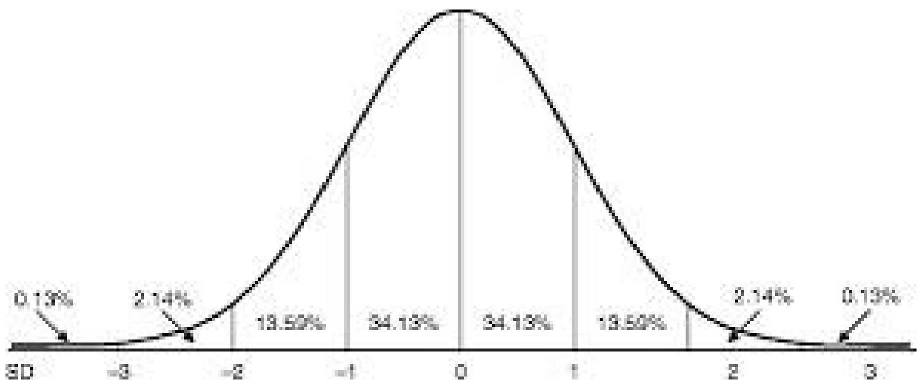
Specific Objective

:

1. To understand the concept of normal distribution
2. To gain knowledge on how use the z-table efficiently
3. To identify and classify some situation pertaining normal distribution
4. To understand the applicability of normal distribution in real life.

If mean and standard deviation are heart and brain of descriptive statistics then perhaps the normal curve is its lifeblood. In the preceding section, we discussed in passing the z-scores, wherein the mean is always zero and the standard deviation is fixed to 1. In this section, it is now proper to finally introduce the normal curve. The normal curve is actually a theoretical distribution. It is a unimodal frequency distribution curve. The scores are scattered on the X axis while the frequency of occurrence is defined by the Y axis.

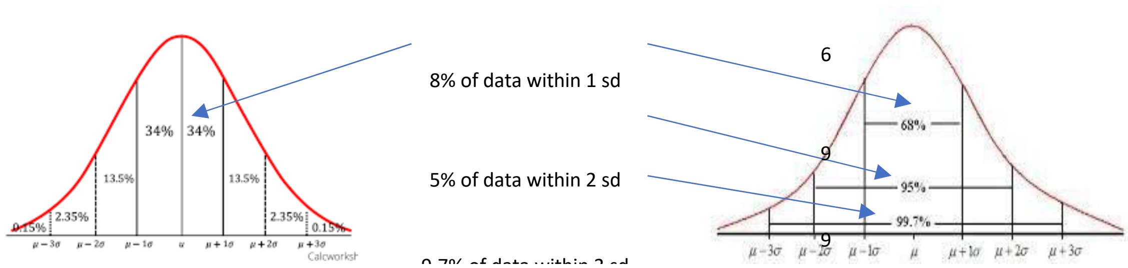
Discussions



Here are some key characteristics of the normal curve.

1. Majority of the scores cluster around the middle of the distribution and fewer scores scattered in both extreme sides or tail ends of the curve.
2. It is always symmetrical and perfectly balanced.
3. Being a theoretical distribution, the mean, median and the mode are all equal.
4. It uses standard deviation along the x-axis.
5. The normal curve is asymptotic to the abscissa and the total area under the curve is approximating 1.0 or 100%
6. The normal curve has a mean of zero and standard deviation of 1 unit.

The Empirical Rule for a normal distribution



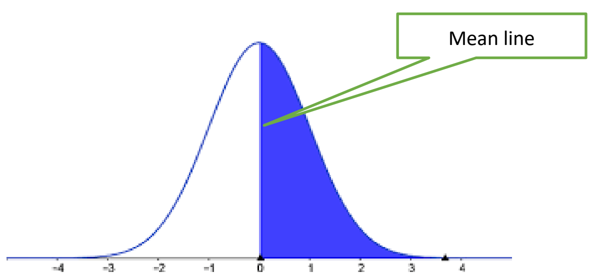
z Scores. The z scores are enormously beneficial in interpreting of relative position of the raw score taking into account the centrality of the distribution and the amount of variability. With the z-score, we can gain understanding of an individual relative performance compared to the performance of the entire group being measured. But before we delve deeper into the concepts of the z score, it is imperative to learn how to use the z-score table. A copy of the z-table can be accessed at this website address:

https://www.calculator.net/z-score-calculator.html

https://www.calculator.net/z-score-calculator.html

https://www.calculator.net/z-score-calculator.html

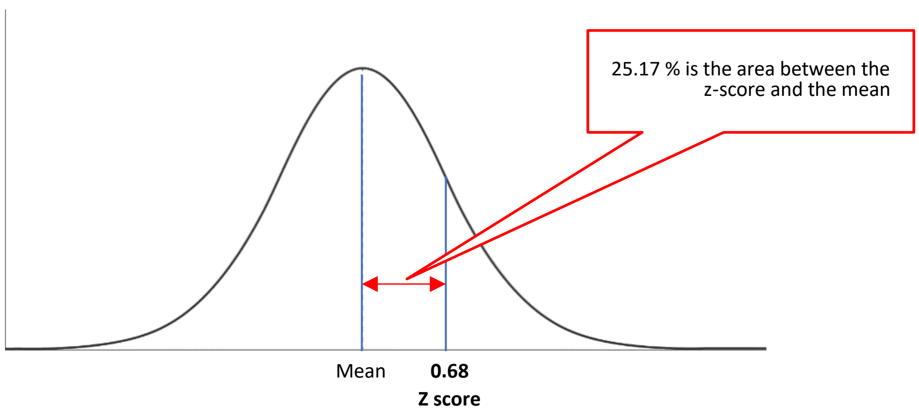
The table we will be using is a right tail z-table. This table is used to find the area between z=0 and any positive value and reference the area to the right side of the standard deviation curve. The z-score table gives only the percentage for the half of the curve. But since the normal curve is symmetrical, a z-score that is given to the right of the mean yields the same percentage as a z score to the left of the mean



For example, to look up a z-score of .68 using the z-score table, look for 0.6 in the far left of the column then look for the second decimal 0.08 in the top row. The table value is 0.25175. It represents a percentage of 25.17 %. It is the percentage of cases falling <u>between</u> the z score and the mean.

Z Table from Mean (0 to Z)

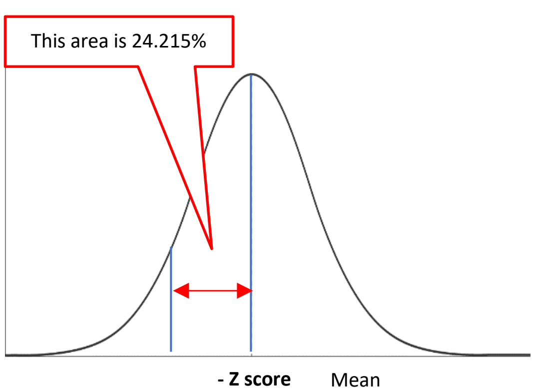
z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.0279	0.03188	0.03586
0.1	0.03983	0.0438	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.1293	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.1591	0.16276	0.1664	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.2054	0.20884	0.21226	0.21566	0.21904	0.2224
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.2549
0.7	0.25804	0.26115	0.26424	0.2673	0.27035	0.27337	0.27637	0.27935	0.2823	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891

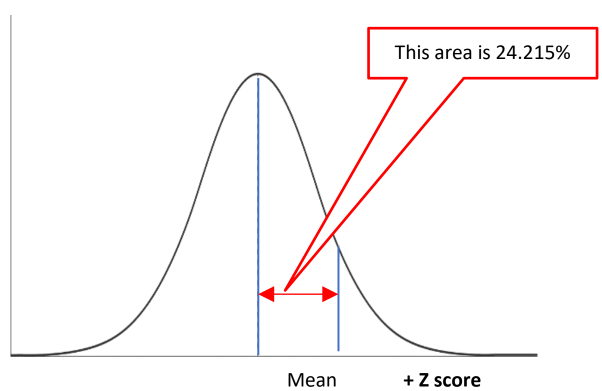


25.17% is the percentage of cases falling <u>between</u> the z score (0.68) and the mean.

Now, let us consider some situations that might possibly occur in using the z-table

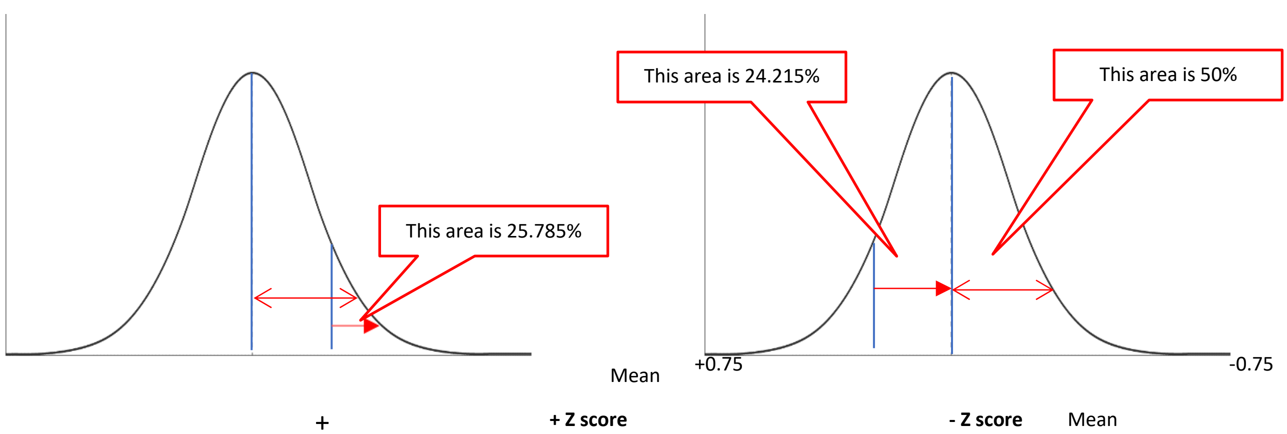
Case 1. Finding percentage of cases falling <u>between</u> z-score and the mean.





As example for Case 1, the z-score of +0.75 will generate a z-table value of 0.24215 or 24.215%. In the same way, the z-score of -0.75 will generate the same value-table value of 0.24215 or 24.215%. Notice that the value is always a positive number since percentage area is always positive.

Case 2. Finding the percentage of cases above the given z-score. It is important to remember for this case that the total area of the normal curve is 1.0 or 100%. It is also essential to keep in mind that the right half of the normal curve is 50% as well as the left half (50%). You also need to consider that the z-table always provide a percentage value in relation to the mean.

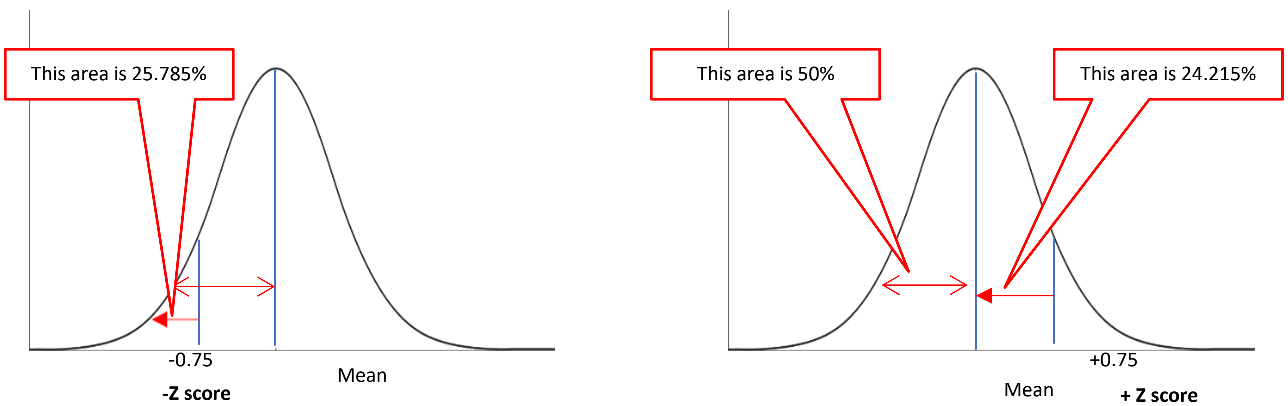


(a) (b)

For Case 2(a), To find the area above the given z-score, the equivalent z-table value must be determined then subtract it from the total area of the right half which is 50%. For example, to find the percentage of cases above the z-score of +0.75. Find the z-table value of +.75 which is 0.24215 (24.215%) then subtract it from the total area of the right half of the normal curve which is 50%. This is 50% - 24.214% = 25.785%

For Case 2(b), in order the determine the area above the given z-score (the z-score here is a negative number because it is situated in the left side of the normal curve) , simply find the equivalent z-table value then add 50%. Again, always keep in mind that the z-table only provide a percentage of cases between the z-score the mean and not the entire right side of the curve. To cite another example, let us find the percentage of cases above the z-score of -0.75. The z-table value of -0.75 is 0.24215. This is equivalent to 24.215%. With this number just add the percentage area of the entire right side which is 50%. So this is 24.215% + 50% =74.215%.

Case 3. Finding the percentage of cases below the given z-score. The principle we made in Case 2 is the same principle that can be applied in Case 3.

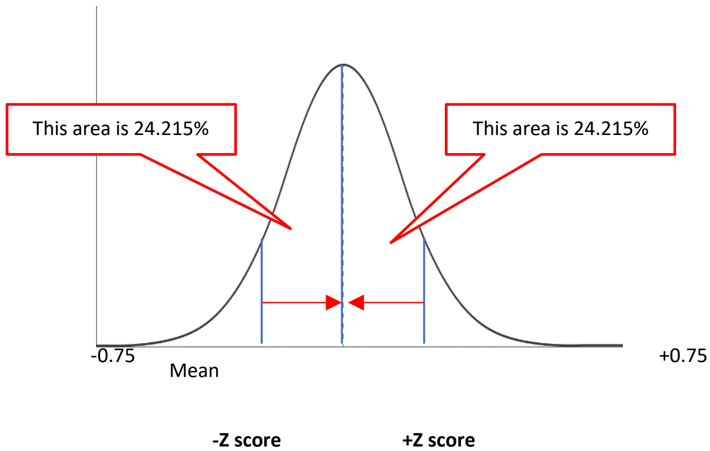


(a) (b)

For case 3(a), try to determine the percentage of cases below the z-score of -0.75. Using similar analysis made in case 2(a), the total area of the left side must be subtracted. If your computation is correct, your answer is 25.785%.

For case 3(b), to determine the percentage of cases below the z-score of +0.75. The z-table value will only cover the percentage of cases between the z-score and the mean, so you need to add 50% which is the l percentage of cases of the left side of the normal curve. Your computation must generate an answer of 74.215%.

Case 4. Finding the percentage of cases between the two z-scores.



To illustrate Case 4, let us try to determine the percentage of cases between the two z-scores. The -0.75 Z-score and +0.75 z-score. The -0.75 z-score generates a z-table value of 24.215%. Also +0.75 z-score generates the same z-table value of 24.215%. Thus, the percentage of cases between -0.75 and +0.75 is simply to add the two percentage of cases and that is (24.215% + 24.215%) 48.43%.

Translating the raw score into the z-score.

We are now familiar with the z-score concepts and having a knowledge about percentages of area above, below and between z-scores. Likewise, we are also equipped with certain knowledge regarding the z-score formula that if the mean and standard deviation are known, we can subtract the mean from the raw score, divide by standard deviation, and obtain the z score.

$$z = \frac{x - \bar{x}}{SD}$$

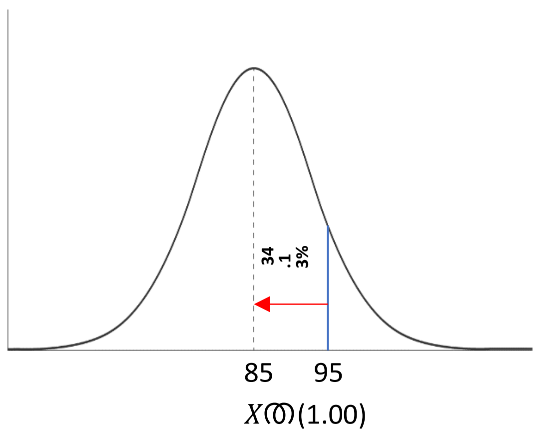
The z-score reveals the location of the raw score from the mean in the standard deviation units. The z score accounts both the mean of the distribution and the amount of variability. Now, let us determine the practical use z-score in the context of normal distribution of raw scores.

Case A. *When the percentage of cases is between the raw score and the mean*. The normal distribution of physics scores has mean of 85 and a standard deviation of 10. What percentage of scores will fall between the physics score of 95 and the mean?

Initially, we need to convert the raw score of 95 into its equivalent z-score.

$$z = \frac{x-\bar{x}}{SD} = \frac{95-85}{10} = 1.0$$

Then draw the normal curve as shown below;



Next is to look up the z-score value in the table (https://www.calculator.net/z-score-calculator.html). The z-table value is 0.34134 or 34.13%. That is the percentage of scores that falling between the physics score of 95 and the mean. This means that around 1 in 3 students (34.13%) fall between the score of 95 and the mean.

Case B. *When the percentage of cases fall below a raw score*. Using the same example, on a normal distribution of scores in physics class, with a mean of 85 and a standard deviation of 10, what percentage of physics scores fall below a score of 95?

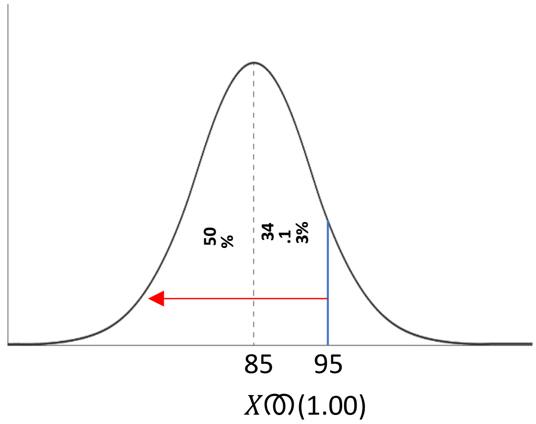
First, convert the raw score of 95 into its equivalent z-score.

$$z = \frac{x-\bar{x}}{SD}$$

95 – 85

= $\frac{}{10}$ = 1.0

Next is to draw the normal curve as already shown below;



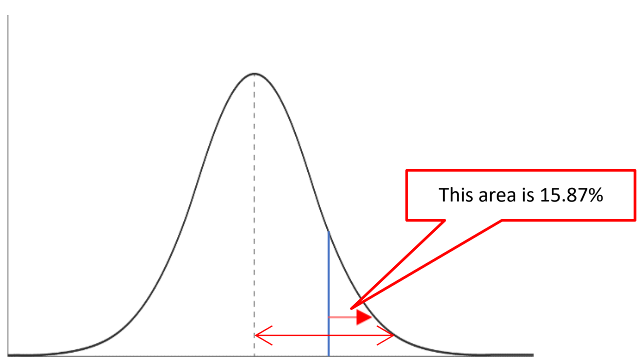
Finally, look up the z-score in the z- table (https://www.calculator.net/z-score-calculator.html)take the right value. It is 0.34134 or 34.13%. Lastly, add the 50% to 34.13% to get the sum 84.13%. The percentage of physics scores fall below a score of 95 is 84.13%. This means that if 100 students took the examination and your score is 95. Then your physics grade surpassed the grade of 84 students.

Case C. When the *percentage of cases is above a raw score*. On a normal distribution of scores in physics class, with a mean of 85 and a standard deviation of 10, what percentage of physics scores above a score of 95?

Again, we need to convert the raw score of 95 into its equivalent z-score.

$$Z = \frac{x - \bar{x}}{SD} = \frac{95 - 85}{10} = 1.0$$

The draw the normal curve as already shown below;



34
.1
3%

85 95

X(0)(1.00)

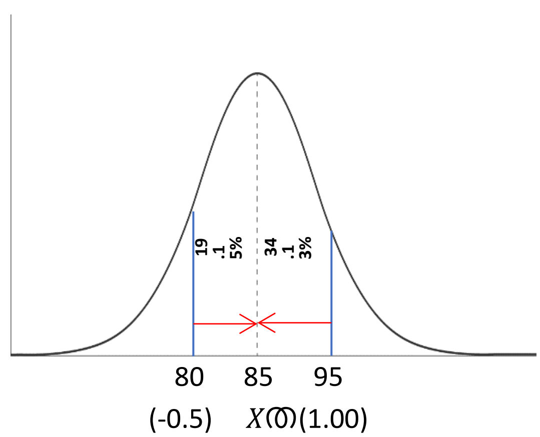
We look up the z-score in the table (https://www.calculator.net/z-score-calculator.html)take the correct value. It is 0.34134 or 34.13%. Then subtract 34.13% from 50%. The answer is 15.87%. This is the percentage of cases above the score of 95. This means that if 100 students took the examination and your score is 95. Then around 15 students surpassed your physics grade of 95.

Case D. When *the percentage of cases is between raw scores*. On a normal distribution of physics scores, the mean is 85 and the standard deviation is 10. Your physics score is 95 and your friends score is 80. You wanted to determine how many students got a score between your friend's score of 80 and your score of 95.

Again, convert the raw score of 95 and the raw score of 80 into its equivalent z-scores.

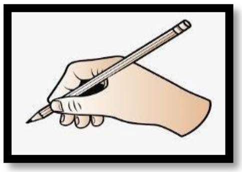
$$z = \frac{x - \bar{x}}{SD} = \frac{95-85}{10} = 1.0 \qquad z = \frac{x - \bar{x}}{SD} = \frac{80-85}{10} = -0.5$$

The draw the normal curve as already shown below;



We look up the z-score in the table (https://www.calculator.net/z-score-calculator.html) and look for z percentage of cases for the z-value 1.0. Also look for the percentage of cases for the z-value -0.5. The percentage of cases is 34.13% and 19.15% respectively. Add the two values to get the percentage of cases between the raw score of 95 and 80. The answer is 53.28%. This means that 1 in 2 students got a score between 95 and 85 (i.e. between your score and your friend’s score).

At this point, we already made a significantly long journey. From the measures of central tendency to the measures of variability and finally to measures of relative position. We are now in the position no longer seeking answers to questions but seeking questions beyond the conventions established by the answers.



Learning Activity 5.5

1. Road test of MG5 Sedan compact car show a fuel mean rating of 20 kilometers per liter in highways, with a standard deviation of 1.5 kilometers per liter. What percentage of these cars (MG5) will achieve results of

   a. More than 25 kilometers per liter?
   b. Less than 17 kilometers per liter?
   c. Between 15 and 24 kilometers per liter?
   d. Between 21 and 24 kilometers per liter?

2. On a normal distribution, at what percentage must
   a. The mean fall?
   b. The median fall?
   c. The mode fall?

Lesson 5.6

The Linear Correlation: Pearson r

Specific Objective

1. To know the characteristics of Pearson r
2. To solve problems dealing with linear correlations
3. To understand the limitations of linear correlations

At the beginning of this course, we defined mathematics as the science of patterns. We realized that nature follows a certain kind of mathematical structure as we observed some patterns and irregularities and whenever we see patterns, irregularity also beg also to be noticed. Also, whenever we see irregularities, some patterns suddenly waving for attention.

The linear correlation is not about patterns, but it is about looking on irregularities and patiently waiting for the patterns to manifest. This lesson deals with determining the connections of the things seemed unrelated and to declare whether some correlations are indeed significant .

Discussions

The Pearson R Linear Correlation.

The Product-Moment Correlation Coefficient or Pearson r is an statistical tool that can determine the linear association between two distributions or groups. This tool can only establish the strength of association or correlation but it can never justify any causal relation that may appear or seemed obvious.

The formula below is the computational method for calculating the Pearson r

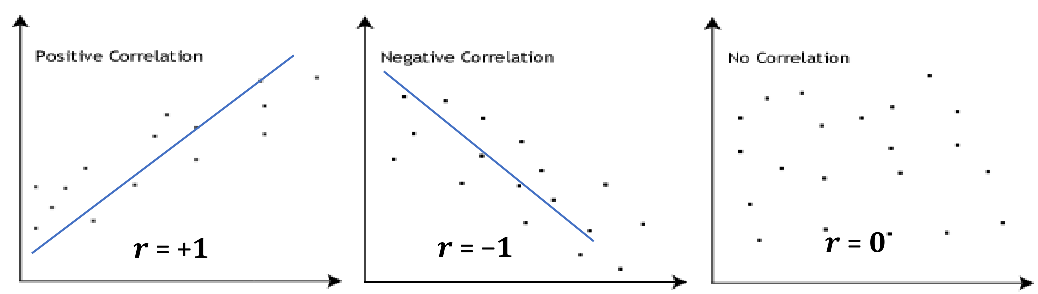
The number of subjects

Means

$$r = \frac{\frac{\Sigma XY}{N} - (x)(y)}{SD_x SD_y}$$

Standard Deviations

The pearson r value may provide three possible scenarios. If the value of r is + then it is a positive correlation. If it is - then it is a negative correlation. If r's value is around "0" then it means that almost no linear correlation found.



An example of positive correlation is height and weight of a person. Under normal circumstances whenever a person gain height it means also a gain in weight. An example of negative correlation is the relationship between length of employment and degree of attractiveness. As you may observe physically attractiveness of an employee is affected by the chronologically advancement of his or her age. An example of zero correlation might be relationship between grade of student living in high land areas and the study habits of students living in the low land areas. You should also remember that Pearson r does not generate a value less than -1 or more than +1. Any answer outside below -1 and above +1 can be attributed to a wrong computation made.

We will explain the nature of linear correlation by using an example. Assuming that we want **to determine if there is a correlation between hours of study and grades of students** last semester. Initially, we need to randomly select students (let say 10) and ask them about their averaged grade last semester as well as the number of hours they spent in studying per week in that semester. Let us presume that right away they provided us these two informations.

Student	Hours of Study (x)	Grade (Y)
A	15	2.75
B	35	1.25
C	05	3.00
D	20	2.50
E	30	1.50
F	40	1.00
G	20	2.25
H	25	1.75
I	25	2.00
J	08	3.00

But before we can immediately use the Pearson r formula, we need to ensure that this is the correct statistical tool in determining the correlation between hours of study and grades. Let us check some basic Pearson r requirements:

1. Random selection of participants.
2. Traits being measured must not depart significantly from normality
3. The measurements on both distributions must be in the form of interval data.
4. Comparing only two groups.
5. And the goal is to determine the linear correlation between two groups.

The formula in solving the Pearson r is…

$$r = \frac{\frac{\Sigma XY}{N} - (\bar{x})(\bar{y})}{SDxSDy}$$

X	refers to one variable and the Y refers to another variable
$\bar{X}$ and $\bar{Y}$	refers to the mean of X and the mean of Y
SD_x and SD_y	refers to the standard deviation of X and Y respectively
N	refers to the numbers of variables
Σ	It is the symbol for summation

Now let us take into account the data below as our example to illustrate the formula.

Student	Hours of Study (x)	x^2	Grade (y)	y^2	xy
A	15	225	2.75	7.56	41.25
B	35	1225	1.25	1.56	43.75
C	05	25	3.00	9.00	15.00
D	20	400	2.50	6.25	50.00
E	30	900	1.50	2.25	45.00
F	40	1600	1.00	1.00	40.00
G	20	400	2.25	5.06	45.00
H	25	625	1.75	3.06	43.75
I	25	625	2.00	4.00	50.00

J	08	64	3.00	9.00	24.00

==

Σx=223 **Σx^2=6089** **Σy=21** **Σy^2 = 48.75** **Σxy=397.75**

$$\overline{x} = \frac{\Sigma x}{N} = \frac{223}{10} = 22.3 \qquad \overline{y} = \frac{\Sigma y}{N} = \frac{21}{10} = 2.1$$

$$SD_x = \sqrt{\frac{\Sigma x^2}{N} - \overline{x}^2} = \sqrt{\frac{6089}{10} - 22.3^2} = 10.56 \qquad SD_y = \sqrt{\frac{\Sigma y^2}{N} - \overline{y}^2} = \sqrt{\frac{48.75}{10} - 2.1^2} = 0.682$$

$$r = \frac{\frac{\Sigma XY}{N} - (\overline{x})(\overline{y})}{SDxSDy}$$

$$r = \frac{\frac{397.75}{10} - (22.3)(2.1)}{(10.56)(0.682)}$$

$$\boldsymbol{r = -0.979}$$

Point to Ponder: Why do you think we generated a negative r value?

Thus, we could say that the correlation between hours of study and grades of students achieved a Pearson r value of -0.979. Do not be confused by the that there is a negative sign in our final answer. This sign provides an idea of the direction of correlation line. You should take into consideration that a grade of 1.0 has a strong academic weight in our grading system but once plug in into the computation it is interpreted by formula as a small number. Nevertheless, with full knowledge of the concept you can always come up with the right interpretation.

Since the distribution exclusively concerns the 10 students and it is not a population sample, then Guilford's suggested interpretation for the values of r can be used without hindrance.

Guilford's Interpretation for the values of r

r value	Interpretation
Less than .20	Almost negligible relationship
.20-.40	Definite but small relationship

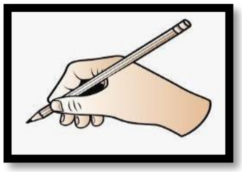
.40-.70	Substantial relationship
.70-.90	Marked relationship
.90-1.00	Very dependable relationship

==

And based on Guilford's suggested interpretation, there is a very dependable relationship between hours of study and grade of students.

Does it mean that better grades can be achieved by spending more time studying?
Does it mean that spending more time studying is a by-product of better grades?
Does it mean that another factor influenced better grades and study habits?

All three of these questions are possible. But the point is that correlation alone is not enough to identify which is the real explanation. Pearson r is not a tool for establishing causation. It can only a tool describe linear correlation between to observed traits.

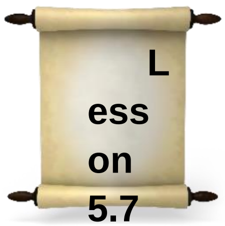


Learning Activity 5.6

Seven randomly selected participants were given both math and music tests. Their scores are as follows:

Math	Music
16	14
6	7
17	15
11	14
12	12
4	6
13	11

Is there math ability related to their music ability?



Lesson 5.7

:

The Least-Squares Regression Line

Specific Objective

1. Define Linear Regression
2. Define Scatter plot
3. Compute for Least Square Regression Line

In the previous lesson, we discussed Pearson r as a powerful tool in determining linear correlation. It is an important tool to investigate associations considering that different mathematical patterns are all around us. Such as, the connection of high tide and low tide in human behavior, the association between height and weight. And the correlation between the metaphoric flap of butterfly in Japan to a weather disturbance in South America a year after.

But correlation entrapped and cloistered us within the parameter of merely associating. Correlation in and by itself cannot establish causation to warrant prediction. But in this lesson of regression analysis, not only that we can connect and associate some observable patterns, it also permits us finally make basic predictions.

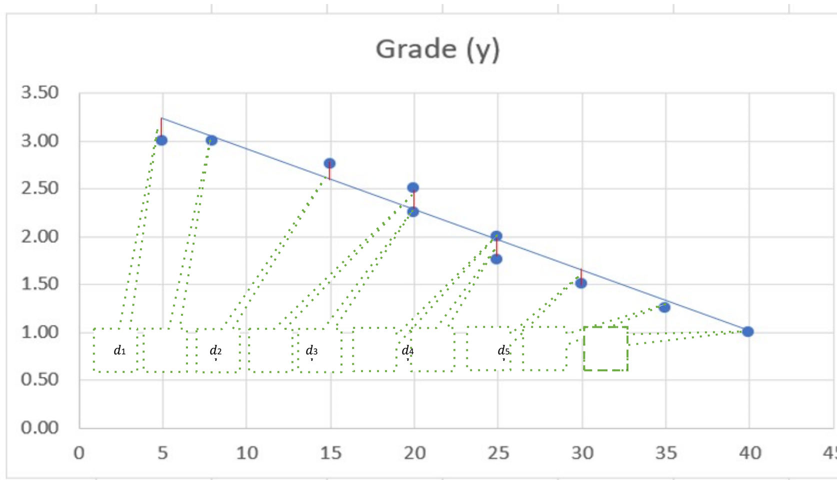
Discussion

Bivariate Scatter Plot

A bivariate simply means that we can graphically represent two variables (x and y) in a scatter plot wherein each point in a scatter plot represent a pair of scores. Scatter plot is necessary in order to determine the regression line. The regression line a generated straight line that lies closest to all the point in the scatter plot.

Our example below illustrates the construction of scatter plot based on some data information regarding the association of our previous example on hours of study and grade.

Student	Hours of Study (x)	Grade (y)
A	15	2.75
B	35	1.25
C	5	3.00
D	20	2.50
E	30	1.50
F	40	1.00
G	20	2.25
H	25	1.75
I	25	2.00
J	8	3.00



As shown in the scatter plot above, the straight line is called the least-squares regression line. This generated line minimizes the sum of the squares of the vertical deviation from each data point to the line. This means that of all the possible lines that can suggest the correlation line strength of all the points, the equation of this generated line has the best fit. The d_n represents the distance from point (x,y) to the line.

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2 + d_8^2 + d_9^2 + d_{10}^2$$

In the least-squares line, this correlation that can be established around the regression line is the basis for resulting prediction. But in order to make predictions, three important ingredients must be on hand: 1. The equation of the best fit line. 2. Slope of the line, and 3. The y-intercept of the line.

The Formula for the Least-Squares Regression Line

There must be n ordered pairs: $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$

$$y = mx + b$$

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2}$$

$$b = \frac{(\Sigma y) - m(\Sigma x)}{n}$$

To apply this formula to our given data, we need to find the value of each summation.

Student	Hours of Study (x)	Grade (y)	x^2	xy
A	15	2.75	225.0	41.25
B	35	1.25	1225.0	43.75
C	5	3.00	25.0	15
D	20	2.50	400.0	50
E	30	1.50	900.0	45
F	40	1.00	1600.0	40
G	20	2.25	400.0	45
H	25	1.75	625.0	43.75
I	25	2.00	625.0	50
J	8	3.00	64.0	24
	Σx = 223	Σy= 21.00	Σx^2 = 6089.0	Σxy = 397.75
	$(\Sigma x)^2$ = 49729			

In finding the value of m :

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} = \frac{10(397.75) - (223)(21)}{10(6089) - 49729} = -0.06321$$

In finding the value of b :

$$b = \frac{(\Sigma y) - m(\Sigma x)}{} = \frac{(21) - (-0.06321)(223)}{} = 3.509$$

n 10

Finally, substituting the values to the given formula:

$$y_{pred} = mx + b$$

$$y_{pred} = -0.06321x + 3.509$$

Slope(m) = -0.06321

y intercept (b) = 3.509

In the preceding lesson, we were able to establish the strength of correlation of this example using Pearson r. We found a very strong relationship between hours of study (x) and grade (y) ($i.e.$ $\mathbf{0.979}$**)**. Now let us predict the grade of students who spent the following weekly study hours: 37, 22, and 8.

Since we have already determined the regression the line, let us just simply plug all the necessary values then "y".

==

$$y_{pred} = mx + b$$

$$y_{pred} = -0.06321x + 3.509$$

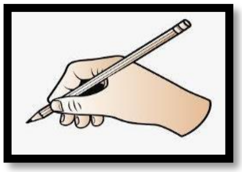
==

x	$-0.06321x$	$-0.06321x + 3.509$ =
37	-2.33877	1.17023
22	-1.39062	2.11838
08	-0.50568	3.00332

y_{pred}

==

The predicted grade of students is around **1.17** for the student who spends 37 hours of study, **2.12** for the one spending 22 hours of study, and just a passing grade of **3.0** for the one engaged for eight hours of study.



Learning Activity 5.7

The research office is interested in the possible relationship anxiety and aptitude scores of randomly selected eight students; they are given both the anxiety test and aptitude test. Their weighted, scaled scores are as follows:

Subjects	Aptitude Test	Anxiety Test
A	10	12
B	7	9
C	13	14
D	8	7
E	11	11
F	6	7
G	10	12
H	11	10

a. What is the correlation between anxiety and aptitude scores.

b. If one of the students receives a score of 12 on the aptitude test, what is your best estimate of the score that student will on anxiety?

Module Five : Project Proposal Requirement

Project Proposal Requirement

For this culminating requirement in Module Five, you need to work together in groups of 3 or 4.

1. Your task is to prepare a proposal study that can contribute to a solution to any social problem.
2. You must use statistical methods for your data processing and analyses.
3. Your final output must be no more than 8 pages that details your project proposal.
4. Please follow the outline provided below:
   a. Title page (not included in the page count)
      -An example of problem to be addressed: In this COVID-19 pandemic, how can we reduce human traffic in wet market places.
   b. Background and Statement of the Problem
   c. Literature Review
   d. Proposed Study with emphasis on how statistics will be used
      i. Data to be collected
      ii. Methods of data collection and data gathering instrument
      iii. Data gathering procedure
      iv. Method of Analyses
   e. Discussion of how your project proposal can address the identified problem.
   f. References (APA or MLA)

5. Below is the format guideline:

Paper Substance (if printed)	Margin	Orientation	Paper Size	Font Type	Font Size	All Line Spacing	Page Number
20	Normal Justified	Portrait	8.5 x 13	Arial	12	1.5	Page x of x

Your project proposal will be graded based on these criteria:

1. **Soundness of the proposal (1/3)**
2. **Appropriate use of statistical method (1/3)**
3. **Coherence (1/3)**

Chapter Test 5

Multiple Choice. Choose the letter of the correct answer and write it on the blank provided at the left side of the test paper.

==

___________ 1. It is a branch of statistics that deals with data analysis and one of its technique is to "describe" data in symbolic form and abbreviated fashion.

a. Inferential Statistics c. Descriptive Statistics
b. Statistics and Probability d. Probability

___________ 2. It is a branch of statistics that has the ability to "infer" and to generalize. It is also the right tool to predict values that are not really known.

a. Inferential Statistics c. Descriptive Statistics
b. Statistics and Probability d. Probability

___________ 3. It is an essential quantifying an observation according to a certain rule. It is also assigning numbers in a prescribed way.

a. Variable c. Data
b. Measurement d. Constant

___________ 4. If the data are labelled 1^{st}, 2^{nd}, 3^{rd}, and so on, in what kind of scale does it falls?

a. Nominal Scale c. Categorical Data
b. Interval Scale d. Ordinal Scale

___________ 5. Personal Biodata falls in what kind of scale?

a. Nominal Scale c. Categorical Data
b. Interval Scale d. Ordinal Scale

___________ 6. It can be used as tools that provide information regarding average, ranking, and category of scores of a large number of scores.

a. Measures of Central Tendency c. Measures of Dispersion
b. Measures of Variability d. Measurement

___________ 7. Mean is the arithmetic average of all scores. In the data: 34, 56, 75, 43, and 67, what is the mean?

a. 53 c. 55
b. 54 d. 56

__________ 8. It is the middle point or midpoint of any distributions. It separates the upper half from the lower half of distribution.

a. Mean c. Mode
b. Median d. Range

__________ 9. The following is a list of retirement ages for the workers in production plant: 65, 64, 65, 61, 62, 64, 65, 63, 63, 65, 64. What is the median?

a. 64 c. 63
b. 65 d. 62

__________ 10. It is the most frequently occurring score in the distribution.

a. Mean c. Mode
b. Median d. Range

__________ 11. These three measures can provide the information about spread of the scores in the distribution.

a. Mean, median and mode c. Range, standard deviation and variance
b. Mean, range, and variance d. Range, median and mode

The grade-point average for the selected university students were computed. The data are as follows:

Student	GPA
1	3.75
2	3.00
3	3.00
4	1.75
5	2.00
6	2.25
7	3.25

__________ 12. What is the range?

a. 1.00 c. 1.75
b. 2.25 d. 2.00

__________ 13. What is the mean (in nearest hundredths)?

a. 2.70 c. 2.72
b. 2.71 d. 2.74

___________ 14. What is the median?

a. 3. 00 c. 1.75
b. 2.25 d. 2.00

___________ 15. If the standard deviation in a distribution is 4, what is the variance?

a. 8 c. 32
b. 16 d. 64

___________ 16. In comparing different groups, there must be a standard scale that can reconcile both means and standard deviation in single standard form. It is only then that direct comparison is possible because transformed scores from different distributions will share common scores and these common scores are called_.

a. Percentile c. T-score
b. Quartile d. Z-score

___________ 17. Jerry took College Admission Test which reflected at 89^{th} percentile, what does it indicates?

a. 89% of those who took the exam did not get it right than Jerry.
b. Out of 100 items of questions, Jerry had 11 mistakes.
c. Jerry answered 89 questions correctly.
d. 89% of those who took the exam get it right than Jerry.

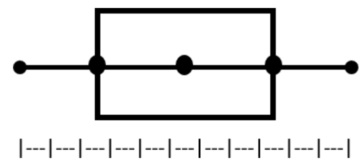
___________ 18. It divides the distribution into quarters.

a. Percentile c. T-score
b. Quartile d. Z-score

___________ 19. The third quartile Q_3 is on what percentile rank?

a. 25^{th} percentile c. 100^{th} percentile
b. 50^{th} percentile d. 75^{th} percentile

___________ 20. In this box-whisker plot



What does the middle dot or point denote?

a. Q_1 c. Q_3
b. Median d. Maximum score

___________ 21. It is a unimodal frequency distribution where the scores are scattered on the X-axis while the frequency of occurrence is defined by the Y-axis.

a. Z-distribution curve c. Normal curve
b. Distribution curve d. X-axis and Y-axis curve

___________ 22. This table only gives the percentage for the half curve but both the right and the left of the mean yields the same percentage since the said curve is symmetrical.

a. Z-table c. T-table
b. Percentage Table d. Normal table

___________ 23. Scores on English Test have an average of 80 with a standard deviation of 6. What is the z-score of the student who earned a 75 on test?

a. -0.97 c. -0.88
b. -0.76 d. -0.83

___________ 24. Group of children compared what they received while trick or treating. The average number of pieces of candy received is 43 with a standard deviation of 2. What is the z-score corresponding to 20 pieces of candy? a. -11.5 c. -9.5
b. -10.5 d. -12.5

___________ 25. The mean growth of the thickness of tree in a forest is found to be 0.5 cm per year with a standard deviation of 0.1 cm per year. What is the z-score corresponding to 1 cm per year?

a. 4 c. 6
b. 5 d. 7

___________ 26. It is a statistical tool that can be used to determine the linear association between two distributions or groups. It can only establish the strength of association or correlation but it can never justify any causal relation that may appear and seemed obvious.

a. Correlation
b. Pearson correlation
c. Pearson R correlation
d. Association

___________ 27. There is no linear correlation found if the value of *r* is____.

a. -1
b. +1
c. 2
d. 0

___________ 28. The linear correlation is said to be substantial relationship if the value of *r* is____.

a. Less than 0.20
b. Between 0.40 and 0.70
c. Between 0.70 and 0.90
d. More than 1

Given the data chart of selected persons with their ages and daily incomes, calculate the Pearson's correlation coefficient.

Person	Age(x)	Income(y)	xy	x^2	y^2
1	20	150	3000	400	2250
2	30	300	90000	900	90000
3	40	500	20000	1600	250000
4	50	750	37500	2500	562500
Total	140	1700	69500	5400	925000

Mean of x = 35
Standard Deviation of x = 11.18
Mean of y = 425
Standard Deviation of y = 225

___________ 29. What is the value of *r*?

a. 1
b. 0.89
c. 0.99
d. 0

___________ 30. What is the interpretation for the *r* value?

a. Very dependable relationship
b. Substantial relationship
c. Almost negligible relationship
d. Marked relationship

___________ 31. It is used in making predictions between two variables.

a. Linear correlation
b. Linear relationship
c. Linear regression
d. Regression correlation

___________ 32. It uses dots to represent values for two different numeric values. It is also important in determining the regression line.

a. Scatter plot
b. Scatter line
c. Box-whisker plot
d. Regression line

The data below pertains to the experience of some workers in a company (number of years) and their performance rating. Estimate the performance rating for a worker with 20 years of experience.

Worker	**Experience(x)**	**Performance(y)**	**xy**	x^2
1	16	87	1392	256
2	12	88	1056	144
3	18	89	1602	324
4	4	68	272	16
5	3	78	234	9
6	10	80	800	100
7	5	75	375	25
8	12	83	996	144
Total	80	648	6727	1018

___________ 2. What is the value for slope of the line?
a. 1.13
b. 1.14
c. 1.15
d. 1.16

___________ 2. What is the value for y-intercept?

a. 69.6
b. 69.7
c. 69.8
d. 69.9

___________ 2. What is the value of regression line?

a. 92.0
b. 92.1
c. 92.2
d. 92.3

Answer: Chapter test

1. C
2. A
3. B
4. D
5. A
6. A
7. C
8. B
9. A
10. C
11. C
12. D
13. B
14. A
15. B
16. D
17. A
18. B
19. D
20. B
21. C
22. A
23. D
24. A
25. B
26. C
27. D
28. B
29. C
30. A
31. C
32. A
33. A
34. B
35. D

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MODULE SIX

Logic

CORE IDEA

Module Six (6) focuses on analyzing information and the relationship between statements, determining the validity of arguments, determining valid conclusions based on given assumptions, and analyzing electronic circuits.

Learning Outcomes

1. Define statements, translate simple and compound statements into symbols, determine the truth value of a statement.
2. Be able to construct truth tables and verify equivalent and tautology statements.
3. Be able to apply logic to switching network and logic gates problems.
4. Be able to translate conditional statements to other equivalent forms of statements and use these to solve NAND gates problems; and
5. Be able to write symbolic argument and determine valid arguments.

Time Allotment: Ten (10) lecture hours

Introduction

According to Immanuel Kant (1785) [2], logic is a science of the necessary laws of thought, without which no employment of the understanding and the reason takes place. Logic can be applied to many disciplines. For instance, lawyers used logic in making arguments and judgment. Referees or umpires in sports use logic in deciding a call in a situation of a match. Engineers use logic in designing computers. And mathematicians use logic to solve problems and make proofs. This section discusses the basic components of logic, the simple and compound statements, how to translate these statements into symbols, and determine its truth value.

The definition of the terms in this section were based mainly on the book: Mathematical Excursions written by Aufmann, *et al.* [1]. Examples to each definition were provided to better understand the concept.

5.1 Logic Statements and Quantifiers

Specific Objectives:

1. Define statements, translate simple and compound statements into symbols; and

2. Determine the truth value of a statement;

5.1.1 Simple and Compound Statements

Like any other discipline, the language of logic has its own syntax. Readers need to learn the rules of the language, symbols used, and the definition of terms. We first give the formal definition of the basic element of logic, the statement.

Definition 5.1 (Statement)**.** A **statement** is a declarative sentence that is either true or false, but not both true and false.

Example 5.1. Determine whether the the following sentences are statements or not a statement.

1. Batangas is a province of the Philippines.

2. People with ages from 21 to 59 years old cannot be infected by COVID-19 virus.

3. $x + 1 = 5$

4. Open the door.

5. What is your mother's name?

Solution.

1.The first sentence is a true sentence. This is a statement. 2.The

second sentence is a false sentence. This a statement.

3. The third sentence is either true or false depending on the value of x. This sentence cannot be both true and false at the same time in a specific value of x. Therefore, Sentence 3 is a statement.

4. Sentence 4 is not a declarative sentence. Hence, this is not a statement.

5.Sentence 5 is not a declarative sentence. This is not a statement.

Sometimes, you may not know whether the given sentence is true or false, but you know that the sentence is either true or false and that it is not both true and false. Examples of this are given below. These two sentences are statements.

Example 5.2. 1.Jaime Mora was the coach of Philippine Sepak Takraw team in 2015 SEA Games.

2.In 2022, the next president of the Philippines is a woman.

There are some sentences that are both true and false at the same time. This kind of sentence is not considered a statement, based on the definition of a statement. For example, consider the sentence below.

"**This is false sentence.**"

The above sentence is not a statement because if we assume it to be a true sentence, then it is false, and if we assume it to be a false sentence, then it is true. Statements cannot be true and false at the same time.

In other references, the term statement is also called proposition.

There two kinds of statements: simple and compound statements. Their definitions are given below.

Definition 5.2. A simple statement is a statement that conveys a single idea. A compound statement is a statement that conveys two or more ideas.

By connecting simple statements with words and phrases such as **and**, **or**, **if . . . then**, and **if and only if** creates a compound statement. These words and phrases are called connectives.

Example 5.3. The following are examples of simple and compound state- ments. The first two statements are simple and the last two statements are compound statements.

1.I will play Mobile Legend.

2.I will go to school.

3.I will play Mobile Legend or I will go to school.

4.I will play Mobile Legend and I will go to school.

NOTATIONS.

For convenience, there are notations used to represent statements and connectives.

Statements are usually represented by a lower-case letter such as p, q, r, and$s$.

The symbols for connectives are as follows:

$\wedge$ - symbol for the connective "**and**"

$\vee$ - symbol for the connective "**or**"

$\longrightarrow$ - symbol for the connective "**if . . . then**"

$\longleftrightarrow$ - symbol for the connective "**if and only if** "

Another important notion here is the negation of a statement. The defi- nition of the negation and its notation is given below.

Definition 5.3. Let be p a statement. The **negation** of p, denoted by $\sim p$, (read as "not p ") is a statement that is false if p is true and it is true if p is false. false.

On the left of the table below are the statements. The corresponding negation of the statement can be found on the right column of the table.

It is worth noting that the negation of the negation of the statement is the original statement. That is, if p is a statement, then $\sim(\sim p) = p$.

The next example tells us how to write compound statements into sym- bolic forms.

Statement	Negation of the statement
1. Rachel is a Blue Badge umpire in table tennis.	1. Rachel is not a Blue Badge umpire in table tennis.
2. The color of the roof is not green.	2. The color of the roof is green.
3. $x \geq 4$	3. $x < 4$

p : I will play table tennis;

q : I will go to school;

r : I will not do my assignment

Example 5.4. Consider the symbols corresponding to the statements below. Let Translate the following compound statements in symbolic form.

1.I will play table tennis or I will go to school.

2.I will go to school and I will not play table tennis. 3.If I

will go to school then I will do my assignment.

4.I will play table tennis if and only if I will not go to school.

Solution. The symbols corresponding to the above statements are given below.

1. $p \vee q$ 2. $q \wedge \sim p$ 3. $q \rightarrow \sim r$ 4. $p \leftrightarrow q$

Type of Statements

A compound statement can be classified according to the connectives used.

1.A statement is called **disjunction** if the connective “or” is used.

2.A statement is called **conjunction** if the connective “and” is used.

3.A statement is called **conditional** if the connective “if . . . only” is used.

4.A statement is called **biconditional** if the connective “ if and only if” is used.

The table below describes logic connectives, statements, symbolic form, and type of statements. Let p and q are simple statements.

Remark 1. In some cases, the word “but” generally means the same as “and”, and the phrase“neither A nor B” is translated as ”not A and not B”.

Statement	*Connective*	*Symbolic form*	*Type of statement*
p and q	and	$p \wedge q$	Conjunction
p or q	or	$p \vee q$	Disjunction
If p then q	If . . . then	$p \rightarrow q$	Conditional
p if and only if q	If and only if	$p \leftrightarrow q$	biconditional
Not p	not	$\sim p$	negation

Learning Activity

1. Determine whether each sentence is a statement.

   (a) Are you going to Manila this Sunday?

   (b) Vaccines for COVID-19 will be available before December 2020.

2. Determine the simple statements in each compound statement.

   (a) The President will visit the campus on Wednesday or Friday.
   (b) If today is Monday, then tomorrow is Sunday.
   (c) Daniel will study the lessons and he will not wash his clothes.

3. Write the negation of each statement.

   (a) The game was shown on ABS-Kapuso TV network.
   (b) Today is a rainy day.
   (c) The dinner does not contain dessert.

4. Write each sentence in symbolic form. Represent each simple statement in the sentence with the letter indicated in the parentheses.

   (a) I will enroll in Abstract Algebra (a) or Calculus (c).
   (b) An angle is a right angle (a) if and only if it measures 90° (b).

Compound Statements and Grouping Symbols

Compound statements may contain more than two simple statements. To avoid confusion with the meaning of the statements, groupings of statements are necessary. Here are the rules in groupings:

Rules of Grouping of a Compound Statement

1. In symbolic form, the parentheses are used to indicate the simple statements that are being grouped together.

2. In sentence form, a comma is used to indicate which simple statements are group together. That is, statements of the same side a comma are group together.

The meaning of the statement is affected by the parenthesis of the sym- bolic statements. For example, the compound statement $\sim(p \wedge q)$ is different from $\sim p \wedge q$. The former is the negation of the compound statement $p \wedge q$. The statement $\sim(p \wedge q)$ is read as " It is not true that, $p \wedge q$". In the latter, the negation is on the statement p only and read as "Not p and q".

Remark 2. Remark. In translating compound statement in symbolic form to an English sentence, the simple statements inside the parentheses in the symbolic form will all be on the same side of the comma that appears in the English sentence.

Example 5.5. Let p, q, and$r$ represent the following statements.

p : John's playing style is the same as Lebron's.

q : Hazel has straight hair.

r : John is a basketball player.

a. Write the symbolic form as an English sentence.

$$(p \wedge q) \rightarrow r$$

b. Write in symbolic form the given statement below.

If John is not a basketball player or Hazel has a straight hair, then John's playing style is not the same as Lebron's.

Solution.

a. Since p and q are grouped together, they must be on the same side of the comma in English sentences. Hence, this becomes, ' If John's playing style is the same as Lebron's and Hazel has straight hair, then John is a basketball player.

b. The statements "John is not a basketball player" and " John's playing style is not the same as Lebron's" are the negation of the statements r and p, respectively. The sentence is a conditional form and the first two statements are of the same side of the comma. Hence, the symbolic form is

$$(r \vee q) \to p.$$

Quantifiers and their Negation

In English sentences, some are true for all or it is true for some conditions. In order that these kinds of sentences are to be statements, quantifiers are needed.

Definition 5.4 (**Existential and Universal Quantifiers**).

- **Existential Quantifiers.** These are used as prefixes to assert the existence of something. These include the word some, and the phrases there exists and at least one.
- **Universal Quantifiers.** These are used as prefixes to assert that every element of a given set satisfies some conditions or to deny the existence of something. These include the words all, every, none, and no.

Example 5.6. The first two statements used existential quantifiers and last two statements used universal quantifiers.

1. Some coffee shops are open.
2. There exists an integer n such that $3n \geq 120$.
3. All players are nice people.
4. No even integers are divisible by 3 .

Negation of statements involving quantifiers

In the previous discussion, it is known that the negation of a statement is false if the statement is true and it is true if the statement is false. This concept must be considered in constructing the negation of a statement involving quantifiers.

For example, let us consider the statement " *Some coffee shops are open*". Suppose this statement is true, this means that some coffee shops are open, and others are not open. To find its negation, we need to write a statement that is false if that given statement is true and false if otherwise. Observe that the statement "*Some coffee shops are not open*" has the same meaning as the first statement. Hence, this statement cannot be the negation of the given statement. Thus, the statement " *No coffee shops are open*" will make the given statement false. Hence, it is the negation of the given statement.

The table below shows the negation of the quantified statements and their negation.

Quantified Statements	Negation
All X are Y	Some X are not Y
No X are Y	Some X are Y
Some X are Y	No X are Y
Some X are not Y	All X are Y

Table 5.1: Quantified statements and their negations

Learning Activity

1. Write each symbolic statement as an English sentence. Use p, q, r, s, and t as defined below.

p : Sarah Geronimo is a singer.

q : Sarah Geronimo not a songwriter.

r : Sarah Geronimo is an actress.

s : Sarah Geronimo plays the piano.

t : Sarah Geronimo does not play the guitar.

(a)$(p \wedge q) \vee r$

(b) $\sim p \rightarrow (p \vee r)$ (c)(p
$\vee s) \rightarrow (q \wedge t)$ (d)($r \wedge$
$p) \leftrightarrow q$
(e) $t \leftrightarrow (\sim r \wedge \sim p)$

2. Write each sentence in symbolic form. Use p, q, $r$and s as defined below.

p : Paul is a table tennis player. q
: Paul is a basketball player. r
: Paul is a rock star.
s : Paul is a mechanical engineer.

(a) Paul is a table tennis player or a basketball player, and he is not an engineer.

(b) If it is not true that Paul is a table tennis player or a rock star, then Paul is a mechanical engineer.

(c) Either Paul is a basketball player and a rock star, or he is a table tennis player.

(d) Paul is a basketball player, if and only if he is not a football player and he is not a rock star.

(e) If Paul is a mechanical engineer, then he is a basketball player and he is not a table tennis player.

3. Write the negation of each quantified statement. Start each negation with "some," "no," or "All." [*Source: Mathematical Excursions by Hoff- mann, et al.*]

(a) Some lions are playful.

(b) All classic movies were first produced in black and white.

(c)No even numbers are odd numbers.

5.2 Truth Tables, Equivalent Statements, and Tautology

Truth Value and Truth Table of a Statement

As discussed in the earlier part of this section, every statement is either true or false. The truth value of a true statement is true (**T**) and the truth value of a false statement is false (**F**). Hence, in determining the truth value of a statement, we need to determine whether the statement is true or false. For example, the statement " *The National hero of the Philippines is Dr. Jose Rizal* " is a true statement. Therefore, its truth value is true. In general, we have the following:

Definition 5.5 (Truth value and Truth table)**.**

- **Truth value**. The truth value of a simple statement is either true (**T**) or false (**F**). The truth value of a compound statement depends on the truth values of its simple statements and its connectives.
- **Truth table**. A truth table is a table that shows the truth value of a compound statement for all possible truth values of its simple statements.

Definition 5.6 (Truth value of the negation)**.** Let p be a statement with its negation $\sim p$. The truth value of $\sim p$ is true if and only if p false.

The truth value of the given statement p and its negation $\sim p$ can be illustrated in Table 5.16.

p	$\sim p$
T	F
F	T

Table 5.2: Truth table for $\sim p$

Here we discuss the truth values of the different compound statements.

The first is the truth value of the conjunction.

Definition 5.7 (Truth Value of the Conjunction)**.** Let p and q are state- ments. The truth value of the conjunction $p \wedge q$ is true if and only if both p and q are true.

For example, the conjunction statement “ *John will go to school and Daniel will study logic*” is true only if both John will go to school and Daniel will study logic are true statements. If at least one of these simple statements is false, then the conjunction statement is false. The truth table for the conjunction statement using p and q as simple statements are given Table reftab:conjunction.

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 5.3: Truth table for $p \land q$

Definition 5.8 (**Truth value of the disjunction**). Let p and q are two statements. The disjunction statement p q is true if and $\lor$ only if p is true, q is true, or p and q are true.

The truth table for the disjunction statement $p \lor q$ is given in Table 5.4.

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 5.4: Truth table for $p \lor q$

Before we continue on the discussion of the truth value of other forms of compound statements, we first apply the truth value of the conjunction and disjunction to determine whether the statement is true or false.

Example 5.7. Determine whether each statement is true or false.

1.3 ≤ 7

2.5 is a prime number and 3 is an odd number.

3.8 f= 5 or 2 is a prime number.

Solution.

1. The inequality 3 7 ≤ means that 3 < 7 or 3 = 7. Using the truth table for disjunction, since 3 < 7 is true and 3 = 7 is false, the truth value of the statement 3 ≤ 7 is true.

2. True. Since both simple statements are true.

3. True. Since 8 *ƒ*= 5 is false, and 2 is a prime number is a true statement.

Here we continue our discussion on the truth value of the other compound statements: the conditional and biconditional statements. First, we give the definition of the conditional statement.

Definition 5.9 (**Conditional Statement**). Let p and q are simple state- ments. The statement of the form p q is called **conditional statement**. The statement p is called **antecedent** and the statement q is called the **consequent**.

For example, the statement "*If you will use my calculator, then you will get a high score in the Exam*" is a conditional statement. The statement "*you will use my calculator* " is the antecedent and the statement "you will get a high score in the Exam" is the consequent of the statement.

The truth value of the conditional statement is given next.

Definition 5.10 (**Truth Value of the Conditional Statement**). The conditional statement is false if is true and is false. It is true in all other cases.

Table 5.5 shows the truth table for the conditional statement.

p	q	$p \to q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 5.5: Truth table for $p \to q$

Example 5.8. Using the truth table above, determine whether each condi- tional statement is true or false.

1.If −3 is an integer, then 2 is a rational number. 2.If 4

≥ 3, then 2 + 5 = 6.

3.If all cats are black, then I am a millionaire.

Solution.

1. Both the antecedent and the consequent are true, the given conditional statement is true.
2. The antecedent is true, and the consequent is false. Using the above truth table, it is false.
3. Since there are cats that are not black, the antecedent is false. Using the truth table for the conditional statement above, in this case, whatever the truth value of the consequent, the conditional statement is true. Therefore, it is true.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5.6: Truth table for $p \leftrightarrow q$

Example 5.9. State whether each biconditional is true or false.

1. $x + 4 = 8$ if and only if $x = 4$.

2. $x^2 = 25$ if and only if $x = 5$.

Solution.

1. If $x+4 = 8$ then $x = 4$ is a true statement. Also, if $x = 4$ then $x+4 = 8$ is a true statement. Therefore, the truth value of this biconditional statement is true.
2. If $x^2 = 25$ then $x = 5$ is a false statement since it possible that other value of x is 5 since (5)^2 = 25 also. On the other hand, If x = 5 then x2=25 is a true statement. Therefore, this is a false statement.

*

Learning Activity

Determine the truth value of each of the given statements.

1.

1.3 ≥ 9 or 4 ≤ 5.

2.$(-2)^2 = 4$ and $-2^2 = 4$.

3. $x \in A$ or $x \notin A$. (A is a set)

4. If all frogs can dance, then today is Monday.

5.If $|x| = 8$ then $x = 8$.

6.If $x < 3$, then $x + 7 < 10$.

7. $|x|$ is a positive number if and only if $x \neq 0$.

8. $x > 0$ if and only if x is positive.

9. $x > 4$ if and only if $x > 8$.

10.3 = 4 if and only if 1 = 2 .

5.3 Truth Tables, Equivalent Statements, and Tautologies

Specific Objectives

At the end of this lesson, the students will be able to:

1.Construct truth tables for compound statements.

2.Define and identify equivalent statements 3.Define

and identify Tautology statements

Introduction

In the previous section, we defined the truth table for the negation of a statement, conjunction, disjunction, conditional and biconditional statements. In this section, we will discuss the construction of the truth table of compound statements involving combinations of these types of compound statements. Then we shall proceed to the definition of the equivalent and tautology statements. After this, we shall use the construction of truth tables in identifying determining equivalent and tautology statements.

Construction of Truth Tables

In computing the truth values of compound statements, the rule is like those used to evaluate algebraic expressions. First, evaluate the expressions within the inner most parentheses, then evaluate the expressions within the next inner most set of parentheses, and so forth until you have the truth values for the complete expression.

Example 5.10. Example 1. Construct the truth table for the statement $(p \vee q) \wedge (\sim q \wedge p)$.

Solution. To construct the truth table, follow the following:

⇒ Set up columns for each simple and compound statements that are found in the given statement. These are columns for p, q, $\sim q$, $p \vee q$, $(\sim q \wedge p)$, and $(p \vee q) \wedge (\sim q \wedge p)$.

⇒ In the p and q columns, fill in all the logical possible combinations of **T**s and **F**s.

⇒ Fill the column of q by the opposite truth values in the column q since this column is the negation of the statement q.

⇒ Use the truth table for disjunction to fill up the column $p \vee q$ using the truth values in columns p and q.

⇒ Use the truth table for conjunction to fill up the truth values of the column $\sim q \wedge p$ using appropriate truth values.

⇒ Finally, use the table for conjunction to fill up the truth values in column $(p \vee q) \wedge (\sim q \wedge p)$ using the corresponding computed truth values of the columns and $p \vee q$ and $\sim q \wedge p$.

The truth table of the statement $(p \vee q) \wedge (\sim q \wedge p)$ is shown in Table 5.7.

p	q	$\sim q$	$p \vee q$	$\sim q \wedge p$	$(p \vee q) \wedge (\sim q \wedge p)$
T	T	F	T	F	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	F

Table 5.7: Truth Table for $(p \vee q) \wedge (\sim q \wedge p)$

The next example is the truth table for compound statements that con- tains three simple statements p, q, and r.

Example 5.11. Construct the truth table for the statement $(p \vee q) \wedge r$.

Solution. Set up columns for $p, q, r, p \vee q$, and $(p \vee q) \wedge r$. Fill in the columns p, q, r with all the logically possible combinations of Ts and Fs, as shown Table 5.8.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Table 5.8: Initial truth table for $(p \vee q) \wedge r$

By similar argument as the solution in Example 1, the truth table for the statement $(p \vee q)\ r$ is given in Table 5.9.

Sometimes the truth value of the simple statements in the compound statement are given. To compute for the truth value of the compound statement, we do not need to complete the truth table. Instead, we set up appropriate columns, use the given truth values for each column and solve for the required truth value of the statement using appropriate truth tables.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	T

Table 5.9: The truth table for $(p \vee q) \wedge r$

Example 5.12. Find the truth value of the compound statement $(p \quad \vee \; q)$ r if the truth values of p, q, and r are T, F, and T, respectively.

Solution. A convenient way solve the above problem is to set up columns $p, q, r,$ $\sim q$, $p \vee q$, and $(p \vee \sim q) \rightarrow r$. Then fill in the given truth value for the columns p, q, r. Since q is T, $\sim q$ is F. Consequently, since p is T, $p \vee q$ is T. By the truth table of conditional statement, the truth value of $(p \vee q) \rightarrow risT$. The summary of this process is shown Table 5.10.

p	q	r	$\sim q$	$(p \vee q)$	$(p \vee \sim q) \rightarrow r$
T	T	T	F	T	T

Table 5.10: Computation of the truth value of $(p \vee \sim q) \rightarrow r$

The following remark determines the required number of rows in the truth table of a compound statement.

Remark 3. If a compound statement consists of k variables, each variable represents a simple statement, then the truth table for the given compound statement contains 2^k rows.

For example, if the compound statement consists of two simple state- ments, p and q, then there the number of rows of its truth table is 2^2. If $k = 3$, then $2^3 = 8$, and so on.

Logically Equivalent Statements

Two statements may be stated in different ways but say the same thing. For example, the statements “ Five is less than 8” and “ 8 is greater than 5” are two different ways saying the same thing. These statements are called logically equivalent (or simply equivalent) statements. We give the definition of equivalent statements.

Definition 5.11 (Equivalent statements)**.** Let p and q are two statements. Then p and q are said to be **logically equivalent** (or simply **equivalent**), denoted by $p \equiv q$, if they both have the same truth values for all possible truth values of their simple statements.

Example 5.13. Show that the statements $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are equiv- alent statements.

Solution. Do the following procedure.

- ⇒ Construct the truth table with one column for the truth values of $\sim(p \wedge q)$ and another column for the truth values of $\sim p \vee \sim q$, as shown in the Table 5.11.
- ⇒ Since the two statements have the same truth value for all possible truth values of their simple statements (see the last two columns of the table), the statements $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are equivalent.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 5.11: Illustrating equivalent statements

We give an illustration to show that the two statements are not logically equivalent.

Example 5.14. Show that the statements $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not equivalent.

Solution. First, construct the truth table. As shown in Table 5.12, it can be observed that the truth values of the two statements are not the same in the second and third row. Hence, the truth values of the two statements are not the same for all possible truth values of their simple statements. That is, when p is T and q is F, or p is F and q is T. Therefore, the statements $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not equivalent.

One of the useful equivalent statements is the De Morgans law.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Table 5.12: Illustrating not equivalent statements

De Morgan's Law

For any statements p and q, the following hold. a.

$\sim (p \wedge q) \equiv \sim p \vee \sim q$

b. $\sim (p \vee r) \equiv \sim p \wedge \sim r$

De Morgan's Law states that the negation of the conjunction statement is equivalent to the disjunction of the negation of each simple statement. And, the negation of the disjunctive statement is equivalent to the conjunction of the negation of each simple statement.

Example 5.15. Use De Morgans Law to write the given statement in an equivalent form.

1. It is not true that, he passed the examination or he played basketball.

2. I did not pass the test and I did not complete the course.

Solution. The equivalent statements are as follows:

1. He did not pass the examination and he did not play basketball. 2.It

is not true that, I passed the test or I completed the course.

Remark 4. De Morgans Law is especially useful in determining the negation of a statement.

Example 5.16. Use De Morgans Law to find the negation of $3 \leq x \leq 8$.

Solution.

The inequality $3 \leq x \leq 8$ means that $3 \leq x$ and $x \leq 8$. By De Morgan's Law, the negation of this is equivalent to $3 \not\leq x$ or $x \not\leq 8$. This implies that $3 > x$ or $x > 8$.

Using the truth table for conditional and biconditional statements, the following are equivalent statements.

Definition 5.12. The equivalent forms of conditional and biconditional statements are given below

(i.)Conditional: $p \to q \equiv \sim p \vee r$

(ii.)Biconditional: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Example 5.17. Write each the following in its disjunctive form 1.If I

pass the try-out, then I will be a varsity player.

2. If there is no vaccine for COVID-19, then classes in the public schools will be blended learning.

Solution. The following statements are equivalent to above example, re- spectively.

1.I cannot pass the try-out or I will be a varsity player.

2.There is a vaccine for COVID-19 or classes in the public schools will be blended learning.

The equivalent disjunctive form of the conditional statement can be used to determine the negation of the statement $p \to q$. Since $p \to q \equiv \sim p \vee r$, the negation of $p \to q$ is equivalent to the negation of $\sim p \vee r$. By De Morgans Law, $\sim (\sim p \vee r) \equiv \sim (\sim p) \wedge \sim r \equiv p \wedge \sim r$. Equivalently, we have the

following.

Negation of the conditional statement

The negattion of the conditional statement is given below.

$$\sim (\sim p \vee r) \equiv p \wedge \sim r.$$

Example 5.18. Write the negation of each conditional statement.

1. If I meet the deadline, I will go out with my friends. 2.If

$x^2 = 16$, then $x = 4$ or $x = -4$.

Solution.

1.I met the deadline and I did not go out with my friends.

2. $x^2 = 16$ and, $x f= 4$ and $x f= -4$.

We give here some known equivalent statements for reference purposes.

List of some equivalent statements

Given any statement variables p, q, and$r$, a tautology t and a contradiction c, the following logical equivalences hold.

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \sim p \equiv t$	$p \wedge \sim p \equiv c$
6. Double negative law:	$\sim (\sim p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. Universal bound laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$

Tautology and Contradictions

There are statements which are always true and there are statements that are always false. Here, we discuss these kinds of statements. First, we give the definition of the following terms.

Definition 5.13. A **tautology** is a statement whose truth value is always true regardless of the truth values of its individual simple statements. A statement which is a tautology is called **tautological statement**. A **con- tradiction** is a statement whose truth value is always false regardless of the truth values of the individual simple statements that is a contradiction is a **contradictory statement**.

The above definition is equivalent to the following remark.

Remark 5. Using the truth table of the given symbolic statement,the fol- lowing hold.

1. If the truth values in the column of the given symbolic statement are all true (T), then the given statement is a tautology.

2. If the truth values in the column of the given symbolic statement are all false (F), then the given statement is a contradiction.

Example 5.19. Show that the statement $p \vee \sim p$ is a tautology.

Solution. First, construct the truth table for the statement $p \lor \sim p$, as shown in Table 5.13. It can be observed that the truth values in the column $p \lor \sim p$ are all true. Therefore, this statement is a tautology.

p	$\sim p$	$p \lor \sim p$
T	F	T
F	T	T

Table 5.13: Truth table for $p \lor \sim p$

Example 5.20. Show that the statement $p \land \sim p$ is a contradiction.

Solution. First, construct the truth table for the statement $p \lor \sim p$, as shown in Table 5.14. It can be observed that the truth values in the column $p \land \sim p$ are all false. Therefore, this statement is a contradiction.

p	$\sim p$	$p \lor \sim p$
T	F	F
F	T	F

Table 5.14: Truth table for $p \land \sim p$

The next example shows the equivalent statements involving tautological and contradictory statements.

Example 5.21. Let p, t, and c are statements. If t is a tautology and c is a contradiction, show that and

i. $p \land t \equiv p$

i. $p \land c \equiv c$

Solution. The solution for (i.)and (ii.) can be done using a single truth table. Create columns $p, c, t, p \land t$ and $p \land c$. The truth values of column of p is either T or F. Since t is a tautology, the truth values column of t are all T. On the other hand, the truth values of column of c are all F since c is a contradiction. Then compute the truth values of $p \quad t \quad p \ p \quad c \quad c$. The results are shown in Table 5.15.It can be observed that the truth values of the columns of $p \land t$ and p are identical. Thus, these two statements are equivalent. Similarly, the statements $p \land c$ and c are equivalent.

Based on the Example 5.21, we have the following remark.

p	t	c	$p \land t$	$p \land c$
T	T	F	T	F
F	T	F	F	F

Table 5.15: Illustrating Truth table for $p \land \sim p$

Remark 6. For a given statement, the following statements hold.

1. The conjunction of any given statement and a tautological statement is equivalent to the given statement.

2. The conjunction of any given statement and a contradictory statement is a statement that is a contradiction.

Learning Activity

1. Write truth tables for the following statements

   (a) $\sim p \land q$
   (b) $p \land (q \land r)$
   (c) $(\sim q \land r) \lor [p \land (q \land \sim r)]$
   (d) $(p \rightarrow q) \rightarrow (q \lor r)$
   (e) $(p \land \sim r) \leftrightarrow (q \lor r)$

2. Determine the truth value of the statement given that p is a true (T) statement, q is a false (F) statement, and r is a true (T) statement.

   (a) $p \lor (q \lor \sim r)$
   (b) $(p \lor \sim q) \land (p \lor r)$
   (c) $[(p \land q) \land r] \lor [p \land (q \land \sim r)]$
   (d) $p \rightarrow (q \lor \sim r)$
   (e) $(p \land \sim r) \leftrightarrow (q \lor r)$

3. Determine whether the statements are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer.

(a) $p \vee (p \wedge q)$ and p

(b) $p \vee (q \wedge \sim p)$ and $p \vee q$

(c) $\sim [p \vee (q \wedge r)]$ and $\sim p \wedge (\sim q \vee \sim r)$

(d) $\sim p \rightarrow (p \vee r)$ and r

(e) $\sim (p \rightarrow q)$ and $p \wedge \sim q$

(f) $p \leftrightarrow \sim q$ and $(p \rightarrow \sim q) \wedge (\sim q \rightarrow p)$

4. Use truth tables to determine which of the statement forms are tautologies and which are contradiction.

(a) $(p \wedge q) \vee [p \vee (p \wedge \sim q)]$

(b) $(p \vee q) \vee (\sim p \vee q)$

(c) $p \wedge (\sim p \vee q)$

5.4 Switching Network and Logic Gates

There are several applications of logic. In this section, we shall discuss its application to circuits, particularly, switching networks and Logic gates. This section is divided into two parts, the first part discusses the switching network and the second part on the logic gates.

Switching Network

Switching networks are used in many electrical appliances, as well as in telephone equipment and computers. The idea that a switch has two possible values, either it is *on* or it is *off* is analogous to the statement in logic which is a *true* or a *false*. This analogy between logic in switching networks was introduced by Claud E. Shannon in his thesis in 1939.

The definition of the Switching Network is given below.

Definition 5.14. A **switching network** consists of wires and switches that can open and close.

Example 5.22. A switching network that consists of single switch P that connects two terminals is shown in Figure **??**

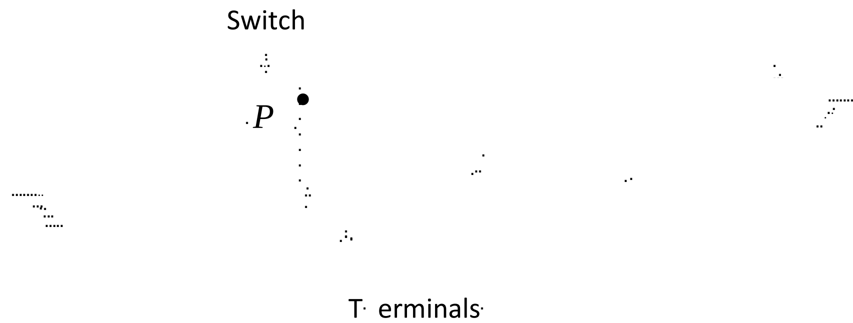


Figure 5.1: Illustrating a switching network with switch P

We say that a switch P is in a **closed** position, if an electric current can flow from one terminal to the other terminal and the switch it is in an **open** position if an electric current cannot flow from one terminal to the other terminal.

Notations

In the discussion of this topic, the following notations will be used for a switch in a switching network.

1. Usually, a switch is denoted by upper case letters in English such as

   $P, Q, R, S, etc.$

2. If two switches are always open at the same time and always closed at the same time, then we will use the same letter to designate both switches.

3. If two switches are of opposite position at the same time, then we the notation P to one switch and $\sim P$ to the other switch is used.

A switching network can either be a *series*, *parallel*, or a combination series and parallel networks.

Example 5.23. Figure 5.2 shows the examples of series, parallel and a com- bination of series and parallel networks.

A switching network can be classified as either an open or closed network.

The definition of these terms is given next.

Definition 5.15 (Open and Closed Network)**.** A network is said to be **closed** if the current can flow between the terminals. If a current cannot flow be- tween the terminals, we say that the network is **open**.

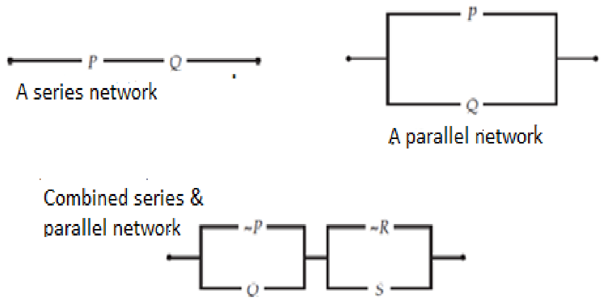


Figure 5.2: Illustrating the different kinds of switching networks

Example 5.24. Consider the networks in Figure 5.2. The series network is closed if and only if both switches P and Q are both closed. And it is open if at least one of P and Q is open. On the other hand, the parallel network is closed if P or Q is closed. This parallel network is open if both P and Q are open.

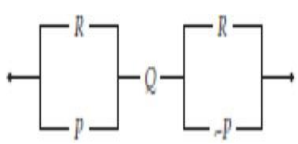
Switching Network and Logic

There is analogous relations between the switching network and a statement in logic.

Remark 7. The following are the analogous relations between switching networks and

- A series network is analogous to the logic statement $p \wedge q$. Thus, two switches P and Q connected in series is denoted by $P \wedge Q$.
- A parallel network is analogous to the logic statement $p \vee q$. Thus, two switches P and Q connected in parallel is denoted $P \vee Q$.

Example 5.25. Consider the switching network below. Write the symbolic statement to represent the switching network below.



Solution. The network is a combination of a series and parallel. R and P are parallel so we have $R\ \ P$. Similarly, R and P are parallel, so $R\ \ \ P$. Since $R\ P$, Q, and $R\ P$ form a series network, the equivalent symbolic statement is $(R\ P\ )\ Q\ (R\ P\ )$.
$\vee\ \ \sim\ \ \vee\ \ \sim\ \ \vee\ \ \vee\ \sim$

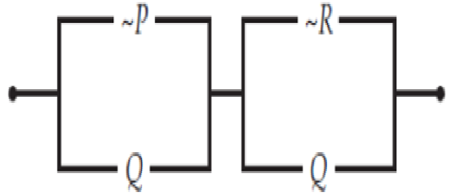
The next example shows how to construct the switching network when symbolic statements representing the switching network are given.

Example 5.26. Example. Consider the symbolic statement $(P \vee Q) \wedge$

$(\sim R \vee Q)$ representing the network. Draw the network.

Solution. In general, $(P\ Q)$ and $(R\ Q)$ are connected in series. But

$\sim P$ and Q are parallel. Also, R and Q are parallel. Thus, the network can be drawn as follows.



Closure Table of a Network

One way to determine that a switching network is closed or open is the use of a closure table. This closure table is analogous to the truth table of a statement in logic. In the closure table, 1 is used to designate that a switch or switching network is closed and a 0 is used to indicate that it is open.

Negation Closure Table

Let P be a switch in a network, the notation P, denotes another switch such that whenever P is closed the switch P is open. And if P is open then
$\sim P$ is closed. Below is the negation closure table.

P	$\sim P$
1	0
0	1

Table 5.16: The negation closure table

Series Network Closure Table

Below is a figure of a series network with two switches P and Q. This network is closed if and only if both P and Q are closed.

.• . P . Q .•.

Figure 5.3:

The series network table is given below.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Table 5.17: The conjunction closure table

Parallel Network Closure Table

The figure below is a parallel network with two switches P and Q. This network is closed if P or Q is closed, or both P and Q are closed.

.P

.•.

.Q

The parallel network table is given below.

Example 5.27. Consider switching network below. Determine the required conditions under which the network is closed.

Solution. First, write the symbolic statement that represents the network. That is, $P \wedge (\sim Q \vee P) \wedge Q$. Then construct the closure table as follows. This statement is equivalent to $[P \wedge (\sim Q \vee P)] \wedge Q$.

As observe from the above table, the network is closed whenever

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Table 5.18: Parallel network closure table

$. \sim . Q$

• $. \sim P$.

$\cdot P$.

P	Q	$\sim P$	$\sim Q$	$\sim Q \vee P$	$\sim P \wedge (\sim Q \vee P)$	$[\sim P \wedge (\sim Q \vee P)] \wedge Q$
1	1	0	0	1	1	1
1	0	0	1	1	1	0
0	1	1	0	0	1	1
0	0	1	1	1	1	1

Table 5.19: The closure table of $\sim P \wedge (\sim q \vee) \wedge Q$

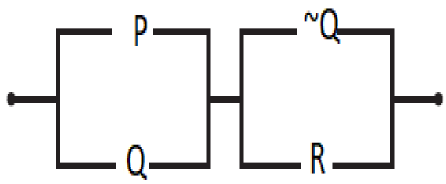
a. P is closed and Q is closed;

b. P is open and Q is closed; or

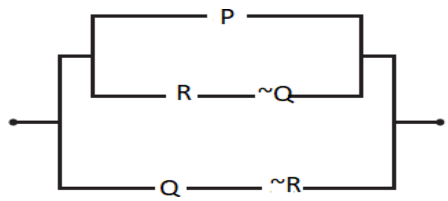
b. P is open and Q is open.

Learning Activity

A. Write a symbolic statement to represent each of the networks.



(A.1)



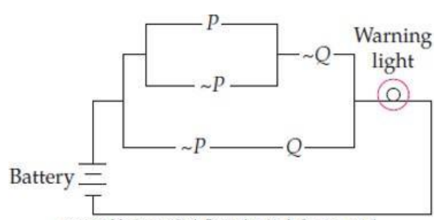
(A.2)

B. Draw a network to represent each statement.

1. $P \wedge [(Q \vee \sim R) \wedge R]$
1.$[P \vee (Q \wedge \sim P)] \wedge \sim Q \vee \sim R \vee (R \wedge \sim Q)$

C. Construct a closure table for each of the switching networks in $A.1$, and $A.2$ above. Use the closure table to determine the required conditions for the network to be closed.

D. The circuit shown below is a switching network, a warning light, and a battery. In each circuit the warning light will turn on only when the switching network is closed. [Source: [1], pp. 120-121]



source: Mathematical Excursion by Aufmann, et al.

For each of the following conditions, determine whether the warning light in the above drawing will be on or off.

1. P is closed and Q is open.
2. P is closed and Q is open.
3. P is open and Q is closed.

Logic Gates

Another application of logic is logic gates. Modern digital computers use gates to process information. These gates are designed to receive two types of electronic signals, which are generally represented as a stream of 1 or a 0. The symbols 0 and 1 are called bits, short for binary digits. This terminology was introduced in 1946 by the statistician John Tukey.

In this section, we discuss three basic circuits. These are known as NOT- gate, AND-gate, and OR-gate. A combination of these gates produces com- plicated circuits.

Definition 5.16. A **NOT-gate** is a circuit with one input signal and one output signal. If the input signal is 1, the output signal is 0. Conversely, if the input signal is 0, then the output signal is 1.

The summary of the action of the NOT-gate is shown in the table below.

Input	Output
P	R
1	0
0	1

Table 5.20: Actions of NOT-gate

Figure 5.4 shows the symbol for action of the NOT-gate with input signal P and output signal R.

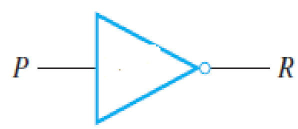


Figure 5.4: Illustrating the NOT-gate

For example, consider the signal 0011 as the input to the NOT-gate. Then the output signal is 1100. The diagram is shown in Figure 5.5.

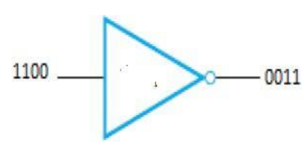


Figure 5.5: Illustrating the action of NOT-gate

Definition 5.17. An **AND-gate** is a circuit with two input signals and one output signal. If both input signals are 1, then the output signal is 1. Otherwise, the output signal is 0.

The summary of the action of the AND-gate is shown in Table 5.21.

Input		Output
P	Q	R
1	1	1
1	0	0
0	1	0
0	0	0

Table 5.21: Actions of AND-gate

Figure 5.6 shows the symbol for action of the AND-gate with input signals P and Q, and output signal R.

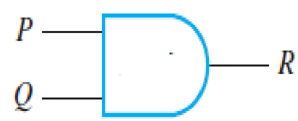


Figure 5.6: Illustrating the AND-gate

For example, consider the two signals P = 11010 and Q = 10111 as the inputs to the AND-gate. Using the definition of AND-gate, the first digit of P is 1 and the first digit of Q is 1, by the definition of AND-gate, the first digit of the output signal is 1. Similarly, the second,third, fourth, and fifth digits of the output are 0, 0, 1, and 0, respectively. Hence, the output signal is 10010. The diagram is shown in Figure 5.7.

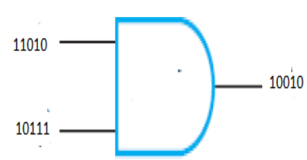


Figure 5.7: Illustrating the action of AND-gate

Definition 5.18. An **OR-gate** also has two input signals and one output signal. If both input signals are 0, then the output signal is 0. Otherwise, the output signal is 1.

The summary of the action of the OR-gate is shown in Table 5.22.

Input		Output
P	Q	R
1	1	1
1	0	1
0	1	1
0	0	0

Table 5.22: Actions of OR-gate

Figure 5.8 shows the symbol for the action of the OR-gate with input signal P and output signal R.

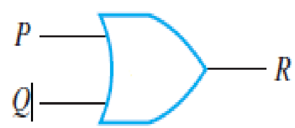


Figure 5.8: Illustrating the OR-gate

For example, consider the two signals P = 110 and Q = 101 as the inputs to the AND-gate. Using the definition of OR-gate, the first digit of P is 1 and the first digit of Q is 1, by the definition of OR-gate, the first digit of the output signal is 1. Similarly, the second and third digits of the output signal are 1 and 1, respectively. Hence, the output signal is 111. The diagram is shown in Figure 5.9.

By identifying the signal 1 with the truth value T of a statement and the signal 0 with the truth value F of a statement, then it can be verified that the actions of Not-gate, AND-gate, and OR-gate are analogous with truth values of the Negation,Conjunction, and Disjunction statements, respectively. The following remark is useful in solving complicated networks.

Remark 8. If P, Q and are given signals, the output for NOT-gate with input P is $\sim P$. The inputs P and Q for AND-gate and OR-gate, the output signal can be represented by $P \wedge Q$ and $P \vee Q$, respectively.

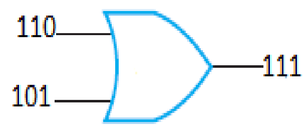


Figure 5.9: Illustrating the action of OR-gate

The NOT, AND, and OR gates may be combined and will results to a more complicated circuit. In this case, one must observe the symbols used for each of these gates so that appropriate actions for each stage will be done.

For example, the circuit in Figure 5.10 shows a network that consists of a OR -gate, NOT- gate and AND- gate with input signals P, Q and R and an output signal S.

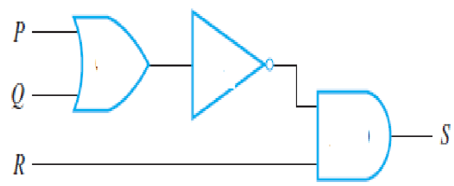


Figure 5.10: Circuit with a combination of gates

Example. Consider the circuit in Figure 5.10. Suppose P = 10, Q = 11 and R = 01.

Solution. Then the computation for the output is shown in Figure 5.11.

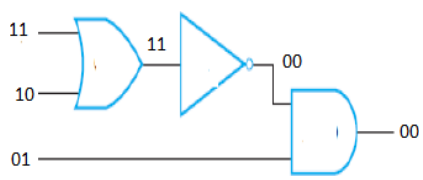


Figure 5.11: Illustrating the computation of an output signal

Boolean Expression

Definition 5.19. Any variable, such as a statement variable or an input signal, that can take one of only two values is called a **Boolean variable**. An expression composed of Boolean variables and the connectives ∼, ∧, and ∧ is called a **Boolean expression**.

Example. Using the circuit in Figure 5.10, determine the Boolean expression.

Solution. There are three gates. The first gate is an OR-gate with inputs P and Q. Hence its output is $P \vee Q$. This signal $P \vee Q$ is then an input to the second gate, which is the NOT-gate. So the output of this second gate is $\sim(P \vee Q)$. This output of the second gate is now an input to the third gate, which is an AND-gate, with another input R. Since the third gate is AND-gate, the value of S can be computed using $\sim(P \vee Q) \wedge R$. This process is shown in Figure 5.12. The symbol $\sim(P \vee Q) \wedge R$ is called the **Boolean Expression** of the circuit.

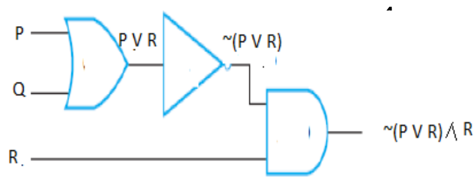
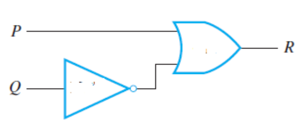


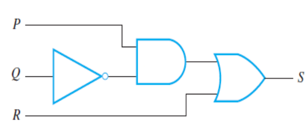
Figure 5.12: Illustrating the computation of an output signal

Learning Activity

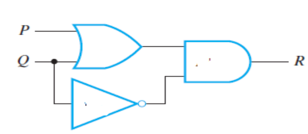
A.Give the output signals for the following circuits if the input signals are as indicated.

1. 

Input signal: P= 1011; Q= 0010

2. 

Input signals: P= 011011; Q= 100110; R= 101011

3. 

input signals: $P = 110$ and $Q = 101$

B.F ind the Boolean expression that corresponds to the circuit in the above exercise.

(a) Circuit in Item 1
(b)Circuit in Item 2
(c)Circuit in Item 3

C.Construct circuits for each of the following Boolean expressions.

(a) $\sim P \vee Q$
(b)$(P \wedge Q) \vee (P \wedge Q)$
(c) $p \vee (\sim Q \vee \sim P)$
(d)$(P \wedge \sim Q) \vee (\sim P \wedge R)$

5.5 The Conditional and Related Statements

In the previous sections, we learned how to determine that the two statements are equivalent. One of these is the equivalent disjunctive form of a conditional statement. Here we discuss the different equivalent forms of conditional statements.

Equivalent Forms of the Conditional

A conditional statement can stated in many equivalent forms.

Table 5.23 gives some of the equivalent statements that can be used to state a conditional statement.

Example. Write the following in “If p, then q” statement.

1.A number is divisible by 3 , only if the sum of its digits is divisible by 3.

2.Every square is a quadrilateral.

Solution. The first statement is of the form “q, only if p” and the second is of the form “every p is a q.”

1. If the sum of its digits of a number is divisible by 3 , then the number is divisible by 3.

2.If it is a square, then it is a quadrilateral.

Every conditional statement $p \to q$ can be written in the following forms.
If p, then q If p, q. p only if q. p implies q. Not p or q. Every p is a q. q, if p. q provided that p. q is a necessary condition for p. p is a sufficient condition for q.

Table 5.23: Common forms of $p \to q$

The Converse, the Inverse, and the Contrapositive

There are three statements related to conditional statements. These are the *converse*, the *inverse*, and the *contrapositive*.

Definition 5.20. Let p and q are statements. Then the following are the statements related to $p \to q$.

1. The **converse** of $p \to q$ is $q \to p$.

2. The **inverse** of $p \to q$ is $\sim p \to \sim q$.

3. The **ccontrapositive** of $p \to q$ is $q \to p$.

Example. Write the converse, inverse, and contrapositive of

If S is a square, then S is a rectangle.

Solution.

1. *Converse:* If S is a rectangle, then S is a square.
2. *Inverse:* If S is not a square, then S is a not a rectangle.
3. *Contrapositive:* If S is not a rectangle, then S is not a square.

All three related statements of the conditional statement $p \rightarrow q$ are not all equivalent to $p \rightarrow q$. However, using the truth table for each related statements, it can be verified that some of them are related to one another. These are the following.

Remark 9. Let p and q are statements. Then the following hold:

1. $p \rightarrow q \equiv \sim q \rightarrow \sim p$; and 2.

$\sim p \rightarrow \sim q \equiv \sim q \rightarrow \sim p$.

Example. Determine whether each pair of statements are equivalent.

1. If you see a man and a woman holding each others hands, then are they are in a relationship with each other.
   If the man and a woman are not in a relationship with each other, then they are not holding each other's hands.

2. If 3 x = 15, then x = 5. If
   $3x \neq 15$ then $x \neq 5$

Solution.

1. The second statement is the converse of the first statement. By Remark 5.5, they are equivalent.

2. The second statement is the inverse of the first statement. They are not equivalent,in view of Remark 5.5.

Learning Activity

A. Write each statement in "If p, then q" statement.

1. Every odd prime is an odd integer.
2. The university will conduct face-to-face classes provided that there is already vaccines for COVID-19.
3. I will be able to buy a car only if I have enough savings.
4. Every parallelogram is a quadrilateral.
5. If I got a high score in the last quiz, I would be exempted to take the final examination.

6. In an equilateral triangle, all angles are equal.

B. Determine whether the given pair of statements are equivalent.

1. If you understand logic, you can remember logic.

   If you do not understand logic, you cannot remember logic.

2. If $|x| > 1$, then $x > 1$ or $x < -1$.

   If $-1 \le x \le 1$, then $|x| \le 1$.

3. If the measure of one angle of a triangle is 90 degrees, then the triangle is a right angle.
   If a triangle is a right triangle, then the measure of one angle is 90 degrees.

4. If $2x + 5 = 15$, then $x = 5$ If
   $2x + 5 \neq 15$, then $x \neq 5$

C. Answer the following:

1. Determine the original statement if the converse is:

   If I do not have enough time, I cannot do my my assignments.

2. Give an example of true conditional statement whose

   i. converse is true.

   ii. converse is false.

   iii. inverse is true.

D. Puzzle. This puzzle was taken from the book written by Aufmann [1].

This puzzle was written by Lewis Carroll.

The Dodo says that the Hatter tells lies.

The Hatter says that the March Hare tells lies.

The March Hare says that both the Dodo and the Hatter tell lies. Who is telling the truth?

Hint: Consider the three different cases in which only one of the charac- ters is telling the truth. In only one of these cases can all three of the statements be true.

5.6 Symbolic Arguments

This section gives the definition of an argument. Also, this will discuss the conditions to determine whether the argument is valid or invalid.

First, we present the definition of an argument.

Definition 5.21. An **argument** consists of a set of statements called **premises** and another statement called the **conclusion**.

Example. The set of statements below is an example of an argument.

If I am going to join the basketball try-out, then I will buy a new pair of shoes. I will join the try-out. Therefore, I will buy a new pair shoes.

The above argument consists of two premises and a conclusion. Usually, the premises and conclusion are written in the following manner.

First Premise:	If I am going to join the basketball try-out, then I will buy a new pair of shoes.
Second Premise:	I will join the try-out.
Conclusion:	Therefore, I will buy a new pair shoes.

In writing an argument in symbolic form, each simple statement in the argument must be represented by a variable, usually by lower case letters in the English alphabet. For instance, consider the argument above. Let the following notations corresponds to each simple statement of the above argument.

x : I am going to join the basketball try-out

y : I will buy a new pair of shoes.

Then the argument can be written in the following form:

$$\begin{array}{l} x \to y \\ x \\ \hline \therefore y \end{array}$$

Definition 5.22. An argument is **valid** if the conclusion is true whenever all the premises are assumed to be true. An argument is **invalid** if it is not a valid argument.

Example. Determine whether the given argument is valid or invalid.

If Rea does not have a quarantine pass, she will stay at home. She did not stay at home. Therefore, she has a quarantine pass.

Solution. First, write the argument in symbolic form. If x and y denotes the following simple statements.

x : Rea has a quarantine pass.

y : Rea will stay at home

Then the symbolic form of the argument can be written as:

$$\begin{array}{l} \sim x \rightarrow y \\ \sim y \\ \hline \therefore x \end{array}$$

Next, we construct the truth table showing the truth value of each premise and the truth value of the conclusion for all truth values of the simple statements, as follows:

x	y	$\sim x$	First premise $\sim x \rightarrow y$	Second premise $\sim y$	conclusion x
T	T	F	T	F	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	T	F

It can be observed from the truth table that that there is only one row that all the premises are true, that is the second row (red). The conclusion in this row is also true. Hence, the argument is valid.

The following remark is useful in determining the invalid argument.

Remark 10. If the conclusion is false in any row in which all of the premises are true, the argument is invalid.

The following are standard form of valid arguments:

Modus ponens	Modus tolens	Law of syllogism	Disjunctive syllogism
$p \to q$ p ——— $\therefore q$	$p \to q$ $\sim q$ ——— $\therefore \sim p$	$p \to q$ $q \to r$ ——— $\therefore p \to r$	$p \vee q$ $\sim p$ ——— $\therefore q$

Table 5.24: Standard Forms of Four Valid Arguments

Learning Activity

A. Write each argument in symbolic form, using the letters p, q or r.

1. If the demand for face masks increase, the manufacturer produces more face masks. The demand for face masks does not increase. Therefore, the manufacturer does not produce more face masks.
2. If it rains, the soil is wet. It does not rain. Therefore the soil is not wet.

B. Use a truth table to determine whether the argument is valid or invalid.

1.	2.	3.	4.
$p \vee q$ $q\ q$ ——— $\therefore p$	$p \to q$ p ——— $\therefore q$	$p \vee q$ q ——— $\therefore p$	$p \to$ $q \to r$ ——— $\therefore p \to r$

Chapter Test 6

1. Determine whether each sentence is a statement.

   (a) Manny Paquiao is the greatest Filipino boxer of all time.
   (b) In Batangas State University, there are more males than females.
   (c)Are you going to Batangas ?
   (d)Vaccines for COVID-19 will not be available before December 2020.

2. Determine the simple statements in each compound statement.

   (a) An integer is even if and only if it is divisible by 2. (b)John will go to church and Paul will not play basketball.

3. Write the negation of each statement.

   (a) Juan is taller than Hazel.
   (b)The color of the gate is blue.

4. Write each sentence in symbolic form. Represent each simple statement in the sentence with the letter indicated in the parentheses.

   (a) If an integer is divisible by 8 (p) then it is divisible by 4 (q).
   (b)Neither John eats breakfast (p) nor he goes to school (q).

5. Write each symbolic statement as an English sentence. Use p, q, r, s, and t as defined below.

   (a) $\sim p \rightarrow (p \vee r)$
   (b)$(r \wedge p) \leftrightarrow q$

p : Lea Salonga is a singer.

q : Lea Salonga not a songwriter.

r : Lea Salonga is an actress.

s : Lea Salonga plays the piano.

t : Lea Salonga does not play the guitar.

p : Paul is a table tennis player. q
: Paul is a basketball player. r
: Paul is a rock star.
s : Paul is a mechanical engineer.

(c) $t \leftrightarrow (\sim r \wedge \sim p)$

6. Write each sentence in symbolic form. Use p, q, $r$and s as defined below.
   (a)If it is not true that Paul is a table tennis player or a rock star, then Paul is a mechanical engineer.
   (b) Paul is a basketball player, if and only if he is not a football player and he is not a rock star.
   (c) If Paul is a mechanical engineer, then he is a basketball player and he is not a table tennis player.

7. Write the negation of each quantified statement. Start each negation with "some," "no," or "All." [*Source: Mathematical Excursions by Hoffmann, et al.*]
   (a) Some dogs are not friendly.
   (b)Everybody enjoyed the dinner.
   (c)No mamals are numbers are odd bird.

8. Write each statement in "If p, then q" statement.
   (a) The university will conduct face-to-face classes provided that there is already vaccines for COVID-19.
   (b) Every parallelogram is a quadrilateral.
   (c) In an equilateral triangle, all angles are equal.

9. Determine whether the given pair of statements are equivalent.

(a) If $|x| > 1$, then $x > 1$ or $x < -1$.

If $-1 \leq x \leq 1$, then $|x| \leq 1$.

(b) If $2x + 5 = 15$, then $x = 5$ If
$2x + 5 \neq 15$, then $x \neq 5$

10. Give an example of true conditional statement whose

(a) converse is true.

(b)converse is false.

(c)inverse is true.

11. Puzzle. This puzzle was taken from the book written by Aufmann [1].

This puzzle was written by Lewis Carroll.

The Dodo says that the Hatter tells lies.

The Hatter says that the March Hare tells lies.

The March Hare says that both the Dodo and the Hatter tell lies. Who is telling the truth?

Hint: Consider the three different cases in which only one of the charac- ters is telling the truth. In only one of these cases can all three of the statements be true.

12. Write truth tables for the following statements

(a) $\sim (p \wedge q) \vee (p \wedge q)$

(b) $p \wedge (\sim q \vee r)$

(c) $(p \vee q) \vee (\sim p \wedge q) \rightarrow q$

(d) $[(p \wedge \sim q) \vee (\sim r \wedge p)] \rightarrow (r \vee \sim q)$

(e) $\sim (p \vee r) \leftrightarrow q$

13. Determine the truth value of the statement given that p is a true (T) statement, q is a false (F) statement, and r is a true (T) statement.

(a) $(p \wedge q) \vee \sim r$

(b) $[(p \wedge \sim q) \vee \sim r] \wedge (p \wedge r)$

(c) $(p \wedge \sim q) \rightarrow r$

(d) $(p \vee r) \rightarrow (q \wedge \sim r)$

(e) $(p \wedge \sim p) \leftrightarrow (p \rightarrow q)$

14. Determine whether the statements are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer.

    (a) $\sim (p wedge\ q)$ and $\sim p wedge \sim q$

    (b) $(p \vee q) \wedge r$ and $(p \vee r) \wedge (q \wedge r)$

    (c) $p \rightarrow \sim r$ and $r \vee \sim p$

    (d) $p \rightarrow q$ and $q \rightarrow p$

    (e) $p \rightarrow (q \vee r)$ and $(p \rightarrow q) \vee (p \rightarrow r)$

15. Use truth tables to determine which of the statement forms are tautologies and which are contradiction.

    (a) $(p \wedge \sim q) \wedge (\sim p \vee q)$

    (b) $(\sim p \vee q) \vee (p \wedge \sim q)$

    (c) $(p \wedge q) \quad \vee \ (\sim p \vee \quad \sim q)$

16. Write each argument in symbolic form, using the letters p, q or r.

    (a) If the demand for face masks increase, the manufacturer produces more face masks. The demand for face masks does not increase. Therefore, the manufacturer does not produce more face masks.

    (b) If it rains, the soil is wet. It does not rain. Therefore the soil is not wet.

17. Use a truth table to determine whether the argument is valid or invalid.

$$\begin{array}{r} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

MODULE SEVEN

Mathematics of Graphs

CORE IDEA

Module Seven (7), mathematics of graphs focuses on the application of graph theory to some real-world problems. This includes

Learning Outcomes

1. Define and construct graphs, model a certain real-life situations using graph, identify eulerian graphs and use these to Postman Tour problems.
2. Define and construct weighted graphs, use greedy and edge-picking algorithm in determining the shortest/economical route of a particular travel.
3. Define graph coloring, determine the chromatic number of a graph and use these concepts in scheduling problems.

Time Allotment: Ten (10) lecture hours

Introduction

Graph theory is an area of mathematics that focuses in the study of struc- tured graphs used to model pair wise relations between objects. The subject of graph theory had its beginnings in recreational math problems, but it has grown into significant area of mathematical research with applications in various field.

The concept of graph theory was first studied by the famous mathemati- cian Leonard Euler in 1735 by giving the solution to the famous problem known as the *Seven Bridges of Konisberg*.

Graph theory has many applications in many areas. It can be used in network science study, engineering, social science, *etc.*

In this module we shall study the basic concepts of graph theory and use these concepts as a tool in solving problems related to Travelers, Assignment , and Scheduling problems.

For interactive learning on the topics to be discussed here, students are advised to visit this site: https://mathigon.org/course/graph-theory/introduction

5.7 Graphs and Eulerian Circuit

Graph and Graph Models

Definition 5.23. A **graph** G is composed of two finite sets: a nonempty set V of vertices and a set E of edges, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**.

Example 5.28. Let G a graph with $V = \{a, b, c, d, \}$ and $E = \{[a, b], [b, d],$ $[a, d], [b, c], [c, d]\}$. This graph consists of 4 vertices and 5 edges.

The sets V and E are called the **vertex set** and **edge set** of G, re- spectively. Vertices are sometimes called **points** or **nodes** and edges are sometimes called **lines**.

Although G is the common symbol to use for a graph, other symbols such as F, H, G_1, G_2, *etc.* can be used. Sometimes, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex set and edge set of a particular graph G. Vertices are sometimes called **points** or **nodes** and edges are sometimes called **lines**.

If x, y are vertices of G, we adopt the notation $[x, y]$ to denote an edge of a graph with endpoints x and y. If $[x, y]$ is an edge of a graph, we say that x is **adjacent** to y or y is adjacent to x. An edge $[x, y]$ is said to be **incident** on the vertices x and vertex y.

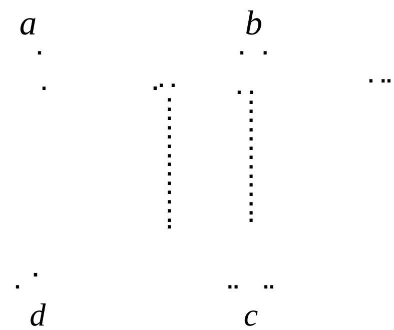
The number of elements of V is called the **order** of a graph and the number of elements of E s called the **size** of a graph.

Pictorial Representation of a Graph

A graph can be represented by a diagram in the plane where vertices are usually represented by small circles (open or solid) and whose edges by line segment or curve between the two points in the plane corresponding to the appropriate vertices.

A graph and its pictorial representation is given next.

Example 5.29. Let G be a graph where $V(G) = \{a, b, c, d, \}$ and $E(G) = \{[a, b], [b, d], [a, d], [b, c], [c, d]\}$.

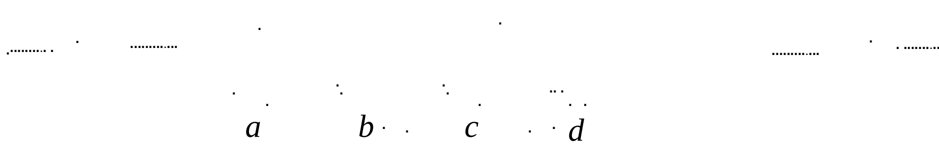


G

Figure 5.13: Pictorial representation of the graph G

The graph G in Figure 5.13 has order 4 and its size is 5. Since $[b, c]$ is an edge of the graph G, we say that a is is adjacent to b.

The pictorial representation of a graph is not unique. The graph in Figure 5.13 can be drawn in another way, as shown in Figure 5.14.



G

Figure 5.14: Another pictorial representation of the graph G

Remark 11. Important reminders in dealing in dealing with the pictorial representation graphs are the following:

1. The placement of the vertices has nothing to do with location, and length of the edges are irrelevant.

2. The important information is which vertices are connected by edges.

An edge of the form $[x, x]$ is called a **loop**. Two edges with the same end points are said to be **parallel edges**. Two edges that are incidence on the same endpoint are called **adjacent edges**. A graph without loops and multiple edges is called **simple graph**.

In Figure 5.15, the graph G_1 is simple but the graphs G_2 and G_3 are not simple since G_2 contains a loop and G_3 contains multiple edges.

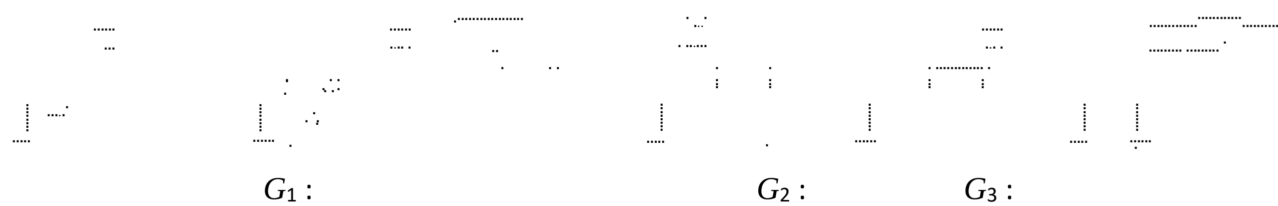


Figure 5.15: Graphs with a loop and multiple edges

Equal and Equivalent Graphs

Two graphs are equal if their vertex set and edge set are equal. For instance, the two graphs in Figure 5.16 are equal graphs since the edge sets and vertex sets of these two graphs are equal. Actually, these are the two pictorial representations of the graph in the previous example.

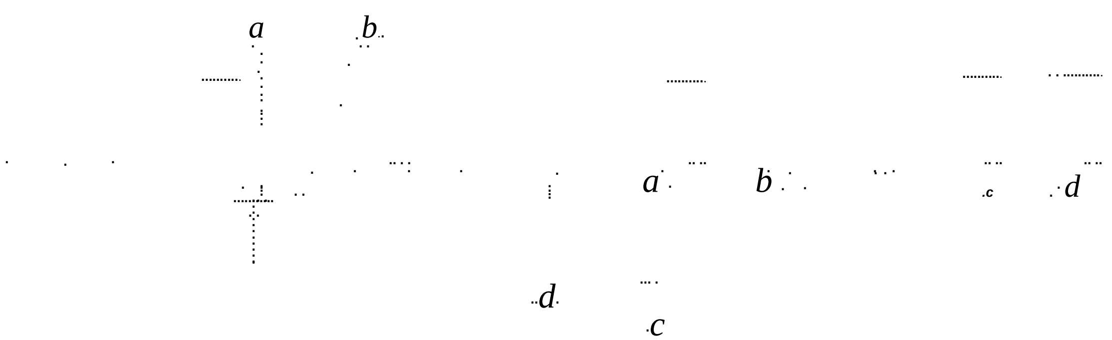


Figure 5.16: Illustrating equal graphs

The graphs in Figure 5.17 are not equal because neither vertex sets nor edge sets are equal. These graphs are called **equivalent** (or isomorphic) graphs.

It can be noted that equal graphs are equivalent but not all equivalent graphs are equal graphs.

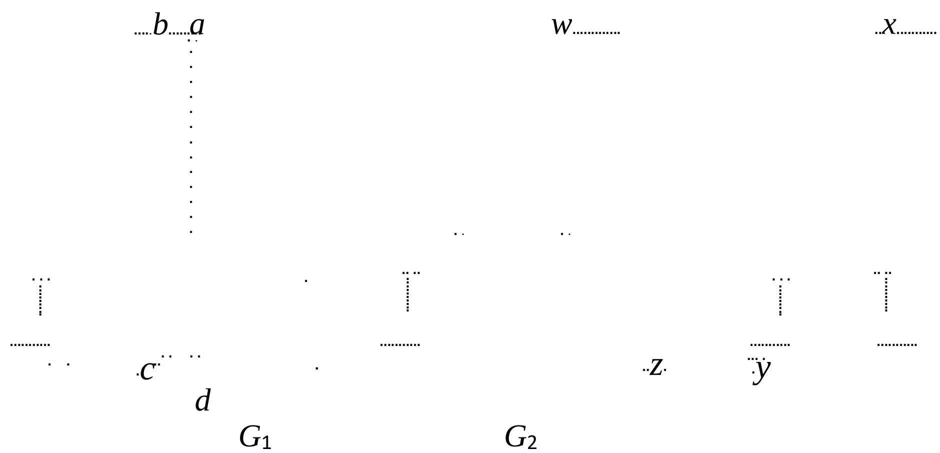


Figure 5.17: Illustrating equivalent graphs

Remark 12. Two graphs are equivalent if one graph can be re-drawn to form the other graph.

Example 5.30. The graphs below are equivalent graphs since H can be re-drawn to form graph G.

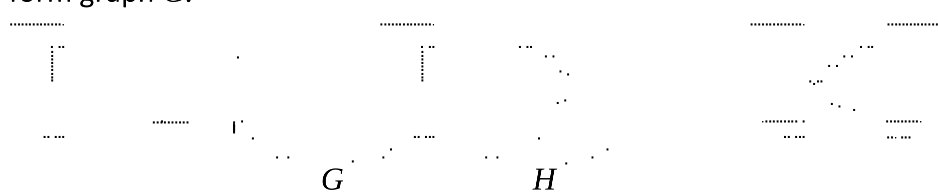


Figure 5.18: Illustrating euivalent graphs

Graph Models

Graph model is a graph representing a certain situations. There are many real-life situations that can be modeled by graphs. Consider the example below.

Example 5.31. The following table lists five students at a college. An “x“ indicates that the two students belong in the same student organization this semester. Draw a graph that represents this information where each vertex represents a student and an edge connects two vertices if the corresponding students belong to the student organization.

Solution. Since the veritces are the students, the graph consists of five vertices. Label each vertex by the following:John, Paul, Hazel, Daniel, and Camille. Start with vertex John, this vertex is adjacent to the vertices corre- sponding to Paul and Daniel, respectively. So, there are edges joining John

	John	Paul	Hazel	Daniel	Camille
John	–	X		X	
Paul	X	–	X	X	
Hazel		X	–		X
Daniel	X	X		–	
Camille			X		–

and Paul, and John and daniel. Next, at Paul, observe that he shares the same organizations with John, Hazel and Daniel. Since there is already an edge joining Paul and John, only the edges joining Paul and Hazel, and Paul and Daniel. By applying the same argument for Hazel and Camille, we have the graph in Figure 5.19.

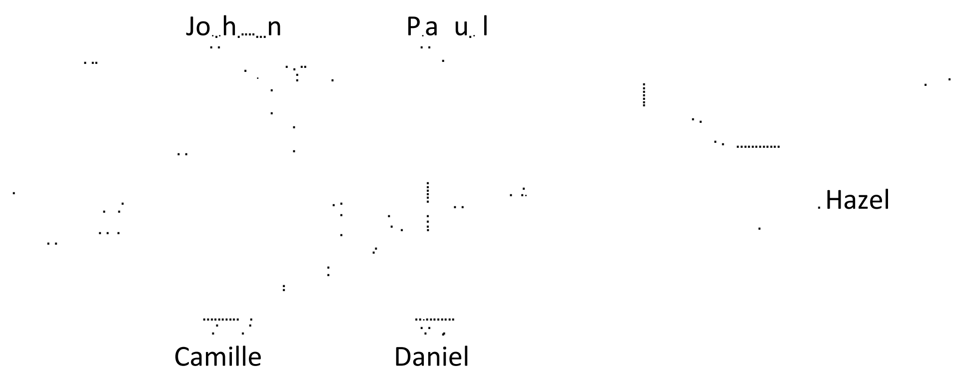


Figure 5.19: Graph model of 5 students

Example 5.32. A floor plan of a museum is shown in Figure 5.7. Draw a graph that represent this information using the rooms as vertices and two two rooms are joined by an edge if there is a doorway between them.

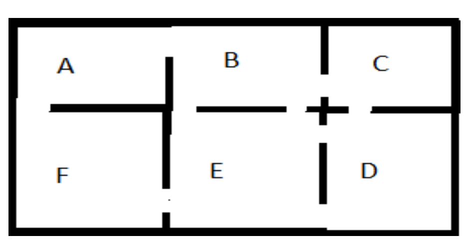


Figure 5.20: A Floor plan

Solution. The vertices of the graph are the rooms , $\{A, B, C, D, E\}$, & F , and two rooms are adjacent if there is a doorway between them. The graph model is shown in Figure 5.21.

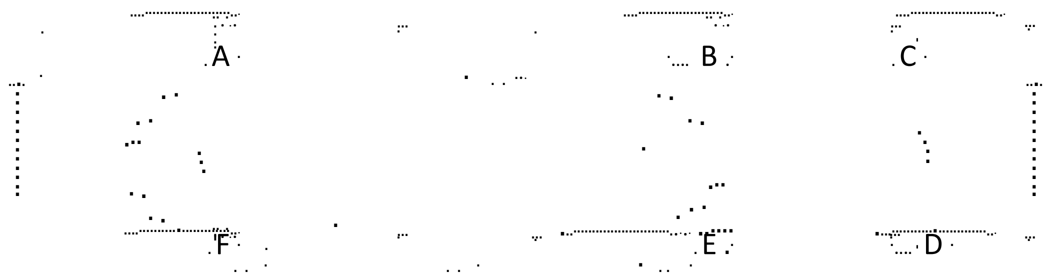


Figure 5.21: Graph model of the floor plan

The graph in Figure 5.19 contains parallel edges since there are two doors between Rooms A and B, and there are also two doors connecting Rooms D and E.

Learning Activity

A. Let G be graph with $V(G) = \{1, 2, 3, 4, 5\}$ and

$E(G) = \{[1, 2], [3, 4], [1, 4], [2, 5], [3, 5], [1, 3], [1, 5]\}$.

1. Make a pictorial representation of G.

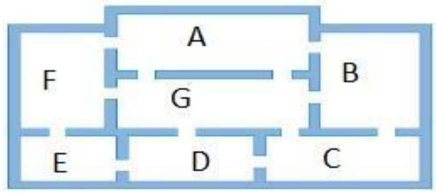
2. Determine the order and the size of G.

3. List the vertices of G that are adjacent to the vertex 1. 4.What are the edges that is (are) adjacent to the edge [2 , 5]

B. Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that two elements i, j of S are adjacent if $i + j$ or $i\,j$ is an element of S.

C. Draw a graph that represents the floor plan, where each vertex represents a room and an edge connects two vertices if there is a doorway between the two rooms.

D. In your class section, divide the class into Five (5) groups, in any group- ings you want. In each group, construct a graph for each of the following



conditions: the vertices are the group members and two members, say A and B, are adjacent if:

1. A and B are friends in Facebook.
2. A and B belong to the same strand (i.e. STEM, ABM, etc.) during high school
3. A and B share a common friend.

Subgraphs and Degree of a vertex

Subgraph

A **subgraph** can be obtained by removing vertices or edges from a graph. For example, the graphs H_1, H_2 and H_3 are subgraphs of G.

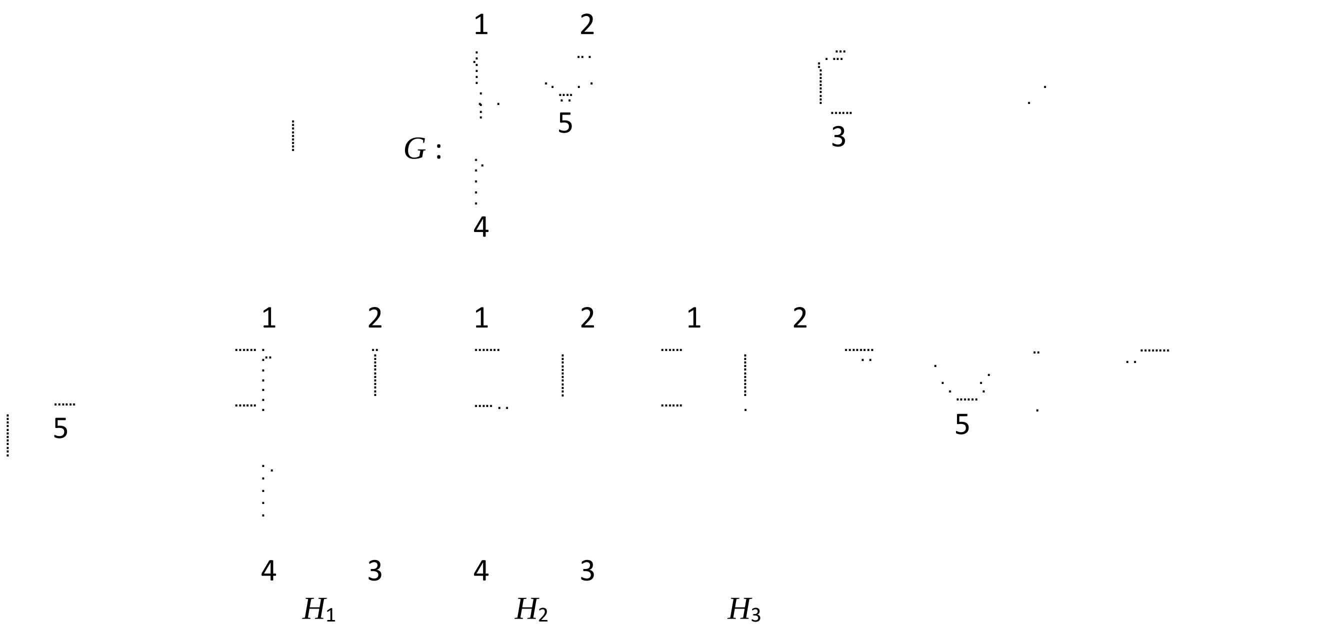
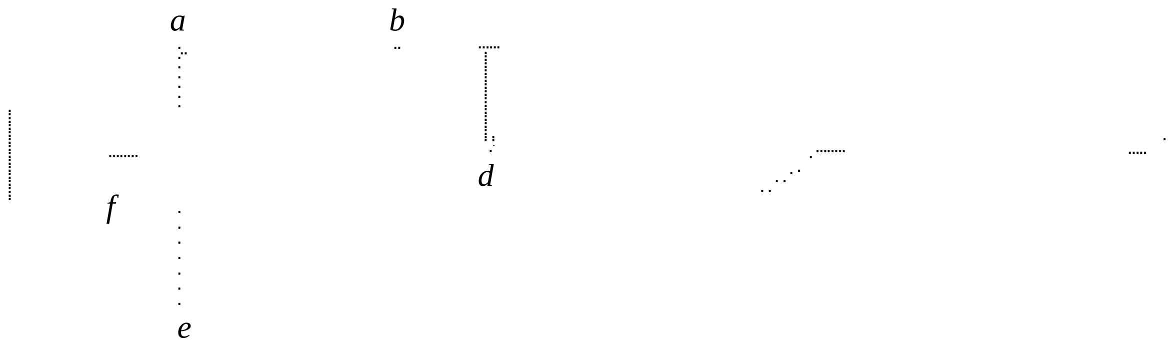


Figure 5.22: A graph and its subgraphs

Degree of a vertex

The **degree** of a vertex *x* in a graph *G* is the number of edges incident with *x*, denoted by deg(*x*).

Consider the graph below, deg(*a*) = 2, deg(*d*) = 3, deg(*c*) = 1.



A vertex with degree 0 is called **isolated vertex**. In the above graph, the vertex *f* is an isolated vertex.

Walks and Connected graphs

Definition 5.24. A **walk** in a graph is a sequence of vertices such that consecutive vertices in the sequence are adjacent. The number of edges in the walk is called **length** of the graph. A walk is a **closed walk** f the first vertex and the last vertex in the sequence are the same.

A walk in a graph is usually represented by $W: v_1, v_2, \ldots, v_k$, where $v_1, v_2, \ldots, v_k$ are vertices of a graph G and $[v_1, v_2], [v_2, v_3], \ldots, [v_{k-1}, v_k]$ are edges of G.

Example 5.33. Refer to the graph Figure 5.23, the following sequences are walks of different lengths in the graph *G*, , except (d.).

a. The sequence W_1 : 1, 2, 3, 4 is a walk of length 3.

b. The sequence W_2 : 1, 5, 3, 5, 2, 1 is a walk of length 5. This is a closed walk.

c. The sequence W_3 : 5 is a walk of length 0. This is called the trivial walk.

d. The sequence W_4 : 1, 3, 2 is not a walk since 1 and 3 are consecutive vertices in W_4 and [1, 3] is not an edge of the graph.

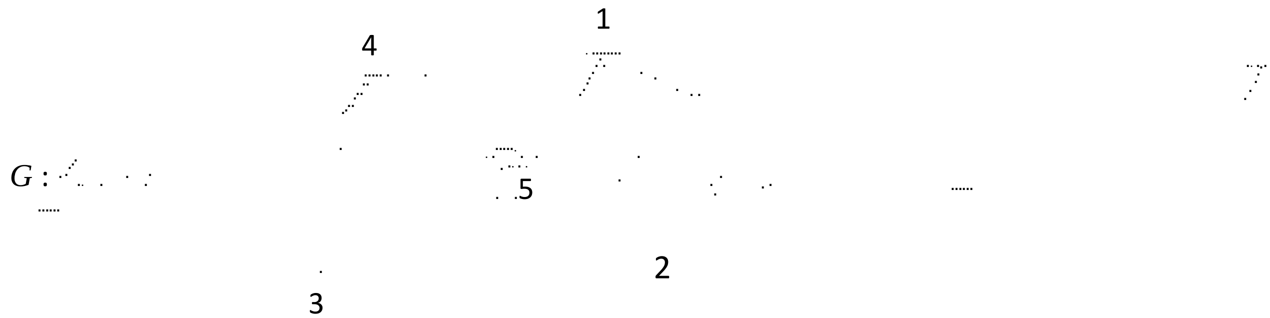


Figure 5.23: Illustrating walks in a graph

Path and Trail in a graph

A walk W is called a **path** if it contains distinct vertices. A walk W is called a **trail** if no edge is repeated.

For example, consider the graph in Figure 5.24. The walk $T : c, d, f, d$ is a trail since no edge is repeated. The walk $P : b, d, e, f$ is a path since no vertex is repeated.

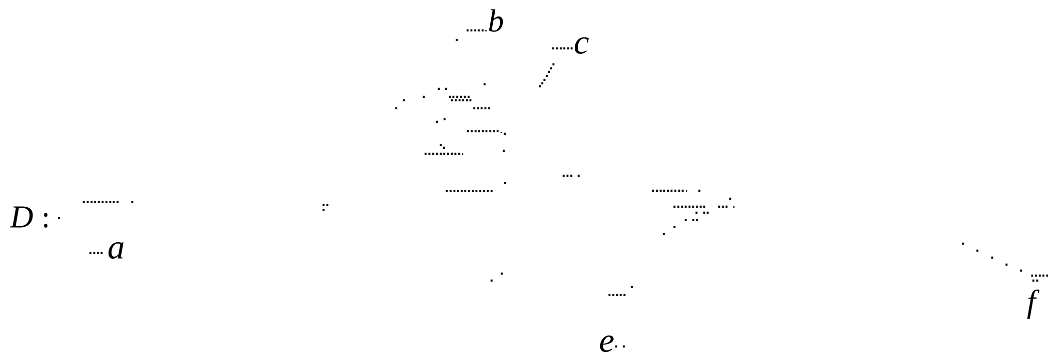


Figure 5.24: Illustrating Path and Trails

Remark 13. All paths are trails but not all trails are paths.

Circuits and Cycles in a graph

A closed walk such that no edge is repeated is called **circuit**. A closed walk that repeats no vertex is called **cycle** .

Example 5.34. 1.The closed walk $C : c, d, f, e, d, b, c$ is a circuit since no edge is repeated.

2.The closed walk $D : d, e, f, d$ is a cycle since no vertex is repeated.

Connected graph

A graph G is said to be **connected** if for every pair of vertices in G, there is a path joining x and y. A graph which is not connected is said to be **disconnected**.

In Figure 5.25, the graph on the left is connected and the graph H on the right is disconnected.

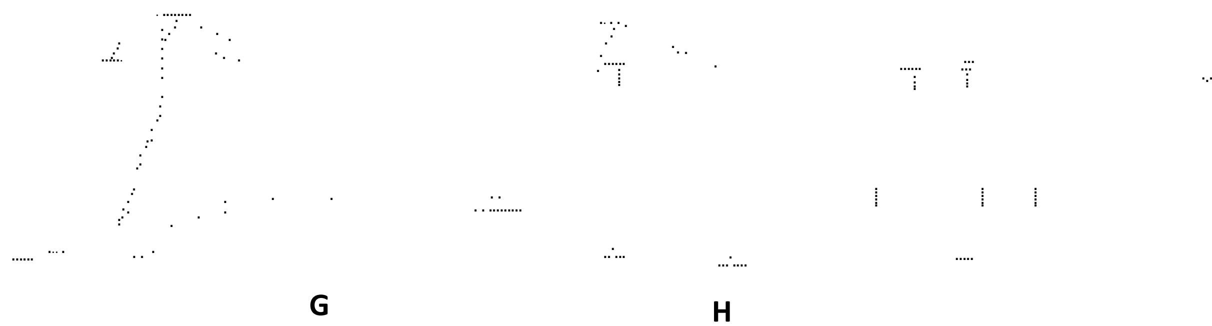
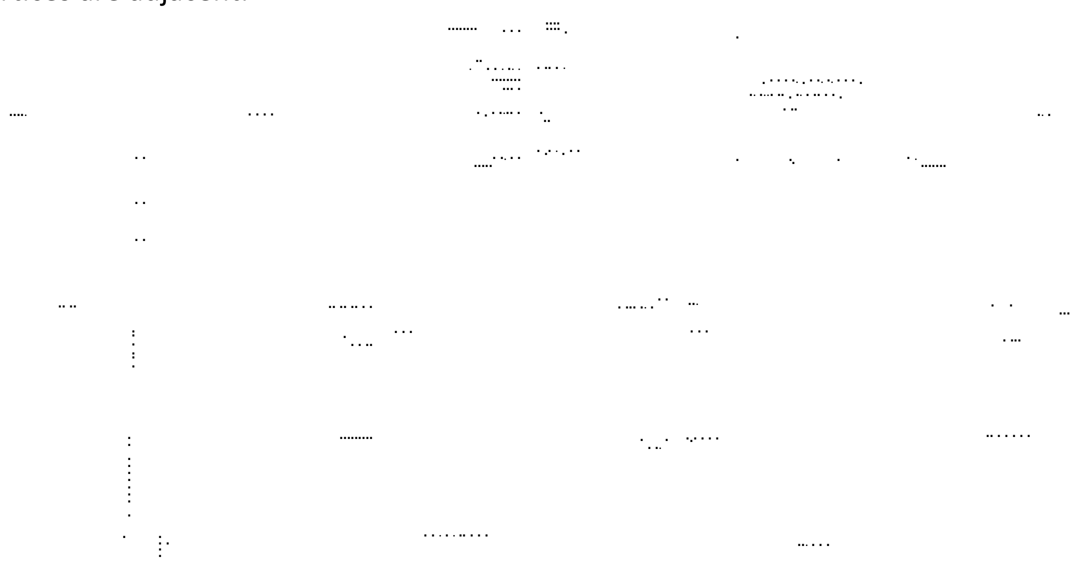


Figure 5.25: Connected and disconnected graphs

Complete graph and Handshake Problem

A **complete graph** of order n, denoted by K_n, is a graph where every pair of vertices are adjacent.



Connected graph

Connected graph

K_3 K_5 K_8

Remark 14. The size of a complete graph of order n is $\frac{n(n-1)}{2}$.

The property of a graph can be used to solve problems like the famous Handshake problem.

Connected graph

Example 5.35 (Handshake Problem)**.** If 10 people meet each other and each shakes hands only once with each of the others, how many handshakes will there be?

Solution. We can represent the handshake problem by a graph. The vertices are the people and edges are the handshakes. Thus the number of edges of the graph represents the number handshakes. There are 10 people, so the graph consists of 10 vertices. Since each person shake hands only once with each of the other, each person will shake hands once to each of the other 9 persons. This means that each vertex of the graph is adjacent to the other 9 vertices, as shown in Figure 5.26. Observe that the graph formed is a complete graph.

By Remark 14, the $= \frac{10(10-1)}{2}$ = 45. Thus,

number of edges is $n(n$

$- 1)$

2

there will be 45 handshakes.

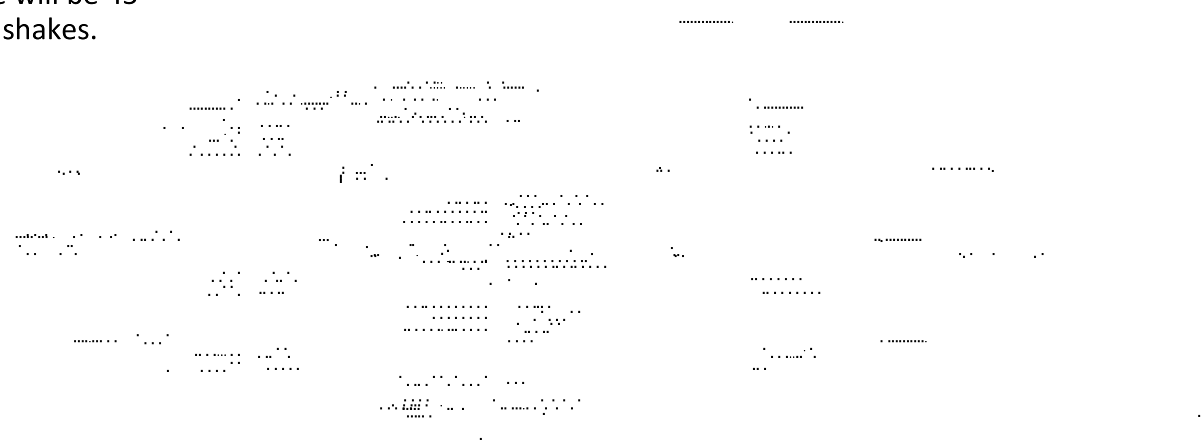


Figure 5.26: Illustrating the number of handshakes of 10 people

Learning Activity

A. Use graph theory to solve the following problems. Identify the vertex set and determine when the two vertices are adjacent.

1. There are Eight (8) couples at a party. Suppose each one will will shake hands with every other person once, except for each couple, they will

not shake hands with each other. How many shake hands are there?

2. There are 11 teams in a basketball tournament. How many games must be played so that each team will play against every other team exactly once?

Example 5.35 (Handshake Problem)**.** If 10 people meet each other and each

Eulerian Circuit and Eulerian Graph

Definition 5.25. A circuit that uses every edge of the graph (but never uses the same edge twice) is called **euler circuit**.

Example 5.36. Find an eulerian circuit in the graph below

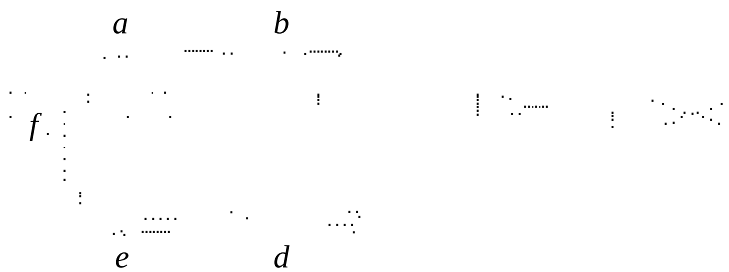


Figure 5.27: Illustrating eulerian circuit

Solution. The walk $W : a, f, e, b, d, e, a, d, c, b, a$ is a circuit since no edge is repeated. Since every edge in the graph is also an edge in the circuit, the circuit is Eulerian circuit.

Definition 5.26. A connected graph that contains an eulerian circuit is called an *eulerian graph*.

Example 5.37. The graph in Figure 5.27 is an eulerian graph, since there is an eulerian circuit, as shown in the solution of Example 5.36.

The graph below is NOT an eulerian graph since it has no eulerian circuit.

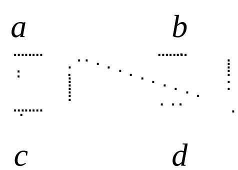


Figure 5.28: Illustrating not eulerian graph

The following theorem gives the characterization of an eulerian graph.

Eulerian Graph Theorem (Euler, 1736)
A connected graph is eulerian if and only if every vertex of the graph is of even degree.

Equivalently, the theorem above is equivalent to the following remark.

Remark 15. If the graph contains a vertex whose degree is odd, then the graph is not eulerian.

The graph in Figure 5.28 is not eulerian because deg(a) = deg(d) = 3 is odd.

Example 5.38. Which of the following graphs has an Euler circuit?

G H

Figure 5.29: The graphs G and H

Solution.

⇒ The degree of each vertex of the graph H is even, by Eulerian graph theorem, it contains an eulerian circuit.

⇒ The graph G contains a vertex of odd degree . By Eulerian graph theorem, this graph is not eulerian. Hence it has no eulerian circuit.

Application of Eulerian graph

The concept of eulerian graphs can be applied to real-life situations. Below are some examples.

Example 5.39. The map below shows a portion of the major roads in Batan- gas City. Suppose a delivery boy needs to travel the full length of each major road. Is it possible to plan a journey that traverses the roads and returns to the starting point without traveling through any portion of a road more than once? (*Note. full length means from one intersection to another intersection*)

Solution.

⇒ We can model the road map by graph, with a vertex at each intersec- tion. An edge represents a road that runs between two intersections.

Figure 5.30: Map of some major roads in Batangas City

- ⇒ Note that the vertex representing the intersection of the National Road, Alangilan National, and P. Herera St., has degree 3, which is odd.
- ⇒ The graph cannot be Eulerian, and it is impossible for the inspector not to travel at least one road twice.

Some graphs has no eulerian circuits, but there is possibility of obtaining a trail having all the edges of the graph.

Definition 5.27. A trail that uses every edge of the graph (but never uses the same edge twice) is called an **Euler trail**.

Example 5.40. Find an Euler trail in the graph below.

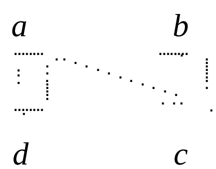


Figure 5.31: Illustrating Euler trail

Solution. Obeserve that the graph is not eulerian graph since it contains vertices with odd degree (vertices a and c). However, the walk W

$$W : a, d, c, a, b, c$$

is a trail since no edge is repeated. Moreover, every edge in the graph contains in the trail W. Therefore, W is an Eulerian trail.

The theorem below gives the conditions for a graph to have an Eulerian trail. In addition, it also gives a method to determine an eulerian trail in a graph.

To illustrate the theorem, consider the graph in Figure 5.31. The graph has

Eulerian Trail Theorem

A connected graph contains an Euler trail if and only if the graph has exactly two vertices of odd degree with all other vertices of even degree.

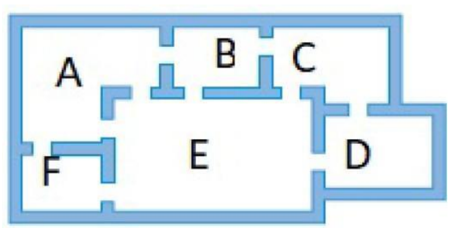
Every Euler trail must start at one of the vertices of odd degree and end at the other.

exactly two vertices with odd degree, vertices a and c, so that it contains an eulerian trail W. Observe further that the trail W starts at vertex a and ends at vertex c.

Application of Euler trail

The next example is an application of euler trail, this example is found in [1].

Example 5.41. The floor plan of an art gallery is pictured below. The rooms are labeled by $A, B, C, D, E, \&F$. Is it possible to take a stroll that passes through every doorway without going through the same doorway twice?



Solution. First, draw a graph that represents the floor plan, where vertices correspond to rooms and edges correspond to doorways, as shown below.

Next, we determine the degree of each vertex of the graph, as shown in Table 5.25.

Observe that there are exactly two vertices with odd degree, vertex B and vertex C. All other vertices have even degree. By Euler trail theorem,

Vertex	Degree
A	4
B	3
C	3
D	2
E	6
F	2

Table 5.25: Illustarating the degree of each vertex

the graph has an Euler trail. In determining the Euler trail, start at one vertex with odd degree and end at the other vertex with odd degree. An Euler trail that starts at B and ends at C is given below.

$$T : B, A, F, E, A, E, D, C, E, B, C.$$

Weighted Graphs and Hamiltonian Cycle

This section discusses the hamiltonian cycle in a graph, weighted graphs and some applications of these concepts to traveller-sales man problem.

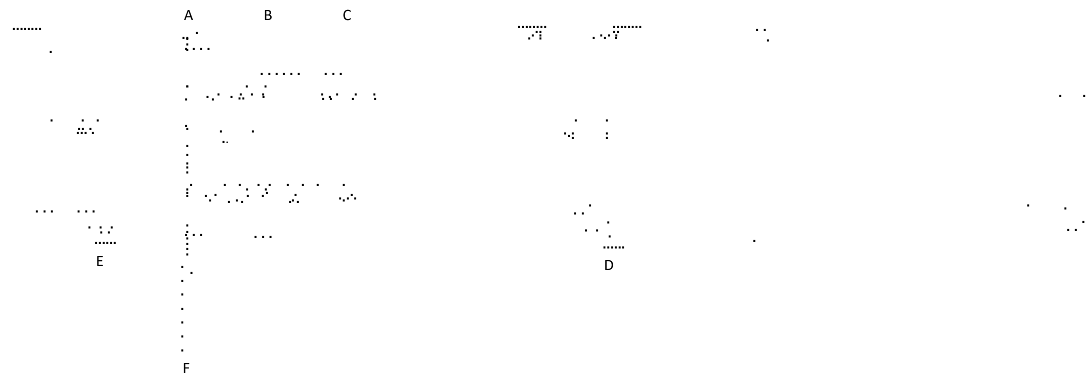
Definition 5.28. A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle**.

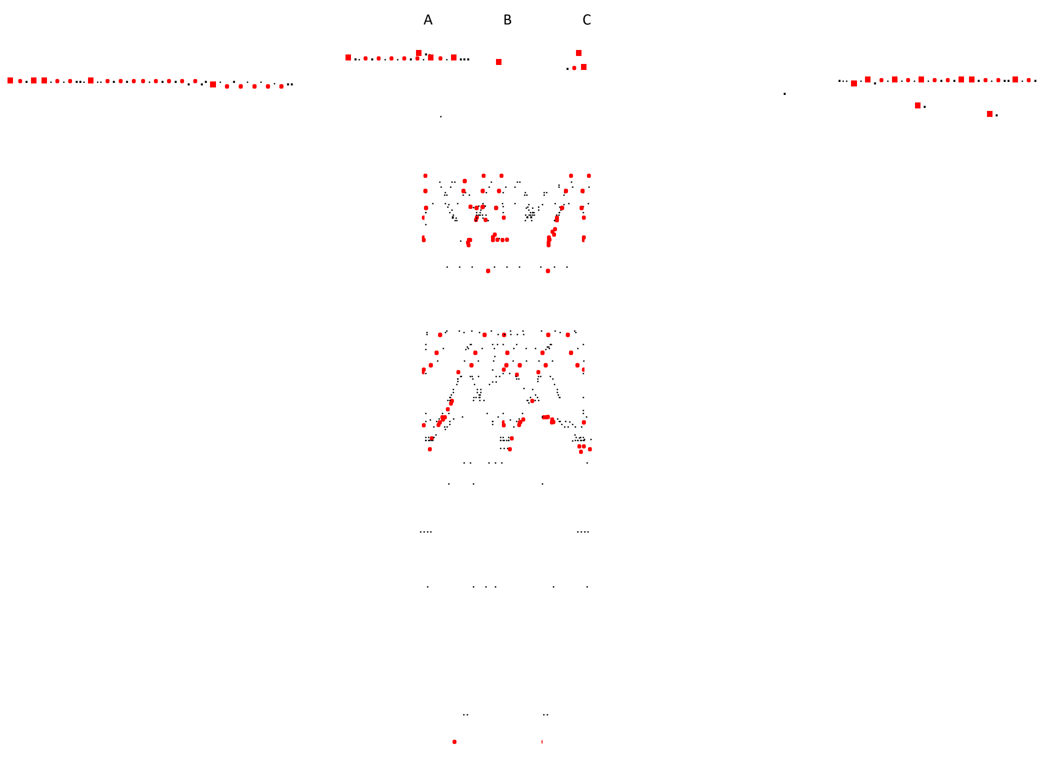
Example 5.42. Find a Hamiltonian cycle in the graph below.

Solution The cycle A, F, B, E, C, D, A is a Hamiltonian cycle since it con- tains every vertex of the graph. In Figure 5.32, the color red forms the hamiltonian cycle.

Definition 5.29. A **Hamiltonian graph** is a graph that contains a hamil- tonian cycle.

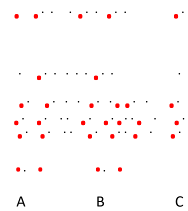
A B C


A
B
C
E
D
F


A
B
C

A B C

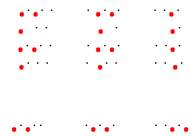
A B C


A
B
C

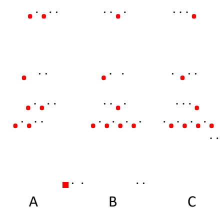
A B C

. . .

A B C



A B C


A
B
C

E F D

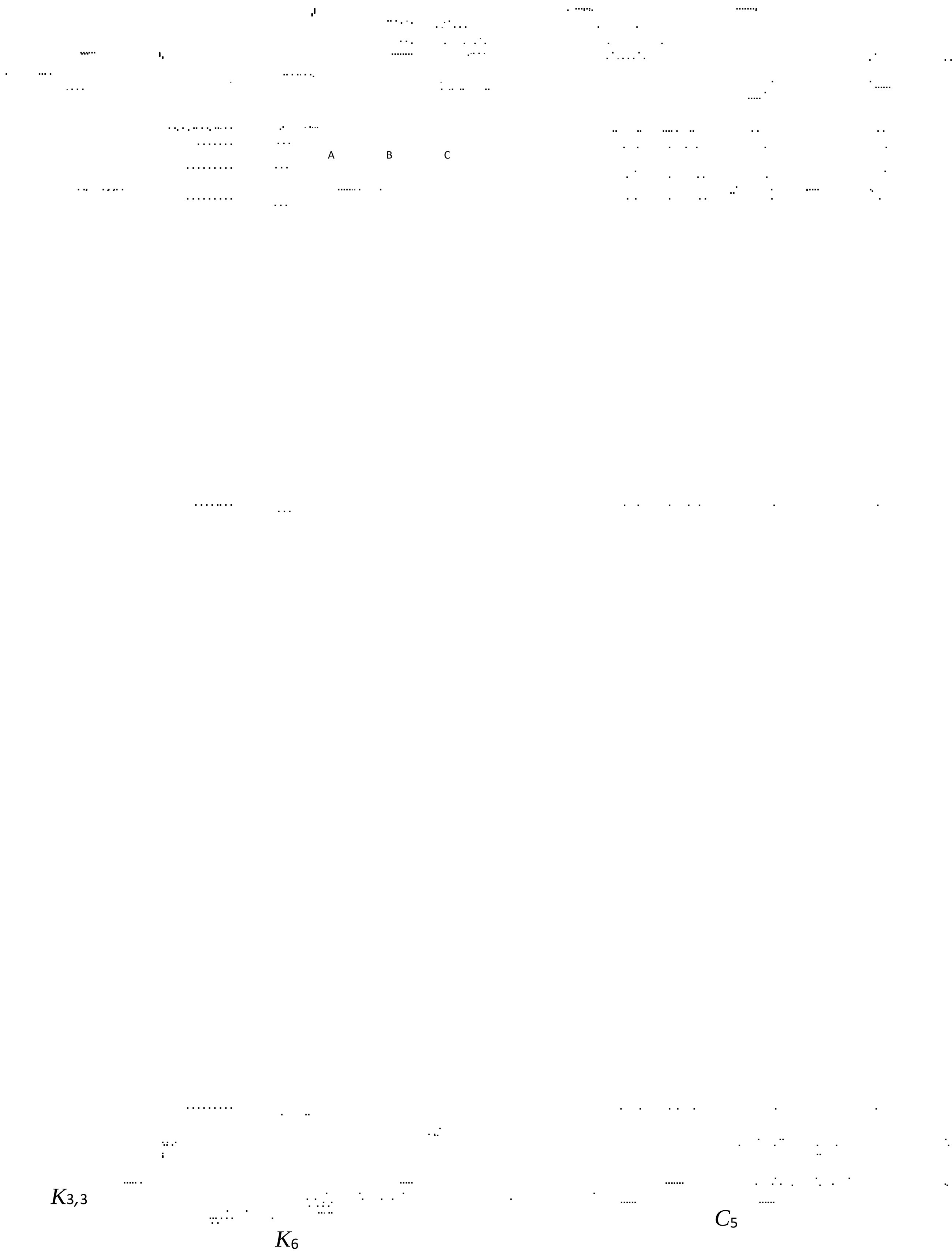
Figure 5.32: Illustrating hamiltonian cycle

A B C

Example 5.43. The three graphs below are Hamiltonian graphs

. .

. A B C



To date, there is no straightforward criterion to guarantee that a graph is Hamiltonian, but the next theorem is helpful.

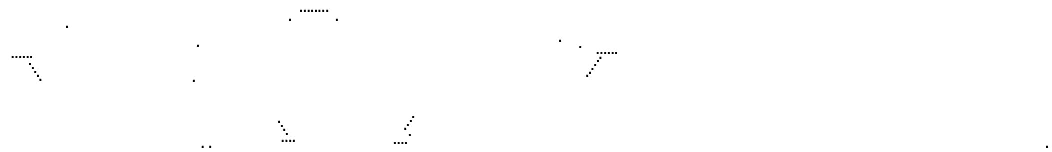
Dirac's Theorem

Consider a connected graph with at least three vertices and no multiple edges. Let n be the number of vertices in the graph. If every vertex has degree of at least $n/2$, then the graph is a Hamiltonian.

The converse of Dirac's theorem is not true. That is, there are some Hamiltonian graphs whose degree of each vertex is less than $n/2$. An example is the graph in Figure 5.33, where the degree of each vertex is 2, which is less that $5/2$.

Application of Dirac's Theorem

A B C

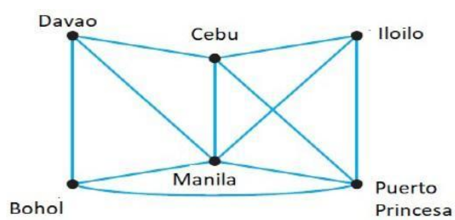


C_5

Figure 5.33: A hamiltonian graph

Example 5.44. The graph below shows the available flights of a Cebu-PAL airlines. An edge between two vertices in the graph means that the airline has direct flights between the two corresponding cities.

1. Apply Dirac's theorem to verify that the following graph is Hamilto- nian.
2. Find a Hamiltonian cycle.
3. What does the Hamiltonian cycle represent in terms of flights?

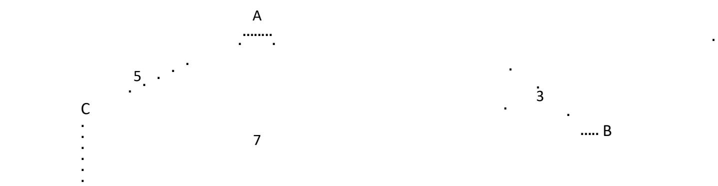


Solution.

1. There are six vertices in the graph, so $n = 6$, and every vertex has a degree of at least $n/2 = 3$. So, by Dirac's theorem, the graph is Hamiltonian.
2. One Hamiltonian cycle is **Manila-Bohol-Puerto Princesa-Iloilo- Cebu-Davao-Manila**
3. This Hamiltonian cycle represents a sequence of flights that visits each city and returns to the starting city without visiting any city twice.

Definition 5.30 (Weighted Graph)**.** A **weighted graph** is a connected graph in which each edge is assigned a number (value), called the **weight**.

Example 5.45. The graph below is an example of a weighted graph.



K_3

Weight of a Graph If G is a graph, the **weight** of G, denoted by w(G), is the sum of weights of its edges.

In the graph below, the weight of K_3 is

$$w(K_3) = 3 + 5 + 7 = 15.$$

K_3

Application of Weighted Graph

Example 5.46. The table below lists the actual flight distances in kilometers between six cities. that the airline flies to. Suppose a traveler would like to start in Manila, visit the other five cities this airline flies to, and return to Manila. Find three different routes that the traveler could follow, and find the total distance flown for each route.

Solution

⇒ Draw a graph where vertices are the cities and two cities are adjacent if there is a flight between the corresponding cities.

	Bohol	Cebu	Davao	Iloilo	Manila	Puerto Princesa
Bohol	–	no flights	465	no flights	429	560
Cebu	no flights	–	253	151	560	580
Davao	465	253	–	no flights	950	no flights
Iloilo	no flights	151		–	465	427
Manila	429	560	950	465	–	580
Puerto Princesa	560	580	no flights	427	580	–

Table 5.26: Flights distances between cities

⇒ Label each edge with a weight that represents the number of miles between the two cities, as shown Figure 5.34

⇒ A route that visits each city just once corresponds to a Hamiltonian cycle.

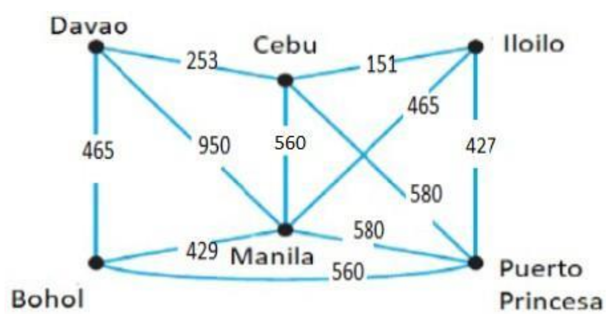


Figure 5.34: Weighted graph of flight distances between cities

Solution

The 3 routes and there corresponding weight are the following:

1. H_1 : Manila, Davao, Cebu, Iloilo, Puerto Princesa, Bohol, Manila

$w(H_1)$ = 950 + 253 + 151 + 427 + 560 + 429 = **2,770**

2. H_2 : Manila, Puerto Princesa, Iloilo, Cebu, Davao, Bohol, Manila

$w(H_2)$ = 580 + 427 + 151 + 253 + 465 + 429 = **2,305**

3. H_3 : Manila, Iloilo, Puerto Princesa, Cebu, Davao, Bohol, Manila

$$w(H_3) = 465 + 427 + 580 + 253 + 465 + 429 = \mathbf{2{,}619}$$

It can be observed that each of the three routes forms a Hamiltonian cycle with different weights. The weight of each hamiltonian cycle represents the total distance traveled by the traveler. Although, the traveler visits each city for each route, the route H_2 gives the minimum weight. That means, among the three routes, for the traveler to get the minimum distance traveled, he will follow the route H_2.

In the above example, we cannot say that the route H_2 has the smallest weight, unless we take the weights of all hamiltonian routes from Manila going to each city and going back to Manila. But this method is very tedious.

The next topic will helps us the algorithm to find the hamiltonian cycle in a complete graph with minimum the weight.

Greedy and Edge Picking Algorithms

There are two algorithms that can be used to find the minimum weight of a Hamiltonian circuit in a complete graph: Greedy and Edge Picking algorithms.

Geedy Algorithm

Choose a vertex to start at, then travel along the connected edge that has the smallest weight. (If two or more edges have the same weight, pick any one.)

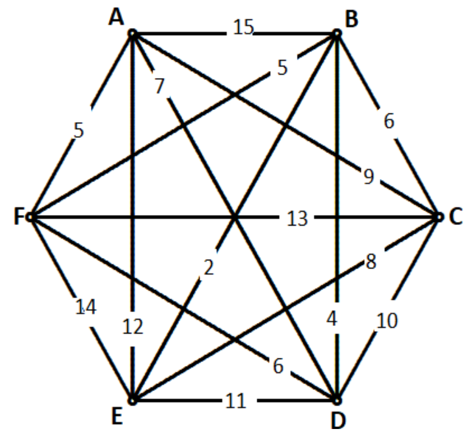
After arriving at the next vertex, travel along the edge of smallest weight that connects to a vertex not yet visited. Continue this process until you have visited all vertices.

Return to the starting vertex.

According to Aufmann [1], the greedy algorithm is so called because it has us choose the "cheapest" option at every chance we get.

Example 5.47. Use the greedy algorithm to find a Hamiltonian cycle in the weighted graph below starting at vertex A.

Solution.



- Begin at A. The weights of the edges incident on A are 5, 12, , 7, 9, and 15. The smallest is 5. Connect A to F. By applying Step 2, we need to travel to a vertex not yet visited through an edge with the smallest weight.

⇒ At F, the edge with the smallest weight is $[F, B]$. Connect F and B.

⇒ At B, the edge with the smallest weight is $[B, E]$. Connect B and E.

⇒ At E, the edge with the smallest weight is $[E, C]$. Connect E and C.

⇒ At C, the only vertex not yet visited is D. Connect C and D.

⇒ Finally, by Step 3, connect D and A, as shown in Figure 5.35

If we denote by H the obtained hamiltonian cycle, Then the hamiltonian cycle is $H : A, F, B, E, C, D, A$. The weight of the hamiltonian cycle is

$$w(H) = 5 + 5 + 2 + 8 + 10 + 7 = 37.$$

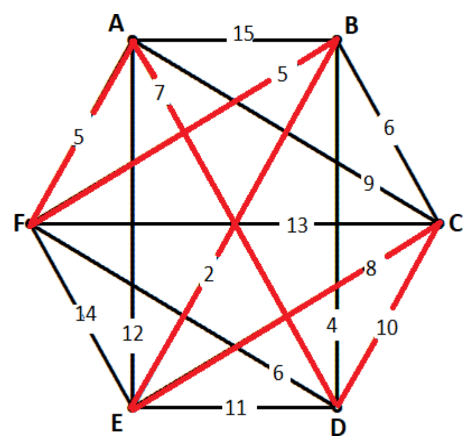


Figure 5.35: The obtained hamiltonian cycle

Edge-Picking Algorithm

Mark the edge of smallest weight in the graph. (If two or more edges have the same weight, pick any one.)

Mark the edge of next smallest weight in the graph, as long as it does not form a cycle and does not add a third edge to a single vertex.

Continue this process until you can no longer mark any edges. Then mark the final edge that completes the Hamiltonian cycle.

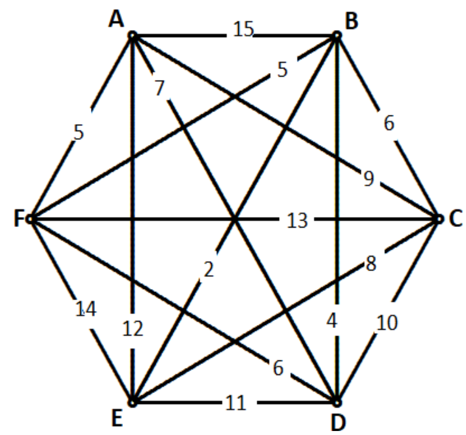
Example 5.48. Use the Edge-Picking algorithm to find a Hamiltonian cycle of the graph shown below.

Solution.

- First, highlight the edge of smallest weight, in this case edge $[B, E]$ with weight 2.

⇒ The edge with next smallest weight is $[B, D]$ with weight 4. Mark this edge.

⇒ There are two edges with weight 5, the next smallest weight, the edges $[B, F]$ and $[A, F]$. We cannot use the edge $[B, F]$ since it will add a third edge to B. So we highlight $[A, F]$



- ⇒ There are also two edges with weight 6, the next smallest weight, the edges $[B, C]$ and $[D, F]$. We cannot use both edges since each will add a third edge to a vertex. So we highlight the next edge with next smallest weight, the edge $[A, D]$, with weight 7.

- ⇒ The next smallest is $[A, D]$ with weight 7. we can again use this edge, so highlight this edge.

- ⇒ The next smallest is $[E, C]$ with weight 8. we can use this edge, so highlight this edge.

- ⇒ At this stage, any edge we mark will either complete a cycle or add a third edge to a vertex. So we highlight the final edge$[F, C]$. The highlighted edges which forms a hamiltonian cycle is shown below. The highlighted edges that forms a hamiltonian cycle is given in Figure 5.36.

Beginning at vertex A, the Hamiltonian cycle is A, D, B, E, C, F, A. The weight of the Hamiltonian cycle is

$$7 + 4 + 2 + 8 + 13 + 5 = 39.$$

Observe from the two examples that the two algorithms gave different Hamiltonian cycles on the same weighted graph. The following remark is useful.

Remark 16. 1.The hamiltonian cycles determined by the two algorithms do not guarantee that these are the most efficient route. There is no known efficient methods for finding the very best route.

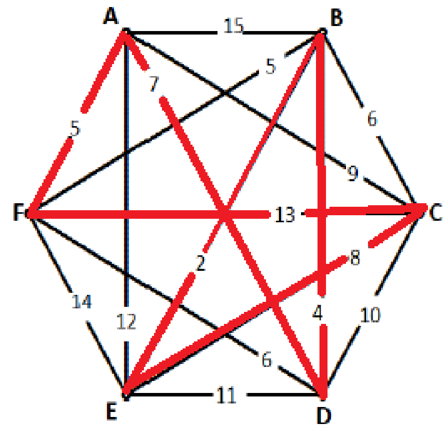
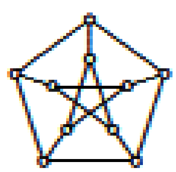


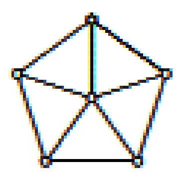
Figure 5.36: Illustrating Edge-Picking Algorithm

2.There is no evidence that Greedy Algorithm gives more efficient route than the Edge-Picking Algorithm and *vice-versa*.

Learning Activity

A. Use Dirac's theorem to verify that the graph is hamiltonian or not hamiltonian.

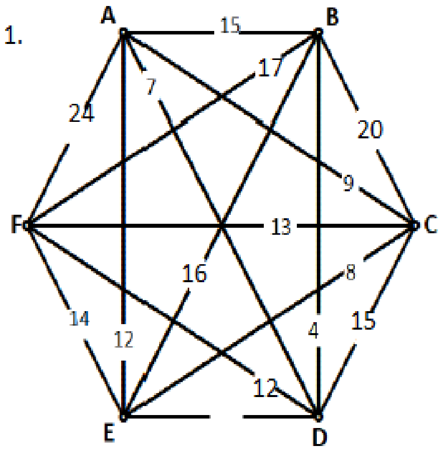
1. 

2. 

B. Use greedy algorithm and edge-picking algorithm to find Hamiltonian cycle starting at vertex A in the weighted graph.

C. Rea wants to tour Asia. She will to start and end the journey in Manila. She will visit the Japan, Hongkong, Thailand, Seoul, and Beijing. The airfares (in US-Dollar) available to her between cities are given in the table. Draw a weighted graph that represents the travel costs between cities and use the greedy and edge-picking algorithms to find a low-cost route.

1.



	Manila	Japan	Hongkong	Thailand	Seoul	Beijing
Manila	–	180	120	90	144	220
Japan	180	–	650	350	470	780
Hongkong	120	650	–	120	480	120
Thailand	90	350	120	–	420	290
Seoul	144	470	480	420	–	215
Beijing	220	780	280	290	215	–

5.8 Graph Coloring

If the map is divided into regions in some manner, what is the minimum number of colors required if the neighboring regions are to be colored differently? This question originated not with map-makers but with a mathematician. In 1852, Francis Guthrie, a recent graduate of the University of London, observed that the countries of England could be colored with four colors so that neighboring countries were colored differently. Francis Guthrie tried to color other maps, he found that three colors were not enough but he felt that four colors were enough for all maps. This observation became known as the *four-color problem*. This problem motivated many mathematicians to work but it takes over 100 years until it was proven.

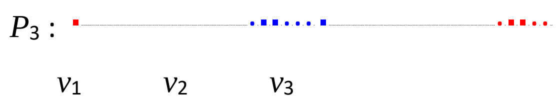
There is a connection between map coloring and graph theory. Maps can be modeled by graphs using the countries as the vertices and two vertices (countries) are adjacent if they share a common boundary.

The process of finding the solution of the four -color problem gives rise to many concepts in graph theory. One of these is the graph coloring. In this context, each vertex of a graph will be assigned one color in such away that no two adjacent vertices have the same color. The interesting idea here is to determine the minimum number of colors to be used so that we can color each vertex of a graph with no two adjacent vertices have the same color.

Graph Coloring and Chromatic Number

Definition 5.31. Let G be a graph. A (vertex) *coloring* of G is an assign- ment of colors to the vertices of G so that adjacent vertices are assigned different colors.

Example 5.49. Consider the path path P_3 with vertices v_1, v_2 and v_3. Using the colors red and blue, if v_1 is red, since v_2 is adjacent to v_1, then v_2 must be blue. Obviously, since v_3 is adjacent to v_2, v_3 must be blue. The figure below represents the coloring of P_3 using the two colors red and blue.



Another coloring of the path P_3 using the three colors red, blue and green.

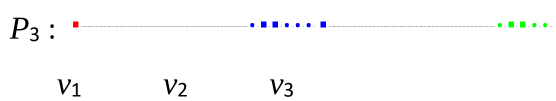


Figure 5.37: Different coloring of a graph

By Definition 5.31, the graph above can not be colored by only one color.

Remark 17. For convenient, the set $\{1, 2, 3\} \ldots$ may be used to color the vertices of a graph.

Using the above remark, we use 1 for the color red, 2 for blue, and 3 for green. Then the different coloring of the graph in Example 5.49, can be shown in Figure 5.38.

Definition 5.32. Let G be a graph. A $k-$ *coloring* of G is a coloring which consists of k different colors and in this case G is said to be $k-$ colorable.

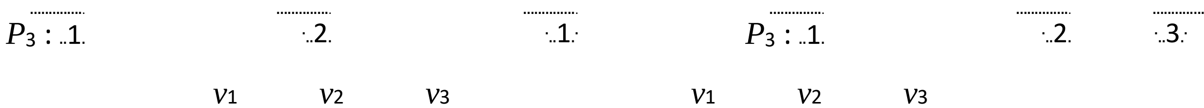


Figure 5.38: Some coloring of P_3

In Example 5.49, the path P_3 is 2 − colorable since it can be colored using 2 colors. Moreover, it is also 3 − colorable but not 1 − colorable graph.

Definition 5.33. Let G be a graph. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k for which there is a k coloring of the graph G.

In Example 5.49, it can be observed that the chromatic number of the path P_3 is 2, that is, $\chi(P_3) = 2$.

Example 5.50. The complete graph K_5 is a graph with 5 vertices and each vertex is adjacent to another vertex. Hence each vertex must have unique color. Thus $\chi(K_5) = 5$.

A pictorial representation of K_5 is shown in Figure 5.39.

Figure 5.39: The graph K_5

Some Known Theorems in Graph Coloring

The following known theorems are helpful in finding the chromatic number of a graph.

Theorem 1 (Four-Color Theorem)**.** Every planar graph is 4 − colorable

Theorem 2. If a graph G contains a subgraph isomorphic to the complete graph K_r, then $\chi(G) \geq r$.

Theorem 3 (Two-Colorable Graph Theorem)**.** A graph is 2 – colorable if and only if it has no circuits that consist of an odd number of vertices.

Theorem 3 is equivalent to the statement that if a graph G has a circuit that consist of an odd number of vertices, then $\chi(G) \geq 3$.

Remark 18 (Chartrand)**.** To show that $\chi(G) = k$ for some graph G and some integer $k \geq 3$ we must show that

1. at least k colors are needed to color G and
2. there is a k – coloring of G.

Example 5.51. The graph G shown in Figure 5.40 is 3 – chromatic.

Solution. Since G contains a subgraph that is isomorphic to the complete graph K_3, by Theorem 2, it follows that $\chi(G) \geq 3$. On the other hand, a 3 – coloring of G is shown in Figure 5.40. This implies that $\chi(G) \leq 3$. Therefore, $\chi(G) = 3$.

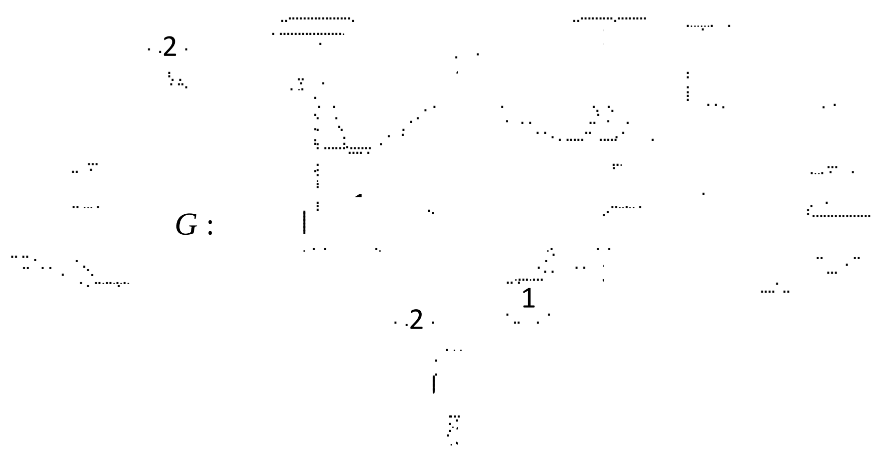
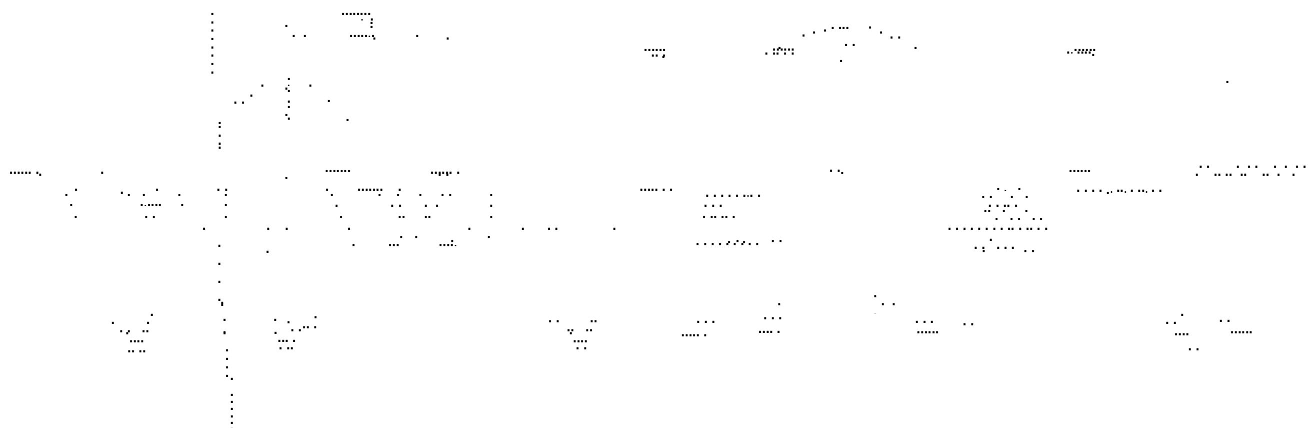


Figure 5.40: A 3 – chromatic graph G

Exercises 5.1. Determine the chromatic number of the different graphs below:



G_1 G_2 G_3

Application of Graph Coloring

Graph coloring is applicable in certain kinds of scheduling problems.

Example 5.52. The ten (10) students of the mathematics department of a certain college are requesting from the department to offer the indicated courses they plan to take this Summer(see Table 5.27). What is the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day? Of course, two classes having no students in common can be taught during the same period.

Learsi:	Linear Algebra (LA) , Statistics (S)
Jason:	Modern Algebra (MA), Linear Algebra(LA), Geometry (G)
Rensie:	Modern Algebra (MA), Geometry (G), Linear Algebra (LA)
Jomari:	Graph Theory (GT), Linear Algebra (LA), Advanced Calculus (AC) Eugene:
	Advanced Calculus (AC), Linear Algebra (LA), Statistics (S) Christian:
	Modern Algebra (MA),Geometry (G), Graph Theory (GT)
Jaime:	Graph Theory (GT), Modern Algebra (MA), Linear Algebra (LA) Ann:
	Linear Algebra (LA), Graph Theory (GT), Statistics (S)
Ronald:	Advanced Calculus (AC), Statistics (S), Linear Algebra (LA)
Christopher:	Graph Theory (GT), Statistics (S)

Table 5.27: Distribution of courses to be taken by the 10 students

Solution.

⇒ First, identify all the courses that are requested by the 7 students. These are the following:

LA	- Linear Algebra
S	- Statistics
MA	- Modern Algebra
G	-Geometry
GT	-Graph Theory
AC	- Advanced Calculus

⇒ Identify the list of students who requested each of the courses, as fol- lows:

LA	- Linear Algebra	: Learsi, Jason, Rensie, Jomarie, Eugene S
	- Statistics	: Learsi, Eugene
MA	- Modern Algebra	: Jason, Rensie, Christian G
	-Geometry	: Jason, Rensie, Christian
GT	-Graph Theory	: jomari, Christian
AC	- Advanced Calculus	: Jomari, Eugene

⇒ Next, construct a graph whose vertices are the six courses. Two vertices (courses) are joined by an edge if some student is taking classes in these two courses, as shown in Figure 5.41.

⇒ Finally, determine the chromatic number of the graph formed, as shown in Figure 5.42.

Therefore, the chromatic number is 3. This means the minimum number of time periods needed is 3. In addition, the vertices with the same color will have the same time period. The schedule may be done as follows:

First period	Linear Algebra
Second Period	Statistics, Geometry, & Graph Theory
Third period	Modern Algebra, Advanced Calculus

Learning Activity Solve the following problems.

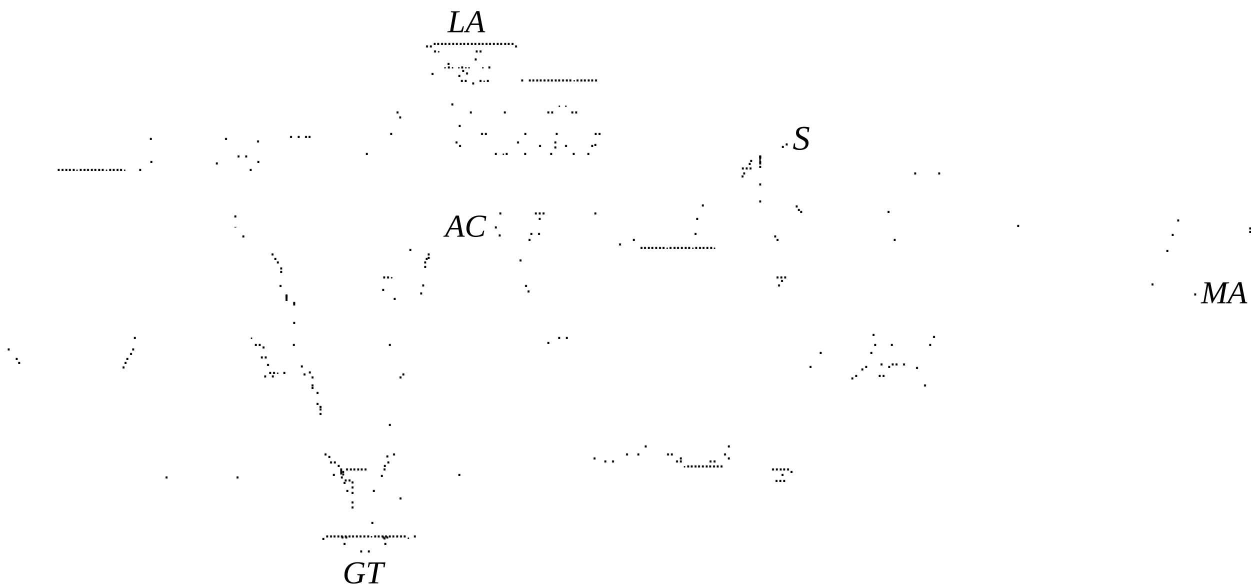
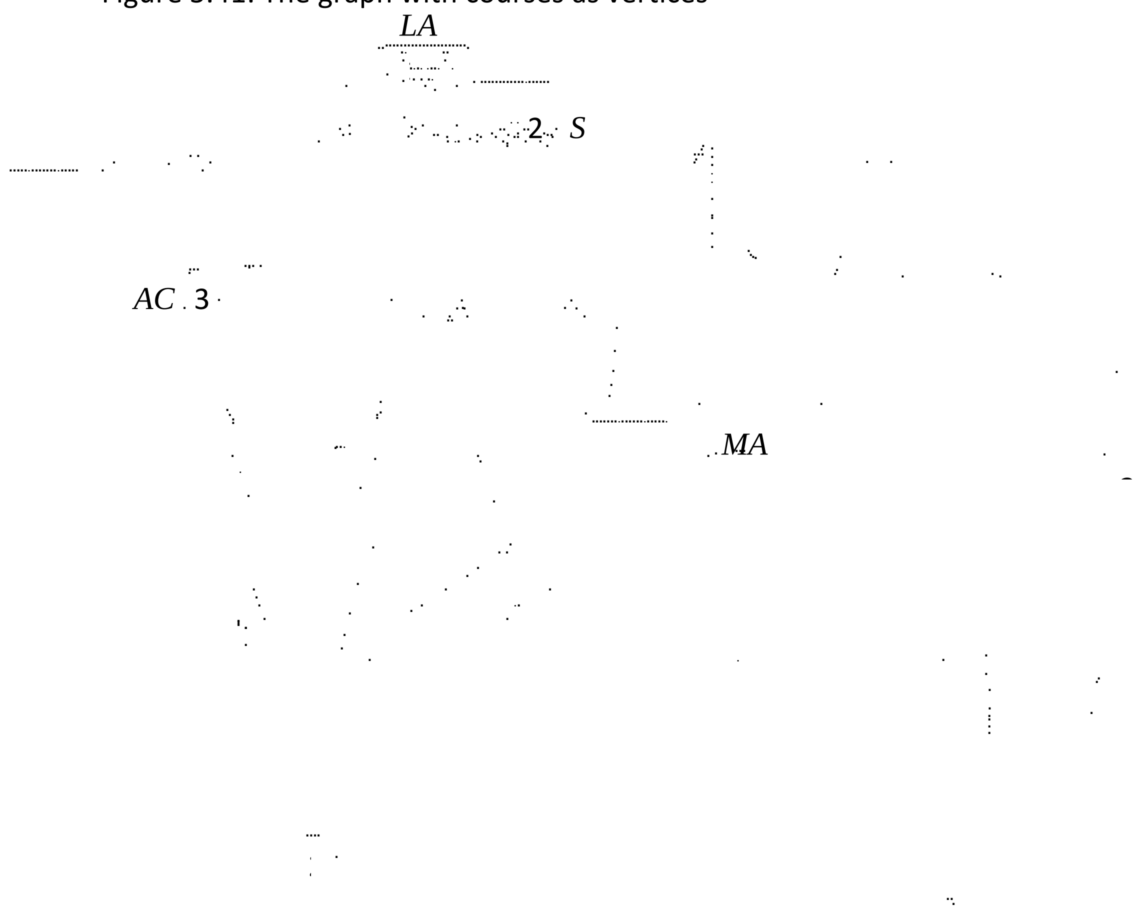
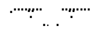


Figure 5.41: The graph with courses as vertices



G



GT

Figure 5.42: Illustrating the chromatics number of the graph

1. Eight mathematics majors at a small college are permitted to attend a meeting dealing with undergraduate research during final exam week

provided they make up all the exams missed on the Monday after they return. The possible time periods for these on Monday are:

(1) 8:00-10:00	(2)10:15-12:15	(3) 12:30-2:30
(4) 2:45-4:45	(5) 5:00-7:00	(6) 7:15-9:15

2. Use graph theory to determine the earliest time on Monday that all eight students can finish their exams if two exams cannot be given dur- ing the same time period if some student must take both exams. the eight students and the courses [Advanced Calculus (AC), Differential Equations (DE), Geometry (G), Graph Theory (GT), Linear Program- ming (LP) Modern Algebra (MA), Statistics (S), Topology (T) each student is taking are listed below:

Alicia: AC,DE, LP
Brian:AC, G,LP Carla:
G, LP, MA
Diane: GT, LP, MA
Edward: DE, GT, LP
Faith: DE, GT,T
Grace: DE, S, T
Henry: AC, DE, S

CHAPTER TEST 7

A.Give the definition of the following:

1.Graph
2.Degree of a vertex
3.Walk
4.Path
5.Trail
6.Parallel edges
7.Loop 8.Equivalent graphs 9.Complete graph

B.Let G be graph with $V(G)$ = $\{1, 2, 3, 4, 5\}$ and

$E(G)$ = $\{[1, 2], [3, 4], [1, 4], [2, 5], [3, 5], [1, 3], [1, 5]\}$.

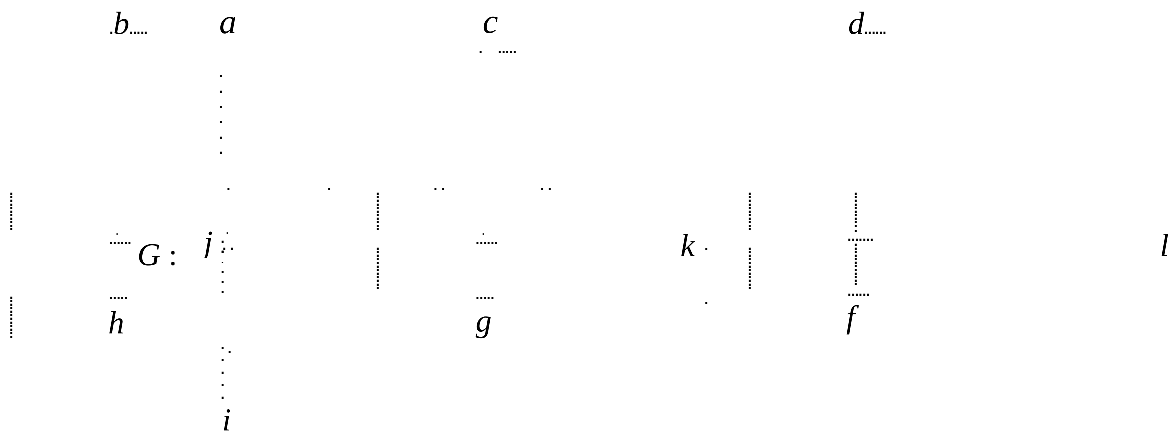
(a) Make a pictorial representation of G.
(b)Determine the order and the size of G.
(c)Construct a table for the degree of each vertex

C.Give an example of a graph G of order 6 and size 10 such that the minimum and maximum degree of a vertex are 3 and 4 respectively.

D.Consider the graph G below. (2 pts each)

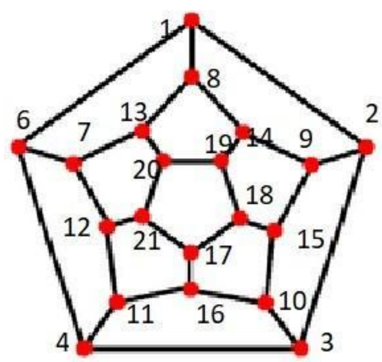
i.Find a walk of length 10 .
ii. Find a closed walk of length 8 .
iii. Find a path of length 5 .
iv. Find a cycle of length 6 .
v. Find a shortest path from a to e.

E.G ive the definition of the following:



i. Eulerian Circuit
ii. Eulerian Graphs.
iii. Hamiltonian Cycle
iv. Hamiltonian Graphs

F. Find a hamiltonian cycle in the given graph below. (10 pts each)

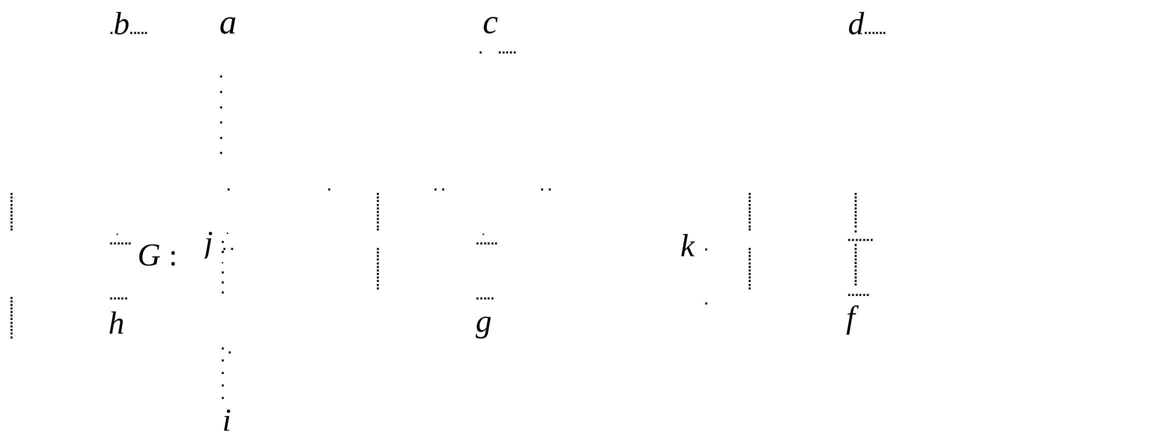


G. Give one example to each of the following graphs: (5 pts each) a. Eulerian Graph

b. Hamiltonian Graph

H. Determine whether the graph below is a eulerian or not. Explain. (10 pts)

I. Susan needs to mail a package at the post office, pick up several items at the grocery store, return a rented video, and make a deposit



	Home	Post office	Grocery store	Video rental store	Bank
Home	—	14	12	20	23
Post office	14	—	8	12	21
Grocery store	12	8	—	17	11
Video rental store	20	12	17	—	18
Bank	23	21	11	18	—

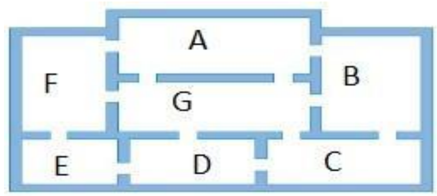
Figure 5.43:

at her bank. The estimated driving time, in minutes, between each of these locations is given in the table beow.

Use both of the algorithms(Greedy & Edge-Picking algorithms) to design routes for Susan to follow that will help minimize her total driving time. Assume she must start from home and return home when her errands are done.

J. i.Draw a graph that represents the floor plan, where each vertex represents a room and an edge connects two vertices if there is a doorway between the two rooms.

ii. Is it possible to walk through the museum and pass through each doorway without going through any doorway twice? Jus- tify your conclusion.

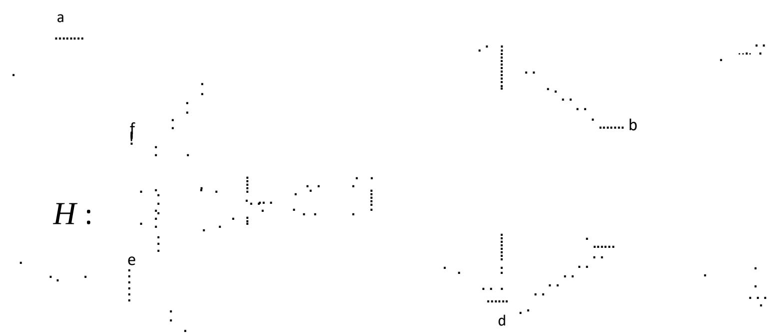


K.Give the definition of the following: (3 pts each)

a.vertex coloring;

b.chromatic number.

L.Determine the chromatic number of the graph below. (10 pts)



M.Eight different school clubs want to schedule meetings on the last day of the semester. Some club members, however, belong to more than one of these clubs, so clubs that share members cannot meet at the same time. How many different time slots are required so that all members can attend all meetings? Clubs that have a member in common are indicated with an ??in the table below.

	Ski club	Student government	Debate club	Honor society	Student newspaper	Community outreach	Campus Democrats	Campus Republicans
Ski club	—	X		X			X	X
Student government	X	—	X	X	X			
Debate club		X	—	X		X		X
Honor society	X	X	X	—	X	X		
Student newspaper		X		X	—	X	X	
Community outreach			X	X	X	—	X	X
Campus Democrats	X				X	X	—	
Campus Republicans	X		X			X		—

N.Six different groups of children would like to visit the zoo and feed different animals. (Assume that each group will visit the zoo on only one day.)

- Group 1 would like to feed the bears, dolphins, and gorillas.
- Group 2 would like to feed the bears, elephants,and hippos.
- Group 3 would like to feed the dolphins and elephants.
- Group 4 would like to feed the dolphins, zebras, and hippos.
- Group 5 would like to feed the bears and hippos.
- Group 6 would like to feed the gorillas, hippos, and zebras.

Use graph coloring to find the minimum number of days that are required so that all groups can feed the animals they would like to feed but no animals will be fed twice on the same day. Design a schedule to accomplish this goal.

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