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NONLINEAR ANALYSIS

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Nikolaos S. Papageorgiou

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To Prof. Zdzisław Denkowski

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Preface

Linear functional analysis deals with infinite dimensional topological vector spaces (which mix in a fruitful way the linear (algebraic) structure with topological one) and the linear operators acting between them. The effort was to extend standard results of linear analysis to an infinite dimensional context. The first half of the twentieth century is marked by intensive theoretical investigations in this area, which were also accompanied by detailed treatment of linear mathematical models. With the exception of a short period during the 1930's (compact operators and Leray-Schauder degree), nonlinear operators were out of the emerging picture. However, mounting evidence from diverse other fields such as physics, engineering, economics, biology and others suggested that there should be an effort to extend the linear theory to various kinds of nonlinear operators. Systematic efforts in this direction started in the early 1960's and mark the beginning of what is known today as "Nonlinear Analysis." Since then several theories have been developed in this respect and today some of them are well established approaching their limits, while others are still the object of intense research activity. It is not a coincidence that simultaneously with the advent of nonlinear analysis, we have the appearance of nonsmooth analysis and of multivalued analysis, both of which were motivated by concrete needs in applied areas such as control theory, optimization, game theory and economics. Their development provided nonlinear analysis with new concepts, tools and theories that enriched the subject considerably. Today nonlinear analysis is a well established mathematical discipline, which is characterized by a remarkable mixture of analysis, topology and applications. It is exactly the fact that the subject combines in a beautiful way these three items that makes it attractive to mathematicians. The notions and techniques of nonlinear analysis provide the appropriate tools to develop more realistic and accurate models describing various phenomena. This gives nonlinear analysis a rather interdisciplinary character. Today the more theoretically inclined nonmathematician (engineer, economist, biologist or chemist) needs a working knowledge of at least a part of nonlinear analysis in order to be able to conduct a complete qualitative analysis of his models. This supports a high demand for books on nonlinear analysis. Of course the subject is big (vast is maybe a more appropriate word) and no single book can cover all its theoretical and applied parts. In this volume, we have focused on those topics of nonlinear analysis which are pertinent to the theory of boundary value problems and their applications such as control theory and calculus of variations.

In Chapter 1 we deal with Hausdorff measures and capacities, which provide the means to estimate the "size" or "dimension" of "thin" or "highly irregular" sets. The recent development of fractal geometry and its uses in a variety of applied areas (such as Brownian motion of particles, turbulence in fluids, geographical coastlines and surfaces etc) renewed the interest on Hausdorff measures, which for a long period were a topic of secondary importance within measure theory. In this chapter we also have our first encounter with Lipschitz and locally Lipschitz functionals which will be examined again in Chapter 4. At this point we prove the celebrated "Rademacher's theorem."

Chapter 2 deals with certain classes of function spaces, which arise naturally in the study of boundary value problems. These are the Lebesgue-Bochner spaces (the suitable spaces for the analysis of evolution equations) and the Sobolev spaces (the suitable spaces for weak solutions of elliptic equations). We conduct a detailed study of these spaces with special emphasis on compactness and embedding results. Also using the tool of Hausdorff measures and capacities, we investigate the fine properties of Sobolev functions and also introduce and study functionals of bounded variation which are useful in theoretical mechanics.

In Chapter 3, we deal with certain large classes of nonlinear operators which arise often in applications. We examine compact operators for which we develop in parallel the corresponding linear theory, with one of the main results being the spectral theorem for compact self-adjoint operators on a Hilbert space. We also investigate nonlinear operators of monotone type which have their roots in the calculus of variations and exhibit remarkable surjectivity properties. Monotone operators lead to accretive operators, the two families being identical in the context of Hilbert spaces. Accretive operators are closely connected with the generation theory of semigroups of operators. We also examine both linear and nonlinear semigroups. Semigroups are basic tools in the study of evolution equations. In addition, we examine the Nemytskii operator which is a nonlinear operator encountered in almost all problems. Finally, in the last section of the chapter, we discuss Young measures which provide the right framework to examine the limit behavior of the minimizing sequence of variational problems which do not have a solution. Young measures are used in optimal control and in the calculus of variations in connection with the so-called "relaxation method."

Chapter 4 presents the calculus of smooth and of certain broad classes of nonsmooth functions. We start with the Gâteaux and Fréchet derivatives. We discuss the generic differentiability of continuous convex functions (Mazur's theorem) and extend Rademacher's theorem to locally Lipschitz functions between certain Banach spaces by using the notion of Haar-null sets. Then we pass to nondifferentiable functions and develop the duality properties and subdifferential theory of convex functions and the generalized subdifferential of locally Lipschitz functions. We also examine integral functionals and discuss the celebrated Ekeland variational principle establishing its equivalence with some other geometric results of nonlinear analysis.

In Chapter 5 we present the critical point theory of C^1-functions defined on a Banach space. This theory is in the core of the variational methods used in the study of boundary value problems. We follow the deformation approach which leads to minimax characterizations of the critical values. We also study the structure of the set of critical points and derive results on the existence of multiple critical points. Next we present the Lusternik-Schnirelman theory which extends to nonlinear eigenvalue problems the corresponding linear theory of R. Courant.

Chapter 6 uses the abstract results of Chapter 5 as well as results from earlier chapters to develop the spectrum of linear elliptic differential operators, of the partial p-Laplacian (with Dirichlet and Neumann boundary conditions) and of the scalar and vector ordinary p-Laplacian (with Dirichlet, Neumann and periodic boundary conditions). We also present linear and nonlinear maximum principles and comparison results, which are useful tools in the study of boundary value problems.

Finally in Chapter 7 we have gathered some basic fixed point theorems. We present results from metric fixed point theory, from topological fixed point theory and fixed point results based on the partial order induced by a closed, convex pointed cone. We also indicate how many of these results can be extended to multifunctions (set-valued functions).

We have tried to make the volume self-contained. For this reason at the end of the book we have included a rather extended appendix for easy reference of the general results used in the book. Nevertheless, within the test whenever we are in the need of using some results not proved in the book, we also give exact references where the interested reader can find additional information.

Now that the project has reached its conclusion, we would like to thank the good people of CRC Press (especially Mrs. Jessica Vakili) for their help and kind cooperation during the preparation of this book. We would like to thank the two editors of this series, Prof. R.P. Agarwal and Prof. D.O'Regan, for supporting this effort.

Chapter 1

Hausdorff Measures and Capacity

During the golden era of measure theory (namely the first two decades of the 20th century), Carathéodory was the first to consider the notion of "length" for sets in $\mathbb{R}^N$. Later, in 1919, Hausdorff, motivated by the ideas of Carathéodory, introduced the measure and dimensional concepts that we shall discuss in this chapter. So in the modern language, the "length" of a set $A \subseteq \mathbb{R}^N$ will be its Hausdorff one-dimensional outer measure (denoted by $\mu^{(1)}$). Following the pioneering works of Carathéodory and Hausdorff, significant contributions to the subject were made by Besicovitch. In fact, in the first decade of development of the subject, the main advances on the subject were made by Besicovitch and his students, since geometric measure theory was not part of the mainstream measure theory. However, since the early 70's, the subject attracted a large number of researchers, due to its fundamental importance in the study of the so-called "Fractal Geometry." Fractal sets arise in many applications, such as turbulence in fluids, geographical coastlines and surfaces, fluctuation of prices in stock exchanges, the Brownian motion of particles and others. Mandelbrojt was the first to emphasize their use to model a variety of phenomenona. There have been many ways to estimate the "size" or "dimension" of small (thin) sets and of highly irregular sets and to generalize the idea that points, curves and surfaces have dimensions 0, 1 and 2 respectively. Hausdorff measure has the advantage of being a measure and together with the notion of Hausdorff dimension can provide a more delicate sense of the size of sets in $\mathbb{R}^N$ than Lebesgue measure provides. To illustrate this, consider in $\mathbb{R}^2$ the set

$$A \stackrel{df}{=} \left\{ \left(t, \sin \frac{1}{t} \right) : \ t \in (0,1) \right\}.$$

Suppose we wish to measure the length of the curve A. A first approximation can be based on the Carathéodory outer measure, which defines:

$$\lambda^1(A) \stackrel{df}{=} \inf_{A \subseteq \bigcup_{n=1}^{\infty} C_n} \sum_{n=1}^{\infty} \delta(C_n),$$

i.e., the infimum is taken over all countable covers of A (by $\delta(A)$ we denote the diameter of the set A; see (1.1)). If we adopt this definition, we see that $\lambda^1(A) < +\infty$, while we know that the length of A is infinite. The reason for

this is that in the definition of $\lambda^1(A)$, the covers of A are not forced to follow the geometry of A. For this reason the Hausdorff s-dimensional measures $\mu^{(s)}(A)$ are defined as limits of outer measures $\mu_\delta^{(s)}$ which follow the local geometry of A (see Definition 1.3.5).

As another illustrative example, consider the unit square S in $\mathbb{R}^2$ (i.e., square of side length equal to 1) and define

$$\lambda^1(S) \stackrel{df}{=} \inf_{S \subseteq \bigcup_{n=1}^{\infty} C_n} \sum_{n=1}^{\infty} \delta(C_n),$$

i.e., again the infimum is taken over all countable covers of S. We observe that we can do no better than cover S itself. Indeed, if we cover S with smaller squares of diameter less or equal to $\frac{1}{n}$, then we see that we need at least n^2 squares to achieve the covering and so the approximation of $\lambda^1(S)$ obtained this way exceeds $n\sqrt{2}$. So the smaller the squares we use to cover, the bigger the estimate for $\lambda^1(S)$. Therefore, small squares are irrelevant in the calculation of $\lambda^1(S)$ and yet it is precisely them that should have an influence on the evaluation of $\lambda^1(S)$. We expect $\lambda^1(S) = 0$, since the diameter is a one-dimensional concept and it is used to measure a square in $\mathbb{R}^2$, which is a two dimensional concept. For this we need a definition which takes into account the local geometry of the set under consideration.

In this chapter, in Section 1.1 we recall some basic definitions and facts from measure theory, which will be needed in what follows. In Section 1.2, we discuss some "covering theorems." Covering results play a central role in geometric measure theory. In Section 1.3 we introduce and study Hausdorff measures and the Hausdorff dimension of sets. Among other things we calculate the Hausdorff dimension of some classical irregular sets in $\mathbb{R}$ (Cantor-like sets). From these calculations, the reader will realize that the Hausdorff measure and the Hausdorff dimension of sets (even of simple ones) may be hard to calculate. For this reason sometimes other notions may be more suitable (such as capacity; see Section 1.6). In Section 1.4 we discuss the differentiation of Hausdorff measures and derive the Lebesgue-Besicovitch differentiation theorem. In Section 1.5, using the tools of Hausdorff measures, we study the geometry of Lipschitz continuous functions. Among other things, we obtain the "area and coarea formulas" and the associated with them "change of variables formulas." Finally in Section 1.6, we present an alternative analytical notion measuring small sets in $\mathbb{R}^N$, namely the p-capacity. We derive some basic properties of the p-capacities and compare them to the Hausdorff measures.

1.1 Measure Theoretical Background

In this section we recall some basic definitions and facts from measure theory, which we shall need in the sequel. Let us start with the concept of outer measure, which, when restricted to a suitable σ-field of sets, leads to a measure.

DEFINITION 1.1.1 *Let X be a set. A map $\mu\colon 2^X \longrightarrow [0,+\infty]$ is said to be an **outer measure**, if*

(a) $\mu(\emptyset) = 0$;

(b) $A \subseteq B \Longrightarrow \mu(A) \leqslant \mu(B)$ *(monotonicity);*

(c) *for any sequence of sets $\{A_n\}_{n\geqslant 1} \subseteq 2^X$, we have*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$$

(subadditivity).

*For a given outer measure μ on X and $A \in 2^X$, we define the **restriction** of μ on A, denoted by $\mu\lfloor A$, by*

$$(\mu\lfloor A)(B) \stackrel{df}{=} \mu(A \cap B) \qquad \forall\ B \in 2^X.$$

*We say that μ is a **finite outer measure** if $\mu(X) < +\infty$ (i.e., μ has values in $\mathbb{R}_+$).*

REMARK 1.1.2 Note that $\mu\lfloor A$ is an outer measure on X, while we define $\mu|_A$ to be the restriction of μ (as a function) on 2^A, i.e., $\mu|_A\colon 2^A \longrightarrow [0,+\infty]$ is defined by

$$\big(\mu|_A\big)(B) \stackrel{df}{=} \mu(B) \qquad \forall\ B \in 2^A \subseteq 2^X.$$

Outer measures are useful because they lead to measures when restricted to suitably defined σ-fields. These σ-fields can be quite large.

DEFINITION 1.1.3 *Let X be a set and μ an outer measure on X. A set $A \in 2^X$ is said to be μ-**measurable**, if*

$$\mu(B) \;=\; \mu(A \cap B) + \mu(B \setminus A) \qquad \forall\ B \in 2^X,$$

i.e., A "decomposes" every set B additively.

REMARK 1.1.4 Let X be a set and μ an outer measure on X.

(a) By virtue of the subadditivity property of an outer measure, to show that $A \in 2^X$ is μ-measurable, it is enough to check that

$$\mu(B) \geqslant \mu(A \cap B) + \mu(B \setminus A) \qquad \forall\, B \in 2^X.$$

(b) Clearly, if $A \in 2^X$ and $\mu(A) = 0$, then A is μ-measurable.

(c) If $A \in 2^X$, then any μ-measurable set is also $\mu\lfloor A$-measurable.

(d) A is μ-measurable if and only if $A^c = X \setminus A$ is μ-measurable.

It is straightforward to check the following result.

PROPOSITION 1.1.5
If X is a set and μ is an outer measure on X, then the collection Σ_μ of all μ-measurable sets is a σ-field and μ restricted on Σ_μ is a measure.

REMARK 1.1.6 While Definition 1.1.3 involves only additivity of μ, the conclusion in Proposition 1.1.5 is about σ-additivity of μ on Σ_μ. This reveals the power of Definition 1.1.3. Note that from Remark 1.1.4**(b)**, it follows that the σ-field Σ_μ is μ-complete.

DEFINITION 1.1.7 *Let X be a nonempty Hausdorff topological space and let μ be an outer measure on X.*

(a) *Let $\mathcal{T}$ be a family of 2^X. We say that μ is $\mathcal{T}$-**regular**, if*

$$\mu(A) = \inf_{\substack{B \in \mathcal{T} \\ A \subseteq B}} \mu(B) \qquad \forall\, A \in 2^X.$$

*If $\mathcal{T} = \Sigma_\mu$, then we simply say that μ is **regular**.*

(b) *We say that μ is a **Borel measure**, if $\mathcal{B}(X) \subseteq \Sigma_\mu$ with $\mathcal{B}(X)$ being the Borel σ-field of X.*

(c) *We say that μ is a **Borel regular measure**, if μ is a Borel measure which is $\mathcal{B}(X)$-regular.*

(d) *We say that μ is a **Radon measure**, if μ is a Borel regular measure and*

$$\mu(K) < +\infty \qquad \forall\, K \subseteq X,\ K\text{-compact}.$$

REMARK 1.1.8 Let X be a Hausdorff topological space and let μ be an outer measure on X.

(a) Note that μ is regular if and only if

$$\forall A \in 2^X \ \exists B \in \Sigma_\mu : \ \mu(A) = \mu(B).$$

(b) If μ is regular on X and $\{A_n\}_{n\geqslant 1} \subseteq 2^X$ is increasing (i.e., $A_n \subseteq A_{n+1}$ for $n \geqslant 1$), then

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \ = \ \sup_{n\geqslant 1} \mu(A_n).$$

PROPOSITION 1.1.9
If X is a Hausdorff topological space, μ is an outer measure on X which is Borel regular and $A \in \Sigma_\mu$ with $\mu(A) < +\infty$,
then $\mu\lfloor A$ is a Radon measure.

PROOF Let

$$\mu_1 \ \stackrel{df}{=} \ \mu\lfloor A.$$

Evidently $\Sigma_\mu \subseteq \Sigma_{\mu_1}$ and so μ_1 is a Borel measure. Also for every compact $K \subseteq X$, we have

$$\mu_1(K) \ < \ +\infty.$$

It remains to show that μ_1 is Borel regular. To this end note that since μ is Borel regular, for a given $A \in 2^X$, we can find $B \in \mathcal{B}(X)$, $A \subseteq B$, such that

$$\mu(A) \ = \ \mu(B) \ < \ +\infty.$$

Because $A \in \Sigma_\mu$, from Definition 1.1.3, we have

$$\mu(B \setminus A) \ = \ \mu(B) - \mu(A) \ = \ 0.$$

Since $A \in \Sigma_\mu$, for every $C \in 2^X$, we have

$$\begin{aligned}(\mu\lfloor B)(C) \ &= \ \mu(B \cap C) \ = \ \mu(B \cap C \cap A) + \mu\big((B \cap C) \setminus A\big) \\ &\leqslant \ \mu(C \cap A) + \mu(B \setminus A) \ = \ \mu(C \cap A) \ = \ (\mu\lfloor A)(C).\end{aligned}$$

As $A \subseteq B$, we infer that

$$\mu\lfloor B \ = \ \mu\lfloor A.$$

So without any loss of generality, we may assume that $A \in \mathcal{B}(X)$.

Let $C \in 2^X$. Since μ is Borel regular, we can find $D \in \mathcal{B}(X)$, such that

$$A \cap C \subseteq D \quad \text{and} \quad \mu(A \cap C) = \mu(D)$$

(see Remark 1.1.8**(a)**). Let us take

$$E \stackrel{df}{=} D \cup (X \setminus A).$$

Evidently $E \in \mathcal{B}(X)$ and

$$C \subseteq (A \cap C) \cup (X \setminus A) \subseteq E.$$

Moreover, since $E \cap A = D \cap A$, we have

$$\mu_1(E) = \mu(E \cap A) = \mu(D \cap A) \leqslant \mu(D) = \mu(A \cap C) = \mu_1(C),$$

so $\mu_1 = \mu \lfloor A$ is Borel regular (see Remark 1.1.8**(a)**), hence Radon. □

We conclude this section, by recalling the following basic measure theoretic approximations.

PROPOSITION 1.1.10
If X is a Hausdorff topological space and μ is an outer measure on X which is Borel,
then

(a) *if $A \in \mathcal{B}(X)$, $\mu(A) < +\infty$ and $\varepsilon > 0$, then we can find an open set $U_\varepsilon \supseteq A$ and a closed set $C_\varepsilon \subseteq A$, such that*

$$\mu(U_\varepsilon \setminus C_\varepsilon) < \varepsilon,$$

i.e.,

$$\mu(A) = \inf_{\substack{U\text{-open} \\ A \subseteq U}} \mu(U) = \sup_{\substack{C\text{-closed} \\ C \subseteq A}} \mu(C).$$

(b) *if μ is Radon, then for every $A \in 2^X$, we have*

$$\mu(A) = \inf_{\substack{U\text{-open} \\ A \subseteq U}} \mu(U)$$

and if $A \in \Sigma_\mu$, then

$$\mu(A) = \sup_{\substack{K\text{-compact} \\ K \subseteq A}} \mu(K).$$

REMARK 1.1.11 Note that in the first part of Proposition 1.1.10**(b)**, the set A need not be μ-measurable.

1.2 Covering Results

One of the main tools in geometric measure theory is the so called Vitali covering theorem. For a given sufficiently large family of sets that cover a given set A, Vitali's covering theorem allows us to select a countable subfamily consisting of distinct sets with exactly the desired approximation properties. The basic principle embodied in the proof of Vitali's covering theorem is illustrated in the next proposition.

In what follows for any subset A of a metric space $(X, d_{_X})$, we define

$$\delta(A) \;=\; \operatorname{diam}(A) \;\stackrel{df}{=}\; \sup_{x,y\in A} d_{_X}(x,y), \tag{1.1}$$

the diameter of A (by convention $\operatorname{diam}\emptyset \stackrel{df}{=} 0$).

PROPOSITION 1.2.1
If $\mathcal{T}$ is a collection of nondegenerate balls in $\mathbb{R}^N$ with

$$\sup_{B\in\mathcal{T}} \delta(B) \;<\; +\infty,$$

then we can find a finite or countable subfamily $\mathcal{F}$ of $\mathcal{T}$ consisting of disjoint balls, such that

$$\bigcup_{B\in\mathcal{T}} B \;\subseteq\; \bigcup_{B\in\mathcal{F}} \widehat{B},$$

with $\widehat{B}$ being the ball concentric with B, but with radius five times the radius of B.

PROOF Let

$$d_0 \;\stackrel{df}{=}\; \sup_{B\in\mathcal{T}} \delta(B),$$

$$\mathcal{T}_n \;\stackrel{df}{=}\; \left\{ B\in\mathcal{T} :\; \frac{d_0}{2^n} < \delta(B) \leqslant \frac{d_0}{2^{n-1}} \right\} \qquad \forall\; n \geqslant 1.$$

Inductively, we generate subfamilies $\mathcal{F}_n \subseteq \mathcal{T}_n$ for $n \geqslant 1$. Namely, let $\mathcal{F}_1$ be any maximal disjoint collection of balls in $\mathcal{T}_1$. Suppose we have selected $\mathcal{F}_1, \ldots, \mathcal{F}_m$. We choose $\mathcal{F}_{m+1}$ to be any maximal disjoint subfamily of

$$\left\{ B\in\mathcal{T}_{m+1} :\; B\cap B' = \emptyset \text{ for all } B' \in \bigcup_{i=1}^{m} \mathcal{F}_k \right\}$$

and finally set

$$\mathcal{F} \;\stackrel{df}{=}\; \bigcup_{m=1}^{\infty} \mathcal{F}_m.$$

Evidently $\mathcal{F} \subseteq \mathcal{T}$ and consists of disjoint balls.

Claim. For each $B \in \mathcal{T}$, we can find $B' \in \mathcal{F}$, such that $B \cap B' \neq \emptyset$ and $\delta(B) \leqslant 2\delta(B')$ (so also $B \subseteq \widehat{B}'$).

For some $m \geqslant 1$, we have $B \in \mathcal{T}_m$. By virtue of the maximality of $\mathcal{F}_m$, we can find $B' \subseteq \bigcup_{k=1}^{m} \mathcal{F}_k$ with $B \cap B' \neq \emptyset$. We have that

$$\frac{d_0}{2^m} \leqslant \delta(B') \quad \text{and} \quad \delta(B) \leqslant \frac{d_0}{2^{m-1}}.$$

So $\delta(B) \leqslant 2\delta(B')$ and this proves the claim.

From the claim it follows at once that $\bigcup_{B\in\mathcal{T}} B \subseteq \bigcup_{B'\in\mathcal{F}} \widehat{B}'$. □

DEFINITION 1.2.2 *Let $A \subseteq \mathbb{R}^N$. A collection $\mathcal{T}$ of sets in $\mathbb{R}^N$ is said to be a **Vitali cover** of A, if $A \subseteq \bigcup_{B\in\mathcal{T}} B$ and for every $x \in A$ and every $\varepsilon > 0$, there exists $B \in \mathcal{T}$, such that $x \in B$ and $0 < \delta(B) < \varepsilon$.*

REMARK 1.2.3 Note that from the second requirement of the above definition it follows that

$$\inf_{B\in\mathcal{T}} \delta(B) = 0.$$

So $\mathcal{T}$ is a Vitali cover of a set A, if every point $x \in A$ is contained in an arbitrary small element of $\mathcal{T}$.

As a straightforward consequence of Proposition 1.2.1 we obtain the following proposition.

PROPOSITION 1.2.4
If $A \subseteq \mathbb{R}^N$, $\mathcal{T}$ is a Vitali cover of A consisting of closed balls, such that

$$\sup_{B\in\mathcal{T}} \delta(B) < +\infty,$$

then there exists a countable family $\mathcal{F} = \{B_n\}_{n\geqslant 1}$ consisting of disjoint balls from $\mathcal{T}$, such that for each $m \geqslant 1$, we have

$$A \subseteq \bigcup_{n=1}^{m} B_n \cup \bigcup_{n=m+1}^{\infty} \widehat{B}_n,$$

where $\widehat{B}_n$ is the closed ball cocentric with B_n and radius five times the radius of B_n.

PROOF Let $\mathcal{F}$ be as in the proof of Proposition 1.2.1. Select $\{B_n\}_{n=1}^m \subseteq \mathcal{F}$. If $A \subseteq \bigcup_{n=1}^m B_n$, then we are done. Otherwise let $x \in A \setminus \bigcup_{n=1}^m B_n$. Since $\mathcal{T}$ is a Vitali cover of A consisting of closed balls, then we can find $B \in \mathcal{T}$, such that $x \in B$ and

$$B \cap B_n \;=\; \emptyset \qquad \forall\, n \in \{1, \ldots, m\}.$$

But from the claim in the proof of Proposition 1.2.1, we can find $B' \in \mathcal{F}$, such that

$$B \subseteq \widehat{B}' \quad \text{and} \quad B \cap B' \neq \emptyset$$

(so $B' \in \{B_n\}_{n=m+1}^{\infty}$). □

Now we are ready to state and prove Vitali's covering theorem. In what follows by λ^N we denote the N-dimensional Lebesgue outer measure.

THEOREM 1.2.5 (Vitali Covering Theorem)
If $A \subseteq \mathbb{R}^N$ with $0 < \lambda^N(A) < +\infty$ and $\mathcal{T}$ is a Vitali cover of A consisting of closed sets,
then we can find a sequence $\{C_n\}_{n\geqslant 1}$ of elements in $\mathcal{T}$, such that $C_n \cap C_m = \emptyset$ for $n \neq m$ and

$$\lambda^N\left(A \setminus \bigcup_{n=1}^{\infty} C_n\right) \;=\; 0.$$

PROOF Without any loss of generality, we can assume that there exists an open set $U \subseteq \mathbb{R}^N$ with $\lambda^N(U) < +\infty$ and

$$C \subseteq U \qquad \forall\, C \in \mathcal{T}.$$

We construct the sequence $\{C_n\}_{n\geqslant 1}$ inductively. Let $C_1 \in \mathcal{T}$. Suppose that $C_1, \ldots, C_n$ are disjoint sets in $\mathcal{T}$. If $A \subseteq \bigcup_{k=1}^n C_k$, then we are finished. If not, setting

$$V_n \;\stackrel{df}{=}\; U \setminus \bigcup_{k=1}^n C_k,$$

we introduce

$$\mathcal{T}_n \;\stackrel{df}{=}\; \big\{C \in \mathcal{T} :\; C \subseteq V_n\big\} \quad \text{and} \quad \delta_n \;\stackrel{df}{=}\; \sup_{C \in \mathcal{T}_n} \lambda^N(C).$$

Because $A \setminus \bigcup_{i=1}^n C_k \neq \emptyset$ and $\mathcal{T}$ is a Vitali cover of A, we see that $\mathcal{T}_n \neq \emptyset$ and so $\delta_n > 0$. We select

$$C_{n+1} \in \mathcal{T} \quad \text{with} \quad \frac{\delta_n}{2} < \lambda^N(C_{n+1}).$$

We continue this process. Then either at some finite step $n \geqslant 1$ we shall have $A \subseteq \bigcup_{k=1}^{n} C_k$, in which case the proof of the theorem is complete or otherwise we produce a sequence $\{C_n\}_{n\geqslant 1} \subseteq \mathcal{T}$ of disjoint sets. Then we have

$$\sum_{n=1}^{\infty} \lambda^N(C_n) \;=\; \lambda^N\Big(\bigcup_{n=1}^{\infty} C_n\Big) \;\leqslant\; \lambda^N(U) \;<\; +\infty. \tag{1.2}$$

For each $n \geqslant 1$ let B_n be a ball with center in C_n and radius equal to $3\delta(C_n)$. We claim that

$$A \setminus \bigcup_{k=1}^{n} C_k \;\subseteq\; \bigcup_{k=n+1}^{\infty} B_k \qquad \forall\, n \geqslant 1. \tag{1.3}$$

Let $x \in A \setminus \bigcup_{k=1}^{n} C_k$. Since $\mathcal{T}$ is a Vitali cover of A, we can find a set $C_x \in \mathcal{T}_n$, such that

$$x \in C_x \quad \text{and} \quad \lambda^N(C_x) > 0.$$

We shall show that

$$C_x \cap C_k \neq \emptyset \quad \text{for some } k > n.$$

Indeed, if this is not the case, then $\lambda^N(C_x) \leqslant \delta_k$ for all $k \geqslant 1$, which contradicts the fact that

$$0 \;\leqslant\; \lim_{k\to+\infty} \delta_k \;\leqslant\; \lim_{k\to+\infty} 2\lambda^N(C_{k+1}) \;=\; 0$$

(recall the choice of C_{k+1} and see (1.2)). Let $m > n$ be the smallest integer, such that

$$C_x \cap C_m \neq \emptyset.$$

Since $C_x \in \mathcal{T}_{m-1}$, we have

$$\lambda^N(C_x) \;\leqslant\; \delta_{m-1} \;<\; 2\lambda^N(C_m)$$

and recalling the choice of B_m, also $C_x \subseteq B_m$. So we have proved (1.3).

Then for any $n \geqslant 1$, we have

$$\lambda^N\Big(A \setminus \bigcup_{k=1}^{\infty} C_k\Big) \;\leqslant\; \lambda^N\Big(A \setminus \bigcup_{k=1}^{n} C_k\Big) \;\leqslant\; \sum_{k=n+1}^{\infty} \lambda^N(B_k). \tag{1.4}$$

Recalling that B_k is a ball of radius $3\delta(C_k)$ and combining (1.2) and (1.4), we conclude that

$$\lambda^N\Big(A \setminus \bigcup_{k=1}^{\infty} C_k\Big) \;=\; 0.$$

□

Vitali's covering theorem may be difficult to digest at first and probably it is necessary to see the lemma in action several times before appreciating it. For this reason we present four simple applications from classical analysis of functions of one-variable.

We start with a definition which establishes the notation for various limits of the difference quotient that we shall use in the sequel. These derivatives are often more useful than the ordinary derivative, since they are defined at every point.

DEFINITION 1.2.6 *For a given function $f\colon [a,b] \longrightarrow \mathbb{R}$, the upper right and lower right derivates of f at $x \in [a,b)$ are defined by*

$$D^+f(x) \stackrel{df}{=} \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

and

$$D_+f(x) \stackrel{df}{=} \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

respectively. Similarly the upper left and lower left derivates of f at $x \in (a,b]$ are defined by

$$D^-f(x) \stackrel{df}{=} \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

and

$$D_-f(x) \stackrel{df}{=} \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

respectively.

REMARK 1.2.7 Evidently, the derivates of a function at a point may be infinite. The function f is differentiable at $x \in (a,b)$, if

$$-\infty \;<\; D^+f(x) \;=\; D_+f(x) \;=\; D^-f(x) \;=\; D_-f(x) \;<\; +\infty.$$

The function f is differentiable at $x = a$ or at $x = b$, if the appropriate two derivates are finite and equal. Also the one-sided derivatives exist at a point x, if

$$D^+f(x) \;=\; D_+f(x) \quad \text{and} \quad D^-f(x) \;=\; D_-f(x).$$

The derivates are also called ***Dini derivates*** and clearly we always have

$$D_+f(x) \;\leqslant\; D^+f(x) \qquad \forall\; x \in [a,b)$$

and

$$D_-f(x) \;\leqslant\; D^-f(x) \qquad \forall\; x \in (a,b].$$

In the literature, sometimes we find the notion of a derived number for a function f at x. So $\beta \in \mathbb{R}^*$ is a derived number for f at x, if there is a sequence

$$\{h_n\}_{n\geqslant 1} \subseteq \mathbb{R},$$

such that

$$h_n \longrightarrow 0,$$

$$h_n \neq 0 \qquad \forall\, n \geqslant 1$$

and

$$\lim_{n\to+\infty} \frac{f(x+h_n)-f(x)}{h_n} = \beta.$$

A function f may have many derived numbers at a point x. Of course f is differentiable at x if and only if all derived numbers of f at x agree and are finite.

EXAMPLE 1.2.8 Consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) \stackrel{df}{=} \begin{cases} x \sin \dfrac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

We can check that

$$D^- f(0) = -1 < D^+ f(0) = 1$$

and every number in $[-1,1]$ is a derived number for f. The function f is not of bounded variation (see Definition A.2.15**(a)**).

LEMMA 1.2.9
If $f\colon [a,b] \longrightarrow \mathbb{R}$ *is nondecreasing,*
then all four derivates of f *are finite almost everywhere on* $[a,b]$.

PROOF Clearly all derivates are nonnegative. So it suffices to show that

$$D^+ f(x) < +\infty \quad \text{and} \quad D^- f(x) < +\infty \quad \text{for a.a. } x \in [a,b].$$

Let

$$A \stackrel{df}{=} \left\{ x \in [a,b] :\ D^+ f(x) = +\infty \right\}$$

and suppose that

$$\lambda^*(A) = \beta > 0,$$

where λ^* is the Lebesgue outer measure on $\mathbb{R}$. Let $M > 0$ be such that

$$f(b) - f(a) < \frac{M\beta}{2}.$$

For every $x \in A$, we can find a decreasing sequence $\{h_n^x\}_{n \geqslant 1}$ with

$$h_n^x \searrow 0,$$

$$h_n^x \neq 0 \qquad \forall\, n \geqslant 1,$$

such that

$$M \leqslant \frac{f(x + h_n^x) - f(x)}{h_n^x}.$$

The collection

$$\left\{[x, x + h_n^x]\right\}_{x \in A, n \geqslant 1}$$

is a Vitali cover of A. By virtue of Vitali's covering theorem (see Theorem 1.2.5), we can find a family of disjoint intervals

$$\left\{[x_n, x_n + h_n]\right\}_{n=1}^m,$$

such that

$$\sum_{n=1}^m h_n > \frac{\beta}{2}.$$

Therefore

$$\sum_{n=1}^m \big(f(x_n + h_n) - f(x_n)\big) \geqslant \sum_{n=1}^m M h_n$$
$$> \frac{M\beta}{2} > f(b) - f(a),$$

a contradiction. This proves that

$$\lambda^*(A) = 0$$

and so

$$D^+ f(x) < +\infty.$$

Analogously we can prove that $D^- f(x) < +\infty$. ∎

Using this lemma and Vitali's covering theorem, we can now prove that a nondecreasing function is differentiable almost everywhere on $[a, b]$.

THEOREM 1.2.10
If $f : [a, b] \longrightarrow \mathbb{R}$ is nondecreasing,
then f is differentiable almost everywhere on $[a, b]$.

PROOF For f to be differentiable at x, we must have that all four derivates at x are finite and equal. By virtue of Lemma 1.2.9, it suffices to show that all four derivates are equal almost everywhere. Let

$$A \stackrel{df}{=} \left\{x \in (a, b) : \; D_+ f(x) < D^+ f(x)\right\}.$$

We show that A is Lebesgue-null. The proof for the other combinations of derivates is similar. Suppose that

$$\lambda^*(A) \ > \ 0.$$

We can find rational numbers r, s, such that the set

$$B \ \stackrel{df}{=} \ \big\{x \in A : \ D_+f(x) < r < s < D^+f(x)\big\}$$

satisfies

$$\lambda^*(B) \ = \ \beta > 0.$$

Let $\varepsilon \in (0, \beta)$. From the regularity of the Lebesgue outer measure λ^*, we know that there exists an open set $U \subseteq (a, b)$, such that

$$B \subseteq U \quad \text{and} \quad \lambda^1(U) - \varepsilon < \beta.$$

For each $x \in B$ and $n \geqslant 1$, we can find

$$h_n^x \ > \ 0, \quad \text{with } h_n^x \searrow 0,$$

such that

$$\big[x, x + h_n^x\big] \subseteq U \quad \text{and} \quad \frac{f(x + h_n^x) - f(x)}{h_n^x} < r.$$

The family

$$\big\{[x, x + h_n^x]\big\}_{x \in B, n \geqslant 1}$$

is a Vitali cover of B. By virtue of Vitali's covering theorem (see Theorem 1.2.5), for a given $\varepsilon > 0$, we can find a disjoint subfamily

$$\big\{[x_n, x_n + h_n]\big\}_{n=1}^m$$

of the Vitali cover, such that

$$\lambda^*\left(B \setminus \bigcup_{n=1}^m [x_n, x_n + h_n]\right) \ < \ \varepsilon.$$

We have

$$\sum_{n=1}^m \big(f(x_n + h_n) - f(x_n)\big) \ < \ r \sum_{n=1}^m h_n \ \leqslant \ r\lambda^1(U) \ < \ r(\beta + \varepsilon). \tag{1.5}$$

Let us set

$$C \ \stackrel{df}{=} \ B \cap \left(\bigcup_{n=1}^m [x_n, x_n + h_n]\right).$$

We have that

$$\beta - \varepsilon \ < \ \lambda^*(C).$$

For every $y \in C$ and $k \geqslant 1$, we can find $u_k^y \in \big(y, y + \frac{1}{k}\big)$, such that

$$\frac{f(u_k^y) - f(y)}{u_k^y - y} \;>\; s$$

and

$$[y, u_k^y] \subseteq (x_n, x_n + h_n), \quad \text{for some } n \in \{1, \ldots, m\}.$$

The family

$$\big\{[y, u_k^y]\big\}_{y \in C, k \geqslant 1}$$

is a Vitali cover of C. Invoking Vitali's covering theorem (see Theorem 1.2.5), we can find a disjoint subfamily

$$\big\{[y_k, u_k]\big\}_{k=1}^{l},$$

such that

$$\lambda^*(C) - \varepsilon \;<\; \sum_{k=1}^{l} (u_k - y_k).$$

Hence, we have

$$\begin{aligned} \sum_{k=1}^{l} \big(f(u_k) - f(y_k)\big) \;&>\; s \sum_{k=1}^{l} (u_k - y_k) \\ > s\big(\lambda^*(C) - \varepsilon\big) \;&>\; s(\beta - 2\varepsilon). \end{aligned} \tag{1.6}$$

For each $1 \leqslant n \leqslant m$, let

$$J_n \;\stackrel{df}{=}\; \big\{k \in \{1, \ldots, l\} :\; [y_k, u_k] \subseteq (x_n, x_n + h_n)\big\}.$$

Since f is nondecreasing, using (1.6) and (1.5), we have

$$\begin{aligned} s(\beta - 2\varepsilon) \;&<\; \sum_{k=1}^{l} \big(f(u_k) - f(y_k)\big) \;=\; \sum_{n=1}^{m} \sum_{k \in J_n} \big(f(u_k) - f(y_k)\big) \\ \leqslant \sum_{n=1}^{m} \big(f(x_n + h_n) - f(x_n)\big) \;&<\; r(\beta + \varepsilon). \end{aligned}$$

Let $\varepsilon \searrow 0$, to conclude that $s \leqslant r$, a contradiction. □

COROLLARY 1.2.11
If $f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation, then f is differentiable almost everywhere.

PROOF Recall that $f = f_1 - f_2$ with f_1 and f_2 nondecreasing and apply Theorem 1.2.10. □

THEOREM 1.2.12
If $f\colon [a,b] \longrightarrow \mathbb{R}$ *is absolutely continuous (see Definition A.2.15(b)) and*

$$f'(x) = 0 \quad \text{for a.a. } x \in [a,b],$$

then f *is constant on* $[a,b]$.

PROOF We show that

$$f(u) \;=\; f(a) \qquad \forall\, u \in (a,b].$$

So fix a $u \in (a,b]$ and let

$$A \stackrel{df}{=} \big\{x \in (a,u) :\; f'(x) = 0\big\}$$

and $\varepsilon, \xi > 0$. Because f is absolutely continuous, we can find $\delta > 0$, such that, if $\big\{(r_n, s_n)\big\}_{n=1}^{m}$ is a finite family of disjoint subintervals of $[a,b]$ with

$$\sum_{n=m}^{m} (s_n - r_n) \;<\; \delta,$$

then we have

$$\sum_{n=1}^{m} \big|f(s_n) - f(r_n)\big| \;<\; \varepsilon.$$

We introduce the family

$$\mathcal{T} \stackrel{df}{=} \left\{[x,y] :\; x \in A,\; x < y < u \text{ and } \left|\frac{f(y) - f(x)}{y - x}\right| < \xi\right\}.$$

Clearly $\mathcal{T}$ is a Vitali cover of A. So by Vitali's covering theorem (see Theorem 1.2.5), we can find a finite subfamily $\big\{[x_n, y_n]\big\}_{n=1}^{m}$ of disjoint sets in $\mathcal{T}$, such that

$$\lambda^1\left(A \setminus \bigcup_{n=1}^{m} [x_n, y_n]\right) \;<\; \delta.$$

We may assume that

$$a \;<\; x_1 \;<\; y_1 \;<\; x_2 \;<\; y_2 \;<\; \ldots \;<\; x_m \;<\; y_m \;<\; u.$$

Since

$$\lambda^1\big((a,u) \setminus A\big) \;=\; 0,$$

we have

$$\lambda^1\left((a,u) \setminus \bigcup_{n=1}^{m} [x_n, y_n]\right) \;<\; \delta$$

and

$$\lambda^1\Big((a,u)\setminus\bigcup_{n=1}^{m}[x_n,y_n]\Big) \;=\; (x_1-a)+\sum_{n=2}^{m}(x_n-y_{n-1})+(u-y_m).$$

Then, it follows that

$$\begin{aligned}\big|f(u)-f(a)\big| \;&\leqslant\; \sum_{n=1}^{m}\big|f(y_n)-f(x_n)\big|+\big|f(x_1)-f(a)\big|\\ &+\sum_{n=2}^{m}\big|f(x_n)-f(y_{n-1})\big|+\big|f(u)-f(y_m)\big|\\ &<\; \xi\sum_{n=1}^{m}(y_n-x_n)+\varepsilon \;\leqslant\; \xi(u-a)+\varepsilon.\end{aligned}$$

Let $\varepsilon,\xi\searrow 0$, to obtain that $f(u)=f(a)$. □

COROLLARY 1.2.13
If $f\colon[a,b]\longrightarrow\mathbb{R}$ is absolutely continuous, then f' is Lebesgue integrable on $[a,b]$ and for all $x\in[a,b]$, we have

$$f(x)-f(a) \;=\; \int_a^x f'(s)\,ds.$$

PROOF From the fundamental theorem of Lebesgue calculus (see Theorem A.2.20), we know that

$$\varphi(x) \;=\; \int_a^x f'(s)\,ds$$

is absolutely continuous on $[a,b]$ and

$$\varphi'(t) \;=\; f'(t) \quad \text{for a.a. } t\in[a,b].$$

Then Theorem 1.2.12 implies that $f=\varphi+\xi$, with a real constant ξ. Evaluating at $x=a$, we have

$$f(a) \;=\; \varphi(a)+\xi=\xi.$$

Therefore

$$\int_a^x f'(s)\,ds \;=\; f(x)-f(a) \qquad \forall\; x\in[a,b].$$

□

As a third application of Vitali's covering theorem (see Theorem 1.2.5), we show that a very weak local growth condition on a strictly increasing function f leads to a strong local growth condition. More precisely, we prove the following result.

THEOREM 1.2.14
If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a strictly increasing function, $r > 0$, $A \subseteq [a,b]$ and at each point $x \in A$ there exists a derived number β (see Remark 1.2.7), such that $\beta < r$,
then $\lambda^*\big(f(A)\big) \leqslant r\lambda^*(A)$.

PROOF If $\lambda^*(A) = +\infty$, then the inequality is obvious. So assume that

$$\lambda^*(A) \;<\; +\infty.$$

For a given $\varepsilon > 0$, we can find a bounded open set $U \subseteq \mathbb{R}$, such that

$$A \subseteq U \quad \text{and} \quad \lambda^1(U) - \varepsilon < \lambda^*(A).$$

If $x \in A$, then by hypothesis we can find a sequence $\{h_n\}_{n\geqslant 1} \subseteq \mathbb{R} \setminus \{0\}$, such that

$$h_n \;\longrightarrow\; 0,$$

$$[x, x+h_n] \;\subseteq\; U \qquad \forall\, n \geqslant 1$$

(or $[x+h_n, x] \subseteq U$ in the event $h_n < 0$; but in the sequel for simplicity we shall write $[x, x+h_n]$ for both cases) and

$$\frac{f(x+h_n) - f(x)}{h_n} \;<\; r \qquad \forall\, n \geqslant 1. \tag{1.7}$$

For all $n \geqslant 1$ and $x \in A$, let

$$D_n(x) \;\stackrel{df}{=}\; [x, x+h_n],$$
$$E_n(x) \;\stackrel{df}{=}\; \big[f(x), f(x+h_n)\big].$$

Because f is strictly increasing $E_n(x)$ is a nondegenerate, closed interval and

$$f\big(D_n(x)\big) \;\subseteq\; E_n(x) \qquad \forall\, n \geqslant 1,\; x \in A.$$

Since

$$\lambda^1\big(D_n(x)\big) \;=\; |h_n| \quad \text{and} \quad \lambda^1\big(E_n(x)\big) \;=\; \big|f(x+h_n) - f(x)\big|,$$

from (1.7), we have

$$\lambda^1\big(E_n(x)\big) \;<\; r\lambda^1\big(D_n(x)\big). \tag{1.8}$$

Passing to the limit as $n \to +\infty$, we have

$$|h_n| \longrightarrow 0$$

and so from (1.8), we obtain that

$$\lim_{n\to+\infty} \lambda^1\big(E_n(x)\big) = 0.$$

Let

$$\mathcal{T} \stackrel{df}{=} \big\{E_n(x)\big\}_{x\in A, n\geqslant 1}.$$

Then $\mathcal{T}$ is a Vitali cover of the set $f(A)$. So Vitali's covering theorem (see Theorem 1.2.5) implies the existence a disjoint sequence

$$\big\{E_{n_k}(x_k)\big\}_{k\geqslant 1} \subseteq \mathcal{T},$$

such that

$$\lambda^1\Big(f(A) \setminus \bigcup_{k=1}^{\infty} E_{n_k}(x_k)\Big) = 0. \tag{1.9}$$

Using (1.9) and (1.8), it follows that

$$\begin{aligned} \lambda^*\big(f(A)\big) &\leqslant \lambda^1\Big(\bigcup_{k=1}^{\infty} E_{n_k}(x_k)\Big) \\ = \sum_{k=1}^{\infty} \lambda^1\left(E_{n_k}(x_k)\right) &< r\sum_{k=1}^{\infty} \lambda^1\big(D_{n_k}(x_k)\big). \end{aligned} \tag{1.10}$$

Since f is strictly increasing, we see that $\big\{D_{n_k}(x_k)\big\}_{k\geqslant 1}$ are pairwise disjoint too. So we have

$$\sum_{k=1}^{\infty} \lambda^1\big(D_{n_k}(x_k)\big) = \lambda^1\Big(\bigcup_{k=1}^{\infty} D_{n_k}(x_k)\Big) \tag{1.11}$$

$$\leqslant \lambda^1(U) \leqslant \lambda^*(A) + \varepsilon. \tag{1.12}$$

From (1.10) and (1.11), we infer that

$$\lambda^*\big(f(A)\big) \leqslant r\big(\lambda^*(A) + \varepsilon\big).$$

Let $\varepsilon \searrow 0$, to conclude that

$$\lambda^*\big(f(A)\big) \leqslant r\lambda^*(A).$$

□

In a similar fashion, we can have the following comparison result.

THEOREM 1.2.15
If $f\colon [a,b] \longrightarrow \mathbb{R}$ *is a strictly increasing function,* $s > 0$, $A \subseteq [a,b]$ *and at each point* $x \in A$ *there exists a derived number* γ, *such that* $\gamma > s$,
then

$$\lambda^*\big(f(A)\big) \geqslant s\lambda^*(A).$$

The final application of Vitali's covering theorem (see Theorem 1.2.5) is the following criterion for measurability of sets in $\mathbb{R}$.

THEOREM 1.2.16
If $\mathcal{F}$ *is any collection of intervals in* $\mathbb{R}$ *and*

$$A = \bigcup_{D \in \mathcal{F}} D,$$

then A *is Lebesgue measurable.*

PROOF Let

$$\mathcal{T} \text{ be a collection of all intervals } E,$$

such that $E \subseteq D$ for some $D \in \mathcal{F}$. Evidently

$$\mathcal{T} \text{ is a Vitali cover of } A$$

and so by Vitali's covering theorem (see Theorem 1.2.5), we can find a sequence

$$\{E_n\}_{n \geqslant 1}$$

of disjoint elements in $\mathcal{T}$, such that

$$\lambda^*\left(A \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0.$$

Because each $E_n \subseteq A$, the set

$$A \stackrel{df}{=} \bigcup_{n=1}^{\infty} E_n \cup \left(A \setminus \bigcup_{n=1}^{\infty} E_n\right)$$

is Lebesgue measurable. □

REMARK 1.2.17 Theorem 1.2.16 can be used to show that the upper and lower derivates of an arbitrary function are measurable. In particular then the four derivates of a measurable function are measurable and so is the derivative of a measurable function. We will not go into that here.

When λ^N is replaced by an arbitrary Radon measure μ on $\mathbb{R}^N$, there is no systematic way to control $\mu(\widehat{B})$ in terms of $\mu(B)$. So the proof of Vitali's covering theorem (see Theorem 1.2.5) which uses the principle involved in Proposition 1.2.1, namely the use of suitable expansions of balls, does not work. So we need an analog of Proposition 1.2.1, which does not require enlarging the balls, though. This is done by the so-called "Besicovitch covering theorem."

THEOREM 1.2.18 (Besicovitch Covering Theorem)
If $\mathcal{F}$ is any collection of closed balls in $\mathbb{R}^N$,

$$\sup_{B\in\mathcal{F}} \delta(B) \;<\; +\infty$$

and

$$A \text{ is the set of centers of all balls } B \in \mathcal{F},$$

then there exist a positive integer $k = k(N) \geqslant 1$ and

$$\mathcal{T}_n \subseteq \mathcal{F} \qquad \forall\, n \in \{1, \ldots, k\},$$

such that each $\mathcal{T}_n$ is a countable collection of disjoint balls in $\mathcal{F}$ and

$$A \;\subseteq \bigcup_{n=1}^{k} \bigcup_{B\in\mathcal{T}_n} B.$$

Using the above theorem, we can have the following counterpart of Vitali's covering theorem (see Theorem 1.2.5).

THEOREM 1.2.19
If μ is a Borel measure on $\mathbb{R}^N$, $\mathcal{T}$ is a family of nondegenerate closed balls in $\mathbb{R}^N$,

$$A \text{ is the set of centers of balls in } \mathcal{T},$$

$\mu(A) < +\infty$,

$$\inf_{B_r(a)\in\mathcal{F}} r \;=\; 0 \qquad \forall\, a \in A$$

and $U \subseteq \mathbb{R}^N$ is an open set,
then there exists a countable collection of disjoint balls $\mathcal{F}$ from $\mathcal{T}$, such that

$$\bigcup_{B\in\mathcal{F}} B \;\subseteq U \quad \text{and} \quad \mu\Big((A\cap U)\setminus \bigcup_{B\in\mathcal{F}} B\Big) \;=\; 0.$$

1.3 Hausdorff Measure and Hausdorff Dimension

Hausdorff measures were introduced as certain lower dimensional measures on $\mathbb{R}^N$ which allow us to measure "small" subsets in $\mathbb{R}^N$. The Hausdorff measure and the associated Hausdorff dimension of the set provide a more delicate sense of the size of a set in $\mathbb{R}^N$ than the Lebesgue measure provides.

We start with the introduction of a special class of outer measures, known as metric outer measures.

DEFINITION 1.3.1 *Let (X, d_X) be a metric space (d is the metric function).*

(a) *If $A, B \subseteq X$, then we say that A and B are* ***separated sets****, if*

$$d_X(A,B) \stackrel{df}{=} \inf_{\substack{a \in A \\ b \in B}} d_X(a,b) > 0.$$

(b) *If μ is an outer measure on X, then we say that μ is a* ***metric outer measure****, if*

$$\mu(A \cup B) = \mu(A) + \mu(B) \qquad \forall\ A, B \subseteq X,\ A \text{ and } B \text{ separated.}$$

We show that if μ is a metric outer measure, then $\mathcal{B}(X) \subseteq \Sigma(\mu)$, i.e., μ is Borel. To this end we need the following auxiliary result, known as ***Carathéodory's lemma***. In what follows (X, d) is a metric space.

LEMMA 1.3.2 (Carathéodory Lemma)
If μ is a metric outer measure on X, $U \subseteq X$ is an open subset, $U \neq X$, $A \subseteq U$ and

$$A_n \stackrel{df}{=} \left\{ x \in A :\ d(x, U^c) \geqslant \frac{1}{n} \right\} \qquad \forall\ n \geqslant 1, \tag{1.13}$$

then $\mu(A) = \lim_{n \to +\infty} \mu(A_n)$.

PROOF Note that the sequence $\{A_n\}_{n \geqslant 1}$ is an increasing sequence and so $\lim_{n \to +\infty} \mu(A_n)$ exists. Moreover, since $A_n \subseteq A$ for $n \geqslant 1$, we have

$$\lim_{n \to +\infty} \mu(A_n) \leqslant \mu(A).$$

So we need to show that

$$\mu(A) \leqslant \lim_{n \to +\infty} \mu(A_n). \tag{1.14}$$

Because U is open, we have

$$d(x, U^c) \ > \ 0 \qquad \forall\, x \in A$$

and so we can find $n_0 \geqslant 1$ large enough so that $x \in A_{n_0}$. Therefore, we have

$$A \ = \ \bigcup_{n=1}^{\infty} A_n.$$

For each $n \geqslant 1$, we introduce the set

$$C_n \ \stackrel{df}{=} \ A_{n+1} \setminus A_n \ = \ \left\{ x \in A : \ \frac{1}{n+1} \leqslant d(x, U^c) < \frac{1}{n} \right\}.$$

We have

$$A \ = \ A_{2n} \cup \bigcup_{k=2n}^{\infty} C_k \ = \ A_{2n} \cup \bigcup_{k=n}^{\infty} C_{2k} \cup \bigcup_{k=n}^{\infty} C_{2k+1}$$

and from the subadditivity of μ, it follows that

$$\mu(A) \ \leqslant \ \mu(A_{2n}) + \sum_{k=n}^{\infty} \mu(C_{2k}) + \sum_{k=n}^{\infty} \mu(C_{2k+1}). \tag{1.15}$$

If both series are convergent, then we obtain (1.14). So suppose that this is not true and, say, we have

$$\sum_{k=1}^{\infty} \mu(C_{2k}) \ = \ +\infty. \tag{1.16}$$

Note that

$$d\big(C_{2k}, C_{2k+2}\big) \ \geqslant \ \frac{1}{2k+1} - \frac{1}{2k+2} \qquad \forall\, k \geqslant 1$$

and so the sets $\{C_k\}_{k \geqslant 1}$ are separated. Therefore, we have

$$\mu\Big(\bigcup_{k=1}^{n-1} C_{2k} \Big) \ = \ \sum_{k=1}^{n-1} \mu(C_{2k}) \qquad \forall\, n \geqslant 1. \tag{1.17}$$

Note that

$$\bigcup_{k=1}^{n-1} C_{2k} \ \subseteq \ A_{2n} \qquad \forall\, n \geqslant 1$$

and so

$$\mu\Big(\bigcup_{k=1}^{n-1} C_{2k} \Big) \ \leqslant \ \mu(A_{2n}) \qquad \forall\, n \geqslant 1. \tag{1.18}$$

From (1.17) and (1.18), it follows that

$$\sum_{k=1}^{n-1} \mu(C_{2k}) \;\leqslant\; \mu(A_{2n}).$$

Combining this with (1.16), we infer that

$$\lim_{n\to+\infty} \mu(A_{2n}) \;=\; +\infty$$

and so

$$\mu(A) \;\leqslant\; \lim_{n\to+\infty} \mu(A_{2n}),$$

as desired. Similarly, if $\sum\limits_{k=1}^{\infty} \mu(C_{2k+1}) = +\infty$. □

THEOREM 1.3.3
If μ is an outer measure on X,
then $\mathcal{B}(X) \subseteq \Sigma(\mu)$ (i.e., μ is Borel) if and only if μ is a metric outer measure.

PROOF "$\Longrightarrow$": Let $A_1, A_2 \subseteq X$ be separated sets and let us set

$$\beta \;\stackrel{df}{=}\; d(A_1, A_2) \;>\; 0.$$

For every $x \in A_1$, we define

$$U(x) \;\stackrel{df}{=}\; B_{\frac{\beta}{2}}(x) \;=\; \left\{ y \in X :\; d(y,x) < \frac{\beta}{2} \right\} \quad \text{and} \quad U \;\stackrel{df}{=}\; \bigcup_{x\in A_1} U(x).$$

Evidently U is open, $A_1 \subseteq U$ and $A_2 \cap U = \emptyset$. Since by hypothesis $U \in \Sigma(\mu)$, we have that

$$\mu(A_1 \cup A_2) \;=\; \mu\big((A_1 \cup A_2) \cap U\big) + \mu\big((A_1 \cup A_2) \cap U^c\big). \tag{1.19}$$

Because $A_1 \subseteq U$ and $A_2 \cap U = \emptyset$, from (1.19), it follows that

$$\mu(A_1 \cup A_2) \;=\; \mu(A_1) + \mu(A_2),$$

i.e., μ is metric outer measure.

"$\Longleftarrow$": It suffices to show that $\Sigma(\mu)$ contains all closed sets. So let $C \subseteq X$ be closed and let us set $U \stackrel{df}{=} C^c$. Let $D \subseteq X$, $A \stackrel{df}{=} D \setminus C$ and let $\{A_n\}_{n\geqslant 1}$ be an increasing sequence of subsets of A as in Lemma 1.3.2. Then

$$d(A_n, C) \;\geqslant\; \frac{1}{n} \qquad \forall\, n \geqslant 1$$

and, from Lemma 1.3.2, we have

$$\mu(D \setminus C) \;=\; \mu(A) \;=\; \lim_{n \to +\infty} \mu(A_n). \tag{1.20}$$

Since by hypothesis μ is a metric outer measure and the sets $\{A_n\}_{n \geqslant 1}$ are separated from C, we have

$$\mu(D) \;\geqslant\; \mu\big((D \cap C) \cup A_n\big) \;=\; \mu(D \cap C) + \mu(A_n) \qquad \forall\, n \geqslant 1.$$

Passing to the limit as $n \to +\infty$ and using (1.20), we obtain

$$\mu(D) \;\geqslant\; \mu(D \cap C) + \mu(D \setminus C).$$

The reverse inequality is always true (subadditivity). So we obtain

$$\mu(D) \;=\; \mu(D \cup C) + \mu(D \setminus C) \qquad \forall\, D \subseteq X.$$

Thus $C \in \Sigma(\mu)$ and hence $\mathcal{B}(X) \subseteq \Sigma(\mu)$. □

To introduce the concept of Hausdorff measure, we shall need the following notion. Recall that by (X, d) we denote a metric space.

DEFINITION 1.3.4 *A sequence $\{A_n\}_{n \geqslant 1}$ of subsets of X is a* ***δ-cover*** *of a set C, if*

$$C \subseteq \bigcup_{n=1}^{\infty} A_n \quad \textit{and} \quad \delta(A_n) \leqslant \delta \qquad \forall\, n \geqslant 1.$$

By $\mathcal{T}_\delta(C)$ we denote the family of all δ-covers of the set C.

Using this notion, we can introduce the Hausdorff s-dimensional measure, $s > 0$. As usual, for any $A \subseteq X$,

$$\delta(A) \;=\; \operatorname{diam}(A) \;\stackrel{df}{=}\; \sup_{x,y \in A} d(x, y),$$

the diameter of A (by convention $\operatorname{diam} \emptyset \stackrel{df}{=} 0$).

DEFINITION 1.3.5 *For any $s \geqslant 0$, $0 < \delta \leqslant +\infty$ and $C \subseteq X$, we define*

$$\mu_\delta^{(s)}(C) \;\stackrel{df}{=}\; \inf_{\{A_n\}_{n \geqslant 1} \in \mathcal{T}_\delta(C)} \sum_{n=1}^{\infty} \delta(A_n)^s$$

(as always we use the convention that $\inf \emptyset = +\infty$). The ***Hausdorff s-dimensional outer measure*** *$\mu^{(s)}$ is defined by*

$$\mu^{(s)}(C) \;\stackrel{df}{=}\; \lim_{\delta \searrow 0} \mu_\delta^{(s)}(C) \;=\; \sup_{\delta > 0} \mu_\delta^{(s)}(C).$$

REMARK 1.3.6 It is easily seen that $\mu^{(s)}$ is an outer measure. Moreover, it is a metric outer measure. Indeed, if $\delta > 0$ is less than the positive distance of two separate sets A and C, then no set in $\mathcal{T}_\delta(A \cup C)$ can intersect both A and C and so it follows that

$$\mu_\delta^{(s)}(A \cup C) \;=\; \mu_\delta^{(s)}(A) + \mu_\delta^{(s)}(C).$$

Letting $\delta \searrow 0$, we can obtain the same equality for $\mu^{(s)}$. In addition by Theorem 1.3.3, $\mu^{(s)}$ is Borel. The restriction of $\mu^{(s)}$ on $\Sigma\big(\mu^{(s)}\big)$ is called the ***Hausdorff s-dimensional measure***. Sometimes it is convenient to consider δ-covers consisting of open or alternatively closed sets. In these cases, although a different value of $\mu_\delta^{(s)}$ may be attained for $\delta > 0$, the limit $\mu^{(s)}$ as $\delta \searrow 0$ is the same (see Davies (1970)). However, the limit $\mu^{(s)}$ is different, if we restrict ourselves to δ-covers by balls (see Besicovitch (1928)). In this case the resulting Hausdorff measure is called the ***spherical Hausdorff measure***. Finally, if $X = \mathbb{R}^N$, it is easy to see that $\mu^{(s)}$ remains the same if we consider δ-covers consisting only of convex sets.

Next we show that for any set $C \subseteq X$, there is a critical value s_0, such that for $s > s_0$, the corresponding Hausdorff s-dimensional measure of C is zero, while for $s < s_0$ the Hausdorff s-dimensional measure of C is infinite.

THEOREM 1.3.7
If $A \subseteq \mathbb{R}^N$ *and* $0 \leqslant s < t < +\infty$, *then*

(a) *if* $\mu^{(s)}(A) < +\infty$, *then* $\mu^{(t)}(A) = 0$;

(b) *if* $\mu^{(t)}(A) > 0$, *then* $\mu^{(s)}(A) = +\infty$.

PROOF **(a)** Let $\mu^{(s)}(A) < +\infty$ and $t > s$. Let $\{A_n\}_{n \geqslant 1} \in \mathcal{T}_{\frac{1}{m}}(A)$. Then for any $n \geqslant 1$, we have

$$\frac{\delta(A_n)^t}{\delta(A_n)^s} \;=\; \delta(A_n)^{t-s} \;\leqslant\; \left(\frac{1}{m}\right)^{t-s},$$

so

$$\mu_{\frac{1}{m}}^{(t)}(A) \;\leqslant\; \sum_{n=1}^{\infty} \delta(A_n)^t \;\leqslant\; \left(\frac{1}{m}\right)^{t-s} \sum_{n=1}^{\infty} \delta(A_n)^s$$

and thus

$$\mu_{\frac{1}{t}}^{(t)}(A) \;\leqslant\; \left(\frac{1}{m}\right)^{t-s} \mu_{\frac{1}{m}}^{(s)}(A).$$

Letting $m \to +\infty$, we obtain $\mu^{(t)}(A) = 0$.

(b) Let $\mu^{(t)}(A) > 0$ and $s < t$.

Assuming that $\mu^{(s)}(A) < +\infty$, from **(a)**, we get that $\mu^{(t)}(A) = 0$, a contradiction. ∎

This theorem leads to the following definition.

DEFINITION 1.3.8 *Let $C \subseteq X$. If there is no $s > 0$, such that $\mu^{(s)}(C) = +\infty$, then $\dim C \stackrel{df}{=} 0$. Otherwise, let*

$$\dim C \stackrel{df}{=} \sup_{\substack{s>0 \\ \mu^{(s)}(C)=+\infty}} s.$$

*Then $\dim C$ is called the **Hausdorff dimension** of C.*

Consider the Cantor ternary set C. It is well known that C is a nonempty, bounded, nowhere dense, perfect set in $\mathbb{R}$ which has Lebesgue measure zero. So the Lebesgue measure can contribute no additional information concerning the size of C. On the other hand, as we shall see the Hausdorff dimension provides a more delicate sense of size.

PROPOSITION 1.3.9
If $C \subseteq [0,1]$ is the Cantor ternary set, then $\dim C = \frac{\ln 2}{\ln 3}$.

PROOF We start with two simple observations concerning the Hausdorff s-dimensional outer measure $\mu^{(s)}$ on $\mathbb{R}$. First note that $\mu^{(s)}$ is translation invariant, namely

$$\mu^{(s)}(A) = \mu^{(s)}(A+x) \qquad \forall\, A \subseteq \mathbb{R},\ x \in \mathbb{R}$$

(here $A + x \stackrel{df}{=} \{a + x :\ a \in A\}$). Second, $\mu^{(s)}$ is s-positive homogeneous, i.e., for every $\vartheta > 0$,

$$\mu^{(s)}(\vartheta A) = \vartheta^s \mu^{(s)}(A) \qquad \forall\, \vartheta > 0.$$

In the construction of C we start by removing from $[0,1]$ the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$. The resulting set consists of two closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Let

$$C^1 \stackrel{df}{=} C \cap \left[0, \frac{1}{3}\right] \quad \text{and} \quad C^2 \stackrel{df}{=} C \cap \left[\frac{2}{3}, 1\right].$$

Evidently C^1 and C^2 are translates of a multiple (by $\frac{1}{3}$) of C. So we have

$$\begin{aligned} \mu^{(s)}(C) &= \mu^{(s)}(C^1 \cup C^2) = \mu^{(s)}(C^1) + \mu^{(s)}(C^2) \\ &= 2\mu^{(s)}(C^2) = 2\left(\frac{1}{3}\right)^s \mu^{(s)}(C) \end{aligned} \tag{1.21}$$

(see Remark 1.3.6 and the observations in the beginning of this proof). From (1.21), it follows that

$$\mu^{(s)}(C) = 0 \quad \text{or} \quad \mu^{(s)}(C) = +\infty \quad \text{or} \quad 2\left(\frac{1}{3}\right)^s = 1.$$

From the last possibility, it follows that

$$s = \frac{\ln 2}{\ln 3}.$$

If we can show that $0 < \mu^{(s)}(C) < +\infty$, then $s = \frac{\ln 2}{\ln 3}$ is the Hausdorff dimension of C (see Theorem 1.3.7).

First we show that $\mu^{(s)}(C) > 0$. Note that

$$d(C^1, C^2) \geqslant \frac{1}{3}.$$

Let $\delta \leqslant \frac{1}{3}$. Then any collection $\{A_n\}_{n\geqslant 1} \in \mathcal{T}_\delta(C)$ (which can be taken to consist of open intervals; see Remark 1.3.6) can be decomposed into two subcollections of intervals $\{A_{n,1}\}_{n\geqslant 1} \in \mathcal{T}_\delta(C^1)$ and $\{A_{n,2}\}_{n\geqslant 1} \in \mathcal{T}_\delta(C^2)$, such that

$$\sum_{n=1}^{\infty} \delta(A_n)^s = \sum_{n=1}^{\infty} \delta(A_{n,1})^s + \sum_{n=1}^{\infty} \delta(A_{n,2})^s. \tag{1.22}$$

In the right hand side of (1.22) suppose that the first sum is smaller than the second. Because C^2 is a translate of C^1, the same translation when applied to the intervals $\{A_{n,1}\}_{n\geqslant 1}$ gives a subcollection $\{A'_{n,1}\}_{n\geqslant 1} \in \mathcal{T}_\delta(C^2)$. Also from $\{A_{n,1}\}_{n\geqslant 1}$ we can produce in a similar way a collection $\{A'_n\}_{n\geqslant 1}$ covering C, such that

$$\delta(A'_n) = 3\delta(A'_{n,1}) \qquad \forall\, n \geqslant 1. \tag{1.23}$$

Then, from (1.23) and the choice of s, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \delta(A_n)^s &\geqslant \sum_{n=1}^{\infty} \delta(A_{n,1})^s + \sum_{n=1}^{\infty} \delta(A'_{n,1})^s \\ = 2\sum_{n=1}^{\infty} \delta(A'_{n,1})^s &= 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^s \delta(A'_n)^s = \sum_{n=1}^{\infty} \delta(A'_n)^s. \end{aligned}$$

If any one of the intervals $\{A'_n\}_{n\geqslant 1}$ has length bigger or equal to $\frac{1}{3}$, we have

$$\sum_{n=1}^{\infty} \delta(A_n)^s \geqslant \left(\frac{1}{3}\right)^s = \frac{1}{2}.$$

Because C is compact, we can use only finite coverings and so

$$\min_{n\geqslant 1} \delta(A_n) > 0.$$

The intervals $\{A'_n\}_{n\geqslant 1}$ are multiples (by (1.23)) of a subfamily of the intervals $\{A_n\}_{n\geqslant 1}$, hence we have

$$3 \min_{n\geqslant 1} \delta(A'_n) \geqslant \min_{n\geqslant 1} \delta(A_n).$$

If every interval A'_n has length (diameter) less than $\frac{1}{3}$, we can apply the same process to the cover $\{A'_n\}_{n\geqslant 1}$. After a finite number of such steps, we produce a cover $\{A''_n\}_{n\geqslant 1}$, such that

$$\max_{n\geqslant 1}\delta(A''_n) \;\geqslant\; \frac{1}{3}$$

and

$$\sum_{n=1}^{\infty}\delta(A_n)^s \;\geqslant\; \sum_{n=1}^{\infty}\delta(A''_n)^s,$$

so

$$\sum_{n=1}^{\infty}\delta(A_n)^s \;\geqslant\; \left(\frac{1}{3}\right)^s \;=\; \frac{1}{2}$$

and thus

$$0 \;<\; \mu^{(s)}(C).$$

Next we show that

$$\mu^{(s)}(C) \;<\; +\infty.$$

Let $\{A_n\}_{n\geqslant 1} \in \mathcal{T}_\delta(C)$ consist of open intervals. From this family, as above, we obtain covers $\{A_{n,k}\}_{n\geqslant 1}$ of C^k for $k \in \{1,2\}$, such that

$$\delta(A_{n,k}) \leqslant \frac{\delta}{3} \qquad \forall\; n \geqslant 1.$$

Again from the choice of s, we have

$$\delta(A_n)^s \;=\; \delta(A_{n,1})^s + \delta(A_{n,2})^s,$$

so

$$\mu_\delta^{(s)}(C) \;\geqslant\; \mu_{\frac{\delta}{3}}^{(s)}(C).$$

Because $\mu_\delta^{(s)}$ is nondecreasing in $\delta > 0$, we infer that $\mu_\delta^{(s)}$ is independent of $\delta > 0$. So we can take an open interval of length greater than 1 as an open cover of C and conclude that $\mu^{(s)}(C) \leqslant 1$. This proves that

$$\dim E \;=\; \frac{\ln 2}{\ln 3}.$$

□

One can show that for every $\xi \in [0,1]$, there exists a set $A \subseteq \mathbb{R}$, such that $\dim A = \xi$. This can be done using Cantor-like sets. These are sets which share most of the properties of the Cantor ternary set, but need not be Lebesgue-null. We can construct a Cantor-like set as follows. We start with the interval $[0,1]$ and proceed inductively. We remove an open interval $B_{1,1}$ centered at $\frac{1}{2}$ with length less than 1. We are left with closed intervals

$D_{1,1}$ and $D_{1,2}$ each with length less than $\frac{1}{2}$. At the n-th step of this process we are left with closed intervals $D_{n,1}, D_{n,2}, \ldots, D_{n,2^n}$ each with length less than $\frac{1}{2^n}$. In the $(n+1)$-st step, from each closed interval $D_{n,k}$ we remove an open interval $E_{n+1,k}$ having the same center as $D_{n,k}$ and length less than the length of $D_{n,k}$. We set

$$S_n \stackrel{df}{=} \bigcup_{k=1}^{2^n} D_{n,k} \quad \text{and} \quad S \stackrel{df}{=} \bigcap_{n=1}^{\infty} S_n.$$

The set S is a Cantor-like set. It is known (see Hewitt & Stromberg (1975, p. 71)) that S is nonempty, compact, nowhere dense and perfect (just as the Cantor ternary set). However, unlike the Cantor ternary set, S need not be Lebesgue-null. More precisely, consider a sequence $\{\vartheta_n\}_{n\geqslant 1}$ of positive numbers, such that

$$1 \;>\; 2\vartheta_1 \;>\; 4\vartheta_2 \;>\; \ldots \;>\; 2^n\vartheta_n \;>\; \ldots.$$

Following the construction of S above, we remove from $[0,1]$ an open interval centered at $\frac{1}{2}$ and having length $1-2\vartheta_1$. The remaining closed intervals $D_{1,1}$ and $D_{1,2}$ each have length ϑ_1. Then from each of the intervals $D_{1,1}$ and $D_{1,2}$ we remove cocentric open intervals each of length $\vartheta_1 - 2\vartheta_2$. We are left with closed intervals $D_{2,1}$, $D_{2,2}$, $D_{2,3}$ and $D_{2,4}$ each of length ϑ_2. We continue this way. In the n-th step we are left with 2^n closed intervals each with length ϑ_n. Then we have

$$\lambda^1(S) \;=\; \lim_{n\to+\infty} 2^n\vartheta_n$$

(λ^1 being the Lebesgue measure on $\mathbb{R}$).

If $\vartheta_n = \frac{1}{3^n}$, then $S = C$ is the Cantor ternary set. Although S is nowhere dense, we can have $\lambda^1(S)$ as close to 1 as we choose. Indeed, for a given $\xi \in (0,1)$, let

$$\vartheta_n \stackrel{df}{=} \frac{1}{2^n}\frac{n\xi+1}{n+1} \qquad \forall\, n \geqslant 1.$$

Then we have $\lambda^1(S) = \xi$.

Suppose that in the construction of the Cantor-like set at each step the closed subintervals are divided in the same proportions as the original, namely

$$\delta(D_{1,1}) = \delta(D_{1,2}) = \vartheta$$

$$\delta(D_{2,1}) \;=\; \delta(D_{2,2}) \;=\; \delta(D_{2,3}) \;=\; \delta(D_{2,4}) \;=\; \vartheta^2$$

and in general

$$\delta(D_{n,k}) \;=\; \vartheta^k \qquad \forall\, k \in \{1,\ldots,2^n\}.$$

Then the resulting Cantor-like set is denoted by S_ϑ. Arguing as in the proof of Proposition 1.3.9, we obtain the following Proposition.

PROPOSITION 1.3.10
If $\vartheta \in \left(0, \frac{1}{2}\right)$, *then* $\dim S_\vartheta = -\frac{\ln 2}{\ln \vartheta}$.

REMARK 1.3.11 If $\vartheta = \frac{1}{3}$, then $S = C$ is the Cantor ternary set and Propositions 1.3.9 and 1.3.10 coincide.

COROLLARY 1.3.12
For each $\xi \in [0,1]$, *there exists* $A \subseteq \mathbb{R}$, *such that* $\dim A = \xi$.

PROOF If $\xi = 0$, then we take A to be a singleton.
If $0 < \xi < 1$, then take $\vartheta = \exp\left(-\frac{\ln 2}{\xi}\right) < \frac{1}{2}$ and use Proposition 1.3.10.
If $\xi = 1$, let $A = I = [0,1]$. Then we can easily check that

$$\mu^{(s)}(A) = \begin{cases} +\infty & \text{if} \quad 0 < s < 1, \\ 1 & \text{if} \quad s = 1, \\ 0 & \text{if} \quad s > 1. \end{cases}$$

Therefore $\dim A = 1$. □

REMARK 1.3.13 An alternative way to define the Hausdorff dimension of a set $A \subseteq X$ is by

$$\dim A \stackrel{df}{=} \inf_{\substack{s > 0 \\ \mu^{(s)}(A) = 0}} s.$$

In general the Hausdorff dimension of a set may be any number in $[0, +\infty]$ and need not be an integer. Even if $\dim A$ is an integer and $k = \dim A > 0$, the set A need not be a "k-dimensional surface" in any sense (see Federer (1969)).

Next we turn our attention to the case $X = \mathbb{R}^N$. Let us begin by recalling the definition of the N-dimensional outer measure λ^N.

DEFINITION 1.3.14 **(a)** *We say that* $Q \subseteq \mathbb{R}^N$ *is a* ***closed*** N***-cube***, *if there exist* $a_k < b_k$ *for* $k = 1, \ldots, N$, *such that* $Q = \prod_{k=1}^{N} [a_k, b_k]$. *We set*

$$|Q| \stackrel{df}{=} \prod_{k=1}^{N} (b_k - a_k).$$

(b) *The* ***Lebesgue*** N***-dimensional outer measure*** λ^N, *for all* $A \subseteq \mathbb{R}^N$, *is defined by*

$$\lambda^N(A) \stackrel{df}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : \; A \subseteq \bigcup_{k=1}^{\infty} Q_k, \; Q_k \text{ is closed } N\text{-cube} \right\}.$$

REMARK 1.3.15 Clearly the definitions of λ^1 and $\mu^{(1)}$ on $\mathbb{R}$ coincide. We shall show that for any $N > 1$ the outer measures λ^N and $\mu^{(N)}$ are closely related. In fact they differ by a multiplicative constant. This is not easy to establish and requires some preparation which culminates to the so-called "isodiametric inequality," which says that the set of maximal volume for a given diameter is the sphere.

LEMMA 1.3.16
If $f\colon \mathbb{R}^N \longrightarrow [0, +\infty]$ *is Lebesgue measurable, then the set*

$$H \stackrel{df}{=} \left\{(x, \vartheta) \in \mathbb{R}^N \times \mathbb{R} :\ 0 \leqslant \vartheta \leqslant f(x)\right\}$$

is Lebesgue measurable in $\mathbb{R}^{N+1}$.

PROOF Let

$$A \stackrel{df}{=} \left\{x \in \mathbb{R}^N :\ f(x) = +\infty\right\}.$$

Then A is Lebesgue measurable. Let $g\colon A^c \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by

$$g(x, \vartheta) \stackrel{df}{=} f(x) - \vartheta \qquad \forall\ (x, \vartheta) \in A^c \times \mathbb{R}_+.$$

Evidently g is a Carathéodory function (i.e., it is Lebesgue measurable in $x \in \mathbb{R}^N$ and continuous in $\vartheta \in \mathbb{R}$). Therefore g is Lebesgue measurable on $A^c \times \mathbb{R}_+$ and so

$$H_0 \stackrel{df}{=} \left\{(x, \vartheta) \in A^c \times \mathbb{R}_+ :\ \vartheta \leqslant f(x)\right\}$$

is Lebesgue measurable in $\mathbb{R}^{N+1}$. Finally note that

$$H \ = \ H_0 \cup (A \times \mathbb{R}_+).$$

□

In what follows for $a, b \in \mathbb{R}^N$, $\|a\|_{\mathbb{R}^N} = 1$, we introduce the following objects:

$$L(a, b) \stackrel{df}{=} \left\{b + ta :\ t \in \mathbb{R}\right\}$$

- the line passing from b in the direction of a and

$$P(a) \stackrel{df}{=} \left\{x \in \mathbb{R}^N :\ (x, a)_{\mathbb{R}^N} = 0\right\}$$

- the plane passing from the origin, perpendicular to a.

DEFINITION 1.3.17 *Let $a \in \mathbb{R}^N$ with $\|a\|_{\mathbb{R}^N} = 1$ and $A \subseteq \mathbb{R}^N$. We define the* ***Steiner symmetrization*** *of A with respect to the plane $P(a)$ to be the set*

$$S(a,A) \stackrel{df}{=} \bigcup_{\substack{b \in P(a) \\ A \cap L(a,b) \neq \emptyset}} \left\{ b + ta : \ |t| \leqslant \frac{1}{2}\mu^{(1)}\big(A \cap L(a,b)\big) \right\}.$$

REMARK 1.3.18 The above defined Steiner symmetrization with respect to an $(N-1)$-dimensional subspace Y of $\mathbb{R}^N$ is the operation which associates to each $A \subseteq \mathbb{R}^N$, the set $V \subseteq \mathbb{R}^N$, such that for every L perpendicular to Y either
• $L \cap A = \emptyset$ and $L \cap V = \emptyset$;
or
• $L \cap A \neq \emptyset$ and $L \cap V$ is a closed segment centered in Y and

$$\mu^{(1)}(L \cap A) \ = \ \mu^{(1)}(L \cap V).$$

If A is compact, then V is compact too and

$$\lambda^N(A) \ = \ \lambda^N(V).$$

Also if A is convex, then V is convex too.

The next Proposition summarizes the properties of the Steiner symmetrization.

PROPOSITION 1.3.19
Let $A \subseteq \mathbb{R}^N$ and $a \in \mathbb{R}^N$.

(a) $\delta\big(S(a,A)\big) \leqslant \delta(A)$.

(b) *If $A \subseteq \mathbb{R}^N$ is Lebesgue measurable, then so is $S(a,A)$ and $\lambda^N\big(S(a,A)\big) = \lambda^N(A)$.*

PROOF **(a)** Assume that

$$\delta(A) \ < \ +\infty$$

or otherwise the result is trivial. Also we may assume that A is closed. For a given $\varepsilon > 0$, let $x, y \in S(a,A)$ be such that

$$\delta\big(S(a,A)\big) - \varepsilon \ \leqslant \ \|x - y\|_{\mathbb{R}^N} .$$

Let

$$b \stackrel{df}{=} x - (x,a)_{\mathbb{R}^N} a \quad \text{and} \quad c \stackrel{df}{=} y - (y,a)_{\mathbb{R}^N} a.$$

Then $b, c \in P(a)$. Let us set

$$r \stackrel{df}{=} \inf\{t \in \mathbb{R} : \ b + ta \in A\}, \qquad s \stackrel{df}{=} \sup\{t \in \mathbb{R} : \ b + ta \in A\},$$
$$u \stackrel{df}{=} \inf\{t \in \mathbb{R} : \ c + ta \in A\}, \qquad v \stackrel{df}{=} \sup\{t \in \mathbb{R} : \ c + ta \in A\}.$$

We may assume that without any loss of generality that

$$v - r \ \geqslant \ s - u.$$

So

$$\begin{aligned} v - r \ &\geqslant \ \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \ = \ \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \\ &\geqslant \ \frac{1}{2}\mu^{(1)}\big(A \cap L(a, b)\big) + \frac{1}{2}\mu^{(1)}\big(A \cap L(a, c)\big). \end{aligned}$$

Note that

$$\big|\,(x, a)_{\mathbb{R}^N}\,\big| \ \leqslant \ \frac{1}{2}\mu^{(1)}\big(A \cap L(a, b)\big)$$

and

$$\big|\,(y, a)_{\mathbb{R}^N}\,\big| \ \leqslant \ \frac{1}{2}\mu^{(1)}\big(A \cap L(a, c)\big)$$

(recall that $x, y \in S(a, b)$). It follows that

$$v - r \ \geqslant \ \big|\,(x, a)_{\mathbb{R}^N}\,\big| + \big|\,(y, a)_{\mathbb{R}^N}\,\big| \ \geqslant \ \big|\,(x - y, a)_{\mathbb{R}^N}\,\big|.$$

Hence we have

$$\begin{aligned} \big(\delta\big(S(a, A)\big) - \varepsilon\big)^2 \ &\leqslant \ \|x - y\|^2_{\mathbb{R}^N} \\ \leqslant \ \|b - c\|^2_{\mathbb{R}^N} + \big|\,(x - y, a)_{\mathbb{R}^N}\,\big|^2 \\ \leqslant \ \|b - c\|^2_{\mathbb{R}^N} + (v - r)^2 \\ = \ \big\|(b + ra) - (c + va)\big\|^2_{\mathbb{R}^N} \ &\leqslant \ \delta(A)^2 \end{aligned}$$

(note that A is closed and so $b + ra, c + va \in A$). It follows that

$$\delta\big(S(a, A)\big) - \varepsilon \ \leqslant \ \delta(A).$$

Let $\varepsilon \searrow 0$, to conclude that $\delta\big(S(a, A)\big) \leqslant \delta(A)$.

(b) Recall that the Lebesgue measure λ^N is rotation invariant. So we may take

$$a \ = \ e_N \ = \ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$P(a) \;=\; P(e_N) \;=\; \mathbb{R}^{N-1}.$$

Note that the function $f\colon \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$, defined by

$$f(b) \stackrel{df}{=} \mu^{(1)}\big(A \cap L(a,b)\big) \qquad \forall\, b \in \mathbb{R}^{N-1},$$

is measurable (Fubini's theorem) and

$$\lambda^N(A) \;=\; \int\limits_A f(b) d\lambda^{N-1}(b)$$

(since $\lambda^1 = \mu^{(1)}$; see Remark 1.3.15). So by virtue of Lemma 1.3.16, we have that

$$S(a,b) \stackrel{df}{=} \left\{(b,\vartheta) \in \mathbb{R}^{N-1} \times \mathbb{R} :\; -\frac{f(b)}{2} \leqslant \vartheta \leqslant \frac{f(b)}{2}\right\} \setminus \left\{(b,0) \in \mathbb{R}^{N-1} \times \mathbb{R} :\; A \cap L(a,b) = \emptyset\right\}$$

is Lebesgue measurable in $\mathbb{R}^N$ and, moreover,

$$\lambda^N\big(S(a,A)\big) \;=\; \int\limits_{\mathbb{R}^{N-1}} f(b)\, d\lambda^{N-1}(b) \;=\; \lambda^N(A).$$

□

Now we are properly equipped to prove the so-called "isodiametric inequality," which states that, if in $\mathbb{R}^N$ we consider the family of all sets with given diameter, the one with maximum Lebesgue N-dimensional outer measure (N-volume) is the sphere.

THEOREM 1.3.20 (Isodiametric Inequality)
For all $A \subseteq \mathbb{R}^N$, we have

$$\lambda^N(A) \;\leqslant\; a(N)\left(\frac{\delta(A)}{2}\right)^N,$$

where $a(N) \stackrel{df}{=} \dfrac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$ is the volume of the unit ball in $\mathbb{R}^N$.

PROOF If $\delta(A) = +\infty$, then there is nothing to prove. So suppose that

$$\delta(A) \;<\; +\infty.$$

Let $\{e_k\}_{k=1}^N$ be the standard basis of $\mathbb{R}^N$. We introduce

$$A_1 = S(e_1, A), \quad A_2 = S(e_2, A_1), \quad \ldots, \quad A_N = S(e_N, A_{N-1}).$$

Let us set $A^* = A_N$.

Claim 1. A^* is symmetric with respect to the origin.

By virtue of the definition of the Steiner symmetrization, we have that A_1 is symmetric with respect to the plain $P(e_1)$. Let $1 \leqslant k \leqslant N-1$ and suppose that A_k is symmetric with respect to $P(e_1), \ldots, P(e_k)$. Again A_{k+1} is symmetric with respect to $P(e_{k+1})$. Let us fix $1 \leqslant m \leqslant k$ and let $R_m \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be reflection with respect to $P(e_m)$. Let $b \in P(e_{k+1})$. Because $R_m(A_k) = A_k$, we have

$$\mu^{(1)}\big\{A_k \cap L(e_{k+1}, b)\big\} \;=\; \mu^{(1)}\big\{A_k \cap L(e_{k+1}, R_m(b))\big\},$$

so

$$\Big\{t \in \mathbb{R} : \; b + te_{k+1} \in A_{k+1}\Big\} \;=\; \Big\{t \in \mathbb{R} : \; R_m(b) + te_{k+1} \in A_{k+1}\Big\}$$

and thus

$$R_m(A_{k+1}) \;=\; A_{k+1},$$

i.e., A_{k+1} is symmetric with respect to $P(e_m)$.

It follows that $A^* = A_N$ is symmetric with respect to $P(e_1), \ldots, P(e_N)$, hence it is symmetric with respect to the origin.

Claim 2. $\lambda^N(A^*) \leqslant \dfrac{\pi^{\frac{N}{2}}}{\big(\frac{N}{2}\big)!}\left(\dfrac{\delta(A^*)}{2}\right)^N$.

Let $x \in A^*$. Then because of Claim 1, we have $-x \in A^*$ and so

$$2\,\|x\|_{\mathbb{R}^N} \;\leqslant\; \delta(A^*).$$

Hence

$$A^* \;\subseteq\; \overline{B}_{\frac{\delta(A^*)}{2}}(0) \;=\; \left\{y \in \mathbb{R}^N : \; \|y\|_{\mathbb{R}^N} \leqslant \frac{\delta(A^*)}{2}\right\}$$

and so

$$\lambda^N(A^*) \;\leqslant\; \lambda^N\left(\overline{B}_{\frac{\delta(A^*)}{2}}(0)\right) \;\leqslant\; \frac{\pi^{\frac{N}{2}}}{\big(\frac{N}{2}\big)!}\left(\frac{\delta(A^*)}{2}\right)^N.$$

Using Claim 2, we can have the isodiametric inequality. Note that $\overline{A} \subseteq \mathbb{R}^N$ is Lebesgue measurable and so by Proposition 1.3.19, we have

$$\lambda^N\big(\overline{A}^*\big) \;=\; \lambda^N\big(\overline{A}\big) \quad \text{and} \quad \delta\big(\overline{A}^*\big) \;\leqslant\; \delta\big(\overline{A}\big).$$

Using Claim 2, it follows that

$$\lambda^N(A) \leqslant \lambda^N(\overline{A}) = \lambda^N(\overline{A}^*) \leqslant \frac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!}\left(\frac{\delta(\overline{A}^*)}{2}\right)^N$$
$$\leqslant \frac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!}\left(\frac{\delta(\overline{A})}{2}\right)^N = \frac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!}\left(\frac{\delta(A)}{2}\right)^N.$$

□

THEOREM 1.3.21

If $A \subseteq \mathbb{R}^N$, *then* $\lambda^N(A) = c_N \mu^{(N)}(A)$, *with* $c_N \stackrel{df}{=} \dfrac{\pi^{\frac{N}{2}}}{2^N (\frac{N}{2})!}$.

PROOF For a given $\varepsilon > 0$, we can find a cover $\{C_n\}_{n \geqslant 1}$ of A consisting of closed, convex sets, such that

$$\sum_{n=1}^{\infty} \delta(C_n)^N \leqslant \mu^{(N)}(A) + \varepsilon.$$

By virtue of Theorem 1.3.20, we have

$$\lambda^N(C_n) \leqslant c_N \delta(C_n)^N \qquad \forall\, n \geqslant 1.$$

So

$$\lambda^N(A) \leqslant \sum_{n=1}^{\infty} \lambda^N(C_n) \leqslant c_N \sum_{n=1}^{\infty} \delta(C_n)^N \leqslant c_N \mu^{(N)}(A) + c_N \varepsilon.$$

Let $\varepsilon \searrow 0$ to conclude that

$$\lambda^N(A) \leqslant c_N \mu^{(N)}(A). \tag{1.24}$$

To prove the opposite inequality, first we show that $\mu^{(N)}$ is absolutely continuous with respect to λ^N (see Definition A.2.22). Note that for any N-cube Q, we have

$$\lambda^N(Q) = |Q| \leqslant \left(\frac{\delta(Q)}{\sqrt{N}}\right)^N.$$

So for a given $\delta > 0$, we have

$$\mu_\delta^{(N)}(A) \leqslant \inf_{\substack{Q_n \text{ -}N\text{-cube} \\ A \subseteq \bigcup_{n=1}^{\infty} Q_n \\ \delta(Q_n) \leqslant \delta}} \sum_{n=1}^{\infty} \delta(Q_n) \leqslant \sqrt{N}^N \lambda^N(A).$$

Let $\delta \searrow 0$, to conclude that $\mu^{(N)}$ is absolutely continuous with respect to λ^N (see Definition A.2.22).

Next for a given $\varepsilon, \delta > 0$, we can find

$$\text{a cover } \{Q_n\}_{n\geqslant 1} \text{ of } A$$

consisting of N-cubes, such that

$$\delta(Q_n) \;<\; \delta \qquad \forall\, n \geqslant 1$$

and

$$\sum_{n=1}^{\infty} \lambda^N(Q_n) \;\leqslant\; \lambda^N(A) + \varepsilon. \tag{1.25}$$

We may suppose that N-cubes are open by expanding them slightly so that the above inequality remains valid. Invoking Vitali's covering theorem (see Theorem 1.2.5), for every $n \geqslant 1$ we can find disjoint balls $\{B_{n,k}\}_{k\geqslant 1}$ contained in Q_n, such that

$$\delta(B_{n,k}) \;\leqslant\; \delta \quad \text{and} \quad \lambda^N\Big(Q_n \setminus \bigcup_{k=1}^{\infty} B_{n,k}\Big) \;=\; 0.$$

By virtue of the absolute continuity of $\mu^{(N)}$ with respect to λ^N, we have

$$\mu^{(N)}\Big(Q_n \setminus \bigcup_{k=1}^{\infty} B_{n,k}\Big) \;=\; 0 \quad \text{and} \quad \mu_\delta^{(N)}\Big(Q_n \setminus \bigcup_{k=1}^{\infty} B_{n,k}\Big) \;=\; 0.$$

Therefore, using (1.25), we have

$$\begin{aligned}
\mu_\delta^{(N)}(A) \;&\leqslant\; \sum_{k=1}^{\infty} \mu_\delta^{(N)}(Q_n) \;\leqslant\; \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_\delta^{(N)}(B_{n,k}) + \sum_{n=1}^{\infty} \mu_\delta^{(N)}\Big(Q_n \setminus \bigcup_{k=1}^{\infty} B_{n,k}\Big) \\
&\leqslant\; \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \delta(B_{n,k})^N \;=\; \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{c_N}\lambda^N(B_{n,k}) \\
&\leqslant\; \frac{1}{c_N}\sum_{n=1}^{\infty} \lambda^N(Q_n) \;\leqslant\; \frac{1}{c_N}\lambda^N(A) + \frac{\varepsilon}{c_N}.
\end{aligned}$$

Let $\varepsilon, \delta \searrow 0$, to conclude that

$$c_N \mu^{(N)}(A) \;\leqslant\; \lambda^N(A). \tag{1.26}$$

From (1.24) and (1.26), we conclude that

$$\lambda^N \;=\; c_N \mu^{(N)}.$$

□

REMARK 1.3.22 Some authors, in order to get rid of the multiplicative constant c_N, normalize the definition of the Hausdorff measures on $\mathbb{R}^N$. So if $C \subseteq \mathbb{R}^N$, $0 \leqslant s < +\infty$, $0 < \delta \leqslant +\infty$, they set

$$\mu_\delta^{(s)}(C) \stackrel{df}{=} \inf_{\substack{C \subseteq \bigcup_{n=1}^{\infty} A_n \\ \delta(A_n) \leqslant \delta}} \sum_{n=1}^{\infty} a(s) \left(\frac{\delta(A_n)}{2} \right)^s,$$

where $a(s) \stackrel{df}{=} \dfrac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$. Here

$$\Gamma(s) \stackrel{df}{=} \int_0^{+\infty} x^{s-1} e^{-x}\, dx$$

is the gamma Euler function. The Hausdorff s-dimensional outer measure $\mu^{(s)}$ is defined by

$$\mu^{(s)}(C) \;=\; \lim_{\delta \searrow 0} \mu_\delta^{(s)}(C) \;=\; \sup_{\delta > 0} \mu_\delta^{(s)}(C)$$

(cf., e.g., Evans & Gariepy (1992, p. 60)) .

Recall that

$$\lambda^N\big(\overline{B}(x,r)\big) \;=\; a(N) r^N \qquad \forall\, x \in \mathbb{R}^N.$$

In this case Theorem 1.3.21 says that

$$\lambda^N \;=\; \mu^{(N)}.$$

Note that $\mu^{(0)}$ is the counting measure.

Let us prove some further properties of the Hausdorff measures on $\mathbb{R}^N$.

PROPOSITION 1.3.23
Let $0 \leqslant s < +\infty$. We have

(a) $\mu^{(s)}(A) = 0$ *for all* $A \subseteq \mathbb{R}^N$ *and all* $s > N$.

(b) $\mu^{(s)}(\xi A) = \xi^s \mu^{(s)}(A)$ *for all* $A \subseteq \mathbb{R}^N$ *and all* $\xi \geqslant 0$.

(c) $\mu^{(s)}\big(K(A)\big) = \mu^{(s)}(A)$ *for all* $A \subseteq \mathbb{R}^N$ *and for any affine isometry* $K : \mathbb{R}^N \longrightarrow \mathbb{R}^N$.

PROOF **(a)** Let $Q = (0,1)^N$ and let $m > 1$ be an integer. For

$$k \;=\; (k_i)_{i=1}^N \;\in\; K \stackrel{df}{=} \{0, \ldots, m-1\}^N,$$

we set

$$Q_k \stackrel{df}{=} \prod_{i=1}^N \left[\frac{k_i}{m}, \frac{k_i+1}{m} \right].$$

Note that

$$Q \;=\; \bigcup_{k\in K} Q_k \quad \text{and} \quad \delta(Q_k) \;=\; \frac{\sqrt{N}}{m}.$$

So we have

$$\mu^{(s)}_{\frac{\sqrt{N}}{m}}(Q) \;\leqslant\; \sum_{k\in K} \delta(Q_k)^s \;=\; m^{N-s}\sqrt{N}^{\,s}.$$

Letting $m \to +\infty$, since $s > N$, we obtain

$$\mu^{(s)}(Q) \;=\; 0,$$

from which it follows that

$$\mu^{(s)}(\mathbb{R}^N) \;=\; 0.$$

(b) Note that for all $C \subseteq \mathbb{R}^N$, we have

$$\delta(\xi C) \;=\; \xi\delta(C).$$

So the result follows at once from Definition 1.3.5.

(c) Note that for all $C \subseteq \mathbb{R}^N$, we have

$$\delta\big(K(C)\big) \;=\; \delta(C).$$

Again the result follows from Definition 1.3.5. □

The next Proposition suggests a convenient way to check that $\mu^{(s)}$ vanishes on a set.

PROPOSITION 1.3.24
If $A \subseteq \mathbb{R}^N$, $0 < \delta \leqslant +\infty$ *and* $0 \leqslant s < +\infty$ *are such that* $\mu^{(s)}_\delta(A) = 0$, *then* $\mu^{(s)}(A) = 0$.

PROOF If $s = 0$, then $\mu^{(0)}_\delta(A) = 0$ implies that $A = \emptyset$ and so $\mu^{(0)}(A) = 0$. So suppose that $s > 0$. For a given $\varepsilon > 0$, we can find $\{C_n\}_{n\geqslant 1}$, such that

$$A \subseteq \bigcup_{n=1}^{\infty} C_n, \quad \delta(C_n) \leqslant \delta \quad \text{and} \quad \sum_{n=1}^{\infty} \delta(C_n)^s \leqslant \varepsilon.$$

Evidently

$$\delta(C_n)^s \;\leqslant\; \varepsilon \qquad \forall\; n \geqslant 1$$

and so $\mu^{(s)}_\varepsilon(A) \leqslant \varepsilon$. Let $\varepsilon \searrow 0$, to conclude that

$$\mu^{(s)}(A) \;=\; 0.$$

□

Taking into account that for a Lipschitz continuous function with constant $c > 0$, for every $A \subseteq \mathbb{R}^N$, we have

$$\delta\big(f(A)\big) \leqslant c\delta(A),$$

and we obtain the following result.

PROPOSITION 1.3.25
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a Lipschitz continuous function with Lipschitz constant $c > 0$ (see Definition 1.5.1), $A \subseteq \mathbb{R}^N$ and $0 \leqslant s < +\infty$,
then $\mu^{(s)}\big(f(A)\big) \leqslant c^s \mu^{(s)}(A)$.

We conclude this section by returning to the notion of Hausdorff dimension (see Definition 1.3.8) and having a second look at this concept.

The Hausdorff dimension has an intuitive appeal when familiar objects are under consideration. So for example $\dim \mathbb{R}^N = N$ (see Theorem 1.3.21). Suppose we want to determine the Hausdorff dimension of a curve $C \subseteq \mathbb{R}^3$. Our first guess will be that $\dim C = 1$. But recall that there are curves in $\mathbb{R}^3$ which fill the unit cube. Such a curve must have Hausdorff dimension 3. Therefore we must proceed with caution.

DEFINITION 1.3.26 *Let (X, d) be a metric space.*

(a) *By a **curve** in X we mean the image $f\big([0,1]\big)$ of a continuous function $f\colon [0,1] \longrightarrow X$.*

(b) *The **length of a curve** $C = f\big([0,1]\big)$ is defined by*

$$l(C) \stackrel{df}{=} \sup \sum_{k=1}^{m} d\big(f(x_{k-1}), f(x_k)\big),$$

where the supremum is taken over all partitions

$$0 = x_0 < x_1 < \ldots < x_m = 1 \text{ of } [0,1].$$

(c) *The curve C is said to be **rectifiable**, if $l(C) < +\infty$.*

REMARK 1.3.27 A curve C is a continuum, i.e., a compact and connected set in X. In particular then a curve is a Borel set; hence it is also $\mu^{(s)}$-measurable. Moreover, if in Definition 1.3.26**(a)** f is injective, then f^{-1} exists and is continuous and so C is the homeomorphic image of $[0,1]$. Also in Definition 1.3.26**(a)**, we can replace $[0,1]$ by any closed bounded interval $[a,b]$. Some authors require f to be injective.

PROPOSITION 1.3.28
If (X,d) *is a metric space,* $f\colon [0,1] \longrightarrow X$ *is a nonconstant curve with length* l *and* $C = f\big([0,1]\big)$,
then

(a) $0 < \mu^{(1)}(C) \leqslant l$;

(b) *if* f *is injective, then* $\mu^{(1)}(C) = l$.

Therefore, if l *is rectifiable (i.e.,* $l < +\infty$*), then* $\dim C = 1$.

PROOF **(a)** First we show that

$$\mu^{(1)}(C) \;\leqslant\; l.$$

Assume that $l < +\infty$ or otherwise there is nothing to prove. Let $\{A_k\}_{k=1}^m$ be a collection of closed subarcs of C, such that

$$C \;=\; \bigcup_{k=1}^{m} A_k, \quad \delta(A_k) \;\leqslant\; \frac{1}{n} \quad \text{and} \quad \mu^{(1)}_{\frac{1}{n}}(C) \;\leqslant\; \sum_{k=1}^{m} \delta(A_k). \tag{1.27}$$

Let us explicitly construct the subarcs A_k for $k \in \{1,\dots,m\}$. Note that f is uniformly continuous and so we can find $\eta > 0$, such that

$$d\big(f(x), f(y)\big) < \frac{1}{n} \qquad \forall\; x,y \in [0,1],\ |x-y| < \eta.$$

Consider a partition

$$0 \;=\; x_0 \;<\; x_1 \;<\; \ldots \;<\; x_n \;=\; 1 \quad \text{of} \quad [0,1],$$

such that

$$|x_k - x_{k-1}| \;<\; \eta \qquad \forall\; k \in \big\{1,\dots,m\big\}.$$

Let

$$A_k \;\stackrel{df}{=}\; f\big([x_{k-1}, x_k]\big), \qquad \forall\; k \in \big\{1,\dots,m\big\}.$$

Evidently the subarcs $\{A_k\}_{k=1}^m$ cover C and

$$d\big(f(x_{k-1}), f(x_k)\big) \;\leqslant\; \delta(A_k) \;<\; \frac{1}{n} \qquad \forall\; k \in \{1,\dots,m\}.$$

Note that every A_k is compact and so we can find points $y_k, z_k \in [x_{k-1}, x_k]$, $y_k \leqslant z_k$, such that

$$d\big(f(y_k), f(z_k)\big) \;=\; \delta(A_k).$$

We generate the finer partition

$$0 \;\leqslant\; y_1 \;\leqslant\; z_1 \;\leqslant\; y_2 \;\leqslant\; z_2 \;\leqslant\; \ldots \;\leqslant\; y_m \;\leqslant\; z_m \;\leqslant\; 1.$$

From (1.27), we have

$$\mu^{(1)}_{\frac{1}{n}}(C) \;\leqslant\; \sum_{k=1}^{m} \delta(A_k) \;=\; \sum_{k=1}^{m} d\big(f(y_k), f(z_k)\big) \;\leqslant\; l.$$

Passing to the limit as $n \to +\infty$, we obtain that $\mu^{(1)}(C) \leqslant l$.

Next we show that $0 < \mu^{(1)}(C)$. To this end note that if $0 \leqslant a < b \leqslant 1$, then

$$d\big(f(a), f(b)\big) \;\leqslant\; \mu^{(1)}\big(f([a,b])\big). \tag{1.28}$$

To see this let $h\colon E \stackrel{df}{=} f([a,b]) \longrightarrow \mathbb{R}$ be the function

$$h(u) \;\stackrel{df}{=}\; d\big(u, f(a)\big).$$

Evidently h is a Lipschitz continuous function with Lipschitz constant 1 and

$$J \;\stackrel{df}{=}\; \big[0, h(b)\big] \;=\; \big[0, d\big(f(a), f(b)\big)\big] \;\subseteq\; h(E).$$

So, from Proposition 1.3.25, we have

$$d\big(f(a), f(b)\big) \;=\; \lambda^1(J) \;=\; \mu^{(1)}(J) \;\leqslant\; \mu^{(1)}\big(h(E)\big) \;\leqslant\; \mu^{(1)}(E).$$

This proves inequality (1.28). But from (1.28) and since for appropriately chosen a, b we have

$$d\big(f(a), f(b)\big) \;>\; 0$$

(recall that the curve is nonconstant), we conclude that $0 < \mu^{(1)}(C)$.

(b) Now suppose that f is injective. Let $0 = x_0 < x_1 < \ldots < x_m = 1$ be a partition of $[0,1]$. The sets

$$A_k \;\stackrel{df}{=}\; f\big([x_{k-1}, x_k]\big)$$

are pairwise disjoint Borel subsets of X. Using inequality (1.28) on each subarc, we obtain

$$\begin{aligned}\sum_{k=1}^{m} d\big(f(x_{k-1}), f(x_k)\big) \;&\leqslant\; \sum_{k=1}^{m} \mu^{(1)}\big(f\big([x_{k-1}, x_k]\big)\big) \\ =\; \mu^{(1)}\Big(\bigcup_{k=1}^{m} f\big([x_{k-1}, x_k]\big)\Big) \;&=\; \mu^{(1)}\big(f\big([0,1]\big)\big) \;=\; \mu^{(1)}(C).\end{aligned}$$

Since the partition of $[0,1]$ was arbitrary, it follows that $l \leqslant \mu^{(1)}(C)$. Combining this with **(a)**, we obtain that $l = \mu^{(1)}(C)$. ∎

1.4 Differentiation of Hausdorff Measures

From the general measure theory, we know that the differentiation theory of real functions can be extended to a theory of differentiation for measures, which has many similar features and interesting problems. For the Lebesgue measures λ^N, $N \geqslant 1$, one of the basic results of this theory is the so-called ***Lebesgue density theorem***, which we recall here.

THEOREM 1.4.1 (Lebesgue Density Theorem)
If $A \subseteq \mathbb{R}^N$ *is a Lebesgue measurable set,*
then

$$\lim_{r \searrow 0} \frac{\lambda^N(\overline{B}_r(x) \cap A)}{\lambda^N(\overline{B}_r(x))} = \begin{cases} 1 & \text{for } \lambda^N\text{-a.a. } x \in A, \\ 0 & \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N \setminus A. \end{cases}$$

DEFINITION 1.4.2 *Let* $A \subseteq \mathbb{R}^N$ *and* $x \in \mathbb{R}^N$. *We say that:*

(a) x *is a* ***point of density*** *of* A, *if*

$$\lim_{r \searrow 0} \frac{\lambda^N(\overline{B}_r(x) \cap A)}{\lambda^N(\overline{B}_r(x))} = 1;$$

(b) x *is a* ***point of dispersion*** *of* A, *if*

$$\lim_{r \searrow 0} \frac{\lambda^N(\overline{B}_r(x) \cap A)}{\lambda^N(\overline{B}_r(x))} = 0.$$

REMARK 1.4.3 According to Theorem 1.4.1, we see that λ^N-almost every point of A is a point of density of A and λ^N-almost every point of $\mathbb{R}^N \setminus A$ is a point of dispersion of A. We can think that the point of density of a set A form a kind of measure theoretic interior of A, while the points of dispersion of A form a kind of measure theoretic exterior of A.

The purpose of this section is to establish analogs of Theorem 1.4.1 for lower dimensional Hausdorff measures. In what follows we work in $\mathbb{R}^N$ and $1 < s < N$.

THEOREM 1.4.4
If $A \subseteq \mathbb{R}^N$ *is* $\mu^{(s)}$*-measurable and* $\mu^{(s)}(A) < +\infty$,
then

$$\lim_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} = 0 \quad \text{for } \mu^{(s)}\text{-a.a. } x \in \mathbb{R}^N \setminus A.$$

PROOF For every $t > 0$, let

$$C_t \stackrel{df}{=} \left\{ x \in \mathbb{R}^N \setminus A : \limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} > t \right\}.$$

To finish the proof it is enough to show that

$$\mu^{(s)}(C_t) = 0 \qquad \forall\, t > 0.$$

Fix $\varepsilon > 0$. We know that

$$\mu^{(s)} \lfloor A \text{ is a Radon measure}$$

(see Proposition 1.1.9). So we can find $K \subseteq A$ compact, such that

$$\mu^{(s)}(A \setminus K) \leqslant \varepsilon$$

(see Proposition 1.1.10**(b)**). Let

$$U \stackrel{df}{=} \mathbb{R}^N \setminus K.$$

Then U is open and

$$C_t \subseteq U.$$

For fixed $\delta > 0$, we consider the family of closed balls

$$\mathcal{T} \stackrel{df}{=} \left\{ \overline{B}_r(x) : \overline{B}_r(x) \subseteq U,\ 0 < r < \delta,\ \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} > t \right\}.$$

Without any loss of generality we may assume that $\mathcal{T} \neq \emptyset$ or otherwise $C_t = \emptyset$ and so

$$\mu^{(s)}(C_t) = 0.$$

Invoking Proposition 1.2.1, we can find a sequence $\left\{ \overline{B}_{r_n}(x_n) \right\}_{n \geqslant 1}$ of disjoint elements in $\mathcal{T}$, such that

$$C_t \subseteq \bigcup_{n=1}^{\infty} \overline{B}_{5r_n}(x_n).$$

Then we have

$$\mu^{(s)}_{10\delta}(C_t) \leqslant \sum_{n=1}^{\infty} (10 r_n)^s \leqslant \frac{5^s}{t} \sum_{n=1}^{\infty} \mu^{(s)}\left(\overline{B}_{r_n}(x_n) \cap A\right)$$
$$\leqslant \frac{5^s}{t} \mu^{(s)}(U \cap A) = \frac{5^s}{t} \mu^{(s)}(A \setminus K) \leqslant \frac{5^s \varepsilon}{t}.$$

Let $\delta \searrow 0$, to obtain

$$\mu^{(s)}(C_t) \leqslant \frac{5^s \varepsilon}{t}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu^{(s)}(C_t) = 0$. ∎

To have a complete analog of Theorem 1.4.1, we need to check and see if something can be said about the density of A at its points. To do this we will make use of Proposition 1.2.4.

THEOREM 1.4.5
If $A \subseteq \mathbb{R}^N$ *is* $\mu^{(s)}$*-measurable and* $\mu^{(s)}(A) < +\infty$, *then*

$$\frac{1}{2^s} \leqslant \limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} \leqslant 1 \quad \text{for } \mu^{(s)}\text{-a.a. } x \in A.$$

PROOF First we show that

$$\limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} \leqslant 1 \quad \text{for } \mu^{(s)}\text{-a.a. } x \in A. \tag{1.29}$$

To this end, for every $t > 1$, we introduce the set $C_t \subseteq A$ defined by

$$C_t \stackrel{df}{=} \left\{ x \in A : \limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} > t \right\}.$$

Fix $\varepsilon > 0$. Again $\mu^{(s)} \lfloor A$ is a Radon measure (see Proposition 1.1.9). We can find an open set $U \subseteq \mathbb{R}^N$, such that

$$C_t \subseteq U$$

and

$$\mu^{(s)}(U \cap A) - \varepsilon \leqslant \mu^{(s)}(C_t) \tag{1.30}$$

(see Proposition 1.1.10**(b)**).

We introduce the family $\mathcal{T}$ of closed balls defined by

$$\mathcal{T} \stackrel{df}{=} \left\{ \overline{B}_r(x) : \overline{B}_r(x) \subseteq U,\ 0 < r < \delta,\ \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} > t \right\}.$$

By virtue of Proposition 1.2.4, we can find a sequence $\left\{\overline{B}_{r_n}(x_n)\right\}_{n \geqslant 1}$ of disjoint balls in $\mathcal{T}$, such that

$$C_t \subseteq \bigcup_{n=1}^{m} \overline{B}_{r_n}(x_n) \cup \bigcup_{n=m+1}^{\infty} \overline{B}_{5r_n}(x_n) \qquad \forall\ m \geqslant 1.$$

Then for $\delta > 0$, we have

$$\begin{aligned} \mu_{10\delta}^{(s)}(C_t) &\leqslant \sum_{n=1}^{m} (2r_n)^s + \sum_{n=m+1}^{\infty} (10r_n)^s \\ &\leqslant \frac{1}{t} \sum_{n=1}^{m} \mu^{(s)}\left(\overline{B}_{r_n} \cap A\right) + \frac{5^s}{t} \sum_{n=m+1}^{\infty} \mu^{(s)}\left(\overline{B}_{r_n}(x_n) \cap A\right) \\ &\leqslant \frac{1}{t}\mu^{(s)}(U \cap A) + \frac{5^s}{t}\mu^{(s)}\left(\bigcup_{n=m+1}^{\infty} \left(\overline{B}_{r_n}(x_n) \cap A\right)\right) \qquad \forall\ m \geqslant 1. \end{aligned}$$

Using (1.30) and letting $m \to +\infty$, we obtain

$$\mu^{(s)}_{10\delta}(C_t) \;\leqslant\; \frac{1}{t}\mu^{(s)}(U \cap A) \;\leqslant\; \frac{1}{t}\big(\mu^{(s)}(C_t) + \varepsilon\big).$$

Letting $\delta \searrow 0$, we see that

$$\mu^{(s)}(C_t) \;\leqslant\; \frac{1}{t}\big(\mu^{(s)}(C_t) + \varepsilon\big).$$

Since $\varepsilon > 0$ was arbitrary, we finally have that

$$\mu^{(s)}(C_t) \;\leqslant\; \frac{1}{t}\mu^{(s)}(C_t),$$

i.e.,

$$\mu^{(s)}(C_t) \;=\; 0$$

(recall that $t > 1$). This proves (1.29).

Next we show that

$$\frac{1}{2^s} \;\leqslant\; \limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} \quad \text{for } \mu^{(s)}\text{-a.a. } x \in A. \tag{1.31}$$

For a given $\xi, \delta \in (0,1)$, we introduce the set $A(\delta,\xi) \subseteq A$, defined by

$$A(\delta,\xi) \;\stackrel{df}{=}\; \Big\{ x \in A :\; \mu^{(s)}_\delta(C \cap A) \leqslant \xi\delta(C)^s \text{ for all } C \subseteq \mathbb{R}^N, \\ \text{with } \delta(C) \leqslant \delta \text{ and } x \in C \Big\}.$$

Let $\{C_n\}_{n\geqslant 1}$ be a δ-cover of $A(\delta,\xi)$, such that

$$A(\delta,\xi) \subseteq \bigcup_{n=1}^{\infty} C_n$$

and

$$\delta(C_n) \leqslant \delta, \quad \text{and} \quad C_n \cap A(\delta,\xi) \neq \emptyset \qquad \forall\; n \geqslant 1.$$

So

$$\mu^{(s)}_\delta\big(A(\delta,\xi)\big) \;\leqslant\; \sum_{n=1}^{\infty} \mu^{(s)}_\delta\big(C_n \cap A(\delta,\xi)\big) \\ \leqslant\; \sum_{n=1}^{\infty} \mu^{(s)}_\delta(C_n \cap A) \;\leqslant\; \sum_{n=1}^{\infty} \xi\delta(C_n)^s$$

and from Definition 1.3.5, we see that

$$\mu^{(s)}_\delta\big(A(\delta,\xi)\big) \;\leqslant\; \xi\mu^{(s)}_\delta\big(A(\delta,\xi)\big).$$

Since

$$0 \ < \ \xi \ < \ 1 \quad \text{and} \quad \mu_\delta^{(s)}\big(A(\delta,\xi)\big) \ < \ +\infty,$$

we have

$$\mu_\delta^{(s)}\big(A(\delta,\xi)\big) \ = \ 0.$$

In particular, from Proposition 1.3.24, we see that

$$\mu^{(s)}\big(A(\delta,1-\delta)\big) \ = \ 0. \tag{1.32}$$

Set

$$D_\infty \ \stackrel{df}{=} \ \left\{x \in A : \ \limsup_{r \searrow 0} \frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} < \frac{1}{2^s}\right\}.$$

If $x \in D_\infty$, then we can find $\delta > 0$, such that

$$\frac{\mu^{(s)}(\overline{B}_r(x) \cap A)}{(2r)^s} \ \leqslant \ \frac{1-\delta}{2^s} \qquad \forall \ r \in (0,\delta]. \tag{1.33}$$

For any $C \subseteq \mathbb{R}^N$, with

$$x \in C \cap A \quad \text{and} \quad \delta(C) \leqslant \delta,$$

from (1.33), we have

$$\mu_\delta^{(s)}(C \cap A) \ \leqslant \ \mu^{(s)}(C \cap A) \ \leqslant \ \mu^{(s)}\big(\overline{B}_{\delta(C)}(x) \cap A\big) \ \leqslant \ (1-\delta)\delta(C)^s.$$

So it follows that $x \in A(\delta, 1-\delta)$. Therefore, we have

$$D_\infty \ \subseteq \ \bigcup_{n=1}^{\infty} A\Big(\frac{1}{n}, 1-\frac{1}{n}\Big),$$

and, using also (1.32), we have

$$\mu^{(s)}(D_\infty) \ = \ 0.$$

Thus we infer that (1.31) is true. □

For a given locally integrable function, we can establish the Hausdorff measure of the set where the function is locally large. To do this we shall need the so-called ***Lebesgue differentiation theorem*** or ***Lebesgue-Besicovitch differentiation theorem***

THEOREM 1.4.6 (Lebesgue-Besicovitch Differentiation Theorem) *If* $f \in L^1_{\text{loc}}\big(\mathbb{R}^N;\mathbb{R}^M\big)$, *then*

$$\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - f(x)\big\|_{\mathbb{R}^M} \, d\lambda^N(y) \ = \ 0 \quad \textit{for } \lambda^N\text{-a.a. } \ x \in \mathbb{R}^N.$$

PROOF Let $D = \{u_k\}_{k \geqslant 1}$ be a dense subset of $\mathbb{R}^M$. Then by the classical differentiation theorem of Lebesgue (see for example Cohn (1980, p. 190)), we have

$$\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - u_n\big\|_{\mathbb{R}^M} \, d\lambda^N(y) \;=\; \big\|f(x) - u_n\big\|_{\mathbb{R}^M} \quad (1.34)$$

$$\text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N.$$

Suppose that $x \in \mathbb{R}^N$ is such a differentiability point for which (1.34) is valid for all $n \geqslant 1$. For a given $\varepsilon > 0$, we can choose u_n, such that

$$\big\|f(x) - u_n\big\|_{\mathbb{R}^N} \;<\; \varepsilon.$$

Then we have

$$\begin{aligned} &\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - f(x)\big\|_{\mathbb{R}^M} \, d\lambda^N(y) \\ \leqslant\; &\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - u_n\big\|_{\mathbb{R}^M} \, d\lambda^N(y) + \big\|u_n - f(x)\big\|_{\mathbb{R}^M} \;<\; 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - f(x)\big\|_{\mathbb{R}^M} \, d\lambda^N(y) \;=\; 0.$$

□

COROLLARY 1.4.7
If $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^M)$, *then*

$$\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} f(y) \, d\lambda^N(y) \;=\; f(x) \quad \textit{for } \lambda^N\textit{-a.a. } x \in \mathbb{R}^N.$$

PROOF Note that

$$\begin{aligned} &\left\| \lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} f(y) \, d\lambda^N(y) - f(x) \right\|_{\mathbb{R}^M} \\ \leqslant\; &\lim_{r \searrow 0} \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|f(y) - f(x)\big\|_{\mathbb{R}^M} \, d\lambda^N(y). \end{aligned}$$

So the corollary follows at once from Theorem 1.4.6. □

REMARK 1.4.8 Theorem 1.4.6 and Corollary 1.4.7 remain valid if λ^N is replaced by any Radon measure on $\mathbb{R}^N$. Also we may replace the ball $\overline{B}_r(x)$ by any other measurable sets $S_r(x)$ containing x which shrink to the point $x \in \mathbb{R}^N$ as $r \searrow 0$. For example we can take $S_r(x)$ to be N-cube with edges equal to $2r$. If $N = 1$, we may take for example the intervals

$$[x-h, h], \quad [x, x+h] \quad \text{or} \quad [x-h, x+h].$$

In Proposition 2.1.22 we shall see that the results are also valid for Banach space valued functions, i.e., $\mathbb{R}^M$ is replaced by a Banach space.

Now we are ready to estimate the Hausdorff measure of the set where a function $f \in L^1_{\text{loc}}\big(\mathbb{R}^N;\mathbb{R}\big)$ is locally large.

THEOREM 1.4.9
If $f \in L^1_{\text{loc}}\big(\mathbb{R}^N;\mathbb{R}\big)$, $0 \leqslant s < N$ *and*

$$C_s \stackrel{df}{=} \left\{ x \in \mathbb{R}^N : \limsup_{r \searrow 0} \frac{1}{r^s} \int\limits_{\overline{B}_r(x)} \big|f(y)\big| \, d\lambda^N(y) > 0 \right\},$$

then $\mu^{(s)}(C_s) = 0$.

PROOF It is clear that without any loss of generality, we may assume that $f \in L^1(\mathbb{R}^N;\mathbb{R})$. By virtue of Corollary 1.4.7, we have that

$$\lim_{r \to 0} \frac{1}{r^s} \int\limits_{\overline{B}_r(x)} \big|f(y)\big| \, d\lambda^N(y) \;=\; 0 \quad \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N$$

(recall that $0 \leqslant s < N$). So

$$\lambda^N(C_s) \;=\; 0.$$

Let $\varepsilon > 0$, $\delta > 0$ and $\xi > 0$ be given. Since $f \in L^1(\mathbb{R}^N;\mathbb{R})$, from the absolute continuity of the Lebesgue integral, we know that we can find $\vartheta > 0$, such that

$$\int\limits_A \big|f(y)\big| \, d\lambda^N(y) \;<\; \xi \qquad \forall\, A \subseteq \mathbb{R}^N, \lambda^N(A) < \vartheta.$$

We introduce the set

$$C_s^{\varepsilon} \;\subseteq\; C_s$$

defined by

$$C_s^{\varepsilon} \stackrel{df}{=} \left\{ x \in C_s : \limsup_{r \searrow o} \frac{1}{r^s} \int\limits_{\overline{B}_r(x)} \big|f(y)\big| \, d\lambda^N(y) > \varepsilon \right\}.$$

We have that

$$\lambda^N(C_s^\varepsilon) \;=\; 0.$$

So we can find an open set $U \subseteq \mathbb{R}^N$, such that

$$\lambda^N(U) \;<\; \vartheta.$$

Let us set

$$\mathcal{T} \;\stackrel{df}{=}\; \Big\{\overline{B}_r(x) :\; x \in C_s^\varepsilon,\; 0 < r < \delta,\; \overline{B}_r(x) \subseteq U \text{ and } \int\limits_{\overline{B}_r(x)} \big|f(y)\big|\, d\lambda^N(y) > \varepsilon r^s\Big\}.$$

Invoking Proposition 1.2.1, we can find a sequence $\big\{B_{r_n}(x_n)\big\}_{n\geqslant 1} \subseteq \mathcal{T}$ of disjoint balls, such that

$$C_s^\varepsilon \;\subseteq\; \bigcup_{n=1}^{\infty} B_{5r_n}(x_n).$$

From this it follows that

$$\begin{aligned}
\mu_{10\delta}^{(s)}(C_s^\varepsilon) \;&\leqslant\; \sum_{n=1}^{\infty}(10r_n)^s \\
&\leqslant\; \frac{10^s}{\varepsilon}\sum_{n=1}^{\infty}\int\limits_{\overline{B}_{r_n}(x_n)} \big|f(y)\big|\, d\lambda^N(y) \\
&\leqslant\; \frac{10^s}{\varepsilon}\int\limits_{U} \big|f(y)\big|\, d\lambda^N(y) \;\leqslant\; \frac{10^s}{\varepsilon}\xi.
\end{aligned}$$

Let $\delta \searrow 0$ and then $\xi \searrow 0$, to conclude that

$$\mu^{(s)}(C_s^\varepsilon) \;=\; 0.$$

Since

$$C_s \;=\; \bigcup_{n=1}^{\infty} C_s^{\frac{1}{n}},$$

we conclude that

$$\mu^{(s)}(C_s) \;=\; 0.$$

□

1.5 Lipschitz Functions

In this section we derive some basic properties relating to the behaviour of Lipschitz continuous functions. A first such result was already established in Proposition 1.3.25.

DEFINITION 1.5.1 *Let $C \subseteq \mathbb{R}^N$.*

(a) *A function $f\colon C \longrightarrow \mathbb{R}^M$ is said to be **Lipschitz continuous**, if there exists a constant $c \geqslant 0$, such that*

$$\big\|f(x) - f(y)\big\|_{\mathbb{R}^M} \;\leqslant\; c\,\|x-y\|_{\mathbb{R}^N} \qquad \forall\; x, y \in C.$$

(b) *If $f\colon C \longrightarrow \mathbb{R}^M$ is Lipschitz continuous, then the **Lipschitz constant** $\mathrm{Lip}(f) \geqslant 0$ of f is defined by*

$$\mathrm{Lip}(f) \;\stackrel{df}{=}\; \sup_{\substack{x,\,y \in C \\ x \neq y}} \frac{\|f(x) - f(y)\|_{\mathbb{R}^M}}{\|x-y\|_{\mathbb{R}^N}}.$$

(c) *If $U \subseteq \mathbb{R}^N$ is open, a function $f\colon U \longrightarrow \mathbb{R}^M$ is said to be **locally Lipschitz**, if for every $x \in U$, we can find a neighbourhood $V \subseteq U$ of x, such that $f|_V$ is Lipschitz continuous.*

THEOREM 1.5.2
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, $f = (f_i)_{i=1}^M$ and $A \subseteq \mathbb{R}^N$ with $\lambda^N(A) > 0$, then

(a) $\dim\big(\mathrm{Gr}\,(f|_A)\big) \geqslant N$, *where $\mathrm{Gr}\,(f|_A)$ is the graph of f over A, defined by*

$$\mathrm{Gr}\,(f|_A) \;\stackrel{df}{=}\; \big\{(x,y) \in A \times \mathbb{R}^M :\; y = f(x)\big\};$$

(b) *if f is Lipschitz continuous, then* $\dim\big(\mathrm{Gr}\,(f|_A)\big) = N$.

PROOF **(a)** Let $P\colon \mathbb{R}^{N+M} \longrightarrow \mathbb{R}^N$ be the projection operator. Operator P is Lipschitz continuous with $\mathrm{Lip}(P) = 1$. By virtue of Theorem 1.3.21 and Proposition 1.3.25, we have that

$$0 \;<\; \frac{1}{c_N}\lambda^N(A) \;=\; \mu^{(N)}(A) \;=\; \mu^{(N)}\big(P\big(\mathrm{Gr}\,(f|_A)\big)\big) \;\leqslant\; \mu^{(N)}\big(\mathrm{Gr}\,(f|_A)\big)$$

and so $\dim\big(\mathrm{Gr}\,(f|_A)\big) \geqslant N$ (see Definition 1.3.8).

(b) Let Q be the N-cube with side length 1. We subdivide Q into subcubes each with side length $\frac{1}{k}$. Denote these subcubes by $Q_1, \ldots, Q_{k^N}$. Note that $\delta(Q_j) = \frac{\sqrt{N}}{k}$ for $j \in \{1, \ldots, k^N\}$. We set

$$\begin{cases} \vartheta_{ij} \stackrel{df}{=} \min\limits_{x \in Q_j} f_i(x) \\ \xi_{ij} \stackrel{df}{=} \max\limits_{x \in Q_j} f_i(x) \end{cases} \qquad \forall\ i \in \{1, \ldots, M\},\ j \in \{1, \ldots, k^N\}.$$

Because f is Lipschitz continuous, we have

$$|\xi_{ij} - \vartheta_{ij}| \ \leqslant\ \mathrm{Lip}(f)\delta(Q_{ij}) \ =\ \mathrm{Lip}(f)\frac{\sqrt{N}}{k}.$$

Let

$$C_j \ \stackrel{df}{=}\ Q_j \times \prod_{i=1}^{M} (\vartheta_{ij}, \xi_{ij}).$$

Evidently, we have

$$\mathrm{Gr}\,(f|_{A \cap Q_j}) \ \subseteq\ C_j \quad \text{and} \quad \delta(C_j) \ \leqslant\ \frac{\mathrm{Lip}(f)}{k}.$$

Because $\mathrm{Gr}\,(f|_{A \cap Q}) \subseteq \bigcup\limits_{j=1}^{k^N} C_j$, we have that

$$\mu^{(N)}_{\frac{\mathrm{Lip}(f)}{k}}\big(\mathrm{Gr}\,(f|_{A \cap Q})\big) \ \leqslant\ \sum_{j=1}^{k^N} \delta(C_j)^N \ \leqslant\ k^N \frac{(\mathrm{Lip}(f))^N}{k^N} \ =\ \big(\mathrm{Lip}(f)\big)^N.$$

Let $k \to +\infty$, to obtain that

$$\mu^{(N)}\big(\mathrm{Gr}\,(f|_{A \cap Q})\big) \ \leqslant\ \mathrm{Lip}(f)^N \ <\ +\infty$$

and so $\dim\big(\mathrm{Gr}\,(f|_{A \cap Q})\big) \leqslant N$. Since Q was any N-cube of side length 1, we conclude that $\dim\big(\mathrm{Gr}\,(f|_A)\big) \leqslant N$. This estimate combined with part **(a)** implies that $\dim \mathrm{Gr}\,(f|_A) = N$. □

REMARK 1.5.3 Later in this section, as an application of the so-called ***area formula*** (see Theorem 1.5.21), we will produce an integral formula for $\mu^{(N)}\big(\mathrm{Gr}\,(f|_A)\big)$ (see Example 1.5.24**(b)**) which is well known from multivariable calculus.

For a given $A \subseteq \mathbb{R}^N$ and a Lipschitz continuous function $f\colon A \longrightarrow \mathbb{R}^M$, often it is useful to know whether we can extend f to all of $\mathbb{R}^N$, preserve the Lipschitz property and when $M = 1$ not increase the Lipschitz constant $\mathrm{Lip}(f)$.

THEOREM 1.5.4
If X is a Banach space, $A \subseteq X$ is nonempty and $f\colon A \longrightarrow \mathbb{R}^M$ is Lipschitz continuous,
then there exists a Lipschitz continuous function $\widehat{f}\colon X \longrightarrow \mathbb{R}^M$, such that $\widehat{f}|_A = f$ and $\mathrm{Lip}(\widehat{f}) \leqslant \sqrt{M}\mathrm{Lip}(f)$.

PROOF First suppose that $M = 1$ and define

$$\widehat{f}(x) \;=\; \inf_{a\in A}\big\{f(a) + \mathrm{Lip}(f)\,\|x-a\|_X\big\}. \tag{1.35}$$

Suppose that $y \in A$. Because f is Lipschitz continuous on A, we have

$$f(y) \;\leqslant\; f(a) + \mathrm{Lip}(f)\,\|y-a\|_X \qquad \forall\, a \in A,$$

so

$$f(y) \;\leqslant\; \widehat{f}(y).$$

On the other hand it is clear from (1.35) that $\widehat{f}(y) \leqslant f(y)$ for all $y \in A$. Therefore we infer that $\widehat{f}|_A = f$.

Next let $x, z \in \mathbb{R}^N$. Then for all $a \in A$, we have that

$$\begin{aligned}\widehat{f}(x) &\leqslant f(a) + \mathrm{Lip}(f)\,\|x-a\|_X \\ &\leqslant f(a) + \mathrm{Lip}(f)\,\|x-z\|_X + \mathrm{Lip}(f)\,\|z-a\|_X\,,\end{aligned}$$

so, from (1.35), we have

$$\widehat{f}(x) \;\leqslant\; \widehat{f}(z) + \mathrm{Lip}(f)\,\|x-z\|_X\,. \tag{1.36}$$

By interchanging the roles of x and z, we also obtain

$$\widehat{f}(z) \;\leqslant\; \widehat{f}(x) + \mathrm{Lip}(f)\,\|x-z\|_X\,. \tag{1.37}$$

From (1.36) and (1.37), we conclude that

$$\big|\widehat{f}(x) - \widehat{f}(z)\big| \;\leqslant\; \mathrm{Lip}(f)\,\|x-z\|_X \qquad \forall\, x, z \in \mathbb{R}^N.$$

If $M > 1$, then we define $\widehat{f} = \big(\widehat{f}_i\big)_{i=1}^M$ (f_i are the component functions of f). We have

$$\big\|\widehat{f}(x) - \widehat{f}(z)\big\|_{\mathbb{R}^M}^2 \;=\; \sum_{i=1}^M \big|\widehat{f}_i(x) - \widehat{f}_i(z)\big|^2 \;\leqslant\; M\big(\mathrm{Lip}(f)\big)^2\,\|x-z\|_{\mathbb{R}^N}^2\,,$$

so

$$\mathrm{Lip}\big(\widehat{f}\big) \;\leqslant\; \sqrt{M}\mathrm{Lip}(f).$$

□

REMARK 1.5.5 Let $f\colon X \longrightarrow \mathbb{R}$ and $\overline{f}\colon X \longrightarrow \mathbb{R}$ be defined by

$$\overline{f}(x) \stackrel{df}{=} \begin{cases} f(x) & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus A, \end{cases}$$

then as we shall see in Chapter 4, $\widehat{f} = \overline{f} \oplus \mathrm{Lip}(f)\,\|\cdot\|_X$, where $\oplus$ denotes the operation of infimal convolution (see Definition 4.4.6**(b)**). Since this operation preserves convexity, then if $A \subseteq X$ is convex and $f\colon A \longrightarrow \mathbb{R}$ is Lipschitz continuous and convex, then so is $\widehat{f}\colon X \longrightarrow \mathbb{R}$. Also note that the extension $\widehat{f}$ obtained in Theorem 1.5.4 is maximal in the sense that if $g\colon X \longrightarrow \mathbb{R}$ is any Lipschitz continuous function with $\mathrm{Lip}(g) \leqslant \mathrm{Lip}(f)$, such that $g|_A = f$, then $g \leqslant \widehat{f}$. Indeed note that

$$g(x) - f(y) \;\leqslant\; \mathrm{Lip}(f)\,\|x - y\|_X \qquad \forall\, x \in X,\ y \in A,$$

hence $g(x) \leqslant \widehat{f}(x)$. A minimal such extension can be obtained by considering the function

$$\widetilde{f}(x) \stackrel{df}{=} \sup_{a \in A} \left[f(a) - \mathrm{Lip}(f)\,\|x - a\|_X \right].$$

This extension is known as the ***McShane extension of*** f and was obtained by McShane (1934) who was the first to study the problem of extension of Lipschitz continuous functions. Finally we mention that Kirszbraun (1934) produced an extension $\widehat{f}$ of a Lipschitz continuous function $f\colon A \longrightarrow \mathbb{R}^M$, such that $\mathrm{Lip}(\widehat{f}) = \mathrm{Lip}(f)$ (see also Federer (1969, p. 201)).

One of the main theorems concerning Lipschitz continuous functions is the so-called ***Rademacher's theorem***, which asserts that a Lipschitz continuous function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is differentiable almost everywhere. This is the starting point for extending the subdifferential theory beyond the family of convex functions (see Chapter 4). First let us recall the following basic definition from multivariable calculus.

DEFINITION 1.5.6 *Let $U \subseteq \mathbb{R}^N$ be an open set. We say that a function $f\colon U \longrightarrow \mathbb{R}^M$ is* ***differentiable*** *(or* ***Fréchet differentiable****) at $x \in U$, if there exists $L(x) \in \mathcal{L}(\mathbb{R}^N; \mathbb{R}^M)$, such that*

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L(x)h}{\|h\|_X} \;=\; 0.$$

REMARK 1.5.7 Evidently $L(x)$ is unique, usually is denoted by $Df(x)$ or $f'(x)$ and it is called the derivative of f at x. From multivariable calculus, we know that if $M = 1$, then

$$Df(x)u \;=\; \sum_{k=1}^{N} \frac{\partial f}{\partial x_k}(x) u_k \;=\; \sum_{k=1}^{N} f'(x; e_k) u_k \qquad \forall\, u \in \mathbb{R}^N,$$

where $f'(x;v)$ is the ***directional*** or ***Gâteaux derivative*** of f at x in the direction v, defined by

$$f'(x;v) \stackrel{df}{=} \lim_{\lambda \to 0} \frac{f(x+\lambda v) - f(x)}{\lambda}$$

and $\{e_k\}_{k=1}^N$ is the canonical basis of $\mathbb{R}^N$.

THEOREM 1.5.8 (Rademacher Theorem)
If $U \subseteq \mathbb{R}^N$ is an open set and $f\colon U \longrightarrow \mathbb{R}^M$ is a Lipschitz continuous function,
then f is differentiable at λ^N-almost all $x \in U$.

PROOF Clearly we may assume that $M = 1$. For any $u \in \mathbb{R}^N$ with $\|u\|_{\mathbb{R}^N} = 1$, we set

$$f'(x;u) \stackrel{df}{=} \lim_{\lambda \to 0} \frac{f(x+\lambda u) - f(x)}{\lambda} \qquad \forall\, x \in U,$$

provided this limit exists.

Claim 1. $f'(x;u)$ exists for λ^N-almost all $x \in U$.
Let

$$f'_+(x;u) \stackrel{df}{=} \limsup_{\lambda \to 0} \frac{f(x+\lambda u) - f(x)}{\lambda} \qquad \forall\, x \in U$$
$$f'_-(x;u) \stackrel{df}{=} \liminf_{\lambda \to 0} \frac{f(x+\lambda u) - f(x)}{\lambda} \qquad \forall\, x \in U.$$

Evidently if

$$C_u \stackrel{df}{=} \big\{x \in U :\ f'(x;u) \text{ does not exist}\big\},$$

then

$$C_u = \big\{x \in U :\ f'_-(x;u) < f'_+(x;u)\big\}.$$

Note that

$$f'_+(x;u) = \inf_{k \geqslant 1} \sup_{\substack{0 < |\lambda| < \frac{1}{k} \\ \lambda \in \mathbb{Q}}} \frac{f(x+\lambda u) - f(x)}{\lambda},$$

so $f'_+(\cdot;u)$ is a Borel measurable function.

Similarly we show that $f'_-(\cdot;u)$ is a Borel measurable function. It follows that $C_u \in \mathcal{B}(U)$ (i.e., $C_u \subseteq U$ is a Borel measurable set).

Next for every $x, u \in \mathbb{R}^N$ with $\|u\|_{\mathbb{R}^N} = 1$, let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\varphi(\lambda) \stackrel{df}{=} f(x+\lambda u) \qquad \forall\, \lambda \in \mathbb{R}.$$

The function φ is Lipschitz continuous, hence absolutely continuous and so by fundamental theorem of Lebesgue calculus (see Theorem A.2.20), it is differentiable at almost every $\lambda \in \mathbb{R}$. Therefore

$$\mu^{(1)}(C_u \cap L) \;=\; 0,$$

for every line L parallel to the direction u. Hence by Fubini's theorem, we have

$$\lambda^N(C_u) \;=\; 0$$

and this proves the claim.

From Claim 1 and Remark 1.5.7, we see that

$$\nabla f(x) \;=\; \left(\frac{\partial f}{\partial x_k}(x)\right)_{k=1}^{N} \quad \text{exists} \quad \text{for } \lambda^N\text{-a.a. } x \in U.$$

Claim 2. $f'(x;u) = \big(u, \nabla f(x)\big)_{\mathbb{R}^N}$ for λ^N-almost all $x \in U$.
Let $\vartheta \in C_c^\infty(U)$. We have

$$\int_U \frac{f(x+\lambda u) - f(x)}{\lambda}\vartheta(x)\, d\lambda^N(x) \;=\; -\int_U f(x)\frac{\vartheta(x) - \vartheta(x-\lambda u)}{\lambda}\, d\lambda^N(x).$$

Let $\lambda = \frac{1}{k}$ for $k \geqslant 1$. Since f is Lipschitz continuous, we have

$$\left|\frac{f(x+\frac{1}{k}u) - f(x)}{\frac{1}{k}}\right| \;\leqslant\; \mathrm{Lip}(f)\,\|u\|_{\mathbb{R}^N} \;=\; \mathrm{Lip}(f).$$

Therefore when $k \to +\infty$, from the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that

$$\begin{aligned}
&\int_U f'(x;u)\vartheta(x)\, d\lambda^N(x) \;=\; -\int_U f(x)\vartheta'(x;u)\, d\lambda^N(x)\\
&= -\sum_{k=1}^{N} u_k \int_U f(x)\frac{\partial \vartheta}{\partial x_k}(x)\, d\lambda^N(x) \;=\; \sum_{k=1}^{N} u_k \int_U \frac{\partial f}{\partial x_k}(x)\vartheta(x)\, d\lambda^N(x)\\
&= \int_U \big(u, \nabla f(x)\big)_{\mathbb{R}^N}\vartheta(x)\, d\lambda^N(x).
\end{aligned}$$

Because $\vartheta \in C_c^\infty(U)$ is arbitrary, it follows that

$$f'(x;u) \;=\; \big(u, \nabla f(x)\big)_{\mathbb{R}^N} \quad \text{for } \lambda^N\text{-a.a. } x \in U.$$

This proves the second claim.

Let $\{u_n\}_{n\geqslant 1}$ be a dense subset of $\partial B_1(0)$. For $n \geqslant 1$, we define

$$E_n \stackrel{df}{=} \left\{x \in U : \ f'(x;u_n) \text{ and } \nabla f(x) \text{ exist and } f'(x;u_n) = \big(u_n, \nabla f(x)\big)_{\mathbb{R}^N}\right\}$$

and

$$E \stackrel{df}{=} \bigcap_{n=1}^{\infty} E_n.$$

By virtue of Claim 2, we have that $\lambda^N(U \setminus E) = 0$.

Claim 3. f is differentiable at every $x \in E$.

Let $x \in E$, $u \in \partial B_1(0)$, $\lambda \in \mathbb{R} \setminus \{0\}$ and set

$$\eta(x,u,\lambda) \stackrel{df}{=} \frac{f(x+\lambda u) - f(x)}{\lambda} - (u, \nabla f(x))_{\mathbb{R}^N}.$$

If $v \in \partial B_1(0)$, we have

$$\begin{aligned} &\big|\eta(x,u,\lambda) - \eta(x,v,\lambda)\big| \\ \leqslant\ &\left|\frac{f(x+\lambda u) - f(x+\lambda v)}{\lambda}\right| + \big|\big(u-v, \nabla f(x)\big)_{\mathbb{R}^N}\big| \\ \leqslant\ &\mathrm{Lip}(f)\,\|u-v\|_{\mathbb{R}^N} + \big\|\nabla f(x)\big\|_{\mathbb{R}^N}\|u-v\|_{\mathbb{R}^N}. \end{aligned}$$

Note that

$$\mathrm{Lip}\left(\frac{\partial f}{\partial x_k}\right) \leqslant 2\mathrm{Lip}(f) \qquad \forall\ k \in \{1,\dots,N\}$$

and so

$$\mathrm{Lip}(\nabla f) \leqslant 2\sqrt{N}\mathrm{Lip}(f)$$

(see the proof of Theorem 1.5.4). Therefore, we have

$$\big|\eta(x,u,\lambda) - \eta(x,v,\lambda)\big| \leqslant \big(1+2\sqrt{N}\big)\mathrm{Lip}(f)\,\|u-v\|_{\mathbb{R}^N}. \tag{1.38}$$

Let $\varepsilon > 0$ be given. We can choose $l \geqslant 1$ large enough so that

$$\forall v \in \partial B_1(0)\ \exists k \in \{1,\dots,l\} : \ \|v-u_k\|_{\mathbb{R}^N} \leqslant \frac{\varepsilon}{2(1+2\sqrt{N})\mathrm{Lip}(f)}. \tag{1.39}$$

As $x \in E$, we have

$$\lim_{\lambda\to 0} \eta(x,u_k,\lambda) = 0$$

(recall that $x \in E$). So we can find $\delta > 0$, such that

$$\big|\eta(x,u_k,\lambda)\big| < \frac{\varepsilon}{2} \qquad \forall\ 0 < |\lambda| < \delta,\ k \in \{1,\dots,l\}. \tag{1.40}$$

Thus from (1.38), (1.39) and (1.40), for every $v \in \partial B_1(0)$, we can find $k \in \{1, \ldots, l\}$, such that for all $0 < |\lambda| < \delta$, we have

$$\big|\eta(x, v, \lambda)\big| \;\leqslant\; \big|\eta(x, u_k, \lambda)\big| + \big|\eta(x, v, \lambda) - \eta(x, u_k, \lambda)\big| \;<\; \varepsilon. \tag{1.41}$$

We emphasize that $\delta > 0$ is independent of $v \in \partial B_1(0)$. Let $y \in U$, $y \neq x$ and let us set

$$v \;\stackrel{df}{=}\; \frac{y - x}{\|y - x\|_{\mathbb{R}^N}} \;\in\; \partial B_1(0).$$

We have $y = x + \lambda v$, with $\lambda = \|y - x\|_{\mathbb{R}^N}$. Then

$$\begin{aligned} f(y) - f(x) - \big(\nabla f(x), y - x\big)_{\mathbb{R}^N} \;&=\; f(y) - f(x) + \lambda\big(\nabla f(x), v\big)_{\mathbb{R}^N} \\ =\; o(\lambda) \;&=\; o\big(\,\|y - x\|_{\mathbb{R}^N}\,\big) \quad \text{as} \quad y \to x. \end{aligned}$$

Therefore f is differentiable at $x \in E$ with $Df(x) = \nabla f(x)$. This proves the claim and the theorem. ∎

COROLLARY 1.5.9
If $U \subseteq \mathbb{R}^N$ is an open set and $f\colon U \longrightarrow \mathbb{R}^M$ is a locally Lipschitz function, then f is differentiable at λ^N-almost all $x \in U$.

PROOF Again we may assume that $M = 1$. Note that since f is locally Lipschitz, it is Lipschitz continuous when restricted to any compact set $K \subseteq U$. Indeed, if this is not true, then we can find a compact set $K \subseteq U$ and two sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subseteq K$, such that

$$n\,\|x_n - y_n\|_{\mathbb{R}^N} \;<\; \big|f(x_n) - f(y_n)\big| \qquad \forall\; n \geqslant 1.$$

Note that

$$\|x_n - y_n\|_{\mathbb{R}^N} \;\leqslant\; \frac{2}{n} \max_K |f| \qquad \forall\; n \geqslant 1.$$

Since K is compact, we can produce two subsequences $\{x_{n_k}\}_{k\geqslant 1}$ of $\{x_n\}_{n\geqslant 1}$ and $\{y_{n_k}\}_{k\geqslant 1}$ of $\{x_n\}_{n\geqslant 1}$, such that

$$x_{n_k} \;\longrightarrow\; v \quad \text{and} \quad y_{n_k} \;\longrightarrow\; v,$$

for some $v \in K$, which contradicts the fact that in a neighbourhood of x, the function f is Lipschitz continuous.

Next let $\{U_n\}_{n\geqslant 1}$ be an increasing sequence of bounded open subsets of U, such that $U = \bigcup_{n=1}^{\infty} U_n$, for example let

$$U_n \;\stackrel{df}{=}\; \left\{ x \in U :\; \|x\|_{\mathbb{R}^N} < n,\; \frac{1}{n} < d(x, \partial U) \right\} \qquad \forall\; n \geqslant 1.$$

Then by virtue of Theorem 1.5.8, f is differentiable at λ^N-almost all $x \in U_n$, for $n \geqslant 1$. Therefore f is differentiable λ^N-almost everywhere on U. ∎

COROLLARY 1.5.10
If $U \subseteq \mathbb{R}^N$ is an open set, $f\colon U \longrightarrow \mathbb{R}^M$ is a locally Lipschitz function and

$$Z \stackrel{df}{=} \big\{x \in U :\ f(x) = 0\big\},$$

then $Df(x) = 0$ for λ^N-almost all $x \in Z$.

PROOF As before we may assume that $M = 1$. Also suppose that $\lambda^N(Z) > 0$ or otherwise there is nothing to prove. Then by virtue of Theorems 1.4.1 and 1.5.8, we can choose $x \in Z$, such that $Df(x)$ exists and

$$\lim_{r \searrow 0} \frac{\lambda^N(\overline{B}_r(x) \cap Z)}{\lambda^N(\overline{B}_r(x))} \ = \ 1. \tag{1.42}$$

We have

$$f(y) \ = \ \big(\nabla f(x), y - x\big)_{\mathbb{R}^N} + o\big(\|y - x\|_{\mathbb{R}^N}\big) \quad \text{as } y \to x. \tag{1.43}$$

Suppose that

$$v \stackrel{df}{=} \nabla f(x) \neq 0.$$

We introduce the set

$$C \stackrel{df}{=} \left\{u \in \partial B_1(0) :\ (v, u)_{\mathbb{R}^N} \geqslant \frac{\|v\|_{\mathbb{R}^N}}{2}\right\}.$$

For a given $u \in C$ and $t > 0$, in (1.43), we set $y = x + tu$ and we have

$$f(x + tu) \ = \ t\big(\nabla f(x), u\big)_{\mathbb{R}^N} + o(t\,\|u\|_{\mathbb{R}^N}) \ \geqslant \ \frac{t\,\|v\|_{\mathbb{R}^N}}{2} + o(t),$$

so

$$\frac{f(x + tv)}{t} \ \geqslant \ \frac{\|v\|_{\mathbb{R}^N}}{2} + \frac{o(t)}{t}$$

and

$$f(x + tu) \ > \ 0 \qquad \forall\ t \in (0, t^*),\ u \in C,$$

for some $t^* > 0$, a contradiction to (1.42). ∎

COROLLARY 1.5.11
If $f_1, f_2\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ are locally Lipschitz and

$$E \stackrel{df}{=} \big\{x \in \mathbb{R}^N :\ (f_2 \circ f_1)(x) = x\big\},$$

then $D(f_2 \circ f_1)(x) = Df_2\big(f_1(x)\big)Df_1(x) = id_{\mathbb{R}^N}$ for λ^N-almost all $x \in E$.

PROOF Let

$$\operatorname{dom} f_i \stackrel{df}{=} \big\{x \in \mathbb{R}^N : \ Df_i(x) \text{ exists}\big\}, \quad \text{for } i = 1, 2.$$

We set

$$C \stackrel{df}{=} E \cap \operatorname{dom} f_1 \cap f_1^{-1}(dom f_2).$$

If $x \in C \setminus f_1^{-1}(\operatorname{dom} f_2)$, then

$$f_1(x) \ \in \ \mathbb{R}^N \setminus \operatorname{dom} f_2$$

and so

$$x \ = \ f_2\big(f_1(x)\big) \ = \ \big(f_2 \circ f_1\big)(x) \ \in \ f_2\big(\mathbb{R}^N \setminus \operatorname{dom} f_2\big).$$

It follows that

$$E \setminus C \ \subseteq \ \big(\mathbb{R}^N \setminus \operatorname{dom} f_1\big) \cup f_2\big(\mathbb{R}^N \setminus \operatorname{dom} f_2\big). \tag{1.44}$$

Invoking Theorem 1.5.8, from (1.44), we infer that

$$\lambda^N(C \setminus E) \ = \ 0$$

(recall that a Lipschitz continuous function maps Lebesgue-null sets to Lebesgue-null sets). If $x \in C$, then from the definition of C, we see that $Df_1(x)$ and $Df_2\big(f_1(x)\big)$ exist and so

$$Df_2\big(f_1(x)\big)Df_1(x) \ = \ D(f_2 \circ f_1)(x)$$

(chain rule). Since $(f_2 \circ f_1)(x) - x = 0$, from Corollary 1.5.10, we infer that

$$Df_2\big(f_1(x)\big)Df_1(x) \ = \ id \quad \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N.$$

□

Continuing with our investigation of Lipschitz continuous maps $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, we aim at deriving ***change of variables*** formulas. We distinguish two cases. In the first case $N \leqslant M$ and the change of variables formula is obtained via the so-called ***area formula***, which asserts that N-dimensional measure of $f(A)$ can be calculated by integrating a suitable Jacobian. In the second case $M \leqslant N$ and the change of variables formula passes through the so-called ***coarea formula***, which asserts that the integral of the $(N-M)$-dimensional measure of the level sets of f is computed by integrating a suitable Jacobian.

First we derive the ***area formula*** and for this we need some preparation. We start with a result from linear algebra, known as ***polar decomposition***. It produces for a linear operator $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ an analog of the polar representation $z = re^{i\vartheta}$ of a complex number. First some definitions.

DEFINITION 1.5.12 **(a)** *An operator $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is said to be* ***orthogonal****, if*

$$\big(Ux, Uy\big)_{\mathbb{R}^M} \;=\; (x, y)_{\mathbb{R}^N} \qquad \forall\; x, y \in \mathbb{R}^N.$$

(b) *For a given operator $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, its* ***adjoint*** *$L^*\colon \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is defined by*

$$\big(Lx, y\big)_{\mathbb{R}^M} \;=\; \big(x, L^*y\big)_{\mathbb{R}^N} \qquad \forall\; x \in \mathbb{R}^N,\; y \in \mathbb{R}^M.$$

(c) *An operator $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is* ***self-adjoint****, if $L^* = L$.*

(d) *An operator $S\colon \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is said to be* ***positive*** *(we write $S \geqslant 0$), if it is self-adjoint (i.e., $S = S^*$) and*

$$\big(Sx, x\big)_{\mathbb{R}^N} \;\geqslant\; 0 \qquad \forall\; x \in \mathbb{R}^N.$$

REMARK 1.5.13 If $N = M$, then $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is orthogonal if and only if $U^* = U^{-1}$ (hence U is invertible). In general, if $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is orthogonal, then $N \leqslant M$ and

$$U^* \circ U \;=\; id_{\mathbb{R}^N}.$$

Also $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is orthogonal if and only if it is an isometry. Finally, if $S\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is self-adjoint, then we can find an orthogonal operator $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and a diagonalizable operator $D\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$, such that

$$S \;=\; U \circ D \circ U^{-1}.$$

A positive operator $S\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ has a unique positive square root $T\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$, i.e., $T^2 = S$.

THEOREM 1.5.14 (Polar Decomposition Theorem)
Let $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ be a linear operator.

(a) *If $N \leqslant M$, then there exist a positive operator $S\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and an orthogonal operator $U\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, such that*

$$L \;=\; U \circ S.$$

(b) *If $M \leqslant N$, then there exists a positive operator $S\colon \mathbb{R}^M \longrightarrow \mathbb{R}^M$ and an orthogonal operator $U\colon \mathbb{R}^M \longrightarrow \mathbb{R}^N$, such that*

$$L \;=\; S \circ U^*.$$

PROOF **(a)** Since $L^{**} = L$, the operator $L^* \circ L \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is clearly positive. So it admits a unique square root $S \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ (see Remark 1.5.13). For each $y = Sx \in R(S)$, we write $Uy = Lx$, motivated by the fact that eventually we must have $L = U \circ S$. First, we need to show that with this definition U is unambiguously defined on $R(S)$, that is

$$\text{if } Sx_1 = Sx_2, \text{ then } Lx_1 = Lx_2.$$

Note that $Sx_1 = Sx_2$ is equivalent to saying that $\big\|S(x_1 - x_2)\big\|_{\mathbb{R}^N} = 0$ and this condition implies that

$$\big\|L(x_1 - x_2)\big\|_{\mathbb{R}^M} \;=\; 0.$$

Therefore U is well defined on $R(S)$ and its range equals $R(L)$. Note that L and S have the same kernel. So

$$\dim R(L) \;=\; \dim R(S) \quad \text{and} \quad \dim R(L)^{\perp} \;=\; \dim R(S)^{\perp}.$$

Therefore there exists an isometric isomorphism $U_0 \colon R(S)^{\perp} \longrightarrow R(L)^{\perp}$. We extend U on $R(S)^{\perp}$ by setting it equal to U_0. Since

$$\mathbb{R}^N \;=\; R(S) \oplus R(S)^{\perp},$$

every $y \in \mathbb{R}^N$ can be written in a unique way as

$$y = Sx + u, \quad \text{with} \quad u \in R(S)^{\perp}.$$

We set $Uy = Lx + U_0 u$ and we have $U \colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, which is linear and well defined. Also, exploiting the orthogonality of $R(S)$ and $R(S)^{\perp}$, we have

$$\begin{aligned} \big(Uy, Uy\big)_{\mathbb{R}^N} \;&=\; \big(Lx + U_0 u, Lx + U_0 u\big)_{\mathbb{R}^M} \\ =\; \big(Lx, Lx\big)_{\mathbb{R}^M} + \big(U_0 u, U_0 u\big)_{\mathbb{R}^M} \;&=\; \big(Sx, Sx\big)_{\mathbb{R}^N} + \big(u, u\big)_{\mathbb{R}^N} \;=\; \big(y, y\big)_{\mathbb{R}^N}, \end{aligned}$$

so U is orthogonal and $U \circ S = L$.

(b) Follows if we apply **(a)** to the operator $L^* \colon \mathbb{R}^M \longrightarrow \mathbb{R}^N$. □

REMARK 1.5.15 In a polar decomposition $L = U \circ S$, the positive operator S is unique. Indeed suppose that $U \circ S = U_1 \circ S_1$. Then by taking adjoints, we obtain $S \circ U^* = S_1 \circ U_1^*$ and so

$$S^2 \;=\; S \circ U^* \circ U \circ S \;=\; S_1 \circ U_1^* \circ U_1 \circ S_1 \;=\; S_1^2.$$

The positive operator $S^2 = S_1^2$ has a unique square root, hence $S = S_1$. Moreover, if $N = M$ and the operator L is invertible, then in the polar decomposition $L = U \circ S$, the orthogonal operator U is unique too. Indeed, since L is invertible, so is S (since $S = U^{-1} \circ L$). Then from $U \circ S = U_1 \circ S_1$ and since $S^{-1} = S_1^{-1}$, we have that

$$U \;=\; U_1 \circ S_1 \circ S^{-1} \;=\; U_1 \circ S_1 \circ S_1^{-1} \;=\; U_1.$$

We can use Theorem 1.5.14 to define the Jacobian of a Lipschitz continuous map $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$.

DEFINITION 1.5.16 *Let $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ be a linear operator.*

(a) *If $N \leqslant M$ and $L = U \circ S$ is a polar decomposition of L (see Theorem 1.5.14), then we define the* ***Jacobian*** *of L to be*

$$\operatorname{jac} L \stackrel{df}{=} \big|\det S\big|.$$

(b) *If $M \leqslant N$ and $L = S \circ U^*$ is a polar decomposition of L (see Theorem 1.5.14), then we define the* ***Jacobian*** *of L to be*

$$\operatorname{jac} L \stackrel{df}{=} \big|\det S\big|.$$

(c) *If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is Lipschitz continuous and*

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix}$$

is the $M \times N$-gradient matrix, then the ***Jacobian*** *of f is defined by*

$$Jf(x) \stackrel{df}{=} \operatorname{jac} Df(x) \quad \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N.$$

REMARK 1.5.17 Since in a polar decomposition the positive operator is uniquely defined (see Remark 1.5.15), then we see that the notions introduced in Definition 1.5.16 are well defined. If $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a linear operator, then we can easily check that if $N \leqslant M$, we have

$$\operatorname{jac} L = \det(L^* \circ L),$$

while if $M \leqslant N$, we have

$$\operatorname{jac} L = \det(L \circ L^*).$$

Another expression computing $\operatorname{jac} L^2$ is given by the so-called ***Binet-Cauchy formula***. So let $N \leqslant M$ and set

$$\Theta(N, M) \stackrel{df}{=} \big\{\theta\colon \{1, \ldots, N\} \longrightarrow \{1, \ldots, M\} \text{ is increasing}\big\}.$$

For each $\theta \in \Theta$, we define $P_\theta\colon \mathbb{R}^M \longrightarrow \mathbb{R}^N$ by

$$P_\theta(x_1, \ldots, x_M) \stackrel{df}{=} (x_{\theta(1)}, \ldots, x_{\theta(N)}).$$

Clearly P_θ is the projection operator of $\mathbb{R}^M$ on some N-dimensional subspace $V = \operatorname{span}\{e_{\theta(k)}\}_{k=1}^N$. Then, if $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a linear operator then a straightforward but cumbersome proof gives the ***Binet-Cauchy formula***:

$$\operatorname{jac} L^2 \;=\; \sum_{\theta\in\Theta(N,M)} \det(P_\theta \circ L)^2.$$

For details we refer to Evans & Gariepy (1992, p. 89).

LEMMA 1.5.18
If $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a linear operator, $N \leqslant M$ and $A \subseteq \mathbb{R}^N$, then $\mu^{(N)}\big(L(A)\big) = \operatorname{jac} L \cdot \lambda^N(A)$.

PROOF Let $L = U\circ S$ be a polar decomposition of L (see Theorem 1.5.14). We know that $\operatorname{jac} L = \big|\det S\big|$ (see Definition 1.5.16**(a)**).

If $\operatorname{jac} L = 0$, then $\det S = 0$ and so S is not surjective, i.e., $\dim R(S) \leqslant N-1$. It follows that $\mu^{(N)}\big(L(A)\big) = 0$ (see, e.g., Proposition 1.3.23).

If $\operatorname{jac} L > 0$, then using the orthogonality of U and the facts that $\mu^{(N)} = \lambda^N$ in $\mathbb{R}^N$, $L = U \circ S$ and $U^* \circ U = id_{\mathbb{R}^N}$, we have

$$\begin{aligned}\frac{\mu^{(N)}(L(\overline{B}_r(x)))}{\lambda^N(\overline{B}_r(x))} &= \frac{\lambda^N((U^*\circ L)(\overline{B}_r(x)))}{\lambda^N(\overline{B}_r(x))} = \frac{\lambda^N((U^*\circ U\circ S)(\overline{B}_r(x)))}{\lambda^N(\overline{B}_r(x))}\\ = \frac{\lambda^N(S(\overline{B}_r(x)))}{\lambda^N(\overline{B}_r(x))} &= \frac{\lambda^N(S(\overline{B}_1(0)))}{a(N)} = \big|\det S\big| = \operatorname{jac} L, \qquad (1.45)\end{aligned}$$

$a(N) \overset{df}{=} \dfrac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$ is the volume of the unit ball in $\mathbb{R}^N$. Finally let

$$\vartheta(A) \;\overset{df}{=}\; \mu^{(N)}\big(L(A)\big) \qquad \forall\, A \subseteq \mathbb{R}^N.$$

Then ϑ is a Radon measure and $\vartheta \prec\prec \lambda^N$. So the Radon-Nikodym derivative of ϑ with respect to λ^N (see Theorem A.2.24 and Remark A.2.25) exists and is given by

$$\frac{d\vartheta}{d\lambda^N}(x) \;=\; \lim_{r\searrow 0} \frac{\vartheta(\overline{B}_r(x))}{\lambda^N(\overline{B}_r(x))} \;=\; \operatorname{jac} L$$

(see (1.45) and Widom (1969, p. 119)). From the Radon-Nikodym theorem (see Theorem A.2.24), we infer that for all Borel sets $A \subseteq \mathbb{R}^N$, we have

$$\mu^{(N)}\big(L(A)\big) \;=\; \operatorname{jac} L \cdot \lambda^N(A). \qquad (1.46)$$

Because ϑ and λ^N are both Radon measures, we conclude that (1.46) holds for all $A \subseteq \mathbb{R}^N$. ∎

LEMMA 1.5.19
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is Lipschitz continuous, $N \leqslant M$ and $A \subseteq \mathbb{R}^N$ is Lebesgue measurable,
then

(a) *$f(A)$ is a $\mu^{(N)}$-measurable set;*

(b) *the map $y \longmapsto \mu^{(0)}\big(A \cap f^{-1}(y)\big)$ is $\mu^{(N)}$-measurable on $\mathbb{R}^M$;*

(c) $\displaystyle\int_{\mathbb{R}^M} \mu^{(0)}\big(A \cap f^{-1}(y)\big)\mu^{(N)}(y) \leqslant \big(\mathrm{Lip}(f)\big)^N \lambda^N(A).$

PROOF Clearly we can assume with any loss of generality that A is bounded (if not, consider instead $A \cap B_r(0)$).

(a) From the regularity of the Lebesgue measure, we know that for every $i \geqslant 1$, we can find compact set $K_i \subseteq A$, such that

$$\lambda^N(A \setminus K_i) \;\leqslant\; \frac{1}{i} \qquad \forall\, i \geqslant 1.$$

Because f is Lipschitz continuous, $f(K_i) \subseteq \mathbb{R}^M$ is compact and so it is $\mu^{(N)}$-measurable. Then

$$f\Big(\bigcup_{i=1}^{\infty} K_i\Big) \;=\; \bigcup_{i=1}^{\infty} f(K_i) \quad \text{is a } \mu^{(N)}\text{-measurable set.}$$

Also, using Proposition 1.3.25, we have

$$\mu^{(N)}\Big(f(A) \setminus f\Big(\bigcup_{i=1}^{\infty} K_i\Big)\Big) \;\leqslant\; \mu^{(N)}\Big(f\Big(A \setminus \bigcup_{i=1}^{\infty} K_i\Big)\Big)$$
$$\leqslant\; \mathrm{Lip}(f)^N \lambda^N\Big(A \setminus \bigcup_{i=1}^{\infty} K_i\Big) \;=\; 0,$$

so $f(A)$ is $\mu^{(N)}$-measurable.

(b) For $n \geqslant 1$, we introduce the following families of N-cubes

$$\mathcal{F}_k \;\stackrel{df}{=}\; \Big\{Q \subseteq \mathbb{R}^N :\; Q = \prod_{j=1}^{N} \Big(\frac{c_j}{k}, \frac{c_j+1}{k}\Big],\; c_j \text{ are integers},\; j \in \{1,\dots,N\}\Big\}.$$

Let

$$h_k \;\stackrel{df}{=}\; \sum_{Q \in \mathcal{F}_k} \chi_{f(A \cap Q)}.$$

Then by part **(a)**, h_k is $\mu^{(N)}$-measurable and for every $y \in \mathbb{R}^N$, $h_k(y)$ is the number of cubes $Q \in \mathcal{F}_k$, such that

$$(A \cap Q) \cap f^{-1}\big(\{y\}\big) \neq \emptyset.$$

Hence for all $y \in \mathbb{R}^N$, we have

$$h_k(y) \nearrow \mu^{(0)}\big(A \cap f^{-1}\big(\{y\}\big)\big) \quad \text{as } k \to +\infty,$$

so the function $y \longmapsto \mu^{(0)}\big(A \cap f^{-1}(y)\big)$ is $\mu^{(N)}$-measurable.

(c) Using the monotone convergence theorem (see Theorem A.2.10) and using Proposition 1.3.25 and the fact that

$$\mathbb{R}^N = \bigcup_{Q \in \mathcal{F}_k} Q \qquad \forall\, k \geqslant 1,$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^M} \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y) &= \lim_{k \to +\infty} \int_{\mathbb{R}^M} h_k(y)\, d\mu^{(N)}(y) \\
= \lim_{k \to +\infty} \sum_{Q \in \mathcal{F}_k} \mu^{(N)}\big(f(A \cap Q)\big) &\leqslant \limsup_{k \to +\infty} \sum_{Q \in \mathcal{F}_k} \big(\mathrm{Lip}(f)\big)^N \lambda^N(A \cap Q) \\
= \big(\mathrm{Lip}(f)\big)^N \lambda^N(A).
\end{aligned}$$

□

LEMMA 1.5.20
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is Lipschitz continuous, $t > 1$ and

$$C \stackrel{df}{=} \big\{x \in \mathbb{R}^N :\ Df(x) \text{ exists and } Jf(x) > 0\big\},$$

then there exists a sequence $\{E_i\}_{i \geqslant 1}$ of Borel subsets of $\mathbb{R}^N$, such that

(a) $C = \bigcup_{i=1}^{\infty} E_i$;

(b) $f|_{E_i}$ *is injective for all $i \geqslant 1$;*

(c) *for each $i \geqslant 1$ there exists a self-adjoint isomorphism $L_i\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$, such that*

$$\mathrm{Lip}\big((f|_{E_i}) \circ L_i^{-1}\big) \leqslant t, \quad \mathrm{Lip}\big(L_i^{-1} \circ (f|_{E_i})\big) \leqslant t,$$

and

$$\frac{|\det L_i|}{t^N} \leqslant Jf|_{E_i} \leqslant t^N |\det L_i|.$$

PROOF Choose $\varepsilon > 0$ so that $\frac{1}{t} + \varepsilon < 1 < t - \varepsilon$. Let E be a countable dense subset of C and let G be a countable dense subset of the space of self-adjoint isomorphisms of $\mathbb{R}^N$.

For each $u \in E$, $L \in G$ and $k \geqslant 1$, we set $E(u, L, k)$ to be the set of all $x \in C \cap \overline{B}_{\frac{1}{k}}(u)$, such that

$$\left(\frac{1}{t} + \varepsilon\right) \|Lh\|_{\mathbb{R}^N} \leqslant \big\|Df(x)h\big\|_{\mathbb{R}^M} \leqslant (t - \varepsilon)\, \|Lh\|_{\mathbb{R}^N} \qquad \forall\, h \in \mathbb{R}^N \quad (1.47)$$

and

$$\big\|f(y) - f(x) - Df(x)(y-x)\big\|_{\mathbb{R}^M} \leqslant \varepsilon \big\|L(y-x)\big\|_{\mathbb{R}^N} \qquad \forall\, y \in \overline{B}_{\frac{2}{k}}(x). \quad (1.48)$$

Since $x \longmapsto Df(x)$ is a Borel function, we see that $E(u, L, k)$ is a Borel subset of $\mathbb{R}^N$. From (1.47) and (1.48), it follows that

$$\frac{1}{t}\big\|L(y-x)\big\|_{\mathbb{R}^N} \leqslant \big\|f(y) - f(x)\big\|_{\mathbb{R}^M} \leqslant t\big\|L(y-x)\big\|_{\mathbb{R}^N} \qquad \forall\, x \in E(u, L, k),\ y \in \overline{B}_{\frac{2}{k}}(x). \quad (1.49)$$

Claim 1. If $x \in E(u, L, k)$, then

$$\left(\frac{1}{t} + \varepsilon\right)^N \big|\det L\big| \leqslant Jf(x) \leqslant (t - \varepsilon)^N \big|\det L\big|.$$

Let

$$Df(x) = L = U \circ S$$

(see Theorem 1.5.14**(a)**). According to Definition 1.5.16**(c)**,

$$Jf(x) = \operatorname{jac} Df(x) = \big|\det S\big|.$$

From (1.47), we have

$$\left(\frac{1}{t} + \varepsilon\right) \|Lh\|_{\mathbb{R}^N} \leqslant \big\|(U \circ S)h\big\|_{\mathbb{R}^M} = \|Sh\|_{\mathbb{R}^N} \leqslant (t - \varepsilon)\, \|Lh\|_{\mathbb{R}^N} \quad \forall h \in \mathbb{R}^N.$$

Since $L \in G$, we have

$$\left(\frac{1}{t} + \varepsilon\right) \|h\|_{\mathbb{R}^N} \leqslant \big\|(S \circ L^{-1})h\big\|_{\mathbb{R}^N} \leqslant (t - \varepsilon)\, \|h\|_{\mathbb{R}^N} \qquad \forall\, h \in \mathbb{R}^N,$$

so

$$(S \circ L^{-1})\big(\overline{B}_1(0)\big) \subseteq \overline{B}_{t-\varepsilon}(0),$$

thus

$$\big|\det(S \circ L^{-1})\big|\, a(N) \leqslant \lambda^N\big(\overline{B}_{t-\varepsilon}(0)\big) = a(N)(t - \varepsilon)^N,$$

$a(N) \stackrel{df}{=} \dfrac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!}$ is the volume of the unit ball in $\mathbb{R}^N$. Finally

$$\big|\det S\big| \;\leqslant\; (t-\varepsilon)^N \big|\det L\big|.$$

Similarly, we prove the other inequality of the claim. So Claim 1 is proved.

Let $\{E_i\}_{i\geqslant 1}$ be an enumeration of the countable set

$$\big\{E(u,L,k):\; u\in E,\; L\in G,\; k\geqslant 1\big\}.$$

(a) Let $x\in C$, while

$$Df(x) \;=\; U\circ S$$

(by Theorem 1.5.14**(a)**). Select $L\in G$, such that

$$\mathrm{Lip}(L\circ S^{-1}) \;=\; \big\|L\circ S^{-1}\big\|_{\mathcal{L}} \;\leqslant\; \left(\frac{1}{t}+\varepsilon\right)^{-1}$$

and

$$\mathrm{Lip}(S\circ L^{-1}) \;=\; \big\|S\circ L^{-1}\big\|_{\mathcal{L}} \;\leqslant\; t-\varepsilon.$$

Note that because $x\in C$, we have that S is invertible. Also select $k\geqslant 1$ and $u\in E$, such that

$$\|x-u\|_{\mathbb{R}^N} \;<\; \frac{1}{k}$$

and

$$\begin{aligned}&\big\|f(y)-f(x)-Df(x)(y-x)\big\|_{\mathbb{R}^M}\\ \leqslant\;& \frac{\varepsilon}{\mathrm{Lip}(L^{-1})}\,\|y-x\|_{\mathbb{R}^N} \;=\; \varepsilon\big\|L(y-x)\big\|_{\mathbb{R}^N} \qquad \forall\; y\in \overline{B}_{\frac{2}{k}}(x).\end{aligned}$$

We infer that $x\in E(u,L,k)$ and so $x\in E_i$ for some $i\geqslant 1$. Because $x\in C$ was arbitrary, we have proved statement **(a)**.

(b) Choose E_i from the countable collection $\{E_i\}_{i\geqslant 1}$. We have

$$E_i \;=\; E(u,L,k)$$

with some $u\in E$, $L\in G$ and $k\geqslant 1$. Let us set

$$L_i \;=\; L.$$

From (1.49), we have

$$\frac{1}{t}\big\|L_i(y-x)\big\|_{\mathbb{R}^N} \;\leqslant\; \big\|f(y)-f(x)\big\|_{\mathbb{R}^M} \;\leqslant\; t\big\|L_i(y-x)\big\|_{\mathbb{R}^N} \qquad \forall\; y\in \overline{B}_{\frac{2}{k}}(x).$$

Since $E_i \subseteq \overline{B}_{\frac{1}{k}}(u) \subseteq \overline{B}_{\frac{2}{k}}(x)$, we have that

$$\begin{aligned} \frac{1}{t}\big\|L_i(y-x)\big\|_{\mathbb{R}^N} &\leqslant \big\|f(y)-f(x)\big\|_{\mathbb{R}^M} \\ &\leqslant t\big\|L_i(y-x)\big\|_{\mathbb{R}^N} \qquad \forall\, x,y \in E_i, \end{aligned} \tag{1.50}$$

so $f|_{E_i}$ is injective.

(c) From (1.50), it follows that

$$\mathrm{Lip}\left(\left(f|_{E_i}\right)\circ L_i^{-1}\right) \leqslant t \quad \text{and} \quad \mathrm{Lip}\left(L_i \circ \left(f|_{E_i}\right)\right) \leqslant t.$$

Moreover, from Claim 1 and letting $\varepsilon \searrow 0$, we obtain

$$\frac{|\det L_i|}{t^N} \leqslant Jf|_{E_i} \leqslant t^N |\det L_i|.$$

□

Now we are ready for the ***area formula***.

THEOREM 1.5.21 (Area Formula)
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a Lipschitz continuous function, $N \leqslant M$ and $A \subseteq \mathbb{R}^N$ is a Lebesgue measurable set,
then

$$\int_A Jf(x)\, d\lambda^N(x) = \int_{\mathbb{R}^M} \mu^{(0)}\left(A \cap f^{-1}(\{y\})\right)\, d\mu^{(N)}(y).$$

PROOF By virtue of Theorem 1.5.8, without any loss of generality we may assume that $Df(x)$ (and so $Jf(x)$ too) exists for all $x \in A$. Also as before we may suppose that $\lambda^N(A) < +\infty$.

Case 1. $A \subseteq \{Jf > 0\}$.

In this case we may use Lemma 1.5.20 and produce a sequence $\{E_i\}_{i\geqslant 1}$ of Borel subsets of $\mathbb{R}^N$ which satisfy the postulates of Lemma 1.5.20. We may additionally assume that the sets $\{E_i\}_{i\geqslant 1}$ are disjoint. Let $\mathcal{F}_k$ be the following family of N-cubes

$$\mathcal{F}_k \stackrel{df}{=} \left\{Q \subseteq \mathbb{R}^N :\ Q = \prod_{j=1}^{N}\left(\frac{c_j}{k}, \frac{c_j+1}{k}\right],\ c_j \text{ are integers},\ j \in \{1,\dots,N\}\right\}$$

(compare with the proof of Lemma 1.5.19**(b)**). We set

$$F_{i,n}^k \stackrel{df}{=} E_i \cap Q_n^k \cap A \quad \text{with} \quad Q_n^k \in \mathcal{F}_k, \qquad \forall\, i \geqslant 1,\ n \geqslant 1.$$

Evidently the sets $F_{i,n}^k$ are disjoint and

$$A \;=\; \bigcup_{i,n=1}^{\infty} F_{i,n}^k \qquad \forall\, k \geqslant 1.$$

First we show that

$$\lim_{k\to+\infty} \sum_{i,n=1}^{\infty} \mu^{(N)}\big(f(F_{i,n}^k)\big) \;=\; \int_{\mathbb{R}^N} \mu^{(0)}\left(A \cap f^{-1}(\{y\})\right) d\mu^{(N)}(y). \tag{1.51}$$

To this end, we introduce

$$h_k \;\stackrel{df}{=}\; \sum_{i,n=1}^{\infty} \chi_{f(F_{i,n}^k)} \qquad \forall\, k \geqslant 1.$$

Therefore $h_k(y)$ is the number of sets from the sequence $\left\{F_{i,n}^k\right\}_{i,n\geqslant 1}$ which intersect $f^{-1}(\{y\})$. Note that

$$h_k(y) \;\nearrow\; \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big) \quad \text{as } k \to +\infty.$$

Then (1.51) follows from the monotone convergence theorem (see Theorem A.2.10).

Let $t > 1$. Because of Lemma 1.5.20 and Proposition 1.3.25, we have

$$\mu^{(N)}\big(f(F_{i,n}^k)\big) \;=\; \mu^{(N)}\left(\left(f|_{E_i} \circ L_i^{-1} \circ L_i\right)(F_{i,n}^k)\right) \;\leqslant\; t^N \lambda^N\big(L_i(F_{i,n}^k)\big) \tag{1.52}$$

and

$$\begin{aligned} \lambda^N\big(L_i(F_{i,n}^k)\big) \;&=\; \mu^{(N)}\left(\left(L_i \circ \left(f|_{E_i}\right)^{-1} \circ f\right)(F_{i,n}^k)\right) \\ &\leqslant\; t^N \mu^{(N)}\big(f(F_{i,n}^k)\big). \end{aligned} \tag{1.53}$$

So, using (1.52), (1.53) and Lemmas 1.5.18 and 1.5.20**(c)**, it follows that

$$\begin{aligned} &\frac{1}{t^{2N}}\mu^{(N)}\big(f(F_{i,n}^k)\big) \;\leqslant\; \frac{1}{t^N}\big(L_i(F_{i,n}^k)\big) \\ =\; &\frac{1}{t^N}\big|\det L_i\big|\lambda^N(F_{i,n}^k) \;\leqslant\; \int_{F_{i,n}^k} Jf(x)\, d\lambda^N(x) \\ \leqslant\; &t^N\big|\det L_i\big|\lambda^N(F_{i,n}^k) \;=\; t^N\lambda^N\big(L_i(F_{i,n}^k)\big) \;\leqslant\; t^{2N}\mu^{(N)}\big(f(F_{i,n}^k)\big). \end{aligned}$$

We take the sum for the parameters $i, n \geqslant 1$. Recalling that the sets $F_{i,n}^k$ are disjoint and since $f|_{E_i}$ is injective (see Lemma 1.5.20**(b)**), we obtain

$$\frac{1}{t^{2N}} \sum_{i,n=1}^{\infty} \mu^{(N)}\big(f(F_{i,n}^k)\big) \;\leqslant\; \int_A Jf(x)\, d\lambda^N(x) \;\leqslant\; t^{2N} \sum_{i,n=1}^{\infty} \mu^{(N)}\big(F_{i,n}^k\big).$$

Let $k \to +\infty$ and use (1.51) to write that

$$\frac{1}{t^{2N}} \int_{\mathbb{R}^N} \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y) \;\leqslant\; \int_A Jf(x)\, d\lambda^N(x)$$
$$\leqslant\; t^{2N} \int_{\mathbb{R}^N} \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y).$$

Since $t > 1$ was arbitrary, we let $t \searrow 1$ and obtain the "area formula" for the case when

$$A \;\subseteq\; \big\{Jf > 0\big\}.$$

Case 2. $A \subseteq \big\{Jf = 0\big\}$.

Let $\varepsilon > 0$. We write

$$f \;=\; p \circ g,$$

where

$$g\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M \times \mathbb{R}^N \quad \text{and} \quad p\colon \mathbb{R}^M \times \mathbb{R}^N \longrightarrow \mathbb{R}^M$$

are defined by

$$g(x) \overset{df}{=} \big(f(x), \varepsilon x\big) \quad \text{and} \quad p(y,z) \;=\; y \qquad \forall\, x, z \in \mathbb{R}^N,\; y \in \mathbb{R}^M.$$

We show that there exists $\xi > 0$, such that

$$0 \;<\; Jg(x) \;\leqslant\; \xi\varepsilon \qquad \forall\, x \in A. \tag{1.54}$$

Note that

$$Dg(x) \;=\; \begin{bmatrix} Df(x) \\ \varepsilon I_M \end{bmatrix}_{(N+M)\times M},$$

where I_M is $M \times M$-identity matrix. Then by virtue of the Binet-Cauchy Formula (see Remark 1.5.17), we have that

$$Jg(x)^2 \;=\; \text{"sum of squares of } (N \times N)\text{-subdeterminants of } Dg(x)\text{"}$$
$$\geqslant\; \varepsilon^{2N} \;>\; 0.$$

Moreover, since

$$\|Df(x)\|_{\mathcal{L}} \;\leqslant\; \mathrm{Lip}(f) \quad \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N,$$

once again from the Binet-Cauchy Formula, we have that

$$Jg(x)^2 \;=\; Jf(x)^2 + \left\{\begin{matrix} \text{sum of squares of terms each} \\ \text{involving at least one } \varepsilon > 0 \end{matrix}\right\} \;\leqslant\; \xi\varepsilon^2,$$

for some $\xi > 0$ and all $x \in A$.

This shows that (1.54) is true. Then, using Proposition 1.3.25, Case 1, (1.54) and the fact that $\|p\|_{\mathcal{L}} = 1$, we have

$$\begin{aligned} \mu^{(N)}\big(f(A)\big) &\leqslant \mu^{(N)}\big(g(A)\big) \\ &\leqslant \int\limits_{\mathbb{R}^{N+M}} \mu^{(0)}\big(A \cap g^{-1}(\{y,z\})\big)\, d\mu^{(N)}(y,z) \\ &= \int\limits_{A} Jg(x)\, d\lambda^N(x) \;\leqslant\; \sqrt{\xi}\varepsilon\lambda^N(A). \end{aligned}$$

Let $\varepsilon \searrow 0$ to conclude that

$$\mu^{(N)}\big(f(A)\big) \;=\; 0.$$

Note that

$$\operatorname{supp} \mu^{(0)}\big(A \cap f^{-1}(\{\cdot\})\big) \;\subseteq\; f(A).$$

Hence

$$\int\limits_{\mathbb{R}^N} \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y) \;=\; \int\limits_{A} Jf(x)\, d\lambda^N(x) \;=\; 0.$$

This proves Case 2.

Finally for the general case, we write

$$A \;=\; A_0 \cup A_1$$

with

$$A_0 \subseteq \big\{Jf = 0\big\} \quad \text{and} \quad A_1 \subseteq \big\{Jf > 0\big\}$$

and apply the result on each set separately. □

REMARK 1.5.22 The function

$$y \longmapsto \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)$$

on $\mathbb{R}^M$ is called the ***multiplicity function***. Also note that from Theorem 1.5.21, we infer

$$f^{-1}(\{y\}) \text{ is at most countable for } \mu^{(N)}\text{-almost all } y \in \mathbb{R}^M.$$

THEOREM 1.5.23 (Change of Variables Formula I)
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a Lipschitz continuous function, $N \leqslant M$ and $g \in L^1(\mathbb{R}^N)$, then

$$\int\limits_{\mathbb{R}^N} g(x) Jf(x)\, d\lambda^N(x) \;=\; \int\limits_{\mathbb{R}^M} \Bigg[\sum_{x \in f^{-1}(\{y\})} g(x) \Bigg] d\mu^{(N)}(y).$$

PROOF First assume that $g \geqslant 0$. Let

$$A_1 \stackrel{df}{=} \left\{ x \in \mathbb{R}^N : \ g(x) \geqslant 1 \right\}$$

and inductively define

$$A_n \stackrel{df}{=} \left\{ x \in \mathbb{R}^N : \ g(x) \geqslant \frac{1}{n} + \sum_{i=1}^{n-1} \frac{1}{i} \chi_{A_i}(x) \right\} \qquad \forall \ n \geqslant 2.$$

Then

$$g \ \geqslant \ \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}.$$

If $g(x) = +\infty$, we see that

$$x \in A_n \qquad \forall \ n \geqslant 1.$$

On the other hand, if $0 < g(x) < +\infty$, then $x \notin A_n$ for infinitely many $n \geqslant 1$. So for infinitely many n, we have

$$0 \ < \ g(x) - \sum_{i=1}^{n-1} \frac{1}{i} \chi_{A_i}(x) \ \leqslant \ \frac{1}{n}$$

and so we conclude that

$$g \ = \ \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}.$$

From the monotone convergence theorem (see Theorem A.2.10) and Theorem 1.5.21, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x) Jf(x) \, d\lambda^N(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}^N} \chi_{A_n}(x) Jf(x) \, d\lambda^N(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_{A_n} Jf(x) \, d\lambda^N(x) \ = \ \sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}^M} \mu^{(0)}\left(A_n \cap f^{-1}(\{y\})\right) d\mu^{(N)}(y) \\
&= \int_{\mathbb{R}^M} \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in f^{-1}(\{y\})} \chi_{A_n}(x) \right) d\mu^{(N)}(y) \\
&= \int_{\mathbb{R}^M} \left(\sum_{x \in f^{-1}(\{y\})} \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}(x) \right) d\mu^{(N)}(y) \\
&= \int_{\mathbb{R}^M} \left(\sum_{x \in f^{-1}(\{y\})} g(x) \right) d\mu^{(N)}(y).
\end{aligned}$$

In the general case let $g = g^+ - g^-$ and apply the first part of the proof on each component function $g^+ \geqslant 0$, $g^- \geqslant 0$. □

EXAMPLE 1.5.24 **(a)** Let $N = 1$, $M \geqslant 1$. Suppose that

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}^M$$

is Lipschitz continuous and injective. We have

$$f \;=\; (f_k)_{k=1}^M$$

and

$$Df \;=\; (f_k')_{k=1}^M$$

with $' = \frac{d}{dt}$. Let

$$-\infty \;<\; a \;<\; b \;<\; +\infty$$

and

$$C \;=\; f\big([a,b]\big)$$

(the curve defined by f). Using Theorem 1.5.21, we have

$$\mu^{(1)}(C) \;=\; \int_a^b \big|f'(t)\big|\, d\lambda^1(t),$$

the ***length*** of C.

(b) Let $N \geqslant 1$, $M = N + 1$. Suppose that

$$g\colon \mathbb{R}^N \longrightarrow \mathbb{R}$$

is Lipschitz continuous and let $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N+1}$ be the Lipschitz continuous function defined by

$$f(x) \;\stackrel{df}{=}\; \big(x, g(x)\big) \qquad \forall\; x \in \mathbb{R}^N.$$

We have

$$Df(x) \;=\; \begin{pmatrix} I_N \\ \nabla g(x) \end{pmatrix}_{(N+1)\times N} \qquad \text{for } \lambda^N\text{-a.a. } x \in \mathbb{R}^N,$$

where I_N is the $N \times N$-identity matrix. Therefore

$$(Jf)^2 \;=\; \left\{ \begin{matrix} \text{sum of squares of} \\ N \times N\text{-subdeterminants} \end{matrix} \right\} \;=\; 1 + \|\nabla g\|_{\mathbb{R}^N}^2 .$$

Then, if

$$G \;\stackrel{df}{=}\; \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R} :\; y = g(x)\big\}$$

(the graph of g), from Theorem 1.5.21, we have

$$\mu^{(N)}(G) \;=\; \int\limits_{\mathbb{R}^N} \sqrt{1 + \big\|\nabla g(x)\big\|^2_{\mathbb{R}^N}}\, d\lambda^N(x),$$

the ***surface area*** of G.

(c) Let $N \geqslant 1$, $M = N+1$. Suppose that $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N+1}$ is Lipschitz continuous and injective. Then

$$f \;=\; (f_k)_{k=1}^{N+1} \quad \text{and} \quad Df \;=\; \left(\frac{\partial f_k}{\partial x_i}\right)_{\substack{k\,=\,1,\,\ldots\,,\,N\,+\,1\\ i\,=\,1,\,\ldots\,,\,N}}.$$

So

$$(Jf)^2 \;=\; \sum_{k=1}^{N+1} \left[\frac{\partial(f_1, \ldots, f_{k-1}, f_{k+1}, f_{N+1})}{\partial(x_1, \ldots, x_N)}\right]^2,$$

the sum of squares of $N \times N$-subdeterminants. Therefore, if $U \subseteq \mathbb{R}^N$ is any open set and $A = f(U) \subseteq \mathbb{R}^{N+1}$, then by Theorem 1.5.21, we have

$$\mu^{(N)}(A) \;=\; \int\limits_U \left(\sum_{k=1}^{N+1} \left|\frac{\partial(f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{N+1})}{\partial(x_1, \ldots, x_N)}\right|^2 \right)^{\frac{1}{2}} d\lambda^N(x).$$

In Theorem 1.5.21, we proved that if $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is Lipschitz continuous, $N \leqslant M$ and $A \subseteq \mathbb{R}^N$ is Lebesgue measurable, then the Jacobian integral

$$\int\limits_A Jf(x)\, d\lambda^N(x)$$

equals the N-dimensional Hausdorff area of $f|_A$, given by

$$\int\limits_{\mathbb{R}^M} \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y).$$

If $N \geqslant M$, then the Jacobian integral equals the "coarea" of $f|_A$, defined by

$$\int\limits_{\mathbb{R}^M} \mu^{(N-M)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(N)}(y).$$

This result is known as the ***coarea formula***.

THEOREM 1.5.25 (Coarea Formula)
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is Lipschitz continuous, $M \leqslant N$ and $A \subseteq \mathbb{R}^N$ is Lebesgue measurable, then

$$\int\limits_A Jf(x)\, d\lambda^N(x) \;=\; \int\limits_{\mathbb{R}^M} \mu^{(N-M)}\big(A \cap f^{-1}(\{y\})\big)\, d\lambda^M(y).$$

As was the case with the area formula, the coarea formula leads to a change of variables formula.

THEOREM 1.5.26 (Change of Variables Formula II)
If $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ *is a Lipschitz continuous function,* $N \geqslant M$ *and* $g \in L^1(\mathbb{R}^N)$,
then

$$\int\limits_{\mathbb{R}^N} g(x) Jf(x)\, d\lambda^N(x) \;=\; \int\limits_{\mathbb{R}^M} \left[\int_{f^{-1}(\{y\})} g(x)\, d\mu^{(N-M)}(x) \right] d\lambda^M(y).$$

PROOF First assume that $g \geqslant 0$. As in the proof of Theorem 1.5.23, we can find Lebesgue measurable sets $\{A_n\}_{n\geqslant 1} \subseteq \mathbb{R}^N$, such that

$$g \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}.$$

Invoking the monotone convergence theorem (see Theorem A.2.10) and Theorem 1.5.25, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} g(x) Jf(x)\, d\lambda^N(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int\limits_{A_n} Jf(x)\, d\lambda^N(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int\limits_{\mathbb{R}^M} \mu^{(N-M)}\big(A_n \cap f^{-1}(\{y\})\big)\, d\lambda^M(y) \\
&= \int\limits_{\mathbb{R}^M} \left[\sum_{n=1}^{\infty} \frac{1}{n} \mu^{(N-M)}\big(A_n \cap f^{-1}(\{y\})\big) \right] d\lambda^M(y) \\
&= \int\limits_{\mathbb{R}^M} \left[\int_{f^{-1}(\{y\})} g(x)\, d\mu^{(N-M)}(y) \right] d\lambda^M(y).
\end{aligned}$$

For the general case let

$$g \;=\; g^+ - g^-$$

and apply the first part to each component function

$$g^+ \;\geqslant\; 0 \quad \text{and} \quad g^- \;\geqslant\; 0.$$

∎

EXAMPLE 1.5.27 **(a)** Let $N \geqslant 1$, $M = 1$. Suppose that $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is defined by

$$f(x) \;=\; \|x\|_{\mathbb{R}^N}$$

and $g \in L^1(\mathbb{R}^N)$. Then, we have

$$Df(x) \;=\; \frac{x}{\|x\|_{\mathbb{R}^N}} \quad \text{and} \quad Jf(x) \;=\; 1 \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}.$$

From Theorem 1.5.26, we have

$$\begin{aligned} \int\limits_{\mathbb{R}^N} g(x)\, d\lambda^N(x) \;&=\; \int\limits_0^{+\infty} \left[\int\limits_{\partial \overline{B}_r(0)} g(x)\, d\mu^{(N-1)}(x) \right] d\lambda^1(r) \\ &=\; \int\limits_0^{+\infty} r^{N-1} \left[\int\limits_{\partial \overline{B}_1(0)} g(rx)\, d\mu^{(N-1)} \right] d\lambda^1(r). \qquad (1.55) \end{aligned}$$

In particular if

$$g \;=\; \chi_{\overline{B}_1(0)},$$

from (1.55), it follows that

$$a(N) \;=\; \frac{1}{N} \mu^{(N-1)}\big(\partial \overline{B}_1(0)\big),$$

$a(N) \overset{df}{=} \dfrac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$ is the volume of the unit ball in $\mathbb{R}^N$.

(b) Let $N \geqslant 1$ and $M = 1$. Suppose that $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Lipschitz continuous function. Then

$$Jf \;=\; \|Df\|_{\mathbb{R}^N}$$

and so from Theorem 1.5.25, we have that

$$\int\limits_{\mathbb{R}^N} \big\|Df(x)\big\|_{\mathbb{R}^N} d\lambda^N(x) \;=\; \int\limits_{-\infty}^{+\infty} \mu^{(N-1)}\big(\{f = t\}\big)\, d\lambda^1(t).$$

We conclude this section with some additional useful results involving the multiplicity function

$$y \;\longmapsto\; \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)$$

of a Lipschitz continuous function f.

PROPOSITION 1.5.28
If X, Y are separable metric spaces, ξ is an outer measure on Y, $f\colon X \longrightarrow Y$ is a map such that for every Borel set $B \subseteq X$, the set $f(B)$ is ξ-measurable, $\vartheta\colon 2^X \longrightarrow \overline{R} = \mathbb{R} \cup \{+\infty\}$ is the outer measure on X, defined by

$$\vartheta(A) \stackrel{df}{=} \xi\big(f(A)\big) \qquad \forall\ A \subseteq X$$

and $\widehat{\vartheta}$ is the Borel measure resulting from ϑ by the Carathéodory construction,

then for every Borel set $B \subseteq X$, we have

$$\widehat{\vartheta}(B) \ = \ \int_Y \mu^{(0)}\big(B \cap f^{-1}(\{y\})\big)\, d\xi(y).$$

PROOF Let $\{\mathfrak{B}_k\}_{k \geqslant 1}$ be a sequence of Borel partitions of B, such that every member of $\mathfrak{B}_k$ is the union of some subcollection in $\mathfrak{B}_{k+1}$ and

$$\sup_{A \in \mathfrak{B}_k} \delta(A) \ \longrightarrow \ 0 \quad \text{as } k \to +\infty,$$

i.e.,

$$\mathfrak{B} \ = \ \bigcup_{k=1}^{\infty} \mathfrak{B}_k \text{ is a Vitali cover of } B.$$

Note that if

$$h_k(y) \stackrel{df}{=} \sum_{A \in \mathfrak{B}_k} \chi_{f(A)}(y) \qquad \forall\ k \geqslant 1,\ y \in Y,$$

then

$$h_k(y) \nearrow \mu^{(0)}\big(B \cap f^{-1}(\{y\})\big) \quad \text{as } k \to +\infty.$$

So by the monotone convergence theorem (see Theorem A.2.10), we have that

$$\widehat{\vartheta}(B) \ = \ \lim_{k \to +\infty} \sum_{A \in \mathfrak{B}_k} \xi\big(f(A)\big) \ = \ \lim_{k \to +\infty} \int_Y \sum_{A \in \mathfrak{B}_k} \chi_{f(A)}(y)\, d\xi(y)$$

$$= \ \int_Y \mu^{(0)}\big(B \cap f^{-1}(\{y\})\big)\, d\xi(y).$$

□

REMARK 1.5.29 Recall that if X is a separable metric space, Y is a Hausdorff topological space, $f\colon X \longrightarrow Y$ is a continuous map, ξ is a Borel outer measure on Y, then for every Borel set $B \subseteq X$, the set $f(B)$ is ξ-measurable. This fact is essentially the starting point for the theory of Souslin sets (see Definition A.2.29**(b)**).

PROPOSITION 1.5.30
If X is a Polish space (see Definition A.2.29(a)), Y is a separable metric space, $f\colon X \longrightarrow Y$ is a Lipschitz continuous function, $0 \leqslant k < +\infty$ and $A \subseteq X$ is a Borel set,
then

$$\int_Y \mu^{(0)}\big(A \cap f^{-1}(\{y\})\big)\, d\mu^{(k)}(y) \;\leqslant\; \big(\mathrm{Lip}(f)\big)^k \mu^{(k)}(A).$$

PROOF From Proposition 1.3.25, we know that

$$\vartheta(A) \stackrel{df}{=} \mu^{(k)}\big(f(A)\big) \;\leqslant\; \big(\mathrm{Lip}(f)\big)^k \mu^{(k)}(A) \qquad \forall\, A \subseteq X.$$

Then apply Proposition 1.5.28 on the outer measure ϑ. □

PROPOSITION 1.5.31
If X is a separable metric space,
then for every connected set $C \subseteq X$, we have

$$\delta(C) \;\leqslant\; \mu^{(1)}(C).$$

PROOF Clearly we may assume that

$$\mu^{(1)}(C) \;<\; +\infty$$

or otherwise the inequality is obvious. Since $\mu^{(1)}$ is a Borel measure, we can find a Borel set $B \supseteq C$, such that

$$\mu^{(1)}(B) = \mu^{(1)}(C).$$

Let $u, v \in C$ and let $f\colon X \longrightarrow \mathbb{R}$ be defined by

$$f(x) \stackrel{df}{=} d_{_X}(x, u) \qquad \forall\, x \in X,$$

where $d_{_X}$ is the metric in X. Since f is Lipschitz continuous with

$$\mathrm{Lip}(f) \;=\; 1,$$

$$f(u), f(v) \;\in\; f(C) \;=\; [a, b],$$

from Proposition 1.5.30, we have that

$$\begin{aligned} \mu^{(1)}(C) \;&=\; \mu^{(1)}(B) \\ &\geqslant \int_{\mathbb{R}} \mu^{(0)}\big(B \cap f^{-1}(\{y\})\big)\, d\mu^{(1)}(y) \\ &\geqslant\; \mu^{(1)}\big(f(B)\big) \;\geqslant\; d_{_X}(v, u). \end{aligned}$$

□

1.6 Capacity

The notion of capacity plays a crucial role in the study of local properties of Sobolev functions. In a sense it takes the place of measure and it is used to characterize the smallness of subsets in $\mathbb{R}^N$. For this reason, it is indispensable in the study of the continuity properties of Sobolev functions. We shall deal with these issues in Section 2.7. Moreover, the concept of capacity enters the study of obstacle problems. In this section we develop the theory of the so-called "p-capacity" (variational capacity).

The development of the theory of the p-capacity requires knowledge of the definition of Sobolev spaces and some results from their theory. To make this section self-contained, we state here the necessary material from the theory of Sobolev spaces, but we postpone the proofs until Section 2.4, where we conduct a more systematic study of Sobolev spaces.

DEFINITION 1.6.1 *Let $U \subseteq \mathbb{R}^N$ be a nonempty open set. By $z = (z_k)_{k=1}^N$, we denote a generic point of U.*

(a) *Suppose that $f \in L^1_{\mathrm{loc}}(U)$. We say that $g_k \in L^1_{\mathrm{loc}}(U)$ is the **distributional** (or **weak**) **partial derivative** of f with respect to z_k (with $k \in \{1, \dots, N\}$) in U, if*

$$\int_U f \frac{\partial \varphi}{\partial z_k}\, dz \;=\; -\int_U g_k \varphi \, dz \qquad \forall\ \varphi \in C_c^\infty(Z)$$

(here $C_c^\infty(Z)$ is the space of all $C^\infty(Z)$-functions with compact supports, i.e., the space of test functions). We write

$$g_k \;=\; \frac{\partial f}{\partial z_k} \;=\; D_k f \qquad \forall\ k \in \{1, \dots, N\}.$$

*If all of the distributional (weak) partial derivatives $D_k f$ exist for $k = 1, \dots, N$, then $Df \stackrel{df}{=} (Df_k)_{k=1}^N$ is the **distributional (weak) derivative** of f.*

(b) *Let $p \in [1, +\infty]$. We define the **Sobolev space** $W^{1,p}(U)$, by*

$$W^{1,p}(U) \;\stackrel{df}{=}\; \{f \in L^p(U) :\ Df \in L^p(U; \mathbb{R}^N)\}.$$

Also we define

$$W^{1,p}_{\mathrm{loc}}(U) \;\stackrel{df}{=}\; \{f : U \longrightarrow \mathbb{R} :\ f|_V \in W^{1,p}(V) \text{ for all } V \subset\subset U\},$$

*where $V \subset\subset U$ means that V is a bounded open subset of U such that $\overline{V} \subseteq U$. The elements of $W^{1,p}_{\mathrm{loc}}(U)$ are called **Sobolev functions**.*

(c) *Let $p \in [1, N)$. We define the* ***critical Sobolev exponent***

$$p^* \stackrel{df}{=} \frac{Np}{N-p}$$

and the space

$$K^p \stackrel{df}{=} \left\{ f \in L^{p^*}(\mathbb{R}^N) : \ f \geqslant 0, \ Df \in L^p\big(\mathbb{R}^N; \mathbb{R}^N\big) \right\}.$$

(d) *Let $p \in [1, N)$ and $C \subseteq \mathbb{R}^N$. The* ***p-capacity*** *of C is defined by*

$$\mathrm{cap}_p(C) \stackrel{df}{=} \inf_{\substack{f \in K^p \\ C \subseteq \mathrm{int}\{f \geqslant 1\}}} \big\|Df\big\|_p^p.$$

REMARK 1.6.2 **(a)** Clearly if the distributional (weak) partial derivative $D_k f$ exists, it is uniquely defined modulo a Lebesgue-null set in $\mathbb{R}^N$.

(b) If $f \in W^{1,p}(U)$, we define

$$\|f\|_{1,p} \stackrel{df}{=} \left(\|f\|_p^p + \|Df\|_p^p \right)^{\frac{1}{p}} \qquad \forall \ p \in [1, +\infty)$$

and

$$\|f\|_{1,\infty} \stackrel{df}{=} \|f\|_\infty + \|Df\|_\infty .$$

These are norms in $W^{1,p}(U)$ for $p \in [1, +\infty)$ and $W^{1,\infty}(U)$ respectively. Normed this way the Sobolev spaces are Banach spaces.

(c) Although $p < p^*$, we do not have $L^{p^*}(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N)$ and so we cannot say that $K^p \subseteq W^{1,p}(\mathbb{R}^N)$.

(d) If $K \subseteq \mathbb{R}^N$ is a compact set, then by using standard regularization (via mollification; see also Definition 2.4.10) of the characteristic function χ_K, we can check that

$$\mathrm{cap}_p(K) \ = \ \inf_{\substack{f \in C_c^\infty(\mathbb{R}^N) \\ f \geqslant \chi_K}} \|Df\|_p^p .$$

(e) Evidently, if $C_1 \subseteq C_2$, then

$$\mathrm{cap}_p(C_1) \ \leqslant \ \mathrm{cap}_p(C_2)$$

(monotonicity).

(f) Because $p < p^*$, the elements of K^p are Sobolev functions.

As we already mentioned, for easy reference, we present four results from the theory of Sobolev spaces, which will be used in the sequel. The proofs of these results will be given in Section 2.4.

PROPOSITION 1.6.3
If $U \subseteq \mathbb{R}^N$ is open, $p \in [1, +\infty)$ and $f \in W^{1,p}(U)$, then we can find a sequence

$$\{f\}_{n\geqslant 1} \subseteq W^{1,p}(U) \cap C^\infty(U),$$

such that

$$f_n \longrightarrow f \quad \text{in } W^{1,p}(U).$$

PROPOSITION 1.6.4
Let $U \subseteq \mathbb{R}^N$ be an open set and let $p \in [1, +\infty)$.

(a) *If $f, g \in W^{1,p}(U)$, then*

$$h_0 \stackrel{df}{=} \min\{f, g\} \in W^{1,p}(U), \quad h_1 \stackrel{df}{=} \max\{f, g\} \in W^{1,p}(U)$$

and

$$Dh_0(x) = \begin{cases} Df(x) & \text{for } \lambda^N\text{-a.a. } x \in \{f \leqslant g\}, \\ Dg(x) & \text{for } \lambda^N\text{-a.a. } x \in \{f \geqslant g\}, \end{cases}$$

$$Dh_1(x) = \begin{cases} Dg(x) & \text{for } \lambda^N\text{-a.a. } x \in \{f \leqslant g\}, \\ Df(x) & \text{for } \lambda^N\text{-a.a. } x \in \{f \geqslant g\}. \end{cases}$$

In particular

$$f^+, \ f^-, \ |f| \in W^{1,p}(U).$$

(b) *If $\{f_n\}_{n\geqslant 1} \subseteq W^{1,p}(U)$ is a sequence, then*

$$h \stackrel{df}{=} \sup_{n\geqslant 1} f_n \in W^{1,p}(U), \quad u \stackrel{df}{=} \sup_{n\geqslant 1} \big\|Df_n\big\|_{\mathbb{R}^N} \in L^p(U)$$

and

$$\big\|Df_n(z)\big\|_{\mathbb{R}^N} \leqslant u(z) \quad \text{for } \lambda^N\text{-a.a. } z \in U.$$

PROPOSITION 1.6.5
If $U \subseteq \mathbb{R}^N$ is a bounded open set with a C^1-boundary and $p \in [1, +\infty)$, then there exists $E \subseteq \mathcal{L}\big(W^{1,p}(U), W^{1,p}(\mathbb{R}^N)\big)$, such that

$$E(f)|_U = f.$$

REMARK 1.6.6 The function $E(f)$ is called an ***extension*** of f on $\mathbb{R}^N$.

Finally we mention two basic inequalities. The first is known as the "Sobolev inequality" (or "Sobolev-Nirenberg-Gagliardo inequality") and the second is known as the "Poincaré-Wirtinger inequality."

THEOREM 1.6.7 (Sobolev-Nirenberg-Gagliardo Inequality)
If $p \in [1, +\infty)$,
then there exists $C = C(N,p) > 0$, *such that*

$$\|f\|_{p^*} \leqslant C \|Df\|_p \qquad \forall\ f \in W^{1,p}(\mathbb{R}^N).$$

THEOREM 1.6.8 (Poincaré-Wirtinger Inequality)
If $U \subseteq \mathbb{R}^N$ *is bounded, connected and open set (i.e., a bounded domain in* $\mathbb{R}^N$*) with a* C^1*-boundary and* $p \in [1, +\infty)$,
then there exists $C_0 = C_0(N,p) > 0$, *such that*

$$\left\|f - \overline{f}\right\|_p \leqslant C_0 \|Df\|_p \qquad \forall\ f \in W^{1,p}(U),$$

with

$$\overline{f} = \frac{1}{\lambda^N(U)} \int_U f(z)\, dz.$$

If $p < N$, *then*

$$\left\|f - \overline{f}\right\|_{p^*} \leqslant C_0 \|Df\|_p .$$

A Sobolev inequality is also valid for the elements in K^p.

PROPOSITION 1.6.9
If $f \in K^p$,
then there exists $C = C(N,p) > 0$, *such that*

$$\|f\|_{p^*} \leqslant C \|Df\|_p \qquad \forall\ f \in K^p.$$

PROOF First we produce a sequence $\{\varphi_n\}_{n \geqslant 1} \subseteq C_c^\infty\left(\mathbb{R}^N\right)$, such that

$$0 \leqslant \varphi_n < 1 \qquad \forall\ n \geqslant 1,$$

$$\varphi_n(z) \nearrow 1 \quad \text{for a.a. } z \in \mathbb{R}^N,$$

$$\varphi_n(z) = 1 \qquad \forall\ \|z\|_{\mathbb{R}^N} < n$$

and

$$\sup_{n \geqslant 1} \|D\varphi_n\|_{\mathbb{R}^N} < +\infty.$$

To this end, let $\varphi \in C_c^\infty\left(B_2(0)\right)$, such that

$$0 \leqslant \varphi \leqslant 1 \quad \text{and} \quad \varphi|_{\overline{B}_1(0)} = 1.$$

Let us set

$$\varphi_n(z) \;=\; \varphi\left(\frac{z}{n}\right), \qquad \forall\, z \in \mathbb{R}^N,\; n \geqslant 1.$$

This is the desired sequence. Note that

$$\varphi_n f \;\in\; W^{1,p}(\mathbb{R}^N)$$

(recall that $p < p^*$ and use the product rule). Invoking the Sobolev-Nirenberg-Gagliardo inequality (see Theorem 1.6.7), we can find $C > 0$, such that for all $n \geqslant 1$, we have

$$\|\varphi_n f\|_{p^*} \;\leqslant\; C\,\|D(\varphi_n f)\|_p \;\leqslant\; C\,\|Df\|_p + C\,\|fD\varphi_n\|_p\,;$$

thus by the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\|f\|_{p^*} \;\leqslant\; C\,\|Df\|_p + C \liminf_{n\to+\infty} \|fD\varphi_n\|_p\,. \tag{1.56}$$

Using Hölder's inequality (see Theorem A.2.27) (as $\frac{p}{p^*} + \frac{1}{(\frac{p^*}{p})'} = 1$), the fact that $\varphi_n|_{\overline{B}_n(0)} \equiv 1$ and since $p\left(\frac{p^*}{p}\right)^N = 1$ and $\sup\limits_{n\geqslant 1} \|D\varphi_n\|_{\mathbb{R}^N} < +\infty$, for every $n \geqslant 1$, we have

$$\begin{aligned}
\|fD\varphi_n\|_p^p \;&=\; \int_{\mathbb{R}^N} \left|f(z)D\varphi_n(z)\right|^p dz \;=\; \int_{\{\|z\|_{\mathbb{R}^N}>n\}} \left|f(z)D\varphi_n(z)\right|^p dz \\
&\leqslant \left(\int_{\{\|z\|_{\mathbb{R}^N}>n\}} \left|f(z)\right|^{p^*} dz\right)^{\frac{p}{p^*}} \left(\int_{\mathbb{R}^N} \left\|D\varphi_n(z)\right\|_{\mathbb{R}^N}^{p\left(\frac{p^*}{p}\right)'} dz\right)^{1-\frac{p}{p^*}} \\
&\leqslant C_1 \left(\int_{\{\|z\|_{\mathbb{R}^N}>n\}} \left|f(z)\right|^{p^*} dz\right)^{\frac{p}{p^*}},
\end{aligned}$$

for some $C_1 > 0$. Since $|f|^{p^*} \in L^1(\mathbb{R}^N)$, we have

$$\lim_{n\to+\infty} \left\|fD\varphi_n\right\|_p^p \;\leqslant\; C_1 \lim_{n\to+\infty} \left(\int_{\{\|z\|_{\mathbb{R}^N}>n\}} \left|f(z)\right|^{p^*} dz\right)^{\frac{p}{p^*}} \;=\; 0.$$

Using this in (1.56), we conclude that

$$\|f\|_{p^*} \;\leqslant\; C\,\|Df\|_p\,.$$

□

We use this inequality to establish that the p-capacity cap_p is an outer measure on $\mathbb{R}^N$.

THEOREM 1.6.10
If $p \in [1, N)$, *then* cap_p *is an outer measure on* $\mathbb{R}^N$.

PROOF Clearly $\operatorname{cap}_p(\emptyset) = 0$ and cap_p is monotone (see Remark 1.6.2**(e)**). So it remains to show that if $\{C_n\}_{n \geqslant 1}$ is a sequence of subsets of $\mathbb{R}^N$ and $C = \bigcup_{n=1}^{\infty} C_n$, then

$$\operatorname{cap}_p(C) \leqslant \sum_{n=1}^{\infty} \operatorname{cap}_p(C_n)$$

(see Definition 1.1.1). We assume that

$$\sum_{n=1}^{\infty} \operatorname{cap}_p(C_n) < +\infty$$

or otherwise the inequality is obvious. According to Definition 1.6.1**(d)**, for a given $\varepsilon > 0$, we can find $f_n \in K^p$, such that

$$C_n \subseteq \operatorname{int}\{f_n \geqslant 1\}$$

and

$$\|Df_n\|_p^p \leqslant \operatorname{cap}_p(C_n) + \frac{\varepsilon}{2^n} \qquad \forall\, n \geqslant 1. \tag{1.57}$$

Let $h = \sup_{n \geqslant 1} f_n$. Evidently $C \subseteq \operatorname{int}\{h \geqslant 1\}$. Also, using the monotone convergence theorem (see Theorem A.2.10), Proposition 1.6.3 and (1.57), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} h(z)^{p^*}\, dz &= \int_{\mathbb{R}^N} \sup_{n \geqslant 1} f_n(z)^{p^*}\, dz \leqslant \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} f_n(z)^{p^*}\, dz \\
&\leqslant C \sum_{n=1}^{\infty} \|Df_n\|_p^{p^*} \leqslant C \sum_{n=1}^{\infty} \left(\operatorname{cap}_p(C_n) + \frac{\varepsilon}{2^n}\right)^{\frac{p^*}{p}} \\
&\leqslant C_1 \left[\sum_{n=1}^{\infty} \left(\operatorname{cap}_p(C_n) + \frac{\varepsilon}{2^n}\right)\right]^{\frac{p^*}{p}} < +\infty,
\end{aligned}$$

for some $C_1 > 0$. As $h \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and $p < p^*$, we have $h \in L^{p^*}(\mathbb{R}^N)$.

Also if $u = \sup_{n \geqslant 1} \|Df_n\|_{\mathbb{R}^N}$, from (1.57), we have

$$\int_{\mathbb{R}^N} |u(z)|^p\, dz \leqslant \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} \|Df_n(z)\|_{\mathbb{R}^N}^p\, dz < +\infty,$$

so $u \in L^p(\mathbb{R}^N)$.

By Proposition 1.6.4**(b)**, we have that $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $\|Dh(z)\| \leqslant u(z)$ for almost all $z \in Z$. Therefore $Dh \in L^p\big(\mathbb{R}^N;\mathbb{R}^N\big)$ and so $h \in K^p$. By virtue of Definition 1.6.1**(d)**, the monotone convergence theorem (see Theorem A.2.10) and (1.57), we have

$$\begin{aligned}\mathrm{cap}_p(C) \;&\leqslant\; \int\limits_{\mathbb{R}^N} \big\|Dh(z)\big\|^p_{\mathbb{R}^N}\,dz \;\leqslant\; \int\limits_{\mathbb{R}^N} u(z)^p\,dz\\ \leqslant\; \sum_{n=1}^{\infty} \int\limits_{\mathbb{R}^N} \big\|Df_n(z)\big\|^p_{\mathbb{R}^N}\,dz \;&\leqslant\; \sum_{n=1}^{\infty} \mathrm{cap}_p(C_n) + \varepsilon.\end{aligned}$$

Let $\varepsilon \searrow 0$, to conclude that

$$\mathrm{cap}_p(C) \;\leqslant\; \sum_{n=1}^{\infty} \mathrm{cap}_p(C_n).$$

□

In the next Theorem, we have collected the basic properties of the p-capacity cap_p.

THEOREM 1.6.11
If $p \in [1,N)$ and $A \subseteq C \subseteq \mathbb{R}^N$, then

(a) $\mathrm{cap}_p(A) = \inf\limits_{\substack{A \subseteq U\\ U \text{ is open}}} \mathrm{cap}_p(U)$.

(b) $\mathrm{cap}_p(\xi A) = \xi^{N-p}\mathrm{cap}_p(A)$ *for all* $\xi > 0$.

(c) $\mathrm{cap}_p\big(L(A)\big) = \mathrm{cap}_p(A)$ *for every affine isometry* $L\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$.

(d) $\mathrm{cap}_p(A) \leqslant C\mu^{(N-p)}(A)$ *for some* $C = C(N,p) > 0$.

(e) $\lambda^N(A) \leqslant C\big(\mathrm{cap}_p(A)\big)^{\frac{N}{N-p}}$ *for some* $C = C(N,p) > 0$.

(f) $\mathrm{cap}_p(A \cap B) + \mathrm{cap}_p(A \cup B) \leqslant \mathrm{cap}_p(A) + \mathrm{cap}_p(B)$.

(g) *if* $\{A_n\}_{n\geqslant 1}$ *is an increasing sequence (i.e.,* $A_n \subseteq A_{n+1}$ *for all* $n \geqslant 1$*), then*

$$\lim_{n\to+\infty} \mathrm{cap}_p(A_n) \;=\; \mathrm{cap}_p\Big(\bigcup_{n=1}^{\infty} A_n\Big).$$

(h) *if* $\{A_n\}_{n\geqslant 1}$ *is a decreasing sequence (i.e.,* $A_n \supseteq A_{n+1}$ *for all* $n \geqslant 1$*) of compact sets in* $\mathbb{R}^N$*, then*

$$\lim_{n\to+\infty} \mathrm{cap}_p(A_n) \;=\; \mathrm{cap}_p\Big(\bigcap_{n=1}^{\infty} A_n\Big).$$

PROOF **(a)** From the monotonicity of the p-capacity, we have

$$\operatorname{cap}_p(A) \leqslant \inf_{\substack{A \subseteq U \\ U \text{ is open}}} \operatorname{cap}_p(U). \tag{1.58}$$

For a given $\varepsilon > 0$, we can find $f \in K^p$, such that $A \subseteq \operatorname{int}\{f \geqslant 1\}$ and

$$\|Df\|_p^p \leqslant \operatorname{cap}_p(A) + \varepsilon. \tag{1.59}$$

Let $U = \operatorname{int}\{f \geqslant 1\}$. Then from Definition 1.6.1**(d)** and (1.58), we have

$$\operatorname{cap}_p(U) \leqslant \|Df\|_p^p \leqslant \operatorname{cap}_p(A) + \varepsilon.$$

Let $\varepsilon \searrow 0$, to obtain that

$$\operatorname{cap}_p(U) \leqslant \operatorname{cap}_p(A).$$

Combining this with (1.58), we conclude that

$$\operatorname{cap}_p(A) = \inf_{\substack{A \subseteq U \\ U \text{ is open}}} \operatorname{cap}_p(U).$$

(b) Let $\varepsilon > 0$ be given. Then we can find $f \in K^p$, such that $A \subseteq \operatorname{int}\{f \geqslant 1\}$ and

$$\|Df\|_p^p \leqslant \operatorname{cap}_p(A) + \varepsilon.$$

Let $\xi > 0$ and $h(z) \stackrel{df}{=} f\left(\frac{z}{\xi}\right)$. We have $h \in K^p$ and $\xi A \subseteq \operatorname{int}(h \geqslant 1)$. So

$$\operatorname{cap}_p(\xi A) \leqslant \|Dh\|_p^p = \xi^{N-p}\|Df\|_p^p \leqslant \xi^{N-p}\big(\operatorname{cap}_p(A) + \varepsilon\big).$$

Let $\varepsilon \searrow 0$ to obtain

$$\operatorname{cap}_p(\xi A) \leqslant \xi^{N-p}\operatorname{cap}_p(A). \tag{1.60}$$

Using (1.60), we see that

$$\operatorname{cap}_p(A) = \operatorname{cap}_p\left(\frac{1}{\xi}(\xi A)\right) \leqslant \frac{1}{\xi^{N-p}}\operatorname{cap}_p(\xi A),$$

so

$$\xi^{N-p}\operatorname{cap}_p(A) \leqslant \operatorname{cap}_p(\xi A). \tag{1.61}$$

From (1.60) and (1.61), we conclude that

$$\operatorname{cap}_p(\xi A) = \xi^{N-p}\operatorname{cap}_p(A) \qquad \forall\ \xi > 0.$$

(c) The proof is similar to that of **(b)**.

(d) Let $\delta > 0$ and suppose that $A \subseteq \bigcup_{n=1}^{\infty} \overline{B}_{r_n}(x_n)$, with $2r_n < \delta$ for all $n \geqslant 1$. Since cap_p is an outer measure (see Theorem 1.6.10), using also **(b)** and **(c)** as $\overline{B}_{r_n}(x_n) = x_n + r_n\overline{B}_1(0)$, we have

$$\mathrm{cap}_p(A) \;\leqslant\; \sum_{n=1}^{\infty} \mathrm{cap}_p\left(\overline{B}_{r_n}(x_n)\right) \;=\; \mathrm{cap}_p\big(\overline{B}_1(0)\big) \sum_{n=1}^{\infty} r_n^{N-p},$$

so

$$\mathrm{cap}_p(A) \;\leqslant\; C\mu^{(N-p)}(A).$$

(e) Let $\varepsilon > 0$ and select $f \in K^p$, such that $A \subseteq \mathrm{int}\left\{f \geqslant 1\right\}$ and

$$\left\|Df\right\|_p^p \;\leqslant\; \mathrm{cap}_p(A) + \varepsilon. \tag{1.62}$$

Using Proposition 1.6.9 and (1.62), we obtain

$$\lambda^N(A)^{\frac{1}{p^*}} \;\leqslant\; \|f\|_{p^*} \;\leqslant\; C\,\|Df\|_p \;\leqslant\; C\big(\mathrm{cap}_p(A) + \varepsilon\big)^{\frac{1}{p}},$$

for some $C > 0$ and so

$$\lambda^N(A) \;\leqslant\; C\mathrm{cap}_p(A)^{\frac{p^*}{p}} \;=\; C\mathrm{cap}_p(A)^{\frac{N}{N-p}},$$

for some $C > 0$.

(f) Let $\varepsilon > 0$ and select $f, g \in K^p$, such that

$$A \subseteq \mathrm{int}\left\{f \geqslant 1\right\}, \quad B \subseteq \mathrm{int}\left\{g \geqslant 1\right\}$$

and

$$\|Df\|_p^p \;\leqslant\; \mathrm{cap}_p(A) + \varepsilon, \quad \|Dg\|_p^p \;\leqslant\; \mathrm{cap}_p(B) + \varepsilon. \tag{1.63}$$

Let

$$h_0 \overset{df}{=} \min\left\{f, g\right\} \quad \text{and} \quad h_1 \overset{df}{=} \max\left\{f, g\right\}.$$

Using Proposition 1.6.4**(a)**, we see that $h_0, h_1 \in K^p$. Also, we have

$$\begin{aligned} &\left\|Dh_0(z)\right\|_{\mathbb{R}^N}^p + \left\|Dh_1(z)\right\|_{\mathbb{R}^N}^p \\ =\; &\left\|Df(z)\right\|_{\mathbb{R}^N}^p + \left\|Dg(z)\right\|_{\mathbb{R}^N}^p \quad \text{for } \lambda^N\text{-a.a. } z \in \mathbb{R}^N \end{aligned} \tag{1.64}$$

and

$$A \cap B \;\subseteq\; \mathrm{int}\left\{h_0 \geqslant 1\right\}, \quad A \cup B \;\subseteq\; \mathrm{int}\left\{h_1 \geqslant 1\right\}. \tag{1.65}$$

Therefore from (1.65), (1.64), (1.63) and since $h_1, h_1 \in K^p$, we obtain

$$\begin{aligned} &\mathrm{cap}_p(A \cap B) + \mathrm{cap}_p(A \cup B) \;\leqslant\; \|Dh_0\|_p^p + \|Dh_1\|_p^p \\ =\; &\|Df\|_p^p + \|Dg\|_p^p \;\leqslant\; \mathrm{cap}_p(A) + \mathrm{cap}_p(B) + 2\varepsilon, \end{aligned}$$

so

$$\mathrm{cap}_p(A\cap B)+\mathrm{cap}_p(A\cup B)\ \leqslant\ \mathrm{cap}_p(A)+\mathrm{cap}_p(B).$$

(g) We do the proof for the case $p\in(1,N)$. For the case $p=1$ we refer to Federer & Ziemer (1972).

By virtue of the monotonicity property, we have

$$\lim_{n\to+\infty}\mathrm{cap}_p(A_n)\ \leqslant\ \mathrm{cap}_p\Big(\bigcup_{n=1}^{\infty}A_n\Big). \tag{1.66}$$

Suppose that

$$\mathrm{cap}_p\Big(\bigcup_{n=1}^{\infty}A_n\Big)\ <\ +\infty,$$

as otherwise there is nothing to prove. Thus also

$$\lim_{n\to+\infty}\mathrm{cap}_p(A_n)\ <\ +\infty.$$

Let $\varepsilon>0$ and for every $n\geqslant 1$ let us select $f_n\in K^p$, such that

$$A_n\ \subseteq\ \mathrm{int}\big\{f_n\geqslant 1\big\}\quad\text{and}\quad \|Df_n\|_p^p\ \leqslant\ \mathrm{cap}_p(A_n)+\frac{\varepsilon}{2^n}. \tag{1.67}$$

Let us set $h_0\overset{df}{=}0$ and $h_k\overset{df}{=}\max\limits_{1\leqslant n\leqslant k}f_n$. We know that $\{h_k\}_{k\geqslant 0}\subseteq K^p$, $h_k=\max\big(f_k,h_{k-1}\big)$ and $A_{k-1}\subseteq\mathrm{int}\big\{\min\big\{f_k,h_{k-1}\big\}\geqslant 1\big\}$. So, using (1.67), we have

$$\begin{aligned}
&\|Dh_k\|_p^p+\mathrm{cap}_p(A_{k-1})\\
\leqslant\ &\big\|D\big(\max\{f_k,h_{k-1}\}\big)\big\|_p^p+\big\|D\big(\min\{f_k,h_{k-1}\}\big)\big\|_p^p\\
=\ &\|Df_k\|_p^p+\|Dh_{k-1}\|_p^p\ \leqslant\ \mathrm{cap}_p(A_k)+\frac{\varepsilon}{2^k}+\|Dh_{k-1}\|_p^p,
\end{aligned}$$

so

$$\|Dh_k\|_p^p-\|Dh_{k-1}\|_p^p\ \leqslant\ \mathrm{cap}_p(A_k)-\mathrm{cap}_p(A_{k-1})+\frac{\varepsilon}{2^k}.$$

Adding and recalling that $h_0=0$, we obtain

$$\|Dh_k\|_p^p\ \leqslant\ \mathrm{cap}_p(A_k)+\varepsilon\qquad\forall\ k\geqslant 1. \tag{1.68}$$

Let $u\overset{df}{=}\lim\limits_{k\to+\infty}h_k$. Evidently

$$\bigcup_{k=1}^{\infty}A_k\ \subseteq\ \mathrm{int}\big\{u\geqslant 1\big\} \tag{1.69}$$

and so, by the monotone convergence theorem (see Theorem A.2.10), Proposition 1.6.9 and (1.68), we have

$$\begin{aligned} \|u\|_{p^*} &= \lim_{k\to+\infty} \|h_k\|_{p^*} \leqslant C \liminf_{k\to+\infty} \|Dh_k\|_p \\ &\leqslant C\left(\lim_{k\to+\infty} \operatorname{cap}_p(A_k) + \varepsilon\right)^{\frac{1}{p}}. \end{aligned} \tag{1.70}$$

So at least for a subsequence of $\{Dh_k\}_{k\geqslant 1} \subseteq L^p\big(\mathbb{R}^N;\mathbb{R}^N\big)$, we have that it is bounded. Hence we may assume that

$$Dh_k \xrightarrow{w} Du \quad \text{in } L^p\big(\mathbb{R}^N;\mathbb{R}^N\big)$$

(recall that we have assumed that $p > 1$). Then from (1.70) and since

$$\|Du\|_p \leqslant \liminf_{k\to+\infty} \|Dh_k\|_p, \tag{1.71}$$

we infer that $f \in K^p$. Therefore, using (1.69), (1.71) and (1.70), we have

$$\operatorname{cap}_p\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \|Du\|_p^p \leqslant \lim_{n\to+\infty} \operatorname{cap}_p(A_n). \tag{1.72}$$

From (1.66) and (1.72), it follows that

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n) = \operatorname{cap}_p\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(h) Note that due to the monotonicity property, we have

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n) \geqslant \operatorname{cap}_p\left(\bigcap_{n=1}^{\infty} A_n\right). \tag{1.73}$$

Let U be an open set such that $\bigcap_{n=1}^{\infty} A_n \subseteq U$. The set $\bigcap_{n=1}^{\infty} A_n$ is compact and so for some $n_0 \geqslant 1$, we have that $A_{n_0} \subseteq U$, hence $A_n \subseteq U$ for all $n \geqslant n_0$. It follows that

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n) \leqslant \operatorname{cap}_p(U)$$

and so, using also **(a)**, we have

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n) \leqslant \inf_{\substack{\bigcap A_n \subseteq U \\ U \text{ is open}}} \operatorname{cap}_p(U) = \operatorname{cap}_p\left(\bigcap_{n=1}^{\infty} A_n\right). \tag{1.74}$$

From (1.73) and (1.74), we conclude that

$$\lim_{n\to+\infty} \operatorname{cap}_p(A_n) = \operatorname{cap}_p\left(\bigcap_{n=1}^{\infty} A_n\right).$$

□

REMARK 1.6.12 The monotonicity of cap_p together with properties **(g)** and **(h)** in Theorem 1.6.11 imply that the set-function $A \longmapsto \mathrm{cap}_p(A)$ is a "Choquet capacity" (see Definition A.2.37). Using Choquet's capacitability theorem (see Theorem A.2.39 or cf. Choquet (1955)), we can say that for all Souslin (analytic) subsets A (see Definition A.2.29**(b)** and Remark A.2.30) of $\mathbb{R}^N$ (in particular then for all Borel sets A of $\mathbb{R}^N$), we have

$$\mathrm{cap}_p(A) \;=\; \sup_{\substack{K \subseteq A \\ K \text{ is compact}}} \mathrm{cap}_p(K).$$

In Theorem 1.6.11**(d)** we obtained a first relation between p-capacity and Hausdorff measures, both of which measure small sets in $\mathbb{R}^N$. We can improve this result as follows.

THEOREM 1.6.13
Let $p \in (1, N)$ and $A \subseteq \mathbb{R}^N$.

(a) *If* $\mu^{(N-p)}(A) < +\infty$, *then* $\mathrm{cap}_p(A) = 0$.

(b) *If* $\mathrm{cap}_p(A) = 0$, *then* $\mu^{(s)}(A) = 0$ *for all* $s > N - p$.

PROOF **(a)** Clearly we may assume that $A \subseteq \mathbb{R}^N$ is compact.

Claim 1. We can find $C = C(N, p, A) > 0$, such that, if $V \subseteq \mathbb{R}^N$ is open with $A \subseteq V$, then we can find an open set $U \subseteq \mathbb{R}^N$ and $f \in K^p$, such that

$$A \subseteq U \subseteq \big\{f = 1\big\}, \quad \mathrm{supp}\, f \subseteq V \quad \text{and} \quad \|Df\|_p^p \leqslant C.$$

Let $V \subseteq \mathbb{R}^N$ be an open set, such that $A \subseteq V$. Let us set

$$\delta \;\stackrel{df}{=}\; \frac{d(A, V^c)}{2} > 0.$$

Because A is compact and

$$\mu^{(N-p)}(A) \;<\; +\infty,$$

we can find

$$\{z_k\}_{k=1}^m \;\subseteq\; A$$

and $\{r_k\}_{k=1}^m \subseteq \mathbb{R}_+ \setminus \{0\}$, such that

$$2r_k < \delta, \quad A \subseteq \bigcup_{k=1}^m B_{r_k}(z_k) \quad \text{and} \quad \sum_{k=1}^m r_k^{N-p} \;\leqslant\; \mu^{(N-p)}(A) + 1. \tag{1.75}$$

Let us set

$$U \stackrel{df}{=} \bigcup_{k=1}^{m} B_{r_k}(z_k)$$

and let $f_k \in K^p$ be defined by

$$f_k(z) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad \|z - z_k\|_{\mathbb{R}^N} < r_k, \\ 2 - \frac{\|z-z_k\|_{\mathbb{R}^N}}{r_k} & \text{if} \quad r_k \leqslant \|z - z_k\|_{\mathbb{R}^N} \leqslant 2r_k, \\ 0 & \text{if} \quad 2r_k < \|z - z_k\|_{\mathbb{R}^N}. \end{cases}$$

Using Proposition 1.6.4**(a)**, we see that

$$\|Df_k\|_p^p \leqslant Cr_k^{N-p} \qquad \forall\, k \in \{1, \dots, m\}. \tag{1.76}$$

Let us set $f \stackrel{df}{=} \max_{1 \leqslant k \leqslant m} f_k$. Then $f \in K^p$, $U \subseteq \{f = 1\}$, $\operatorname{supp} f \subseteq V$ and from (1.76) and (1.75), we have

$$\|Df\|_p^p \leqslant \sum_{k=1}^{m} \|Df_k\|_p^p \leqslant C \sum_{k=1}^{m} r_k^{N-p} \leqslant C\big(\mu^{(N-p)}(A) + 1\big),$$

which proves Claim 1.

We use the claim inductively and produce a sequence $\{U_n\}_{n \geqslant 1}$ of open sets in $\mathbb{R}^N$ and functions $\{f_n\}_{n \geqslant 1} \subseteq K^p$, such that

$$A \subseteq U_{n+1} \subseteq U_n, \quad \overline{U}_{n+1} \subseteq \operatorname{int}\{f_n = 1\} \qquad \forall\, n \geqslant 1$$

and

$$\operatorname{supp} f_n \subseteq U_n, \quad \|Df_n\|_p^p \leqslant C \qquad \forall\, n \geqslant 1.$$

Let

$$S_m \stackrel{df}{=} \sum_{n=1}^{m} \frac{1}{n} \quad \text{and} \quad h_m \stackrel{df}{=} \frac{1}{S_m} \sum_{n=1}^{m} \frac{1}{n} f_n.$$

We have

$$h_m \in K^p \quad \text{and} \quad h_m \geqslant 1 \quad \text{on } U_{m+1}.$$

Also because $\operatorname{supp}\big(\|Df_n(\cdot)\|_{\mathbb{R}^N}\big) \subseteq U_n \setminus \overline{U}_{n+1}$ and $p > 1$, we see that

$$\begin{aligned} \operatorname{cap}_p(A) &\leqslant \big\|Dh_m\big\|_p^p \leqslant \frac{1}{S_m^p} \sum_{n=1}^{m} \frac{1}{n^p} \big\|Df_n\big\|_p^p \\ &\leqslant \frac{C}{S_m^p} \sum_{n=1}^{m} \frac{1}{n^p} \longrightarrow 0 \quad \text{as } m \to +\infty. \end{aligned}$$

(b) Since $\mathrm{cap}_p(A) = 0$, for every $n \geqslant 1$ we can find $f_n \in K^p$, such that $A \subseteq \mathrm{int}\{f_n \geqslant 1\}$ and

$$\|Df_n\|_p^p \leqslant \frac{1}{2^n}. \tag{1.77}$$

Let us set $h \stackrel{df}{=} \sum_{n=1}^{\infty} f_n$. From (1.77), we have

$$\|Dh\|_p \leqslant \sum_{n=1}^{\infty} \|Df_n\|_p < +\infty. \tag{1.78}$$

Using Proposition 1.6.9 and (1.78), we have that

$$\|h\|_{p^*} \leqslant \sum_{n=1}^{\infty} \|f_n\|_{p^*} \leqslant \sum_{n=1}^{\infty} C\,\|Df_n\|_p < +\infty,$$

so $h \in K^p$.

Observe that $A \subseteq \mathrm{int}\{h \geqslant m\}$ for all $m \geqslant 1$. Let $z_0 \in A$. For $r > 0$ small, we have $\overline{B}_r(z_0) \subseteq \mathrm{int}\{h \geqslant m\}$, hence

$$\overline{h}_{z_0,r} \stackrel{df}{=} \frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z_0)} h(z)\,dz \geqslant m,$$

which implies that

$$\overline{h}_{z_0,r} \longrightarrow +\infty \quad \text{as } r \searrow 0. \tag{1.79}$$

Claim 2. For every $z_0 \in A$, we have

$$\lim_{r \searrow 0} \frac{1}{r^s} \int\limits_{\overline{B}_r(z_0)} \big\|Dh(z)\big\|_{\mathbb{R}^N}^p\,dz = +\infty,$$

for any $s > N - p$.

To prove this claim, we proceed by contradiction. Let $z_0 \in A$ and suppose that

$$\lim_{r \searrow 0} \frac{1}{r^s} \int\limits_{\overline{B}_r(z_0)} \big\|Dh(z)\big\|_{\mathbb{R}^N}^p\,dz < +\infty.$$

Then we can find $M < +\infty$, such that for all $r \in (0, 1]$, we have

$$\frac{1}{r^s} \int\limits_{\overline{B}_r(z_0)} \big\|Dh(z)\big\|_{\mathbb{R}^N}^p\,dz \leqslant M.$$

Invoking the Poincaré-Wirtinger inequality (see Theorem 1.6.8), we have

$$\begin{aligned} &\frac{1}{\lambda^N(\overline{B}_r(z_0))}\int\limits_{\overline{B}_r(z_0)} \left|h(z)-\overline{h}_{z_0,r}\right| dz \\ &\leqslant \frac{C}{\lambda^N(\overline{B}_r(z_0))}\int\limits_{\overline{B}_r(z_0)} \left\|Dh(z)\right\|^p_{\mathbb{R}^N} dz \;\leqslant\; C_1 r^{\vartheta}, \end{aligned} \tag{1.80}$$

for some $C, C_1 > 0$, $\vartheta = s-(N-p)$ and for all $r \in (0,1]$ (recall that $\lambda^N\big(\overline{B}_r(z_0)\big) = a(N)r^N$). Since

$$\lambda^N\big(\overline{B}_r(z_0)\big) \;=\; 2^N\lambda^N\big(\overline{B}_{\frac{r}{2}}(z_0)\big)$$

and using Jensen's inequality (see Theorem A.2.26) and (1.80), we have that

$$\begin{aligned} \left|\overline{h}_{z_0,\frac{r}{2}}-\overline{h}_{z_0,r}\right| \;&=\; \frac{1}{\lambda^N(\overline{B}_{\frac{r}{2}}(z_0))}\left|\int\limits_{\overline{B}_{\frac{r}{2}}(z_0)} \big(h(z)-\overline{h}_{z_0,r}\big)\, dz\right| \\ &\leqslant \frac{2^N}{\lambda^N(\overline{B}_r(z_0))}\int\limits_{\overline{B}_r(z_0)} \left|h(z)-\overline{h}_{z_0,r}\right| dz \\ &\leqslant \frac{2^N}{\lambda^N(\overline{B}_r(z_0))^{\frac{1}{p}}}\left(\int\limits_{\overline{B}_r(z_0)} \left|h(z)-\overline{h}_{z_0,r}\right|^p dz\right)^{\frac{1}{p}} \;\leqslant\; C_2 r^{\frac{\vartheta}{p}}, \end{aligned} \tag{1.81}$$

for some $C_2 > 0$. Therefore, using (1.81), for $k > i$, we have

$$\left|\overline{h}_{z_0,\frac{1}{2^k}}-\overline{h}_{z_0,\frac{1}{2^i}}\right| \;\leqslant\; \sum_{l=i+1}^{k}\left|\overline{h}_{z_0,\frac{1}{2^l}}-\overline{h}_{z_0,\frac{1}{2^{l-1}}}\right| \;\leqslant\; C_2\sum_{l=i+1}^{k}\left(\frac{1}{2^{l-1}}\right)^{\frac{\vartheta}{p}},$$

so $\left\{\overline{h}_{z_0,\frac{1}{2^n}}\right\}_{n\geqslant 1}$ is a Cauchy sequence and this contradicts the fact that $\overline{h}_{z_0,\frac{1}{2^n}} \longrightarrow +\infty$ (see (1.79)). This proves Claim 2.

Then we have

$$\begin{aligned} A &\subseteq \left\{z_0 \in \mathbb{R}^N :\; \limsup_{r\searrow 0}\frac{1}{r^s}\int\limits_{\overline{B}_r(z_0)} \left\|Dh(z)\right\|^p_{\mathbb{R}^N} dz = +\infty\right\} \\ &\subseteq \left\{z_0 \in \mathbb{R}^N :\; \limsup_{r\searrow 0}\frac{1}{r^s}\int\limits_{\overline{B}_r(z_0)} \left\|Dh(z)\right\|^p_{\mathbb{R}^N} dz > 0\right\} \;=\; C_s. \end{aligned}$$

But from Theorem 1.4.9, we have that $\mu^{(s)}(C_s) = 0$, hence $\mu^{(s)}(A) = 0$. □

REMARK 1.6.14 If $p = 1$ and $A \subseteq \mathbb{R}^N$, it can be shown that $\text{cap}_1(A) = 0$ if and only if $\mu^{(1)}(A) = 0$. The proof of this result, which uses functions of bounded variations and the isoperimetric inequality, can be found in Evans & Gariepy (1992, p. 193).

PROPOSITION 1.6.15
If $T \subseteq (0,1)$ *is such that* $\lambda^1(T) > 0$, $p \in [N-1, N)$, $A \subseteq \overline{B}_1(0) \subseteq \mathbb{R}^N$ *and for each* $r \in T$ *there exists unique* $z_r \in \partial\overline{B}_r(0)$, *such that* $z_r \in A$,
then $\text{cap}_p(A) > 0$.

PROOF Let $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ be defined by

$$f(z) \stackrel{df}{=} \|z\|^2_{\mathbb{R}^N} = \sum_{n=1}^{N} z_n^2 \qquad \forall\, z = (z_n)_{n=1}^N \in \mathbb{R}^N.$$

Evidently f is Lipschitz continuous with $\text{Lip}(f) = 1$. So by Proposition 1.3.25, we have

$$\mu^{(1)}\big(f(A)\big) \leqslant \mu^{(1)}(A).$$

Note that $T = f(A)$. So by virtue of Theorem 1.3.21, we have that

$$0 < \lambda^1(T) = \mu^{(1)}\big(f(A)\big) \leqslant \mu^{(1)}(A).$$

If for some $p \in [N-1, N)$, we have that

$$\text{cap}_p(A) = 0,$$

then from Theorem 1.6.13**(b)** (see also Remark 1.6.14), we have $\mu^{(1)}(A) = 0$ (note that $1 > N - p$), a contradiction. So

$$\text{cap}_p(A) > 0 \qquad \forall\, p \in [N-1, N).$$

□

The next result provides a kind of Chebyshev inequality in terms of p-capacities.

PROPOSITION 1.6.16
If $p \in [1, N)$, $f \in K^p$, $\varepsilon > 0$ *and*

$$A \stackrel{df}{=} \left\{ z_0 \in \mathbb{R}^N : \frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_t(z_0)} f(z)\, dz > \varepsilon \text{ for some } r > 0 \right\},$$

then there exists a constant $C = C(N,p)$, *such that*

$$\text{cap}_p(A) \leqslant \frac{C}{\varepsilon^p} \|Df\|_p^p.$$

PROOF First, we show that the set $A \subseteq \mathbb{R}^N$ is open. Let $z_0 \in A$. Then for some $r > 0$ and $\xi > 0$, we have

$$\frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z_0)} f(z)\, dz \;=\; \varepsilon + \xi.$$

Since $f \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^N\big)$, exploiting the absolute continuity of the Lebesgue integral, we can find $\vartheta > 0$ small enough so that if $\lambda^N(B) < \vartheta$, then

$$\frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{B} f(z)\, dz \;<\; \xi.$$

Also let $\delta > 0$ be such that

$$\lambda^N\left(\overline{B}_r(z) \,\triangle\, \overline{B}_r(z_0)\right) \;<\; \vartheta \qquad \forall\ \|z - z_0\|_{\mathbb{R}^N} < \delta,$$

where $X \,\triangle\, Y \stackrel{df}{=} (X \setminus Y) \cup (Y \setminus X)$ is the symmetric difference of X and Y. So, if $\|z - z_0\|_{\mathbb{R}^N} < \delta$, we have

$$\begin{aligned}
&\frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z)} f(y)\, dy \\
&= \frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z_0)} f(y)\, dy + \frac{1}{\lambda^N(\overline{B}_r(z_0))} \left[\int\limits_{\overline{B}_r(z)} f(y)\, dy - \int\limits_{\overline{B}_r(z_0)} f(y)\, dy \right] \\
&\geqslant \varepsilon + \xi - \frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z) \nabla \overline{B}_r(z_0)} f(y)\, dy \;>\; \varepsilon + \xi - \xi \;=\; \varepsilon,
\end{aligned}$$

so $z \in A$ and we infer that A is open.

Next let $z_0 \in A$ and let $r > 0$ be such that

$$\frac{1}{\lambda^N(\overline{B}_r(z_0))} \int\limits_{\overline{B}_r(z_0)} f(y)\, dy \;>\; \varepsilon.$$

Then, by Jensen's inequality (see Theorem A.2.26), we have

$$\begin{aligned}
a(N) r^N \varepsilon \;&<\; \int\limits_{\overline{B}_r(z_0)} f(y)\, dy \\
&\leqslant \left(a(N) r^N\right)^{1 - \frac{1}{p^*}} \left(\int\limits_{\overline{B}_r(z_0)} f(y)^{p^*}\, dy \right)^{\frac{1}{p^*}} \;\leqslant\; \left(a(N) r^N\right)^{1 - \frac{1}{p^*}} \|f\|_{p^*},
\end{aligned}$$

so $r \leqslant C_1$ for some $C_1 > 0$ independent of z_0.

Invoking the Besicovitch covering theorem (see Theorem 1.2.18), we can find $k = k(N) \geqslant 1$, a positive integer and countable collection $\{\mathcal{T}_n\}_{n=1}^{k}$ of closed balls, such that

$$A \subseteq \sum_{n=1}^{k} \sum_{B \in \mathcal{T}_n} B$$

and

$$\frac{1}{\lambda^N(B)} \int_B f(y)\, dy \;>\; \varepsilon \qquad \forall\; B \in \bigcup_{n=1}^{k} \mathcal{T}_n. \tag{1.82}$$

Let $\{B_i^{(n)}\}_{i \geqslant 1}$ be an enumeration of the elements in the countable collection $\mathcal{T}_n$ for $n = 1, \ldots, k$. Using Proposition 1.6.4**(a)**, we can check that

$$\left(\frac{1}{\lambda^N(B_i^{(n)})} \int_{B_i^{(n)}} f(y)\, dy - f \right)^+ \;\in\; W^{1,p}\big(B_i^{(n)}\big).$$

Then Poincaré's inequality (see Theorem 1.6.8) implies that

$$\left\| \left(\frac{1}{\lambda^N(B_i^{(n)})} \int_{B_i^{(n)}} f(y)\, dy - f \right)^+ \right\|_{W^{1,p}(B_i^{(n)})} \;\leqslant\; C_2 \left\| Df \right\|_{L^p(B_i^{(n)};\mathbb{R}^N)},$$

for some $C_2 > 0$. Invoking Proposition 1.6.5, we can find $g_i^{(n)} \in W^{1,p}(\mathbb{R}^N)$, such that

$$g_i^{(n)} \geqslant 0, \quad g_i^{(n)}(z) = \left(\frac{1}{\lambda^N(B_i^{(n)})} \int_{B_i^{(n)}} f(y)\, dy - f \right)^+ (z) \quad \text{for a.a. } z \in B_i^{(n)}$$

and

$$\left\| g_i^{(n)} \right\|_{W^{1,p}(\mathbb{R}^N)} \;\leqslant\; C_3 \left\| Df \right\|_{L^p(B_i^{(n)};\mathbb{R}^N)}, \tag{1.83}$$

for some $C_3 = C_3(N,p) > 0$. From (1.82), we have

$$f + g_i^{(n)} \;\geqslant\; \frac{1}{\lambda^N(B_i^{(n)})} \int_{B_i^{(n)}} f(y)\, dy \;>\; \varepsilon.$$

Also if

$$g \overset{df}{=} \sup_{\substack{i \geqslant 1 \\ n\,=\,1,\ldots,k}} g_i^{(n)},$$

we have $g \geqslant 0$. We claim that $h \in K^p$. To this end, using (1.83), note that

$$\int_{\mathbb{R}^N} \sup_{\substack{i \geqslant 1 \\ n=1,\dots,k}} \left|g_i^{(n)}(y)\right|^p dy \leqslant \sum_{n=1}^{k} \sum_{i=1}^{\infty} \int_{\mathbb{R}^N} \left|g_i^{(n)}(y)\right|^p dy$$
$$\leqslant \sum_{n=1}^{k} C_3^p \sum_{i=1}^{\infty} \int_{B_i^{(n)}} \|Df(y)\|_{\mathbb{R}^N}^p \, dy \leqslant k C_3^p \|Df\|_p^p,$$

so $g \in L^p(\mathbb{R}^N)$.

Also, we have

$$\int_{\mathbb{R}^N} \sup_{\substack{i \geqslant 1 \\ n=1,\dots,k}} \left|Dg_i^{(n)}\right|^p dy \leqslant \sum_{n=1}^{k} \sum_{i=1}^{\infty} \left\|Dg_i^{(n)}\right\|_p^p$$
$$\leqslant \sum_{n=1}^{k} C_3^p \sum_{i=1}^{\infty} \int_{B_i^{(n)}} \|Df(y)\|_{\mathbb{R}^N}^p \, dy \leqslant k C_3^p \|Df\|_p^p. \tag{1.84}$$

From Proposition 1.6.4**(b)**, we infer that

$$\|Dg(y)\|_{\mathbb{R}^N} \leqslant \sup_{\substack{i \geqslant 1 \\ n=1,\dots,k}} \left|Dg_i^{(n)}(y)\right|^p \quad \text{for a.a. } y \in \mathbb{R}^N,$$

so by (1.84), we have that $Dg \in L^p\big(\mathbb{R}^N; \mathbb{R}^N\big)$ and thus $g \in K^p$.

Because $f + g > \varepsilon$ almost everywhere on A and A is open, using also (1.84), it follows that

$$\operatorname{cap}_p(A) \leqslant \int_{\mathbb{R}^N} \left\|\frac{1}{\varepsilon} D(f+g)(y)\right\|_{\mathbb{R}^N}^p dy$$
$$\leqslant \frac{C_4}{\varepsilon^p} \left(\|Df\|_p^p + \|Dg\|_p^p\right) \leqslant \frac{C_5}{\varepsilon^p} \|Df\|_p^p,$$

for some $C_4, C_5 = C_5(N,p) > 0$. Setting $C = C_5$, we obtain the result. ∎

DEFINITION 1.6.17 *A function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be **p-quasicontinuous**, if for each $\varepsilon > 0$, we can find an open set $U \subseteq \mathbb{R}^N$, such that $\operatorname{cap}_p(U) < \varepsilon$ and $f|_{\mathbb{R}^N \setminus U}$ is continuous.*

We have all the necessary tools to prove the following "differentiability" result for Sobolev functions. As we already said a systematic study of Sobolev functions and of their differentiability properties will be conducted in Chapter 2. Here we state a result which says that up to a set A of p-capacity zero, a function $f \in W^{1,p}(\mathbb{R}^N)$ can be represented by a p-quasicontinuous function.

THEOREM 1.6.18
If $p \in [1, N)$ *and* $f \in W^{1,p}(\mathbb{R}^N)$,
then there exists $f^* \colon \mathbb{R}^N \longrightarrow \mathbb{R}$ *a p-quasicontinuous function, such that*

(a) *there exists a Borel set* $A \subseteq \mathbb{R}^N$ *with* $\mathrm{cap}_p(A) = 0$, *such that*

$$\lim_{r \to 0} \frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} f(y)\, dy \;=\; f^*(z) \qquad \forall\, z \in \mathbb{R}^N \setminus A;$$

(b) *for each* $z \in \mathbb{R}^N \setminus A$, *we have*

$$\lim_{r \to 0} \frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} \left| f(y) - f^*(y) \right|^{p^*} dy \;=\; 0.$$

PROOF **(a)** Let

$$C \stackrel{df}{=} \left\{ \limsup_{r \to 0} \frac{1}{r^{N-p}} \int\limits_{\overline{B}_r(z)} \|Df(y)\|^p_{\mathbb{R}^N}\, dy > 0 \right\}.$$

From Theorem 1.4.9, we have that

$$\mu^{(N-p)}(C) \;=\; 0$$

and this by Theorem 1.6.13**(a)** implies that

$$\mathrm{cap}_p(C) \;=\; 0.$$

Moreover, from the Poincaré-Wirtinger inequality (see Theorem 1.6.8), we see that

$$\lim_{r \to 0} \frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} \left| f(y) - \overline{f}_{z,r} \right| dy \;=\; 0 \qquad \forall\, z \in \mathbb{R}^N \setminus C. \tag{1.85}$$

By virtue of Proposition 1.6.3, we can find a sequence

$$\{f_n\}_{n \geqslant 1} \;\subseteq\; W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N),$$

such that

$$\|Df - Df_n\|^p_p \;\leqslant\; \frac{1}{2^{(p+1)n}} \qquad \forall\, n \geqslant 1. \tag{1.86}$$

For $n \geqslant 1$, we introduce the sets

$$E_n \stackrel{df}{=} \left\{ z \in \mathbb{R}^N : \left| \frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} |f(y) - f_n(y)|\, dy \right| > \frac{1}{2^n} \text{ for some } r > 0 \right\}.$$

From Proposition 1.6.16 and (1.86), we have that

$$\frac{\mathrm{cap}_p(E_n)}{2^{pn}} \leqslant C\,\|Df - Df_n\|_p^p \leqslant \frac{C}{2^{(p+1)n}},$$

for some $C = C(N,p) > 0$, so

$$\mathrm{cap}_p(E_n) \leqslant \frac{c}{2^n}. \tag{1.87}$$

Moreover, we have

$$\begin{aligned}\left|\overline{f}_{z,r} - f_n(z)\right| \leqslant \frac{1}{\lambda^N(B_r(z))}\Bigg\{ &\int\limits_{B_r(z)} \left|f(y) - \overline{f}_{z,r}\right| dy + \int\limits_{B_r(z)} |f(y) - f_n(y)|\, dy \\ &+ \int\limits_{B_r(z)} |f_n(y) - f_n(z)|\, dy\Bigg\},\end{aligned}$$

so from (1.85), we have

$$\lim_{r\to 0}\left|\overline{f}_{z,r} - f_n(z)\right| \leqslant \frac{1}{2^n} \qquad \forall\ z \in C \cup E_n. \tag{1.88}$$

We set

$$A_k \overset{df}{=} C \cup \left(\bigcup_{n=k}^{\infty} E_n\right) \qquad \forall\ k \geqslant 1.$$

Evidently A_k is Borel for $k \geqslant 1$ and we have that

$$\mathrm{cap}_p(A_k) \leqslant \mathrm{cap}_p(C) + \sum_{n=k}^{\infty} \mathrm{cap}_p(E_n) \leqslant \sum_{n=k}^{\infty} \frac{1}{2^n}. \tag{1.89}$$

Then, if

$$A \overset{df}{=} \bigcap_{k=1}^{\infty} A_k,$$

from (1.89), we have that

$$\mathrm{cap}_p(A) \leqslant \lim_{k\to+\infty} \mathrm{cap}_p(A_k) = 0.$$

Note that, if $z \in \mathbb{R}^N \setminus A_k$ and $n, m \geqslant k$, from (1.88), we have

$$\left|f_n(z) - f_m(z)\right| \leqslant \limsup_{r\to 0}\left|\overline{f}_{z,r} - f_n(z)\right| + \limsup_{r\to 0}\left|\overline{f}_{z,r} - f_m(z)\right| \leqslant \frac{1}{2^n} + \frac{1}{2^m},$$

so the sequence $\{f_n\}_{n\geqslant 1}$ converges uniformly on $\mathbb{R}^N \setminus A_k$ to some $h \in C\big(\mathbb{R}^N\big)$.

Also we have

$$\limsup_{r\to 0}\left|h(z) - \overline{f}_{z,r}\right| \leqslant |h(z) - f_n(z)| + \limsup_{r\to 0}\left|f_n(z) - \overline{f}_{z,r}\right|,$$

so, from (1.87), we have

$$h(z) \;=\; \lim_{r\to 0} \overline{f}_{z,r} \;=\; f^*(z) \qquad \forall\, z \in \mathbb{R}^N \setminus A_k,\; k \geqslant 1$$

hence

$$f^*(z) \;=\; \lim_{r\to 0} \overline{f}_{z,r} \qquad \forall\, z \in \mathbb{R}^N \setminus A.$$

We need to show that f^* is p-quasicontinuous. For this purpose let $\varepsilon > 0$ be given. We choose $k \geqslant 1$, such that

$$\mathrm{cap}_p(A_k) \;<\; \frac{\varepsilon}{2}$$

(see (1.89)). From Theorem 1.6.11**(a)**, we know that we can find open set $U \supseteq A_k$ with

$$\mathrm{cap}_p(A) \;<\; \varepsilon.$$

Because the sequence $\{f_n\}_{n\geqslant 1}$ converges to f^* uniformly on $\mathbb{R}^N \setminus U$, we infer that the function $f^*|_{\mathbb{R}^N\setminus U}$ is continuous, hence f^* is p-quasicontinuous.

(b) Note that $C \subseteq A$ and so from (1.85), we see that for all $z \in \mathbb{R}^N \setminus A$, we have that

$$\begin{aligned} &\lim_{r\to 0} \left(\frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} |f(y) - f^*(z)|^{p^*}\, dy \right)^{\frac{1}{p^*}} \\ &\leqslant \lim_{r\to 0} \left|\overline{f}_{z,r} - f^*(z)\right| + \lim_{r\to 0} \left(\frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} \left|f(y) - \overline{f}_{z,r}\right|^{p^*} dy \right)^{\frac{1}{p^*}} \;=\; 0. \end{aligned}$$

□

REMARK 1.6.19 By virtue of Theorem 1.6.13**(b)**, we have that

$$\mu^{(s)}(A) \;=\; 0 \qquad \forall\, s > N - p.$$

Hence $\dim A \leqslant N - p$. On the other hand this last inequality does not necessarily imply that

$$\mu^{(N-p)}(A) \;<\; +\infty,$$

which in turn gives us that $\mathrm{cap}_p(A) = 0$ (see Theorem 1.6.13**(a)**). Therefore the conclusion $\mathrm{cap}_p(A) = 0$ in the statement of Theorem 1.6.18 is stronger than the dimensionality condition $\dim A \leqslant N - p$.

1.7 Remarks

1.1: The necessary measure theoretic background is standard and can be found in any books on abstract measure theory. We mention the books of Ash (1972), Denkowski, Migórski & Papageorgiou (2003*a*, Chapter 2), Dudley (1989), Hewitt & Stromberg (1975) and Royden (1968). Outer measures were first introduced by Carathéodory (1914), who also gave the definition of a μ-measurable set (see Definition 1.1.3) and proved that $\Sigma(\mu)$ is a σ-field and the outer measure μ restricted on $\Sigma(\mu)$ is a measure.

For a proof of Proposition 1.1.10 we refer to Denkowski, Migórski & Papageorgiou (2003*a*, p. 184-186) and Evans & Gariepy (1992, p. 6–9).

1.2: In some books a Vitali cover (see Definition 1.2.2) is called "fine cover" (see, e.g., Evans & Gariepy (1992)). The original version of Theorem 1.2.5 (Vitali covering theorem) is due to Vitali (1908), who employed closed cubes. The first to study the differentiability of monotone functions (or more generally of functions of bounded variation, in particular then of absolutely continuous functions) was Lebesgue (1904, 1910). Evans & Gariepy (1992, p. 30), Hardt (1979), Simon (1983) and Ziemer (1989) contain the proof of Theorem 1.2.18. For the proof of Theorem 1.2.19 see Evans & Gariepy (1992, p. 35). In general covering theorems are useful in harmonic analysis and in geometric measure theory. More about them can be found in de Guzman (1975).

1.3: Carathéodory (1914), working with outer measures, was the first to introduce "Hausdorff measures." More precisely, he introduced "1-dimensional" (or "linear") measures in $\mathbb{R}^N$ and also indicated that similarly one can define k-dimensional measures in $\mathbb{R}^N$ for any integer $k \geqslant 1$. Hausdorff (1919) realized that Carathéodory's definition can be used also for noninteger $s \geqslant 0$. He then went on to show that Cantor's ternary set has fractional dimension $\frac{\ln 2}{\ln 3}$. An extension of the theory can be achieved by replacing $\delta(A_n)^s$ in the definition of the Hausdorff measure, by $\xi(A_n)$, where $\xi\colon 2^X \longrightarrow \mathbb{R}_+$ is any premeasure, i.e., $\xi(\emptyset) = 0$ and if $U \subseteq V$ then $\xi(U) \leqslant \xi(V)$ (monotonicity). Of special interest are premeasures resulting from Hausdorff functions h. Namely we consider a function $h\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying:

(a) $h(t) > 0$ for all $t \geqslant 0$,

(b) if $t \leqslant s$, then $h(t) \leqslant h(s)$,

(c) h is right continuous at every $t \geqslant 0$.

Such a function is called ***Hausdorff function***. For such a function and a positive constant ϑ, we define a premeasure ξ on the metric space X, by

$$\xi(A) \stackrel{df}{=} \begin{cases} \min\{h(\delta(A)), h(\vartheta)\} & \text{if} \quad A \neq \emptyset, \\ 0 & \text{if} \quad A = \emptyset. \end{cases}$$

Then ξ is the premeasure defined by h and the cut-off level ϑ. For more details about this generalization, we refer to Davies (1970) and Davies & Samuels (1974). In the presentation of the isodiametric inequality (see Theorem 1.3.20) and of the fact that $\mu^{(N)}$ is a multiple of the Lebesgue measure λ^N, we follow Evans & Gariepy (1992, Chapter 3). We refer also to Falconer (1985, Section 1.6), Federer (1969, Section 2.10.33) and Hardt (1979). It will be a grave omission not to mention the fundamental contributions on the field of Hausdorff measures made by Besicovitch. We mention the works of Besicovitch (1945, 1946) related to Theorem 1.2.18 (the covering theorem bearing his name). A more complete list of the works of Besicovitch can be found in the book of Falconer (1985).

1.4: The intuitive meaning of Theorem 1.4.1 is that for λ^N-almost all $x \in A$, small balls centered at x consist predominantly of points of A. Theorems 1.4.1 and 1.4.6 are due to Lebesgue (1910). They were generalized by Besicovitch (1945, 1946), who replaced the Lebesgue measure λ^N by a Radon measure on $\mathbb{R}^N$. Another source of information for the differentiation of measures in $\mathbb{R}^N$ is the book of Widom (1969).

1.5: Theorem 1.5.4 was originally proved by McShane (1934), who produced the minimal Lipschitz extension of f. Theorem 1.5.8 was originally proved by Rademacher (1919). The proof that is given here is essentially due to Morrey (1966, Theorem 3.1.6). It can be found also in Evans & Gariepy (1992), Simon (1983) and Ziemer (1989). If we employ the notion of Haar-null set, we can also have an extension of Rademacher's theorem to locally Lipschitz functions between Banach spaces.

DEFINITION 1.7.1 *Let $(G,+)$ be an abelian Polish group and d an invariant metric on G compatible with the topology (therefore automatically complete). A universally measurable set $A \subseteq G$ is a* ***Haar-null set****, if there exists a probability measure μ on G (not unique), such that $\chi_A \star \mu = 0$, where χ_A is the characteristic function of the set A and $(\chi_A \star \mu)(x) = \int_G \chi_A(x+y)\, d\mu(y)$.*

REMARK 1.7.2 The above definition is equivalent to the requirement that every translate of the set A is a zero set for the measure μ. The measure μ is usually called test measure.

The next theorem is the extension of Theorem 1.5.8 to functions between Banach spaces. It will be proved in Section 4.3 (see Theorem 4.3.17).

THEOREM 1.7.3
If X is a separable Banach space, Y is a Banach space with the RNP (see Section 2.1) and $f\colon X \longrightarrow Y$ is locally Lipschitz,
then there exists a universally measurable set $D \subseteq X$, such that $X \setminus D$ is Haar-null and $f|_D$ is differentiable in the sense of Gâteaux.

Our derivation of the area formula (see Theorem 1.5.21) is based on Evans & Gariepy (1992, Section 3.3) (see also Federer (1969, Section 3.2) and Hardt (1979)). Lipschitz continuous functions and their properties are discussed in Federer (1969, Section 3.3). If $N = M$, from the two change of variables results (see Theorem 1.5.23 and Theorem 1.5.26), we obtain the following change of variables formula.

THEOREM 1.7.4
If $U, V \subseteq \mathbb{R}^N$ are open sets, $f\colon U \longrightarrow V$ is a locally Lipschitz homeomorphism and $u \in L^1(V)$,
then

$$v \;=\; (u \circ f)\big|\det J_f\big| \;\in\; L^1(U)$$

and

$$\int_U u\big(f(x)\big)\big|\det J_f(x)\big|\, d\lambda^N(x) \;=\; \int_V u(y)\, d\lambda^N(y).$$

The proof of Theorem 1.5.25 can be found in Evans & Gariepy (1992, Section 3.4) or Federer (1969, Subsection 3.2.11).

1.6: Our treatment of capacity follows Evans & Gariepy (1992, Section 4.7) (see also Federer & Ziemer (1972)). There are other notions of capacity as for example Bessel capacity, Riesz capacity, etc., which are discussed in Stein (1970) and Ziemer (1989). The abstract theory of capacities in Banach spaces can be found in Fowler (1973). Moreover, for the use of capacities in the convergence of obstacles, we refer to Dal Maso (1985).

Chapter 2

Lebesgue-Bochner and Sobolev Spaces

The functional-analytic approach to the solution of (partial) differential equations requires knowledge of the properties of spaces of functions of one or several real variables.

A large class of infinite dimensional dynamical systems (evolution systems) can be modelled as an abstract differential equation defined on a suitable Banach space or on a suitable manifold therein. The advantage of such an abstract formulation lies not only on its generality but also in the insight that can be gained about the many common unifying properties that tie together apparently diverse problems. It is clear that such a study relies on the knowledge of various spaces of vector valued functions (i.e., of Banach space valued functions).

For this reason Section 2.1 deals with vector valued functions. We introduce the various notions of measurability for such functions and then based on them we define the different integrals corresponding to them. The emphasis is on the so-called Bochner integral, which generalizes in a very natural way the classical Lebesgue integral to vector valued functions.

In Section 2.2 we continue with vector valued functions and introduce the so-called Lebesgue-Bochner spaces, which extend to vector valued functions the well known Lebesgue L^p-spaces. We also consider evolution triples and the function spaces associated with them. Evolution triples provide a suitable analytical framework for the study of a large class of linear and nonlinear evolution equations.

In Section 2.3 we have compactness results for the spaces introduced and studied in the previous section. The compactness results refer to both the strong and the weak topologies on the spaces under consideration.

Thus far we are dealing with function spaces arising in evolutionary problems. In Section 2.4 we study Sobolev spaces, which are the main tools in the analysis of both stationary and nonstationary equations. Sobolev spaces play a central role in the modern theory of partial differential equations and they allow us to broaden significantly the notion of solution of a boundary value problem. They provide a natural functional analytical framework for the study of weak solutions of elliptic boundary value problems. No specific applications to problems in partial differential equations are discussed. Instead the section aims to serve as a concise introduction to the properties of

the Sobolev spaces (of both one and several variables).

In Section 2.5 we present some fundamental inequalities associated with Sobolev functions, the celebrated embedding theorems for the Sobolev spaces and some of their consequences. The embedding theorems are arguably the most important results in this theory and the reason why Sobolev spaces are so effective in dealing with boundary value problems.

Finally in Section 2.6 we establish some fine properties of Sobolev spaces and introduce functions of bounded variation (BV-functions). These are functions whose weak first partial derivatives are Radon measure and this is essentially the weakest measurable theoretic sense in which a function can be differentiable. They are particularly useful in theoretical mechanics.

2.1 Vector-Valued Functions

In this section we deal with functions which take values in Banach spaces. For such functions we define the various notions of measurability and different integrals corresponding to them. The domain of a function is a finite measure space (Ω, Σ, μ) and the range is a Banach space X. By X^* we denote the topological dual of X and by $\langle \cdot, \cdot \rangle_X$ the duality brackets for the pair (X^*, X). By $\mathcal{B}(X)$ we denote the Borel σ-field of X.

DEFINITION 2.1.1 *Let $f\colon \Omega \longrightarrow X$ be a function.*

(a) *Function f is said to be a **simple function**, if it takes only finite number of values, say $x_1, \dots, x_N$ and*

$$C_k \;=\; f^{-1}\big(\{x_k\}\big) \;\in\; \Sigma \qquad \forall\, k \in \big\{1, \dots, N\big\}.$$

*The formula $s = \sum_{k=1}^{N} x_k \chi_{C_k}$ is called the **standard representation** of f.*

(b) *Function f is said to be **strongly measurable** (or **Bochner measurable**), if there exists a sequence $\big\{s_n\colon \Omega \longrightarrow X\big\}_{n \geqslant 1}$ of simple functions, such that*

$$s_n(\omega) \;\longrightarrow\; f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

where $\longrightarrow$ denotes the convergence in the norm topology of X, i.e.,

$$\big\|f(\omega) - s_n(\omega)\big\|_X \;\longrightarrow\; 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

(c) *Function f is said to be **weakly measurable**, if for all $x^* \in X^*$, the function $\omega \longrightarrow \big\langle x^*, f(\omega) \big\rangle_X$ is Σ-measurable.*

(d) *Function $f\colon \Omega \longrightarrow X^*$ is said to be* ***weak*-measurable****, if for all $x \in X$, the function $\omega \longmapsto \big\langle f(\omega), x\big\rangle_X$ is Σ-measurable.*

REMARK 2.1.2 Evidently strong measurability of a function $f\colon \Omega \longrightarrow X$ implies its weak measurability. Also strong measurability implies that for every $B \in \mathcal{B}(X)$, we have that $f^{-1}(B) \in \Sigma$ (i.e., f is Borel measurable). Moreover, adapting the proof of the classical result, which asserts that a measurable $\mathbb{R}$-valued function is the μ-almost everywhere limit of a sequence of simple functions, we see that if X is separable, then $f\colon \Omega \longrightarrow X$ is strongly measurable if and only if it is Borel measurable. In fact, in the next theorem, known as the ***Pettis measurability theorem***, when X is separable, the situation simplifies considerably.

THEOREM 2.1.3
A function $f\colon \Omega \longrightarrow X$ is strongly measurable if and only if it is weakly measurable and μ-almost separably valued (i.e., there exists a set $A \in \Sigma$ with $\mu(A) = 0$, such that $f(\Omega \setminus A)$ is separable in X).

PROOF "$\Longrightarrow$": Let $f\colon \Omega \longrightarrow X$ be a strongly measurable function. As we already pointed out, f is weakly measurable. Also since f is strongly measurable, we can find a sequence $\{s_n\}_{n\geqslant 1}$ of X-valued simple functions and a μ-null set $A \in \Sigma$, such that

$$s_n(\omega) \longrightarrow f(\omega) \quad \text{in } X, \text{ for all } \omega \in \Omega \setminus A.$$

Let $\{y_n\}_{n\geqslant 1}$ be a sequence of all the values taken by the sequence $\{s_n\}_{n\geqslant 1}$ (clearly the set is countable). Let

$$Y \stackrel{df}{=} \overline{\text{span}}\,\{y_n\}_{n\geqslant 1}.$$

Then Y is a closed separable subspace of X. Moreover, $f(\Omega \setminus A) \subseteq Y$ and so f is μ-almost separably valued.

"$\Longleftarrow$": Let $f\colon \Omega \longrightarrow X$ be a μ-almost separably valued function. Without any loss of generality, we may assume that f is separably valued. Then replacing X by $Y = \overline{\text{span}}\, f(\Omega)$, which is separable, we see that we may assume that X is separable. Let $\{x_n^*\}_{n\geqslant 1}$ be dense in $\partial \overline{B}_1^{X^*}(0)$, where

$$\partial \overline{B}_1^{X^*}(0) = \big\{x^* \in X^* : \|x^*\|_{X^*} = 1\big\}.$$

Then

$$\big\|f(\omega)\big\|_X = \sup_{n\geqslant 1} \big|\big\langle x_n^*, f(\omega)\big\rangle_X\big|.$$

But for each $n \geqslant 1$,

$$\text{the function } \omega \longmapsto \big\langle x_n^*, f(\omega)\big\rangle_X \text{ is } \Sigma\text{-measurable,}$$

hence

$$\text{the function } \omega \longmapsto \big\|f(\omega)\big\|_X \text{ is } \Sigma\text{-measurable.}$$

Let

$$C_0 \stackrel{df}{=} \big\{\omega \in \Omega : \ \big\|f(\omega)\big\|_X > 0\big\}.$$

We have that $C_0 \in \Sigma$ and for every $y \in X$,

$$\text{the function } \omega \longmapsto f(\omega) - y \text{ is } \Sigma \cap C_0\text{-measurable.}$$

Therefore,

$$\text{the function } \omega \longmapsto \big\|f(\omega) - y\big\|_X \text{ is } \Sigma \cap C_0\text{-measurable.}$$

Let $\{z_n\}_{n\geqslant 1}$ be dense in $f(\Omega)$. For a given $\varepsilon > 0$, we define

$$D_n \stackrel{df}{=} \big\{\omega \in C_0 : \ \|f(\omega) - z_n\|_X < \varepsilon\big\}.$$

Evidently

$$D_n \ \in \ \Sigma \cap C_0 \quad \text{and} \quad C_0 \ = \ \bigcup_{n=1}^{\infty} D_n.$$

Let

$$E_n \stackrel{df}{=} D_n \setminus \bigcup_{i=1}^{n-1} D_i.$$

Then $\{E_n\}_{n\geqslant 1} \subseteq \Sigma \cap C_0$ is a sequence of disjoint sets and

$$C_0 \ = \ \bigcup_{n=1}^{\infty} E_n.$$

We define

$$f_\varepsilon(\omega) \stackrel{df}{=} \begin{cases} z_n & \text{if} \quad \omega \in E_n, \ n \geqslant 1, \\ 0 & \text{if} \quad \omega \in \Omega \setminus C_0. \end{cases}$$

Clearly $f_\varepsilon : \Omega \longrightarrow X$ is Σ-measurable, countably-valued (i.e., takes countably many values) and

$$\|f(\omega) - f_\varepsilon(\omega)\|_X \ < \ \varepsilon \qquad \forall\, \omega \in \Omega.$$

Taking $\varepsilon = \frac{1}{k}$, $k \geqslant 1$, we see that f is the uniform limit of a sequence $\big\{f_{\frac{1}{k}}\big\}_{k\geqslant 1}$ of countably-valued functionals, hence f is strongly measurable. □

An interesting byproduct of the previous proof is the following result.

COROLLARY 2.1.4

A function $f : \Omega \longrightarrow X$ is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued, Σ-measurable functions.

By virtue of Theorem 2.1.3, we see that the measurability situation of X-valued functions simplifies considerably when X is separable.

THEOREM 2.1.5
If X is separable and $f\colon \Omega \longrightarrow X$,
then the following three properties are equivalent:

(a) *f is strongly measurable;*

(b) *f is Borel measurable;*

(c) *f is weakly measurable.*

REMARK 2.1.6 The usual facts regarding the stability of strongly measurable functions under sum, scalar multiplication and pointwise μ-almost everywhere limits hold. Also by just replacing absolute values by norms in the proof of the classical Egorov's theorem (see Theorem A.2.12), we see that the result generalizes to X-valued functions. Finally for any Banach space X and a strongly measurable function $f\colon \Omega \longrightarrow X$, the function $\omega \longmapsto \|f(\omega)\|_X$ is Σ-measurable. Indeed, if $\{s_n\}_{n\geqslant 1}$ is the sequence of X-valued simple functions, such that

$$s_n(\omega) \longrightarrow f(\omega) \quad \text{in } X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then

$$\big|\|f(\omega)\|_X - \|s_n(\omega)\|_X\big| \leqslant \|f(\omega) - s_n(\omega)\|_X \longrightarrow 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and so the function $\omega \longmapsto \|f(\omega)\|_X$ is Σ-measurable.

EXAMPLE 2.1.7 It can be shown that weak measurability does not imply strong measurability. Because of Theorem 2.1.5, we look for functions with values in a nonseparable Banach space. So consider the nonseparable Hilbert space $X = l^2\big([0,1]\big)$ and let $\{e_t\}_{t\in[0,1]}$ be an orthonormal basis. The function $f\colon [0,1] \longrightarrow l^2\big([0,1]\big)$ defined by

$$f(t) = e_t$$

is weakly measurable, since

$$\big(x^*, f(t)\big)_X = \big(x^*, e_t\big)_X = 0 \qquad \forall\, x^* \in l^2\big([0,1]\big)^* = l_2\big([0,1]\big).$$

On the other hand, if $A \subseteq [0,1]$, then $f\big([0,1] \setminus A\big)$ is separable if and only if $[0,1] \setminus A$ is countable and so we cannot have

$$\lambda^1(A) = 0.$$

Therefore by virtue of Corollary 2.1.4, f is not strongly measurable.

Now we are ready to define the Bochner integral for strongly measurable functions.

DEFINITION 2.1.8 **(a)** *Let*

$$s(\omega) \stackrel{df}{=} \sum_{k=1}^{N} x_k \chi_{C_k}(\omega), \quad x_k \in X, \quad C_k \in \Sigma$$

be an X-valued simple function. The ***Bochner integral*** *of s is defined by*

$$\int_{\Omega} s(\omega)\, d\mu(\omega) \stackrel{df}{=} \sum_{k=1}^{N} \mu(C_k) x_k.$$

(b) *A function $f\colon \Omega \longrightarrow X$ is said to be* ***Bochner integrable****, if there exists a sequence $\{s_n\}_{n\geqslant 1}$ of simple functions, such that*

$$\lim_{n\to+\infty} \int_{\Omega} \big\|f(\omega) - s_n(\omega)\big\|_X \, d\mu(\omega) \;=\; 0.$$

If $A \in \Sigma$, we define the ***Bochner integral*** $\int_A f(\omega)\, d\mu(\omega)$ *of f on A, by*

$$\int_A f(\omega)\, d\mu(\omega) \;=\; \lim_{n\to+\infty} \int_{\Omega} \chi_A(\omega) s_n(\omega)\, d\mu(\omega). \tag{2.1}$$

Instead of $\int_{\Omega} f(\omega)\, d\mu(\omega)$*, we will often write*

$$\int_{\Omega} f(\omega)\, d\mu \text{ or even } \int_{\Omega} f\, d\mu,$$

when no confusion is possible.

REMARK 2.1.9 It is easy to verify that in Definition 2.1.8**(b)**, the limit in (2.1) exists and is independent of the sequence of simple functions $\{s_n\}_{n\geqslant 1}$ with the properties postulated there.

The next theorem gives a necessary and sufficient condition for the Bochner integrability of a function $f\colon \Omega \longrightarrow X$.

PROPOSITION 2.1.10
A strongly measurable function $f\colon \Omega \longrightarrow X$ is Bochner integrable if and only if the function $\omega \longmapsto \big\|f(\omega)\big\|_X$ is Lebesgue integrable (i.e., $\big\|f(\cdot)\big\|_X \in L^1(\Omega)$).

PROOF "$\Longrightarrow$": Let $\{s_n\}_{n\geqslant 1}$ be a sequence of X-valued simple functions, such that

$$\int_\Omega \big\|f(\omega)-s_n(\omega)\big\|_X\,d\mu \;\longrightarrow\; 0.$$

Then for any $n\geqslant 1$, we have

$$\int_\Omega \big\|f(\omega)\big\|_X\,d\mu \;\leqslant\; \int_\Omega \big\|f(\omega)-s_n(\omega)\big\|_X\,d\mu+\int_\Omega \big\|s_n(\omega)\big\|_X\,d\mu \;<\; +\infty,$$

so $\|f(\cdot)\|_X\in L^1(\Omega)$.

"$\Longleftarrow$": Since $f\colon\Omega\longrightarrow X$ is strongly measurable, we can find a sequence $\{s_n\}_{n\geqslant 1}$ of X-valued, simple functions, such that

$$\lim_{n\to+\infty}\big\|f(\omega)-s_n(\omega)\big\|_X \;=\; 0 \qquad \forall\,\omega\in\Omega\setminus A,$$

with $\mu(A)=0$. Hence

$$\lim_{n\to+\infty}\big\|s_n(\omega)\big\|_X \;=\; \big\|f(\omega)\big\|_X \qquad \forall\,\omega\in\Omega\setminus A.$$

Let

$$h_n\colon\Omega\longrightarrow X \qquad \forall\,n\geqslant 1$$

be defined by

$$h_n(\omega)\;\stackrel{df}{=}\;\begin{cases} s_n(\omega) & \text{if } \|s_n(\omega)\|_X<2\,\|f(\omega)\|_X\,,\\ 0 & \text{otherwise.}\end{cases}$$

Evidently for every $n\geqslant 1$, h_n is an X-valued simple function. Also

$$\lim_{n\to+\infty}\big\|f(\omega)-h_n(\omega)\big\|_X \;=\; 0 \qquad \forall\,\omega\in\Omega\setminus A$$

and

$$\big\|f(\omega)-h_n(\omega)\big\|_X \;\leqslant\; 3\big\|f(\omega)\big\|_X \qquad \forall\,\omega\in\Omega\setminus A.$$

So by the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\lim_{n\to+\infty}\int_\Omega \big\|f(\omega)-h_n(\omega)\big\|_X\,d\mu \;=\; 0,$$

so f is Bochner integrable (see Definition 2.1.8**(b)**). □

COROLLARY 2.1.11
If $f\colon\Omega\longrightarrow X$ is a Bochner integrable function and $A\in\Sigma$, then

$$\left\|\int_A f(\omega)\,d\mu\right\|_X \;\leqslant\; \int_A \big\|f(\omega)\big\|_X\,d\mu.$$

PROOF It is clear that the corollary holds for any $s\colon \Omega \longrightarrow X$ simple function. Then use Proposition 2.1.10. □

It is a direct consequence of Definition 2.1.8**(b)** that the Bochner integral is a linear operator. Namely we have the following proposition.

PROPOSITION 2.1.12
If $f, g\colon \Omega \longrightarrow X$ are two Bochner integrable functions, $A \in \Sigma$ and $\xi \in \mathbb{R}$, then $f + \xi g$ is Bochner integrable too and

$$\int_{\Omega} \big(f + \xi g\big)(\omega)\, d\mu \;=\; \int_{A} f(\omega)\, d\mu + \xi \int_{A} g(\omega)\, d\mu.$$

The Lebesgue dominated convergence theorem (see Theorem A.2.2) applies also to Bochner integrable functions.

PROPOSITION 2.1.13
If $f\colon \Omega \longrightarrow X$ is a strongly measurable function, $f_n\colon \Omega \longrightarrow X$, $n \geqslant 1$ are Bochner integrable,

$$f_n(\omega) \;\longrightarrow\; f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and there exists $h \in L^1(\Omega)_+$, such that

$$\big\|f_n(\omega)\big\|_X \;\leqslant\; h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \text{ and all } n \geqslant 1,$$

then f is Bochner integrable and we have

$$\int_{A} f(\omega)\, d\mu \;=\; \lim_{n \to +\infty} \int_{A} f_n(\omega)\, d\mu \qquad \forall\ A \in \Sigma.$$

PROOF Clearly

$$\big\|f(\omega)\big\|_X \;\leqslant\; h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and the function

$$\omega \longmapsto \big\|f(\omega) - f_n(\omega)\big\|_X$$

is Σ-measurable for every $n \geqslant 1$. Since

$$\big\|f(\omega) - f_n(\omega)\big\|_X \;\leqslant\; 2h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

we have that

$$\big\|f(\cdot) - f_n(\cdot)\big\|_X \in L^1(\Omega) \qquad \forall\ n \geqslant 1.$$

Thus by the Lebesgue dominated convergence theorem (see Theorem A.2.2) for $\mathbb{R}$-valued functions, we have

$$\int_\Omega \big\|f(\omega) - f_n(\omega)\big\|_X \, d\mu \;\longrightarrow 0. \tag{2.2}$$

By virtue of Definition 2.1.8**(b)**, for each $n \geqslant 1$, we can find an X-valued step function s_n, such that

$$\int_\Omega \big\|f_n(\omega) - s_n(\omega)\big\|_X \, d\mu \;<\; \frac{1}{n}.$$

We have

$$\begin{aligned}
&\int_\Omega \big\|f(\omega) - s_n(\omega)\big\|_X \, d\mu \\
&\leqslant \int_\Omega \big\|f(\omega) - f_n(\omega)\big\|_X \, d\mu + \int_\Omega \big\|f_n(\omega) - s_n(\omega)\big\|_X \, d\mu \;\longrightarrow\; 0 \quad \text{as } n \to +\infty,
\end{aligned}$$

so f is Bochner integrable.

Moreover, from Corollary 2.1.11 and (2.2), we have

$$\begin{aligned}
\left\| \int_A f(\omega)\, d\mu - \int_A s_n(\omega)\, d\mu \right\| &\leqslant \int_A \big\|f(\omega) - s_n(\omega)\big\|_X \, d\mu \\
&\leqslant \int_\Omega \big\|f(\omega) - s_n(\omega)\big\|_X \, d\mu \;\longrightarrow\; 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

□

Also we have a version of Fatou's lemma (see Theorem A.2.1).

PROPOSITION 2.1.14
If $f_n \colon \Omega \longrightarrow X$, $n \geqslant 1$, are Bochner integrable, $\big\{\int_\Omega \|f_n\| \, d\mu\big\}_{n\geqslant 1}$ is bounded and

$$f_n(\omega) \overset{w}{\longrightarrow} f(\omega) \quad \textit{for a.a. } \omega \in \Omega,$$

then f is Bochner integrable and

$$\int_\Omega \big\|f(\omega)\big\|_X \, d\mu \;\leqslant\; \liminf_{n\to+\infty} \int_\Omega \big\|f_n(\omega)\big\|_X \, d\mu.$$

PROOF Evidently f is weakly measurable. Also by Theorem 2.1.3 for every $n \geqslant 1$, we can find $A_n \in \Sigma$ which is μ-null and $f_n(\Omega \setminus A)$ is separable. Let $C \in \Sigma$ be the μ-null set, such that for $\omega \in \Omega \setminus C$, we have

$$f_n(\omega) \xrightarrow{w} f(\omega).$$

Let

$$A \stackrel{df}{=} \left(\bigcup_{n=1}^{\infty} A_n \right) \cup C.$$

Then $A \in \Sigma$ and it is μ-null. Let

$$Y \stackrel{df}{=} \overline{\text{span}} \bigcup_{n=1}^{\infty} f_n(\Omega \setminus A_n).$$

Evidently Y is a separable Banach subspace of X and

$$f(\Omega \setminus C) \subseteq Y.$$

So by virtue of the weak lower semicontinuous of the norm functional in a Banach space, we have

$$\|f(\omega)\|_X \leqslant \liminf_{n \to +\infty} \|f_n(\omega)\|_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Since $\|f_n(\omega)\|_X$, $n \geqslant 1$, is Lebesgue integrable (see Proposition 2.1.10), by the Fatou's lemma (see Theorem A.2.1), we have

$$\int_{\Omega} \|f(\omega)\|_X \, d\mu \leqslant \liminf_{n \to +\infty} \int_{\Omega} \|f_n(\omega)\|_X \, d\mu.$$

□

DEFINITION 2.1.15 *A set function $m : \Sigma \longrightarrow X$ is said to be a **vector measure**, if for all sequences $\{A_n\}_{n \geqslant 1} \subseteq \Sigma$ of pairwise disjoint sets, we have*

$$m\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} m(A_n),$$

where the series converges in the norm topology of X.

The next proposition shows that the indefinite Bochner integral

$$A \longmapsto \int_A f \, d\mu$$

of a Bochner integrable function $f : \Omega \longrightarrow X$ is a vector measure which is absolutely continuous with respect to μ (i.e., $m \prec\prec \mu$).

PROPOSITION 2.1.16
If $f\colon \Omega \longrightarrow X$ is a Bochner integrable function, then the set function $m\colon \Sigma \longrightarrow X$ defined by

$$m(A) \stackrel{df}{=} \int_A f(\omega)\, d\mu \qquad \forall\ A \in \Sigma$$

is a vector measure and $m \prec\prec \mu$, i.e.,

$$\lim_{\mu(A)\searrow 0} m(A) \ = \ 0.$$

PROOF Let $\{A_n\}_{n\geqslant 1} \subseteq \Sigma$ be a sequence of pairwise disjoint sets. Since

$$\left\| \int_{A_n} f(\omega)\, d\mu \right\|_X \ \leqslant \ \int_{A_n} \left\| f(\omega) \right\|_X d\mu \qquad \forall\ n \geqslant 1,$$

the series

$$\sum_{n=1}^{\infty} \int_{A_n} f(\omega)\, d\mu$$

is dominated term-by-term by the convergent series of positive terms

$$\sum_{n=1}^{\infty} \int_{A_n} \left\| f(\omega) \right\|_X d\mu \ \leqslant \ \int_{\Omega} \left\| f(\omega) \right\|_X d\mu \ < \ +\infty$$

(see Proposition 2.1.10). Therefore the series

$$\sum_{n=1}^{\infty} \int_{A_n} f(\omega)\, d\mu$$

is absolutely convergent. Moreover, for all $k \geqslant 1$, we have

$$\left\| \int_{\bigcup_{n=1}^{\infty} A_n} f(\omega)\, d\mu - \sum_{n=1}^{k} \int_{A_n} f(\omega)\, d\mu \right\|_X \ = \ \left\| \int_{\bigcup_{n=k+1}^{\infty} A_n} f(\omega)\, d\mu \right\|_X$$

$$\leqslant \int_{\bigcup_{n=k+1}^{\infty} A_n} \left\| f(\omega) \right\|_X d\mu \ \longrightarrow \ 0 \quad \text{as } k \to +\infty,$$

so

$$m\left(\bigcup_{n=1}^{\infty} A_n \right) \ = \ \sum_{n=1}^{\infty} m(A_n),$$

i.e., m is a vector measure.

Since $\big\|f(\cdot)\big\|_X \in L^1(\Omega)$, from the absolute continuity of the Lebesgue integral, we have

$$\lim_{\mu(A)\searrow 0} \int_A \big\|f(\omega)\big\|_X \, d\mu \;=\; 0.$$

From Corollary 2.1.11, we have

$$\lim_{\mu(A)\searrow 0} \big\|m(A)\big\|_X \;=\; \lim_{\mu(A)\searrow 0} \left\| \int_A f(\omega)\, d\mu \right\|_X \;\leqslant\; \lim_{\mu(A)\searrow 0} \int_A \big\|f(\omega)\big\|_X \, d\mu \;=\; 0$$

and so $m \prec\prec \mu$. □

Thus far the theory of Bochner integration is a straightforward extension of the theory of Lebesgue integration, with the absolute values replaced by norms. The next theorem exhibits a strong property of the Bochner integral that has no counterpart in the theory of Lebesgue integration.

THEOREM 2.1.17

If Y is another Banach space, $L\colon X \supseteq D \longrightarrow Y$ is a closed linear operator and $f\colon \Omega \longrightarrow X$ is a Bochner integrable function, such that $L\big(f(\cdot)\big)\colon \Omega \longrightarrow Y$ is Bochner integrable too,
then

$$L\bigg(\int_A f(\omega)\, d\mu \bigg) \;=\; \int_A L\big(f(\omega)\big)\big)\, d\mu \qquad \forall\ A \in \Sigma.$$

PROOF Let

$$C_0 \;\stackrel{df}{=}\; \Big\{ \omega \in \Omega :\ \big\|f(\omega)\big\|_X > 0 \Big\} \;\in\; \Sigma.$$

By Corollary 2.1.4, for a given $\varepsilon > 0$, we can find countably valued functions

$$h_\varepsilon\colon C_0 \longrightarrow X \quad \text{and} \quad g_\varepsilon\colon C_0 \to Y,$$

such that

$$\sup_{\omega\in\Omega\setminus E} \big\|f(\omega) - h_\varepsilon(\omega)\big\|_X \;<\; \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{\omega\in\Omega\setminus E} \big\|L\big(f(\omega)\big) - g_\varepsilon(\omega)\big\|_X \;<\; \frac{\varepsilon}{2},$$

with $E \in \Sigma$ being a μ-null set. Let $\{B_n\}_{n\geqslant 1} \subseteq \Sigma \cap C_0$ be a common refinement of the subdivisions corresponding to h_ε and g_ε and let

$$\omega_n \in B_n \qquad \forall\ n \geqslant 1.$$

We introduce the function $u_\varepsilon\colon \Omega \longrightarrow X$, defined by

$$u_\varepsilon(\omega) \overset{df}{=} \begin{cases} f(\omega_n) & \text{if} \quad \omega \in B_n,\ n \geqslant 1, \\ 0 & \text{if} \quad \omega \in \Omega \setminus C_0. \end{cases}$$

Then, we have

$$\int_\Omega \big\|f(\omega) - u_\varepsilon(\omega)\big\|_X \, d\mu \ < \ \varepsilon\mu(\Omega) \tag{2.3}$$

and

$$\int_\Omega \big\|L\big(f(\omega)\big) - L\big(u_\varepsilon(\omega)\big)\big\|_Y \, d\mu \ < \ \varepsilon\mu(\Omega). \tag{2.4}$$

Also for every $A \in \Sigma$, we have

$$\int_A u_\varepsilon(\omega)\, d\mu \ = \ \sum_{n=1}^{\infty} f(\omega_n)\mu(B_n \cap A) \ = \ \lim_{N\to+\infty} \sum_{n=1}^{N} f(\omega_n)\mu(B_n \cap A) \tag{2.5}$$

and

$$\begin{aligned} \int_A L\big(u_\varepsilon(\omega)\big)\, d\mu \ &= \ \sum_{n=1}^{\infty} L\big(f(\omega_n)\big)\mu(B_n \cap A) \\ &= \ \lim_{N\to+\infty} \sum_{n=1}^{N} L\big(f(\omega_n)\big)\mu(B_n \cap A). \end{aligned} \tag{2.6}$$

Since by hypothesis L is a closed, linear operator, from (2.5) and (2.6), we have that

$$\Big(\int_A u_\varepsilon \, d\mu, \int_A L\big(u_\varepsilon\big)\, d\mu\Big) \ \in \ \operatorname{Gr} L.$$

Consider a sequence $\varepsilon_n \searrow 0$. From (2.3) and (2.4), we have

$$\int_A u_{\varepsilon_n}\, d\mu \ \longrightarrow \ \int_A f\, d\mu \quad \text{and} \quad \int_A L(u_{\varepsilon_n})\, d\mu \ \longrightarrow \ \int_A L(f)\, d\mu.$$

Since

$$L\left(\int_A u_{\varepsilon_n}\, d\mu\right) \ = \ \int_A L(u_{\varepsilon_n})\, d\mu \qquad \forall\ n \geqslant 1$$

and L is closed, it follows that

$$L\Big(\int_A f(\omega)\, d\mu\Big) \ = \ \int_A L\big(f(\omega)\big)\, d\mu \qquad \forall\ A \in \Sigma.$$

□

REMARK 2.1.18 If $f\colon \Omega \longrightarrow X$ is a Bochner integrable function and

$$L \in \mathcal{L}(X;Y),$$

then $L(f)$ is Bochner integrable, since

$$\big\|L(f(\omega))\big\|_Y \leqslant \|L\|_{\mathcal{L}}\, \|f(\omega)\|_X \qquad \forall\, \omega \in \Omega$$

COROLLARY 2.1.19
If $f, g\colon \Omega \longrightarrow X$ are two Bochner integrable functions and

$$\int_A f(\omega)\, d\mu = \int_A g(\omega)\, d\mu \qquad \forall\, A \in \Sigma,$$

then

$$f(\omega) = g(\omega) \text{ for } \mu\text{-almost all } \omega \in \Omega.$$

PROOF We may assume that $g = 0$ and that X is separable (see Theorem 2.1.3). Then the ball

$$\overline{B}_1^* = \big\{x^* \in X^* :\ \|x^*\|_{X^*} \leqslant 1\big\}$$

furnished with the relative weak*-topology is compact, metrizable (see Alaoglu theorem; Theorem A.3.9 and Theorem A.3.13).

Let $\{x_n^*\}_{n\geqslant 1}$ be a countable w^*-dense subset of $\overline{B}_1^*$. By Theorem 2.1.17, for every $n \geqslant 1$ and $A \in \Sigma$, we have

$$\int_A \big\langle x_n^*, f(\omega)\big\rangle_X\, d\mu = \left\langle x_n^*, \int_A f(\omega)\, d\mu \right\rangle_X = 0,$$

so

$$\big\langle x_n^*, f(\omega)\big\rangle_X = 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Since

$$\big\|f(\omega)\big\|_X = \sup_{x^* \in \overline{B}_1^*} \big|\, \langle x^*, f(\omega)\rangle_X \,\big|,$$

we have

$$\big\|f(\omega)\big\|_X = 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and so

$$f(\omega) = 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

□

A similar proof gives us the following result.

COROLLARY 2.1.20
If $f, g\colon \Omega \longrightarrow X$ *are two strongly measurable functions and*

$$\langle x^*, f(\omega)\rangle_X \;=\; \langle x^*, g(\omega)\rangle_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } x^* \in X^*$$

(the exceptional μ*-null set may depend on* $x^* \in X^*$*),*
then

$$f(\omega) \;=\; g(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

The next result can be viewed as a kind of mean value theorem for the Bochner integral.

PROPOSITION 2.1.21
If $f\colon \Omega \longrightarrow X$ *is a Bochner integrable function and* $A \in \Sigma$ *with* $\mu(A) > 0$*, then*

$$\frac{1}{\mu(A)} \int_A f(\omega)\, d\mu \;\in\; \overline{\operatorname{conv}}\, f(A).$$

PROOF We proceed by contradiction. Suppose that

$$\frac{1}{\mu(A)} \int_A f(\omega)\, d\mu \;\notin\; \overline{\operatorname{conv}}\, f(A).$$

Then by the strong separation theorem for convex sets (see Theorem A.3.2), we can find $x^* \in X^* \setminus \{0\}$ and $\vartheta \in \mathbb{R}$, such that

$$\left\langle x^*, \frac{1}{\mu(A)} \int_A f(\omega)\, d\mu \right\rangle_X \;<\; \vartheta \;\leqslant\; \langle x^*, f(\omega)\rangle_X \qquad \forall\, \omega \in A,$$

so using Theorem 2.1.17, we have

$$\frac{1}{\mu(A)} \int_A \langle x^*, f(\omega)\rangle_X\, d\mu \;<\; \vartheta \;\leqslant\; \langle x^*, f(\omega)\rangle_X \qquad \forall\, \omega \in A.$$

Integrating this inequality over A, we obtain

$$\int_A \langle x^*, f(\omega)\rangle_X\, d\mu \;<\; \vartheta\mu(A) \;\leqslant\; \int_A \langle x^*, f(\omega)\rangle_X\, d\mu,$$

a contradiction. □

Also for Bochner integrable functions, the Lebesgue differentiation theorem holds (see Theorem 1.4.6). So we have the following result.

PROPOSITION 2.1.22

If $Z \subseteq \mathbb{R}^N$ is a bounded open set and $f\colon Z \longrightarrow X$ is a Bochner integrable function,
then

$$\lim_{r \searrow 0} \frac{1}{a(N) r^N} \int\limits_{\overline{B}_r(x)} \big\| f(y) - f(x) \big\|_X \, d\lambda^N(y) \;=\; 0 \quad \textit{for } \lambda^N\textit{-a.a. } x \in Z,$$

where

$$a(N) \;\stackrel{df}{=}\; \frac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$$

is the volume of the unit ball in $\mathbb{R}^N$.

PROOF Invoking Theorem 2.1.3, we may assume that X is separable. Let $\{x_n\}_{n \geqslant 1}$ be a dense set in X. Then by Theorem 1.4.6, we have

$$\begin{aligned} &\lim_{r \searrow 0} \frac{1}{a(N) r^N} \int\limits_{\overline{B}_r(x)} \big\| f(y) - x_n \big\|_X \, d\lambda^N(y) \\ =\; &\big\| f(x) - x_n \big\|_X \quad \text{for } \lambda^N\text{-a.a. } x \in Z \text{ and all } n \geqslant 1. \end{aligned} \tag{2.7}$$

Let $x \in Z$ be a point where (2.7) is valid. Then for a given $\varepsilon > 0$, we can select x_n, such that

$$\big\| f(x) - x_n \big\|_X \;<\; \varepsilon.$$

We have

$$\begin{aligned} &\limsup_{r \searrow 0} \frac{1}{a(N) r^N} \int\limits_{\overline{B}_r(x)} \big\| f(y) - f(x) \big\|_X \, d\lambda^N(y) \\ \leqslant\; &\limsup_{r \searrow 0} \frac{1}{a(N) r^N} \int\limits_{\overline{B}_r(x)} \Big[\big\| f(y) - x_n \big\|_X + \big\| x_n - f(x) \big\|_X \Big] \, d\lambda^N(y) \\ <\; &2\varepsilon, \end{aligned}$$

so

$$\limsup_{r \searrow 0} \frac{1}{a(N) r^N} \int\limits_{\overline{B}_r(x)} \big\| f(y) - f(x) \big\|_X \, d\lambda^N(y) \;=\; 0 \quad \text{for } \lambda^N\text{-a.a. } x \in Z.$$

□

We conclude this section by introducing three weaker integrals for Banach space valued functions.

DEFINITION 2.1.23 *Let $f\colon \Omega \longrightarrow X$ be a function.*

(a) *Suppose that $f\colon \Omega \longrightarrow X$ is weakly measurable. We say that f is **Pettis integrable**, if for each $A \in \Sigma$, there exists $x_A \in X$, such that*

$$\big\langle x^*, x_A \big\rangle_X \;=\; \int\limits_A \big\langle x^*, f(\omega)\big\rangle_X \, d\mu \qquad \forall\; x^* \in X^*.$$

Then we write

$$x_A \;=\; (P)\text{-}\int\limits_A f(\omega)\, d\mu.$$

(b) *Suppose that $f\colon \Omega \longrightarrow X$ is weakly measurable. We say that f is **Dunford integrable**, if for each $A \in \Sigma$, there exists $x_A^{**} \in X^{**}$, such that*

$$\big\langle x_A^{**}, x^* \big\rangle_{X^*} \;=\; \int\limits_A \big\langle x^*, f(\omega)\big\rangle_X \, d\mu \qquad \forall\; x^* \in X^*.$$

Then we write

$$x_A^{**} \;=\; (D)\text{-}\int\limits_A f(\omega)\, d\mu.$$

(c) *Suppose that $f\colon \Omega \longrightarrow X^*$ is w^*-measurable. We say that f is **Gelfand integrable**, if for each $A \in \Sigma$, there exists $x_A^* \in X^*$, such that*

$$\big\langle x_A^*, x \big\rangle_X \;=\; \int\limits_A \big\langle f(\omega), x\big\rangle_X \, d\mu \qquad \forall\; x \in X.$$

Then we write

$$x_A^* \;=\; (G)\text{-}\int\limits_A f(\omega)\, d\mu.$$

REMARK 2.1.24 Clearly we have that Bochner integrability implies Pettis integrability and Pettis integrability implies Dunford integrability. The reverse implications need not be true. Of course if X is reflexive, then the Pettis and Dunford integrals coincide. Finally note that the Gelfand integral is actually the Pettis integral for X^*-valued functions.

For a Pettis integrable function $f\colon \Omega \longrightarrow X$, we consider the set-valued function $A \longmapsto (\mathrm{P})\text{-}\int\limits_A f\, d\mu$. We want to know if this is a μ-continuous vector measure, as was the case with the Bochner integral (see Proposition 2.1.16). To answer this we need some preparation.

DEFINITION 2.1.25 *Let* $\sum_{n=1}^{\infty} x_n$ *be a series of elements of* X.

(a) *We say that the series* $\sum_{n=1}^{\infty} x_n$ *is* ***unconditionally convergent to*** x, *if for all permutations* π *of* $\mathbb{N}$, *the series* $\sum_{n=1}^{\infty} x_{\pi(n)}$ *converges to* x.

(b) *We say that the series* $\sum_{n=1}^{\infty} x_n$ *is* ***weakly subseries convergent to*** x, *if for every strictly increasing sequence* $\{n_k\}_{k\geqslant 1}$ *of integers, the series* $\sum_{k=1}^{\infty} x_{n_k}$ *is weakly convergent.*

REMARK 2.1.26 If $\sum_{n=1}^{\infty} x_n$ is absolutely convergent (i.e., the series $\sum_{n=1}^{\infty} \|x_n\|_X$ is convergent), then it is unconditionally convergent. Also unconditional convergence is equivalent to the subseries convergence (in the norm topology of X) and implies convergence.

The next result is known as the Orlicz-Pettis theorem.

THEOREM 2.1.27 (Orlicz-Pettis Theorem)
A formal series $\sum_{n=1}^{\infty} x_n$ *in* X *is unconditionally convergent if and only if it is weakly subseries convergent.*

REMARK 2.1.28 An interesting consequence of the above theorem is that if $m\colon \Sigma \longrightarrow X$ is a weakly countably additive set function, then it is a vector measure.

PROPOSITION 2.1.29
If $f\colon \Omega \longrightarrow X$ *is Pettis integrable,*
then the function

$$\Sigma \ni A \longmapsto m(A) = (P)\text{-}\int_A f\, d\mu$$

is a vector measure.

PROOF Let $\{A_n\}_{n\geqslant 1}$ be a sequence of pairwise disjoint sets in Σ. For every $x^* \in X^*$, we have

$$\left\langle x^*, (\mathrm{P})\text{-}\int_{\bigcup_{n=1}^{\infty} A_n} f(\omega)\,d\mu \right\rangle_X = \int_{\bigcup_{n=1}^{\infty} A_n} \langle x^*, f(\omega)\rangle_X\,d\mu$$

$$= \sum_{n=1}^{\infty} \int_{A_n} \langle x^*, f(\omega)\rangle_X\,d\mu = \sum_{n=1}^{\infty} \left\langle x^*, (\mathrm{P})\text{-}\int_{\bigcup_{n=1}^{\infty} A_n} f(\omega)\,d\mu \right\rangle_X,$$

so the function

$$\Sigma \ni A \longmapsto m(A) = (\mathrm{P})\text{-}\int_A f\,d\mu$$

is weakly countably additive.

Of course the same argument applies to any subsequence of $\{A_n\}_{n\geqslant 1}$. So we can invoke Theorem 2.1.27 and conclude that the function

$$\Sigma \ni A \longmapsto m(A) = (\mathrm{P})\text{-}\int_A f\,d\mu$$

is a vector measure. □

REMARK 2.1.30 The result is not true for the Dunford integral, which is not even strongly additive (see Diestel & Uhl (1977, p. 53)).

The next result provides an easy test for checking the Gelfand integrability of $f\colon \Omega \longrightarrow X^*$.

PROPOSITION 2.1.31
If $f\colon \Omega \longrightarrow X^*$ *has the following property*

$$\langle f(\cdot), x\rangle_X \in L^1(\Omega) \qquad \forall\ x \in X,$$

then f *is Gelfand integrable.*

PROOF Let $A \in \Sigma$ and let $L\colon X \longrightarrow L^1(\Omega)$ be defined by

$$L(x) \stackrel{df}{=} \langle f(\cdot), x\rangle_X \qquad \forall\ x \in X.$$

We claim that the linear operator L has a closed graph. To this end suppose that

$$x_n \longrightarrow x \quad \text{in } X$$

and

$$\langle f(\cdot), x_n\rangle_X \longrightarrow g \quad \text{in } L^1(\Omega).$$

Then by passing to a suitable subsequence of $\{x_n\}_{n\geqslant 1}$ if necessary, we may assume that

$$\langle f(\omega), x_n\rangle_X \longrightarrow g(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Therefore

$$g(\omega) = \langle f(\omega), x\rangle_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

hence $(x, g) \in \operatorname{Gr} L$, i.e., L has closed graph. By the closed graph theorem (see Theorem A.3.7), L is continuous and if $I_A\colon L^1(\Omega) \longrightarrow \mathbb{R}$ is the integral operator, defined by

$$I_A(g) \stackrel{df}{=} \int_A g(\omega)\, d\mu \qquad \forall\, A \in \Sigma,$$

we have that $I_A \circ L \in X^*$ and so there exists $x_A^* \in X^*$, such that

$$\langle x_A^*, x\rangle_X = \int_A \langle f(\omega), x\rangle_X\, d\mu \qquad \forall\, x \in X.$$

Therefore f is Gelfand integrable. □

REMARK 2.1.32 The same closed graph argument shows that if $f\colon \Omega \longrightarrow X$ is such that

$$\langle x^*, f(\cdot)\rangle_X \in L^1(\Omega) \qquad \forall\, x^* \in X^*,$$

then f is Dunford integrable. The situation with the Pettis integrability is less satisfactory and more sophisticated criteria are needed to establish it (see Diestel & Uhl (1977, pp. 54–56)).

COROLLARY 2.1.33
If $f\colon \Omega \longrightarrow X^*$ is a w^*-measurable function and has range which is norm bounded in X^*,
then f is Gelfand integrable.

The same argument as in the proof of Proposition 2.1.21 gives the following mean value theorem for the Gelfand integral.

THEOREM 2.1.34
If $f\colon \Omega \longrightarrow X^*$ is a Gelfand integrable function and $A \in \Sigma$,
then

$$\frac{1}{\mu(A)}(G)\text{-}\int_A f\, d\mu \in \overline{\operatorname{conv}}^{w^*} f(A).$$

2.2 Lebesgue-Bochner Spaces and Evolution Triples

Using the Bochner integral introduced in the previous section, we can introduce generalizations of the classical Lebesgue spaces to Banach space valued functions.

As in the previous section (Ω, Σ, μ) is a finite measure space and X is a Banach space. Additional hypotheses will be introduced as needed.

DEFINITION 2.2.1 *Let $p \in [1, +\infty]$. By $L^p(\Omega; X)$ we denote the space of equivalence classes of strongly measurable functions $f : \Omega \longrightarrow X$, such that $\|f(\cdot)\|_X \in L^p(\Omega)$. Also we introduce their respective norms by*

$$\|f\|_p \stackrel{df}{=} \left(\int_\Omega \|f(\omega)\|_X^p \, d\mu \right)^{\frac{1}{p}} \quad \text{if } p \in [1, +\infty)$$

and

$$\|f\|_\infty \stackrel{df}{=} \operatorname*{esssup}_{\omega \in \Omega} \|f(\omega)\|_X .$$

REMARK 2.2.2 As with $\mathbb{R}$-valued functions, the equivalence relation used in the above definition is the following:

$$f \sim g \quad \text{if and only if} \quad f(\omega) = g(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

It is routine to check the following facts.

PROPOSITION 2.2.3
(a) *$\left(L^p(\Omega; X),\ \|\cdot\|_p\right)$ is a Banach space for $p \in [1, +\infty]$.*

(b) *If $p \in [1, +\infty)$, Σ is countably generated and X is separable, then $L^p(\Omega; X)$ is separable.*

(c) *If $p \in (1, +\infty)$ and X is reflexive, then $L^p(\Omega; X)$ is reflexive.*

(d) *If X is a Hilbert space, then $L^2(\Omega; X)$ is a Hilbert space too with inner product*

$$(f, g)_2 = \int_\Omega \big(f(\omega), g(\omega)\big)_X \, d\mu.$$

REMARK 2.2.4 The σ-field Σ is ***countably generated*** if there exists a countable subfamily $\mathcal{T}$, such that $\Sigma = \sigma(\mathcal{T})$. If Ω is an open or closed subset of $\mathbb{R}^N$, then the Borel σ-field $\mathcal{B}(\Omega)$ is countably generated. Also clearly if $p \in [1, +\infty)$ and $L^p(\Omega; X)$ is separable, then X is separable. Additional conditions on X usually translate to corresponding properties of the Lebesgue-Bochner space $L^p(\Omega; X)$. So if $p \in (1, +\infty)$, then $L^p(\Omega; X)$ is uniformly convex if and only if X is uniformly convex (see Day (1955, 1973)). Moreover, as for the Lebesgue spaces, simple functions are dense in $L^p(\Omega; X)$ and if $Z \subseteq \mathbb{R}^N$ is a bounded open set then $C^\infty\big(\overline{Z}; X\big)$ is dense in $L^p(\Omega; X)$ for $p \in [1, +\infty)$.

PROPOSITION 2.2.5
If Y is another Banach space, $X \subseteq Y$ and the embedding is continuous, $p, r \in [1, +\infty]$, $p \leqslant r$,
then $L^r(\Omega; X) \subseteq L^p(\Omega; Y)$ and the embedding is continuous.

PROOF Let $f \in L^r(\Omega; X)$. Since the embedding $X \subseteq Y$ is continuous, using Hölder's inequality (see Theorem A.2.27; as $p \leqslant r$), we have

$$\left(\int_\Omega \big\|f(\omega)\big\|_Y^p \, d\mu\right)^{\frac{1}{p}} \leqslant c_1 \left(\int_\Omega \big\|f(\omega)\big\|_X^p \, d\mu\right)^{\frac{1}{p}} \leqslant c_2 \left(\int_\Omega \|f(\omega)\|_X^r \, d\mu\right)^{\frac{1}{r}},$$

for some $c_1, c_2 > 0$.

So $L^r(\Omega; X) \subseteq L^p(\Omega; Y)$ and the embedding is continuous. □

We want to identify the dual of $L^p(\Omega; X)$ for $p \in [1, +\infty)$. First a definition which is motivated by the fact that the proof of the classical Riesz representation theorem (see Theorem A.3.24) uses the Radon-Nikodym theorem (see Theorem A.2.24).

DEFINITION 2.2.6 **(a)** *Let $m \colon \Sigma \longrightarrow X$ be a vector measure (see Definition 2.1.15). We say that m is of **bounded variation**, if $|m|(\Omega) < +\infty$, where*

$$|m|(A) = \sup_{\mathcal{T}_A} \sum_{C \in \mathcal{T}_A} \big\|m(C)\big\|_X \qquad \forall\, A \in \Sigma,$$

*with $\mathcal{T}_A$ running through the set of all finite Σ-partitions of A. The quantity $|m| \colon \Sigma \longrightarrow \mathbb{R}_+$ is called the **variation** of m and is a measure.*

(b) *A Banach space X is said to have the **Radon-Nikodym property** (**RNP** for short), if for every probability space (Ω, Σ, μ) and every vector measure $m \colon \Sigma \longrightarrow X$ of bounded variation such that $m \prec\prec \mu$ (i.e., if $\mu(A) = 0$ then $m(A) = 0$), there exists $f \in L^1(\Omega; X)$, such that*

$$m(A) = \int_A f(\omega) \, d\mu \qquad \forall\, A \in \Sigma.$$

REMARK 2.2.7 The RNP is not a property that every Banach space has. Suppose that

$$X_1 \;=\; c_0.$$

On $\big([0,1], \mathcal{B}([0,1]), \lambda^1\big)$ (here $\mathcal{B}\big([0,1]\big)$ is the Borel σ-field of $[0,1]$ and λ^1 is the Lebesgue measure) consider the vector measure $m\colon \mathcal{B}\big([0,1]\big) \longrightarrow c_0$, defined by

$$m(A) \;=\; \left\{ \int_A \cos nt \, dt \right\}_{n \geqslant 1} \qquad \forall\; A \in \mathcal{B}\big([0,1]\big).$$

The Riemann-Lebesgue Lemma guarantees that

$$m(A) \in c_0 \qquad \forall\; A \in \mathcal{B}\big([0,1]\big).$$

Also

$$m \;\prec\!\prec\; \lambda^1.$$

However, m cannot have a Radon-Nikodym derivative (see Theorem A.2.24 and Remark A.2.25) in $L^1\big([0,1]; c_0\big)$, since

$$\{\cos nt\}_{n \geqslant 1} \notin \; c_0 \quad \text{for a.a. } t \in [0,1].$$

Therefore c_0 lacks the RNP.

However, there are two large classes of Banach spaces which have the RNP.

PROPOSITION 2.2.8
If X is reflexive or it is a separable dual space,
then X has the RNP.

Now we state the Riesz representation theorem for the Lebesgue-Bochner spaces.

THEOREM 2.2.9 (Riesz Representation Theorem for the Lebesgue-Bochner Spaces)
If $p \in [1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$,
then $\big(L^p(\Omega; X)\big)^* = L^{p'}\big(\Omega; X^*\big)$ if and only if X^* has the RNP and the duality pairing is given by

$$\big\langle g, f \big\rangle_{L^p(\Omega;X)} \;=\; \int_\Omega \big\langle g(\omega), f(\omega) \big\rangle_X \, d\mu \qquad \forall\; f \in L^p(\Omega; X),\; g \in L^{p'}\big(\Omega; X^*\big).$$

What can be said if X^* does not have the RNP (for example when $X = C\big([0,1]\big)$)? We can still have a representation theorem for $L^1(\Omega; X)$. First a definition.

DEFINITION 2.2.10 *By $L^\infty\big(\Omega; X^*_{w^*}\big)$ we denote the space of all w^*-measurable functions $g\colon \Omega \longrightarrow X^*$, such that there exists $c > 0$ with*

$$\big|\big\langle g(\omega), x\big\rangle_X\big| \;\leqslant\; c\,\|x\|_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } x \in X \tag{2.8}$$

(the exceptional μ-null set may depend on x). Two functions g, h are equivalent in $L^\infty\big(\Omega; X^*_{w^*}\big)$ (denoted by $g \approx h$) if

$$\big\langle g(\omega), x\big\rangle_X \;=\; \big\langle h(\omega), x\big\rangle_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } x \in X.$$

The infimum of all $c > 0$ for which the above inequality (2.8) is true is denoted by $\|g\|_{L^\infty(\Omega; X^*_{w^*})}$ and we have

$$\big|\big\langle g(\omega), x\big\rangle_X\big| \;\leqslant\; \|g\|_{L^\infty(\Omega; X^*_{w^*})}\,\|x\|_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

We can easily check that $\|\cdot\|_{L^\infty(\Omega; X^*_{w^*})}$ is a norm.

REMARK 2.2.11 **(a)** The equivalence relation in $L^\infty\big(\Omega; X^*_{w^*}\big)$ does not coincide with the usual one in the L^p-space, since

$$\big\langle g(\omega), x\big\rangle_X \;=\; 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } x \in X$$

does not necessarily imply that

$$g(\omega) \;=\; 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Indeed let

$$\Omega \;=\; [0,1] \quad \text{and} \quad X \;=\; l^2\big([0,1]\big)$$

(it is a nonseparable Hilbert space). Then

$$L^\infty\big(\Omega; X^*_{w^*}\big) \;=\; L^\infty(\Omega; X_w)$$

and let $g(\omega) = \big(g_a(\omega)\big)_{a\in[0,1]}$ with

$$g_a(\omega) \;=\; \begin{cases} 1 & \text{if} \quad \omega = t, \\ 0 & \text{if} \quad \omega \neq t. \end{cases}$$

Then $g \approx 0$ in $L^\infty(\Omega; X_w)$, but

$$\big\|g(\omega)\big\|_X \;=\; 1 \quad \text{for a.a. } \omega \in [0,1].$$

However, if X is sparable and $g \in L^\infty\big(\Omega; X^*_{w^*}\big)$, then the function

$$\omega \;\longmapsto\; \big\|g(\omega)\big\|_{X^*}$$

is measurable, essentially bounded and

$$\|g\|_{L^\infty(\Omega; X^*_{w^*})} \;=\; \operatorname*{esssup}_{\omega\in\Omega} \big\|g(\omega)\big\|_{X^*}.$$

(b) In general, we have $L^\infty\big(\Omega; X^*_{w^*}\big) \neq L^\infty(\Omega; X^*)$, even if X is separable. To see this let $\Omega = [0,1]$ and $X = C\big([0,1]\big)$. We know that $X^* = M\big([0,1]\big)$, the space of finite Borel measures on $[0,1]$ equipped with the total variation norm. Let $g\colon \Omega \longrightarrow X^*$ be defined by

$$g(\omega) \stackrel{df}{=} \delta_\omega,$$

the Dirac measure at $\omega \in [0,1]$. Then $g \in L^\infty\big(\Omega; X^*_{w^*}\big)$, but it is not strongly measurable, nor equivalent to any strongly measurable function. To see this, note that due to the separability of X, $g \approx h$ if and only if $g(\omega) = h(\omega)$ for almost all $\omega \in \Omega$ (with h being strongly measurable). Then g is strongly measurable too and so by virtue of Corollary 2.1.4, there exists a countably-valued function u, such that

$$\big\|g(\omega) - u(\omega)\big\|_{X^*} \;<\; \frac{1}{2} \qquad \forall\, \omega \in \Omega.$$

Let $A \in \Sigma = \mathcal{B}\big([0,1]\big)$ be a set of positive Lebesgue measure, such that $u|_A$ is constant. We have

$$\begin{aligned}\big\|g(\omega') - g(\omega)\big\|_{X^*} \;\leqslant\; & \big\|g(\omega') - u(\omega')\big\|_{X^*} + \big\|u(\omega') - u(\omega)\big\|_{X^*} \\ & + \big\|u(\omega) - g(\omega)\big\|_{X^*} \;<\; 1 \qquad \forall\, \omega', \omega \in A,\end{aligned}$$

a contradiction, since $\big\|g(\omega') - g(\omega)\big\|_{X^*} = 1$ for all $\omega' \neq \omega$.

The next representation theorem is sometimes known as the Dinculeanu-Foias theorem.

THEOREM 2.2.12 (Dinculeanu-Foias Theorem) *$\big(L^1(\Omega; X)\big)^* = L^\infty\big(\Omega; X^*_{w^*}\big)$ and the duality pairing between the two spaces is given by*

$$\big\langle g, f\big\rangle_{L^1(\Omega;X)} \;=\; \int_\Omega \big\langle g(\omega), f(\omega)\big\rangle_X \, d\mu \qquad \forall\; f \in L^1(\Omega; X),\; g \in L^\infty\big(\Omega; X^*_{w^*}\big).$$

REMARK 2.2.13 If X^* is separable and $g\colon \Omega \longrightarrow X^*$ is w^*-measurable, then g is strongly measurable. Indeed we can find a sequence $\{x^*_n\}_{n\geqslant 1} \subseteq \overline{B}^*_1 \stackrel{df}{=} \big\{y^* \in X^* :\; \|y^*\|_{X^*} \leqslant 1\big\}$, such that

$$\|x\|_X \;=\; \sup_{n\geqslant 1} \big|\big\langle x^*_n, x\big\rangle_X\big| \qquad \forall\; x \in X.$$

Therefore for every $y^* \in X^*$, the function $\omega \longmapsto \big\|g(\omega) - y^*\big\|_{X^*}$ is Σ-measurable and then from the proof of Theorem 2.1.3, we can infer that the function $\omega \longmapsto g(\omega)$ is strongly measurable. Hence $L^\infty\big(\Omega; X^*_{w^*}\big) = L^\infty(\Omega; X^*)$ and Theorem 2.2.12 coincides with Theorem 2.2.9.

In complete analogy with the case of $\mathbb{R}$-valued functions, we introduce the notion of absolutely continuous X-valued function.

DEFINITION 2.2.14 *A function $f\colon T = [0,b] \longrightarrow X$ is said to be **absolutely continuous**, if for every $\varepsilon > 0$, we can find $\delta(\varepsilon) > 0$, such that for each sequence $\{(a_n, b_n)\}_{n\geqslant 1}$ of pairwise disjoint intervals in T with $\sum\limits_{n=1}^{\infty}(b_n - a_n) < \delta$, we have*

$$\sum_{n=1}^{\infty} \big\|f(b_n) - f(a_n)\big\|_X \ < \ \varepsilon.$$

Also for a function $f\colon T = [0,b] \longrightarrow X$ and a partition

$$P\colon \ 0 \ = \ x_0 \ < \ \ldots \ < \ x_n \ = \ b \quad \text{of } T,$$

we define

$$V(f,P) \ \stackrel{df}{=} \ \sum_{k=1}^{m} \big\|f(x_k) - f(x_{k-1})\big\|_X.$$

*The **variation** of f on T is defined by*

$$V(f)(b) \ \stackrel{df}{=} \ \sup\big\{V(f,P)\colon \ P \text{ is a partition of } T\big\}.$$

*When $V(f)(b)$ is finite, we say that f is of **bounded variation**.*

REMARK 2.2.15 Clearly the function $t \longmapsto V(f)(t)$ is an increasing function and if $f\colon T = [0,b] \longrightarrow X$ is absolutely continuous, then it is of bounded variation. The converse is not true.

It is well known that an $\mathbb{R}$-valued, absolutely continuous function is almost everywhere differentiable on T and it is the indefinite integral of its derivative. The result is no longer true for X-valued in general.

EXAMPLE 2.2.16 Let $X = L^1[0,1]$ and consider the function $f\colon [0,1] \longrightarrow X$, defined by

$$f(t) \ \stackrel{df}{=} \ \chi_{[0,t]} \qquad \forall\ t \in [0,1].$$

It is easy to see that f is absolutely continuous. However, f is nowhere differentiable on $[0,1]$. Indeed, if f is differentiable at $t = t_0 \in [0,1]$, then for every $g \in L^{\infty}[0,1] = \big(L^1[0,1]\big)^*$, the function

$$t \ \longmapsto \ \vartheta(t) \ \stackrel{df}{=} \ \big\langle g, f(t)\big\rangle_{L^1[0,1]} \ = \ \int_0^1 g(s)f(t)(s)\,ds \ = \ \int_0^t g(s)ds$$

is differentiable at $t = t_0$. Let

$$g(s) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad s \leqslant t_0, \\ -1 & \text{if} \quad s > t_0. \end{cases}$$

We have

$$\vartheta(t) \stackrel{df}{=} \begin{cases} t & \text{if} \quad t \leqslant t_0, \\ 2t_0 - t & \text{if} \quad t \geqslant t_0, \end{cases}$$

and ϑ clearly is not differentiable at $t = t_0$. Note that in this example $X = L^1[0,1]$ does not have the RNP.

THEOREM 2.2.17
If X is reflexive and $f\colon T = [0,b] \longrightarrow X$ is absolutely continuous, then f is differentiable at almost all $t \in T$ and

$$f(t) \;=\; f(0) + \int_0^t f'(s)\,ds \qquad \forall\, t \in T.$$

PROOF Because of Theorem 2.1.3, we may assume that X is also separable. Since f is absolutely continuous, it is of bounded variation and

$$\text{the function } t \longmapsto V(f)(t) \text{ is increasing on } T = [0,b]$$

(see Definition 2.2.14 and Remark 2.2.15). For $0 \leqslant t \leqslant t+h \leqslant b$, we have

$$\big\|f(t+h) - f(t)\big\|_X \;\leqslant\; V(f)(t+h) - V(f)(t),$$

so

$$\frac{\|f(t+h) - f(t)\|_X}{h} \;\leqslant\; \frac{1}{h}\big(V(f)(t+h) - V(f)(t)\big) \qquad \forall\, h > 0$$

and

$$\limsup_{h\to 0} \frac{\|f(t+h) - f(t)\|_X}{h} \;\leqslant\; \frac{d}{dt}V(f)(t) \;<\; +\infty \quad \text{for a.a. } t \in T. \tag{2.9}$$

Since X is separable, reflexive, X^* is separable too (see Remark A.3.14). Let $\{x_n^*\}_{n\geqslant 1}$ be a dense sequence in X^*. For every $n \geqslant 1$,

$$\text{the function } t \longmapsto \big\langle x_n^*, f(t)\big\rangle_X \text{ is differentiable at every point of } T \setminus D_n,$$

with $\lambda^1(D_n) = 0$ (as before λ^1 denotes the Lebesgue measure on T). Also let

$$D_0 \stackrel{df}{=} \left\{ t \in T : \limsup_{h\to 0} \frac{\|f(t+h) - f(t)\|_X}{h} = +\infty \right\}$$

and let us set

$$D \stackrel{df}{=} \bigcup_{n=0}^{\infty} D_n.$$

From (2.9), we have that $\lambda^1(D) = 0$. Then for $\varepsilon > 0$ small enough and $t \in T \setminus D$, the family

$$\left\{ \frac{\|f(t+h) - f(t)\|_X}{h} \,:\, |h| \leqslant \varepsilon \text{ and } t \in T \setminus D \right\}$$

is bounded. Since for every $n \geqslant 1$ and every $t \in T \setminus D$,

$$\text{the limit } \lim_{n \to +\infty} \left\langle x_n^*, \frac{f(t+h) - f(t)}{h} \right\rangle_X \text{ exists,}$$

we infer that there exists $u(t) \in X$, such that for all $x^* \in X^*$ and all $t \in T \setminus D$, we have

$$\lim_{h \to 0} \left\langle x_n^*, \frac{f(t+h) - f(t)}{h} \right\rangle_X = \langle x^*, u(t) \rangle_X,$$

so f is weakly differentiable at every $t \in T \setminus D$.

Let f' be the weak derivative of f (i.e., $f'(t) = u(t)$ for all $t \in T \setminus D$). Clearly f' is weakly measurable and so by Theorem 2.1.3 it is also strongly measurable. Moreover, from the weak lower semicontinuity of the norm in a Banach space, we have

$$\|f'(t)\|_X \leqslant \liminf_{h \to 0} \frac{\|f(t+h) - f(t)\|_X}{h} \qquad \forall\, t \in T \setminus D. \tag{2.10}$$

Then from (2.10) and Fatou's lemma (see Theorem A.2.1), we have that

$$\int_0^b \|f'(t)\|_X \, dt \leqslant V(f)(b),$$

i.e., $f' \in L^1(T; X)$. Also

$$\langle x^*, f(t) - f(0) \rangle_X = \int_0^t \langle x^*, f'(s) \rangle_X \, ds \qquad \forall\, x^* \in X^*,\ t \in T,$$

so from Theorem 2.1.17, we have

$$f(t) - f(0) = \int_0^t f'(s)\, ds \qquad \forall\, t \in T \tag{2.11}$$

and finally f is almost everywhere strongly differentiable with

$$\frac{df}{dt} = f' \in L^1(T; X)$$

and (2.11) holds. ∎

REMARK 2.2.18 The result is more generally true if we assume that X has the RNP. This follows from the fact the RNP is passed to closed linear subspaces of X and if X is a separable Banach space with the RNP, then it has the separable dual (see Diestel & Uhl (1977, pp. 217–218)). So a careful reading of the previous proof reveals that it remains valid if instead we assume only that X has the RNP.

The next result is an extension of the so-called "Lagrange lemma" and "DuBois-Reymond lemma" (see Denkowski, Migórski & Papageorgiou (2003*b*, p. 673)) to Banach space valued functionals.

PROPOSITION 2.2.19
Let $f \in L^1(T;X)$ (with $T = [0,b]$).

(a) *If*

$$\int_0^b f(t)\vartheta(t)\,dt \;=\; 0 \qquad \forall\, \vartheta \in C_c^\infty\big((0,b)\big),$$

then $f = 0$.

(b) *If*

$$\int_0^b f(t)\vartheta'(t)\,dt \;=\; 0 \qquad \forall\, \vartheta \in C_c^\infty\big((0,b)\big),$$

then f is constant.

PROOF **(a)** By virtue of Theorem 2.1.3, we may assume that X is separable. Then $X^*_{w^*}$ (the dual space X^* furnished with the w^*-topology) is w^*-separable (in fact $X^*_{w^*}$ is a Souslin space; see Definition A.2.29**(b)** and Remark A.2.30). Let $\{x_n^*\}_{n\geqslant 1}$ be w^*-dense in X^*. Then for all $n \geqslant 1$ and all $\vartheta \in C_c^\infty\big((0,b)\big)$, we have

$$\int_0^b \vartheta(t)\big\langle x_n^*, f(t)\big\rangle\,dt \;=\; \left\langle x_n^*, \int_0^b f(t)\vartheta(t)\,dt \right\rangle_X \;=\; 0,$$

so by the Lagrange lemma, we have

$$\big\langle x_n^*, f(t)\big\rangle_X \;=\; 0 \quad \text{for a.a. } t \in T \text{ and all } n \geqslant 1$$

and since $\overline{\{x_n^*\}_{n\geqslant 1}}^{w^*} = X^*$, we obtain that

$$f(t) \;=\; 0 \quad \text{for a.a. } t \in T.$$

(b) The proof is similar, using this time the DuBois-Reymond lemma. □

The next proposition permits the identification of the space of X-valued absolutely continuous functions with a vector Sobolev space.

PROPOSITION 2.2.20
If $f, g \in L^1(T;X)$ *(with* $T=[0,b]$*),*
then the following conditions are equivalent:

(a) $f(t) = v + \int_0^t g(s)\,ds$, $v \in X$, *for almost all* $t \in T$;

(b) $\int_0^b f(t)\vartheta'(t)\,dt = -\int_0^b g(t)\vartheta(t)\,dt$ *for all* $\vartheta \in C_c^\infty\big((0,b)\big)$;

(c) *for every* $x^* \in X^*$,

$$\frac{d}{dt}\big\langle x^*, f(\cdot)\big\rangle_X \;=\; \big\langle x^*, g(\cdot)\big\rangle_X$$

in the distributional sense on $(0,b)$ *(see Definition 1.6.1***(a)***).*

PROOF "**(a)**$\Longrightarrow$**(b)**,**(c)**": These implications follow from a simple integration by parts.

"**(c)**$\Longrightarrow$**(b)**": From the definition of distributional derivative (see Definition 1.6.1**(a)**), for all $\vartheta \in C_c^\infty\big((0,b)\big)$ and all $x^* \in X^*$, we have

$$\begin{aligned}\int_0^b \big\langle x^*, f(t)\big\rangle_X \vartheta'(t)\,dt \;&=\; -\int_0^b \frac{d}{dt}\big\langle x^*, f(t)\big\rangle_X \vartheta(t)\,dt \\ &=\; \int_0^b \big\langle x^*, g(t)\big\rangle_X \vartheta(t)\,dt,\end{aligned}$$

so

$$\begin{aligned}&\int_0^b \big\langle x^*, \vartheta'(t)f(t) + \vartheta(t)g(t)\big\rangle_X\,dt \\ &= \left\langle x^*, \int_0^b \big(\vartheta'(t)f(t) + \vartheta(t)g(t)\big)\,dt\right\rangle_X \;=\; 0 \qquad \forall\, x^* \in X^*\end{aligned}$$

and thus

$$\int_0^b f(t)\vartheta'(t)\,dt \;=\; -\int_0^b g(t)\vartheta(t)\,dt \qquad \forall\, \vartheta \in C_c^\infty\big((0,b)\big).$$

"**(b)**$\Longrightarrow$**(a)**": Let

$$\widehat{f}(t) \stackrel{df}{=} \int_0^b g(s)\,ds \qquad \forall\, t \in T.$$

Evidently $\widehat{f}$ is absolutely continuous and

$$\widehat{f}'(t) \;=\; g(t) \quad \text{for a.a. } t \in T.$$

Let

$$h \stackrel{df}{=} f - \widehat{f}.$$

We have

$$\int_0^b h(t)\vartheta'(t)\,dt \;=\; 0 \qquad \forall\, \vartheta \in C_c^\infty\big((0,b)\big),$$

so, using Proposition 2.2.19**(b)**, we have

$$h(t) \;=\; v \;\in\; X \qquad \forall\, t \in T$$

and finally

$$f(t) = v + \int_0^b g(s)\,ds \qquad \forall\, t \in T.$$

□

COROLLARY 2.2.21
If $f, g \in L^1(T;X)$ ($T = [0,b]$) and one of the equivalent statements **(a)**, **(b)** *or* **(c)** *in Proposition 2.2.20 holds,*
then f is almost everywhere equal to an absolutely continuous function $f_1 : T \longrightarrow X$.

Extending the notion of distributional (weak) derivative and the resulting Sobolev spaces (see Definition 1.6.1) to X-valued functions, we make the following definitions.

DEFINITION 2.2.22 **(a)** *Let $f, g \in L^1(T;X)$ (with $T = [0,b]$). We say that g is the* ***distributional (weak)*** *derivative of f, if*

$$\int_0^b f(t)\vartheta'(t)\,dt \;=\; -\int_0^b g(t)\vartheta(t)\,dt \qquad \forall\, \vartheta \in C_c^\infty\big((0,b)\big).$$

We denote this derivative of f by Df.

(b) *Let $p \in [1, +\infty]$ and $T = [0, b]$. We define*

$$W^{1,p}\big((0,b);X\big) \stackrel{df}{=} \Big\{ f \in L^p(T;X) : \ Df \in L^p(T;X) \Big\}.$$

(c) *Let $p \in [1, +\infty]$ and $T = [0, b]$. We define*

$$AC^{1,p}\big(T,X\big) \stackrel{df}{=} \Big\{ f\colon T \longrightarrow X : \ f \text{ is absolutely continuous, differentiable almost everywhere with derivative } f' \in L^p(T;X) \Big\}.$$

REMARK 2.2.23 According to Theorem 2.2.17 (see also Remark 2.2.18), if X is reflexive (or more generally if X has RNP), then $f \in AC^{1,p}\big(T,X\big)$ if and only if there exists a function $g \in L^p(T;X)$, such that

$$f(t) \ = \ f(0) + \int_0^t g(s)\, ds \qquad \forall\ t \in T.$$

Invoking Proposition 2.2.20, we see that the spaces $W^{1,p}\big((0,b);X\big)$ and $AC^{1,p}\big(T,X\big)$ (for $p \in [1, +\infty]$) can be identified.

THEOREM 2.2.24
If $p \in [1, +\infty]$ and $f \in L^p(T;X)$ (with $T = [0, b]$), then the following statements are equivalent:

(a) *$f \in W^{1,p}(T;X)$;*

(b) *there exists $f_1 \in AC^{1,p}\big(T,X\big)$, such that $f(t) = f_1(t)$ for almost all $t \in T$.*

REMARK 2.2.25 In Section 2.4, we shall see that this property distinguishes Sobolev functions of one variable (i.e., defined on $(0, b)$) from Sobolev functions of several variables (i.e., functions defined on an open set $Z \subseteq \mathbb{R}^N$ with $N > 1$).

PROPOSITION 2.2.26
If X is reflexive, $p \in (1, +\infty)$ and $f \in L^p(T;X)$ (with $T = [0, b]$), then the following two conditions are equivalent:

(a) *$f \in W^{1,p}\big((0,b);X\big)$ (or there exists $f_1 \in AC^{1,p}\big(T,X\big)$, such that $f(t) = f_1(t)$ for almost all $t \in T$);*

(b) $\displaystyle\int_0^{b-h} \big\|f(t+h) - f(t)\big\|_X^p\, dt \leqslant ch^p$ *for some $c > 0$ and all $h \in (0, b)$.*

PROOF "**(a)**$\Longrightarrow$**(b)**": By Theorem 2.2.24, we have

$$f(t+h) - f(t) \;=\; \int_t^{t+h} Df_1(s)\,ds \qquad \forall\, t, t+h \in T = [0,b].$$

By Jensen inequality (see Theorem A.2.26), we have

$$\big\|f(t+h) - f(t)\big\|_X^p \;\leqslant\; h^{p-1} \int_t^{t+h} \big\|Df_1(s)\big\|_X^p\,ds,$$

so

$$\int_0^{b-h} \big\|f(t+h) - f(t)\big\|_X^p\,dt \;\leqslant\; h^{p-1} \int_0^{b-h}\int_0^{t+h} \big\|Df_1(s)\big\|_X^p\,ds\,dt. \tag{2.12}$$

Note that

$$\int_0^{b-h} \frac{1}{h} \int_t^{t+h} \big\|Df_1(s)\big\|_X^p\,ds\,dt \;\longrightarrow\; \int_0^b \big\|Df_1(s)\big\|_X^p\,ds \quad \text{as } h \to 0$$

(see Proposition 2.1.22). So from (2.12), we conclude that

$$\int_0^{b-h} \big\|f(t+h) - f(t)\big\|_X^p\,dt \;\leqslant\; ch^p \qquad \forall\, h \in (0,b),$$

for some constant $c > 0$.

"**(b)**$\Longrightarrow$**(a)**": For every $n \geqslant 1$, let

$$g_n(t) \;\stackrel{df}{=}\; \chi_{[0,b-\frac{1}{n}]}(t)\frac{f(t+\frac{1}{n}) - f(t)}{\frac{1}{n}}.$$

By virtue of condition **(b)**, the sequence $\{g_n\}_{n\geqslant 1} \subseteq L^p(T;X)$ is bounded. Since $p \in (1,+\infty)$ and X is reflexive, the Lebesgue-Bochner space $L^p(T;X)$ is reflexive too (see Proposition 2.2.3**(c)**). So by the Eberlein-Smulian theorem (see Theorem A.3.8), we may assume that

$$g_n \;\stackrel{w}{\longrightarrow}\; g \quad \text{in } L^p(T;X),$$

for some $g \in L^p(T;X)$.

For every $\vartheta^* \in C_c^\infty\big((0,b);X^*\big)$, we have

$$\begin{aligned}
&\int_0^b \big\langle \vartheta^*(t), g(t)\big\rangle_X \, dt \\
&= \lim_{n\to+\infty} \int_0^b \big\langle \vartheta^*(t), g_n(t)\big\rangle_X \, dt \\
&= \lim_{n\to+\infty} \int_0^b \Big\langle \vartheta^*(t), \frac{f(t+\frac{1}{n}) - f(t)}{\frac{1}{n}} \Big\rangle_X \, dt \qquad (2.13) \\
&= \lim_{n\to+\infty} \Big[\int_0^{b-\frac{1}{n}} \Big\langle \frac{\vartheta^*(t-\frac{1}{n}) - \vartheta^*(t)}{\frac{1}{n}}, f(t) \Big\rangle_X \, dt - n \int_{b-\frac{1}{n}}^b \big\langle \vartheta^*(t), f(t)\big\rangle_X \, dt \Big].
\end{aligned}$$

Because $\vartheta^* \in C_c^\infty\big((0,b);X^*\big)$, for $n \geqslant 1$ large enough, we have

$$\int_{b-\frac{1}{n}}^b \big\langle \vartheta^*(t), f(t)\big\rangle_X \, dt \;=\; 0.$$

Also

$$\int_0^{b-\frac{1}{n}} \Big\langle \frac{\vartheta^*(t-\frac{1}{n}) - \vartheta^*(t)}{\frac{1}{n}}, f(t) \Big\rangle_X \, dt \;\longrightarrow\; -\int_0^b \big\langle \vartheta^{*\prime}(t), f(t)\big\rangle_X \, dt.$$

So from (2.13), we have

$$\int_0^b \big\langle \vartheta^*(t), g(t)\big\rangle_X \, dt \;=\; -\int_0^b \big\langle \vartheta^{*\prime}(t), f(t)\big\rangle_X \, dt \qquad \forall\ \vartheta^* \in C_c^\infty\big((0,b);X\big)$$

and finally

$$Df \;=\; g \quad \text{in } L^p(T;X),$$

i.e.,

$$f \;\in\; W^{1,p}\big((0,b);X\big).$$

□

To prove the next result concerning X-valued functions, we shall need the following general result about embeddings of Banach spaces, which will also be helpful in our discussion of evolution triples later in this section.

LEMMA 2.2.27
If Y is another Banach space, such that $X \subseteq Y$, the embedding is continuous and X is dense in Y,
then

(a) *the embedding $Y^* \subseteq X^*$ is continuous;*

(b) *if X is reflexive, then Y^* is dense in X^*.*

PROOF **(a)** Since by hypothesis X is embedded continuously in Y, there exists $c_1 > 0$, such that

$$\|x\|_Y \leqslant c_1 \|x\|_X \qquad \forall\, x \in X.$$

Let $y^* \in Y^*$. Then

$$\left| \langle y^*, x \rangle_Y \right| \leqslant \|y^*\|_{Y^*} \|x\|_Y \leqslant c_1 \|y^*\|_{Y^*} \|x\|_X . \tag{2.14}$$

Let $\widehat{y}^* = y^*|_X$. Then from (2.14), we have $\widehat{y}^* \in X^*$ and

$$\|\widehat{y}^*\|_{X^*} \leqslant c_1 \|y^*\|_{Y^*} . \tag{2.15}$$

We show that $\widehat{y}^* = 0$ implies that $y^* = 0$. Indeed for all $x \in X$, we have

$$0 = \langle \widehat{y}^*, x \rangle_X \leqslant \langle y^*, x \rangle_X .$$

Because X is dense in Y, it follows that $y^* = 0$. So the map

$$i^* : Y^* \longrightarrow X^*,$$

defined by

$$i^*(y^*) \stackrel{df}{=} \widehat{y}^*,$$

is continuous, injective. Hence y^* can be identified with $\widehat{y}^*$ and so $Y^* \subseteq X^*$ with continuous injection (see (2.15)).

(b) Suppose that assertion is not true. Then

$$\overline{Y^*}^{\|\cdot\|_{X^*}} \neq X^*$$

and so by the Hahn-Banach theorem, we can find $u \in X^{**} = X$ (since X is reflexive), $u \neq 0$, such that

$$\langle x^*, u \rangle_X = 0 \qquad \forall\, x^* \in Y^*.$$

It follows that $u = 0$, a contradiction. ∎

PROPOSITION 2.2.28
If X is reflexive, Y is another Banach space, $X \subseteq Y$, the embedding is continuous and $f \in L^\infty(T;X) \cap C\big(T;Y_w\big)$ ($T = [0,b]$ and Y_w is the Banach space Y equipped with the weak topology),
then $f \in C\big(T;X_w\big)$ (where X_w is the Banach space X equipped with the weak topology).

PROOF By replacing Y with $\overline{X}^{\|\cdot\|_Y}$ if necessary, we may assume that X is dense in Y. So by virtue of Lemma 2.2.27**(b)**, $Y^* \subseteq X^*$ and the embedding is continuous and dense. From Corollary 2.1.4, we know that there exists a sequence $\{f_n\}_{n\geqslant 1}$ of X-valued, countably valued functions on T, such that

$$f_n \longrightarrow f \quad \text{uniformly on } T \text{ in } X.$$

We know that

$$\big\|f_n(t)\big\|_X \leqslant c_1 \|f\|_{L^\infty(T;X)} \qquad \forall\, t \in T,\ n \geqslant 1,$$

for some $c_1 > 0$ and

$$\big\langle y^*, f_n(t)\big\rangle_X \longrightarrow \big\langle y^*, f(t)\big\rangle_X \qquad \forall\, y^* \in Y^*,\ t \in T.$$

It follows that

$$\big|\big\langle y^*, f_n(t)\big\rangle_X\big| \leqslant c_1 \|y^*\|_{X^*} \|f_n\|_{L^\infty(T;X)} \qquad \forall\, n \geqslant 1,\ t \in T,$$

thus

$$\big|\big\langle y^*, f(t)\big\rangle_X\big| \leqslant c_1 \|y^*\|_{X^*} \|f\|_{L^\infty(T;X)} \qquad \forall\, t \in T$$

and so

$$f(t) \in X \quad \text{and} \quad \big\|f(t)\big\|_X \leqslant c_1 \|f\|_{L^\infty(T;X)} \qquad \forall\, t \in T. \tag{2.16}$$

Next let $x^* \in X^*$. We can find a sequence $\{y_m^*\}_{m\geqslant 1} \subseteq Y^*$, such that

$$y_m^* \longrightarrow x^* \quad \text{in } X^*.$$

Also let $t_n \to t$ in T. Because $f \in C\big(T;Y_w\big)$, we have

$$\begin{aligned}\big\langle y_m^*, f(t_n)\big\rangle_X &= \big\langle y_m^*, f(t_n)\big\rangle_Y \longrightarrow \big\langle y_m^*, f(t)\big\rangle_Y \\ &\text{as } n \to +\infty, \text{ for all } m \geqslant 1 \text{ and all } t \in T.\end{aligned} \tag{2.17}$$

Also we have

$$\begin{aligned}\big\langle y_m^*, f(t)\big\rangle_Y &= \big\langle y_m^*, f(t)\big\rangle_X \longrightarrow \big\langle x^*, f(t)\big\rangle_X \\ &\text{as } m \to +\infty, \text{ for all } t \in T.\end{aligned} \tag{2.18}$$

From (2.17) and (2.18), via the double limit lemma (see Proposition A.2.35), we deduce that there exists a sequence $\{m(n)\}_{n\geqslant 1}$ increasing (not necessarily strictly) to $+\infty$ such that

$$\langle y^*_{m(n)}, f(t_n)\rangle_X \longrightarrow \langle x^*, f(t)\rangle_X. \tag{2.19}$$

From (2.16) and (2.19), we have

$$\begin{aligned}
&\left|\langle x^*, f(t_n)\rangle_X - \langle x^*, f(t)\rangle_X\right| \\
\leqslant\ &\left|\langle x^*, f(t_n)\rangle_X - \langle y^*_{m(n)}, f(t_n)\rangle_X\right| + \left|\langle y^*_{m(n)}, f(t_n)\rangle_X - \langle x^*, f(t)\rangle_X\right| \\
\leqslant\ &\|x^* - y^*_{m(n)}\|_{X^*}\|f(t_n)\|_X + \left|\langle y^*_{m(n)}, f(t_n)\rangle_X - \langle x^*, f(t)\rangle_X\right| \longrightarrow 0,
\end{aligned}$$

so $f \in C(T; X_w)$. □

The next lemma is crucial in obtaining compactness theorems for function spaces which arise in the study of evolution equations.

LEMMA 2.2.29
If X, Y, Z are three Banach spaces, such that $X \subseteq Y \subseteq Z$ with the first embedding compact and the second continuous,
then for every $\xi > 0$, we can find $c(\xi) > 0$, such that

$$\|x\|_Y \leqslant \xi\|x\|_X + c(\xi)\|x\|_Z \qquad \forall\, x \in X.$$

PROOF Suppose the lemma is not true. Then we can find $\xi > 0$ and a sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$\|x_n\|_Y > \xi\|x_n\|_X + n\|x_n\|_Z \qquad \forall\, n \geqslant 1.$$

Let $y_n \overset{df}{=} \frac{x_n}{\|x_n\|_X}$ for all $n \geqslant 1$. We have

$$\|y_n\|_Y > \xi + n\|y_n\|_Z \qquad \forall\, n \geqslant 1. \tag{2.20}$$

Since $\|y_n\|_X = 1$ for all $n \geqslant 1$ and the embedding $X \subseteq Y$ is compact, from (2.20), we have that

$$\|y_n\|_Z \longrightarrow 0 \tag{2.21}$$

and also the sequence $\{y_n\}_{n\geqslant 1} \subseteq Y$ is relatively compact.

Thus we can find a subsequence $\{y_{n_k}\}_{k\geqslant 1}$ of $\{y_n\}_{n\geqslant 1}$, such that

$$y_{n_k} \longrightarrow u \quad \text{in } Y.$$

Since Y is embedded continuously in Z, we have also that

$$y_{n_k} \longrightarrow u \quad \text{in } Z.$$

Because of (2.21), we have that $u = 0$. On the other hand from (2.20) in the limit as $k \to +\infty$, we have $\|u\|_Y \geqslant \xi > 0$, a contradiction. This proves the lemma. □

Let X, Y, Z be three Banach spaces, with X, Y reflexive. Assume that

$$X \subseteq Y \subseteq Z,$$

with the embeddings being continuous. Moreover, we suppose that the first embedding is compact. Let $T = [0, b]$ and $1 < p, r$. We introduce the space

$$W_{\rm pr}(T) \stackrel{df}{=} \big\{u \in L^p(T; X) : u' = Du \in L^r(T; Z)\big\}.$$

Here $u' = Du$ denotes the derivative in the distributional sense in Z, i.e.,

$$\int_0^b u(t)\vartheta'(t)\,dt = -\int_0^b u'(t)\vartheta(t)\,dt \quad \text{in } Z \qquad \forall\ \vartheta \in C_c^\infty\big((0,b)\big).$$

We furnish $W_{\rm pr}(T)$ with the norm

$$\|u\|_{pr} = \|u\|_p + \|u'\|_r.$$

Clearly $W_{\rm pr}(T)$ normed this way is a Banach space. Indeed, consider the isomorphism

$$\eta\colon W_{\rm pr}(T) \longrightarrow L^p(T; X) \times L^r(T; Z),$$

given by

$$\eta(x) \stackrel{df}{=} (x, x') \qquad \forall\ x \in W_{\rm pr}(T)$$

and view $W_{\rm pr}(T)$ as a closed subspace of $L^p(T; X) \times L^r(T; Z)$. Moreover, if X and Z are separable, then so is $W_{\rm pr}(T)$ and finally if X and Z are reflexive, then $W_{\rm pr}(T)$ is reflexive too. It is evident that

$$W_{\rm pr}(T) \subseteq L^p(T; X) \subseteq L^p(T; Z) \subseteq L^s(T; Z),$$

with $s = \min\{p, r\}$. Then

$$W_{\rm pr}(T) \subseteq W^{1,s}\big((0,b); Z\big)$$

and so every $u \in W_{\rm pr}(T)$ viewed as a Z-valued function belongs in $AC^{1,s}\big(T, Z\big)$ (see Theorem 2.2.24). Therefore the derivative $u' = Du$ is actually a strong derivative in Z almost everywhere, i.e.,

$$u' = \frac{du}{dt}.$$

We note that

$$W_{\rm pr}(T) \subseteq L^p(T; X) \subseteq L^p(T; Y)$$

and clearly the embeddings are continuous.

We can say more about the embedding $W_{\rm pr}(T) \subseteq L^p(T; Y)$, provided we strengthen our conditions on the spaces X, Y and Z.

THEOREM 2.2.30
If X, Y, Z are Banach spaces, with X, Z being reflexive, the embeddings $X \subseteq Y \subseteq Z$ being continuous and the embedding $X \subseteq Y$ being compact, then the embedding $W_{\mathrm{pr}}(T) \subseteq L^p(T;Y)$ is compact.

PROOF Let $\{u_n\}_{n \geqslant 1} \subseteq W_{\mathrm{pr}}(T)$ be a bounded sequence. We need to show that it has a subsequence which converges strongly in $L^p(T;Y)$. Note that $W_{\mathrm{pr}}(T)$ is reflexive. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_{\mathrm{pr}}(T).$$

This means that

$$u_n \xrightarrow{w} u \quad \text{in } L^p(T;X)$$

and

$$u_n' \xrightarrow{w} u' \quad \text{in } L^r(T;Z).$$

Recall that $W_{\mathrm{pr}}(T) \subseteq C(T;Z)$.

Claim 1. The embedding $W_{\mathrm{pr}}(T) \subseteq C(T;Z)$ is continuous.

To see this suppose that

$$u_n \longrightarrow u \quad \text{in } W_{\mathrm{pr}}(T). \tag{2.22}$$

Then

$$u_n(t) \longrightarrow u(t) \quad \text{in } X \qquad \forall\, t \in T \setminus D,$$

with $\lambda^1(D) = 0$. Evidently

$$u_n(t) \longrightarrow u(t) \quad \text{in } Z \qquad \forall\, t \in T \setminus D. \tag{2.23}$$

For $t \in T$ and $s \in T \setminus D$, from Proposition 2.2.20, we have

$$\big\|u_n(t) - u(t)\big\|_Z \leqslant \big\|u_n(s) - u(s)\big\|_Z + c_1 \big\|u_n' - u'\big\|_{L^r(T;Z)} \qquad \forall\, n \geqslant 1,$$

for some $c_1 > 0$. For every $n \geqslant 1$, we choose $t_n \in T$, such that

$$\big\|u_n - u\big\|_{C(T;Z)} = \big\|u_n(t_n) - u(t_n)\big\|_Z.$$

So from (2.22) and (2.23), we have

$$\big\|u_n - u\big\|_{C(T;Z)} \leqslant \big\|u_n(s) - u(s)\big\|_Z + c_1 \big\|u_n' - u'\big\|_{L^r(T;Z)} \longrightarrow 0.$$

This proves Claim 1.

Let

$$v_n \stackrel{df}{=} u_n - u \qquad \forall\, n \geqslant 1.$$

Then from Claim 1, it follows that we can find $c_2 > 0$, such that

$$\|v_n\|_{C(T;Z)} \;=\; \|u_n - u\|_{C(T;Z)} \;\leqslant\; c_2 \qquad \forall\, n \geqslant 1. \tag{2.24}$$

We claim that

$$v_n(t) \;\longrightarrow\; 0 \quad \text{in } Z \qquad \forall\, t \in T.$$

We shall prove this for $t = 0$. The proof is similar for any other $t \in T$. We have

$$v_n(0) \;=\; v_n(t) - \int_0^t v_n'(\tau)\, d\tau,$$

so

$$v_n(0) \;=\; \frac{1}{s}\int_0^s v_n(t)\, dt - \frac{1}{s}\int_0^s \int_0^t v_n'(\tau)\, d\tau\, dt,$$

thus

$$v_n(0) \;=\; \xi_n + \eta_n \qquad \forall\, n \geqslant 1,$$

with

$$\xi_n \;=\; \frac{1}{s}\int_0^s v_n(t)\, dt \quad \text{and} \quad \eta_n \;=\; -\frac{1}{s}\int_0^s \int_0^t v_n'(\tau)\, d\tau\, dt \qquad \forall\, n \geqslant 1.$$

Note that

$$\eta_n \;=\; -\frac{1}{s}\int_0^s (s-t) v_n'(t)\, dt.$$

For a given $\varepsilon > 0$, select $s \in T$ so that

$$\|\eta_n\|_Z \;\leqslant\; \int_0^s \big\|v_n'(t)\big\|_Z\, dt \;\leqslant\; \frac{\varepsilon}{2} \qquad \forall\, n \geqslant 1.$$

For this fixed $s \in T$, note that

$$\xi_n \;\overset{w}{\longrightarrow}\; 0 \quad \text{in } X$$

and so

$$\xi_n \;\longrightarrow\; 0 \quad \text{in } Z$$

(since X is embedded compactly in Z). So for $n \geqslant 1$ large enough, we have

$$\|\xi_n\|_Z \;\leqslant\; \frac{\varepsilon}{2}.$$

This means that

$$v_n(t) \;\longrightarrow\; 0 \quad \text{in } Z \qquad \forall\, t \in T.$$

Because of (2.24), we can apply Proposition 2.1.13 and infer that

$$v_n \longrightarrow 0 \quad \text{in } L^p(T;Z).$$

By virtue of Lemma 2.2.29, for a given $\gamma > 0$ we can find $c(\gamma) > 0$, such that

$$\|v_n\|_{L^p(T;Y)} \leqslant \gamma \|v_n\|_{L^p(T;X)} + c(\gamma) \|v_n\|_{L^p(T;Z)},$$

so

$$\|v_n\|_{L^p(T;Y)} \leqslant \gamma c_3 + c(\gamma) \|v_n\|_{L^p(T;Z)} \qquad \forall\, n \geqslant 1, \tag{2.25}$$

for some $c_3 > 0$. Since $\gamma > 0$ was arbitrary and $\|v_n\|_{L^p(T;Z)} \longrightarrow 0$, from (2.25) we infer that

$$\limsup_{n \to +\infty} \|v_n\|_{L^p(T;Y)} \leqslant 0,$$

i.e., $v_n \to 0$ in $L^p(T;Y)$. □

Now we are about to introduce a notion that plays a central role in the study of evolution equations. The modern strategy in studying parabolic equations is to make use of many different function spaces. The concept of evolution triple, which we define next, provides an appropriate analytical framework to realize this strategy.

DEFINITION 2.2.31 *A triple of spaces (X, H, X^*) is said to be an **evolution triple**, if the following are true:*

(a) *X is a separable, reflexive Banach space;*

(b) *H is a separable Hilbert space;*

(c) *the embedding $X \subseteq H$ is continuous and dense.*

REMARK 2.2.32 By virtue of Lemma 2.2.27**(b)**, the embedding $H^* \subseteq X^*$ is continuous and dense. Since by the Riesz-Fréchet representation theorem (see, e.g., Denkowski, Migórski & Papageorgiou (2003*a*, p. 316)) we can assume that $H = H^*$, then we have that all embeddings $X \subseteq H \subseteq X^*$ are continuous and dense. For all $h \in H$ and all $x \in X$, we have

$$\langle h, x \rangle_X = (h, x)_H,$$

i.e., $\langle \cdot, \cdot \rangle_X |_{H \times X} = (\cdot, \cdot)_H$. Also for $x^* \in X^*$, $x \in X$, we have

$$\langle x^*, x \rangle_X = \lim_{\substack{h \xrightarrow{\|\cdot\|_{X^*}} x^* \\ h \in H}} (h, x)_H$$

(since H is dense in X^*). Therefore if X is a Hilbert space too, we do not represent the elements of X^* using the inner product of X (the Riesz-Fréchet theorem), but using the inner product of H.

EXAMPLE 2.2.33 If $Z \subseteq \mathbb{R}^N$ is a bounded open set with smooth boundary and $p \in [2, +\infty)$, then as we shall see in Section 2.4, the spaces

$$X = W^{1,p}(Z), \quad H = L^2(Z) \quad \text{and} \quad X^* = \left(W^{1,p}(Z)\right)^*$$

form an evolution triple.

For the evolution triple (X, H, X^*), we can consider the reflexive Banach space

$$W_{pp'}(T) \stackrel{df}{=} \left\{ u \in L^p(T; X) : \ u' \in L^{p'}\left(T; X^*\right) \right\},$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, introduced earlier. In the next proposition we establish a regularity property for the elements of $W_{pp'}(T)$ and also derive an "integration by parts formula," which is crucial in the treatment of evolution equations.

PROPOSITION 2.2.34 (Integration by Parts Formula) *If (X, H, X^*) is an evolution triple and $1 < p, p' < +\infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then*

(a) *$W_{pp'}(T) \subseteq C(T; H)$ and the embedding is continuous;*

(b) *for all $u, v \in W_{pp'}(T)$ and all $0 \leqslant s \leqslant t \leqslant b$, we have*

$$\big(u(t), v(t)\big)_H - \big(u(s), v(s)\big)_H = \int_s^t \left[\big\langle u'(\tau), v(\tau) \big\rangle_X + \big\langle u(\tau), v'(\tau) \big\rangle_X \right] d\tau.$$

PROOF **(a)** Note that by the generalized Weierstrass approximation theorem , the space of X-valued polynomials is dense in $W_{pp'}(T)$. In particular then

$$\text{the embedding } C^1(T; X) \subseteq W_{pp'}(T) \text{ is dense.}$$

Now let $u, v \in C^1(T; X)$. We have

$$\frac{d}{dt}\big(u(t), v(t)\big)_H = \big(u'(t), v(t)\big)_H + \big(u(t), v'(t)\big)_H \qquad \forall\ t \in T.$$

Thus

$$\begin{aligned} &\big(u(t), v(t)\big)_H - \big(u(s), v(s)\big)_H \\ &= \int_s^t \left[\big(u'(\tau), v(\tau)\big)_H + \big(u(\tau), v'(\tau)\big)_H \right] d\tau \qquad \forall\ 0 \leqslant s \leqslant t \leqslant b, \end{aligned}$$

so

$$\big(u(t), v(t)\big)_H - \big(u(s), v(s)\big)_H$$

$$= \int_s^t \Big[\big\langle u'(\tau), v(\tau) \big\rangle_X + \big\langle u(\tau), v'(\tau) \big\rangle_X \Big] \, d\tau \qquad \forall\ 0 \leqslant s \leqslant t \leqslant b. \tag{2.26}$$

Choose $\vartheta \in C^1(\mathbb{R})$, such that

$$\vartheta(s) = 0, \quad \vartheta(t) = 1 \quad \text{and} \quad |\vartheta| + |\vartheta'| \leqslant 1 \quad \text{on } \mathbb{R}.$$

Let

$$v \stackrel{df}{=} \vartheta u.$$

Then

$$v' = \vartheta' u + \vartheta u'$$

and using Hölder's inequality (see Theorem A.2.27) from (2.26), we obtain

$$\big|u(t)\big|^2 \leqslant c_1 \, \|u\|^2_{pp'} \qquad \forall\ t \in T,$$

for some $c_1 > 0$ and so

$$\|u\|_{C(T;H)} \leqslant \sqrt{c_1} \, \|u\|_{pp'} \qquad \forall\ u \in C^1(T;X). \tag{2.27}$$

Therefore the identity map

$$i \colon \big(C^1(T;X), \|\cdot\|_{pp'}\big) \longrightarrow C(T;H)$$

is continuous. But as we said in the beginning of the proof

$$\text{the embedding } C^1(T;X) \subseteq W_{\mathrm{pp'}}(T) \text{ is dense.}$$

So we can extend i continuously on $W_{\mathrm{pp'}}(T)$. Hence

$$\text{the embedding } W_{\mathrm{pp'}}(T) \subseteq C(T;H) \text{ is continuous.}$$

(b) The integration by parts formula follows from (2.26) and the density of the embedding $C^1(T;X) \subseteq W_{\mathrm{pp'}}(T)$ □

REMARK 2.2.35 Even if the embedding $X \subseteq H$ is compact,

$$\text{the embedding } W_{\mathrm{pp'}}(T) \subseteq C(T;H) \text{ is not compact}$$

(see Migórski (1994)). In general, if X and Z are Banach spaces, the embedding $X \subseteq Z$ is continuous (a special case is if $X = Z$), $p, r \in [1, \infty]$ and

$$W_{\mathrm{pr}}(T) \stackrel{df}{=} \big\{ u \in L^p(T;X) : \ u' \in L^r(T;Z) \big\},$$

then $W_{\mathrm{pr}}(T) \subseteq C(T;Z)$. This inclusion as well as that of Proposition 2.2.34**(a)** means that if $u \in W_{\mathrm{pp'}}(T)$ (respectively $u \in W_{\mathrm{pr}}(T)$), then there exists $u_1 \in C(T;H)$ (respectively $u_1 \in C(T;Z)$), such that

$$u(t) = u_1(t) \quad \text{for a.a. } t \in T.$$

2.3 Compactness Results

In this section we prove compactness and weak compactness results for subsets of $C(T;X)$ and $L^p(T;X)$ $(p \in [1,+\infty))$. Throughout this section $T=[0,b]$ $(b<+\infty)$ and X is a Banach space. Additional hypotheses will be introduced as needed. We start with the classical "Arzela-Ascoli theorem" which characterizes the compact subsets of $C(T;X)$. In its proof we shall need the following lemma.

LEMMA 2.3.1
If $K \subseteq X$ is a nonempty set and for every $\varepsilon > 0$ there exists a relatively compact set $K_\varepsilon \subseteq X$, such that for every $x \in K$ we can find $x_\varepsilon \subseteq K_\varepsilon$, such that $\|x - x_\varepsilon\|_X < \varepsilon$,
then K is relatively compact.

PROOF Let $\varepsilon > 0$. Choose $K_{\frac{\varepsilon}{2}} \subseteq X$ to be the relatively compact subset postulated by the hypothesis of the lemma. We can find $\{x_\varepsilon^k\}_{k=1}^n \subseteq K_{\frac{\varepsilon}{2}}$, such that

$$K_{\frac{\varepsilon}{2}} \subseteq \bigcup_{k=1}^{n} B_{\frac{\varepsilon}{2}}(x_\varepsilon^k).$$

By hypothesis for every $x \in K$, there exists $x_{\frac{\varepsilon}{2}} \subseteq K_{\frac{\varepsilon}{2}}$, such that

$$\left\|x - x_{\frac{\varepsilon}{2}}\right\|_X < \frac{\varepsilon}{2}.$$

But $x_{\frac{\varepsilon}{2}} \in B_{\frac{\varepsilon}{2}}(x_\varepsilon^{k_0})$ for some $k_0 \in \{1,\dots,n\}$. Hence $\left\|x_{\frac{\varepsilon}{2}} - x_\varepsilon^{k_0}\right\|_X < \frac{\varepsilon}{2}$ and so finally $\left\|x - x_\varepsilon^{k_0}\right\|_X < \varepsilon$, which proves that $K \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_\varepsilon^k)$. This means that $K \subseteq X$ is relatively compact. □

Using this lemma we can prove the ***Arzela-Ascoli theorem***.

THEOREM 2.3.2 (Arzela-Ascoli Theorem)
A set $K \subseteq C(T;X)$ is relatively compact if and only if

(a) *for every $t \in (0,b)$, the set $K(t) = \{u(t) : u \in K\}$ is relatively compact in X; and*

(b) *K is uniformly equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that, if $t,s \in T$ and $|t-s| < \delta$, then*

$$\|u(t) - u(s)\|_X < \varepsilon \qquad \forall\, u \in K$$

(the equicontinuity is uniform in $t \in T$ since T is compact).

PROOF "$\Longrightarrow$": Property **(a)** follows from the fact that for every $t \in T$ the evaluation at t map

$$e_t \colon C(T;X) \ni u \longmapsto u(t) \in X$$

is continuous. To prove property **(b)** (the equicontinuity property), we proceed as follows. Let $\varepsilon > 0$. Because K is relatively compact in $C(T;X)$, we can find $\{u_k\}_{k=1}^n \subseteq K$, such that

$$K \subseteq \bigcup_{k=1}^{n} B_{\frac{\varepsilon}{3}}(u_k).$$

If $t \in T$, there is a $\delta = \delta(\varepsilon) > 0$, such that if $s \in T$ and $|t-s| < \delta$, then

$$\big\|u_k(t) - u_k(s)\big\|_X < \frac{\varepsilon}{3} \qquad \forall\ k \in \{1,\dots,n\}$$

(recall that the functions $\{u_k\}_{k=1}^n$ are uniformly continuous on T since T is compact). Now let $s \in T$ with $|t-s| < \delta$ and $u \in K$. Choose $k_0 \in \big\{1,\dots,n\big\}$, such that $\|u - u_{k_0}\|_\infty < \frac{\varepsilon}{3}$. We have

$$\begin{aligned} &\big\|u(t) - u(s)\big\|_X \\ \leqslant\ &\big\|u(t) - u_{k_0}(t)\big\|_X + \big\|u_{k_0}(t) - u_{k_0}(s)\big\|_X + \big\|u_{k_0}(s) - u(s)\big\|_X \\ \leqslant\ &\|u - u_{k_0}\|_\infty + \frac{\varepsilon}{3} + \|u_{k_0} - u\|_\infty \ <\ \varepsilon, \end{aligned}$$

so K is equicontinuous.

"$\Longleftarrow$": First note that $K(0)$ and $K(b)$ are both relatively compact. Indeed, for a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that if $0 < s < \delta$, then

$$\big\|u(s) - u(0)\big\|_X \ <\ \varepsilon \qquad \forall\ u \in K.$$

Since by hypothesis $K(s) \subseteq X$ is relatively compact, from Lemma 2.3.1, it follows that $K(0) \subseteq X$ is relatively compact. Similarly for $K(b) \subseteq X$.

For every integer N, let $u_N \colon T \longrightarrow X$ be the function equal to $u \in K$ at the points $t_k = \frac{kb}{N}$, $k = 0,\dots,N$ and linear between these points. Then the set $K_N \stackrel{df}{=} \big\{u_N :\ u \in K\big\}$ is isomorphic to

$$\prod_{k=0}^{N} K\left(\frac{kb}{N}\right) \ \subseteq\ X^{N+1},$$

which is relatively compact (Tychonoff's theorem). Therefore $K_N \subseteq C(T;X)$ is relatively compact. Also if $N > \frac{b}{\delta}$, then by property **(b)**, we have $\|u - u_N\|_\infty < \varepsilon$. So by Lemma 2.3.1, we conclude that $K \subseteq C(T;X)$ is relatively compact. ∎

We can have a "weak" variant of the Arzela-Ascoli theorem. First a definition.

DEFINITION 2.3.3 *A set $K \subseteq C\big(T; X_w\big)$ is **weakly equicontinuous**, if for every $\varepsilon > 0$ and $x^* \in X^*$, we can find $\delta = \delta(\varepsilon, x^*) > 0$, such that if $t, s \in T$ and $|t - s| < \delta$, then*

$$\big|\big\langle x^*, u(t) - u(s)\big\rangle_X\big| \;<\; \varepsilon \qquad \forall\; u \in K.$$

*Also we say that a sequence of functions $u_n \colon T \longrightarrow X$, $n \geqslant 1$ converges **weakly uniformly** to $u \colon T \longrightarrow X$, if for every $\varepsilon > 0$ and $x^* \in X^*$, we can find $n_0 = n_0(\varepsilon, x^*) \geqslant 1$, such that*

$$\big|\big\langle x^*, u_n(t) - u(t)\big\rangle_X\big| \;<\; \varepsilon \qquad \forall\; t \in T,\; n \geqslant n_0.$$

THEOREM 2.3.4
If X^* is separable, $\{u_n\}_{n\geqslant 1} \subseteq C\big(T; X_w\big)$, for every $t \in T$, the set $\overline{\{u_n(t)\}_{n\geqslant 1}}^{w}$ is weakly compact in X and the sequence $\{u_n\}_{n\geqslant 1}$ is weakly equicontinuous,
then we can find $u \in C\big(T; X_w\big)$ and a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$ such that

$$u_{n_k} \;\longrightarrow\; u \quad \text{weakly uniformly in } T.$$

PROOF Let C^* be a countable dense subset on X^*. We introduce $D^* \stackrel{df}{=} \operatorname{span}_{\mathbb{Q}} C^*$, the set of linear combinations with rational coefficients of the elements of C^*. Evidently D^* is countable and dense in X^*. Using the classical Arzela-Ascoli theorem on $C(T)$ together with the Cantor diagonal process, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$, such that

$$\big\langle x^*, u_{n_k}(\cdot)\big\rangle_X \;\longrightarrow\; v(x^*)(\cdot) \quad \text{in } C(T) \quad \text{as } k \to +\infty.$$

Note that $x^* \longmapsto v(x^*)$ is a map defined on D^* with values in $C(T)$. Moreover, we have

$$\big|\big\langle x^* - y^*, u_n(t)\big\rangle_X\big| \;\leqslant\; c_1 \left\|x^* - y^*\right\|_{X^*} \qquad \forall\; t \in T,\; x^*, y^* \in D^*,$$

for some $c_1 > 0$, so

$$\big|v(x^*)(t) - v(y^*)(t)\big| \;\leqslant\; c_1 \left\|x^* - y^*\right\|_{X^*} \qquad \forall\; t \in T,\; x^*, y^* \in D^*$$

and thus

$$\big\|v(x^*) - v(y^*)\big\|_{C(T)} \;\leqslant\; c_1 \left\|x^* - y^*\right\|_{X^*}.$$

Therefore the map $v \colon D^* \longrightarrow C(T)$ is uniformly continuous. Thus it can be extended to a unique continuous map $\widehat{v} \colon \overline{D^*}^{\|\cdot\|_{X^*}} = X^* \longrightarrow C(T)$. Clearly $\widehat{v}$ is continuous. This together with the fact that $K(t) \subseteq X$ is weakly compact imply that we can find $u \colon T \longrightarrow X$, such that

$$\big\langle x^*, u(t)\big\rangle_X \;=\; \widehat{v}(x^*)(t) \qquad \forall\; x^* \in X^*,\; t \in T.$$

We conclude that $u \in C\big(T; X_w\big)$ and $u_{n_k} \longrightarrow u$ weakly uniformly on T. $\square$

DEFINITION 2.3.5 *A subset $K \subseteq L^p(T;X)$ $(p \in [1,+\infty))$ is said to be* ***p-equiintegrable****, if it is uniformly integrable (see Definition A.2.3) and*

$$\lim_{h \searrow 0} \int_0^{b-h} \big\|u(t+h) - u(t)\big\|_X^p \, dt \;=\; 0 \quad \textit{uniformly for all } u \in K.$$

In the next theorem we present a characterization of relatively compact sets of the Lebesgue-Bochner spaces $L^p(T;X)$ $(p \in [1,+\infty])$ and also obtain an alternative criterion for compactness in $C(T;X)$ in which the compactness condition of the Arzela-Ascoli theorem (see Theorem 2.3.2) is replaced by a similar one for integrals. In what follows

$$\tau_h(u)(t) \stackrel{df}{=} u(t+h) \qquad \forall\ h > 0.$$

So if u is defined on T, this translated version $\tau_h(u)$ is defined on $[-h, b-h]$. Note that the definition of p-equiintegrability is equivalent to saying that

$$\big\|\tau_h(u) - u\big\|_{L^p(T_h;X)} \;\longrightarrow\; 0 \quad \text{as } h \to 0, \quad \text{uniformly for all } u \in K,$$

with $T_h \stackrel{df}{=} [0, b-h]$.

THEOREM 2.3.6
$K \subseteq L^p(T;X)$, $p \in [1,+\infty)$ (respectively $K \subseteq C(T;X)$) is relatively compact if and only if

(a) *for all $t, s \in (0,b)$, $s < t$, we have that the set* $\left\{ \int_s^t u(\tau)\, d\tau : \; u \in K \right\}$ *is relatively compact in X; and*

(b) *K is p-equiintegrable (respectively $\lim_{h \to +\infty} \|\tau_h(u) - u\|_{L^\infty(T_h;X)} = 0$ uniformly in $u \in K$).*

PROOF Suppose that $K \subseteq L^p(T;X)$, $p \in [1,+\infty)$.

"$\Longrightarrow$": Since the map $u \longmapsto \int_s^t u(\tau)\, d\tau$ is continuous from $L^p(T;X)$ into X, property **(a)** is satisfied. Due to the relative compactness of $K \subseteq L^p(T;X)$, we can find a sequence $\{u_k\}_{k=1}^n \subseteq L^p(T;X)$, such that for every $u \in K$, we can find $k \in \{1, \ldots, n\}$, such that $\|u - u_k\|_p < \frac{\varepsilon}{3}$. Because the embedding $C(T;X) \subseteq L^p(T;X)$ is dense, we can assume that $u_k \in C(T;X)$. Then we can find $h_k > 0$, such that for $h \in (0, h_k)$, we have

$$\big\|\tau_k(u_k) - u_k\big\|_{L^p(T_h;X)} \;<\; \frac{\varepsilon}{3}.$$

Let $\widehat{h} = \min\limits_{1\leqslant k\leqslant n} h_k$. We have

$$\tau_h(u) - u \;=\; \tau_h(u-u_k) - (u-u_k) + \big(\tau_h(u_k) - u_k\big)$$

and so

$$\|\tau_h(u) - u\|_{L^p(T_h;X)} \;<\; \varepsilon \qquad \forall\; h \leqslant \widehat{h},\; u \in K,$$

so

$$\lim_{h\searrow 0} \|\tau_h(u) - u\|_{L^p(T_h;X)} \;=\; 0 \quad \text{uniformly for } u \in K.$$

"$\Longleftarrow$": Let $u \in K$ and $r > 0$. We set

$$M_r(u)(t) \;\stackrel{df}{=}\; \frac{1}{r}\int_t^{t+r} u(s)\,ds.$$

We have $M_r(u) \in C\big(T_r;X\big)$ with $T_r = [0, b-r]$. For every $t, s \in [0, b-r]$, $s \leqslant t$, we have

$$\begin{aligned}\big\|M_r(u)(t) - M_r(u)(s)\big\|_X \;&=\; \left\|\frac{1}{r}\int_s^{s+r} \big(\tau_{t-s}(u) - u\big)(\tau)\,d\tau\right\|_X \\ &\leqslant\; \frac{1}{r}\big\|\tau_{t-s}(u) - u\big\|_{L^1(T_{t-s};X)},\end{aligned}$$

so

$$M_rK \;=\; \big\{M_ru :\; u \in K\big\} \;\subseteq\; C\big(T_r;X\big)$$

is uniformly equicontinuous (see condition **(b)**).

Also from condition **(a)**, we see that for every $t \in (0, b-r)$, the set $\big(M_rK\big)(t) \subseteq X$ is relatively compact. So by Theorem 2.3.2, we have that $M_rK \subseteq C\big(T_r;X\big)$ is relatively compact. Note that

$$M_r(u)(t) - u(t) \;=\; \frac{1}{r}\int_0^r \big(\tau_h(u) - u\big)(t)\,dh \qquad \forall\; t \in T_r,$$

so

$$\big\|M_r(u) - u\big\|_{L^p(T_r;X)} \;\leqslant\; \max_{h\in[0,r]} \big\|\tau_h(u) - u\big\|_{L^p(T_r;X)}.$$

But because of condition **(b)**, for all $\widehat{b} < b$, K is the uniform limit of M_rK in $L^p\big(\widehat{T};X\big)$ with $\widehat{T} = \big[0,\widehat{b}\big]$ as $r \to 0$, $r \leqslant b - \widehat{b}$. But since M_rK is relatively compact in $C\big(\widehat{T};X\big)$ and the embedding $C\big(\widehat{T};X\big) \subseteq L^p\big(\widehat{T};X\big)$ is continuous, we see that K is relatively compact in $L^p\big(\widehat{T};X\big)$.

Conditions **(a)** and **(b)** remain valid if one changes the time direction. Namely if $\overline{u}(t) = u(b-t)$, then the set $\overline{K} = \big\{\overline{u} :\; u \in K\big\}$ still satisfies

conditions **(a)** and **(b)**. Then from the previous argument we have that $\overline{K}$ is relatively compact in $L^p\big(\widehat{T};X\big)$. It follows that K is relatively compact in $L^p\big(\big[\widehat{b},b\big];X\big)$. Setting for example $\widehat{b}=\frac{b}{2}$, we obtain the relative compactness of K in $L^p(T;X)$.

The proof of the case when $K\subseteq C(T;X)$ is similar. □

REMARK 2.3.7 The restriction $p<+\infty$ is necessary, because if $K=\{u\}$ with u bounded but discontinuous from T into X, then K is compact in $L^\infty(T;X)$ but condition **(b)** is not satisfied.

COROLLARY 2.3.8
If $u\in L^p(T;X)$, $p\in[1,+\infty)$, *then*

$$\big\|\tau_h(u)-u\big\|_{L^p(T_h;X)}\longrightarrow 0\quad \textit{as } h\searrow 0.$$

When $X=\mathbb{R}$, we have the so-called "Riesz-Kolmogorov theorem" for $p\in[1,+\infty)$.

COROLLARY 2.3.9 (Riesz-Kolmogorov Theorem)
$K\subseteq L^p(T)$, $p\in[1,+\infty)$ (respectively $K\subseteq C(T)$) is relatively compact if and only if

(a) *there exist $t,s\in(0,b)$, $s<t$, such that the set*

$$\left\{\int_s^t u(\tau)\,d\tau:\ u\in K\right\}\ \subseteq\ X$$

is bounded; and

(b) $\displaystyle\int_0^{b-h}\big|u(t+h)-u(t)\big|^p\,dt\longrightarrow 0$ *as* $h\searrow 0$ *uniformly for* $u\in K$.

PROOF From **(a)** and **(b)** it follows that for all $t,s\in K$, $s<t$, we have that the set $\left\{\displaystyle\int_s^t u(\tau)\,d\tau:\ u\in K\right\}\subseteq X$ is bounded. So we can apply Theorem 2.3.6. □

REMARK 2.3.10 Theorem 2.3.6 and Corollary 2.3.9 provide characterizations of compact sets in $C(T;X)$ and $C(T)$ respectively. Compared with the classical Arzela-Ascoli theorem (see Theorem 2.3.2), we see that condition **(a)** (the space criterion) is now a condition on integrals, while condition **(b)** (the time criterion) remains the same.

Next we shall characterize sets which are bounded in $L^p(T;X)$ and compact in $L^r(T;X)$ with $r < p$. Such results are known as "partial compactness" results, since the compactness is not achieved for the larger order p for which the set is actually bounded. First we obtain two auxiliary results relating compactness with time-local compactness.

LEMMA 2.3.11
The set $K \subseteq L^p(T;X)$ (with $p \in [1,+\infty)$) is relatively compact if and only if

(a) *$K \subseteq L^p_{\rm loc}(T;X)$ is relatively compact (i.e., for all $s,t \in (0,b)$, $s<t$, the set $K|_{[s,t]} \subseteq L^p\big([s,t];X\big)$ is relatively compact); and*

(b) $\int\limits_0^h \big\|u(t)\big\|_X^p\,dt + \int\limits_{b-h}^b \big\|u(t)\big\|_X^p\,dt \longrightarrow 0$ *as* $h \to 0$ *uniformly for* $u \in K$.

PROOF "$\Longrightarrow$": Condition **(a)** is automatically true. Let $\overline{u}$ be the extension by 0 outside T of u. Then the set

$$\overline{K} \;=\; \big\{\overline{u}:\; u \in K\big\}$$

is relatively compact in $L^p\big([-b,2b];X\big)$. As

$$\begin{aligned}&\big\|\tau_h(\overline{u}) - \overline{u}\big\|_{L^p([-b,2b];X)}\\ &= \int\limits_0^h \big\|u(t)\big\|_X^p\,dt + \int\limits_0^{b-h} \big\|u(t+h)-u(t)\big\|_X^p\,dt + \int\limits_{b-h}^b \big\|u(t)\big\|_X^p\,dt,\end{aligned}$$

applying Theorem 2.3.6, we obtain

$$\int\limits_0^h \big\|u(t)\big\|_X^p\,dt + \int\limits_{b-h}^b \big\|u(t)\big\|_X^p\,dt \;\longrightarrow\; 0 \quad \text{as } h \searrow 0 \quad \text{uniformly for } u \in K.$$

"$\Longleftarrow$": Let

$$u_h \;\stackrel{df}{=}\; \chi_{[h,b-h]}u \quad \text{and} \quad K_h \;\stackrel{df}{=}\; \big\{u_h:\; u \in K\big\}.$$

Condition **(b)** implies that for a given $\varepsilon > 0$, we can find $h > 0$ small enough, such that

$$\|u_h - u\|_{L^p(T;X)} \;<\; \varepsilon \qquad \forall\, u \in K.$$

Since $K_h \subseteq L^p(T;X)$ is relatively compact (see condition **(a)**), from Lemma 2.3.1, we infer that $K \subseteq L^p(T;X)$ is relatively compact. ∎

LEMMA 2.3.12
If $K \subseteq L^p(T;X)$ *(with* $p \in (1,+\infty]$*) is bounded and* $K \subseteq L^1_{\mathrm{loc}}(T;X)$ *is relatively compact (i.e., for all* $t,s \in (0,b)$*,* $s<t$*,* $K|_{L^1([s,t];X)}$ *is relatively compact),*
then $K \subseteq L^r(T;X)$ *is relatively compact for all* $r \in (1,p)$*.*

PROOF For every $h \leqslant b$ and every $u \in K$, from Hölder's inequality (see Theorem A.2.27), we have

$$\int_0^h \|u(t)\|_X \, dt + \int_{b-h}^b \|u(t)\|_X \, dt \;\leqslant\; 2h^{\frac{1}{q'}} \|u\|_q ,$$

so $K \subseteq L^1(T;X)$ is relatively compact (see Lemma 2.3.11).

So for a given $\varepsilon > 0$, we can find $\{u_k\}_{k=1}^n \subseteq K$, such that for every $u \in K$, there exists $k \in \{1,\dots,n\}$, such that

$$\|u-u_k\|_1 \;<\; \frac{\varepsilon^{\frac{1}{\vartheta}}}{\delta^{\frac{1-\vartheta}{\vartheta}}},$$

where $\delta = \operatorname{diam}_{L^p(T;X)} K$ and $\vartheta \in (0,1)$ is such that $\frac{1}{r} = \frac{1-\vartheta}{p} + \vartheta$. Using the interpolation inequality (see Theorem A.2.28) we have

$$\|u-u_k\|_r \;\leqslant\; \|u-u_k\|_p^{1-\vartheta} \|u-u_k\|_1^{\vartheta} \;<\; \delta^{1-\vartheta} \frac{\varepsilon}{\delta^{1-\vartheta}} \;=\; \varepsilon.$$

So an appeal to Lemma 2.3.1 implies that the set $K \subseteq L^r(T;X)$ is relatively compact. □

Now we are ready to state the partial compactness result.

THEOREM 2.3.13
If $K \subseteq L^p(T;X)$ *(with* $p \in (1,+\infty]$*) is bounded,*
then $K \subseteq L^r(T;X)$ *(with* $r \in (1,p)$*) is relatively compact if and only if for all* $t,s \in (0,b)$*,* $s<t$*,*

(a) $\left\{ \int_s^t u(\tau)\, d\tau : \; u \in K \right\} \subseteq X$ *is relatively compact; and*

(b) $\|\tau_h(u) - u\|_{L^1([s,t];X)} \longrightarrow 0$ *as* $h \searrow 0$ *uniformly for* $u \in K$*.*

PROOF Theorem 2.3.6 implies that $K|_{[s,t]} \subseteq L^1\big([s,t];X\big)$ is relatively compact, hence $K \subseteq L^1_{\mathrm{loc}}(T;X)$ is relatively compact. An appeal to Lemma 2.3.12 finishes the proof. □

REMARK 2.3.14 The restriction $r < p$ cannot be improved to $r = p$.

THEOREM 2.3.15
If Y is a Banach space, such that the embedding $X \subseteq Y$ is compact and $K \subseteq L^p(T;Y)$ (with $p \in [1,+\infty]$) is such that:

(i) $K \subseteq L^1_{\text{loc}}(T;X)$ is bounded; and

(ii) $\|\tau_h(u) - u\|_{L^p(T_h;Y)} \longrightarrow 0$ as $h \searrow 0$ uniformly for all $u \in K$,

then K is relatively compact in $L^p(T;Y)$ if $p \in [1,+\infty)$ and in $C(T;Y)$ if $p = +\infty$.

PROOF For every $t, s \in (0,b)$, $s < t$, $K \subseteq L^1\big([s,t];X\big)$ is bounded. Then the set $\left\{ \int_s^t u(\tau)\,d\tau : u \in K \right\} \subseteq X$ is bounded and so because X is embedded compactly into Y, the set $\left\{ \int_s^t u(\tau)\,d\tau : u \in K \right\}$ is relatively compact in Y. This fact together with hypothesis (ii) permit the use of Theorem 2.3.6, which gives the desired compactness results in $L^p(T;Y)$ (for $p \in [1,+\infty)$) and in $C(T;Y)$ (for $p = +\infty$). □

REMARK 2.3.16 A careful look at the proof reveals that hypothesis (i) can be replaced by a weaker one which says that for all $s, t \in (0,b)$, $s < t$, the set $\left\{ \int_s^t u(\tau)\,d\tau : u \in K \right\} \subseteq X$ is bounded.

If we want compactness in a lower order Lebesgue-Bochner space $L^r(T;Y)$ (with $r \in (1,p)$), then we can weaken the hypotheses of Theorem 2.3.15. The result is an immediate consequence of Theorem 2.3.13.

THEOREM 2.3.17
If Y is a Banach space, such that the embedding $X \subseteq Y$ is compact, $p \in (1,+\infty]$,

$$K \subseteq L^1_{\text{loc}}(T;X) \cap L^p(T;Y)$$

is a bounded set and for all $t, s \in (0,b)$, $s < t$,

$$\big\|\tau_h(u) - u\big\|_{L^1([s,t];Y)} \longrightarrow 0 \quad \text{as } h \searrow 0 \text{ uniformly for } u \in K,$$

then $K \subseteq L^r(T;Y)$ is relatively compact for all $r \in (1,p)$.

Next we introduce a third space Z containing $Y \supseteq X$ and see what can be said. This setting (which includes as a special case evolution triples) permits the weakening of the time-criterion, but at the expense of a strengthened space criterion. The result is based on the following lemma.

LEMMA 2.3.18
If Y, Z are Banach spaces, $X \subseteq Y \subseteq Z$ with continuous embeddings, the embedding of X into Y is compact, $K \subseteq L^p(T;X)$ (with $p \in [1,+\infty]$) is bounded and $K \subseteq L^p(T;Z)$ is relatively compact,
then $K \subseteq L^p(T;Y)$ is relatively compact.

PROOF Since $K \subseteq L^p(T;Z)$ is relatively compact, for a given $\varepsilon > 0$, we can find $\{u_k\}_{k=1}^n \subseteq K$, such that for each $u \in K$ there exists $k \in \{1,\dots,n\}$, such that $\|u - u_k\|_{L^p(T;Z)} < \varepsilon$. Invoking Lemma 2.2.29, for every $\xi > 0$, we can find $c = c(\xi) > 0$, such that

$$\begin{aligned} \|u - u_k\|_{L^p(T;Y)} &\leqslant \xi \|u - u_k\|_{L^p(T;X)} + c \|u - u_k\|_{L^p(T;Z)} \\ &\leqslant \xi\delta_X + c\varepsilon, \end{aligned} \tag{2.28}$$

where $\delta_X = \operatorname{diam}_{L^p(T;X)} K$. For a given $\varepsilon' > 0$, select $\xi \stackrel{df}{=} \frac{\varepsilon'}{2\delta_X}$ and $\varepsilon \stackrel{df}{=} \frac{\varepsilon'}{2c}$. Then from (2.28), we have

$$\|u - u_k\|_{L^p(T;Y)} \leqslant \varepsilon',$$

so $K \subseteq L^p(T;Y)$ is relatively compact. □

Based on the above lemma, we can have the following compactness result for an intermediate space.

THEOREM 2.3.19
If Y, Z are Banach spaces, the embeddings $X \subseteq Y \subseteq Z$ are continuous, with the first embedding compact, $p \in [1,+\infty]$ and

(i) $K \subseteq L^p(T;X)$ is bounded,

(ii) $\|\tau_h(u) - u\|_{L^p(T_h;Z)} \longrightarrow 0$ as $h \searrow 0$ uniformly for $u \in K$,

then K is relatively compact in $L^p(T;Y)$ if $p \in [1,+\infty)$ and in $C(T;Y)$ if $p = +\infty$.

PROOF Because of the compactness of the embedding $X \subseteq Y$ and of Theorem 2.3.6, the set $K \subseteq L^p(T;Z)$ is relatively compact. An application of Lemma 2.3.18 finishes the proof. □

This result permits an extension of Theorem 2.2.30.

THEOREM 2.3.20
If Y, Z are Banach spaces, the embeddings $X \subseteq Y \subseteq Z$ are continuous with the first embedding compact,
then

(a) *if $K \subseteq L^p(T;X)$ (with $p \in [1, +\infty)$) is bounded and the set*

$$K' \stackrel{df}{=} \{u' = Du : \ u \in K\} \subseteq L^1(T;Z)$$

is bounded, we have that $K \subseteq L^p(T;Y)$ is relatively compact;

(b) *if $K \subseteq L^\infty(T;X)$ is bounded and $K' \stackrel{df}{=} \{u' = Du : \ u \in K\} \subseteq L^r(T;Z)$ (with $r > 1$) is bounded, we have that $K \subseteq C(T;Y)$ is relatively compact.*

Now let us look at weakly compact subsets of $L^1(\Omega;X)$. To describe a large class of such sets in $L^1(\Omega;X)$, we shall need two results which for easy reference we state here without proofs. The first is the celebrated James theorem.

THEOREM 2.3.21 (James Theorem)
A nonempty, weakly closed and bounded subset of a Banach space X is weakly compact if and only if every $x^* \in X^*$ attains its maximum on the set.

The second result is a remarkable consequence of the property of decomposability. If (Ω, Σ, μ) is a finite measure space and X is a Banach space, a set $K \subseteq L^1(\Omega;X)$ is said to be ***decomposable*** if and only if $\chi_A u_1 + \chi_{A^c} u_2 \in K$ for all $(u_1, u_2, A) \in K \times K \times \Sigma$.

PROPOSITION 2.3.22
If (Ω, Σ, μ) is a finite measure space, X is a separable Banach space, $\varphi : \Omega \times X \longrightarrow \overline{R} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is jointly measurable, $F : \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is graph measurable (i.e., $\operatorname{Gr} F \stackrel{df}{=} \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-field of X),

$$I_\varphi(u) \ = \ \int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu$$

is defined (maybe $+\infty$ or $-\infty$) for all $u \in S^1_F$ with

$$S^1_F \stackrel{df}{=} \{u \in L^1(\Omega;X) : \ u(\omega) \in F(\omega) \text{ for } \mu\text{-a.a. } \omega \in \Omega\}$$

and there exists $u_0 \in S^1_F$, such that

$$I_\varphi(u_0) \ > \ -\infty,$$

then

$$\sup_{u \in S^1_F} I_\varphi(u) \ = \ \int_\Omega \sup_{x \in F(\omega)} \varphi(\omega, x)\, d\mu.$$

Using these results, we can identify a large class of weakly compact subsets of $L^1(\Omega; X)$.

THEOREM 2.3.23
If (Ω, Σ, μ) *is a finite measure space,* X *is a separable Banach space,* $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ *is graph measurable, for* μ*-almost all* $\omega \in \Omega$, $F(\omega)$ *is weakly compact, convex and there exists* $h \in L^1(\Omega)_+$, *such that*

$$\sup_{x \in F(\omega)} \|x\|_X \leqslant h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then

$$S_F^1 = \big(u \in L^1(\Omega; X) : \ u(\omega) \in F(\omega) \text{ for } \mu\text{-a.a. } \omega \in \Omega\big)$$

is weakly compact and convex.

PROOF Convexity of S_F^1 is obvious. Moreover, because of the boundedness by $h \in L^1(\Omega)_+$, we have $S_F^1 \neq 0$ (see Denkowski, Migórski & Papageorgiou (2003*a*, p. 432)). So we show that S_F^1 is weakly compact in $L^1(\Omega; X)$. According to Theorem 2.3.21, it suffices to show that every $u^* \in \big(L^1(\Omega; X)\big)^*$ attains its supremum on S_F^1. From Theorem 2.2.12, we know that

$$\big(L^1(\Omega; X)\big)^* = L^\infty\big(\Omega; X^*_{w^*}\big)$$

and the duality pairing is given by

$$\langle u^*, u\rangle_{L^1(\Omega;X)} = \int_\Omega \big\langle u^*(\omega), u(\omega)\big\rangle_X \, d\mu.$$

From Proposition 2.3.22, we have

$$\sup_{u \in S_F^1} \langle u^*, u\rangle_{L^1(\Omega;X)} = \sup_{u \in S_F^1} \int_\Omega \big\langle u^*(\omega), u(\omega)\big\rangle_X \, d\mu$$

$$= \int_\Omega \sup_{x \in F(\omega)} \big\langle u^*(\omega), x\big\rangle_X \, d\mu.$$

Let

$$M(\omega) \stackrel{df}{=} \left\{ y \in F(\omega) : \ \big\langle u^*(\omega), y\big\rangle_X = \sup_{x \in F(\omega)} \big\langle u^*(\omega), x\big\rangle_X \right\}.$$

Since $F(\omega)$ is μ-almost everywhere w-compact, we see that

$$M(\omega) \neq 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

By setting $F(\omega) = \{0\}$ on the exceptional Lebesgue-null set, we can say that

$$M(\omega) \neq 0 \qquad \forall\, \omega \in \Omega.$$

Also from Denkowski, Migórski & Papageorgiou (2003*a*, p. 433), we know that we can find a sequence of Σ-measurable functions $f_n\colon \Omega \longrightarrow X$, such that

$$f_n(\omega) \in F(\omega) \qquad \forall\, \omega \in \Omega,\ n \geqslant 1$$

and

$$F(\omega) = \overline{\{f_n(\omega)\}_{n\geqslant 1}}^{\|\cdot\|_X} \qquad \forall\, \omega \in \Omega.$$

Hence

$$\sup_{x\in F(\omega)} \langle u^*(\omega), x\rangle_X = \sup_{n\geqslant 1} \langle u^*(\omega), f_n(\omega)\rangle_X$$

and so

$$\omega \longmapsto \sup_{x\in F(\omega)} \langle u^*(\omega), x\rangle_X = m^*(\omega) \quad \text{is } \Sigma\text{-measurable.}$$

Then

$$\begin{aligned} \operatorname{Gr} M &= \big\{(\omega, x) \in \Omega \times X :\ x \in M(\omega)\big\} \\ &= \big\{(\omega, x) \in \Omega \times X :\ \langle u^*(\omega), x\rangle_X = m^*(\omega)\big\}. \end{aligned}$$

Since u^* is w^*-measurable, it follows that $\operatorname{Gr} M \in \Sigma \times \mathcal{B}(X)$. So we can apply the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33) and obtain a strongly measurable function $u_0\colon \Omega \longrightarrow X$, such that

$$u_0(\omega) \in M(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Evidently

$$u_0 \in S^1_F$$

and

$$\langle u^*, u_0\rangle = \sup_{u\in S^1_F} \langle u^*, u\rangle_X .$$

Since $u^* \in L^\infty\big(\Omega; X^*_{w^*}\big) = \big(L^1(\Omega; X)\big)^*$ was arbitrary and clearly S^1_F is weakly closed and bounded, from Theorem 2.3.21, we conclude that $S^1_F \subseteq L^1(\Omega; X)$ is weakly compact. □

A classical theorem of Dunford-Pettis isolates the relatively weakly compact subsets of $L^1(\Omega)$ as the bounded, uniformly integrable subsets (see Definition A.2.3). If X is reflexive, the original proof for $L^1(\Omega)$ extends with only notational changes to the present vector valued setting. So we have

THEOREM 2.3.24 (Dunford-Pettis Theorem)
If (Ω, Σ, μ) is a finite measure space, X is reflexive and $K \subseteq L^1(\Omega; X)$ is bounded,
then K is relatively weakly compact in $L^1(\Omega; X)$ if and only if it is uniformly integrable.

The relative weak compactness in $L^1(\Omega;X)$ is closely related with the so-called "biting convergence," which is useful in the calculus of variations and in optimal control.

DEFINITION 2.3.25 *A sequence $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega;X)$ is said to converge to $u \in L^1(\Omega;X)$ in the **biting sense**, denoted by $u_n \xrightarrow{\text{b}} u$, if there exists an increasing sequence $\{C_m\}_{m\geqslant 1} \subseteq \Sigma$, such that*

$$\mu(C_m) \nearrow \mu(\Omega) \quad \text{as } m \to +\infty$$

and

$$u_n \xrightarrow{w} u \quad \text{in } L^1(C_m;X) \qquad \forall\, m \geqslant 1.$$

The so-called "Chacon Biting Lemma" says that if X is a reflexive Banach space, then every bounded sequence in $L^1(\Omega;X)$ has a subsequence converging in $L^1(\Omega;X)$ in the biting sense. The next result is a slightly stronger version of the original biting lemma.

THEOREM 2.3.26
If (Ω,Σ,μ) is a finite measure space, X is a Banach space and $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega;X)$ is bounded,
then there exists a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$ and an increasing sequence $\{C_m\}_{m\geqslant 1} \subseteq \Sigma$, such that $\mu(C_m) \nearrow \mu(\Omega)$ as $m \to +\infty$ and $\left\{\chi_{C_k} u_{n_k}\right\}_{k\geqslant 1}$ is uniformly integrable.

PROOF Let $m \geqslant 1$ and define

$$h_m(t) \stackrel{df}{=} \sup_{n\geqslant m} \int\limits_{\{\|u_m\|_X > t\}} \left\|u_n(\omega)\right\|_X d\mu \qquad \forall\, t \geqslant 0.$$

Note that $h_m\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and it is decreasing. So $\lim\limits_{t\to+\infty} h_m(t)$ exists for all $m \geqslant 1$. Let

$$\xi \stackrel{df}{=} \lim_{t\to+\infty} h_1(t).$$

Since for every $m > 1$, $\{u_1,\ldots,u_{m-1}\}$ is uniformly integrable, it follows that

$$\lim_{t\to+\infty} h_m(t) = \xi.$$

Let $\{t_i\}_{i\geqslant 1} \subseteq \mathbb{R}_+$ be a sequence increasing to $+\infty$, such that

$$\xi \leqslant h_1(t_i) \leqslant \xi + \frac{1}{i} \qquad \forall\, i \geqslant 1.$$

Because

$$\xi \leqslant h_m(t) \qquad \forall\, m \geqslant 1,\ t \in \mathbb{R}_+,$$

we can find a strictly increasing sequence $\{l_i\}_{i\geqslant 1}$, such that

$$\int\limits_{\{\|u_{l_i}\|_X > t_i\}} \|u_{l_i}(\omega)\|_X \, d\mu \;\geqslant\; \xi - \frac{1}{i}.$$

Set $D_i \stackrel{df}{=} \{\|u_{l_i}\|_X > t_i\}$. Then

$$t_i \mu(D_i) \;\leqslant\; \sup_{n\geqslant 1} \|u_n\|_1$$

and so

$$\mu(D_i) \;\longrightarrow\; 0 \quad \text{as } i \to +\infty.$$

Let $C_i \stackrel{df}{=} \Omega \setminus D_i$. We claim that the sequence $\{\chi_{C_i} u_{l_i}\}_{i\geqslant 1}$ is uniformly integrable. To this end let

$$h(t) \;\stackrel{df}{=}\; \sup_{i\geqslant 1} \int\limits_{C_i \cap \{\|u_{l_i}\|_X > t\}} \|u_{l_i}(\omega)\|_X \, d\mu.$$

We need to show that $h(t) \longrightarrow 0$ as $t \to +\infty$. We have

$$\begin{aligned} h(t_r) \;&=\; \sup_{i\geqslant r} \int\limits_{\{t_r < \|u_{l_i}\|_X \leqslant t_i\}} \|u_{l_i}(\omega)\|_X \, d\mu \\ &=\; \sup_{i\geqslant r} \left[\int\limits_{\{\|u_{l_i}\|_X > t_r\}} \|u_{l_i}(\omega)\|_X \, d\mu - \int\limits_{\{\|u_{l_i}\|_X > t_i\}} \|u_{l_i}(\omega)\|_X \, d\mu \right] \\ &\leqslant\; \sup_{i\geqslant r} \left[\left(\xi + \frac{1}{r} - \left(\xi - \frac{1}{i} \right) \right) \right] \;\leqslant\; \frac{2}{r}, \end{aligned}$$

so $h(t) \longrightarrow 0$ as $t \to +\infty$.

Finally replace $\{D_i\}_{i\geqslant 1}$ by a sequence decreasing to $\emptyset$ (or a μ-null set). Then there exists a strictly increasing sequence $\{i_k\}_{k\geqslant 1}$, such that

$$\mu\big(D_{i_k}\big) \;\leqslant\; \frac{1}{2^k} \qquad \forall\, k \geqslant 1.$$

Set $A_k \stackrel{df}{=} \bigcup\limits_{j=k}^{\infty} D_{i_j}$ for $k \geqslant 1$. Then $\{A_k\}_{k\geqslant 1}$ decreases to a μ-null set. The subsequence of the statement of the theorem is defined by setting $u_{n_k} = u_{l_i}$ with $i = i_k$. Also $C_k = \Omega \setminus A_k$ for $k \geqslant 1$. The uniform integrability of the sequence $\{\chi_{C_k} u_{n_k}\}_{k\geqslant 1}$ follows from the inclusion $\Omega \setminus A_k \subseteq \Omega \setminus D_{i_k}$. ∎

EXAMPLE 2.3.27 In the previous theorem, it is necessary to pass to a subsequence. To see this let

$$\Omega \;=\; [0,1] \quad \text{and} \quad \mu \;=\; \lambda^1$$

(the Lebesgue measure on $\mathbb{R}$). If $n = 2^k + i$ with $k \geqslant 1$, $i \in \{0, \dots, 2^k - 1\}$, we set

$$u_n(\omega) \;\stackrel{df}{=}\; \begin{cases} 2^k & \text{if } \omega \in \left[\dfrac{i}{2^k}, \dfrac{i+1}{2^k}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Then there is no increasing sequence $\{C_m\}_{m\geqslant 1}$ with $\Omega = \bigcup\limits_{m=1}^{\infty} C_m$, such that $\{\chi_{C_m} u_n\}_{n\geqslant 1}$ is uniformly integrable. Indeed, if

$$\lambda^1(\Omega \setminus C_m) \;\leqslant\; \frac{1}{2},$$

then for all $k \geqslant 1$, there exists $i \in \{0, \dots, 2^k - 1\}$, such that

$$\lambda^1\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right) \cap C_m\right) \;\geqslant\; \frac{1}{2^{k+1}}.$$

COROLLARY 2.3.28
If X is reflexive and $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega; X)$ is bounded, then we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$ and $u \in L^1(\Omega; X)$, such that

$$u_{n_k} \;\xrightarrow{\text{b}}\; u \quad \text{as } k \to +\infty.$$

REMARK 2.3.29 As we shall see in Section 2.5, some of the ideas involved in the "Biting Lemma" are common in the "concentration compactness theorem" (see Theorem 2.5.30).

To have an analogous result for the spaces $L^p(T; X)$, with $p \in (1, +\infty)$, we need the following result which is useful in many situations since it provides information about the pointwise behaviour of a weakly convergent sequence in $L^p(T; X)$ for $p \in [1, +\infty)$. First a definition-notation.

DEFINITION 2.3.30 *If X is a Banach space and $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$. We set*

$$w\text{-}\limsup_{n\to+\infty} A_n \;\stackrel{df}{=}\; \left\{x \in X :\; x = w\text{-}\lim_{k\to+\infty} x_{n_k},\; x_{n_k} \in A_{n_k},\; n_1 < n_2 < \dots\right\}.$$

Here w stands for the weak topology on X.

PROPOSITION 2.3.31
If (Ω, Σ, μ) *is a finite measure space,* X *is a Banach space,* $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega; X)$ *and* $u \in L^p(\Omega; X)$ *with* $p \in [1, +\infty)$,

$$u_n \xrightarrow{w} u \quad \text{in } L^p(\Omega; X)$$

and for μ*-almost all* $\omega \in \Omega$, *the sequence* $\{u_n(\omega)\}_{n\geqslant 1}$ *is relatively weakly compact,*
then

$$u(\omega) \in \overline{\text{conv}}\, w\text{-}\limsup_{n\to+\infty} \{u_n(\omega)\} \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Using this Proposition we can have the following result for bounded sequences in $L^p(T; X)$ (with $p \in (1, +\infty)$).

THEOREM 2.3.32
If (Ω, Σ, μ) *is a finite measure space,* X *is a reflexive Banach space,* $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega; X)$ *(with* $p \in (1, +\infty)$*) is bounded and*

$$u_n(\omega) \xrightarrow{w} u(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \quad \text{in } X, \tag{2.29}$$

then $u \in L^p(\Omega; X)$ *and*

$$u_n \xrightarrow{w} u \quad \text{in } L^p(\Omega; X).$$

PROOF From Proposition 2.2.3**(c)**, we know that $L^p(\Omega; X)$ is reflexive. So by the Eberlein-Smulian theorem (see Theorem A.3.8), we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$, such that

$$u_{n_k} \xrightarrow{w} \widehat{u} \quad \text{in } L^p(\Omega; X).$$

Using Proposition 2.3.31 and (2.29), we infer that

$$u = \widehat{u} \in L^p(\Omega; X).$$

So every subsequence of $\{u_n\}_{n\geqslant 1}$ has a further subsequence weakly convergent in $L^p(\Omega; X)$ to u and from this it follows that

$$u_n \xrightarrow{w} u \quad \text{in } L^p(\Omega; X).$$

□

Another notion related to the weak convergence in $L^1(T; X)$ ($T = [0, b]$) is given in the next definition.

DEFINITION 2.3.33 *Let* $T = [0, b]$ *(*$b < +\infty$*) and* X *a Banach space. The* ***weak norm*** *on* $L^1(T; X)$ *is defined by*

$$\|u\|_w \stackrel{df}{=} \max_{0\leqslant s\leqslant t\leqslant b} \left\| \int_s^t u(\tau)\, d\tau \right\|_X \qquad \forall\ u \in L^1(T; X).$$

REMARK 2.3.34 Equivalently we can define

$$\|u\|_w \;=\; \max_{t\in T}\left\|\int_0^t u(\tau)\,d\tau\right\|_X \qquad \forall\, u\in L^1(T;X).$$

Evidently $\|\cdot\|_w$ is a norm on $L^1(T;X)$ weaker than the usual norm

$$\|u\|_1 \;=\; \int_0^b \|u(t)\|_X\,dt \qquad \forall\, u\in L^1(T;X).$$

We shall show that for a broad class of subsets of $L^1(T;X)$, the topology generated by the weak norm $\|\cdot\|_w$ and the weak $L^1(T;X)$-topology coincide. For this purpose we introduce the following property for subsets of $L^1(T;X)$.

DEFINITION 2.3.35 *Let $T=[0,b]$ and X a Banach space. We say that $K\subseteq L^1(T;X)$ has **property** U, if*

(a) *K is uniformly integrable; and*

(b) *for every $\varepsilon>0$, there exists a compact set $C_\varepsilon\subseteq X$, such that for every $u\in K$ there exists a Lebesgue measurable set $A_{\varepsilon,u}\subseteq T$, such that*

$$\lambda^1\big(T\setminus A_{\varepsilon,u}\big) \;<\; \varepsilon$$

and

$$u(t) \;\in\; C_\varepsilon \qquad \forall\, t\in A_{\varepsilon,u}$$

(here λ^1 stands for the Lebesgue measure on T).

REMARK 2.3.36 Since the Lebesgue measure λ^1 is nonatomic (see Theorem A.2.5 and Remark A.2.6), the uniform integrability property implies that K is bounded. Also if $K\subseteq L^1(T;X)$ has property U, then K is relatively w-compact (see Bourgain (1979)).

THEOREM 2.3.37
If $T=[0,b]$, X is a Banach space and $K\subseteq L^1(T;X)$ has property U, then the weak $L^1(T;X)$-topology and $\|\cdot\|_w$-norm topology on K coincide. Moreover, K is relatively $\|\cdot\|_w$-compact.

PROOF For every $n\geqslant 1$, let $C_{\frac{1}{n}}\subseteq X$ be the compact set postulated by Definition 2.3.35. The set

$$C \;\stackrel{df}{=}\; \bigcup_{n=1}^{\infty} C_{\frac{1}{n}}$$

is separable in X and note that

$$u(t) \;\in\; C \qquad \forall\, t\in T\setminus A_u,$$

with $\lambda^1(A_u) = 0$. So by replacing X by $\overline{\operatorname{span}} C$ if necessary, we may assume that X is a separable Banach space. Then the dual unit ball

$$\overline{B}_1^* \;=\; \big\{x^* \in X^* :\; \|x^*\|_{X^*} \leqslant 1\big\}$$

furnished with the relative w^*-topology is compact metrizable. Let

$$\{t_n\}_{n\geqslant 1} \;\subseteq\; T$$

be a dense set and consider a w^*-dense set $\{x_n^*\}_{n\geqslant 1} \subseteq \overline{B}_1^*$. The family

$$\Big\{\chi_{[t_m,t_k]} x_n^* :\; n,m,k \geqslant 1,\; t_m < t_k\Big\}$$

is countable and so it can be enumerated as $\{\varphi_i\}_{i\geqslant 1}$. We have

$$\|u\|_w \;=\; \sup_{i\geqslant 1} \left| \int_0^b \big\langle \varphi_i(t), u(t)\big\rangle_X \, dt \right|$$

(see Definition 2.3.35). Let $S\colon L^1(T;X) \longrightarrow l^\infty$ be the continuous, linear operator defined by

$$S(u) \;\stackrel{df}{=}\; \left\{ \int_0^b \big\langle \varphi_i(t), u(t)\big\rangle_X \, dt \right\}_{i\geqslant 1}.$$

Note that

$$\big\|S(u)\big\|_{l^\infty} \;=\; \|u\|_w\,.$$

We claim that $S(K)$ is relatively strongly compact in l^∞. First suppose that there exists a norm compact set $D \subseteq X$, such that

$$K \;\subseteq\; S_D^1 \;=\; \Big\{u \in L^1(T;X) :\; u(t) \in D \text{ for a.a. } t \in T\Big\}.$$

Let $\{e_i\}_{i\geqslant 1}$ be the standard basis in l^1 and $C(D)$ the space of continuous $\mathbb{R}$-valued functions on D. Let $\widehat{S}\colon l^1 \longrightarrow L^1\big(T;C(D)\big)$ be defined by

$$\widehat{S}(e_i) \;\stackrel{df}{=}\; \varphi_i \qquad \forall\; i \geqslant 1$$

and on all of l^1 by linearity and continuity. Using Theorem 2.3.6, we can see that $\{\varphi_i\}_{i\geqslant 1} \subseteq L^1\big(T;C(D)\big)$ is relatively norm-compact. Hence $\widehat{S}$ is a compact operator and then by Schauder's theorem, the adjoint operator

$$\widehat{S}^*\colon L^1\big(T;C(D)\big)^* = L^\infty\big(T;M(D)_{w^*}\big) \;\longrightarrow\; l^\infty$$

is compact (here by $M(D)_{w^*}$ we denote the space of Radon measure furnished with w^*-topology; recall that by the Riesz-Markov theorem, $C(D)^* = M(D)$;

see Theorem A.3.25). For every $g \in L^\infty\big(T; M(D)_{w^*}\big) = L^1\big(T; C(D)\big)^*$, we have

$$\widehat{S}^*(g) \;=\; \left\{ \int_0^b \big\langle g(t), \varphi_i(t) \big\rangle_{C(D)}\, dt \right\}_{i \geqslant 1},$$

where

$$\big\langle g(t), \varphi_i(t) \big\rangle_{C(D)} \;=\; \int_D \varphi_i(t)(x)\, dg(t)(x),$$

for the measure $g(t) \in M(D)$. If for every $t \in T$, $g(t)$ is Dirac measure concentrated on $u(t) \in D \subseteq X$, then

$$g \;\in\; L^\infty\big(T; M(D)_{w^*}\big)$$

and

$$\widehat{S}^*(g) \;=\; \left\{ \int_0^b \big\langle \varphi_i(t), u(t) \big\rangle_X\, dt \right\} \;=\; S(u).$$

Therefore we see that the action of the operator $\widehat{S}$ on K can be identified with the action of the operator $\widehat{S}^*$ and so $S(K) \subseteq l^\infty$ is relatively compact.

Now we pass to the general case and assume that K has property U. For $\varepsilon > 0$ consider the set

$$K_\varepsilon \;\stackrel{df}{=}\; \Big\{ \chi_{A_{\varepsilon,u}} u : \; u \in K \Big\},$$

where $A_{\varepsilon,u} \subseteq T$ is the Lebesgue measurable set postulated by Definition 2.3.35. By virtue of the uniform integrability of K, for each $\delta > 0$, we can find $\varepsilon > 0$, such that

$$\inf_{v \in K_\varepsilon} \|u - v\|_1 \;<\; \delta \qquad \forall\, u \in K.$$

Note that $\|\cdot\|_w \leqslant \|\cdot\|_1$. Therefore from the definition of the operator S, we have

$$\inf_{y \in S(K_\varepsilon)} \big\|S(u) - y\big\|_{l^\infty} \;<\; \delta.$$

But the set $S(K_\varepsilon) \subseteq l^\infty$ is relatively norm-compact. Hence $S(K) \subseteq l^\infty$ is relatively norm-compact. Since $S \in \mathcal{L}\big(L^1(T;K); l^\infty\big)$, we also have that S is weak-to-weak continuous. The weak and norm topologies coincide on $S(K)$. Thus $S|_K$ is weak-to-norm continuous. Recall that because K has property U, it is relatively w-compact (see Remark 2.3.36). Since without any loss of generality we may assume that K is convex, the linear map $S\colon K \longrightarrow S(K)$ is a weak-to-norm homeomorphism. The norm topology of l^∞ on $S(K)$ is the norm topology of $\big(L^1(T;X), \|\cdot\|_{w^*}\big)$ on K. Therefore the set K is relatively $\|\cdot\|_w$-compact and the proof of the theorem is finished. ∎

Next we ask the question when a weakly convergent sequence in L^p is strongly convergent. If $p \in (1, +\infty)$ and X is uniformly convex (hence reflexive; see Remark A.3.22), the Lebesgue-Bochner space $L^p(\Omega; X)$ is uniformly convex and so we have the Kadec-Klee property which says that if

$$u_n \xrightarrow{w} u \quad \text{in } L^p(\Omega; X)$$

and

$$\|u_n\|_p \longrightarrow \|u\|_p,$$

then

$$u_n \longrightarrow u \quad \text{in } L^p(\Omega; X).$$

This is no longer true for $L^1(\Omega; X)$. The next proposition illustrates the difference between weak and strong convergence in $L^1(\Omega)$. A sequence $\{u_n\}_{n \geqslant 1} \subseteq L^1(\Omega)$ which converges weakly but not strongly oscillates violently around its weak limit.

PROPOSITION 2.3.38
If (Ω, Σ, μ) is a finite measure space, $\{u_n\}_{n \geqslant 1} \subseteq L^1(\Omega)$, $u_n \xrightarrow{w} u$ in $L^1(\Omega)$ and

$$u(\omega) \leqslant \liminf_{n \to +\infty} u_n(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then $u_n \longrightarrow u$ in $L^1(\Omega)$.

PROOF Without any loss of generality, we may assume that $u = 0$. From Theorem 2.3.24, we know that the sequence $\{u_n\}_{n \geqslant 1} \subseteq L^1(\Omega)$ is uniformly integrable. So given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$, such that if $A \in \Sigma$, $\mu(A) < \delta$, then

$$\int_A |u_n| \, d\mu < \varepsilon \qquad \forall\, n \geqslant 1.$$

For every $N \geqslant 1$, let

$$\Omega_N \stackrel{df}{=} \left\{ \omega \in \Omega : \inf_{n \geqslant N} u_n(\omega) \geqslant -\frac{\varepsilon}{\mu(\Omega)} \right\}.$$

Because of our hypothesis and since we have assumed that $u = 0$, we can find $N \geqslant 1$ large so that $\mu\big(\Omega_N^c\big) < \delta$. Also since

$$u_n \xrightarrow{w} u = 0 \quad \text{in } L^1(\Omega),$$

we can find $N_1 \geqslant N$, such that for all $n \geqslant N_1$, we have

$$\left| \int_{\Omega_N} u_n \, d\mu \right| < \varepsilon.$$

So for all $n \geqslant N_1$, we have

$$\begin{aligned}\int_\Omega |u_n|\,d\mu &= \int_{\Omega_N} |u_n|\,d\mu + \int_{\Omega_N^c} |u_n|\,d\mu \\ &\leqslant \int_{\Omega_N} \left|u_n + \frac{\varepsilon}{\mu(\Omega)}\right| d\mu + \int_{\Omega_N} \frac{\varepsilon}{\mu(\Omega)}\,d\mu + \int_{\Omega_N^c} |u_n|\,d\mu \leqslant 4\varepsilon,\end{aligned}$$

so

$$\int_\Omega |u_n|\,d\mu \longrightarrow 0 \quad \text{as } n \to +\infty,$$

i.e.,

$$u_n \longrightarrow u = 0 \quad \text{in } L^1(\Omega).$$

□

When we deal with $\mathbb{R}^N$-valued functions, an extremality condition replaces the inequality hypothesis in the previous proposition. The result is due to Visintin (1984), where the reader can find the proof.

PROPOSITION 2.3.39
If (Ω, Σ, μ) is a finite measure space, $\{f_n\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ is a sequence such that

$$f_n \xrightarrow{w} f \quad \text{in } L^1\big(\Omega;\mathbb{R}^N\big),$$

for some $f \in L^1\big(\Omega;\mathbb{R}^N\big)$ and

$$f(\omega) \in \operatorname{ext}\left(\overline{\operatorname{conv}} \limsup_{n\to+\infty} \{f_n(\omega)\}\right) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then

$$f_n \longrightarrow f \quad \text{in } L^1\big(\Omega;\mathbb{R}^N\big).$$

We conclude this section with a brief look at the space of Radon measures, which appears in applications (such as optimal control, game theory, mathematical economics etc.) and also is useful in the study of Sobolev spaces (see Section 2.4).

So let Z be a locally compact, σ-compact metric space. We consider the following three spaces of continuous functions on Z:

$$\begin{aligned}C_c(Z) &\stackrel{df}{=} \big\{u\colon Z \longrightarrow \mathbb{R} \text{ continuous with compact support}\big\},\\ C_0(Z) &\stackrel{df}{=} \big\{u\colon Z \longrightarrow \mathbb{R} \text{ continuous and vanishes at infinity,}\\ &\qquad \text{i.e., for all } \varepsilon > 0 \text{ there exists a compact set } K_\varepsilon \subseteq Z,\\ &\qquad \text{such that } \big|u(z)\big| < \varepsilon \text{ for all } z \notin K_\varepsilon\big\},\\ C_b(Z) &\stackrel{df}{=} \big\{u\colon Z \longrightarrow \mathbb{R} \text{ continuous and bounded}\big\}.\end{aligned}$$

Evidently we have the following inclusions:

$$C_c(Z) \subseteq C_0(Z) \subseteq C_b(Z).$$

If Z is compact, then these three spaces coincide. If Z is not compact, each inclusion is strict. We can define a norm on $C_b(Z)$ by setting

$$\|u\|_{C_b(Z)} \stackrel{df}{=} \|u\|_\infty = \sup_{z\in Z} |u(z)|.$$

By restriction, this norm also passes to the spaces $C_c(Z)$ and $C_0(Z)$.

PROPOSITION 2.3.40
The space $C_b(Z)$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space. The space $C_0(Z)$ is a closed subspace of this Banach space (hence itself a Banach space). The space $C_c(Z)$ is $\|\cdot\|_\infty$-dense in $C_0(Z)$.

PROOF The first two statements are obvious. Only the third requires some work. Since Z is a locally compact, σ-compact metric space, $Z = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n\geqslant 1}$ is a sequence of compact sets with $C_n \subseteq \operatorname{int} C_{n+1}$ for all $n \geqslant 1$. Let $\{\vartheta_n\}_{n\geqslant 1}$ and $\{\xi_n\}_{n\geqslant 1}$ be continuous partitions of unit subordinate to the open covers $\{\operatorname{int} C_n\}_{n\geqslant 1}$ and $\{C_n^c\}_{n\geqslant 1}$ respectively. We have $\vartheta_n + \xi_n = 1$ on Z and so $\vartheta_n = 1$ on C_n for $n \geqslant 1$. Let $u \in C_0(Z)$ and set $u_n \stackrel{df}{=} \vartheta_n u$. Then

$$u_n \in C_c(Z) \quad \text{and} \quad \|u - u_n\|_\infty = \|\xi_n u\|_\infty \qquad \forall\, n \geqslant 1.$$

Since $\operatorname{supp} \xi_n \subseteq C_n^c$ and

$$u(z) \longrightarrow 0 \quad \text{as } z \text{ tends to infinity}$$

(the one point Alexandrov compactification of Z; see Theorem A.1.3 and Remark A.1.4), we conclude that

$$\|u - u_n\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty.$$

□

Also by $M(Z)$ we denote the space of all signed measures $m\colon \mathcal{B}(Z) \longrightarrow \mathbb{R}$ (with $\mathcal{B}(Z)$ being the Borel σ-field of Z) that have bounded variation. Since Z is a metric space such measures are regular. The measures in $M(Z)$ are known as ***Radon measures***.

The Riesz-Markov representation theorem says that $M(Z)$ is the dual space of $C_0(Z)$.

THEOREM 2.3.41 (Riesz-Markov Representation Theorem) *If X is a locally compact, σ-compact metric space, then $C_0(Z)^* = M(Z)$ and the duality pairing is given by*

$$\langle \mu, u \rangle_{C_0(Z)} = \int_Z u(z)\, d\mu \qquad \forall\ u \in C_0(Z),\ \mu \in M(Z).$$

Using the three spaces of continuous functions on Z introduced earlier, we can define three different notions of convergence for sequences of Radon measures.

DEFINITION 2.3.42 *Let Z be a locally compact, σ-compact metric space and $\{\mu_n\}_{n\geqslant 1} \subseteq M(Z)$.*

(a) *We say that the sequence $\{\mu_n\}_{n\geqslant 1}$* ***converges vaguely*** *to $\mu \in M(Z)$ if and only if*

$$\int_Z u(z)\, d\mu_n \longrightarrow \int_Z u\, d\mu \qquad \forall\ u \in C_c(Z).$$

(b) *We say that the sequence $\{\mu_n\}_{n\geqslant 1}$* ***converges weakly*** *to $\mu \in M(Z)$ if and only if*

$$\int_Z u(z)\, d\mu_n \longrightarrow \int_Z u\, d\mu \qquad \forall\ u \in C_0(Z).$$

We denote this convergence by $\mu_n \xrightarrow{w} \mu$.

(c) *We say that the sequence $\{\mu_n\}_{n\geqslant 1}$* ***converges narrowly*** *to $\mu \in M(Z)$ if and only if*

$$\int_Z u(z)\, d\mu_n \longrightarrow \int_Z u\, d\mu \qquad \forall\ u \in C_b(Z).$$

We denote this convergence $\mu_n \xrightarrow{n} \mu$.

REMARK 2.3.43 Evidently we have that

- the norm convergence in $M(Z)$ implies the narrow convergence in $M(Z)$;
- the narrow convergence in $M(Z)$ implies the weak convergence in $M(Z)$;
- the weak convergence in $M(Z)$ implies the vague convergence in $M(Z)$.

In functional analytic terms the weak convergence is actually the weak*-convergence in the Banach space $M(Z)$ (see Theorem 2.3.41). The term weak convergence originates from probability theory. Also the term narrow convergence is the English translation of the term "convergence étroite" first used by Bourbaki (1969).

PROPOSITION 2.3.44
If Z is a locally compact, σ-compact metric space, $\{u_n\}_{n\geqslant 1} \subseteq C_0(Z)$ is a sequence and $u \in C_0(Z)$,
then

$$u_n \xrightarrow{w} u \quad \text{in } C_0(Z)$$

if and only if

$$\sup_{n\geqslant 1} \|u_n\|_\infty < \infty \quad \text{and} \quad u_n(z) \longrightarrow u(z) \qquad \forall\, z \in Z.$$

PROOF "$\Longrightarrow$": A weakly convergent sequence in a Banach space is bounded. So $\sup_{n\geqslant 1} \|u_n\|_\infty < +\infty$. Also if $\mu = \delta_z$ is the Dirac measure concentrated at $z \in Z$, then

$$\langle \delta_z, u_n \rangle \longrightarrow \langle \delta_z, u \rangle .$$

But

$$\langle \delta_z, u_n \rangle = u_n(z) \quad \text{and} \quad \langle \delta_z, u \rangle = u(z).$$

So

$$u_n(z) \longrightarrow u(z) \qquad \forall\, z \in Z.$$

"$\Longleftarrow$": This is an immediate consequence of the Lebesgue dominated convergence theorem (see Theorem A.2.2). □

PROPOSITION 2.3.45
If Z is a locally compact, σ-compact metric space and $\{\mu_n\}_{n\geqslant 1} \subseteq M(Z)$ is a sequence,
then

(a) *if*

$$\mu_n \longrightarrow \mu \quad \text{vaguely in } M(Z)$$

and for every $\varepsilon > 0$ there exists compact set $K_\varepsilon \subseteq Z$, such that

$$|\mu_n|\big(K_\varepsilon^c\big) < \varepsilon \qquad \forall\, n \geqslant n_0,$$

then

$$\mu_n \longrightarrow \mu \quad \text{narrowly in } M(Z);$$

(b) *if $\mu_n \geqslant 0$ for all $n \geqslant 1$,*

$$\mu_n \longrightarrow \mu \quad \text{vaguely in } M(Z),$$

and

$$\mu_n(Z) \longrightarrow \mu(Z),$$

then

$$\mu_n \longrightarrow \mu \quad \text{narrowly in } M(Z).$$

PROOF **(a)** Let $u \in C_b(Z)$ and $\varepsilon > 0$. Let $K_\varepsilon \subseteq Z$ be the compact set postulated by the hypotheses. We take

$$\xi_\varepsilon \in C_c(Z),$$

such that $\xi_\varepsilon|_{K_\varepsilon} = 1$. Evidently

$$u \;=\; \xi_\varepsilon u + v$$

with $\operatorname{supp} v \subseteq K_\varepsilon^c$. So we have

$$\int_Z u \, d\mu_n \;=\; \int_Z \xi_\varepsilon u \, d\mu_n + \int_Z v \, d\mu_n.$$

Since $\mu_n \longrightarrow \mu$ vaguely in $M(Z)$ and $\xi_\varepsilon u \in C_c(Z)$, we have

$$\int_Z \xi_\varepsilon u \, d\mu_n \;\longrightarrow\; \int_Z \xi_\varepsilon u \, d\mu.$$

Also

$$\left| \int_Z v \, d\mu_n \right| \;=\; \left| \int_{K_\varepsilon^c} v \, d\mu_n \right| \;\leqslant\; \|v\|_\infty \, |\mu_n|\big(K_\varepsilon^c\big) \;\leqslant\; \varepsilon \, \|u\|_\infty .$$

So we obtain

$$\limsup_{n \to +\infty} \int_Z u \, d\mu_n \;\leqslant\; \int_Z \xi_\varepsilon u \, d\mu + \varepsilon \, \|u\|_\infty \tag{2.30}$$

and

$$\liminf_{n \to +\infty} \int_Z u \, d\mu_n \;\geqslant\; \int_Z \xi_\varepsilon u \, d\mu - \varepsilon \, \|u\|_\infty .$$

Since $\varepsilon > 0$ was arbitrary and $\xi_\varepsilon \longrightarrow 1$, we obtain that

$$\int_Z u \, d\mu_n \;\longrightarrow\; \int_Z u \, d\mu,$$

i.e.,

$$\mu_n \;\longrightarrow\; \mu \quad \text{narrowly in } M(Z).$$

(b) Every measure $\mu' \in M(Z)$, $\mu' \geqslant 0$ is tight. So given $\varepsilon > 0$, we can find a compact set $K_\varepsilon \subseteq Z$, such that $\mu'\big(K_\varepsilon^c\big) < \varepsilon$. Let $u \in C_c(Z)$ be such that

$$\operatorname{supp} u \;\subseteq\; K_\varepsilon, \quad 0 \;\leqslant\; u \;\leqslant\; 1 \quad \text{and} \quad \|\mu\|_* - \varepsilon \;<\; \int_Z u \, d\mu.$$

Let $n_0 \geqslant 1$ be such that

$$\big|\mu_n(Z) - \mu(Z)\big| \;<\; \varepsilon \quad \text{and} \quad \left|\int_Z u\,d\mu_n - \int_Z u\,d\mu\right| \;<\; \varepsilon \qquad \forall\, n \geqslant n_0. \tag{2.31}$$

Then for $n \geqslant n_0$, from (2.31), we have

$$\mu_n\big(K_\varepsilon^c\big) \;\leqslant\; \|\mu_n\|_* - \int_Z u\,d\mu_n + \varepsilon - \int_Z u\,d\mu \;\leqslant\; \|\mu_n\|_* - \|\mu\|_* + 2\varepsilon \;<\; 3\varepsilon.$$

So, from part **(a)**, we conclude that

$$\mu_n \;\longrightarrow\; \mu \text{ narrowly in } M(Z).$$

□

We have a compactness result for the weak convergence of Radon measures.

THEOREM 2.3.46
If Z is a locally compact, σ-compact metric space and $\{\mu_n\}_{n\geqslant 1} \subseteq M(Z)$ is bounded,
then there is a subsequence $\{\mu_{n_k}\}_{k\geqslant 1}$ of $\{\mu_n\}_{n\geqslant 1}$ and $\mu \in M(Z)$, such that

$$\mu_{n_k} \;\xrightarrow{w}\; \mu \quad \text{as } k \to +\infty.$$

PROOF Let Z^* be the Alexandrov one-point compactification of Z (see Theorem A.1.3 and Remark A.1.4). Then Z^* is a compact metrizable space and so $C(Z^*)$ is a separable Banach space. Set

$$E \;\stackrel{df}{=}\; \big\{u \in C(Z^*) :\; u(\infty) = 0\big\}.$$

Then this is a closed subspace of $C(Z^*)$; thus E is a separable Banach space too. For every $u \in E$, let $\widehat{u}$ denote the restriction of u to Z. Consider the linear map $L\colon E \longrightarrow C_b(Z)$ defined by

$$L(u) \;\stackrel{df}{=}\; \widehat{u} \qquad \forall\, u \in E.$$

We claim that L is an isometry of E onto $C_0(Z)$. To this end, let $u \in E$. Since u is continuous at $+\infty$, then for every $\varepsilon > 0$, there exists a compact set K_ε, such that

$$\big|u(z) - u(\infty)\big| \;<\; \varepsilon \qquad \forall\, z \in K_\varepsilon^c.$$

This means that $\widehat{u} \in C_0(Z)$. On the other hand let $v \in C_0(Z)$. Then v can be extended to Z^* by setting

$$v_1(\infty) \;=\; 0$$

and

$$v_1(z) \;=\; v(z) \qquad \forall\, z \in Z.$$

Since $v \in C_0(Z)$, we see that

$$v_1 \in C(Z^*)$$

and so $v_1 \in E$. This isometry shows that $C_0(Z)$ is separable. Then the weak*-topology on bounded subsets of $M(Z) = C_0(Z)^*$ is compact (by Alaoglu's theorem; see Theorem A.3.9) and metrizable. This proves the theorem. □

REMARK 2.3.47 Using the compactification technique of the previous proof we can show that if

$$G \stackrel{df}{=} \big\{\mu \in M(Z^*) :\; \mu\big(\{\infty\}\big) = 0\big\}$$

and $S\colon G \longrightarrow M(Z)$ is defined by

$$S(\mu) \stackrel{df}{=} \widehat{\mu} \qquad \forall\, \mu \in M(Z^*)$$

with

$$\widehat{\mu}(A) \;=\; \mu(A) \qquad \forall\, A \in \mathcal{B}(Z) \subseteq \mathcal{B}(Z^*),$$

then S is an isometry of G onto $M(Z)$.

When Z is compact, then Theorem 2.3.46 can be strengthened. In what follows by $M(Z)_+$ we denote the elements μ of $M(Z)$ for which we have $\mu \geqslant 0$ (i.e., they are measures).

THEOREM 2.3.48
If Z is a compact metric space and $\{\mu_n\}_{n\geqslant 1} \subseteq M(Z)_+$ is such that

$$\|\mu_n\| \;=\; r \qquad \forall\, n \geqslant 1,$$

then there exists a subsequence $\{\mu_{n_k}\}_{k\geqslant 1}$ of $\{\mu_n\}_{n\geqslant 1}$ and $\mu \in M(Z)_+$ with $\|\mu\| = r$, such that

$$\mu_{n_k} \stackrel{w}{\longrightarrow} \mu \quad \text{as } k \to +\infty,$$

i.e., the set

$$S_R^+ = \big\{\mu \in M(Z)_+ :\; \|\mu\| = r\big\}$$

is w^*-sequentially compact).

We conclude with a result for sequences of functions which converge simultaneously pointwise and weakly in $L^p(\Omega)$ ($p \in [1, +\infty)$). This result can be viewed as a refinement of Fatou's Lemma.

PROPOSITION 2.3.49
If (Ω, Σ, μ) *is a finite measure space,* $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega)$ $(p \in [1, +\infty))$,

$$u_n \xrightarrow{w} u \quad \text{in } L^p(\Omega)$$

and

$$u_n(\omega) \longrightarrow u(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then

$$\lim_{n\to+\infty} \left(\|u_n\|_p^p - \|u_n - u\|_p^p \right) = \|u\|_p^p.$$

PROOF For a given $\varepsilon > 0$, we can find $c(\varepsilon) > 0$, such that for all $a, b \in \mathbb{R}$, we have

$$||a+b|^p - |a|^p| \leqslant \varepsilon|a|^p + c(\varepsilon)|b|^p. \tag{2.32}$$

We set

$$h_n^\varepsilon \stackrel{df}{=} \left(||u_n|^p - |u_n - u|^p - |u|^p| - \varepsilon|u_n - u|^p\right)^+.$$

Evidently we have

$$h_n^\varepsilon(\omega) \longrightarrow 0 \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and from (2.32),

$$h_n^\varepsilon(\omega) \leqslant \big(1 + c(\varepsilon)\big)\big|u(\omega)\big|^p.$$

So from the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\int_\Omega h_n^\varepsilon \, d\mu \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{2.33}$$

But note that

$$|u_n|^p - |u_n - u|^p - |u|^p \leqslant h_n^\varepsilon + \varepsilon|u_n - u|^p \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Hence, using (2.33), we obtain

$$\limsup_{n\to+\infty} \int_\Omega ||u_n|^p - |u_n - u|^p - |u|^p| \, d\mu \leqslant M\varepsilon,$$

where

$$M \stackrel{df}{=} \sup_{n\geqslant 1} \|u_n - u\|_p^p.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{n\to+\infty} \left(\|u_n\|_p^p - \|u_n - u\|_p^p \right) = \|u\|_p^p.$$

□

REMARK 2.3.50 Note that if $p = 2$, then we do not need the μ-almost everywhere pointwise convergence of the sequence $\{u_n\}_{n\geqslant 1}$ to u.

2.4 Sobolev Spaces

Already in Section 1.6, we introduced the Sobolev space $W^{1,p}(Z)$ (see Definition 1.6.1). Here we introduce Sobolev spaces of any order $m \geqslant 1$ and conduct a systematic study of them, proving among other things those results stated in Section 1.6 without a proof.

Let us start by fixing the notation. An element $\alpha = (\alpha_k)_{k=1}^N \in \mathbb{N}^N$ is said to be a ***multi-index***. Associated to a multi-index α, we have the following symbols:

$$|\alpha| \stackrel{df}{=} \sum_{k=1}^N \alpha_k$$

the length of α, and

$$z^\alpha = z_1^{\alpha_1} \dots z_N^{\alpha_N} \qquad \forall\ z = (z_k)_{k=1}^N \in \mathbb{R}^N.$$

We say that two multi-indices α, β are related by $\alpha \leqslant \beta$, if $\alpha_k \leqslant \beta_k$ for all $k \in \{1, \dots, N\}$. Finally we set

$$D_k \stackrel{df}{=} \frac{\partial}{\partial z_k} \qquad \forall\ k \in \{1, \dots, N\}$$

and

$$D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N} \stackrel{df}{=} \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N}}.$$

DEFINITION 2.4.1 *Let $Z \subseteq \mathbb{R}^N$ be an open set. By $\mathcal{D}(Z)$ we denote the space $C_c^\infty(Z)$ (the space of $C^\infty(Z)$ functions with compact support) equipped with the following convergence notion:*

> *"the sequence $\{\vartheta_n\}_{n\geqslant 1} \subseteq C_c^\infty(Z)$ is said to converge to 0, if there exists a fixed compact set $K \subseteq Z$, such that $\operatorname{supp} \vartheta_n \subseteq K$ for all $n \geqslant 1$ and $\{D^\alpha \vartheta_n\}_{n\geqslant 1}$ converges uniformly to 0 for all $\alpha \in \mathbb{N}^N$."*

*The elements of the space $\mathcal{D}(Z)$ are called **test functions**.*

A linear functional $T\colon \mathcal{D}(Z) \longrightarrow \mathbb{R}$, such that

$$\vartheta_n \longrightarrow 0 \quad \text{in } \mathcal{D}(Z),$$

implies

$$T(\vartheta_n) \longrightarrow 0$$

*is called a **distribution**. The space of distributions is denoted by $\mathcal{D}(Z)^*$.*

REMARK 2.4.2 The convergence notion introduced on $C_c^\infty(Z)$ is actually topological, i.e., corresponds to a topology on $C_c^\infty(Z)$. Therefore $\mathcal{D}(Z)^*$ is the dual of the space of test functions. Recall that $\mathcal{D}(Z)$ is dense in $L^p(\Omega)$ for all $p \in [1, +\infty)$. If $u \in L^1_{\text{loc}}(Z)$ and $T_u \colon \mathcal{D}(Z) \longrightarrow \mathbb{R}$ is defined by

$$T_u(\vartheta) \stackrel{df}{=} \int_\Omega u\vartheta \, dz \qquad \forall\ \vartheta \in \mathcal{D}(Z),$$

then $T_u \in \mathcal{D}(Z)^*$. Moreover, if $u, v \in L^1_{\text{loc}}(Z)$ and

$$u \ = \ v \quad \text{for a.a. } z \in Z,$$

then $T_u = T_v$. In particular, if

$$u(z) \ = \ 0 \quad \text{for a.a. } z \in Z,$$

it defines the zero distribution. In fact the converse is also true. If $T_u = 0$, then $u(z) = 0$ for almost all $z \in Z$, provided that $u \in L^1_{\text{loc}}(Z)$. Distributions resulting from locally integrable functions are usually called ***regular distributions***. Another important distribution is the ***Dirac δ-function***; namely for $z \in Z$, we define

$$\delta_z(\vartheta) \stackrel{df}{=} \vartheta(z) \qquad \forall\ \vartheta \in \mathcal{D}(Z).$$

This distribution is not regular.

DEFINITION 2.4.3 *For every distribution $T \in \mathcal{D}(Z)^*$ and every $\alpha \in \mathbb{N}^N$, the distribution $D^\alpha T$ is defined by*

$$D^\alpha T(\vartheta) \stackrel{df}{=} (-1)^{|\alpha|} T\big(D^\alpha \vartheta\big) \qquad \forall\ \vartheta \in \mathcal{D}(Z).$$

*Then $D^\alpha T$ is the **derivative** of order α of the distribution T. For given two functions $u, v \in L^1_{\text{loc}}(Z)$ and $\alpha \in \mathbb{N}^N$, we write*

$$v \ = \ D^\alpha u$$

to express the fact that $D^\alpha T_u = T_v$. This is equivalent to saying that

$$\int_\Omega v\vartheta \, dz \ = \ (-1)^{|\alpha|} \int_\Omega u D^\alpha \vartheta \, dz.$$

The function $v = D^\alpha u$ is the derivative of order α in the sense of distributions of the function u.

REMARK 2.4.4 If $u \in C^{|\alpha|}(Z)$, then the distributional derivative $\mathcal{D}^\alpha u$ coincides with the classical partial derivative $\dfrac{\partial^{|\alpha|} u}{\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N}}$.

Now we are ready to give the definition of Sobolev space.

DEFINITION 2.4.5 *Let $Z \subseteq \mathbb{R}^N$ be an open set. The Sobolev space $W^{m,p}(Z)$ for $m \in \mathbb{N}_0$, $p \in [1, +\infty]$, is defined by*

$$W^{m,p}(Z) \stackrel{df}{=} \big\{u \in L^p(Z) : \ D^\alpha u \in L^p(Z) \text{ for all } \alpha \in \mathbb{N}^N \text{ with } |\alpha| \leqslant m\big\}.$$

For every $u \in W^{m,p}(Z)$, we define

$$\|u\|_{W^{m,p}(Z)} \stackrel{df}{=} \left(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} \quad \text{if } p \in [1, +\infty)$$

and

$$\|u\|_{W^{m,\infty}(Z)} \stackrel{df}{=} \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_\infty .$$

Clearly this is a norm on $W^{m,p}(Z)$.

Finally we set

$$W_0^{m,p}(Z) \stackrel{df}{=} \overline{\mathcal{D}(Z)}^{\|\cdot\|_{W^{m,p}(Z)}},$$

for $p \in [1, +\infty)$.

REMARK 2.4.6 Evidently

$$u_n \longrightarrow u \quad \text{in } W^{m,p}(Z)$$

if and only if for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant m$, we have

$$D^\alpha u_n \longrightarrow D^\alpha u \quad \text{in } L^p(Z).$$

Let

$$r \stackrel{df}{=} \text{card}\big\{\alpha : \ \alpha \text{ is multi-index, } |\alpha| \leqslant m\big\}$$

and consider the map

$$L\colon W^{m,p}(Z) \longrightarrow \big(L^p(Z)\big)^r,$$

defined by

$$L(u) \stackrel{df}{=} \big(D^\alpha u\big)_{|\alpha| \leqslant m} \qquad \forall\ u \in W^{m,p}(Z).$$

It is easily seen that L is an isometric isomorphism. Based on this observation, we can state the following result.

PROPOSITION 2.4.7

The spaces $\big(W^{m,p}(Z), \|\cdot\|_{W^{m,p}(Z)}\big)$ (with $p \in [1, +\infty]$, $m \in \mathbb{N}_0$) are Banach spaces, which are separable for $p \in [1, +\infty)$, reflexive and uniformly convex for $p \in (1, +\infty)$.

COROLLARY 2.4.8
For every $m \in \mathbb{N}_0$ and $p \in [1, +\infty]$, the space $W_0^{m,p}(Z)$ is a closed subspace of $W^{m,p}(Z)$.

PROOF We need to show that

$$W_0^{m,p}(Z) \subseteq W^{m,p}(Z).$$

So let $u \in W_0^{m,p}(Z)$ and let $\{\vartheta_n\}_{n\geqslant 1} \subseteq \mathcal{D}(Z)$ be such that

$$\vartheta_n \longrightarrow u \quad \text{in } W^{m,p}(Z).$$

From Proposition 2.4.7, it follows that $u \in W^{m,p}(Z)$. □

REMARK 2.4.9 The space $p = 2$ is important and we reserve a special notation for it. We set

$$H^m(Z) \stackrel{df}{=} W^{m,2}(Z) \quad \text{and} \quad H_0^m(Z) \stackrel{df}{=} W_0^{m,2}(Z).$$

For $u, v \in H^m(Z)$, we define

$$(u,v)_{H^m(Z)} \stackrel{df}{=} \sum_{|\alpha|\leqslant m} (D^\alpha u, D^\alpha v)_{L^2(Z)} = \sum_{|\alpha|\leqslant m} \int_\Omega D^\alpha u \, D^\alpha v \, dz.$$

Clearly $(\cdot,\cdot)_{H^m(Z)}$ defines an inner product on $H^m(Z)$ which generates the norm $\|\cdot\|_{W^{m,2}(Z)}$. From Proposition 2.4.7, it follows that $H^m(Z)$ and $H_0^m(Z)$ are Hilbert spaces.

From now on $m = 1$. So we examine the first Sobolev spaces $W^{1,p}(Z)$ and $W_0^{1,p}(Z)$, with $p \in [1, +\infty]$. Next we derive ways to approximate the elements of the Sobolev space $W^{1,p}(Z)$ by smooth functions. For this purpose we introduce certain regularizing sequences known as ***mollifiers***.

DEFINITION 2.4.10 *Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi \geqslant 0$ be such that*

$$\operatorname{supp}\varphi \subseteq \left\{z \in \mathbb{R}^N : \|z\|_{\mathbb{R}^N} \leqslant 1\right\} \quad \text{and} \quad \int_{\mathbb{R}^N} \varphi(z)\, dz = 1.$$

A possible choice is the function

$$\varphi(z) \stackrel{df}{=} \begin{cases} c \exp\left(\frac{1}{\|z\|_{\mathbb{R}^N}^2 - 1}\right) & \text{if} \quad \|z\|_{\mathbb{R}^N} < 1, \\ 0 & \text{if} \quad \|z\|_{\mathbb{R}^N} \geqslant 1, \end{cases}$$

with $c > 0$ chosen in such a way so that

$$\int_{\mathbb{R}^N} \varphi(z)\, dz = 1.$$

If $\varepsilon > 0$, we define

$$\varphi_\varepsilon(z) \;=\; \frac{1}{\varepsilon^N}\varphi\left(\frac{z}{\varepsilon}\right).$$

Then $\varphi_\varepsilon \in C_c^\infty\big(\mathbb{R}^N\big)$, $\varphi_\varepsilon \geqslant 0$ is such that

$$\operatorname{supp}\varphi_\varepsilon \;\subseteq\; \big\{z \in \mathbb{R}^N :\; \|z\|_{\mathbb{R}^N} \leqslant \varepsilon\big\} \quad \textit{and} \quad \int\limits_{\mathbb{R}^N} \varphi_\varepsilon(z)\,dz \;=\; 1.$$

The function φ_ε is called a ***mollifier*** *and given $u \in L^1_{\rm loc}(Z)$, the* ***mollification*** *(or* ***regularization****) of u corresponding to $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is given by*

$$u_\varepsilon(z) \;\stackrel{df}{=}\; \int\limits_{\mathbb{R}^N} u(z-y)\varphi_\varepsilon(y)\,dy \;=\; \int\limits_{\Omega} u(y)\varphi_\varepsilon(z-y)\,dy,$$

where we have extended u to all of $\mathbb{R}^N$ as zero (i.e., $u_\varepsilon = u \star \varphi_\varepsilon$ with $\star$ denoting the convolution operation).

REMARK 2.4.11 Note that

$$\operatorname{supp} u_\varepsilon \;\subseteq\; \operatorname{supp} u + \big\{z \in \mathbb{R}^N :\; \|z\|_{\mathbb{R}^N} \leqslant \varepsilon\big\}.$$

The next proposition summarizes the approximations achieved via mollification.

PROPOSITION 2.4.12
If $Z \subseteq \mathbb{R}^N$ is an open set,
then

(a) *for every $u \in L^1_{\rm loc}(Z)$ and every $\varepsilon > 0$, $u_\varepsilon \in C^\infty\big(Z_{-\varepsilon}\big)$, where $Z_{-\varepsilon} \stackrel{df}{=} \big\{z \in Z :\; d_Z(z, \partial Z) > \varepsilon\big\}$;*

(b) *if $u \in C(Z)$, then*

$$u_\varepsilon \;\longrightarrow\; u \quad \textit{as } \varepsilon \searrow 0$$

uniformly on compact subsets of Z;

(c) *if $u \in L^p_{\rm loc}(Z)$ for some $p \in [1, +\infty)$, then*

$$u_\varepsilon \;\longrightarrow\; u \quad \textit{in } L^p_{\rm loc}(Z);$$

(d) *if $u \in W^{1,p}(Z)$, $p \in [1, +\infty]$, then*

$$D_i u_\varepsilon \;=\; \varphi_\varepsilon \star D_i u \qquad \forall\; i \in \{1, \ldots, N\};$$

(e) *if $u \in W^{1,p}(Z)$, $p \in [1, +\infty)$, then*

$$u_\varepsilon \;\longrightarrow\; u \quad \textit{in } W^{1,p}(Z).$$

PROOF **(a)** Note that u_ε is defined on $Z_{-\varepsilon}$ (see Remark 2.4.11). Let $z \in Z_{-\varepsilon}$, $i \in \{1, \dots, N\}$ and $e_1, \dots, e_N$ be the canonical basis of $\mathbb{R}^N$. For $|t|$ small enough, we have that

$$z + te_i \ \in \ Z_{-\varepsilon} \qquad \forall\, i \in \{1, \dots, N\}.$$

So if $t_k \longrightarrow 0$, we can assume that

$$z + t_k e_i \ \in \ Z_{-\varepsilon} \qquad \forall\, k \geqslant 1.$$

We set

$$h(z, y) \ = \ \frac{1}{\varepsilon^N} \varphi\left(\frac{z - y}{\varepsilon}\right) u(y)$$

and

$$f_k(z, y) \ = \ \frac{1}{t_k}\big(h(z + t_k e_i - y) - h(z, y)\big).$$

We have

$$\frac{1}{t_k}\big(u_\varepsilon(z + t_k e_i) - u_\varepsilon(z)\big) \ = \ \int\limits_{Z_{-\varepsilon}} f_k(z, y)\, dy.$$

Note that

$$f_k(z, y) \ \longrightarrow \ \frac{\partial \varphi_\varepsilon}{\partial z_i}(z - y) \quad \text{as } k \to +\infty, \qquad \forall\, y \in Z_{-\varepsilon}.$$

Moreover, by the mean value theorem , we have

$$\big|f_k(z, y)\big| \ \leqslant \ \frac{1}{\varepsilon^{N+1}} \|D\varphi\|_\infty |u| \ \in \ L^1\big(Z_{-\varepsilon}\big).$$

So by the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\frac{\partial u_\varepsilon}{\partial z_i}(z) \ = \ \lim_{k \to +\infty} \frac{1}{t_k}\big(u_\varepsilon(z + t_k e_i) - u_\varepsilon(z)\big) \ = \ \int\limits_Z \frac{\partial \varphi_\varepsilon}{\partial z_i}(z - y) u(y)\, dy.$$

In a similar way we show that the partial derivatives of u_ε of all orders exist and are continuous on Z_ε. Therefore

$$u_\varepsilon \ \in \ C^\infty\big(Z_{-\varepsilon}\big).$$

Note that we have

$$\frac{\partial}{\partial z_i}\big(u \star \varphi_\varepsilon\big) \ = \ u \star \frac{\partial \varphi_\varepsilon}{\partial z_i} \qquad \forall\, i \in \{1, \dots, N\}.$$

(b) Let K be a compact subset of Z. Take $z \in K$ and set

$$x \ \stackrel{df}{=} \ \frac{y - z}{\varepsilon}, \quad y \in Z.$$

Note that

$$\varphi(-x) \;=\; \varphi(x).$$

Then we have

$$\big|u_\varepsilon(z) - u(z)\big| \;\leqslant\; \frac{1}{\varepsilon^N} \int\limits_{B_\varepsilon(z)} \varphi\left(\frac{z-y}{\varepsilon}\right) \big|u(y) - u(z)\big|\, dy \tag{2.34}$$

$$\leqslant \int\limits_{B_1(0)} \varphi(x) \big|u(z+\varepsilon x) - u(z)\big|\, dz \;\leqslant\; \xi(\varepsilon) \int\limits_{B_1(0)} \varphi(x)\, dx \;=\; \xi(\varepsilon),$$

where

$$\xi(\varepsilon) \;=\; \sup_{\substack{(y,\,v)\,\in\,\overline{K}_\varepsilon \times \overline{K}_\varepsilon \\ \|y-v\|\,\leqslant\,\varepsilon}} \big|u(y) - u(v)\big|.$$

Since u is uniformly continuous on the compact subsets of Z, we have that

$$\lim_{\varepsilon \searrow 0} \xi(\varepsilon) \;=\; 0.$$

So from (2.34), we conclude that

$$u_\varepsilon \;\longrightarrow\; u \quad \text{as } \varepsilon \searrow 0 \quad \text{uniformly on compact subsets of } Z.$$

(c) Let K be a compact subset of Z and $0 < \varepsilon < d\big(K, \partial Z\big)$. Then for $z \in K$, we have

$$\big|u_\varepsilon(z)\big|^p \;\leqslant \int\limits_{B_1(0)} \vartheta(y) \big|u(z+\varepsilon y)\big|^p\, dy. \tag{2.35}$$

To see (2.35) note that it is clearly true if $p = 2$. So suppose that $p \in (1, +\infty)$. If $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\vartheta(y) u(z+\varepsilon y) \;=\; \vartheta(y)^{\frac{1}{p}} \vartheta(y)^{\frac{1}{p'}} u(z+\varepsilon y).$$

Invoking Hölder's inequality (see Theorem A.2.27), we have

$$\big|u_\varepsilon(z)\big| \;\leqslant\; \left(\int\limits_{B_1(0)} \vartheta(y)\, dy \right)^{\frac{1}{p'}} \left(\int\limits_{B_1(0)} \vartheta(y) \,|u(z+\varepsilon y)|^p \; dy \right)^{\frac{1}{p}}. \tag{2.36}$$

Since

$$\int\limits_{B_1(0)} \vartheta(y)\, dy \;=\; 1,$$

from (2.36), we obtain (2.35). Invoking (2.35), we have

$$\int\limits_K \big|u_\varepsilon(z)\big|^p\, dz \;\leqslant\; \int\limits_K \left(\int\limits_{B_1(0)} \vartheta(y) \big|u(z+\varepsilon y)\big|^p\, dy \right) dz;$$

thus by Fubini's theorem, we have

$$\int_K |u_\varepsilon(z)|^p\,dz \;\leqslant\; \int_{B_1(0)} \vartheta(y) \int_K |u(z+\varepsilon y)|^p\,dz\,dy \;\leqslant\; \int_{B_1(0)} \vartheta(y) \int_{K_\varepsilon} |u(v)|^p\,dv\,dy,$$

where

$$K_\varepsilon \stackrel{df}{=} \big\{v \in \mathbb{R}^N :\; d(v,K) \leqslant \varepsilon\big\}.$$

Since $\int_{B_1(0)} \vartheta(y)\,dy = 1$, we have

$$\|u_\varepsilon\|_{L^p(K)} \;\leqslant\; \int_{K_\varepsilon} |u(v)|^p\,dv, \tag{2.37}$$

so $u_\varepsilon \in L^p(K)$.

Let V be a bounded open set, such that

$$K_\varepsilon \;\subseteq\; V \;\subseteq\; \overline{V} \;\subseteq Z \qquad \forall\, \varepsilon > 0 \text{ small enough.}$$

Let $\delta > 0$ be given and select $h \in C\big(\overline{V}\big)$, such that

$$\|u-h\|_{L^p(V)} \;<\; \delta.$$

Here we exploit the density of the embedding

$$C\big(\overline{V}\big) \;\subseteq\; L^p(V).$$

Then, using (2.37), for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\|u-u_\varepsilon\|_{L^p(K)} &\leqslant \|u-h\|_{L^p(K)} + \|h-h_\varepsilon\|_{L^p(K)} + \|(h-v)_\varepsilon\|_{L^p(K)}\\ &\leqslant \|u-h\|_{L^p(K)} + \|h-h_\varepsilon\|_{L^p(K)} + \|h-v\|_{L^p(K_\varepsilon)} \;\leqslant\; 3\delta,\end{aligned}$$

so

$$u_\varepsilon \;\longrightarrow\; u \quad \text{in } L^p_{\mathrm{loc}}(Z) \quad \text{as } \varepsilon \searrow 0.$$

(d) Suppose that $u \in W^{1,p}(Z)$, $p \in (1,+\infty)$. From the proof of part **(a)** and integrating by parts, we know that

$$\begin{aligned}D_i u(z) \;&=\; \int_Z \frac{\partial \varphi_\varepsilon}{\partial z_i}(z-y)u(y)\,dy \;=\; -\int_Z \frac{\partial \varphi_\varepsilon}{\partial y_i}(z-y)u(y)\,dy\\ &=\; \int_Z \varphi_\varepsilon(z-y) D_i u(y)\,dy \;=\; \big(\varphi_\varepsilon \star D_i u\big)(z),\end{aligned}$$

for $z \in Z_{-\varepsilon}$, $\varepsilon > 0$ and $i \in \{1,\dots,N\}$.

(e) This follows from **(c)** and **(d)**. ∎

The next theorem shows that smooth functions are dense in $W^{1,p}(Z)$, $p \in [1, +\infty]$. So equivalently the space $W^{1,p}(Z)$ can be defined as the closure in the $\|\cdot\|_{W^{1,p}(Z)}$-norm of $C^\infty(Z) \cap W^{1,p}(Z)$. The result is known in the literature as the ***Meyers-Serrin theorem***.

THEOREM 2.4.13 (Meyers-Serrin Theorem)
If $p \in [1, +\infty)$,
then the embedding

$$C^\infty(Z) \cap W^{1,p}(Z) \ \subseteq \ W^{1,p}(Z)$$

is dense.

PROOF We define

$$Z_{-n} \stackrel{df}{=} \left\{ z \in Z : \ d\big(z, \partial Z\big) > \frac{1}{n}, \ \|z\|_{\mathbb{R}^N} < n \right\} \qquad \forall \ n \geqslant 1$$

and

$$Z_0 \stackrel{df}{=} \emptyset.$$

Set

$$U_n \stackrel{df}{=} Z_{-(n+1)} \setminus \overline{Z}_{-(n-1)} \qquad \forall \ n \geqslant 1.$$

The collection $\{U_n\}_{n \geqslant 1}$ is an open cover of Z. So we can find a smooth partition of unity $\{\xi_n\}_{n \geqslant 1}$ subordinate to $\{U_n\}_{n \geqslant 1}$. Then

$$\xi_n \in C_c^\infty(U_n), \quad 0 \leqslant \xi \leqslant 1 \qquad \forall \ n \geqslant 1$$

and

$$\sum_{n=1}^{\infty} \xi_n(z) \ = \ 1 \qquad \forall \ z \in Z.$$

Let $u \in W^{1,p}(Z)$ and $\delta > 0$. We have

$$\xi_n u \in W^{1,p}(Z) \quad \text{and} \quad \text{supp}\,\big(\xi_n u\big) \ \subseteq \ U_n \qquad \forall \ n \geqslant 1.$$

Thus by virtue of Proposition 2.4.12**(e)**, there exist $\varepsilon_n > 0$, such that

$$\text{supp}\,\big(\vartheta_{\varepsilon_n} \star (\xi_n u)\big) \ \subseteq \ U_n$$

and

$$\|\vartheta_{\varepsilon_n} \star (\xi_n u) - \xi_n u\| \ < \ \frac{\delta}{2^n}. \tag{2.38}$$

Define

$$u_\delta \stackrel{df}{=} \sum_{n=1}^{\infty} \vartheta_{\varepsilon_n} \star (\xi_n u).$$

In some neighbourhood of each point $z \in U$, there are only finitely many nonzero terms in the sum. Hence $u_\delta \in C^\infty(Z)$. Next note that

$$u \;=\; \sum_{n=1}^{\infty} \xi_n u.$$

From (2.38), we have

$$\begin{aligned} \|u_\delta - u\|_{W^{1,p}(Z)} \;\leqslant\; & \sum_{n=1}^{\infty} \left(\int_Z \left| \vartheta_{\varepsilon_n} \star \big((\xi_n u) - \xi_n u\big) \right|^p \, dz \right)^{\frac{1}{p}} \\ & + \sum_{n=1}^{\infty} \left(\int_Z \left| \vartheta_{\varepsilon_n} \star \big(D(\xi_n u) - Du\big) \right|^p \, dz \right)^{\frac{1}{p}} \;<\; \delta, \end{aligned}$$

so $u_\delta \in C^\infty(Z) \cap W^{1,p}(Z)$ and

$$u_\delta \;\longrightarrow\; u \quad \text{in } W^{1,p}(Z) \quad \text{as } \delta \searrow 0.$$

□

REMARK 2.4.14 The result is true for all Sobolev spaces $W^{m,p}(Z)$, $m \geqslant 1$. We emphasize that in the above approximation result, we do not claim that the approximating functions belong in $C^\infty\big(\overline{Z}\big)$. To obtain this we need to strengthen the geometry of the boundary ∂Z of Z.

DEFINITION 2.4.15 *A bounded open set $Z \subseteq \mathbb{R}^N$ is said to be **Lipschitz**, if for each $z \in \partial U$, there exists a neighbourhood U of z, such that*

$$Z \cap U \;=\; \big\{ y = (z_k)_{k=1}^N \in \mathbb{R}^N : \; \eta(y_1, \ldots, y_{N-1}) < y_N \big\} \cap U,$$

where $\eta \colon \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ is a Lipschitz continuous function and $\{y_k\}_{k=1}^N$ is a system of Cartesian coordinates of $\mathbb{R}^N$.

REMARK 2.4.16 From this definition it follows that ∂Z locally has a representation of the form

$$y_N \;=\; \eta(y_1, \ldots, y_{N-1}),$$

i.e., near $z \in \partial Z$, the boundary ∂Z is the graph of a Lipschitz continuous function. By Rademacher's theorem (see Theorem 1.5.8), the outer unit normal $n(z)$ to the domain Z exists for $\mu^{(N-1)}$-almost all $z \in \partial Z$. If $\overline{Z}$ is a bounded polyhedron, then Z is Lipschitz. Also if $\overline{Z}$ is a C^∞-submanifold with C^∞-boundary ∂Z, then Z is Lipschitz. Every Lipschitz open set $Z \subseteq \mathbb{R}^N$ is locally star-shaped.

For Lipschitz $Z \subseteq \mathbb{R}^N$ we can improve the conclusion of Theorem 2.4.13.

THEOREM 2.4.17
If $Z \subseteq \mathbb{R}^N$ is a bounded open set, which is Lipschitz and $p \in [1, +\infty)$, then the embedding $C^\infty(\overline{Z}) \subseteq W^{1,p}(Z)$ is dense.

REMARK 2.4.18 Theorem 2.4.17 implies that for any bounded, open, Lipschitz set $Z \subseteq \mathbb{R}^N$ and any given $u \in W^{1,p}(Z)$ $(p \in [1, +\infty))$, there exists a sequence $\{u_n\}_{n \geqslant 1} \subseteq \mathcal{D}(\mathbb{R}^N)$, such that

$$u_n|_Z \longrightarrow u \quad \text{in } W^{1,p}(Z).$$

In general for any open set $Z \subseteq \mathbb{R}^N$ and $u \in W^{1,p}(Z)$ $(p \in [1, +\infty))$, we can say that there exists a sequence $\{u_n\}_{n \geqslant 1} \subseteq \mathcal{D}(\mathbb{R}^N)$, such that

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } L^p(Z) \\ D_i x_n|_{Z'} &\longrightarrow D_i x|_{Z'} \quad \text{in } L^p(Z'), \qquad \forall\, i \in \{1, \dots, N\},\ Z' \subset\subset Z \end{aligned}$$

(i.e., Z' is a bounded open set with $\overline{Z'} \subseteq Z$). This result is known in the literature as ***Friedrich's theorem***. Theorem 2.4.17 holds for all Sobolev spaces $W^{m,p}(Z)$, $m \geqslant 1$.

None of these approximation results (Theorems 2.4.13 and 2.4.17) is true for $p = +\infty$. Indeed consider the following examples.

EXAMPLE 2.4.19 **(a)** Let $Z = \mathbb{R}^N$. We know that

$$\overline{C_c(\mathbb{R}^N)}^{\|\cdot\|_\infty} = C_0(\mathbb{R}^N).$$

Thus while $u \equiv 1 \in W^{1,\infty}(\mathbb{R}^N)$, it can not be approximated by functions in $C_c^\infty(\mathbb{R}^N)$.

(b) Let $Z = (-1, 1)$ and consider the function

$$u(z) \overset{df}{=} \begin{cases} 0 & \text{if} \quad z \leqslant 0, \\ z & \text{if} \quad z > 0. \end{cases}$$

Then u is absolutely continuous. Its derivative in the sense of distribution is given by

$$Du(z) \overset{df}{=} \begin{cases} 0 & \text{if} \quad z < 0, \\ 1 & \text{if} \quad z > 0. \end{cases}$$

Let $\vartheta \in C^\infty(Z)$ be such that $\|\vartheta' - u'\|_\infty < \varepsilon$. So if $z < 0$, then $|\vartheta'(z)| < \varepsilon$ and if $z > 0$, then $|\vartheta'(z) - 1| < \varepsilon$, hence $1 - \varepsilon < \vartheta'(z)$. By continuity, we obtain

$$\vartheta'(0) \leqslant \varepsilon \quad \text{and} \quad \vartheta'(0) \geqslant 1 - \varepsilon.$$

If $\varepsilon < \frac{1}{2}$, we reach a contradiction. This shows that u cannot be approximated in $W^{1,\infty}(Z)$ by smooth functions.

The next Proposition proves a simple characterization of the elements in $W^{1,p}(Z)$ $(p \in (1,+\infty])$.

PROPOSITION 2.4.20
If $Z \subseteq \mathbb{R}^N$ is open and $u \in L^p(Z)$ with $p \in (1,+\infty]$, then the following statements are equivalent:

(a) $u \in W^{1,p}(Z)$;

(b) *there exists a constant $c > 0$, such that*

$$\left| \int_Z u \frac{\partial \vartheta}{\partial z_k}\, dz \right| \leqslant c \|\vartheta\|_{p'} \qquad \forall\ \vartheta \in C_c^\infty(Z),\ k \in \{1,\dots,N\},$$

with $\frac{1}{p} + \frac{1}{p'} = 1$;

(c) *there exists a constant $c > 0$, such that for all $Z' \subset\subset Z$ (i.e., Z' is a bounded open set such that $\overline{Z'} \subseteq Z$) and*

$$\left\|\tau_y(u) - u\right\|_{L^p(Z')} \leqslant c \|y\|_{\mathbb{R}^N} \quad \forall y \in \mathbb{R}^N,\ \text{with}\ \|y\|_{\mathbb{R}^N} < d_{\mathbb{R}^N}(Z', Z^c).$$

Moreover, in both **(b)** *and* **(c)** *we can take $c = \|Du\|_p$.*

PROOF "**(a)**$\Longrightarrow$**(b)**": Obvious.

"**(b)**$\Longrightarrow$**(a)**": Let $L_k \colon C_c^\infty(Z) \longrightarrow \mathbb{R}$ be defined by

$$L_k(\vartheta) \stackrel{df}{=} \int_Z u \frac{\partial \vartheta}{\partial z_k}\, dz \qquad \forall\ k \in \{1,\dots,N\}.$$

Evidently L_k is linear, $L^{p'}$-continuous. Since the embedding $C_c^\infty(Z) \subseteq L^{p'}(Z)$ is dense, we can extend L_k continuously on all of $L^{p'}(Z)$. So $L_k \in L^{p'}(Z)^*$ and by the Riesz representation theorem (see Theorem A.3.24), we can find $h \in L^p(Z)$, such that

$$L_k(v) = \int_Z hv\, dz \qquad \forall\ v \in L^{p'}(Z),$$

so

$$\int_Z u \frac{\partial \vartheta}{\partial z_k}\, dz = \int_Z h\vartheta\, dz \qquad \forall\ \vartheta \in C_c^\infty(Z)$$

and

$$D_k u = h \qquad \forall\ k \in \{1,\dots,N\},$$

hence $u \in W^{1,p}(Z)$.

"**(a)**$\Longrightarrow$**(c)**": First suppose that $u \in C_c^\infty(Z)$. Let $y \in \mathbb{R}^N$ and set

$$v(t) \stackrel{df}{=} u(z+ty) \qquad \forall\, t \in \mathbb{R}.$$

Then from the chain rule, we have

$$v'(t) \;=\; (Du(z+ty), y)_{\mathbb{R}^N}\,.$$

Integrating, we obtain

$$u(z+y) - u(z) \;=\; v(1) - v(0) \;=\; \int_0^1 v'(t)\,dt \;=\; \int_0^1 (Du(z+ty), y)_{\mathbb{R}^N}\; dt$$

and so

$$\begin{aligned}
&\|\tau_y(u) - u\|^p_{L^p(Z')} \\
&\leqslant\; \|y\|^p_{\mathbb{R}^N} \int_{Z'} \int_0^1 \|Du(z+ty)\|^p_{\mathbb{R}^N}\; dt\, dz \\
&=\; \|y\|^p_{\mathbb{R}^N} \int_0^1 \int_{Z'} \|Du(z+ty)\|^p_{\mathbb{R}^N}\; dz\, dt \\
&=\; \|y\|^p_{\mathbb{R}^N} \int_0^1 \int_{Z'+ty} \|Du(r)\|^p_{\mathbb{R}^N}\; dr\, dt.
\end{aligned}$$

If

$$\|y\|_{\mathbb{R}^N} \;<\; d_{\mathbb{R}^N}\big(Z', Z^c\big),$$

we can find a bounded open set Z'', such that

$$\overline{Z}'' \;\subseteq\; Z \quad \text{and} \quad Z' + ty \;\subseteq\; Z'' \qquad \forall\, t \in [0,1].$$

Therefore

$$\|\tau_y(u) - u\|^p_{L^p(Z')} \;\leqslant\; \|y\|^p_{\mathbb{R}^N} \int_{Z''} \|Du\|^p_{\mathbb{R}^N}\; dz. \tag{2.39}$$

For the general case, suppose that $u \in W^{1,p}(Z)$, $p \in (1, +\infty]$. Then we can find a sequence $\{u_n\}_{n \geqslant 1} \subseteq C_c^\infty(\mathbb{R})$, such that

$$\begin{aligned}
u_n &\longrightarrow u \quad \text{in } L^p(Z), \\
Du_n &\longrightarrow Du \quad \text{in } L^p(Z'),
\end{aligned}$$

for any $Z' \subset\subset Z$ (see Remark 2.4.18). From (2.39), we have

$$\|\tau_y(u_n) - u_n\|^p_{L^p(Z')} \;\leqslant\; \|y\|^p_{\mathbb{R}^N} \int_{Z'} \|Du_n\|^p_{\mathbb{R}^N}\; dz,$$

so

$$\|\tau_y(u) - u\|^p_{L^p(Z')} \;\leqslant\; \|y\|^p_{\mathbb{R}^N} \int\limits_{Z'} \|Du\|^p_{\mathbb{R}^N}\, dz. \tag{2.40}$$

If $p = +\infty$, we obtain (2.40) for $p < +\infty$ and then let $p \to +\infty$.

"**(c)**$\Longrightarrow$**(b)**": Let $\vartheta \in C^\infty_c(Z)$ and consider an open set Z', such that

$$\operatorname{supp}\vartheta \;\subseteq\; Z' \;\subset\subset\; Z.$$

Let $y \in \mathbb{R}^N$ with

$$\|y\|_{\mathbb{R}^N} \;<\; d\big(Z', Z^c\big).$$

By hypothesis we have

$$\left| \int\limits_Z \big(\tau_y(u) - u\big)\vartheta\, dz \right| \;\leqslant\; c\,\|y\|_{\mathbb{R}^N}\,\|\vartheta\|_{p'}\,.$$

Note that

$$\int\limits_Z \big(u(z+y) - u(z)\big)\vartheta(z)\, dz \;=\; \int\limits_Z u(z)\big(\vartheta(z-y) - \vartheta(z)\big)\, dz,$$

so

$$\left| \int\limits_Z u(z)\frac{\vartheta(z-y) - \vartheta(z)}{\|y\|_{\mathbb{R}^N}}\, dz \right| \;\leqslant\; c\,\|\vartheta\|_{p'}\,.$$

Let $y = te_k$, $t \in \mathbb{R}$, $k \in \{1, \dots, N\}$. Passing to the limit as $t \to 0$, we obtain

$$\left| \int\limits_Z u\frac{\partial\vartheta}{\partial z_k}\, dz \right| \;\leqslant\; c\,\|\vartheta\|_{p'} \qquad \forall\ \vartheta \in C^\infty_c(Z),\ k \in \{1, \dots, N\}.$$

Finally it is clear from the above proofs that in **(b)** and **(c)**, the constant $c > 0$ can be taken as $c = \|Du\|_p$. □

REMARK 2.4.21 If $p = 1$, then we have **(a)** $\Longrightarrow$ **(b)** $\Longrightarrow$ **(c)**. From the implication **(a)** $\Longrightarrow$ **(c)**, we can see that if $Z \subseteq \mathbb{R}^N$ is an open set and $u \in W^{1,\infty}(Z)$, then

$$\big|u(z) - u(y)\big| \;\leqslant\; \|Du\|_\infty\,\|z - y\|_{\mathbb{R}^N} \qquad \forall\ z, y \in Z.$$

So $W^{1,\infty}(Z)$ is the space of Lipschitz continuous functions on Z. In particular

$$W^{1,\infty}(Z) \;\subseteq\; C\big(\overline{Z}\big).$$

More generally, it is easy to show that if $u\colon Z \longrightarrow \mathbb{R}$ is locally Lipschitz, then $u \in W^{1,p}_{\mathrm{loc}}(Z)$ $(p \in [1, +\infty])$.

PROPOSITION 2.4.22
If $Z \subseteq \mathbb{R}^N$ is an open set and $u, v \in W^{1,p}(Z) \cap L^\infty(Z)$ with $p \in [1, +\infty]$, then $uv \in W^{1,p}(Z)$ and $D(uv) = uDv + vDu$ (product rule).

PROOF If $p = +\infty$, then u, v are Lipschitz continuous functions and so differentiable for almost all $z \in Z$ (see Theorem 1.5.8). Clearly uv is Lipschitz continuous too, hence in $W^{1,\infty}(Z)$ and the product rule results from the usual product rule of differentiable functions.

So suppose that $p \in [1, +\infty)$. We assume that

$$\big|u(z)\big|, \big|v(z)\big| \leqslant 1 \quad \text{for a.a. } z \in Z.$$

Invoking Theorem 2.4.13, we can find sequences

$$\{\widehat{u}_n\}_{n \geqslant 1}, \{\widehat{v}_n\}_{n \geqslant 1} \subseteq C^\infty(Z) \cap W^{1,p}(Z),$$

such that

$$\begin{aligned}
\widehat{u}_n &\longrightarrow u \quad \text{in } W^{1,p}(Z), \\
\widehat{u}_n(z) &\longrightarrow u(z) \quad \text{for a.a. } z \in Z, \\
\widehat{v}_n &\longrightarrow v \quad \text{in } W^{1,p}(Z), \\
\widehat{v}_n(z) &\longrightarrow v(z) \quad \text{for a.a. } z \in Z.
\end{aligned}$$

Let

$$\begin{aligned}
u_n &\stackrel{df}{=} \max\big\{-1, \min\{\widehat{u}_n, 1\}\big\}, \\
v_n &\stackrel{df}{=} \max\big\{-1, \min\{\widehat{v}_n, 1\}\big\}.
\end{aligned}$$

Then $u_n v_n$ is locally Lipschitz. Moreover, we have

$$D(u_n v_n) = u_n Dv_n + v_n Du_n \in L^p(Z),$$

so

$$u_n v_n \in W^{1,p}(Z) \qquad \forall\, n \geqslant 1.$$

Note that

$$\begin{aligned}
u_n &\longrightarrow u \quad \text{in } W^{1,p}(Z), \\
u_n(z) &\longrightarrow u(z) \quad \text{for a.a. } z \in Z, \\
v_n &\longrightarrow v \quad \text{in } W^{1,p}(Z), \\
v_n(z) &\longrightarrow v(z) \quad \text{for a.a. } z \in Z.
\end{aligned}$$

We have

$$\begin{aligned}
&\|u_n v_n - uv\|_p^p \\
&\leqslant \int_Z \big|v_n(z)\big|^p \big|u_n(z) - u(z)\big|^p \, dz + \int_Z \big|u(z)\big|^p \big|v_n(z) - v(z)\big|^p \, dz \longrightarrow 0.
\end{aligned}$$

In addition

$$\begin{aligned}
&\left\|u_nDv_n + v_nDu_n - \big(uDv + vDu\big)\right\|_p^p \\
\leqslant\ &\|u_nDv_n - uDv\|_p^p + \|v_nDu_n - vDu\|_p^p \\
\leqslant\ &\int_Z \left|u_n(z)\right|^p \|Dv_n - Dv\|_{\mathbb{R}^N}^p \, dz \\
&+ \int_Z \|Dv(z)\|_{\mathbb{R}^N}^p \left|u_n(z) - u(z)\right|^p dz \\
&+ \int_Z \left|v_n(z)\right|^p \|Du_n - Du\|_{\mathbb{R}^N}^p \, dz \\
&+ \int_Z \|Du(z)\|_{\mathbb{R}^N}^p \left|v_n(z) - v(z)\right|^p dz \longrightarrow 0.
\end{aligned}$$

Therefore we conclude that $uv \in W^{1,p}(Z)$ and

$$D(uv) \ = \ uDv + vDu.$$

□

In fact a careful reading of this proof reveals that the following result is also true.

PROPOSITION 2.4.23
If $Z \subseteq \mathbb{R}^N$ is an open set, $u \in W^{1,p}(Z)$, $p \in [1, +\infty]$ and $v \in W^{1,\infty}(Z)$, then $uv \in W^{1,p}(Z)$ and

$$D(uv) \ = \ uDv + vDu.$$

Next we prove a chain rule for Sobolev functions.

PROPOSITION 2.4.24 (Chain Rule for Sobolev Functions)
If $Z \subseteq \mathbb{R}^N$ is an open set, $\xi \in C^1(\mathbb{R})$, $\xi' \in L^\infty(\mathbb{R})$, $\xi(0) = 0$ and $u \in W^{1,p}(Z)$ with $p \in [1, +\infty]$,
then $\xi \circ u \in W^{1,p}(Z)$ and

$$D(\xi \circ u) \ = \ \xi'(u)Du.$$

Moreover, if Z is bounded, then we can drop the condition that

$$\xi(0) \ = \ 0.$$

PROOF Let $p \in [1, +\infty)$ and $\vartheta \in C_c^\infty(Z)$ be such that

$$\operatorname{supp} \vartheta \ \subseteq \ V \ \subset\subset \ Z.$$

Using Proposition 2.4.12**(e)** (twice) and integration by parts, we have

$$\begin{aligned}\int_Z \xi(u)\frac{\partial\vartheta}{\partial z_k}\,dz &= \int_V \xi(u)\frac{\partial\vartheta}{\partial z_k}\,dz\\ &= \lim_{\varepsilon\searrow 0}\int_V \xi(u_\varepsilon)\frac{\partial\vartheta}{\partial z_k}\,dz = -\lim_{\varepsilon\searrow 0}\int_V \xi'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial z_k}\vartheta\,dz\\ &= -\int_V \xi'(u)(D_k u)\vartheta\,dz = -\int_Z \xi'(u)(D_k u)\vartheta\,dz,\end{aligned}$$

so

$$D_k(\xi\circ u) = \xi'(u)D_k u \qquad \forall\, k\in\{1,\dots,N\}.$$

It is clear from the above argument (see second equality) that if Z is bounded, then the condition $\xi(0)=0$ can be dropped.

If $p=+\infty$, then

$$\xi\circ u \text{ is a Lipschitz continuous function}$$

(see Remark 2.4.21) and the result follows from the classical chain rule. □

In fact there is a stronger version of the previous proposition. It is due to Marcus & Mizel (1972), where the interested reader can find the proof.

PROPOSITION 2.4.25
If $Z\subseteq\mathbb{R}^N$ is an open set, $\xi\colon\mathbb{R}\longrightarrow\mathbb{R}$ is Lipschitz continuous, $\xi(0)=0$ and $u\in W^{1,p}(Z)$, $p\in[1,+\infty]$,
then $\xi\circ u\in W^{1,p}(Z)$ and $D(\xi\circ u)=\xi^*(u)Du$ almost everywhere on Z with $\xi^*\colon\mathbb{R}\longrightarrow\mathbb{R}$ being any bounded Borel function such that

$$\xi^*(z) = \xi'(z) \quad \text{for a.a. } z\in Z.$$

REMARK 2.4.26 The function f^* can always be taken to be bounded, by virtue of the following result due to Stampacchia (1966):

> "If $u\in W^{1,p}(Z)$ and $A\subseteq\mathbb{R}$ is a Lebesgue-null set, then $Du(z)=0$ for almost all $z\in u^{-1}(A)$."

Moreover, note that the chain rule (see Proposition 2.4.25) is also valid for $W_0^{1,p}(Z)$ (see Corollary 2.4.8).

Recall that

$$u^+ = \max\{u,0\} \quad\text{and}\quad u^- = \min\{-u,0\}.$$

We have

$$u = u^+ - u^- \quad\text{and}\quad |u| = u^+ + u^-.$$

Using the general version of the chain rule (see Proposition 2.4.25), we obtain at once the following result.

PROPOSITION 2.4.27
If $Z \subseteq \mathbb{R}^N$ *is an open set and* $u \in W^{1,p}(Z)$, $p \in [1, +\infty]$, *then* $u^+, u^-, |u| \in W^{1,p}(Z)$ *and we have*

$$Du^+ = \begin{cases} Du & \text{for a.a. } z \in \{u > 0\}, \\ 0 & \text{for a.a. } z \in \{u \leqslant 0\}, \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{for a.a. } z \in \{u \geqslant 0\}, \\ -Du & \text{for a.a. } z \in \{u < 0\}, \end{cases}$$

$$D|u| = \begin{cases} Du & \text{for a.a. } z \in \{u > 0\}, \\ 0 & \text{for a.a. } z \in \{u = 0\}, \\ -Du & \text{for a.a. } z \in \{u < 0\}. \end{cases}$$

Using this proposition we can show that the Sobolev spaces $W^{1,p}(Z)$, $p \in [1, +\infty]$ have a lattice structure.

COROLLARY 2.4.28
If $Z \subseteq \mathbb{R}^N$ *is an open set and* $u, v \in W^{1,p}(Z)$, $p \in [1, +\infty]$, *then*

$$h_0 = \min\{u, v\} \in W^{1,p}(Z), \quad h_1 = \max\{u, v\} \in W^{1,p}(Z)$$

and we have

$$Dh_0 \stackrel{df}{=} \begin{cases} Du & \text{for a.a. } z \in \{u \leqslant v\}, \\ Dv & \text{for a.a. } z \in \{u \geqslant v\}, \end{cases}$$

$$Dh_1 \stackrel{df}{=} \begin{cases} Du & \text{for a.a. } z \in \{u \geqslant v\}, \\ Dv & \text{for a.a. } z \in \{u \leqslant v\}. \end{cases}$$

PROOF Note that

$$h_1 = (u - v)^+ + v \quad \text{and} \quad h_0 = u - (u - v)^+.$$

Then the result follows at once from Proposition 2.4.27. □

An immediate consequence of this Corollary is the following particular case of the result of Stampacchia mentioned in Remark 2.4.26.

COROLLARY 2.4.29
If $Z \subseteq \mathbb{R}^N$ *is an open set and* $u \in W^{1,p}(Z)$, $p \in [1, +\infty]$, *then for every* $\eta \in \mathbb{R}$, *we have*

$$Du(z) = 0 \quad \text{for a.a. } z \in \{u = \eta\}.$$

PROPOSITION 2.4.30
If $Z \subseteq \mathbb{R}^N$ *is an open set,* $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(Z)$, $p \in (1,+\infty)$,

$$h \stackrel{df}{=} \sup_{n\geqslant 1} \|Du_n\| \in L^p(Z) \quad \textit{and} \quad g \stackrel{df}{=} \sup_{n\geqslant 1} u_n,$$

then $g \in W^{1,p}(Z)$ *and* $\big\|Dg(z)\big\|_{\mathbb{R}^N} \leqslant h(z)$ *for almost all* $z \in Z$.

PROOF Let

$$g_k \stackrel{df}{=} \max_{1\leqslant n\leqslant k} u_n \qquad \forall\, k \geqslant 1.$$

From Corollary 2.4.28, we have that $g_k \in W^{1,p}(Z)$ and

$$\big\|Dg_k(z)\big\|_{\mathbb{R}^N} \leqslant \max_{1\leqslant n\leqslant k} \big\|Du_n(z)\big\|_{\mathbb{R}^N} \leqslant h(z) \quad \text{for a.a. } z \in Z. \tag{2.41}$$

Evidently the sequence $\{g_k\}_{k\geqslant 1}$ is increasing and

$$g_k(z) \longrightarrow g(z) \quad \text{for a.a. } z \in Z \quad \text{as } k \to +\infty.$$

Also from (2.41), we see that the sequence $\{Dg_k\}_{k\geqslant 1} \subseteq L^p\big(Z;\mathbb{R}^N\big)$ is bounded. Then from the monotone convergence theorem (see Theorem A.2.10) and the Eberlein-Smulian theorem (see Theorem A.3.8), we have

$$\begin{aligned} g_k &\longrightarrow g \quad \text{in } L^p(Z), \\ Dg_{k_m} &\xrightarrow{w} w \quad \text{in } L^p\big(Z;\mathbb{R}^N\big), \end{aligned}$$

with $\{g_{k_m}\}_{m\geqslant 1}$ being a subsequence of $\{g_k\}_{k\geqslant 1}$.

For every $\vartheta \in C_c^\infty(Z)$ and every $i \in \{1,\dots,N\}$, from the definition of the distributional derivative, we have

$$\int_Z (D_i g_k)\vartheta \, dz = -\int_Z g_k D_i \vartheta \, dz,$$

so

$$\int_A w_i \vartheta \, dz = -\int_Z g D\vartheta \, dz,$$

where $w = (w_i)_{i=1}^N \in L^p\big(Z;\mathbb{R}^N\big)$ and finally

$$w_i = D_i g \qquad \forall\, i \in \{1,\dots,N\}.$$

Therefore, we infer that for whole sequence, we have

$$Dg_n \xrightarrow{w} Dg \quad \text{in } L^p\big(Z;\mathbb{R}^N\big).$$

So $g \in W^{1,p}(Z)$ and

$$\|Dg(z)\|_{\mathbb{R}^N} \leqslant h(z) \quad \text{for a.a. } z \in Z.$$

□

From the above proofs, it is clear that we have:

PROPOSITION 2.4.31
If $Z \subseteq \mathbb{R}^N$ is an open set and $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(Z)$, $p \in [1,+\infty)$ is a sequence, such that

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } L^p(Z) \\ Du_n &\xrightarrow{w} w \quad \text{in } L^p\big(Z;\mathbb{R}^N\big), \end{aligned}$$

then $u \in W^{1,p}(Z)$ and

$$Du \;=\; w.$$

PROPOSITION 2.4.32
If $Z \subseteq \mathbb{R}^N$ is an open set, $\{u_n\}_{n\geqslant 1}, \{v_n\}_{n\geqslant 1} \subseteq W^{1,p}(Z)$, $p \in [1,+\infty)$ and

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } W^{1,p}(Z), \\ v_n &\xrightarrow{w} v \quad \text{in } W^{1,p}(Z), \end{aligned}$$

then

$$\begin{aligned} \min\big\{u_n, v_n\big\} &\longrightarrow \min\big\{u, v\big\} \quad \text{in } W^{1,p}(Z), \\ \max\big\{u_n, v_n\big\} &\longrightarrow \max\big\{u, v\big\} \quad \text{in } W^{1,p}(Z). \end{aligned}$$

PROOF It suffices to show that

$$\text{if } u_n \;\longrightarrow\; u \quad \text{in } W^{1,p}(Z), \text{ then } u_n^+ \;\longrightarrow\; u^+ \quad \text{in } W^{1,p}(Z).$$

First note that

$$\big|u_n^+ - u^+\big| \;\leqslant\; \big|u_n - u\big|$$

and so it follows that

$$u_n^+ \;\longrightarrow\; u^+ \quad \text{in } L^p(Z).$$

Next let

$$h \;=\; \chi_{(0,+\infty)}.$$

Using Proposition 2.4.27, we have

$$\begin{aligned} &\big\|Du_n^+ - Du^+\big\|_p^p \\ =\; &\int_Z \|h(u_n)Du_n - h(u)Du\|_{\mathbb{R}^N}^p \; dz \\ \leqslant\; &\|Du_n(z) - Du(z)\|_p^p + \int_Z \|Du(z)\|_{\mathbb{R}^N}^p \big|h(u_n) - h(u)\big|^p \, dz \;\longrightarrow\; 0. \end{aligned}$$

□

Using this, we can conclude the validity of Proposition 2.4.27 and Corollary 2.4.28 for the spaces $W_0^{1,p}(Z)$, $p \in [1, +\infty)$. So $W_0^{1,p}(Z)$, $p \in [1, +\infty)$ has a lattice structure.

PROPOSITION 2.4.33
If $Z \subseteq \mathbb{R}^N$ is an open set and $u, v \in W_0^{1,p}(Z)$, $p \in [1, +\infty)$,
then

$$\max\{u, v\}, \min\{u, v\} \in W_0^{1,p}(Z).$$

In particular

$$u^+, u^-, |u| \in W_0^{1,p}(Z).$$

PROOF Again it suffices to show that $u^+ \in W_0^{1,p}(Z)$. Let $\{\vartheta_n\}_{n \geqslant 1} \subseteq C_c^\infty(Z)$ be such that

$$\vartheta_n \longrightarrow u \quad \text{in } W^{1,p}(Z).$$

From the proof of Proposition 2.4.12**(c)** and **(d)**, it follows that we can find a sequence $\{\psi_{nm}\}_{n \geqslant 1} \subseteq C_c^\infty(Z)$, $\psi_{nm} \geqslant 0$, such that

$$\psi_{nm} \longrightarrow \vartheta_n^+ \quad \text{in } W_0^{1,p}(Z) \quad \text{as } m \to +\infty \qquad \forall\, n \geqslant 1.$$

Since

$$\vartheta_n^+ \longrightarrow u^+ \quad \text{in } W^{1,p}(Z)$$

(see Proposition 2.4.32), via the double limit lemma (see Proposition A.2.35), we can find a sequence $\{m(n)\}_{n \geqslant 1}$ increasing (not necessarily strictly) to $+\infty$, such that

$$\psi_{n\, m(n)} \longrightarrow u^+ \quad \text{in } W^{1,p}(Z).$$

Since $\psi_{n\, m(n)} \in C_c^\infty(Z)$, we deduce that $u^+ \in W^{1,p}(Z)$. □

In fact the previous result can be also obtained by having a chain rule for $W_0^{1,p}(Z)$, $p \in (1, +\infty)$ (see also Proposition 2.4.25). First an auxiliary result which is actually of independent interest.

PROPOSITION 2.4.34
If $Z \subseteq \mathbb{R}^N$ is an open set, $u \in W^{1,p}(Z)$, $p \in [1, +\infty)$ and u vanishes outside a compact $K \subseteq Z$,
then $u \in W_0^{1,p}(Z)$.

PROOF Let Z' be a bounded open set in $\mathbb{R}^N$, such that

$$K \subseteq Z' \subset\subset Z.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, such that

$$\varphi|_K \equiv 1$$

(i.e., φ is what we usually call a cut off function). We have

$$\varphi u \;=\; u.$$

We can find $\{\vartheta_n\}_{n\geqslant 1} \subseteq C_c^\infty(Z)$, such that

$$\vartheta_n \;\longrightarrow\; u \quad \text{in } L^p(Z)$$

and

$$D\vartheta_n \;\longrightarrow\; Du \quad \text{in } L^p\big(Z';\mathbb{R}^N\big)$$

(see Remark 2.4.18). We have

$$\varphi\vartheta_n \;\longrightarrow\; \varphi u \quad \text{in } W^{1,p}(Z)$$

and $\varphi\vartheta_n \in C_c^\infty(Z)$. Therefore

$$\varphi u \;=\; u \in W_0^{1,p}(Z).$$

□

Using this result we can prove the chain rule for the Sobolev spaces $W_0^{1,p}(Z)$, $p \in (1,+\infty)$.

PROPOSITION 2.4.35
If $Z \subseteq \mathbb{R}^N$ is a bounded open set, $\xi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz continuous, $\xi(0) = 0$ and $u \in W_0^{1,p}(Z)$, $p \in (1,+\infty)$,
then

$$\xi \circ u \in W_0^{1,p}(Z) \quad \textit{and} \quad D(\xi \circ u) = \xi'(u)Du.$$

PROOF Let $\{\vartheta_n\}_{n\geqslant 1} \subseteq C_c^\infty(Z)$ be such that

$$\vartheta_n \;\longrightarrow\; u \quad \text{in } W^{1,p}(Z).$$

Set

$$h_n \;\stackrel{df}{=}\; \xi \circ \vartheta_n \qquad \forall\, n \geqslant 1.$$

Evidently h_n is a Lipschitz continuous function and since ϑ_n has compact support, so does h_n. Also

$$\left|\frac{\partial h_n}{\partial z_i}\right| \;\leqslant\; \mathrm{Lip}(h_n) \qquad \forall\, i \in \{1,\dots,N\},\ n \geqslant 1.$$

Because $Z \subseteq \mathbb{R}^N$ is bounded, we infer that

$$\frac{\partial h_n}{\partial z_i} \in L^p(Z),$$

hence $h_n \in W^{1,p}(Z)$ and has compact support. So Proposition 2.4.34 implies that $h_n \in W_0^{1,p}(Z)$. Also we have

$$\begin{aligned} \big|h_n(z) - \xi\big(u(z)\big)\big| &= \big|\xi\big(\vartheta_n(z)\big) - \xi\big(u(z)\big)\big| \\ &\leqslant \mathrm{Lip}(\xi)\big|\vartheta_n(z) - u(z)\big| \quad \text{for a.a. } z \in Z, \end{aligned}$$

so

$$h_n \longrightarrow \xi \circ u \quad \text{in } L^p(Z).$$

Moreover, if $\{e_k\}_{k=1}^N$ is the standard orthonormal basis of $\mathbb{R}^N$, we have

$$\frac{|h_n(z+te_i) - h_n(z)|}{|t|} \leqslant \frac{\mathrm{Lip}(\xi)\,|\vartheta_n(z+te_i) - \vartheta_n(z)|}{|t|},$$

so

$$\limsup_{n\to+\infty} \left\| \frac{\partial h_n}{\partial z_i} \right\|_p \leqslant \mathrm{Lip}(\xi) \limsup_{n\to+\infty} \left\| \frac{\partial \vartheta_n}{\partial z_i} \right\|_p. \tag{2.42}$$

But

$$\frac{\partial \vartheta_n}{\partial z_i} \longrightarrow \frac{\partial u}{\partial z_i} \quad \text{in } L^p(Z) \qquad \forall\, i \in \{1,\dots,N\}.$$

So from (2.42), we infer that the sequence $\left\{\frac{\partial h_n}{\partial z_i}\right\}_{i\geqslant 1} \subseteq L^p(Z)$ is bounded. Since $p \in (1,+\infty)$, by passing to a subsequence if necessary, we may assume that

$$\frac{\partial h_n}{\partial z_i} \xrightarrow{w} w_i \quad \text{in } L^p(Z) \qquad \forall\, i \in \{1,\dots,N\}.$$

From Proposition 2.4.31, we have that $w_i = \frac{\partial \xi(u)}{\partial z_i}$ and so

$$h_n \longrightarrow \xi(u) \quad \text{in } W^{1,p}(Z).$$

Because $h_n \in W_0^{1,p}(Z)$, we conclude that $\xi \circ u = \xi(u) \in W_0^{1,p}(Z)$.

Finally note that

$$Dh_n = \xi'(\vartheta_n) D\vartheta_n$$

and so in the limit we have

$$D(\xi \circ u) = \xi'(u) Du.$$

□

REMARK 2.4.36 If $Z \subseteq \mathbb{R}^N$ is bounded, open, Lipschitz, then in the above proof we can choose $\{\vartheta_n\}_{n\geqslant 1} \subseteq C_c^\infty(Z)$, such that

$$\vartheta_n \longrightarrow u \quad \text{in } W^{1,p}(Z).$$

Then the same proof is valid and so we have a proof of Proposition 2.4.25 with the extra hypothesis that $Z \subseteq \mathbb{R}^N$ is bounded and Lipschitz.

We can also have the product rule for the spaces $W^{1,p}(Z)$, $p \in [1, +\infty]$. The proof is the same as that of Proposition 2.4.22, using this time a sequence $\{\widehat{u}_n\}_{n \geqslant 1} \subseteq C_c^\infty(Z)$.

PROPOSITION 2.4.37
If $Z \subseteq \mathbb{R}^N$ *is an open set and*

$$u \in W_0^{1,p}(Z) \cap L^\infty(Z) \quad \text{and} \quad v \in W^{1,p}(Z) \cap L^\infty(Z),$$

$p \in [1, +\infty]$,
then $uv \in W_0^{1,p}(Z)$ *and*

$$D(uv) = uDv + vDu.$$

Continuing with the Sobolev spaces $W_0^{1,p}(Z)$, $p \in [1, +\infty)$, we have the following results.

PROPOSITION 2.4.38
If $Z \subseteq \mathbb{R}^N$ *is an open set,*

$$u \in W_0^{1,p}(Z) \quad \text{and} \quad v \in W^{1,p}(Z),$$

$p \in [1, +\infty]$ *and*

$$0 \leqslant v(z) \leqslant u(z) \quad \text{for a.a. } z \in Z,$$

then $v \in W_0^{1,p}(Z)$.

PROOF From the proof of Proposition 2.4.33, we know that there exists a sequence $\{\vartheta_n\}_{n \geqslant 1} \subseteq C_c^\infty(Z)$, such that

$$\vartheta_n \geqslant 0 \qquad \forall\, n \geqslant 1$$

and

$$\vartheta_n \longrightarrow u \quad \text{in } W^{1,p}(Z).$$

Let

$$h_n \stackrel{df}{=} \min\{v, \vartheta_n\} \qquad \forall\, n \geqslant 1.$$

Evidently h_n has compact support and so by Proposition 2.4.34,

$$h_n \in W_0^{1,p}(Z).$$

Moreover, from Proposition 2.4.32, we have that

$$h_n \longrightarrow \min\{v, u\} = v \quad \text{in } W^{1,p}(Z).$$

So $v \in W_0^{1,p}(Z)$. ∎

PROPOSITION 2.4.39
If $Z \in \mathbb{R}^N$ *is an open set,* $u \in W_0^{1,p}(Z)$, $v \in W^{1,p}(Z)$, $p \in [1, +\infty)$ *and*

$$|v(z)| \leqslant |u(z)| \quad \text{for a.a. } z \in Z \setminus K,$$

where K *is a compact subset of* Z,
then $v \in W_0^{1,p}(Z)$.

PROOF Let $\varphi \in C_c^\infty(Z)$ be such that

$$0 \leqslant \varphi \leqslant 1 \quad \text{and} \quad \varphi|_K = 1$$

(a cut off function). We set

$$\widehat{u} \stackrel{df}{=} (1-\varphi)|u| + \varphi v^+.$$

From Propositions 2.4.33 and 2.4.34, we have that

$$\widehat{u} \in W_0^{1,p}(Z) \quad \text{and} \quad 0 \leqslant v^+ \leqslant \widehat{u}.$$

So

$$v^+ \in W_0^{1,p}(Z)$$

(see Proposition 2.4.38).
Similarly we show that $v^- \in W_0^{1,p}(Z)$. Hence $v \in W_0^{1,p}(Z)$. □

We can improve Proposition 2.4.34 and motivate the discussion of trace which follows.

PROPOSITION 2.4.40
If $Z \subseteq \mathbb{R}^N$ *is a bounded open set,* $u \in W^{1,p}(Z)$, $p \in [1, +\infty)$ *and*

$$\lim_{z \to y} u(z) = 0 \qquad \forall\, y \in \partial Z,$$

then $u \in W_0^{1,p}(Z)$.

PROOF Since

$$u = u^+ - u^-,$$

we may assume that $u \geqslant 0$. For $\varepsilon > 0$ let $u_\varepsilon \in W^{1,p}(Z)$ (see Proposition 2.4.27) and u_ε has compact support. Therefore by Proposition 2.4.34, we have that $u_\varepsilon \in W_0^{1,p}(Z)$. Now note that

$$u_\varepsilon \longrightarrow u \quad \text{in } W^{1,p}(Z) \quad \text{as } \varepsilon \searrow 0.$$

Thus $u \in W_0^{1,p}(Z)$. □

So roughly speaking a function $u \in W^{1,p}(Z)$ belongs to $W_0^{1,p}(Z)$, if u is vanishing on ∂Z. But it is not meaningful to talk of values of u on a set of measure zero. Hence we must be more careful on how we assign boundary values to Sobolev functions. Trace theory does exactly this, namely defines and studies the concept of boundary values for the Sobolev spaces $W^{1,p}(Z)$, $p \in [1, +\infty)$. The trace of a Sobolev function is an extension of the restriction of a continuous function on ∂Z. We start with a simple lemma.

LEMMA 2.4.41
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz, then for all $p \in [1, +\infty)$ there exists $c > 0$, such that

$$\int_{\partial Z} |u|^p \, d\mu^{(N-1)} \leqslant c \, \|u\|^p_{W^{1,p}(Z)} \qquad \forall \; u \in C^1(\overline{Z}).$$

PROOF Since by hypothesis the boundary ∂Z is Lipschitz, for any $z = (z_k)_{k=1}^N \in \partial Z$ we can find $r > 0$ and a Lipschitz continuous function

$$\eta \colon \mathbb{R}^{N-1} \longrightarrow \mathbb{R},$$

such that (upon rotating and relabelling the coordinate axes if necessary), we have

$$Z \cap C_r(z) = \left\{ y = (y_k)_{k=1}^N \in \mathbb{R}^N : \; \eta(y_1, \ldots, y_{N-1}) < y_N \right\} \cap C_r(z),$$

where

$$C_r(z) \stackrel{df}{=} \left\{ y = (y_k)_{k=1}^N \in \mathbb{R}^N : \; |y_k - z_k| < r, \; k = 1, \ldots, N \right\}.$$

First assume that $u|_{Z \setminus C_r(z)}$. If $\{e_k\}_{k=1}^N$ is the standard orthonormal basis of $\mathbb{R}^N$ and $n(\cdot)$ is the outward unit normal vector on ∂Z, then we have

$$-(e_N, n)_{\mathbb{R}^N} \geqslant \left(1 + \mathrm{Lip}(\eta)^2\right)^{-\frac{1}{2}} > 0 \quad \text{for } \mu^{(N-1)}\text{-a.a. } z \in \partial Z \cap C_r(z). \tag{2.43}$$

Let $\varepsilon > 0$ be given and set

$$\xi_\varepsilon(t) \stackrel{df}{=} (t^2 + \varepsilon^2)^{\frac{1}{2}} - \varepsilon \qquad \forall \; t \in \mathbb{R}.$$

Using the Gauss-Green theorem (see Theorem A.4.1) and since $\left|\xi_\varepsilon'\right| \leqslant 1$ for

all $t \in \mathbb{R}$, we have

$$\begin{aligned}
\int\limits_{\partial Z} \xi_\varepsilon\big(u(y)\big)\, d\mu^{(N-1)} &= \int\limits_{\partial Z \cap C_r(z)} \xi_\varepsilon\big(u(y)\big)\, d\mu^{(N-1)} \\
&\leqslant c \int\limits_{\partial Z \cap C_r(z)} \xi_\varepsilon\big(u(y)\big)\, (-e_N, n(y))_{\mathbb{R}^N}\, d\mu^{(N-1)} \\
&= -c \int\limits_{\partial Z \cap C_r(z)} \frac{\partial}{\partial y_N} \big(\xi_\varepsilon\big(u(y)\big)\big)\, dy \\
&\leqslant c \int\limits_{\partial Z \cap C_r(z)} \big|\xi'_\varepsilon\big(u(y)\big)\big|\, \big\|Du(y)\big\|_{\mathbb{R}^N}\, dy \leqslant c \int\limits_Z \big\|Du(y)\big\|_{\mathbb{R}^N},
\end{aligned}$$

with $c > 0$ independent of u (see (2.43)).

Note that

$$\xi_\varepsilon(u) \longrightarrow |u| \quad \text{as } \varepsilon \searrow 0,$$

so in the limit we obtain

$$\int\limits_{\partial Z} |u|\, d\mu^{(N-1)} \leqslant c \int\limits_Z \|Du(z)\|_{\mathbb{R}^N}\, dy. \tag{2.44}$$

Now we remove the extra hypothesis that $u|_{Z \setminus C_r(z)} = 0$. In the general case, we can cover ∂Z by a finite number of such cubes $C_{r_i}(z_i) = C_i$ for $i = 1, \ldots, m$. Then we can find smooth functions $\{\xi_i\}_{i=0}^m$, such that

$$0 \leqslant \xi_i \leqslant 1, \quad \operatorname{supp} \xi_i \subseteq C_i \qquad \forall\, i \in \{1, \ldots, N\},$$

$$0 \leqslant \xi_0 \leqslant 1, \quad \operatorname{supp} \xi_0 \subseteq Z$$

and

$$\sum_{i=0}^m \xi_i(z) = 1 \qquad \forall\, z \in Z.$$

We set

$$u_i \stackrel{df}{=} \xi_i u \qquad \forall\, i \in \{0, 1, \ldots, m\}.$$

Evidently each $u_i|_{Z \setminus C_i} = 0$ and so using (2.44), we obtain

$$\int\limits_{\partial Z} |u|\, d\mu^{(N-1)} \leqslant c \int\limits_Z \big(|u(z)| + \|Du(z)\|_{\mathbb{R}^N}\big)\, dy \qquad \forall\, u \in C^1(\overline{Z}). \tag{2.45}$$

If $p \in (1, +\infty)$, then we use (2.45) with $|u|^p$ replacing $|u|$. So finally

$$\int\limits_{\partial Z} \big|u(z)\big|^p\, d\mu^{(N-1)} \leqslant c\, \|u\|^p_{W^{1,p}(Z)} \qquad \forall\, p \in [1, +\infty).$$

□

Using this Lemma, we can state and prove the ***trace theorem*** which gives meaning to the concept of boundary values for Sobolev spaces.

THEOREM 2.4.42 (Trace Theorem)
If $Z \subseteq \mathbb{R}^N$ is bounded, open set which is Lipschitz and $p \in [1, +\infty)$, then there exists a unique continuous linear map

$$\gamma_0 \colon W^{1,p}(Z) \longrightarrow L^p\big(\partial Z, \mu^{(N-1)}\big),$$

such that

$$\gamma_0(u) \;=\; u|_{\partial Z} \qquad \forall\, u \in C^1(\overline{Z}).$$

PROOF By virtue of Theorem 2.4.17, the embedding $C^1(\overline{Z}) \subseteq W^{1,p}(Z)$ is dense. From Lemma 2.4.41, we know that

$$\|u|_{\partial Z}\|^p_{L^p\left(\partial Z, \mu^{(N-1)}\right)} \;\leqslant\; c\,\|u\|_{W^{1,p}(Z)} \qquad \forall\, u \in C^1(\overline{Z}),$$

for some $c > 0$. So we can extend uniquely to a continuous linear map

$$\gamma_0 \colon W^{1,p}(Z) \longrightarrow L^p\big(\partial Z, \mu^{(N-1)}\big).$$

□

DEFINITION 2.4.43 *For every $u \in W^{1,p}(Z)$, we say that $\gamma_0(u)$ is the **trace** of u on ∂Z.*

PROPOSITION 2.4.44
If $Z \subseteq \mathbb{R}^N$ is bounded, open set which is Lipschitz and $p \in [1, +\infty)$, then

$$\int_Z D_i u(z)\, dz \;=\; \int_{\partial Z} \gamma_0(u) n_i \, d\mu^{(N-1)} \qquad \forall\, u \in W^{1,p}(Z),\; i \in \{1, \ldots, N\}$$

(as before $n = (n_i)_{i=1}^N$ is the outward unit normal on ∂Z).

PROOF Let $\{u_n\}_{n \geqslant 1} \subseteq C^1(\overline{Z})$ be such that

$$\|u - u_n\|_{W^{1,p}(Z)} \;\longrightarrow\; 0$$

(see Theorem 2.4.17). From the divergence theorem of multivariable calculus (see Theorem A.4.1), we have

$$\int_Z D_i u_n(z)\, dz \;=\; \int_{\partial Z} \gamma_0(u_n) n_i \, d\mu^{(N-1)} \qquad \forall\, n \geqslant 1. \tag{2.46}$$

From Theorem 2.4.42, we know that

$$\gamma_0(u_n) \longrightarrow \gamma_0(u) \quad \text{in } L^p\big(\partial Z, \mu^{(N-1)}\big).$$

Also since

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(Z),$$

we have

$$D_i u_n \longrightarrow D_i u \quad \text{in } L^p(Z).$$

So passing to the limit as $n \to +\infty$ in (2.46), we obtain

$$\int_Z D_i u(z)\, dz = \int_{\partial Z} \gamma_0(u) n_i \, d\mu^{(N-1)}.$$

□

This proposition leads to a ***Green's Formula*** for Sobolev functions. First an auxiliary result which provides still another version of the product rule.

LEMMA 2.4.45
If $Z \subseteq \mathbb{R}^N$ is an open set, $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then for all $u \in W^{1,p}(Z)$, $v \in W^{1,p'}(Z)$, we have $uv \in W^{1,1}(Z)$ and

$$D_i(uv) = uD_i v + vD_i u \qquad \forall\, i \in \{1, \dots, N\}.$$

PROOF First assume that $u \in C^1(Z)$ and consider a sequence $\{v_n\}_{n \geqslant 1} \subseteq C^1(Z)$, such that

$$v_n \longrightarrow v \quad \text{in } W^{1,p'}(Z') \qquad \forall\, Z' \subset\subset Z$$

(see Remark 2.4.18). Let $\vartheta \in \mathcal{D}(Z)$ and consider $Z' \subseteq \mathbb{R}^N$ bounded, open set, such that

$$\operatorname{supp} \vartheta \subseteq Z' \subseteq \overline{Z}' \subseteq Z.$$

For every $i \in \{1, \dots, N\}$, we have

$$\int_Z uv_n D_i \vartheta \, dz = \int_{Z'} uv_n D_i \vartheta \, dz = -\int_{Z'} \big(uD_i v_n + v_n D_i u\big)\vartheta \, dz,$$

so

$$\lim_{n \to +\infty} \int_{Z'} \big(uD_i v_n + v_n D_i u\big)\vartheta \, dz = \int_{Z'} (uD_i v + vD_i u)\, \varphi \, dz.$$

Since

$$v_n \longrightarrow v \quad \text{in } W^{1,p'}(Z),$$

we have

$$\lim_{n\to+\infty}\int_Z uv_nD_i\vartheta\,dz \;=\; \int_Z uvD_i\vartheta\,dz.$$

So in the limit as $n\to+\infty$, we obtain

$$\int_Z uvD_i\vartheta\,dz \;=\; -\int_{Z'}\big(uD_iv+vD_iu\big)\vartheta\,dz \qquad \forall\ \vartheta\in\mathcal{D}(Z),$$

so

$$D_i(uv) \;=\; uD_iv+vD_iu \;\in\; L^1(Z),$$

i.e., $uv\in W^{1,1}(Z)$.

Now we remove the restriction that $u\in C^1(Z)$. If $u\in W^{1,p}(Z)$, we can find a sequence $\{u_n\}_{n\geqslant 1}\subseteq C^1(Z)$, such that

$$u_n \;\longrightarrow\; u \quad \text{in } W^{1,p}(Z')$$

for all open sets $Z'\subset\subset Z$. From the first part of the proof we know that

$$D_i(u_nv) \;=\; u_nD_iv+vD_iu_n \qquad \forall\ n\geqslant 1.$$

Hence

$$u_nv \;\longrightarrow\; uv \quad \text{in } L^1(Z)$$

and

$$D_i(u_nv) \;\longrightarrow\; uD_iv+vD_iu \quad \text{in } L^1(Z).$$

Therefore $\{u_nv\}_{n\geqslant 1}$ is a Cauchy sequence in $W^{1,1}(Z)$ and so

$$u_nv \;\longrightarrow\; uv \quad \text{in } W^{1,1}(Z)$$

and

$$D_i(uv) \;=\; uD_iv+vD_iu.$$

□

THEOREM 2.4.46 (Green Formula)
If $Z\subseteq\mathbb{R}^N$ is a bounded, open set which is Lipschitz, $p\in(1,+\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$, then for all $u\in W^{1,p}(Z)$, $v\in W^{1,p'}(Z)$ and $i\in\{1,\dots,N\}$, we have

$$\int_Z uD_iv\,dz+\int_Z vD_iu\,dz \;=\; \int_{\partial Z}\gamma_0(uv)n_i\,d\mu^{(N-1)}.$$

PROOF From Lemma 2.4.45, we know that

$$uv \;\in\; W^{1,1}(Z) \quad\text{and}\quad D_i(uv) \;=\; uD_iv+v_iDu.$$

An application of Proposition 2.4.44 leads to Green's formula. □

COROLLARY 2.4.47
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz, $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$,
then for all $u \in W^{1,p}(Z)$ and $h \in C^1\big(\mathbb{R}^N; \mathbb{R}^N\big)$, we have

$$\int_Z u \operatorname{div} h \, dz + \int_Z \big(Du(z), h(z)\big)_{\mathbb{R}^N} \, dz \;=\; \int_{\partial Z} \gamma_0(u) \big(h, n\big)_{\mathbb{R}^N} \, d\mu^{(N-1)}.$$

Theorem 2.4.42 gives meaning to the quantity $u|_{\partial Z}$ for any $u \in W^{1,p}(Z)$, $p \in [1, +\infty)$. In fact we can do the same thing for $\frac{\partial^m}{\partial n^m}$ for any $m \geqslant 1$. Here $\frac{\partial}{\partial n}$ denotes the outward normal derivative on ∂Z. Also we can give a more precise description of the range of the trace map. To do this we need to introduce Sobolev spaces of fractional order on manifolds.

DEFINITION 2.4.48 *Let M be a compact manifold in $\mathbb{R}^N$. For any $s \in (0,1)$, $p \in [1, +\infty)$ and $u \in C^\infty(M)$, we define*

$$\|u\|_{W^{s,p}(M)} \stackrel{df}{=} \left[\int_M |u(z)|^p \, dz + \int_{M \times M} \frac{|u(z) - u(z')|^p}{\|z - z'\|_{\mathbb{R}^N}^{N-1+sp}} \, dz \, dz' \right]^{\frac{1}{p}}.$$

This is a norm. The completion of $C^\infty(M)$ under this norm is denoted by $W^{s,p}(M)$. For any $s > 0$, we set $s = k + \eta$, with a positive integer k and $\eta \in (0,1)$ (if s is not an integer). We define

$$W^{s,p}(M) \stackrel{df}{=} \big\{ u \in W^{k,p}(M) : \; D^\alpha u \in W^{\eta,p}(M) \text{ for all } |\alpha| = k \big\}.$$

REMARK 2.4.49 The definition makes sense also for any $Z \subseteq \mathbb{R}^N$ bounded and open. Also if $s = 0$, by convention

$$W^{0,p}(M) \;=\; L^p(M).$$

Now we can state the full version of the trace theorem.

THEOREM 2.4.50 (Trace Theorem)
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz, $m \geqslant 1$ is a positive integer and $p \in [1, +\infty)$,
then there exists a unique bounded, linear operator

$$\gamma \;=\; (\gamma_k)_{k=0}^{m-1} \colon W^{m,p}(Z) \longrightarrow L^p(\partial Z)^m,$$

such that

(a) *if $u \in C^\infty\big(\overline{Z}\big)$, then $\gamma_k(u) = \frac{\partial^k u}{\partial n^k}$ for $k = 1, \ldots, m-1$;*

(b) $\operatorname{range} \gamma = \prod_{k=0}^{m-1} W^{m-k-\frac{1}{p'},p}(\partial Z)$;

(c) $\ker \gamma = W_0^{m,p}(Z)$.

Using Theorem 2.4.46 and the continuity of the trace map γ_1, we obtain the following result.

THEOREM 2.4.51
If $Z \subseteq \mathbb{R}^N$ *is a bounded, open set which is Lipschitz and* $u \in H^2(Z)$, $v \in H^1(Z)$,
then

$$\int_Z (\Delta u) v \, dz + \int_Z \big(Du, Dv\big)_{\mathbb{R}^N} dz \;=\; \int_{\partial Z} v \frac{\partial u}{\partial n} \, d\mu^{(N-1)}.$$

REMARK 2.4.52 The equality in the above theorem is sometimes called ***Second Green's Identity***.

We can have a nonlinear extension of this theorem (i.e., $p \neq 2$). For this purpose if $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz and $q \in (1, +\infty)$, we introduce the space:

$$V^q\big(Z, \operatorname{div}\big) \stackrel{df}{=} \big\{v \in L^q\big(Z; \mathbb{R}^N\big) : \operatorname{div} v \in L^q(Z)\big\}.$$

We furnish $V^q\big(Z, \operatorname{div}\big)$ with the norm

$$\|v\|_{V^q(Z,\operatorname{div})} \stackrel{df}{=} \Big[\|v\|^q_{L^q(Z;\mathbb{R}^N)} + \big\|\operatorname{div} v\big\|^q_{L^q(Z)}\Big]^{\frac{1}{q}}.$$

It is easy to see that $V^q\big(Z, \operatorname{div}\big)$ equipped with this norm is a separable, reflexive Banach space and the embedding $C^\infty\big(\overline{Z}; \mathbb{R}^N\big) \subseteq V^q\big(Z, \operatorname{div}\big)$ is dense. The next theorem extends Theorem 2.4.46. For a proof of it we refer to Casas & Fernández (1989) and Kenmochi (1975).

THEOREM 2.4.53
If $Z \subseteq \mathbb{R}^N$ *is a bounded, open set which is Lipschitz,* $p \in (1, +\infty)$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$,
then there exists a unique bounded, linear operator

$$\gamma_n \colon V^{p'}\big(Z, \operatorname{div}\big) \longrightarrow W^{-\frac{1}{p'}, p'}(\partial Z) = W^{\frac{1}{p'}, p}(\partial Z)^*,$$

such that

$$\gamma_n(v) \;=\; (v, n)_{\mathbb{R}^N} \qquad \forall\, v \in C^\infty\big(\overline{Z}; \mathbb{R}^N\big)$$

and

$$\begin{aligned} &\int_Z u \operatorname{div} v \, dz + \int_Z \big(Du, v\big)_{\mathbb{R}^N} dz \\ &= \big(\gamma_n(v), \gamma_0(u)\big)_{W^{\frac{1}{p'}, p}(\partial Z)} \qquad \forall\, v \in V^{p'}\big(Z, \operatorname{div}\big),\; u \in W^{1,p}(Z). \end{aligned}$$

If for $u \in W^{1,p}(Z)$, we set

$$\Delta_p u \stackrel{df}{=} \operatorname{div}\left(\|Du\|^p_{\mathbb{R}^N} Du\right)$$

(the p-Laplacian), then from Theorem 2.4.53, we obtain the following nonlinear extension of Theorem 2.4.51.

THEOREM 2.4.54
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz, $p \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$u \in W^{1,p}(Z) \quad \text{and} \quad \Delta_p u \in L^{p'}(Z),$$

then there exists a unique element of $W^{-\frac{1}{p'},p'}(\partial Z)$, which by extension we denote by $\frac{\partial u}{\partial n_p}$, satisfying for all $v \in W^{1,p}(Z)$,

$$\int_Z (\Delta_p u) v \, dz + \int_Z \|Du\|^{p-2}_{\mathbb{R}^N} (Du, Dv)_{\mathbb{R}^N} \, dz = \left(\frac{\partial u}{\partial n_p}, \gamma_0(v)\right)_{W^{\frac{1}{p'},p}(\partial Z)}.$$

If $u \in W^{1,p}_0(Z)$, then we can extend u to $\widehat{u} \in W^{1,p}(\mathbb{R}^N)$ by simply setting u equal to zero on $\mathbb{R}^N \setminus Z$. It is not clear whether this extension is possible for $u \in W^{1,p}(Z)$. The next theorem shows when this is possible. It is known as ***extension theorem***.

THEOREM 2.4.55 (Extension Theorem)
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz and $\widehat{Z} \supseteq Z$ is open, then there exists an extension operator

$$E\colon W^{1,p}(Z) \longrightarrow W^{1,p}\big(\widehat{Z}\big),$$

such that

$$E(u)|_Z = u, \quad \big\|E(u)\big\|_{L^p(\widehat{Z})} \leqslant c\,\|u\|_{L^p(Z)} \qquad \forall\, u \in W^{1,p}(Z)$$

and

$$\big\|E(u)\big\|_{W^{1,p}\left(\widehat{Z}\right)} \leqslant c\,\|u\|_{W^{1,p}(Z)} \qquad \forall\, u \in W^{1,p}(Z),$$

for some $c = c\big(Z, \widehat{Z}\big) > 0$.

Next let us define the dual of $W^{1,p}(Z)$ for an open set $Z \subseteq \mathbb{R}^N$ and $p \in [1, +\infty)$. By considering the map

$$L_1\colon W^{1,p}(Z) \longrightarrow L^p(Z)^{N+1},$$

defined by

$$L_1(u) \stackrel{df}{=} \big(u, Du\big) \qquad \forall\, u \in W^{1,p}(Z),$$

we see that $W^{1,p}(Z)$ is isometrically isomorphic to a subspace of $L^p(Z)^{N+1}$. So from the Riesz representation theorem (see Theorem A.3.24), we have

THEOREM 2.4.56
If $Z \subseteq \mathbb{R}^N$ is an open set, $p \in [1, +\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $G \in W^{1,p}(Z)^*$, then

$$G(u) = \int_Z (h, Du)_{\mathbb{R}^N} \, dz \qquad \forall\, y \in W^{1,p}(Z),$$

for some $h \in L^{p'}(Z; \mathbb{R}^N)$

The dual of $W^{1,p}(Z)$ is generally more than a space of distributions on Z. Clearly the restriction on $C_c^\infty(Z)$ of an element in $W^{1,p}(Z)^*$ belongs to $\mathcal{D}(Z)^*$. However, this restriction is not injective because $C_c^\infty(Z)$ is not dense in $W^{1,p}(Z)$. The problem is that the elements of $W^{1,p}(Z)$ can have nonzero boundary values (in the sense of trace). On the other hand $C_c^\infty(Z)$ is dense in $W_0^{1,p}(Z)$. So for this Sobolev space the restriction is injective and we can have a convenient description of $W_0^{1,p}(Z)^*$.

THEOREM 2.4.57
If $Z \subseteq \mathbb{R}^N$ is an open set and $p \in [1, +\infty)$, then

$$W_0^{1,p}(Z)^* \stackrel{df}{=} \left\{ G \in \mathcal{D}(Z)^* : \; G = -\sum_{k=1}^N D_k T_{g_k}, \right.$$
$$\left. \text{for some } g = (g_k)_{k=1}^N \in L^{p'}(T)^N \right\}.$$

We set $W^{-1,p'}(Z) \stackrel{df}{=} W_0^{1,p}(Z)^*$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

For Sobolev functions of one variable (i.e., $N = 1$), using Theorem 2.2.24, we have the following convenient characterization.

THEOREM 2.4.58
If $Z = T = [0, b]$ $(b < +\infty)$ and $u \in W^{1,p}(T)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then u admits a representative which is absolutely continuous. Moreover, for $p = +\infty$, the representative is Lipschitz continuous.

REMARK 2.4.59 The result is also true if T is unbounded. In this case the representative is locally absolutely continuous (see Definition A.2.15**(b)**). For $p = +\infty$, again the representative is Lipschitz continuous (see Remark 2.4.21).

2.5 Inequalities and Embedding Theorems

The study of Sobolev spaces is useful because their elements possess special properties. Many of those properties are a consequence of the so-called ***embedding theorems***. Among other things, the embedding theorems establish regularity properties for the Sobolev functions, in addition to the ones implied by their definition. Let us start with a negative observation.

EXAMPLE 2.5.1 $H^1(Z)$ is in general not embedded in $L^\infty(Z)$. To see this let

$$Z \stackrel{df}{=} \left\{(x,y) \in \mathbb{R}^2 : \; x^2+y^2 < \frac{1}{e^2}\right\},$$

$\eta \in \left(0, \frac{1}{2}\right)$ and let $u(x,y) = |\ln r|^\eta$, where $r = \sqrt{x^2+y^2}$ and $(x,y) \in Z$. We have

$$\int\limits_Z |u|^2 \, dx\, dy \;=\; \int\limits_0^{2\pi} \left(\int\limits_0^{\frac{1}{e}} |\ln r|^{2\eta} r \, dr \right) d\vartheta \;\leqslant\; 2\pi \int\limits_0^{\frac{1}{e}} |\ln r|^{2\eta} r \, dr \;<\; +\infty,$$

i.e., $u \in L^2(Z)$. Note that $\frac{\partial u}{\partial x}$ exists on $Z \setminus \left\{(0,0)\right\}$ and we have

$$\frac{\partial u}{\partial x} \;=\; -\eta |\ln r|^{\eta-1} \frac{1}{r} \frac{x}{r} \;=\; -\eta |\ln r|^{\eta-1} \frac{1}{r} \cos\vartheta,$$

so

$$\int\limits_Z \left| \frac{\partial u}{\partial x} \right|^2 dx\, dy \;\leqslant\; \int\limits_0^{2\pi} \left(\int\limits_0^{\frac{1}{e}} \eta^2 |\ln r|^{2\eta-2} \frac{1}{r} \, dr \right) \cos^2\vartheta \, d\vartheta$$

$$\leqslant\; 2\pi\eta^2 \int\limits_0^{\frac{1}{e}} |\ln r|^{2\eta-2} \frac{1}{r} \, dr \;\leqslant\; 2\pi\eta^2 \left[-\frac{(-\ln r)^{2\eta-1}}{2\eta-1} \right]_0^{\frac{1}{e}} \;\leqslant\; \frac{2\pi\eta^2}{1-2\eta}$$

and thus

$$\frac{\partial u}{\partial x} \;\in\; L^2(Z) \quad \text{and} \quad \frac{\partial u}{\partial x} \;\in\; C^1\big(Z \setminus \left\{(0,0)\right\}\big).$$

Let $\vartheta \in \mathcal{D}(Z)$. We have

$$u(\cdot, y) \;\in\; C^1\left(-\frac{1}{e}, \frac{1}{e}\right) \qquad \forall\, y \in \left(-\frac{1}{e}, \frac{1}{e}\right) \setminus \{0\}$$

and so

$$-\int\limits_{-\frac{1}{e}}^{\frac{1}{e}} \frac{\partial u}{\partial x}(x,y)\vartheta(x,y)\, dy \;=\; \int\limits_{-\frac{1}{e}}^{\frac{1}{e}} u(x,y) \frac{\partial \vartheta}{\partial x}(x,y)\, dx.$$

Integrating with respect to $y \in \left(-\frac{1}{e}, \frac{1}{e}\right) \setminus \{0\}$ and using Fubini's theorem, we obtain

$$-\int_Z \frac{\partial u}{\partial x}\vartheta \, dx\, dy \;=\; \int_Z u\frac{\partial \vartheta}{\partial x}\, dx\, dy,$$

so

$$D_x T_u \;=\; T_{\frac{\partial u}{\partial x}}$$

and similarly we show that

$$D_y T_u \;=\; T_{\frac{\partial u}{\partial y}}.$$

This proves that $u \in H^1(Z)$. From this example we see that $H^1(Z) \not\subseteq L^\infty(Z)$.

Next we prove that if $f \in W^{1,p}(\mathbb{R}^N)$, then $f \in L^r(\mathbb{R}^N)$ for a certain range of $r \geqslant 1$ (including $r = p$).

DEFINITION 2.5.2 *If $p \in [1, N)$, then $p^* \stackrel{df}{=} \frac{Np}{N-p}$ is called the **critical Sobolev exponent** corresponding to p.*

The next inequality, known as the "Sobolev-Nirenberg-Gagliardo inequality" (or simply as "Sobolev inequality"; see also Theorem 1.6.7), implies that the embedding $W^{1,p}(\mathbb{R}^N) \subseteq L^r(\mathbb{R}^N)$ is continuous for $r \in [1, p^*]$.

THEOREM 2.5.3 (Sobolev-Nirenberg-Gagliardo Inequality)
If $p \in [1, N)$,
then there exists $c > 0$, such that

$$\|u\|_{p^*} \;\leqslant\; c\,\|Du\|_p \qquad \forall\, u \in W^{1,p}(\mathbb{R}^N).$$

PROOF By virtue of Theorem 2.4.13, we may assume that $u \in C_c^1\big(\mathbb{R}^N\big)$. For every $i \in \{1, \ldots, N\}$, we have

$$u\big(x_1, \ldots, x_i, \ldots, x_N\big) \;=\; \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}\big(x_1, \ldots, t_i, \ldots, x_N\big)\, dt_i,$$

thus

$$\big|u(x)\big| \;\leqslant\; \int_{-\infty}^{+\infty} \big\|Du\big(x_i, \ldots, t_i, \ldots, x_N\big)\big\|_{\mathbb{R}^N}\, dt_i \qquad \forall\, i \in \{1, \ldots, N\}$$

and so

$$\big|u(x)\big|^{\frac{N}{N-1}} \;\leqslant\; \prod_{i=1}^{N} \left(\int_{-\infty}^{+\infty} \big\|Du\big(x_1, \ldots, t_i, \ldots, x_N\big)\big\|_{\mathbb{R}^N}\, dt_i \right)^{\frac{1}{N-1}}.$$

We integrate with respect to $x_1 \in \mathbb{R}$.

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |u|^{1^*}\, dx_1 \\
&\leqslant \Big(\int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dt_1 \Big)^{\frac{1}{N-1}} \int_{-\infty}^{+\infty} \prod_{i=2}^{N} \Big(\int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dt_i \Big)^{\frac{1}{N-1}} dx_1 \\
&\leqslant \Big(\int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dt_1 \Big)^{\frac{1}{N-1}} \Big(\prod_{i=2}^{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dx_1\, dt_i \Big)^{\frac{1}{N-1}}.
\end{aligned}$$

Next we integrate with respect to $x_2 \in \mathbb{R}$.

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u|^{1^*}\, dx_1\, dx_2 \\
&\leqslant \Big(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dx_1\, dx_2 \Big)^{\frac{1}{N-1}} \Big(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dt_1\, dt_2 \Big)^{\frac{1}{N-1}} \times \\
&\quad \times \prod_{i=3}^{N} \Big(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dx_1\, dx_2\, dt_i \Big)^{\frac{1}{N-1}}.
\end{aligned}$$

We continue this way and we obtain

$$\int_{\mathbb{R}^N} |u|^{1^*}\, dx \;\leqslant\; \prod_{i=1}^{N} \Big(\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \|Du\|_{\mathbb{R}^N}\, dx_1 \dots dx_N \Big)^{\frac{1}{N-1}} \;=\; \|Du\|_1^{\frac{N}{N-1}},$$

so

$$\|u\|_{1^*} \;\leqslant\; \|Du\|_1 . \tag{2.47}$$

This proves the theorem for $p = 1$.

Now suppose that $p \in (1, +\infty)$. Set $h = |u|^\eta$ with $\eta > 0$ to be chosen in the process in the proof. Using (2.47) and Hölder's inequality (see Theorem A.2.27), we obtain

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N} |u|^{\frac{\eta N}{N-1}}\, dx \Big)^{\frac{N-1}{N}} \;\leqslant\; \int_{\mathbb{R}^N} \big\| D|u|^\eta \big\|_{\mathbb{R}^N}\, dx \;\leqslant\; \eta \int_{\mathbb{R}^N} |u|^{\eta-1} \|Du\|_{\mathbb{R}^N}\, dx \\
&\leqslant \eta \Big(\int_{\mathbb{R}^N} |u|^{(\eta-1)p'}\, dx \Big) \|Du\|_p ,
\end{aligned}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Choose $\eta > 0$ so that

$$\frac{\eta N}{N-1} \;=\; (\eta - 1)\frac{p}{p-1}.$$

Then

$$\frac{p}{p-1} \;=\; \eta\left(\frac{p}{p-1} - \frac{N}{N-1}\right)$$

and

$$p \;=\; \eta\frac{N-p}{N-1},$$

hence

$$\eta \;=\; \frac{Np-p}{N-p} \;=\; \frac{N-1}{N}p^*.$$

So we have

$$\left(\int\limits_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{N-1}{N}} \;\leqslant\; \eta\left(\int\limits_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{1}{p'}} \|Du\|_p\,,$$

so

$$\|u\|_{p^*} \;\leqslant\; c\,\|Du\|_p\,,$$

with $c = c(N,p) > 0$ (note that $\frac{N-1}{N} \geqslant \frac{N-p}{Np}$). □

The next inequality is known as "Poincaré's inequality" and is very useful in the study of Dirichlet elliptic equations.

THEOREM 2.5.4 (Poincaré's Inequality)
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set and $p \in [1, +\infty)$, then there exists $c = c(Z,p) > 0$, such that

$$\|u\|_p \;\leqslant\; c\,\|Du\|_p \qquad \forall\; u \in W^{1,p}(Z).$$

PROOF Since $Z \subseteq \mathbb{R}^N$ is bounded, we can find $\xi > 0$, such that

$$Z \;\subseteq\; (-\xi, \xi)^N.$$

Let

$$\vartheta \;\in\; \mathcal{D}(Z)$$

and extend it by zero on the whole "cube" $(-\xi,\xi)^N$. For every $z = (z_k)_{k=1}^N$, we have

$$\vartheta(z) \;=\; \int\limits_{-\xi}^{z_N} \frac{\partial \vartheta}{\partial x_N}(x_1, \ldots, x_{N-1}, t)\,dt.$$

By Hölder's inequality (see Theorem A.2.27), with $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\left|\vartheta(z_1,\dots,z_N)\right|^p \leqslant (2\xi)^{\frac{p}{p'}} \int_{-\xi}^{\xi} \left|\frac{\partial\vartheta}{\partial z_N}(z_1,\dots,z_{N-1},t)\right|^p dt,$$

so

$$\int_{(-\xi,\xi)^{N-1}} \left|\vartheta(z_1,\dots,z_N)\right|^p dz_1\dots dz_{N-1}$$

$$\leqslant (2\xi)^{\frac{p}{p'}} \int_{(-\xi,\xi)^{N-1}} \left(\int_{-\xi}^{\xi} \left|\frac{\partial\vartheta}{\partial z_N}(z_1,\dots,z_{N-1},t)\right|^p dt\right) dz_1,\dots dz_{N-1},$$

thus

$$\int_Z \left|\vartheta(z)\right|^p dz \leqslant (2\xi)^{\frac{p}{p'}+1} \int_Z \left\|D\vartheta\right\|_{\mathbb{R}^N}^p dz$$

and since $\frac{p}{p'}+1=p$, we have

$$\left\|\vartheta\right\|_p \leqslant 2\xi \left\|D\vartheta\right\|_p.$$

Since $\mathcal{D}(Z)$ is dense in $W^{1,p}(Z)$, we conclude that

$$\left\|u\right\|_p \leqslant c\left\|Du\right\|_p \qquad \forall\, u\in W^{1,p}(Z),$$

for some $c=c(Z,p)>0$. □

REMARK 2.5.5 In fact the result is true if $Z\subseteq\mathbb{R}^N$ is unbounded but of finite width, namely it lies between two parallel hyperplanes (see Adams (1975, p. 158)). However, the result fails in truly unbounded domains $Z\subseteq\mathbb{R}^N$.

EXAMPLE 2.5.6 Let $Z\subseteq\mathbb{R}^N$ and $\vartheta\in\mathcal{D}(\mathbb{R}^N)$ be such that

$$\vartheta|_{\overline{B}_1(0)}\equiv 1, \quad \vartheta|_{B_2(0)^c}\equiv 0 \quad \text{and} \quad 0\leqslant\vartheta\leqslant 1.$$

Let $\vartheta_m(z)\overset{df}{=}\vartheta\left(\frac{z}{m}\right)$. Then if $N<p$, we have

$$\left\|\vartheta_m\right\|_{W^{1,p}(Z)} \longrightarrow 0 \quad \text{as } m\to+\infty,$$

while

$$\left\|\vartheta_m\right\|_p \geqslant \lambda^N\big(B_m(0)\big) \longrightarrow +\infty \quad \text{as } m\to+\infty.$$

An immediate useful consequence of Theorem 2.5.4, which is a basic tool in the study of Dirichlet elliptic problems, is the following result.

COROLLARY 2.5.7
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set and $p \in [1, +\infty)$, then

$$\|Du\|_p \stackrel{df}{=} \left(\int_{\mathbb{R}^N} \left\|Du(z)\right\|_{\mathbb{R}^N}^p \right)^{\frac{1}{p}} \qquad \forall\ u \in W_0^{1,p}(Z)$$

is a norm on $W_0^{1,p}(Z)$ equivalent to the usual Sobolev norm $\|u\|_{W^{1,p}(Z)}$.

Let us use this opportunity to mention a few equivalent norms for the Sobolev spaces $W^{1,p}(Z)$.

PROPOSITION 2.5.8
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set and $p \in [1, +\infty)$, then the following three norms are equivalent to the original Sobolev norm $\|\cdot\|_{W^{1,p}(Z)}$:

$$\|u\|_{(1)} \stackrel{df}{=} \left(\left\|Du\right\|_p^p + \left| \int_Z u\, dz \right|^p \right)^{\frac{1}{p}}$$

$$\|u\|_{(2)} \stackrel{df}{=} \left(\left\|Du\right\|_p^p + \left| \int_{\partial Z} u\, d\mu^{(N-1)} \right|^p \right)^{\frac{1}{p}}$$

$$\|u\|_{(3)} \stackrel{df}{=} \left(\left\|Du\right\|_p^p + \int_{\partial Z} |u|^p\, d\mu^{(N-1)} \right)^{\frac{1}{p}}.$$

REMARK 2.5.9 If $N = 1$ and $Z = (0, b)$ $(b \in (0, +\infty))$, then

$$\int_{\partial Z} u\, d\mu^{(N-1)} = u(0) + u(b).$$

Before passing to the so-called ***embedding theorems***, let us mention one more inequality, known in the literature as "Morrey's inequality." First a definition.

DEFINITION 2.5.10 *Let $\eta \in (0, 1)$. A function $u\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be **Hölder continuous with exponent** η, if*

$$\sup_{\substack{x,\, y \in \mathbb{R}^N \\ x \neq y}} \frac{|u(x) - u(y)|}{\|x - y\|_{\mathbb{R}^N}^{\eta}} < +\infty.$$

In the proof of Morrey's inequality, we shall use the following lemma.

LEMMA 2.5.11
For every $p \in [1, +\infty)$, there exists $c = c(N,p) > 0$, such that

$$\int\limits_{\overline{B}_r(x)} \big|u(y) - u(z)\big|^p \, dy \;\leqslant\; cr^{N+p-1} \int\limits_{\overline{B}_r(x)} \big\|Du(y)\big\|^p_{\mathbb{R}^N} \big\|y - z\big\|^{1-N}_{\mathbb{R}^N} \, dy,$$

for all $r > 0$, $u \in C^1\big(\overline{B}_r(x)\big)$ and all $y, z \in \overline{B}_r(x)$.

PROOF If $y, z \in \overline{B}_r(x)$, then

$$u(y) - u(z) \;=\; \int\limits_0^1 \frac{d}{dt} u\big(z + t(y-z)\big) \, dt \;=\; \int\limits_0^1 \big(Du(z + t(y-z)), y - z\big)_{\mathbb{R}^N} \, dt,$$

thus

$$\big|u(y) - u(z)\big|^p \;\leqslant\; \|y - z\|^p_{\mathbb{R}^N} \int\limits_0^1 \big\|Du\big(z + t(y-z)\big)\big\|^p_{\mathbb{R}^N} \, dt.$$

So using Proposition 1.3.23**(b)** and **(c)**, for $s > 0$, we have

$$\begin{aligned}
&\int\limits_{\overline{B}_r(x) \cap \partial\overline{B}_s(z)} \big|u(y) - u(z)\big|^p \, d\mu^{(N-1)}(y) \\
&\leqslant\; s^p \int\limits_0^1 \int\limits_{\overline{B}_r(x) \cap \partial\overline{B}_s(z)} \big\|Du\big(z + t(y-z)\big)\big\|^p_{\mathbb{R}^N} \, d\mu^{(N-1)}(y) \, dt \\
&\leqslant\; s^p \int\limits_0^1 \frac{1}{t^{N-1}} \int\limits_{\overline{B}_r(x) \cap \partial\overline{B}_{st}(z)} \big\|Du(w)\big\|^p_{\mathbb{R}^N} \, d\mu^{(N-1)}(w) \, dt \\
&=\; s^{N+p-1} \int\limits_0^1 \int\limits_{\overline{B}_r(x) \cap \partial\overline{B}_{st}(z)} \big\|Du(w)\big\|^p_{\mathbb{R}^N} \big\|w - z\big\|^{1-N}_{\mathbb{R}^N} \, d\mu^{(N-1)}(w) \, dt \\
&=\; s^{N+p-2} \int\limits_{\overline{B}_r(x) \cap \overline{B}_s(z)} \big\|Du(w)\big\|^p_{\mathbb{R}^N} \big\|w - z\big\|^{1-N}_{\mathbb{R}^N} \, dw.
\end{aligned}$$

Then from Example 1.5.27**(a)**, we have

$$\int\limits_{\overline{B}_r(x)} \big|u(y) - u(z)\big|^p \, dy \;\leqslant\; cr^{N+p-1} \int\limits_{\overline{B}_r(x)} \big\|Du(y)\big\|^p_{\mathbb{R}^N} \big\|y - z\big\|^{1-N}_{\mathbb{R}^N} \, dy,$$

with $c = c(N,p) > 0$. □

THEOREM 2.5.12 (Morrey Inequality)
(a) *For every $p \in (N, +\infty)$, there exists $c = c(N,p) > 0$, such that*

$$\big|u(y) - u(z)\big| \;\leqslant\; cr \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big\|Du(w)\big\|^p_{\mathbb{R}^N}\, dw,$$

for all $r > 0$, $u \in W^{1,p}\big(B_r(x)\big)$ and λ^N-almost all $y, z \in \overline{B}_r(x)$.

(b) *If $p \in (N, +\infty)$ and $u \in W^{1,p}(\mathbb{R}^N)$, then the limit*

$$\lim_{r \searrow 0} \overline{u}_{r,z} \;=\; u^*(z)$$

exists for all $z \in \mathbb{R}^N$ and u^* is Hölder continuous with exponent $1 - \frac{N}{p}$; recall that

$$\overline{u}_{r,z} \;=\; \frac{1}{\lambda^N(\overline{B}_r(z))} \int\limits_{\overline{B}_r(z)} u(x)\, d\lambda^N(x).$$

PROOF **(a)** First suppose that $u \in C^1\big(B_r(x)\big)$. Using Lemma 2.5.11 with $p = 1$, Hölder's inequality (see Theorem A.2.27) and recalling that $p' = \frac{p}{p-1}$, for all $y, z \in \overline{B}_r(x)$, we have

$$\begin{aligned}
\big|u(y) - u(z)\big| \;&\leqslant\; \frac{1}{\lambda^N(\overline{B}_r(x))} \int\limits_{\overline{B}_r(x)} \big(\big|u(y) - u(w)\big| + \big|u(w) - u(z)\big|\big)\, dw \\
&\leqslant\; c \int\limits_{\overline{B}_r(x)} \big\|Du(w)\big\|_{\mathbb{R}^N} \Big(\|y - w\|^{1-N}_{\mathbb{R}^N} + \|z - w\|^{1-N}_{\mathbb{R}^N}\Big)\, dw \\
&\leqslant\; c\bigg(\int\limits_{\overline{B}_r(x)} \Big(\|y - w\|^{1-N}_{\mathbb{R}^N} + \|z - w\|^{1-N}_{\mathbb{R}^N}\Big)^{p'}\, dw\bigg)^{\frac{1}{p'}} \bigg(\int\limits_{\overline{B}_r(x)} \|Du\|^p_{\mathbb{R}^N}\, dw\bigg)^{\frac{1}{p}} \\
&\leqslant\; cr^{(N-(N-1)p')\frac{1}{p'}} \bigg(\int\limits_{\overline{B}_r(x)} \big\|Du(w)\big\|^p_{\mathbb{R}^N}\bigg)^{\frac{1}{p}} \\
&\leqslant\; cr^{1-\frac{N}{p}} \bigg(\int\limits_{\overline{B}_r(x)} \big\|Du(w)\big\|^p_{\mathbb{R}^N}\, dw\bigg)^{\frac{1}{p}}.
\end{aligned}$$

Invoking Theorem 2.4.13, we see that the same estimate holds for all $u \in W^{1,p}\big(B_r(x)\big)$ and for λ^N-almost all $y, z \in B_r(x)$.

(b) From part **(a)**, for λ^N-almost all $y, z \in \overline{B}_r(x)$, with $r = \|x-y\|_{\mathbb{R}^N}$, we have

$$\begin{aligned} \big|u(y)-u(z)\big| \;&\leqslant\; c\,\|y-z\|_{\mathbb{R}^N}^{1-\frac{N}{p}} \left(\int\limits_{\overline{B}_r(x)} \big\|Du(w)\big\|_{\mathbb{R}^N}^p \, dw \right)^{\frac{1}{p}} \\ &\leqslant\; c\,\|Du\|_{L^p(Z;\mathbb{R}^N)}\, \|y-z\|_{\mathbb{R}^N}^{1-\frac{N}{p}}, \end{aligned}$$

so u is λ^N-almost everywhere equal to a Hölder continuous function u^* with exponent $1-\frac{N}{p}$. So

$$\lim_{r \searrow 0} \overline{u}_{r,z} \;=\; u^*(z) \qquad \forall\, z \in \mathbb{R}^N.$$

□

REMARK 2.5.13 If $p = +\infty$, then we know that the elements of $W^{1,\infty}\big(\mathbb{R}^N\big)$ are Lipschitz continuous functions (see Remark 2.4.21). From Theorem 2.5.12, it follows that, if $u \in W^{1,p}(\mathbb{R}^N)$, $N < p$, then

$$\lim_{\|z\|_{\mathbb{R}^N} \to +\infty} u(z) \;=\; 0.$$

Already from Theorem 2.5.3, we know that the embedding

$$W^{1,p}(\mathbb{R}^N) \;\subseteq\; L^r(\mathbb{R}^N)$$

is continuous for all $r \in [1, p^*]$. Moreover, from Theorem 2.5.12, we know that if $p > N$, then the embedding

$$W^{1,p}(\mathbb{R}^N) \;\subseteq\; L^\infty(\mathbb{R}^N)$$

is continuous. The next two theorems make these facts much more precise. The first theorem is known as the ***Sobolev embedding theorem***, while the second is known as the ***Rellich-Kondrachov embedding theorem***. First let us introduce a new kind of boundary regularity.

DEFINITION 2.5.14 **(a)** *For given $z \in \mathbb{R}^N$, an open ball B_1 with center z and an open ball B_2 not containing z, the set*

$$C_z \;\stackrel{df}{=}\; \big\{ z + t(y-z) :\; y \in B_2,\; t > 0 \big\} \cap B_1$$

is called a ***finite cone in*** $\mathbb{R}^N$.

(b) *Let $Z \subseteq \mathbb{R}^N$ be an open set. We say that Z has the* ***cone property****, if there exists a finite cone C_0, such that for each $z \in Z$, there exists an orthogonal transformation $U_x \in \mathcal{L}\big(\mathbb{R}^N;\mathbb{R}^N\big)$, for which we have*

$$U_x(C_0) \;\subseteq\; Z.$$

REMARK 2.5.15 If $Z \subseteq \mathbb{R}^N$ is a bounded, open set which is Lipschitz, then it has the cone property (see Adams (1975, p. 51)). Also it is clear that if Z is C^1, then it has the cone property.

THEOREM 2.5.16 (Sobolev Embedding Theorem)
If $Z \subseteq \mathbb{R}^N$ is an open set with the cone property, $p \in [1, +\infty)$, k, m are integers, $k \geqslant 0$, $m \geqslant 1$,
then

(a) *if $mp < N$, then the embedding $W^{k+m,p}(Z) \subseteq W^{k,r}(Z)$ is continuous for all $r \in \left[p, \frac{Np}{N-mp}\right]$ and in particular if $k = 0$, we have that the embedding $W^{m,p}(Z) \subseteq L^r(Z)$ is continuous for all $r \in \left[p, \frac{Np}{N-mp}\right]$;*

(b) *if $mp = N$, then the embedding $W^{k+m,p}(Z) \subseteq W^{k,r}(Z)$ is continuous for all $r \in [p, +\infty)$ and in particular if $k = 0$, we have that the embedding $W^{m,p}(Z) \subseteq L^r(Z)$ is continuous for all $r \in [p, +\infty)$;*

(c) *if $mp > N$, then the embedding $W^{k+m,p}(Z) \subseteq C_b^k(Z)$ is continuous with $C_b^k(Z)$ being the space of all functions $u \in C^k(Z)$, such that $D^\alpha u$ is bounded on Z for all multiindices $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant k$.*

When $Z \subseteq \mathbb{R}^N$ is bounded, the conclusions are stronger.

THEOREM 2.5.17 (Rellich-Kondrachov Embedding Theorem)
If $Z \subseteq \mathbb{R}^N$ is an open, bounded set with the cone property, $p \in [1, +\infty)$, k, m are integers, $k \geqslant 0$, $m \geqslant 1$,
then

(a) *if $mp < N$, then the embedding $W^{k+m,p}(Z) \subseteq W^{k,r}(Z)$ is compact for all $r \in \left[1, \frac{Np}{N-mp}\right)$ and in particular if $k = 0$, we have that the embedding $W^{m,p}(Z) \subseteq L^r(Z)$ is compact for all $r \in \left[1, \frac{Np}{N-mp}\right)$;*

(b) *if $mp = N$, then the embedding $W^{k+m,p}(Z) \subseteq W^{k,r}(Z)$ is compact for all $r \in [1, +\infty)$ and in particular if $k = 0$, we have that the embedding $W^{m,p}(Z) \subseteq L^r(Z)$ is compact for all $r \in [1, +\infty)$;*

(c) *if $mp > N$, then the embedding $W^{k+m,p}(Z) \subseteq C^k(\overline{Z})$ is compact for all $r \in [1, +\infty]$ and in particular if $k = 0$, we have that the embedding $W^{m,p}(Z) \subseteq L^r(Z)$ is compact for all $r \in [1, +\infty]$.*

REMARK 2.5.18 Since $W_0^{1,p}(Z)$ is a closed subspace of $W^{1,p}(Z)$, we see that both embedding theorems (i.e., Theorems 2.5.16 and 2.5.17) are also valid for $W_0^{1,p}(Z)$. In fact in this case the cone property can be dropped.

Using the embedding theorems, we can prove a generalized form of Poincaré's inequality (see Theorem 2.5.4).

THEOREM 2.5.19 (Generalized Poincaré Inequality)
If $Z \subseteq \mathbb{R}^N$ is a bounded, open, connected set with the cone property, $p \in (1, +\infty)$ and V is a closed linear subspace of $W^{1,p}(Z)$, such that the only constant function in V is the zero function,
then

$$\|u\|_p \leqslant c \|Du\|_p \qquad \forall\, u \in V,$$

for some $c > 0$.

PROOF We proceed by contradiction. So suppose that the conclusion of the theorem is not true. We can find a sequence $\{u_n\}_{n \geqslant 1}$, such that

$$\|u_n\|_p > n \|Du_n\|_p \qquad \forall\, n \geqslant 1.$$

Let

$$v_n \stackrel{df}{=} \frac{u_n}{\|u_n\|_p} \qquad \forall\, n \geqslant 1.$$

Then

$$\|v_n\|_p = 1 \quad \text{and} \quad \|Dv_n\|_p < \frac{1}{n},$$

hence

$$Dv_n \longrightarrow 0 \quad \text{in } L^p\big(Z; \mathbb{R}^N\big).$$

Therefore

$$\text{the sequence } \{v_n\}_{n \geqslant 1} W^{1,p}(Z) \text{ is bounded}$$

and so by passing to a subsequence if necessary, we may assume that

$$v_n \stackrel{w}{\longrightarrow} v \quad \text{in } W^{1,p}(Z).$$

Because of Theorem 2.5.17**(a)**, we have that

$$v_n \longrightarrow v \quad \text{in } L^p(Z).$$

Hence

$$\|v\|_p = 1.$$

Also exploiting the weak lower semicontinuity of the norm in a Banach space, we have

$$\|Dv\|_p \leqslant \liminf_{n \to +\infty} \|Dv_n\|_p = 0,$$

so $v \in V$ is constant, thus

$$v = 0,$$

which is a contradiction to the fact that $\|v\|_p = 1$. ∎

COROLLARY 2.5.20
If $Z \subseteq \mathbb{R}^N$ is a bounded, open, connected set which is Lipschitz, $S_0 \subseteq \partial Z$ with $\mu^{(N-1)}(S_0) > 0$, $p \in (1, +\infty)$ and

$$V \stackrel{df}{=} \left\{u \in W^{1,p}(Z) : \ \gamma_0(u) = 0 \text{ on } S_0\right\}$$

(γ_0 being the trace operator on $W^{1,p}(Z)$; see Theorem 2.4.42),
then

$$\|u\|_p \ \leqslant \ c\,\|Du\|_p \qquad \forall\ u \in V,$$

for some $c > 0$.

PROOF Since γ_0 is continuous linear (see Theorem 2.4.42), V is closed, linear subspace of $W^{1,p}(Z)$. Suppose that the constant function $u \equiv c \in V$. We have

$$0 \ = \ \gamma_0(u) \ = \ \gamma_0(c) \ = \ c.$$

This permits the application of Theorem 2.5.19. □

This leads to another fundamental inequality, known as the "Poincaré-Wirtinger inequality." It is an essential tool in the study of periodic ordinary differential equations and Neumann partial differential equations.

THEOREM 2.5.21 (Poincaré-Wirtinger Inequality)
If $Z \subseteq \mathbb{R}^N$ is a bounded, open, connected set with the cone property and $p \in (1, +\infty)$,
then

$$\|u - \overline{u}\|_p \ \leqslant \ c\,\|Du\|_p \qquad \forall\ u \in W^{1,p}(Z),$$

for some $c > 0$, where $\overline{u} \stackrel{df}{=} \frac{1}{\lambda^N(Z)} \int\limits_Z u(z)\,dz$.

PROOF Let

$$V \stackrel{df}{=} \left\{u \in W^{1,p}(Z) : \ \int\limits_Z u(z)\,dz = 0\right\}.$$

Clearly V is a closed, linear subspace of $W^{1,p}(Z)$. If the constant function $u \equiv c \in V$, then

$$\int\limits_Z u(z)\,dz \ = \ c\lambda^N(Z) \ = \ 0,$$

hence $c = 0$. Also

$$u - \overline{u} \ \in \ V \qquad \forall\ u \in W^{1,p}(Z).$$

So an application of Theorem 2.5.19 finishes the proof. □

We already know that for the Sobolev functions of one variable (i.e., $N = 1$), the situation is better (see Theorem 2.4.58). In this case the embedding theorems take a sharper form.

THEOREM 2.5.22
Let $T \subseteq \mathbb{R}$ be an interval.

(a) *If T is open and $p \in [1, +\infty]$, then the embedding $W^{1,p}(T) \subseteq L^\infty(T)$ is continuous;*

(b) *If T is open, bounded and $p \in (1, +\infty]$, then the embedding $W^{1,p}(T) \subseteq C(\overline{T})$ is compact;*

(c) *If T is open, bounded and $p \in [1, +\infty)$, then the embedding $W^{1,1}(T) \subseteq L^p(T)$ is compact.*

PROOF **(a)** Using the extension theorem (see Theorem 2.4.55), we see that without any loss of generality, we may assume that $T = \mathbb{R}$. First let $u \in C_c^1(\mathbb{R})$ and for $p \in [1, +\infty)$ let

$$\xi_p(r) \stackrel{df}{=} |r|^{p-1} r \qquad \forall\, r \in \mathbb{R}.$$

We have that $\xi_p(u) \in C_c^1(\mathbb{R})$ and from the chain rule, we have

$$\frac{d}{dt}\xi_p\big(u(t)\big) = \xi_p'\big(u(t)\big)\frac{d}{dt}u(t) = p\big|u(t)\big|^{p-1}\frac{d}{dt}u(t)$$

(since $\xi_p'(r) = p|r|^{p-1}$). Hence for every $t \in \mathbb{R}$, we have

$$\xi_p\big(u(t)\big) = \int_{-\infty}^{t} p\big|u(s)\big|^{p-1}u'(s)\,ds$$

(since $u \in C_c^1(\mathbb{R})$), so by Hölder's inequality (see Theorem A.2.27), we have

$$\big|\xi_p\big(u(t)\big)\big| = \big|u(t)\big|^p \leqslant \int_{\mathbb{R}} p\big|u(s)\big|^{p-1}\big|u'(s)\big|\,ds$$

$$\leqslant p\,\|u\|_p^{p-1}\,\|u'\|_p.$$

By Young's inequality (see Proposition A.4.5), we obtain

$$\|u\|_\infty \leqslant c\|u\|_{W^{1,p}(T)} \qquad \forall\, u \in C_c^1(\mathbb{R}), \tag{2.48}$$

for some $c > 0$.

Now let $u \in W^{1,p}(\mathbb{R})$, with $p \in [1, +\infty)$. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq C_c^1(\mathbb{R})$, such that

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\mathbb{R}).$$

From (2.48), we have

$$\|u_n - u_m\|_\infty \leqslant c\,\|u_n - u_m\|_{W^{1,p}(T)} \qquad \forall\, n, m \geqslant 1,$$

hence the sequence $\{u_n\}_{n\geqslant 1} \subseteq L^\infty(\mathbb{R})$ is a Cauchy sequence. Therefore

$$u_n \longrightarrow u \quad \text{in } L^\infty(\mathbb{R})$$

and we have proved the continuity of the embedding $W^{1,p}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ for $p \in [1, +\infty)$. Of course the result is trivially true for $p = +\infty$.

(b) Let $\overline{B}_1(0)$ be the closed unit ball in $W^{1,p}(T)$, $p \in (1, +\infty]$. Let $u \in \overline{B}_1(0)$. We have

$$\big|u(t) - u(s)\big| = \left|\int_s^t u'(\tau)\,d\tau\right| \leqslant \|u'\|_p\,|t-s|^{\frac{1}{p'}} \leqslant |t-s|^{\frac{1}{p'}} \qquad \forall\, t, s \in T,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ (see Theorem 2.4.58). Then the Arzela-Ascoli theorem (see Theorem 2.3.2) implies that $\overline{B}_1(0)$ is relatively compact in $C\big(\overline{T}\big)$ and so we have proved the compactness of the embedding $W^{1,p}(T) \subseteq C\big(\overline{T}\big)$ for $p \in (1, +\infty]$ with $T \subseteq \mathbb{R}$ being a bounded, open interval.

(c) Let $\overline{B}_1(0)$ be the closed unit ball in $W^{1,1}(T)$. Let E be an open subset of T, such that $\overline{E} \subseteq T$. Let

$$|h| < d_{\mathbb{R}}\big(E, T^c\big) \quad \text{and} \quad u \in \overline{B}_1(0).$$

From Remark 2.4.21, we know that

$$\|\tau_h(u) - u\|_{L^1(E)} \leqslant |h|\,\|u'\|_{L^1(T)} \leqslant |h|,$$

so

$$\int_E \big|u(t+h) - u(t)\big|^p\,dt \leqslant \Big(2\,\|u\|_{L^\infty(T)}\Big)^{p-1} \int_E \big|u(t+h) - u(t)\big|\,dt \leqslant c|h|,$$

for some $c > 0$ with $p \in [1, +\infty)$. Thus

$$\left(\int_E \big|u(t+h) - u(t)\big|^p\,dt\right)^{\frac{1}{p}} \leqslant c^{\frac{1}{p}}|h|^{\frac{1}{p}}.$$

Invoking Theorem 2.3.6, we infer that $\overline{B}_1(0)$ is relatively compact in $L^p(T)$, $p \in [1, +\infty)$. This proves the compactness of the embedding $W^{1,1}(T) \subseteq L^p(T)$ for $p \in [1, +\infty)$ with $T \subseteq \mathbb{R}$ being a bounded, open interval. ∎

REMARK 2.5.23 The embedding

$$W^{1,1}(T) \subseteq C(\overline{T})$$

is continuous but never compact even if the open interval T is bounded. If $T \subseteq \mathbb{R}$ is a bounded, open interval and $\{u_n\}_{n\geqslant 1} \subseteq W^{1,1}(T)$ is a bounded sequence, then we can extract a subsequence $\{u_{n_k}\}_{k\geqslant 1}$, such that $\{u_{n_k}(t)\}_{k\geqslant 1}$ converges for every $t \in T$ (see Denkowski, Migórski & Papageorgiou (2003*a*, p. 229)). Also if $T \subseteq \mathbb{R}$ is an unbounded, open interval and $p \in (1, +\infty]$, then the embedding

$$W^{1,p}(T) \subseteq L^{\infty}(T)$$

is continuous, but not compact.

To the equivalent Sobolev norms mentioned in Proposition 2.5.8, we can add one more.

THEOREM 2.5.24

(a) *If $T \subseteq \mathbb{R}$ is a bounded, open interval and $r \in [1, +\infty]$, then*

$$\|u\|_{W^{1,p}(T)} \stackrel{df}{=} \|u\|_r + \|u'\|_p$$

is equivalent to the usual norm $\|\cdot\|_{W^{1,p}(T)}$ on $W^{1,p}(T)$;

(b) *If $Z \subseteq \mathbb{R}^N$ is a bounded, open set with the cone property and $p \in [1, +\infty)$, then*

$$\|u\|_{W^{1,p}(T)} \stackrel{df}{=} \|u\|_r + \|Du\|_p$$

is equivalent to the usual norm $\|\cdot\|_{W^{1,p}(Z)}$ on $W^{1,p}(Z)$ provided that $r \in [1, p^*]$ if $p < N$, $r \in [1, +\infty)$ if $p = N$ and $r \in [1, +\infty]$ if $p > N$.

Now let $Z \subseteq \mathbb{R}^N$ be a bounded, open set which is Lipschitz. Consider the Banach space $M(Z)$ of Radon measures on Z with the total variation norm. Recall that $M(Z) = \big(C_0(Z)\big)^*$ (see Theorem 2.3.41). From Theorem 2.4.17 (see also Remark 2.5.18), we know that if $r > N$, then the embedding

$$W_0^{1,r}(Z) \subseteq C_0(\overline{Z}) = \big\{u \in C(\overline{Z}) : u|_{\partial Z} = 0\big\}$$

is continuous and dense. So by virtue of Lemma 2.2.27**(a)**, we have that

$$M(Z) \subseteq W^{-1,r'}(Z),$$

with $\frac{1}{r} + \frac{1}{r'} = 1$ (see Theorem 2.4.57). This observation is crucial in proving the next compactness result.

THEOREM 2.5.25
If $Z \subseteq \mathbb{R}^N$ is a bounded, open set with the cone property and $\{\mu_n\}_{n\geqslant 1} \subseteq M(Z)$ is a bounded sequence,
then the sequence $\{\mu_n\}_{n\geqslant 1}$ is relatively compact in $W^{-1,r}(Z)$ for every $r \in [1, 1^*)$, where $1^* \stackrel{df}{=} \frac{N}{N-1}$.

PROOF By virtue of Theorem 2.3.46, we can find a subsequence $\{\mu_{n_k}\}_{n\geqslant 1}$ of the sequence $\{\mu_n\}_{n\geqslant 1}$ and $\mu \in M(Z)$, such that

$$\mu_{n_k} \stackrel{w}{\longrightarrow} \mu \quad \text{in } M(Z).$$

Let $r' > 0$ be the conjugate exponent of $r \in [1, 1^*)$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Then $r' > N$ and so by Theorem 2.5.17**(c)**, the embedding $W_0^{1,r'}(Z) \subseteq C_0\big(\overline{Z}\big)$ is compact. Let $\overline{B}_1(0)$ be the closed unit ball in $W_0^{1,r'}(Z)$. We see that $\overline{B}_1(0)$ is compact in $C_0\big(\overline{Z}\big)$. So given $\varepsilon > 0$ we can find a finite sequence $\{u_i\}_{i=1}^{m(\varepsilon)}$, such that for every $u \in \overline{B}_1(0)$, we have

$$\min_{1\leqslant i\leqslant m(\varepsilon)} \|u - u_i\|_{C_0\left(\overline{Z}\right)} < \varepsilon. \tag{2.49}$$

So, if $u \in \overline{B}_1(0)$, for some $i \in \{1, \ldots, m(\varepsilon)\}$, we have

$$\begin{aligned}
&\left| \int_Z u \, d\mu_{n_k} - \int_Z u \, d\mu \right| \\
\leqslant\ &\left| \int_Z (u - u_i) \, d\mu_{n_k} \right| + \left| \int_Z u_i \, d\mu_{n_k} - \int_Z u_i \, d\mu \right| + \left| \int_Z (u_i - u) \, d\mu \right| \\
\leqslant\ &2\varepsilon \sup_{k\geqslant 1} |\mu_{n_k}|(Z) + \left| \int_Z u_i \, d\mu_{n_k} - \int_Z u_i \, d\mu \right|.
\end{aligned}$$

Since

$$\mu_{n_k} \stackrel{w}{\longrightarrow} \mu \quad \text{as } k \to +\infty,$$

we have that

$$\left| \int_Z u_i \, d\mu_{n_k} - \int_Z u_i \, d\mu \right| \longrightarrow 0 \quad \text{as } k \to +\infty.$$

Therefore, we conclude that

$$\lim_{k\to+\infty} \sup_{u\in\overline{B}_1(0)} \left| \int_Z u \, d\mu_{n_k} - \int_Z u \, d\mu \right| = 0,$$

so

$$\mu_{n_k} \longrightarrow \mu \quad \text{in } W^{-1,r}(Z).$$

∎

Next we sharpen both the Rellich-Kondrachov theorem (see Theorem 2.5.17) and Egorov's theorem (see Theorem A.2.10). We show that a sequence bounded in $W^{1,r}(Z)$ $(r \in (1,N))$ has a subsequence converging uniformly outside a very small set. The set is not only small in the Lebesgue measure (as the Egorov's theorem postulates; see Theorem A.2.10), but it is also small in p-capacity, for $p \in [1,r)$. First we introduce a notion, which is useful in the study of the pointwise properties of Sobolev functions.

DEFINITION 2.5.26 *Suppose that* $u \in L^1_{\rm loc}\big(\mathbb{R}^N\big)$. *Then*

$$u^*(z) \stackrel{df}{=} \begin{cases} \lim\limits_{r \searrow 0} \dfrac{1}{\lambda^N(\overline{B}_r(z))} \displaystyle\int\limits_{\overline{B}_r(z)} u(y)\,dy & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is the ***precise representative*** *of* u.

REMARK 2.5.27 If $u, v \in L^1_{\rm loc}\big(\mathbb{R}^N\big)$ and

$$u(z) \;=\; v(z) \quad \text{for } \lambda^N\text{-almost all } z \in \mathbb{R}^N,$$

then

$$u^*(z) \;=\; v^*(z) \qquad \forall\, z \in \mathbb{R}^N.$$

Moreover, in view of the Lebesgue differentiation theorem (see Theorem 1.4.6), the limit in the definition of u^* (see Definition 2.5.26) exists for λ^N-almost all $z \in Z$.

In the next theorem, we identify each function in the Sobolev space $W^{1,r}(Z)$ with its precise representative.

THEOREM 2.5.28
If $Z \subseteq \mathbb{R}^N$ *is a bounded, open set which is Lipschitz,* $r \in (1,N)$ *and* $\{u_n\}_{n\geqslant 1} \subseteq W^{1,r}(Z)$ *is bounded,*
then there exist a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ *of* $\{u_n\}_{n\geqslant 1}$ *and* $u \in W^{1,r}(Z)$, *such that for every* $p \in [1,r)$ *and every* $\delta > 0$, *there exists a relatively closed set* $A_\delta \subseteq Z$, *such that*

$$\operatorname{cap}_p\big(U \setminus A_\delta\big) \;\leqslant\; \delta$$

and

$$u_{n_k} \;\longrightarrow\; u \quad \text{uniformly on } A_\delta.$$

PROOF We may assume that $u_k \in W^{1,r}_0(Z)$ for all $k \geqslant 1$. Indeed if this is not the case we choose a bounded, open set $U \supseteq \overline{Z} \supseteq Z$ and a cut-off function $\varphi \in C_c\big(\mathbb{R}^N\big)$, such that

$$\varphi|_{\overline{Z}} \;=\; 1 \quad \text{and} \quad \varphi_{U^c} \;=\; 0.$$

If $u \in W^{1,r}(Z)$ and $E(u) \in W^{1,r}(U)$ is the extension (see Theorem 2.4.55), then

$$\varphi E(u) \ \in \ W_0^{1,r}(U).$$

Because by hypothesis the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,r}(Z)$ is bounded, by passing to a subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } W_0^{1,r}(Z) \\ u_n &\longrightarrow u \quad \text{in } L^r(Z), \end{aligned}$$

with $u \in W_0^{1,r}(Z)$ (see Theorem 2.5.17**(a)**).

Fix $\delta, \varepsilon > 0$ and let

$$C_n^\varepsilon \ \stackrel{df}{=} \ \left\{z \in Z : \ \left|u_n(z) - u(z)\right| \geqslant \varepsilon\right\}$$

and

$$h_n^\varepsilon \ \stackrel{df}{=} \ \frac{2}{\varepsilon}\left(|u_n - u| - \frac{\varepsilon}{2}\right)^+.$$

Note that

$$h_n^\varepsilon \ \in \ W_0^{1,r}(Z)$$

(see Proposition 2.4.33) and $h_n^\varepsilon \geqslant 1$ on C_n^ε. From Definition 1.6.1**(d)**, by Hölder's and Poincaré's inequalities (see Theorem A.2.27), for $p \in [1, r)$, we have

$$\begin{aligned} &\operatorname{cap}_p\left(C_\varepsilon^n\right) \ \leqslant \ \left\|Dh_n^\varepsilon\right\|_p^p \\ \leqslant \ &\left(\frac{2}{\varepsilon}\right)^p \lambda^N\left(\left\{|u_n - u| > \frac{\varepsilon}{2}\right\}\right)^{1-\frac{p}{r}} \left(\|Du_n\|_r^r + \|Du\|_r^r\right)^{\frac{p}{r}} \\ \leqslant \ &c(\varepsilon)\left\|u_n - u\right\|_{L^r(Z)}^{r-p}. \end{aligned}$$

We choose a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$, such that

$$\sum_{k=1}^{\infty} \left\|u_{n_k} - u\right\|_r^{r-p} \ < \ +\infty.$$

We set

$$D_m^i \ \stackrel{df}{=} \ \bigcup_{k=m}^{\infty} C_{n_k}^{\frac{1}{i}}.$$

Then since cap_p is an outer measure on $\mathbb{R}^N$ (see Theorem 1.6.10), we have

$$\begin{aligned} \operatorname{cap}_p(D_m^i) \ &\leqslant \ \sum_{k=m}^{\infty} \operatorname{cap}_p\left(C_{n_k}^{\frac{1}{i}}\right) \\ \leqslant \ c\left(\frac{1}{i}\right) &\sum_{k=1}^{\infty} \left\|u_{n_k} - u\right\|_r^{r-p} \ < \ \frac{\delta}{2^{k+1}}, \end{aligned}$$

provided that $m = m(i) \geqslant 1$ is large enough.

From Theorem 1.6.11**(a)**, we know that we can find an open set $V_m^i \supseteq D_m^i$ with

$$\operatorname{cap}_p(V_m^i) \;<\; \frac{\delta}{2^k}.$$

Set

$$A_\delta \;\stackrel{df}{=}\; Z \setminus \bigcup_{i=1}^{\infty} V_{m(i)}^i.$$

Then $A_\delta \subseteq Z$ is relatively closed,

$$\operatorname{cap}_p(A_\delta) \;\leqslant\; \sum_{i=1}^{\infty} \operatorname{cap}_p\big(V_{m(i)}^i\big) \;\leqslant\; \delta$$

and

$$u_{n_k} \;\longrightarrow\; u \quad \text{uniformly on } A_\delta.$$

□

REMARK 2.5.29 The result in general fails if $p = r$.

Now we shall discuss how the continuous embedding

$$W^{1,p}(Z) \;\subseteq\; L^{p^*}(Z)$$

(with $Z \in \mathbb{R}^N$ and $p \in [1, N)$) fails to be compact (see Theorem 2.5.17). It has to do with the so-called "concentration phenomena." To start having an idea about such situations, recall Proposition 2.3.38. There we saw that if

$$u_n \;\stackrel{w}{\longrightarrow}\; u \quad \text{in } L^1(\mathbb{R})$$

and oscillates rapidly around its weak limit, then the sequence $\{u_n\}_{n\geqslant 1}$ cannot converge strongly in $L^1(Z)$. Moreover, if $p \in (1, +\infty)$,

$$u_n \;\stackrel{w}{\longrightarrow}\; u \quad \text{in } L^p(Z)$$

and we also know that

$$u_n(z) \;\longrightarrow\; u(z) \quad \text{for a.a. } z \in Z;$$

still we cannot in general deduce strong convergence in $L^p(Z)$. The problem is that the mass of $|u_n - u|^p$ may coalesce onto a set of zero Lebesgue measure. This is the problem of "concentration." For this reason, in contrast to the case $p = 1$, for $p > 1$ the best constant in the Sobolev inequality (see Theorem 2.5.3) is never achieved when $Z \subseteq \mathbb{R}^N$, $Z \neq \mathbb{R}^N$ is an open set which is Lipschitz and in particular is never achieved on a bounded Lipschitz domain. For this reason our analysis will be for $Z = \mathbb{R}^N$. So let $N > 1$ and let $p \in [1, N)$ and consider

$$D^{1,p}(\mathbb{R}^N) \;\stackrel{df}{=}\; \big\{u \in L^{p^*}(\mathbb{R}^N) :\; Du \in L^p\big(\mathbb{R}^N; \mathbb{R}^N\big)\big\},$$

(where $p^* \overset{df}{=} \frac{Np}{N-p}$) furnished with the norm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} \;=\; \|Du\|_p\,.$$

That this is a norm on $D^{1,p}(\mathbb{R}^N)$ follows from the Sobolev-Nirenberg-Gagliardo inequality (see Theorem 2.5.3; normed this way $D^{1,p}(\mathbb{R}^N)$ is a separable Banach space, Hilbert space if $p = 2$). The best constant $c > 0$ in that inequality is given by

$$(c^{-1})^p \;=\; S \;=\; \inf_{\substack{u \,\in\, D^{1,p}(\mathbb{R}^N)\\ u \,\neq\, 0}} \frac{\|Du\|_p^p}{\|u\|_{p^*}^p} \;=\; \inf_{\substack{u \,\in\, D^{1,p}(\mathbb{R}^N)\\ \|u\|_{p^*} \,=\, 1}} \|Du\|_p^p\,.$$

The question is whether this infimum is realized by an element in $D^{1,p}(\mathbb{R}^N)$. So consider a minimizing sequence $\{u_n\}_{n\geqslant 1} \subseteq D^{1,p}(\mathbb{R}^N)$, i.e.,

$$\|Du_n\|_p^p \;\longrightarrow\; S,$$

with

$$\|u_n\|_{p^*} \;=\; 1 \qquad \forall\; n \geqslant 1.$$

By passing to a subsequence if necessary, we may assume that

$$u_n \;\overset{w}{\longrightarrow}\; u \quad \text{in } D^{1,p}(\mathbb{R}^N)$$

and so

$$\|Du\|_p^p \;\leqslant\; \liminf_{n\to+\infty} \|Du_n\|_p^p\,.$$

This u is a minimizer provided that $\|u\|_{p^*} = 1$. But since

$$u_n \;\overset{w}{\longrightarrow}\; u \quad \text{in } L^{p^*}(\mathbb{R}^N),$$

we only know that $\|u\|_{p^*} \leqslant 1$. Note that if $v \in D^{1,p}(\mathbb{R}^N)$, $y \in \mathbb{R}^N$ and $\lambda > 0$, then the rescaled function

$$v^{y,\lambda}(z) \;=\; \lambda^{\frac{N-p}{p}} v(\lambda z + y)$$

satisfies

$$\left\|Dv^{y,\lambda}\right\|_p \;=\; \|Dv\|_p \quad \text{and} \quad \left\|v^{y,\lambda}\right\|_{p^*} \;=\; \|v\|_{p^*}\,.$$

So the problem is invariant under translations and dilations. In order to avoid noncompactness of the minimizing sequence (hence achieve $\|u\|_{p^*} = 1$) we need the following result, known as the ***Concentration-Compactness Lemma***. In what follows we shall regard $L^1(\mathbb{R}^N)$ in a natural way as a subset of $M(\mathbb{R}^N)$, by associating to $u \in L^1(\mathbb{R}^N)$ the measure

$$\mu(A) \;=\; \int_A u(z)\,dz,$$

i.e., $d\mu = u\,dz$.

THEOREM 2.5.30
If $p \in [1, N)$,

$$u_n \xrightarrow{w} u \quad \text{in } D^{1,p}(\mathbb{R}^N),$$
$$\widehat{\mu}_n \;=\; \|Du_n\|^p_{\mathbb{R}^N} \;\xrightarrow{w}\; \mu$$

and

$$\widehat{\nu}_n \;=\; |u_n|^{p^*} \;\xrightarrow{w}\; \nu \quad \text{in } M(\mathbb{R}^N),$$

with $\mu, \nu \in M(\mathbb{R}^N)$, $\mu, \nu \geqslant 0$,
then

(a) *there exists an at most countable index set* I, *a family* $\{z_i\}_{i\in I}$ *of distinct points in* $\mathbb{R}^N$ *and nonnegative numbers* $\{\mu_i, \nu_i\}_{i\in I}$, *such that*

$$\mu \;\geqslant\; \|Du\|^p_{\mathbb{R}^N} + \sum_{i\in I} \mu_i \delta_{z_i}$$

and

$$\nu \;=\; |u|^{p^*} + \sum_{i\in I} \nu_i \delta_{z_i};$$

(b) *for all* $i \in I$,

$$S\nu_i^{\frac{p}{p^*}} \;\leqslant\; \mu_i$$

and in particular

$$\sum_{i\in I} \nu_i^{\frac{p}{p^*}} \;<\; +\infty.$$

PROOF First suppose that $u = 0$. Let $\vartheta \in C_c^\infty(\mathbb{R}^N)$. We have

$$\int_{\mathbb{R}^N} |\vartheta u_n|^{p^*} dz \;\leqslant\; S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} \|D(\vartheta u_n)\|^p dz \right)^{\frac{p^*}{p}} \qquad \forall\, n \geqslant 1. \tag{2.50}$$

Since

$$|u_n|^{p^*} \;\xrightarrow{w}\; \nu \quad \text{in } M(\mathbb{R}^N),$$

we have that

$$\int_{\mathbb{R}^N} |\vartheta u_n|^{p^*} dz \;\longrightarrow\; \int_{\mathbb{R}^N} |\vartheta|^{p^*} d\nu. \tag{2.51}$$

Also, using the facts that

$$u_n \;\longrightarrow\; u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N)$$

(see Theorem 2.5.17**(a)**) and

$$\|Du_n\|^p_{\mathbb{R}^N} \;\xrightarrow{w}\; \mu \quad \text{in } M(\mathbb{R}^N),$$

we have that

$$\begin{aligned}
&\liminf_{n\to+\infty}\left(\int_{\mathbb{R}^N}\left\|D(\vartheta u_n)\right\|^p_{\mathbb{R}^N}\,dz\right)^{\frac{p^*}{p}}\\
&=\liminf_{n\to+\infty}\left(\int_{\mathbb{R}^N}|\vartheta|^p\left\|Du_n\right\|^p_{\mathbb{R}^N}\,dz\right)^{\frac{p^*}{p}}\\
&=\int_{\mathbb{R}^N}|\vartheta|^p\,d\mu. \qquad (2.52)
\end{aligned}$$

From (2.50), (2.51) and (2.52), we infer that in the limit as $n\to+\infty$, we have

$$\left(\int_{\mathbb{R}^N}|\vartheta|^{p^*}\,d\nu\right)^{\frac{1}{p^*}}\leqslant S^{-\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\vartheta|^p\,d\mu\right)^{\frac{1}{p}}\qquad\forall\,\vartheta\in C^\infty_c\big(\mathbb{R}^N\big). \qquad (2.53)$$

From (2.53), it follows that for all compact sets $K\subseteq\mathbb{R}^N$, we have

$$\nu(K)^{\frac{1}{p^*}}\leqslant S^{-\frac{1}{p}}\mu(K)^{\frac{1}{p}}. \qquad (2.54)$$

Because the measures are Radon, from (2.54), we deduce that

$$\nu(A)^{\frac{1}{p^*}}\leqslant S^{-\frac{1}{p}}\mu(A)^{\frac{1}{p}}\quad\text{for all Borel sets } A\subseteq\mathbb{R}^N. \qquad (2.55)$$

From Saks Lemma (see Theorem A.2.13), we know that

$$\mu=\mu_0+\mu_a,$$

with μ_0 nonatomic measure, $\mu_0\geqslant 0$, μ_a purely atomic and

$$\mu_a=\sum_{i\in I}\mu_i\delta_{z_i},$$

where I is a countable index set, $\{\mu_i\}_{i\in I}\subseteq\mathbb{R}_+\setminus\{0\}$ and $\{z_i\}_{i\in I}\subseteq\mathbb{R}^N$. Because of (2.55), we see that

$$\nu\prec\!\prec\mu$$

and so from the Radon-Nikodym theorem (see Theorem A.2.24), we have that

$$\nu(A)=\int_A\frac{d\nu}{d\mu}\,d\mu\quad\text{for all Borel sets } A\subseteq\mathbb{R}^N, \qquad (2.56)$$

where

$$\frac{d\nu}{d\mu}\in L^1(\mathbb{R}^N;\mu)$$

is the Radon-Nikodym derivative (see Remark A.2.25). So

$$\frac{d\nu}{d\mu}(z) \;=\; \lim_{r\searrow 0}\frac{\nu(\overline{B}_r(z))}{\mu(\overline{B}_r(z))} \quad \text{for } \mu\text{-a.a. } z\in\mathbb{R}^N. \tag{2.57}$$

From (2.55), we see that

$$\frac{\nu(\overline{B}_r(z))}{\mu(\overline{B}_r(z))} \;\leqslant\; S^{-\frac{p^*}{p}}\mu\big(\overline{B}_r(z)\big)^{\frac{N}{N-p}}, \tag{2.58}$$

provided $\mu\big(\overline{B}_r(z)\big)\neq 0$. From (2.57) and (2.58), it follows that

$$\frac{d\nu}{d\mu}(z) \;=\; 0 \quad \text{for a.a. } z\in \operatorname{supp}\mu_0 = \mathbb{R}^N\setminus\{z_i\}_{i\in I}. \tag{2.59}$$

Set

$$\nu_i \;=\; \frac{d\nu}{d\mu}(z_i)\mu_i.$$

From (2.56), (2.57), (2.58) and (2.59), we see that the theorem holds when $u=0$.

Now suppose that $u\neq 0$. Set

$$w_n \;\overset{df}{=}\; u_n - u.$$

Then the previous calculations apply to $\{w_n\}_{n\geqslant 1}$. Moreover, by virtue of Proposition 2.3.49, we have

$$\|Dw_n\|_p^p \;=\; \|Du_n\|_p^p - \|Du\|_p^p + \varepsilon_n \quad \text{with } \varepsilon_n\searrow 0,$$

so

$$\|Dw_n\|_{\mathbb{R}^N}^p \;\overset{w}{\longrightarrow}\; \mu - \|Du\|_{\mathbb{R}^N}^p \;=\; \widehat{\mu} \;\geqslant\; 0 \quad \text{in } M(\mathbb{R}^N).$$

Similarly, we have

$$|w_n|^{p^*} \;\longrightarrow\; \nu - |u|^{p^*} \;=\; \widehat{\nu} \quad \text{in } M(\mathbb{R}^N).$$

So, we are back to the case $u=0$, with u_n replaced by w_n, μ replaced by $\widehat{\mu}$ and ν replaced by $\widehat{\nu}$. □

COROLLARY 2.5.31

If $p\in[1,+\infty)$,

$$u_n \;\overset{w}{\longrightarrow}\; 0 \quad \textit{in } D^{1,p}(\mathbb{R}^N),$$
$$\mu_n \;=\; \|Du_n\|_{\mathbb{R}^N}^p \;\overset{w}{\longrightarrow}\; \mu$$

and

$$\nu_n \;=\; |u_n|^{p^*} \;\overset{w}{\longrightarrow}\; \nu \quad \textit{in } M(\mathbb{R}^N),$$

with $\mu,\nu\geqslant 0$ *and*

$$\mu(\mathbb{R}^N)^{\frac{1}{p}} \;\leqslant\; S^{\frac{1}{p}}\nu(\mathbb{R}^N)^{\frac{1}{p^*}},$$

then ν *is a Dirac measure.*

PROOF From (2.55) and the hypotheses, we see that

$$\mu(\mathbb{R}^N)^{\frac{1}{p}} \;=\; S^{\frac{1}{p}}\nu(\mathbb{R}^N)^{\frac{1}{p^*}}.$$

Also from (2.53), we have that

$$\left(\int\limits_{\mathbb{R}^N} |\vartheta|^{p^*}\,d\nu\right)^{\frac{1}{p^*}} \;\leqslant\; S^{-\frac{1}{p}}\mu(\mathbb{R}^N)^{\frac{1}{N}}\left(\int\limits_{\mathbb{R}^N} |\vartheta|^{p^*}\,d\mu\right)^{\frac{1}{p^*}} \qquad \forall\ \vartheta \in C_c^\infty\big(\mathbb{R}^N\big).$$

Thus we infer that

$$\nu \;=\; S^{-\frac{p^*}{p}}\mu(\mathbb{R}^N)^{\frac{p}{N-p}}\mu$$

and (2.53) becomes

$$\left(\int\limits_{\mathbb{R}^N} |\vartheta|^{p^*}\,d\nu\right)^{\frac{1}{p^*}}\big(\nu(\mathbb{R}^N)\big)^{\frac{1}{N}} \;\leqslant\; \left(\int\limits_{\mathbb{R}^N} |\vartheta|^{p}\,d\nu\right)^{\frac{1}{p}},$$

so

$$\nu(A)^{\frac{1}{p^*}}\nu(\mathbb{R}^N)^{\frac{1}{N}} \;\leqslant\; \nu(A)^{\frac{1}{p}} \quad \text{for all Borel sets } A \subseteq \mathbb{R}^N.$$

But this is impossible if ν is not a Dirac measure. □

We return to the problem of determining the best Sobolev constant, i.e.,

$$S \;=\; \inf_{\substack{u\,\in\, D^{1,p}(\mathbb{R}^N)\\ \|u\|_{p^*}\,=\,1}} \|Du\|_p^p. \tag{2.60}$$

We have the following existence result for problem (2.60). The result is due to Lions (1985*a*, 1985*a*) and its proof, which also uses Theorem 2.5.30, can be found there.

THEOREM 2.5.32
If $p \in (1, N)$ *and* $\{u_n\}_{n\geqslant 1} \subseteq D^{1,p}(\mathbb{R}^N)$ *is a minimizing sequence for problem (2.60),*
then $\{u_n\}_{n\geqslant 1}$ *up to translation and dilation is relatively compact in* $D^{1,p}(\mathbb{R}^N)$, *i.e., there exists a sequence*

$$\{(z_n, \lambda_n)\}_{n\geqslant 1} \subseteq \mathbb{R}^N \times (\mathbb{R}_+ \setminus \{0\}),$$

such that the sequence

$$u_n^{z_n,\lambda_n}(z) \;=\; \lambda_n^{\frac{N-p}{p}}\, u_n(\lambda_n z + z_n) \qquad \forall\ z \in \mathbb{R}^N$$

is relatively compact in $D^{1,p}(\mathbb{R}^N)$.

REMARK 2.5.33 If $p = 2$, the function

$$u(z) \stackrel{df}{=} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1 + \|z\|^2_{\mathbb{R}^N})^{\frac{N-2}{2}}}$$

is a minimizer for problem (2.60) (see Aubin (1976) and Talenti (1976)).

If $Z \subseteq \mathbb{R}^N$ is any open set (not necessarily equal $\mathbb{R}^N$) and by $S(Z)$ we denote the value of problem (2.60) when $\mathbb{R}^N$ is replaced by Z, then $S(Z) = S$, but $S(Z)$ is never attained if $Z \neq \mathbb{R}^N$.

Finally if $p = 1$, then the best constant for the embedding

$$D^{1,1}(\mathbb{R}^N) \subseteq L^{\frac{N}{N-1}}(\mathbb{R}^N)$$

is attained on the characteristic functions of balls, that is in $BV_{\rm loc}(\mathbb{R}^N)$ (see Section 2.6).

Since we are in the business of determining sharp constants in inequalities, let us check to see what happens with the constant in the Poincaré-Wirtinger inequality (see Theorem 2.5.21) for Sobolev functions of one variable.

Let $T = (0, b)$ $(b < +\infty)$ and $p \in (1, +\infty)$. We introduce the space

$$W^{1,p}_{\rm per}\big(T; \mathbb{R}^N\big) \stackrel{df}{=} \big\{u \in W^{1,p}\big(T; \mathbb{R}^N\big) : \ u(0) = u(b)\big\}.$$

From Theorem 2.5.22**(b)**, we have that $W^{1,p}_{\rm per}\big(T; \mathbb{R}^N\big)$ is embedded continuously (in fact compactly) in $C\big(\overline{T}; \mathbb{R}^N\big)$. Therefore the evaluations at $t = b$ and $t = 0$ make sense.

PROPOSITION 2.5.34

If $u \in W^{1,p}_{\rm per}\big(T; \mathbb{R}^N\big)$ $(p \in (1, +\infty))$ *and* $\int_0^b u(t)\, dt = 0$,

then $\|u\|_\infty \leqslant b^{\frac{1}{p'}} \|u'\|_p$.

PROOF Arguing on each component separately, we may assume without any loss of generality that $N = 1$. Then from the mean value theorem for integrals, we can find $\tau \in T = (0, b)$, such that

$$u(\tau) = \frac{1}{b} \int_0^b u(s)\, ds = 0.$$

By Hölder's inequality (see Theorem A.2.27), with $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\big|u(t)\big| = \left|\int_\tau^t u'(s)\, ds\right| \leqslant \int_0^b \big|u'(s)\big|\, ds \leqslant b^{\frac{1}{p'}} \|u'\|_p \qquad \forall\, t \in [0, b],$$

so $\|u\|_\infty \leqslant b^{\frac{1}{p'}} \|u'\|_p$. ∎

In the case of the Hilbert space $W^{1,2}_{\rm per}(T;\mathbb{R}^N)$, we have the following sharp estimates.

PROPOSITION 2.5.35
If $u \in W^{1,p}_{\rm per}(T;\mathbb{R}^N)$ $(p \in (1,+\infty))$ *and*

$$\int_0^b u(t)\,dt \;=\; 0,$$

then

(a) $\|u\|_2^2 \leqslant \dfrac{b^2}{4\pi^2}\,\|u'\|_2^2$;

(b) $\|u\|_\infty^2 \leqslant \dfrac{b}{12}\,\|u'\|_2^2$.

PROOF Again we may assume that $N = 1$.

(a) We consider the Fourier expansion of u, i.e.,

$$u(t) \;=\; \sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty} a_k \exp\left(\frac{2i\pi kt}{b}\right).$$

Parseval's equality implies that

$$\|u'\|_2^2 \;=\; \sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty} b\frac{4\pi^2k^2}{b^2}|a_k|^2 \;\geqslant\; \frac{4\pi^2}{b^2}\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty} b|a_k|^2 \;=\; \frac{4\pi^2}{b^2}\,\|u\|_2^2\,.$$

(b) Using the Cauchy-Schwarz-Bunyakowski inequality (see Proposition A.4.5 and Remark A.4.6), Parseval's equality and since

$$\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; \frac{\pi^2}{6},$$

for every $t \in [0,b]$, we have

$$\begin{aligned}\big|u(t)\big|^2 \;&\leqslant\; \left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}|a_k|\right)^2\\ &\leqslant \left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\frac{b}{4\pi^2k^2}\right)\left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\frac{4\pi^2k^2}{b}|a_k|^2\right) \;=\; \frac{b}{12}\,\|u'\|_2^2\,.\end{aligned}$$

□

2.6 Fine Properties of Functions and BV-Functions

In this section we establish some further differentiability properties of Sobolev functions and also introduce the space of functions of bounded variation (BV-functions) and establish some of their basic properties.

We start with a result on the L^{p^*}-differentiability of Sobolev functions.

PROPOSITION 2.6.1
If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $p \in [1, N)$,
then for λ^N*-almost all* $z \in \mathbb{R}^N$, *we have*

$$\left(\frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)} \left|u(y) - u(z) - (Du(z), y-z)_{\mathbb{R}^N}\right|^{p^*} dy\right)^{\frac{1}{p^*}} = o(r) \quad \text{as } r \searrow 0.$$

PROOF From Theorem 1.4.6, we know that for λ^N-almost all $z \in \mathbb{R}^N$, we have

$$\lim_{r\searrow 0} \frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)} \left|u(y) - u(z)\right|^p dz = 0$$

and

$$\lim_{r\searrow 0} \frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)} \left\|Du(y) - Du(z)\right\|^p_{\mathbb{R}^N} dz = 0.$$

We fix such a point $z \in \mathbb{R}^N$ (known as a Lebesgue point for the functions u and Du). Clearly exploiting the translation invariance of the Lebesgue measure λ^N, we can take $z = 0$. We choose $\vartheta \in C^1_c\big(\overline{B}_r(0)\big)$ with $\|\vartheta\|_{p'} \leqslant 1$ (here $\frac{1}{p} + \frac{1}{p'} = 1$).

Let φ be a mollifier (see Definition 2.4.10) and for every $\varepsilon > 0$, set

$$u_\varepsilon \stackrel{df}{=} \vartheta_\varepsilon \star u.$$

Choose $y \in \overline{B}_r(0)$ and let $h(t) = u_\varepsilon(ty)$. Then

$$h(1) = h(0) + \int\limits_0^1 h'(s)\, ds,$$

so

$$\begin{aligned} u_\varepsilon(y) &= u_\varepsilon(0) + \int_0^1 \big(Du_\varepsilon(sy), y\big)_{\mathbb{R}^N}\, ds \\ &= u_\varepsilon(0) + \big(Du(0), y\big)_{\mathbb{R}^N} + \int_0^1 \big(Du_\varepsilon(sy) - Du(0), y\big)_{\mathbb{R}^N}\, ds. \end{aligned} \tag{2.61}$$

Using Fubini's theorem and a change of variables, we have

$$\begin{aligned} &\frac{1}{\lambda^N(\overline{B}_r(0))} \int_{\overline{B}_r(0)} \vartheta(y)\left(u_\varepsilon(y) - u_\varepsilon(0) - (Du(0), y)_{\mathbb{R}^N}\right) dy \\ &= \int_0^1 \frac{1}{\lambda^N(\overline{B}_r(0))} \int_{\overline{B}_r(0)} \vartheta(y)\big(Du_\varepsilon(sy) - Du(0), y\big)_{\mathbb{R}^N}\, dy\, ds \\ &= \int_0^1 \frac{1}{s\lambda^N(\overline{B}_{rs}(0))} \int_{\overline{B}_{rs}(0)} \vartheta\left(\frac{y}{s}\right)\big(Du_\varepsilon(y) - Du(0), y\big)_{\mathbb{R}^N}\, dy\, ds. \end{aligned}$$

Letting $\varepsilon \searrow 0$, in the limit we obtain

$$\begin{aligned} &\frac{1}{\lambda^N(\overline{B}_r(0))} \int_{\overline{B}_r(0)} \vartheta(y)\left(u(y) - u(0) - (Du(0), y)_{\mathbb{R}^N}\right) dy \\ &= \int_0^1 \frac{1}{s\lambda^N(\overline{B}_{rs}(0))} \int_{\overline{B}_{rs}(0)} \vartheta\left(\frac{y}{s}\right)\big(Du(y) - Du(0), y\big)_{\mathbb{R}^N}\, dy\, ds \\ &\leqslant r\int_0^1 \left(\frac{1}{\lambda^N(\overline{B}_{rs}(0))} \int_{\overline{B}_{rs}(0)} \left|\vartheta\left(\frac{y}{s}\right)\right|^{p'} dy\right)^{\frac{1}{p'}} \times \\ &\quad \times \left(\frac{1}{\lambda^N(\overline{B}_{rs}(0))} \int_{\overline{B}_{rs}(0)} \|Du(y) - Du(0)\|^p_{\mathbb{R}^N}\, dy\right)^{\frac{1}{p}}. \end{aligned}$$

Note that

$$\frac{1}{\lambda^N(\overline{B}_{rs}(0))} \int_{\overline{B}_{rs}(0)} \left|\vartheta\left(\frac{y}{s}\right)\right|^{p'} dy = \frac{1}{\lambda^N(\overline{B}_r(0))} \int_{\overline{B}_r(0)} |\vartheta(y)|^{p'}\, dy \leqslant \frac{1}{a(N)r^N}$$

(see Remark 1.3.22). So we obtain

$$\frac{1}{\lambda^N(\overline{B}_r(0))}\int\limits_{\overline{B}_r(0)} \vartheta(y)\left(u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N}\right)dy = \varepsilon\left(r^{1-\frac{N}{p'}}\right) \quad \text{as } r\searrow 0.$$

Taking the supremum over all ϑ, we obtain

$$\frac{1}{r^N}\left(\int\limits_{\overline{B}_r(0)} \left|u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N}\right|^p dy\right)^{\frac{1}{p}} = o\left(r^{1-\frac{N}{p'}}\right) \quad \text{as } r\searrow 0,$$

so

$$\left(\frac{1}{\lambda^N(\overline{B}_r(0))}\int\limits_{\overline{B}_r(0)} \left|u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N}\right|^p dy\right)^{\frac{1}{p}} = o(r) \quad \text{as } r\searrow 0. \tag{2.62}$$

Set

$$h(y) \overset{df}{=} u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N},$$

so $h\in W^{1,p}(B_r(0))$ and consider its extension $E(h)\in W^{1,p}(\mathbb{R}^N)$. We have

$$\big\|E(h)\big\|_{W^{1,p}(\mathbb{R}^N)} \leqslant c_1\,\|u\|_{W^{1,p}(B_r(0))}, \tag{2.63}$$

for some $c_1>0$ (see Theorem 2.4.55). Then, via Sobolev's inequality (see Theorem 2.5.3) and (2.63), we have

$$\begin{aligned}\left(\int\limits_{\overline{B}_r(0)} \big|h(y)\big|^{p^*}dy\right)^{\frac{1}{p^*}} &\leqslant \left(\int\limits_{\mathbb{R}^N} \big|E(h)(y)\big|^{p^*}dy\right)^{\frac{1}{p^*}}\\ &\leqslant c_2\left(\int\limits_{\mathbb{R}^N} \big\|DE(h)(y)\big\|^p_{\mathbb{R}^N}dy\right)^{\frac{1}{p}}\\ &\leqslant c_3\left(\int\limits_{\overline{B}_r(0)} \left(\big|h(y)\big|^p+\|Dh(y)\|^p_{\mathbb{R}^N}\right)dy\right)^{\frac{1}{p}}.\end{aligned} \tag{2.64}$$

Therefore using (2.62) and (2.64), we conclude that

$$\begin{aligned}
&\left(\frac{1}{\lambda^N(\overline{B}_r(0))}\int\limits_{\overline{B}_r(0)} \left|u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N}\right|^{p^*}\,dy\right)^{\frac{1}{p^*}}\\
\leqslant\ & c_4 r\left(\frac{1}{\lambda^N(\overline{B}_r(0))}\int\limits_{\overline{B}_r(0)} \big\|Du(y)-Du(0)\big\|^p_{\mathbb{R}^N}\,dy\right)^{\frac{1}{p}}\\
&+ c_4\left(\frac{1}{\lambda^N(\overline{B}_r(0))}\int\limits_{\overline{B}_r(0)} \left|u(y)-u(0)-\big(Du(0),y\big)_{\mathbb{R}^N}\right|^{p}\,dy\right)^{\frac{1}{p}}\\
=\ & o(r) \quad \text{as } r\searrow 0.
\end{aligned}$$

□

For differentiability λ^N-almost everywhere, as probably expected, we consider the case $p \in (N, +\infty]$.

PROPOSITION 2.6.2
If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $p \in (N, +\infty]$,
then u *is differentiable* λ^N*-almost everywhere and the derivative equals the distributional derivative* λ^N*-almost everywhere.*

PROOF Since $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N) \subseteq W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p < +\infty$, we may assume that $p \in (N, +\infty)$. For λ^N-almost all $z \in \mathbb{R}^N$, we have

$$\lim_{r\searrow 0} \frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)} \|Du(y)-Du(z)\|^p_{\mathbb{R}^N}\,dy \;=\; 0. \tag{2.65}$$

Choose $z \in \mathbb{R}^N$, such that (2.65) holds. Set

$$h(y) \stackrel{df}{=} u(y)-u(z)-\big(Du(z),y-z\big)_{\mathbb{R}^N} \qquad \forall\ y \in \overline{B}_r(z).$$

Using Morrey's inequality (see Theorem 2.5.12**(a)**), we have

$$\big|h(y)-h(z)\big| \;\leqslant\; cr\left(\frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)} \big\|Dh(y)\big\|^p_{\mathbb{R}^N}\,dy\right)^{\frac{1}{p}},$$

with $r = \|y-z\|_{\mathbb{R}^N}$. Since

$$h(z) \;=\; 0 \quad \text{and} \quad Dh \;=\; Du - Du(z),$$

using (2.65), we obtain

$$\begin{aligned}&\frac{|u(y)-u(z)-(Du(z),y-z)_{\mathbb{R}^N}|}{\|y-z\|_{\mathbb{R}^N}}\\ &\leqslant c\left(\frac{1}{\lambda^N(\overline{B}_r(z))}\int\limits_{\overline{B}_r(z)}\big\|Dh(y)\big\|^p\,dy\right)^{\frac{1}{p}}\longrightarrow 0\quad\text{as } y\to z,\end{aligned}$$

so u is λ^N-almost everywhere differentiable and

$$\nabla u(z)\;=\;Du(z)\quad\text{for a.a. } z\in\mathbb{R}^N.$$

□

Next we investigate the properties of a Sobolev function u (or more exactly of its precise representation u^* (see Definition 2.5.26)) along lines. In this direction we have the following result.

PROPOSITION 2.6.3
(a) *If* $u\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$, $p\in[1,+\infty)$,
then for each $k\in\{1,\dots,N\}$ *the function*

$$u^*_k(z',t)\;=\;u^*\big(z_1,\dots,z_{k-1},t,z_{k+1},\dots,z_N\big)$$

is locally absolutely continuous in t *for* λ^{N-1}*-almost all* $z'=(z_i)_{i=1,i\neq k}^N\in\mathbb{R}^{N-1}$ *(see Definition A.2.15***(b)***). Moreover* $(u^*_k)'\in L^p_{\rm loc}\big(\mathbb{R}^N\big)$.

(b) *If* $u\in L^p_{\rm loc}\big(\mathbb{R}^N\big)$ *and* $u=h$ λ^N*-almost everywhere where for each* $k\in\{1,\dots,N\}$, *the function*

$$h_k(z',t)\;\stackrel{df}{=}\;h\big(z_1,\dots,z_{k-1},t,z_{k+1},\dots,z_N\big)$$

is locally absolutely continuous in t *for* λ^{N-1}*-almost all* $z'=(z_i)_{i=1,i\neq k}^N\in\mathbb{R}^{N-1}$ *and* $h'_k\in L^p_{\rm loc}\big(\mathbb{R}^N\big)$,
then $u\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$.

PROOF **(a)** Clearly we may assume that $k=N$. Set $u_\varepsilon=\varphi_\varepsilon\star u$ with $\{\varphi_\varepsilon\}_{\varepsilon>0}$ being a family of mollifiers. We know that

$$u_\varepsilon\;\longrightarrow\;u\quad\text{in } W^{1,p}_{\rm loc}(\mathbb{R}^N)$$

(see Proposition 2.4.12**(e)**). For every $M>0$ and λ^{N-1}-almost all $z'=(z_i)_{i=1}^{N-1}\in\mathbb{R}^{N-1}$, from Fubini's theorem, we have

$$\int\limits_{-M}^{M}\left(|u_\varepsilon(z',t)-u(z',t)|^p+\left|\frac{\partial u_\varepsilon}{\partial z_N}(z',t)-\frac{\partial u}{\partial z_N}(z',t)\right|^p\right)dt\;\longrightarrow\;0\quad\text{as } \varepsilon\searrow 0.$$

Let

$$u_{N,\varepsilon}(t) \stackrel{df}{=} u_\varepsilon(z',t).$$

Then

$$u_{N,\varepsilon} \longrightarrow u_N \quad \text{in } W^{1,p}_{\text{loc}}(\mathbb{R}) \quad \text{as } \varepsilon \searrow 0$$

and so also locally uniformly to a locally absolutely continuous function u_N with

$$u'_N(t) = D_N u(z',t) \quad \text{for } \lambda^1\text{-a.a. } t \in \mathbb{R}.$$

Also from Theorems 1.6.18, 1.6.13**(b)** and Remark 1.6.14, we have that

$$u_\varepsilon \longrightarrow u^* \quad \mu^{(N-1)} - a.e.,$$

so from Proposition 1.3.25, we have

$$u_{N,\varepsilon}(t) \longrightarrow u^*(z',t) \quad \text{for } \lambda^{N-1}\text{-a.a. } z' \in \mathbb{R}^{N-1} \text{ and all } t \in \mathbb{R}.$$

Thus

$$u_N(t) = u^*(z',t) \quad \text{for } \lambda^{N-1}\text{-a.a. } z' \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

(b) For each $\vartheta \in C^1_c\big(\mathbb{R}^N\big)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} u \frac{\partial \vartheta}{\partial z_k}\, dz &= \int_{\mathbb{R}^N} h \frac{\partial \vartheta}{\partial z_k}\, dz \\
&= \int_{\mathbb{R}^{N-1}} \left(\int_{-\infty}^{+\infty} h_k(z',t)\vartheta'(z',t)\, dt \right) dz' \\
&= -\int_{\mathbb{R}^{N-1}} \left(\int_{-\infty}^{+\infty} h'_k(z',t)\vartheta(z',t)\, dt \right) dz' = -\int_{\mathbb{R}^{N-1}} h'_k \vartheta\, dz',
\end{aligned}$$

so

$$D_k u(z) = h'_k(z) \quad \text{for } \lambda^N\text{-a.a. } z \in \mathbb{R}^N \text{ and all } k \in \{1,\dots,N\},$$

thus $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$. ∎

Before starting discussing BV-functions, let us prove a result on the superposition operator defined on a Sobolev space. More precisely let $Z \subseteq \mathbb{R}^N$ be an open set and let $\xi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz continuous function. If Z is unbounded, we also assume that $u(0) = 0$. From Proposition 2.4.25, we know that

$$\text{if } u \in W^{1,p}(Z), \text{ then } \xi \circ u \in W^{1,p}(Z).$$

So we can define the map $N_\xi\colon W^{1,p}(Z) \longrightarrow W^{1,p}(Z)$, by

$$N_\xi(u) \stackrel{df}{=} \xi \circ u \qquad \forall\, u \in W^{1,p}(Z).$$

PROPOSITION 2.6.4
If $p \in (1, +\infty)$,
then $N_\xi \colon W^{1,p}(Z) \longrightarrow W^{1,p}(Z)$ *is continuous.*

PROOF Suppose that

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(Z).$$

Then

$$\xi(u_n) \longrightarrow \xi(u) \quad \text{in } L^p(Z).$$

Also from Proposition 2.4.25, we know that

$$D(\xi \circ u_n)(z) = (\xi^* \circ u_n) Du_n(z) \quad \text{for a.a. } z \in Z,$$

with a bounded Borel measurable function $\xi^* \colon \mathbb{R} \longrightarrow \mathbb{R}$, such that

$$\xi^*(z) = \xi'(z) \quad \text{for a.a. } z \in \mathbb{R}.$$

So the sequence

$$\{D(\xi \circ u_n)\}_{n \geqslant 1} \subseteq L^p\big(Z; \mathbb{R}^N\big)$$

is bounded and it follows that

$$D(\xi \circ u_n) \xrightarrow{w} D(\xi \circ u) \quad \text{in } L^p\big(Z; \mathbb{R}^N\big).$$

First suppose that $\xi^* = \chi_A$, with A being a Borel set. Set

$$\eta^*(t) \stackrel{df}{=} \xi^*(t) - \frac{1}{2} \quad \text{and} \quad \eta(t) \stackrel{df}{=} \xi(t) - \frac{1}{2}.$$

We have

$$\int_Z \big\|D(\eta \circ u_n)\big\|^p_{\mathbb{R}^N} \, dz = \int_Z \big\|(\eta^* \circ u_n) Du_n\big\|^p_{\mathbb{R}^N} \, dz$$

$$= \frac{1}{2^p} \int_Z \|Du_n\|^p_{\mathbb{R}^N} \, dz \longrightarrow \frac{1}{2^p} \int_Z \|Du\|^p_{\mathbb{R}^N} \, dz = \int_Z \big\|D(\eta \circ u)\big\|^p_{\mathbb{R}^N} \, dz.$$

Since

$$D(\eta \circ u_n) \xrightarrow{w} D(\eta \circ u) \quad \text{in } L^p\big(Z; \mathbb{R}^N\big),$$

$$\big\|D(\eta \circ u_n)\big\|_p \longrightarrow \big\|D(\eta \circ u)\big\|_p$$

and the uniform convexity of $L^p\big(Z; \mathbb{R}^N\big)$ $(p \in (1, +\infty))$, from the Kadec-Klee property (see Remark A.3.22), we have that

$$D(\xi \circ u_n) \longrightarrow D(\xi \circ u) \quad \text{in } L^p\big(Z; \mathbb{R}^N\big),$$

whenever ξ^* is a characteristic function of a Borel set.

Clearly then the same is true for ξ^* being a countably-valued Borel function.

Now suppose that ξ^* is an arbitrary bounded Borel function. For a given $\varepsilon > 0$, we can find a countably-valued function s^*, such that

$$\sup_{t\in\mathbb{R}} \left|\xi^*(t) - s^*(t)\right| \;\leqslant\; \varepsilon$$

(see Corollary 2.1.4). So using Proposition 2.4.25, we have

$$\begin{aligned} &\left\|D(\xi\circ u_n) - D(\xi\circ u)\right\|_p \\ \leqslant\; &\left\|(s^*\circ u_n)Du_n - (s^*\circ u)Du\right\|_p + \varepsilon\left(\left\|Du_n\right\|_p + \left\|Du\right\|_p\right), \end{aligned}$$

so

$$\limsup_{n\to+\infty} \left\|D(\xi\circ u_n) - D(\xi\circ u)\right\|_p \;\leqslant\; 2\varepsilon\left\|Du\right\|_p.$$

Let $\varepsilon \searrow 0$, to obtain

$$D(\xi\circ u_n) \;\longrightarrow\; D(\xi\circ u) \quad \text{in } L^p\big(Z;\mathbb{R}^N\big),$$

hence

$$\xi\circ u_n \;\longrightarrow\; \xi\circ u \quad \text{in } W^{1,p}(Z).$$

□

REMARK 2.6.5 The result is also true for $p = 1$, but the proof is more involved. We refer to Marcus & Mizel (1979), for details.

The weakest measure theoretic sense in which a function $w \in L^1(Z)$ can be differentiable is to require that its partial derivatives in the sense of distributions are Radon measures. Such functions are called ***functions of bounded variation***. More precisely we make the following definition.

DEFINITION 2.6.6 *Let $Z \subseteq \mathbb{R}^N$ be an open set. A function $u \in L^1(Z)$ is said to be of **bounded variation**, if and only if there exist bounded Borel signed measures*

$$\mu_k\colon \mathcal{B}(Z) \;\longrightarrow\; \mathbb{R}, \quad \textit{for } k \in \{1,\dots,N\},$$

such that

$$\int_Z u D_k\vartheta\, dz \;=\; -\int_Z \vartheta\, d\mu_k \qquad \forall\ \vartheta \in C_c^\infty(Z).$$

The space of functions of bounded variation is denoted by $BV(Z)$.

The next Proposition clarifies the structure of the functions of bounded variation.

PROPOSITION 2.6.7
If $Z \subseteq \mathbb{R}^N$ is an open set, $u \in BV(Z)$ and for $h \in C_c(Z)$, $h \geqslant 0$, we set

$$\|Du\|(h) \stackrel{df}{=} \sup\Big\{ \int_Z u \operatorname{div} \vartheta \, dz : \ \vartheta = (\vartheta_k)_{k=1}^N \in C_c^\infty\big(Z;\mathbb{R}^N\big), \\ \big\|\vartheta(z)\big\|_{\mathbb{R}^N} \leqslant h(z), \ z \in Z \Big\},$$

then $\|Du\|$ is a Radon measure.

PROOF According to the Riesz-Markov representation theorem (see Theorem 2.3.41), we need to show that $\|Du\|$ is a positive linear functional on $C_c(Z)$ which is continuous under monotone convergence, i.e., if $h_n \nearrow h$ in $C_c(Z)$, then

$$\|Du\|(h_n) \longrightarrow \|Du\|(h).$$

To this end let $\mu = (\mu_k)_{k=1}^N = Du$. From Definition 2.6.6, we have that

$$\int_Z u \operatorname{div} \vartheta \, dz = -\int_Z \vartheta \, d\mu \qquad \forall \ \vartheta \in C_c^\infty\big(Z;\mathbb{R}^N\big).$$

Thus, we may write

$$\|Du\|(h) = \sup\Big\{ \int_Z v \, d\mu : \ v = (v_k)_{k=1}^N \in C_c\big(Z;\mathbb{R}^N\big), \\ \big\|v(z)\big\|_{\mathbb{R}^N} \leqslant h(z) \text{ for all } z \in Z \Big\}.$$

We show that $\|Du\|(\cdot)$ is additive. So let $h_1, h_2 \in C_c(Z)$, $h_1, h_2 \geqslant 0$ and suppose that $v \in C_c\big(Z;\mathbb{R}^N\big)$ is such that

$$\big\|v(z)\big\| \leqslant h_1(z) + h_2(z) \qquad \forall \ z \in Z.$$

Let $g = \min\big\{h_1, \|v\|\big\}$ and

$$w(z) \stackrel{df}{=} \begin{cases} g(z)\frac{v(z)}{\|v(z)\|_{\mathbb{R}^N}} & \text{if} \quad v(z) \neq 0, \\ 0 & \text{if} \quad v(z) = 0. \end{cases}$$

Clearly $w \in C_c\big(Z;\mathbb{R}^N\big)$ and

$$\big\|v(z) - w(z)\big\|_{\mathbb{R}^N} = \big\|v(z)\big\|_{\mathbb{R}^N} - g(z) \leqslant h_2(z) \qquad \forall \ z \in Z.$$

Therefore, since

$$\big\|w(z)\big\|_{\mathbb{R}^N} = g(z) \leqslant h_1(z) \qquad \forall \ z \in Z,$$

we have

$$\int_Z v\,d\mu \;=\; \int_Z w\,d\mu + \int_Z (v-w)\,d\mu \;\leqslant\; \|Du\|(h_1) + \|Du\|(h_2),$$

so

$$\|Du\|(h_1+h_2) \;\leqslant\; \|Du\|(h_1) + \|Du\|(h_1).$$

Since the opposite inequality is clearly true, we conclude that $\|Du\|(\cdot)$ is additive. Also it is clearly positively homogeneous. Thus it remains to show that if $h_n \nearrow h$ in $C_c(Z)_+$, then

$$\|Du\|(h_n) \;\longrightarrow\; \|Du\|(h).$$

Let $v \in C_c\big(Z;\mathbb{R}^N\big)$, such that

$$\big\|v(z)\big\|_{\mathbb{R}^N} \;\leqslant\; h(z) \qquad \forall\, z \in Z.$$

Let $g_n \stackrel{df}{=} \min\big\{h_n, \|v\|\big\}$ and

$$w_n(z) \;\stackrel{df}{=}\; \begin{cases} g_n(z)\frac{v(z)}{\|v(z)\|_{\mathbb{R}^N}} & \text{if} \quad v(z)\neq 0, \\ 0 & \text{if} \quad v(z)=0. \end{cases}$$

We have $w_n \in C_c\big(Z;\mathbb{R}^N\big)$,

$$\big\|w_n(z)\big\|_{\mathbb{R}^N} \;=\; g_n(z) \;\leqslant\; h_n(z) \qquad \forall\, z \in Z$$

and

$$\|v-w_n\| \;=\; \|v\| - g_n \;\searrow\; 0.$$

Because

$$\|v-w_n\| \;=\; \|v\| - g_n \;\leqslant\; 2\,\|v\|,$$

by virtue of the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that

$$\int_Z v\,d\mu \;=\; \langle Du, v\rangle_{C_c(Z;\mathbb{R}^N)} \;=\; \lim_{n\to+\infty} \langle Du, w_n\rangle_{C_c(Z;\mathbb{R}^N)} \;\leqslant\; \lim_{n\to+\infty} \|Du\|(g_n),$$

so

$$\|Du\|(h) \;\leqslant\; \lim_{n\to+\infty} \|Du\|(g_n).$$

Since

$$g_n \;\leqslant\; h \qquad \forall\, n \geqslant 1,$$

we have that the opposite inequality also holds, hence

$$\|Du\|(h) \;\geqslant\; \lim_{n\to+\infty} \|Du\|(g_n),$$

so

$$\|Du\|(h) \;=\; \lim_{n\to+\infty} \|Du\|(h_n).$$

□

COROLLARY 2.6.8
If $Z \subseteq \mathbb{R}^N$ *is an open set and* $u \in BV(Z)$,
then there exists a Borel measurable function $\xi\colon Z \longrightarrow \mathbb{R}^N$, *such that*

$$\big\|\xi(z)\big\|_{\mathbb{R}^N} \;=\; 1 \quad \mu = Du\text{-a.e.}$$

and

$$\int\limits_Z u \operatorname{div} \vartheta \, dz \;=\; -\int\limits_Z (\vartheta, \xi)_{\mathbb{R}^N} \, d\,\|Du\| \qquad \forall\ \vartheta \in C_c^1\big(Z; \mathbb{R}^N\big).$$

REMARK 2.6.9 Evidently

$$\xi \;=\; \frac{d(Du)}{d\,\|Du\|}$$

(i.e., the Radon-Nikodym derivative of $\mu = Du$ with respect to $\|Du\|$, since $Du \prec\prec \|Du\|$; see Theorem A.2.24 and Remark A.2.25). So, we have

$$\int\limits_Z u \operatorname{div} \vartheta \, dz \;=\; -\int\limits_Z \vartheta dDu \qquad \forall\ \vartheta \in C_c^1\big(Z; \mathbb{R}^N\big).$$

In the sequel for $u \in L^1_{\mathrm{loc}}(Z)$, we say that $u \in BV_{\mathrm{loc}}(Z)$ (i.e., has locally bounded variation in Z), if for every bounded open set $V \subseteq Z$ with $\overline{V} \subseteq Z$, we have that $u \in BV(V)$. Note that the total variation of $\|Du\|$ is given by

$$\|Du\|(Z) \;=\; \sup\Bigg\{ \int\limits_Z u \operatorname{div} \vartheta \, dz :\ \vartheta = (\vartheta_k)_{k=1}^N \in C_c^\infty\big(Z; \mathbb{R}^N\big), \\ \big\|\vartheta(z)\big\|_{\mathbb{R}^N} \leqslant 1 \text{ for all } z \in Z \Bigg\}.$$

The norm of $BV(Z)$ is given by

$$\|u\|_{BV(Z)} \;=\; \|u\|_1 + \|Du\|$$

and makes $BV(Z)$ a Banach space.

It is also well known that an absolutely continuous function $u\colon \mathbb{R} \longrightarrow \mathbb{R}$ with $u' \in L^1(\mathbb{R})$ is of bounded variation in $\mathbb{R}$. In particular then

$$W^{1,1}(\mathbb{R}) \;\subseteq\; BV(\mathbb{R}).$$

Next we show that the same is true in higher dimensions (i.e., for $N > 1$). First two examples to motivate what follows.

EXAMPLE 2.6.10 **(a)** Let $Z \subseteq \mathbb{R}^N$ be an open set and $u \in W^{1,1}(Z)$, then $u \in BV(Z)$ (i.e., $W^{1,1}(Z) \subseteq BV(Z)$). To see this let $\vartheta \in C^1_c\big(Z;\mathbb{R}^N\big)$ with $\big\|\vartheta(z)\big\|_{\mathbb{R}^N} \leqslant 1$ for all $z \in Z$. We have

$$\int_Z u \operatorname{div} \vartheta \, dz \;=\; -\int_Z \big(Du, \vartheta\big)_{\mathbb{R}^N} dz,$$

so

$$\|Du\| \;=\; \int_Z \big\|Du(z)\big\|_{\mathbb{R}^N} dz$$

and

$$\xi(z) \;=\; \begin{cases} \frac{Du(z)}{\|Du(z)\|_{\mathbb{R}^N}} & \text{if} \quad Du(z) \neq 0 \\ 0 & \text{if} \quad Du(z) = 0 \end{cases} \qquad \text{for } \lambda^N\text{-a.a. } z \in Z.$$

Similarly we show that $W^{1,1}_{\mathrm{loc}}(Z) \subseteq BV_{\mathrm{loc}}(Z)$. In particular then $W^{1,p}_{\mathrm{loc}}(Z) \subseteq BV_{\mathrm{loc}}(Z)$ for all $p \in [1, +\infty)$ and if $Z \subseteq \mathbb{R}^N$ is bounded and open then $W^{1,p}(Z) \subseteq BV(Z)$ for all $n \geqslant 1$.

(b) Let $Z \subseteq \mathbb{R}^N$ be an open set, $U \subseteq \mathbb{R}^N$ another open set with C^2-boundary ∂U, such that

$$\mu^{(N-1)}\big(\partial U \cap K\big) \;<\; +\infty \quad \text{for all compact sets } K \subseteq Z.$$

Then from Proposition 2.4.44, for $\vartheta \in C^1_c\big(Z;\mathbb{R}^N\big)$, we have

$$\int_U \operatorname{div} \vartheta \, dz \;=\; \int_{\partial U} (\vartheta, n)_{\mathbb{R}^N} \, d\mu^{(N-1)}$$

(here n denotes the outward unit normal along ∂U). Hence for any bounded, open set $V \subseteq Z$, with $\overline{V} \subseteq Z$ and for any $\vartheta \in C^1_c\big(V;\mathbb{R}^N\big)$, we have

$$\int_U \operatorname{div} \vartheta \, dz \;=\; \int_{\partial U \cap V} (\vartheta, n)_{\mathbb{R}^N} \, d\mu^{(N-1)} \;\leqslant\; \mu^{N-1}\big(\partial U \cap V\big),$$

so $\chi_U \in BV_{\mathrm{loc}}(Z)$.

Moreover,

$$\|\partial \chi_U\|(Z) \;=\; \mu^{(N-1)}\big(\partial U \cap Z\big).$$

Thus $\|\partial \chi_U\|(Z)$ measures the size of ∂U in Z. Since χ_U is not in general in $W^{1,1}_{\mathrm{loc}}(Z)$, we see that not every function of (locally) bounded variation is a Sobolev function.

Motivated by Example 2.6.10**(b)**, we make the following definition.

DEFINITION 2.6.11 *A Lebesgue measurable set $A \subseteq \mathbb{R}^N$ is said to have **finite perimeter** in an open set $Z \subseteq \mathbb{R}^N$, if $\chi_A \in BV(Z)$.*

REMARK 2.6.12 Some authors call sets of finite perimeter, ***Caccioppoli sets***.

Next we shall establish some elementary properties of BV-functions. The first is the lower semicontinuity of the variational measure.

PROPOSITION 2.6.13
If $Z \subseteq \mathbb{R}^N$ is an open set and $\{u_n\}_{n\geqslant 1} \subseteq BV(Z)$ is such that

$$u_n \longrightarrow u \quad \text{in } L^1_{\text{loc}}(Z),$$

then for every open set $U \subseteq Z$, we have

$$\|Du\|(U) \leqslant \liminf_{n\to+\infty} \|Du_n\|(U).$$

PROOF Let $\vartheta \in C_c^\infty\big(Z;\mathbb{R}^N\big)$ be such that

$$\|\vartheta(z)\|_{\mathbb{R}^N} \leqslant 1 \qquad \forall\, z \in U.$$

We have

$$\int_U u \operatorname{div}\vartheta\, dz = \lim_{n\to+\infty} \int_U u_n \operatorname{div}\vartheta\, dz \leqslant \liminf_{n\to+\infty} \|Du_n\|(U);$$

so from Remark 2.6.9, we have

$$\|Du\|(U) \leqslant \liminf_{n\to+\infty} \|Du_n\|(U).$$

□

REMARK 2.6.14 The above Proposition does not assert that $u \in BV(Z)$. This will be true if $u \in L^1(Z)$ and $\sup\limits_{n\geqslant 1} \|Du_n\|(Z) < +\infty$. To see this let $\vartheta \in C_c^1(Z)$ and $k = 1, \dots, N$. We have

$$\lim_{n\to+\infty} \int_Z \vartheta D_k u_n\, dz = -\lim_{n\to+\infty} \int_Z u_n D_k\vartheta\, dz = -\int_Z u D_k\vartheta\, dz,$$

so

$$\left| \int_Z u D_k\vartheta\, dz \right| \leqslant \|\vartheta\|_\infty \liminf_{n\to+\infty} \|Du_n\|(Z) < +\infty.$$

Because the embedding $C_c^1(Z) \subseteq C_c(Z)$ is dense, we have that

$$D_k u(\vartheta) = -\int_Z u D_k\vartheta\, dz \qquad \forall\, k = 1, \dots, N$$

is a bounded linear functional on $C_c(Z)$, hence a measure.

In the next Proposition, we establish an upper semicontinuity property of the total variation measure.

PROPOSITION 2.6.15
If $Z \subseteq \mathbb{R}^N$ is an open set, $\{u_n\}_{n\geqslant 1} \subseteq BV(Z)$,

$$u_n \longrightarrow u \quad \text{in } L^1_{\mathrm{loc}}(Z)$$

and

$$\|Du\|(Z) = \lim_{n\to+\infty} \|Du_n\|(Z),$$

then

$$\limsup_{n\to+\infty} \|Du_n\|\left(\overline{U}\cap Z\right) \leqslant \|Du\|\left(\overline{U}\cap Z\right) \quad \text{for all open sets } U \subseteq Z.$$

PROOF The set $V = Z \setminus \overline{U}$ is open and so from Proposition 2.6.13, we have that

$$\|Du\|(V) \leqslant \liminf_{n\to+\infty} \|Du_n\|(V). \tag{2.66}$$

Then we have

$$\begin{aligned}
&\|Du\|\left(\overline{U}\cap Z\right) + \|Du\|(V) = \|Du\|(Z) = \lim_{n\to+\infty} \|Du_n\|(Z)\\
&\geqslant \limsup_{n\to+\infty} \|Du_n\|\left(\overline{U}\cap Z\right) + \liminf_{n\to+\infty} \|Du_n\|(V)\\
&\geqslant \limsup_{n\to+\infty} \|Du_n\|\left(\overline{U}\cap Z\right) + \|Du\|(V),
\end{aligned}$$

so

$$\limsup_{n\to+\infty} \|Du_n\|\left(\overline{U}\cap Z\right) \leqslant \|Du\|\left(\overline{U}\cap Z\right).$$

□

Combining Propositions 2.6.13 and 2.6.15, we have the following.

COROLLARY 2.6.16
If $Z \subseteq \mathbb{R}^N$ is an open set, $\{u_n\}_{n\geqslant 1} \subseteq BV(Z)$,

$$u_n \longrightarrow u \quad \text{in } L^1_{\mathrm{loc}}(Z),$$

$$\|Du_n\|(Z) \longrightarrow \|Du\|(Z)$$

and

$$\|Du\|\left(\partial U\right) = 0 \quad \text{for all open sets } U \subseteq Z,$$

then

$$\|Du_n\|(U) \longrightarrow \|Du\|(U).$$

The next theorem is the counterpart for the space $BV(Z)$ of the Meyers-Serrin theorem (see Theorem 2.4.13).

THEOREM 2.6.17
If $Z \subseteq \mathbb{R}^N$ is an open set and $u \in BV(Z)$,
then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq BV(Z) \cap C^\infty(Z)$, such that

$$u_n \longrightarrow u \quad \text{in } L^1(Z) \quad \text{and} \quad \|Du_n\|(Z) \longrightarrow \|Du\|(Z).$$

PROOF Let $\varepsilon > 0$. For a given positive integer $m \geqslant 1$, we define the following open subset of Z:

$$Z_k \stackrel{df}{=} \left\{ z \in Z : \ d(z, \partial Z) > \frac{1}{k+m} \right\} \cap B_{k+m}(0) \qquad \forall\ k \geqslant 1.$$

Choose $m \geqslant 1$ large enough so that

$$\|Du\|(Z \setminus Z_1) \ < \ \varepsilon. \tag{2.67}$$

Setting $Z_0 = \emptyset$, we introduce the following sequence of open sets of Z:

$$V_k \stackrel{df}{=} Z_{k+1} \setminus \overline{Z}_{k-1} \qquad \forall\ k \geqslant 1.$$

Let $\{\xi_k\}_{k\geqslant 1}$ be a C^∞-partition of unity subordinate to the open cover $\{V_k\}_{k\geqslant 1}$ of Z, i.e.,

$$\xi_k \in C_c^\infty(V_k), \quad 0 \leqslant \xi_k \leqslant 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \xi_k = 1 \quad \text{on } Z.$$

Let φ be a mollifier and for each $k \geqslant 1$, choose $\varepsilon_k > 0$, such that

$$\begin{cases} \text{supp } (\varphi_{\varepsilon_k} \star (\xi_k u)) \ \subseteq \ V_k \\ \|\varphi_{\varepsilon_k} \star (\xi_k u) - \xi_k u\|_1 \ < \ \dfrac{\varepsilon}{2^k} \\ \|\varphi_{\varepsilon_k} \star (uD\xi_k) - uD\xi_k\|_1 \ < \ \dfrac{\varepsilon}{2^k}. \end{cases} \tag{2.68}$$

Let

$$u_\varepsilon \stackrel{df}{=} \sum_{k=1}^{\infty} \varphi_{\varepsilon_k} \star (\xi_k u).$$

Then $u_\varepsilon \in C^\infty(Z)$ and because

$$u \ = \ \sum_{k=1}^{\infty} \xi_k u,$$

from (2.68), we have

$$\|u_\varepsilon - u\|_1 \ < \ \varepsilon,$$

so

$$u_\varepsilon \longrightarrow u \quad \text{in } L^1(Z) \quad \text{as } \varepsilon \searrow 0. \tag{2.69}$$

Invoking Proposition 2.6.13, we have

$$\|Du\|(Z) \leqslant \liminf_{\varepsilon \searrow 0} \|Du_\varepsilon\|(Z). \tag{2.70}$$

Now let $\vartheta \in C^1_c\big(Z;\mathbb{R}^N\big)$ be such that

$$\big\|\vartheta(z)\big\|_{\mathbb{R}^N} \leqslant 1 \qquad \forall\ z \in Z.$$

We have

$$\begin{aligned}
\int_Z u_\varepsilon \mathrm{div}\,\vartheta\, dz &= \sum_{k=1}^{\infty} \int_Z \varphi_{\varepsilon_k} \star (\xi_k u)\mathrm{div}\,\vartheta\, dz \\
&= \sum_{k=1}^{\infty} \int_Z \xi_k u \mathrm{div}\,\big(\varphi_{\varepsilon_k} \star \vartheta\big)\, dz \\
&= \sum_{k=1}^{\infty} \int_Z u\mathrm{div}\,\left(\xi_k(\varphi_{\varepsilon_k} \star \vartheta)\right) dz - \sum_{k=1}^{\infty} \int_Z u\left(D\xi_k, (\varphi_{\varepsilon_k} \star \vartheta)\right)_{\mathbb{R}^N} dz \\
&= \sum_{k=1}^{\infty} \int_Z u\mathrm{div}\,\big(\xi_k(\varphi_{\varepsilon_k} \star \vartheta)\big)\, dz - \sum_{k=1}^{\infty} \int_Z (\vartheta, \varphi_{\varepsilon_k} \star (uD\xi_k) - uD\xi_k)_{\mathbb{R}^N} dz \\
&= \eta_{1,\varepsilon} + \eta_{2,\varepsilon}.
\end{aligned}$$

Note that

$$\big\|\xi_k\big(\varphi_{\varepsilon_k} \star \vartheta\big)(z)\big\|_{\mathbb{R}^N} \leqslant 1 \qquad \forall\ z \in Z,\ k \geqslant 1.$$

Also each $z \in Z$ belongs in at most three elements in the cover $\{V_k\}_{k\geqslant 1}$. So we have

$$\begin{aligned}
\big|\eta_{1,\varepsilon}\big| &= \left| \int_Z u\mathrm{div}\,\big(\xi_1(\varphi_{\varepsilon_1} \star \vartheta)\big)\, dz + \sum_{k=2}^{\infty} \int_Z u\,\mathrm{div}\,\big(\xi_k(\varphi_{\varepsilon_k} \star \vartheta)\big)\, dz \right| \\
&\leqslant \|Du\|(Z) + \sum_{k=2}^{\infty} \|Du\|(V_k) \\
&\leqslant \|Du\|(Z) + 3\,\|Du\|(Z \setminus Z_1) \leqslant \|Du\|(Z) + 3\varepsilon.
\end{aligned} \tag{2.71}$$

Also from (2.68), we have that

$$\big|\eta_{2,\varepsilon}\big| < \varepsilon. \tag{2.72}$$

From (2.71) and (2.72), it follows that

$$\int_Z u_\varepsilon \mathrm{div}\,\vartheta\, dz \leqslant \|Du\|(Z) + 4\varepsilon,$$

thus

$$\left\|Du_\varepsilon\right\|(Z) \leqslant \|Du\|(Z) + 4\varepsilon$$

and so

$$\limsup_{\varepsilon\to 0} \left\|Du_\varepsilon\right\|(Z) \leqslant \|Du\|(Z). \tag{2.73}$$

From (2.70) and (2.73), we infer that

$$\left\|Du_\varepsilon\right\|(Z) \longrightarrow \|Du\|(Z) \quad \text{as } \varepsilon \searrow 0.$$

This combined with (2.69) finishes the proof of the theorem. □

REMARK 2.6.18 Note that in the previous "local" approximation result, we do not have that

$$\left\|D(u_\varepsilon - u)\right\|(Z) \longrightarrow 0 \quad \text{as } \varepsilon \searrow 0$$

and so we cannot claim the density of $BV(Z) \cap C^\infty(Z)$ in $BV(Z)$.

COROLLARY 2.6.19
If $Z \subseteq \mathbb{R}^N$ is a bounded open set which is Lipschitz, then $\overline{B}_r$, for $r > 0$, is compact in $L^1(Z)$, where

$$\overline{B}_r = \{u \in BV(Z) : \|u\|_{BV} \leqslant r\}.$$

PROOF Let $\{u_n\}_{n\geqslant 1} \subseteq \overline{B}_r$. By Theorem 2.6.17, we can find $\psi_n \in C^\infty(Z)$, such that

$$\left\|u_n - \psi_n\right\|_1 \leqslant \frac{1}{n} \quad \text{and} \quad \|D\psi_n\| = \int_Z \left\|D\psi_n(z)\right\| dz \leqslant 2.$$

It follows that $\{\psi_n\}_{n\geqslant 1} \subseteq W^{1,1}(Z)$ is bounded. By virtue of Theorem 2.5.17, the sequence $\{\psi_n\}_{n\geqslant 1} \subseteq L^1(Z)$ is relatively compact. So we may assume that

$$\psi_n \longrightarrow u \quad \text{in } L^1(Z).$$

From Remark 2.6.14, we have that $u \in BV(Z)$ and from Proposition 2.6.13, we have that

$$\|u\|_{BV} \leqslant r,$$

i.e., $u \in \overline{B}_r$. □

REMARK 2.6.20 According to Corollary 2.6.19, if $Z \subseteq \mathbb{R}^N$ is a bounded open set which is Lipschitz, then the embedding $BV(Z) \subseteq L^1(Z)$ is compact.

An interesting application of this compact embedding is the following result.

PROPOSITION 2.6.21
If $Z \subseteq \mathbb{R}^N$ *is a bounded open set which is Lipschitz,*

$$\mathcal{T} \stackrel{df}{=} \Big\{ A \subseteq Z : \quad A \text{ is Lebesgue measurable}, \\ \lambda^N(A) = \lambda^N(Z \setminus A) = \frac{1}{2}\lambda^N(Z) \Big\}$$

and

$$P(A, Z) \;=\; \|D\chi_A\| \qquad \forall\, A \in \mathcal{T}$$

(the perimeter with respect to Z *functional),*
then there exists $A^* \in \mathcal{T}$*, such that* $P(A^*, Z) = \inf_{A \in \mathcal{T}} P(A, Z)$.

PROOF Let

$$S \stackrel{df}{=} \{\chi_A : \; A \in \mathcal{T}\} \;\subseteq\; L^1(Z).$$

We furnish S with the relative $L^1(Z)$-topology. Since

$$\|\chi_A\|_1 \;\leqslant\; \lambda^N(Z) \qquad \forall\, A \in \mathcal{T},$$

we see that the functional $\xi \colon S \longrightarrow \mathbb{R}$ defined by

$$\xi(\chi_A) \stackrel{df}{=} \|D\chi_A\| \;=\; P(A, Z)$$

is coercive on S for the $BV(Z)$-norm. Therefore the sub-level sets of ξ are bounded in $BV(Z)$, thus relatively compact in $S \subseteq L^1(Z)$ (note that S is closed in $L^1(Z)$ and see Remark 2.6.20). Also from Proposition 2.6.13, we know that ξ is lower semicontinuous on S. This means that its sub-level sets are compact in $L^1(Z)$. So by the Weierstrass theorem, we can find $A^* \in \mathcal{T}$, such that $P(A^*, Z) = \inf_{A \in \mathcal{T}} P(A, Z)$. □

We can relate the variation measure of u and the perimeters of its super-level sets. The result is actually a "Co-Area Formula" for BV-functions (see also Theorem 1.5.25).

THEOREM 2.6.22
If $Z \subseteq \mathbb{R}^N$ *is an open set,* $u \in L^1(Z)$ *and for every* $r \in \mathbb{R}$ *let*

$$L_r \stackrel{df}{=} \big\{ z \in Z : \; u(z) > r \big\},$$

then

(a) $u \in BV(Z)$ *implies that*

$$\|Du\|(Z) = \int_{-\infty}^{\infty} \big\|D\chi_{L_r}\big\|(Z)\, dr;$$

(b) *if for almost all* $r \in R$, L_r *has a finite perimeter, then* $u \in BV(Z)$.

2.7 Remarks

2.1: To have a good theory of integration, we need a reasonable notion of measurability of functions. In this direction the basic result is the Pettis measurability theorem (see Theorem 2.1.3), which was proved by Pettis (1938*a*). The main integral for vector valued functions, which has a rich enough structure to have significant applications, is the Bochner integral. The Bochner integral can be traced in the works of Bochner (1933) and Dunford (1935) and for this reason is also known as "Dunford's first integral." Most of the properties of the Bochner integral follow from the corresponding properties of the classical Lebesgue integral, by virtue of Proposition 2.1.10. So some analysts say that the Bochner integral is the Lebesgue integral with the absolute value replaced by norms. The Pettis integral has much fewer applications, which require knowledge and use of sophisticated measure theoretic results. The theory of Pettis integration started with the work of Pettis and attracted renewed attention after the paper of Edgar (1977). A detailed study of the Pettis integral with applications can be found in the monograph of Talagrand (1984*a*). On the subject of vector valued functions and their integration, the reader can consult the books of Diestel & Uhl (1977), Dunford & Schwartz (1958) and Hille & Phillips (1957). The proof of the Orlicz-Pettis theorem can be found in Diestel & Uhl (1977, p. 22).

2.2: A reference to Lebesgue-Bochner spaces can be found in every book dealing with infinite dimensional dynamical systems. They are a natural generalization of the classical Lebesgue spaces using the notion of Bochner integral. Vector measures were already considered by Pettis (1938*b*). However, the real expansion on the subject occurred in the late 60s and during the 70s, when there was a systematic study of the geometry of Banach spaces. That is when RNP spaces were introduced and studied in detail. That a reflexive Banach space has the RNP, which was established by Phillips (1940), while the fact that a separable dual Banach space is an RNP space is due to Dunford & Pettis (1940). The proof of Proposition 2.2.8 can be found in Diestel & Uhl (1977, pp. 79 and 82). Theorem 2.2.9 (the Riesz Representation theorem for the Lebesgue Bochner spaces $L^p(\Omega;X)$, $p \in [1,+\infty)$) is essentially due to Bochner & Taylor (1938). Its proof can be found in Diestel & Uhl (1977, p. 97). Its extension (for $p = 1$) mentioned in Theorem 2.2.12 (called Dinculeanu-Foias theorem) is due to Dinculeanu & Foias (1961) and its proof, based on "lifting theory," can be found in Ionescu-Tulcea & Ionescu-Tulcea (1969, p. 93). Absolute continuity of real valued functions (see Definition 2.2.14) was introduced by Vitali (1908), who established the fundamental fact that a real valued function on $[0,1]$ is absolutely continuous if and only if it is the integral of its derivative (the fundamental theorem of Lebesgue calculus). Theorem 2.2.17 is due to Komura (1967). Lemma 2.2.29

can be found in Lions (1969, p. 58) and shows how new inequalities can be derived from the properties of embedding operators. Theorem 2.2.30 is due to Aubin (1963) and plays a central role in the theory of evolution equations. Evolution triples (see Definition 2.2.31) are also known as "Gelfand triples," because of their systematic use by Gelfand & Shilov (1977) (see also Wloka (1987)). Evolution triples and their properties and applications can be found in Denkowski, Migórski & Papageorgiou (2003*a*, 2003*b*), Hu & Papageorgiou (1997, 2000), Lions (1969), Showalter (1997) and Zeidler (1990*a*, 1990*b*). Finally we mention a result on the structure of $L^1(\Omega; X)$ due to Talagrand (1984*b*).

PROPOSITION 2.7.1
If (Ω, Σ, μ) *is a finite measure space and* X *is a Banach space which is weakly sequentially complete,*
then $L^1(\Omega; X)$ *is weakly sequentially complete too.*

2.3: Theorem 2.3.2 is known as the "Arzela-Ascoli theorem" although some authors use only one of the two names. Working on $C\big([0,1]\big)$, Arzela (1889) proved the necessity part, while Ascoli (1883–1884) proved the sufficiency part. A general formulation of this theorem can be found in Denkowski, Migórski & Papageorgiou (2003*a*, p. 73). The results on the compactness of various sets in $L^p(T; X)$ $(p \in [1, +\infty))$ and in $C(T; X)$ (variations of the Arzela-Ascoli theorem) can be found in Simon (1987). They are formulations and extensions of the classical criterion for strong compactness in $L^p(T)$ $(p \in [1, +\infty))$, due to Riesz (1933) and Kolmogorov (1931). James' theorem (see Theorem 2.3.21), due to James (1964), is one of the deepest and most influential results of functional analysis. From it, it follows that a Banach space X is reflexive if and only if every $x^* \in X^*$ attains its supremum on the unit ball of X. For a proof of James' theorem see Holmes (1975, pp. 157–161). Theorem 2.3.21 can be found in Papageorgiou (1985) (see also Denkowski, Migórski & Papageorgiou (2003*a*, p. 462)), where a kind of converse of it can also be found. The proof of Proposition 2.3.22 can be found in Denkowski, Migórski & Papageorgiou (2003*a*, p. 458). Ionescu-Tulcea & Ionescu-Tulcea (1969) were the first to observe that the classical Dunford-Pettis theorem (see Dunford (1935)) can be extended to X-valued functions with X being a reflexive Banach space, after some straightforward modifications in the original proof (see Theorem 2.3.24). The proof of Proposition 2.3.31 can be found in Denkowski, Migórski & Papageorgiou (2003*a*, p. 484). The notion of biting convergence (see Definition 2.3.35) is due to Chacon (see Brooks & Chacon (1980) and Ball & Murat (1989)). In Brooks & Chacon (1980), we can find the original version of Theorem 2.3.26 (Biting Theorem). Property U (see Definition 2.3.33) is natural in the context of solution flows of a differential equation. Theorem 2.3.37 is due to Gutman (1985). Extensions of Proposition 2.3.39 to Banach space valued functions can be found in Rzezuchowski

(1989). The notation for the various spaces of continuous functions is not standard (see, e.g., Hewitt & Stromberg (1975, p. 86)). For a proof of Theorem 2.3.41 (Riesz-Markov representation theorem) we refer to Denkowski, Migórski & Papageorgiou (2003*a*, p. 322). Also the names for the various modes of convergence introduced in Definition 2.3.42 vary among authors. So we caution the reader to be careful. Since we are dealing with the space of measures, let us mention two striking results concerning them. Let (Ω, Σ) be a measurable space and let $\mathrm{ca}(\Sigma)$ be the space of all signed measures on Σ of bounded variation endowed with the total variation norm

$$\|\mu\|_1 \stackrel{df}{=} |\mu|(\Omega) \qquad \forall\ \mu \in \mathrm{ca}(\Sigma).$$

We can also introduce another norm given by

$$\|\mu\|_\infty \stackrel{df}{=} \sup_{A\in\Sigma} \big|\mu(A)\big| \qquad \forall\ \mu \in \mathrm{ca}(\Sigma).$$

Then

$$\|\mu\|_\infty \leqslant \|\mu\|_1 \leqslant 4\,\|\mu\|_\infty \qquad \forall\ \mu \in \mathrm{ca}(\Sigma)$$

(i.e., the two norms are equivalent). The space $\big(\mathrm{ca}(\Sigma), \|\cdot\|_1\big)$ is a Banach space. The first result is a remarkable improvement of the Uniform Boundedness Principle and is known as "Nikodym's boundedness theorem."

PROPOSITION 2.7.2
If $\{\mu_s\}_{s\in S} \subseteq \mathrm{ca}(\Sigma)$ *and*

$$\sup_{s\in S} \big|\mu_s(A)\big| < +\infty \qquad \forall\ A \in \Sigma,$$

then

$$\sup_{\substack{s\,\in\, S\\ A\,\in\,\Sigma}} \big|\mu_s(A)\big| < +\infty.$$

The second result is known as "Nikodym's convergence theorem."

PROPOSITION 2.7.3
If $\{\mu_n\}_{n\geqslant 1} \subseteq \mathrm{ca}(\Sigma)$ *and*

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A) \text{ exists} \qquad \forall\ A \in \Sigma,$$

then

$$\mu \in \mathrm{ca}(\Sigma)$$

and moreover, if $\mu_n \prec\prec \lambda$ *for all* $n \geqslant 1$ *with* $\lambda \in \mathrm{ca}(\Sigma)$, *then* $\mu \prec\prec \lambda$.

Both results can be found in Diestel (1984, pp. 80 and 90) and Dunford & Schwartz (1958, pp. 309 and 321).

For a proof of Theorem 2.3.48 we refer to Denkowski, Migórski & Papageorgiou (2003*a*, p. 198) and Parthasarathy (1967, p. 45). Proposition 2.3.49 is due to Brézis & Lieb (1983). Finally we state a compactness result concerning vector measures. The result is known as "Lyapunov's convexity theorem" and has important ramifications in Control Theory (see Hermes & LaSalle (1969)).

THEOREM 2.7.4
Let (Ω, Σ) be a measurable space.

(a) *If $\mu_k \colon \Sigma \longrightarrow \mathbb{R}$, $k = 1, \dots, N$ are finite nonatomic measures, then $R = \bigcup_{A \in \Sigma} \big(\mu_k(A)\big)_{k=1}^{N}$ is compact and convex in $\mathbb{R}^N$.*

(b) *If X is a Banach space with the RNP and $m \colon \Sigma \longrightarrow X$ is a vector measure which is nonatomic and of bounded variation, then $R = \overline{\bigcup_{A \in \Sigma} m(A)}^{\|\cdot\|}$ is strongly compact and convex.*

2.4: Sobolev spaces were introduced by Sobolev (1963*a*, 1963*b*). Related spaces were also studied by Morrey (1940, 1966) and later by Deny & Lions (1953–1954). Today there are many well known books on the subject. We mention Adams (1975), Brézis (1983), Evans & Gariepy (1992), Kufner, John & Fučik (1977), Lions & Magenes (1972), Maz'ja (1985) and Ziemer (1989). We mention that for functions of several variables (i.e., $N > 1$), when $p = 2$, we use the notation $H^m(Z)$ (respectively $H_0^m(Z)$) for the Sobolev space $W^{m,2}(Z)$ (respectively $W_0^{m,2}(Z)$). However, for functions of one variable (i.e., $N = 1$, hence $Z = T = (a, b)$), we keep the notation $W^{m,2}(T)$ (respectively $W_0^{m,2}(T)$). Theorem 2.4.13 is due to Meyers & Serrin (1964). The result is often called "local approximation theorem." A discussion of the various geometric conditions imposed on the boundary ∂Z can be found in Adams (1975, pp. 66–67). For a proof of the approximation result given in Theorem 2.4.17, we refer to Evans & Gariepy (1992, p. 127). To see that without further conditions on the domain Z, Theorem 2.4.17 is not true, consider the following example.

EXAMPLE 2.7.5 Let

$$Z \stackrel{df}{=} \big\{(z_1, z_2) \in \mathbb{R}^2 : \ 0 < |z_1| < 1, \ 0 < z_2 < 1\big\}$$

and

$$u(z_1, z_2) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad z_1 > 0, \\ 0 & \text{if} \quad z_1 < 0. \end{cases}$$

Clearly $u \in W^{1,p}(Z)$ $(p \in [1, +\infty))$. However, given $\varepsilon > 0$ sufficiently small, it is easy to see that we cannot find $\vartheta \in C^1(\overline{Z})$, such that $\|u - \vartheta\|_{W^{1,p}(Z)} < \varepsilon$. Note that this particular Z lies on both sides of its boundary.

Another approximation result, useful in optimal control problems, is given below. First a definition.

DEFINITION 2.7.6 *Let Z be an open set. We say that $u\colon Z \longrightarrow \mathbb{R}$ is **affine**, if it is the restriction to Z of an affine function over $\mathbb{R}^N$. We say that $u\colon Z \longrightarrow \mathbb{R}$ is **piecewise affine**, if it is continuous and there exists a partition of Z into a Lebesgue-null set and finite number of open sets on which u is affine.*

REMARK 2.7.7 If $u\colon Z \longrightarrow \mathbb{R}$ is affine, then

$$Du \;=\; \text{constant}$$

and the converse is true, if Z is connected. We have

$$u(z) \;=\; \big(Du(z), z\big)_{\mathbb{R}^N} + c,$$

with $c \in \mathbb{R}$.

PROPOSITION 2.7.8
If $Z \subseteq \mathbb{R}^N$ is a bounded open set which is Lipschitz and $u \in W_0^{1,p}(Z)$ $(p \in (1, +\infty))$,
then we can find a sequence $\{u_n\}_{n\geqslant 1}$ of piecewise affine functions over Z, null on ∂Z (i.e., $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$), such that

$$u_n \;\longrightarrow\; u \quad \text{in } W_0^{1,p}(Z).$$

For the case $p = +\infty$, there is the following approximation result.

PROPOSITION 2.7.9
If $Z \subseteq \mathbb{R}^N$ is a bounded open set which is Lipschitz and $u \in W^{1,\infty}(Z)$,
then there exists a sequence $\{u_n, Z_n\}_{n\geqslant 1}$ where $u_n \in W^{1,\infty}(Z)$, $Z_n \subseteq Z$ are open,

$$Z_n \;\subseteq\; Z_{n+1} \qquad \forall\; n \geqslant 1,$$

$$\lambda^N(Z \setminus Z_n) \;\longrightarrow\; 0,$$

$$u_n|_{Z_n} \text{ are piecewise affine,}$$

$$u_n(z) \;=\; u(z) \qquad \forall\; z \in \partial Z,\; n \geqslant 1,$$

$$u_n \;\longrightarrow\; u \quad \text{uniformly on } Z,$$

$$Du_n(z) \;\longrightarrow\; Du(z) \quad \text{for a.a. } z \in Z$$

and

$$\|Du_n\|_\infty \;\leqslant\; \|Du\|_\infty + \varepsilon(n),$$

with $\varepsilon(n) \longrightarrow 0$ as $n \to +\infty$.

REMARK 2.7.10 Recall that, if $u \in W^{1,\infty}(Z)$, then it is Lipschitz continuous on Z and so it can be extended continuously to $\overline{Z}$ (i.e., $W^{1,\infty}(Z) \subseteq C(\overline{Z})$). So the boundary values of u are well defined.

Both the previous approximation results can be found in Ekeland & Temam (1976, pp. 316–317).

A detailed discussion of Sobolev spaces of fractional order and on manifolds can be found in Adams (1975) and Kufner, John & Fučik (1977). Theorem 2.4.54 can be found in Kenmochi (1975) and Casas & Fernández (1989).

Finally we mention a Proposition useful in the interpretation of the variational formulation of various equations, such as the Navier-Stokes equation. The result is due to de Rham (1955).

PROPOSITION 2.7.11
If $Z \subseteq \mathbb{R}^N$ *is an open set and* $u = (u_k)_{k=1}^N \in \mathcal{D}(Z;\mathbb{R}^N)^*$,
then a necessary and sufficient condition that $u = Dh$ *for some* $h \in \mathcal{D}(Z)^*$ *is that*

$$\langle u, \vartheta \rangle \;=\; 0 \qquad \forall\, \vartheta \in V \;\stackrel{df}{=}\; \big\{\vartheta \in \mathcal{D}(Z;\mathbb{R}^N) :\ \operatorname{div}\vartheta = 0\big\}.$$

REMARK 2.7.12 Note that the divergence operator div maps $W_0^{1,p}(Z;\mathbb{R}^N)$ onto the space

$$V \;\stackrel{df}{=}\; \left\{h \in L^p(Z) : \int_Z h(z)\,dz = 0\right\} \;=\; L^p(Z)/\mathbb{R}$$

(recall that $-\operatorname{div}$ is the adjoint of the gradient operator).

For the proof of the trace theorem (see Theorem 2.4.50), we refer to Adams (1975, p. 216) and Kufner, John & Fučik (1977, p. 337) and for the proof of Extension Theorem (see Theorem 2.4.55), we refer to Brézis (1983, p. 158).

2.5: Theorem 2.5.3 is the classical "Sobolev inequality" (see Sobolev (1963*a*, 1963*b*)), which was also developed by Gagliardo (1958), Morrey (1940, 1966) and Nirenberg (1959). The proof given here is due to Nirenberg (1959). For the Poincaré inequality (see Theorem 2.5.4) and the Poincaré-Wirtinger inequality (see Theorem 2.5.21) we refer to Meyers (1978). For the proof of Proposition 2.5.8 we refer to Maz'ja (1985, p. 27). The Sobolev embedding theorem (see Theorem 2.5.16) originated in the work of Sobolev (1963*a*), with important refinements by Morrey (1940) and Gagliardo (1958). The Rellich-Kondrachov embedding theorem (see Theorem 2.5.17) originated in a paper by Rellich (1930) for $p = 2$ and by Kondrachov (1945) for the general case. For the proofs of both theorems 2.5.16 and 2.5.17 we refer to Brézis (1983, pp. 168–170). There are variations of this theorem with interesting applications, like the following one due to Frehse (1984).

PROPOSITION 2.7.13
If $Z \subseteq \mathbb{R}^N$ is a bounded open set, $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(Z)$ (with $p \in [1,+\infty)$) is a bounded sequence and

$$\int_Z \|Du_n\|^{p-2} \left(Du_n, Dh\right)_{\mathbb{R}^N} dz \leqslant M \|h\|_\infty \quad \forall n \geqslant 1,\ h \in W^{1,p}(Z) \cap L^\infty(Z),$$

for some $M > 0$,
then there exist $u \in W^{1,p}(Z)$ and a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$, such that

$$u_n \longrightarrow u \quad \text{in } W^{1,r}(Z) \qquad \forall\ r < p.$$

For the proof of Theorem 2.5.24, we refer to Adams (1975, p. 79). Theorem 2.5.28 is another refinement of the Rellich-Kondrachov theorem (and simultaneously of the Egorov theorem; see Theorem A.2.10) and can be found in Evans (1990, p. 8). Theorem 2.5.30 (the Concentration-Compactness Lemma) is due to Lions (1985*a*, 1985*b*) and is important in the study of elliptic differential equations involving critical exponents. For additional results in this direction we refer to the work of Ben Naoum, Troestler & Willem (1996) and Bianchi, Chabrowski & Szulkin (1995) and the monographs of Evans (1990) and Willem (1996). Propositions 2.5.34 and 2.5.35 can be found in Mawhin & Willem (1989).

2.6: For the L^p-differentiability and λ^N-a.e. differentiability of Sobolev functions we refer to Bagby & Ziemer (1974), Liu (1977) and Resetnjak (1969) and the books of Evans & Gariepy (1992), Federer (1969), Simon (1983), Stein (1970) and Ziemer (1989). Proposition 2.6.3 is due to Marcus & Mizel (1972), where the interested reader can find additional results in this direction. Proposition 2.6.4 is due to Marcus & Mizel (1979), where the authors prove that the result is also valid for $p = 1$. Functions of bounded variation on $\mathbb{R}$ were introduced by Jordan (1881), who placed integration within the context of a "measurable" set. Lebesgue (1910) proved that a function of bounded variation on $\mathbb{R}$ is almost everywhere differentiable (for a proof which does not use measure theory – except sets of measure zero – we refer to Riesz & Nagy (1955, pp. 3–10)). Before the formal introduction of distributions, extensions of the notion of bounded variation to functions of many variables were suggested by Tonelli (1926) and Cesari (1936). It involved consideration of functions along the coordinate axes. Theorem 2.6.17 is due to Krickerberg (1957). The theory of sets of finite perimeter was introduced by Caccioppoli (1953) and De Giorgi (1954, 1955) (where one can find the Co-Area Formula for BV-functions; see Theorem 2.6.22). The proof of Theorem 2.6.22 can be also found in Evans & Gariepy (1992, p. 185) and Ziemer (1989, p. 231). Further contributions were made by Federer (1958), Fleming (1960) and Krickerberg (1957). More details on the space of BV-functions can be found in the books of Evans & Gariepy (1992), Giusti (1984) and Ziemer (1989).

Chapter 3

Nonlinear Operators and Young Measures

In this chapter we study certain nonlinear operators which arise in applications and we also discuss the so-called Young measures, which roughly speaking capture the limits of minimizing sequences in variational problems which do not have a solution. For some cases we also develop the corresponding linear theory in order to have a complete picture of the theory, see the similarities and differences of the two and appreciate the limitations of the nonlinear theory.

In Section 3.1, we consider compact operators. Compactness was introduced as a first attempt to deal with infinite dimensional nonlinear operator equations. By its nature, compactness approximates infinite objects by finite ones. We see that in the context of compact operators (linear and nonlinear alike) this principle is in general true. We also discuss proper maps, the spectral theory of linear, compact, self-adjoint operators on a Hilbert space and Fredholm operators. A broader framework for the analysis of infinite dimensional problems is provided by monotone operators, which extend to an infinite dimensional context, the simple notion of an increasing real function.

In Section 3.2 we examine monotone operators from a Banach space into its dual, with special emphasis on maximal monotone operators, which are a generalization of a continuous increasing real function. Maximal monotone operators have remarkable surjectivity properties. We point out that surjectivity results are important because they correspond to existence results for certain classes of nonlinear operator equations. At the end of the section we also discuss generalizations of the notion of monotonicity. These are the so-called operators of monotone type, the most important of which are the pseudomonotone operators. Monotone operators map a Banach space to its dual. If instead we want to consider nonlinear operators mapping a Banach space to itself, we need to consider accretive and m-accretive operators. Their importance comes from the fact that they are the generators of linear and nonlinear semigroups, which, roughly speaking, are an abstraction of the trajectories of a given differential equation.

In Section 3.3 first we examine accretive operators and then we look at semigroups of operators generated by certain accretive operators. We present in detail both the linear and nonlinear theories. Undoubtedly the most common nonlinear operator is the so-called Nemytskii operator (or superposition)

operator.

In Section 3.4 we examine this operator and we have a first look at integral functionals corresponding to normal integrands. In a variational problem, when the objective functional is not inf-compact, a solution does not exist. Nevertheless, the minimizing sequences (or appropriate subsequences of them) have a limit behaviour (usually more and more oscillating), which is captured by embedding the original functions to the space of Young measures (or parametrized probabilities). This embedding leads to a larger inf-compact problem which has a solution (relaxation). In Section 3.5 we discuss the theory of the Young measures and obtain additional lower semicontinuity results for integral functionals. Some of the topics of this chapter will be revisited in the course of the next chapter.

3.1 Compact and Fredholm Operators

The first efforts to solve nonlinear functional equations involved various aspects of compactness. For this reason compact operators were introduced. They constitute a class of maps to which we can generalize several of the results which are valid for maps between finite dimensional Banach spaces. Degree theory and fixed point theory, which provide important tools for the study of functional equations, depend on the notion of compact maps.

DEFINITION 3.1.1 *Let X, Y be two Banach spaces and let D be a subset of X. We say that $f\colon D \longrightarrow Y$ is* ***compact****, if it is continuous and for every bounded set $B \subseteq D$, the set $\overline{f(B)}$ is compact in Y. We denote the set of compact maps by*

$$K(D;Y).$$

Also if $D = X$, we set

$$\mathcal{L}_c(X;Y) \stackrel{df}{=} K(X;Y) \cap \mathcal{L}(X;Y).$$

REMARK 3.1.2 Evidently $K(D;Y)$ is a linear space which is closed under composition with continuous bounded maps. If $\dim Y < +\infty$, then every continuous bounded map $f\colon D \longrightarrow Y$ is compact. In the sequel we shall see that the space $K(D;Y)$ consists of precisely those maps which can be approximated by mappings with a finite dimensional range (see Theorem 3.1.10). Note that if $L\colon X \longrightarrow Y$ is linear and maps bounded sets in X into relatively compact sets in Y, then

$$L \in \mathcal{L}_c(X;Y)$$

(i.e., L is also continuous). Finally, if $L \in \mathcal{L}_c(X;Y)$, then L has a separable range.

Another notion involving compactness is given in the next definition.

DEFINITION 3.1.3 *Let X, Y be two Banach spaces and let D be a subset of X. We say that $f\colon D \longrightarrow Y$ is* ***completely continuous****, if for every sequence $\{x_n\}_{n\geqslant 1} \subseteq D$, such that*

$$x_n \overset{w}{\longrightarrow} x \quad \text{in } X,$$

for some $x \in D$, we have that

$$f(x_n) \longrightarrow f(x) \quad \text{in } Y$$

(i.e., f is sequentially continuous from D with the relative weak topology of X into Y with the norm topology).

REMARK 3.1.4 A completely continuous linear operator $L\colon X \longrightarrow Y$ is also known as ***Dunford-Pettis operator*** and is of course continuous.

In general the classes of compact maps and completely continuous maps are not comparable. However, for linear operators the situation is better. We can establish that complete continuity actually lies properly between compactness and boundedness.

PROPOSITION 3.1.5
If X, Y are two Banach spaces and

$$L \in \mathcal{L}_c(X;Y) \;=\; K(X;Y) \cap \mathcal{L}(X;Y),$$

then L is completely continuous.

PROOF If

$$x_n \overset{w}{\longrightarrow} x \quad \text{in } X,$$

then the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is bounded. Because $L \in K(X;Y)$, we have that

$$\overline{\{L(x_n)\}}_{n\geqslant 1}^{\|\cdot\|_Y} \quad \text{is compact in } Y.$$

Thus we can find a subsequence $\{x_{n_k}\}_{k\geqslant 1}$ of $\{x_n\}_{n\geqslant 1}$, such that

$$L(x_{n_k}) \longrightarrow y \quad \text{in } Y.$$

But because $L \in \mathcal{L}(X;Y)$, we also have

$$L(x_n) \overset{w}{\longrightarrow} L(x) \quad \text{in } Y.$$

Therefore $y = L(x)$ and so we conclude that

$$L(x_n) \longrightarrow L(x) \quad \text{in } Y,$$

i.e., L is completely continuous. ∎

The converse of the above Proposition is not in general true.

EXAMPLE 3.1.6 Recall that the Banach space l^1 has the Schur property, namely if

$$x_n \xrightarrow{w} x \quad \text{in } l^1,$$

then

$$x_n \longrightarrow x \quad \text{in } l^1.$$

Using this we see that the identity map $i\colon l^1 \longrightarrow l^1$ is a completely continuous linear operator which is not compact.

However, if we strengthen the condition on the space X, the situation improves.

PROPOSITION 3.1.7
If X is a reflexive Banach space, Y is a Banach space, $D \subseteq X$ is a nonempty, closed set, and $f\colon D \longrightarrow Y$ is completely continuous,
then $f \in K(D;Y)$.

PROOF Clearly f is continuous. Let $B \subseteq D$ be a bounded set. We need to show that

$$\overline{f(B)} \text{ is compact in } Y.$$

To this end let

$$\{y_n\}_{n\geqslant 1} \subseteq f(B).$$

Then

$$y_n = f(x_n) \quad \text{with } x_n \in B \qquad \forall\, n \geqslant 1.$$

Since X is reflexive, by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } D.$$

Then

$$f(x_n) \longrightarrow f(x) \quad \text{in } Y$$

and so $\overline{f(B)}$ is indeed compact in Y. ∎

Combining Propositions 3.1.5 and 3.1.7, we have the following.

COROLLARY 3.1.8
If X is a reflexive Banach space, Y is a Banach space and $L \in \mathcal{L}(X;Y)$, then L is compact if and only if L is completely continuous.

REMARK 3.1.9 In both Proposition 3.1.7 and Corollary 3.1.8 the condition that X is reflexive cannot be relaxed (see Example 3.1.6).

The next theorem gives a characterization of compact maps defined on a bounded set, which explains why compact maps are the suitable class to extend the properties of maps between finite dimensional Banach spaces.

THEOREM 3.1.10
If X, Y are two Banach spaces, $D \subseteq X$ is a bounded set and $f\colon D \longrightarrow Y$, then the following are equivalent:

(a) $f \in K(D;Y)$;

(b) *given $\varepsilon > 0$ we can find a continuous, bounded map $f_\varepsilon\colon D \longrightarrow Y$, such that*

$$\big\|f(x) - f_\varepsilon(x)\big\|_Y \;<\; \varepsilon \qquad \forall\, x \in D,$$
$$f_\varepsilon(D) \;\subseteq\; \overline{\operatorname{conv}} f(D) \quad \textit{and} \quad \dim\big(\operatorname{span} f_\varepsilon(D)\big) \;<\; +\infty.$$

PROOF "**(a)$\Longrightarrow$(b)**": Since $f \in K(D;Y)$, we have that

$$\text{the set } \overline{f(D)} \text{ is compact in } Y.$$

So given $\varepsilon > 0$, we can find $\{y_k\}_{k=1}^m \subseteq Y$, such that

$$\overline{f(D)} \;\subseteq\; \bigcup_{k=1}^m B_\varepsilon(y_k).$$

Let

$$a_k(y) \;\stackrel{df}{=}\; \max\big\{\varepsilon - \|y - y_k\|_Y, 0\big\}$$

and

$$\vartheta_k(y) \;\stackrel{df}{=}\; \frac{a_k(y)}{\sum\limits_{k=1}^m a_k(y)} \qquad \forall\, y \in \overline{f(D)}.$$

We define

$$f_\varepsilon(x) \;\stackrel{df}{=}\; \sum_{k=1}^m \vartheta_k\big(f(x)\big) y_k \qquad \forall\, x \in D.$$

Evidently the function $f_\varepsilon\colon D \longrightarrow Y$ is continuous,

$$f_\varepsilon(D) \;\subseteq\; \operatorname{span}\{y_k\}_{k=1}^m,$$

the set $\overline{f_\varepsilon(D)}$ is compact and

$$\begin{aligned}\big\|f(x) - f_\varepsilon(x)\big\|_Y \;&=\; \frac{1}{\sum\limits_{k=1}^m a_k(f(x))}\left\|\sum_{k=1}^m a_k\big(f(x)\big)\big(y_k - f(x)\big)\right\|_Y \\ &<\; \frac{\sum\limits_{k=1}^m a_k\big(f(x)\big)}{\sum\limits_{k=1}^m a_k\big(f(x)\big)}\varepsilon \;=\; \varepsilon \qquad \forall\, x \in D.\end{aligned}$$

"**(b)**$\Longrightarrow$**(a)**": Let $\varepsilon_n \stackrel{df}{=} \frac{1}{n}$ and let $f_{\varepsilon_n} = f_n$ be the continuous, bounded map with finite dimensional range postulated by statement **(b)**. Then f, being the uniform limit of the sequence $\{f_n\}_{n\geqslant 1}$ of continuous maps, is itself continuous. Also let $y = f(x)$ with $x \in D$. We have

$$\|y - y_n\|_Y \ < \ \frac{1}{n} \quad \text{with} \quad y_n \stackrel{df}{=} f_{\varepsilon_n}(x) \ \in \ f_{\varepsilon_n}(D).$$

Since $f_{\varepsilon_n}(D)$ is relatively compact in Y, we can apply Lemma 2.3.1 and conclude that

$$\overline{f(D)} \text{ is compact in } Y.$$

So $f \in K(D;Y)$. □

Compact maps have a useful extension property. First let us recall the so-called "Dugundji extension theorem," which is a generalization of the well known Tietze extension theorem (see Theorem A.1.14).

THEOREM 3.1.11 (Dugundji Extension Theorem)
If X is a metric space, Y is a locally convex space, A is a nonempty closed set and $f\colon A \longrightarrow Y$ is continuous,
then there exists a continuous map $\widehat{f}\colon X \longrightarrow Y$, such that

$$\widehat{f}|_A \ = \ f \quad \text{and} \quad \widehat{f}(X) \subseteq \operatorname{conv} f(A).$$

In the case of compact maps the conclusion of the Theorem 3.1.11 can be strengthened in the following way.

THEOREM 3.1.12
If X, Y are two Banach spaces, $D \subseteq X$ is nonempty, closed and bounded and $f \in K(D;Y)$,
then there exists $\widehat{f} \in K(X;Y)$, such that

$$\widehat{f}|_D \ = \ f \quad \text{and} \quad \widehat{f}(X) \ \subseteq \operatorname{conv} f(D).$$

PROOF By virtue of Theorem 3.1.11, we can find a continuous map $\widehat{f}\colon X \longrightarrow Y$, such that

$$\widehat{f}|_D \ = \ f \quad \text{and} \quad \widehat{f}(X) \ \subseteq \ \operatorname{conv} f(D).$$

But because $f \in K(D;Y)$, we have that $f(D)$ is relatively compact in Y and so by Mazur's theorem (see Theorem A.3.12), $\operatorname{conv} f(D)$ is relatively compact too. This shows that

$$\widehat{f} \ \in \ K(X;Y).$$

□

It is well known that a continuous map between Hausdorff topological spaces maps compact sets to compact sets. Next we look at continuous maps which return compact sets to compact sets, i.e.,

$$f \in C(X;Y)$$

and for every compact set $C \subseteq Y$, we have that

$$f^{-1}(C) \text{ is compact in } X.$$

Of course not every continuous map has this property (consider for example a constant map). The importance of this class of continuous maps resides in the fact that this property restricts the size of the solution set

$$S(y) \stackrel{df}{=} \{x \in X : \ f(x) = y\}.$$

DEFINITION 3.1.13 *Let X, Y be Hausdorff topological spaces and $f \in C(X;Y)$. We say that f is **proper**, if for every compact set $C \subseteq Y$,*

$$\text{the set } f^{-1}(C) \text{ is compact in } X.$$

REMARK 3.1.14 Evidently if X, Y are Banach spaces and

$$L \in \mathcal{L}(X;Y)$$

is proper, then L is bijective and has closed range. Also if

$$f \in C(X;Y)$$

is proper, then f is closed map (i.e., maps closed sets to closed sets).

Properness is related to coercivity.

PROPOSITION 3.1.15
If X, Y are two finite dimensional Banach spaces and $f \in C(X;Y)$, then f is proper if and only if f is weakly coercive, i.e.,

$$\|f(x)\|_Y \longrightarrow +\infty \quad \text{as } \|x\|_X \to +\infty.$$

PROOF "$\Longrightarrow$": Since X, Y are finite dimensional, the properness of f implies that the inverse image of a bounded set in Y is a bounded set in X. But this is actually a restatement of coercivity.

"$\Longleftarrow$": If f is coercive and $C \subseteq Y$ is compact, then

$$f^{-1}(C) \text{ is bounded}$$

(due to the coercivity) and so

$$f^{-1}(C) \text{ is compact in } X$$

(since X is finite dimensional). ∎

In the case of infinite dimensional Banach spaces, we have the following result relating properness and coercivity.

PROPOSITION 3.1.16
If X, Y are infinite dimensional Banach spaces, $f \in C(X;Y)$, f is weakly coercive and either
(i) $f = g + u$ with $g \in C(X;Y)$ being proper and $u \in C(X;Y)$ being compact; or
(ii) X is reflexive and

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

with the sequence $\{f(x_n)\}_{n\geqslant 1}$ being strongly convergent in Y, imply

$$x_n \longrightarrow x \quad \text{in } X,$$

then f is proper.

PROOF First suppose that hypothesis (i) holds. Let $C \subseteq Y$ be a compact set. We need to show that $f^{-1}(C)$ is compact in X. Let $\{x_n\}_{n\geqslant 1} \subseteq f^{-1}(C)$. Then

$$f(x_n) = y_n \in C \qquad \forall\, n \geqslant 1.$$

Because $C \subseteq Y$ is compact, by passing to a suitable subsequence if necessary, we may assume that

$$y_n \longrightarrow y \in C \quad \text{in } Y.$$

The weak coercivity of f implies that the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is bounded. So the sequence $\{u(x_n)\}_{n\geqslant 1} \subseteq Y$ is relatively compact and we may assume that

$$u(x_n) \longrightarrow z \quad \text{in } Y.$$

Then

$$g(x_n) = f(x_n) - u(x_n) \longrightarrow y - z \quad \text{in } Y.$$

Because g is proper, it follows that the sequence $\{x_n\}_{n\geqslant 1}$ has a subsequence $\{x_{n_k}\}_{k\geqslant 1}$, such that

$$x_{n_k} \longrightarrow x \quad \text{in } X.$$

Therefore

$$f(x_{n_k}) \longrightarrow f(x)$$

and so

$$y = f(x),$$

i.e.,

$$x \in f^{-1}(C),$$

which proves the properness of f.

Next suppose that hypothesis (ii) holds. Again

$$f(x_n) \;=\; y_n \;\longrightarrow\; y \quad \text{in } Y$$

and due to the weak coercivity of f, the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is bounded. Because X is reflexive, we may assume that

$$x_n \;\xrightarrow{w}\; x \quad \text{in } X.$$

Then hypothesis (ii) implies that

$$x_n \;\longrightarrow x \quad \text{in } X$$

and so

$$y_n \;=\; f(x_n) \;\longrightarrow\; f(x) \quad \text{in } Y,$$

hence

$$y \;=\; f(x).$$

This proves the properness of f. □

Compactness and properness are related as follows.

PROPOSITION 3.1.17
If X is a Banach space, $D \subseteq X$ is a closed, bounded set and $f \in K(D;X)$, then $id_X - f$ is proper (id_X is the identity operator on X).

PROOF Let $C \subseteq X$ be a compact set and let $\{x_n\}_{n\geqslant 1} \subseteq (id_X - f)^{-1}(C)$. Then

$$x_n - f(x_n) \;=\; c_n, \quad c_n \in C \qquad \forall\; n \geqslant 1.$$

Since C is compact and $f \in K(D;X)$, by passing to a suitable subsequence if necessary, we may assume that

$$c_n \;\longrightarrow\; c \in C \quad \text{and} \quad f(x_n) \;\longrightarrow\; y \quad \text{in } X.$$

Then

$$x_n \;=\; c_n + f(x_n) \;\longrightarrow\; c + y \;=\; x \quad \text{in } X$$

and so

$$f(x_n) \;\longrightarrow\; f(x) \quad \text{in } X.$$

Thus $y = f(x)$ and we have

$$c \;=\; x - f(x),$$

hence

$$x \;\in\; (id_X - f)^{-1}(C),$$

which shows that

$$(id_X - f)^{-1}(C) \text{ is compact.}$$

□

Next we have a closer look at the space of compact linear operators $\mathcal{L}_c(X;Y)$.

PROPOSITION 3.1.18
If X, Y are two Banach spaces,
then $\mathcal{L}_c(X;Y)$ with the operator norm is a Banach space.

PROOF Clearly $\mathcal{L}_c(X;Y)$ is a linear subspace of $\mathcal{L}(X;Y)$ (see also Remark 3.1.2). Because $\mathcal{L}(X;Y)$ with the operator norm is a Banach space, it suffices to show that $\mathcal{L}_c(X;Y)$ is closed in $\mathcal{L}(X;Y)$. So let $\{L_n\}_{n\geqslant 1} \subseteq \mathcal{L}_c(X;Y)$ and suppose that

$$\|L_n - L\|_{\mathcal{L}} \longrightarrow 0. \tag{3.1}$$

We need to show that $L \in \mathcal{L}_c(X;Y)$. Because of (3.1), we have that

$$\sup_{\|x\|_X \leqslant 1} \big\|L_n(x) - L(x)\big\|_Y \longrightarrow 0.$$

So given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geqslant 1$, such that

$$\big\|L_n(x) - L(x)\big\|_Y < \frac{\varepsilon}{2} \qquad \forall\, n \geqslant n_0,\ \|x\|_X \leqslant 1.$$

If

$$B_1^X \stackrel{df}{=} \big\{x \in X :\ \|x\|_X < 1\big\},$$

then the set $L_{n_0}\big(B_1^X\big)$ is relatively compact. So we can find a finite set $F \subseteq Y$, such that

$$L_{n_0}\big(B_1^X\big) \subseteq \bigcup_{y\in F} B_{\frac{\varepsilon}{2}}(y).$$

We claim that

$$L\big(B_1^X\big) \subseteq \bigcup_{y\in F} B_{\varepsilon}(y).$$

Indeed for a given $x \in B_1^X$, we can find $y \in F$, such that

$$\big\|L_{n_0}(x) - y\big\|_Y < \frac{\varepsilon}{2}.$$

Then

$$\big\|L(x) - y\big\|_Y \leqslant \big\|L(x) - L_{n_0}(x)\big\|_Y + \big\|L_{n_0}(x) - y\big\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $L\big(B_1^X\big)$ is totally bounded, thus relatively compact. ∎

REMARK 3.1.19 Since composition of two operators, one of which is compact, is again a compact operator, we infer that $\mathcal{L}_c(X)$ is a closed two-sided ideal of the Banach algebra $\mathcal{L}(X)$. Moreover, it is clear that if $L \in \mathcal{L}_c(X)$ and $\dim X = +\infty$, then L^{-1} does not exist (i.e., L is a singular operator).

The next characterization of the elements in the Banach space $\mathcal{L}_c(X;Y)$ is known as "Schauder's theorem." First a definition.

DEFINITION 3.1.20 *If X, Y are two Banach spaces and $L \in \mathcal{L}(X;Y)$, then its **adjoint** $L^*\colon Y^* \longrightarrow X^*$ is the linear operator given by*

$$L^*(y^*) \stackrel{df}{=} y^*L \qquad \forall\, y^* \in Y^*,$$

i.e.,

$$\big\langle L^*(y^*), x\big\rangle_X \;=\; \big\langle y^*, L(x)\big\rangle_Y \qquad \forall\, x \in X,\; y^* \in Y^*,$$

where by $\langle\cdot,\cdot\rangle_Z$ we denote the duality brackets for the pair (Z, Z^*) for any Banach space Z.

REMARK 3.1.21 Clearly

$$L^* \;\in\; \mathcal{L}(Y^*;X^*) \quad \text{and} \quad \|L\|_{\mathcal{L}} \;=\; \|L^*\|_{\mathcal{L}}\,.$$

So the map $L \longrightarrow L^*$ is an isometric isomorphism from $\mathcal{L}(X;Y)$ into $\mathcal{L}(Y^*;X^*)$. Moreover,

$$(L^{-1})^* \;=\; (L^*)^{-1}$$

and $L^*(Y^*)$ is closed if and only if it is w^*-closed.

THEOREM 3.1.22 (Schauder Theorem)
If X, Y are two Banach spaces and $L \in \mathcal{L}(X;Y)$, then $L \in \mathcal{L}_c(X;Y)$ if and only if $L^* \in \mathcal{L}_c(Y^*;X^*)$.

PROOF "$\Longrightarrow$": We need to show that $L^*\big(B_1^{Y^*}\big)$ is relatively compact, where

$$B_1^{Y^*} \stackrel{df}{=} \big\{y^* \in Y^* :\; \|y^*\|_{Y^*} < 1\big\}.$$

Let $\{y_n^*\}_{n\geqslant 1} \subseteq B_1^{Y^*}$ be a sequence. Consider the elements y_n^* for $n \geqslant 1$, restricted on the set $\overline{L\big(B_1^X\big)}$, which is compact in Y. Clearly the sequence $\{y_n^*\}_{n\geqslant 1} \subseteq C\big(\overline{L\big(B_1^X\big)}\big)$ is bounded and equicontinuous (see Definition A.1.15). So by the Arzela-Ascoli theorem (see Theorem 2.3.2), the sequence $\{y_n^*\}_{n\geqslant 1} \subseteq C\big(\overline{L\big(B_1^X\big)}\big)$ is relatively compact. Hence we can find a subsequence $\big\{y_{n_k}^*\big\}_{k\geqslant 1}$ of $\{y_n^*\}_{n\geqslant 1}$, such that

$$\sup_{x\in B_1^X} \big|\big\langle y_{n_k}^*, L(x)\big\rangle_Y - \big\langle y_{n_m}^*, L(x)\big\rangle_Y\big| \;\longrightarrow\; 0 \quad \text{as } k,m \to +\infty.$$

So

$$\begin{aligned}
&\lim_{k,m\to+\infty} \big\|L^*(y_{n_k}^*) - L^*(y_{n_m}^*)\big\|_{X^*} \\
=\;& \lim_{k,m\to+\infty} \sup_{x\in B_1^X} \big|\big\langle L^*(y_{n_k}^*) - L^*(y_{n_m}^*), x\big\rangle_X\big| \\
=\;& \lim_{k,m\to+\infty} \sup_{x\in B_1^X} \big|\big\langle y_{n_k}^* - y_{n_m}^*, L(x)\big\rangle_Y\big| \;=\; 0.
\end{aligned}$$

Therefore $\{L^*(y^*_{n_k})\}_{k\geqslant 1} \subseteq X^*$ is a Cauchy sequence and so it is convergent. This implies that the set $\overline{L^*\big(B_1^{Y^*}\big)}$ is compact in X^* and so $L^* \in \mathcal{L}_c(Y^*;X^*)$.

"$\Longleftarrow$": Let $r\colon X \longrightarrow X^*$ be the canonical embedding of X into X^{**}. We have

$$L^{**}r \;=\; rL.$$

So identifying X with $r(X)$, we have that

$$L^{**}|_X \;=\; L.$$

From the first part of the proof and since by hypothesis $L^* \in \mathcal{L}_c(Y^*;X^*)$, we have that

$$\overline{L^{**}\big(B_1^{X^{**}}\big)} \;\subseteq\; Y^{**} \quad \text{is compact,}$$

where

$$B_1^{X^{**}} \;\stackrel{df}{=}\; \big\{x^{**} \in X^{**} :\; \|x^{**}\|_{X^{**}} < 1\big\}.$$

Since $\overline{L^{**}\big(B_1^X\big)}$ is a closed subset of $\overline{L^{**}\big(B_1^{X^{**}}\big)}$, it follows that

$$\overline{L^{**}\big(B_1^X\big)} \;\subseteq\; X^{**} \quad \text{is compact.}$$

But as we already established $L^{**}\big(B_1^X\big) = L\big(B_1^X\big)$. So $\overline{L\big(B_1^X\big)} \subseteq X$ is compact and we conclude that $L \in \mathcal{L}_c(X;Y)$. □

DEFINITION 3.1.23 *Let X, Y be two Banach spaces and $L \in \mathcal{L}(X;Y)$. We say that L is a **finite rank operator** (or **finite dimensional operator** or **degenerate operator**), if $\dim L(X) < +\infty$. We denote the space of all finite dimensional operators from X into Y equipped with the norm inherited from $\mathcal{L}(X;Y)$, by $\mathcal{L}_f(X;Y)$. If $L \in \mathcal{L}_f(X;Y)$, then $\operatorname{rank} L \stackrel{df}{=} \dim L(X)$.*

REMARK 3.1.24 Clearly $\mathcal{L}_f(X;Y) \subseteq \mathcal{L}_c(X;Y)$. The inclusion is in general strict as the next example illustrates.

EXAMPLE 3.1.25 Consider the operator $L \in \mathcal{L}(l^2)$, defined by

$$L(x) \;\stackrel{df}{=}\; \Big\{\frac{x_n}{2^n}\Big\}_{n\geqslant 1} \qquad \forall\; x = \{x_n\}_{n\geqslant 1} \in l^2.$$

We claim that $L \in \mathcal{L}_c(l^2) \setminus \mathcal{L}_f(l^2)$. Clearly $L \notin \mathcal{L}_f(l^2)$. So let us show that $L \in \mathcal{L}_c(l^2)$. We need to show that $L\big(B_1^{l^2}\big) \subseteq l^2$ is relatively compact. For a given $\varepsilon > 0$, find $n_0 = n_0(\varepsilon) \geqslant 1$, such that

$$\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \;\leqslant\; \varepsilon.$$

The set

$$C \stackrel{df}{=} \left\{ \left(\frac{x_1}{2}, \frac{x_2}{2^2}, \ldots, \frac{x_{n_0}}{2^{n_0}}, 0, \ldots \right) : \ |x_n| \leqslant 1 \right\}$$

is compact in l^2 (view it as a subset of $\mathbb{R}^{n_0}$). So we can find a finite set $F \subseteq C$, such that

$$C \subseteq \bigcup_{v \in F} B_\varepsilon(v).$$

Let $x \in \overline{B}_1^{l^2}$. We have

$$|x_n| \ \leqslant \ 1 \qquad \forall \ n \geqslant 1$$

and so there exists $v \in F$, such that

$$\sum_{n=1}^{n_0} \left| \frac{x_n}{2^n} - v_n \right|^2 \ < \ \varepsilon^2.$$

Then we have

$$\left\| \left\{ \frac{x_n}{2^n} \right\}_{n \geqslant 1} - v \right\|_{l^2} \ = \ \sum_{n=1}^{n_0} \left| \frac{x_n}{2^n} - v_n \right|^2 + \sum_{n=n_0+1}^{\infty} \left| \frac{x_n}{2^n} \right|^2 \ < \ \varepsilon^2 + \varepsilon^2 \ = \ 2\varepsilon^2,$$

so $L\big(B_1^{l^2}\big)$ is relatively compact in l^2, hence $L \in \mathcal{L}_c(l^2)$.

Making use of a finite basis to describe the finite dimensional range of $L \in \mathcal{L}_f(X;Y)$, we can easily establish the following result.

PROPOSITION 3.1.26
If X, Y are two Banach spaces and $L \in \mathcal{L}(X;Y)$, then $L \in \mathcal{L}_f(X;Y)$ if and only if $L^* \in \mathcal{L}_f(Y^*;X^*)$. Moreover, $\operatorname{rank} L = \operatorname{rank} L^*$.

From Theorem 3.1.10 we know that every compact map can be uniformly approximated locally by maps with range in a finite dimensional space. Motivated by this fact, it is natural to ask whether

$$\mathcal{L}_c(X;Y) \ = \ \overline{\mathcal{L}_f(X;Y)}^{\|\cdot\|_{\mathcal{L}}}.$$

In fact for a long time this was one of the major open problems in Banach space theory. But let us formulate the problem precisely. We start with a definition.

DEFINITION 3.1.27 *A Banach space Y has the **approximation property**, if for every Banach space X, we have that*

$$\mathcal{L}_c(X;Y) \ = \ \overline{\mathcal{L}_f(X;Y)}^{\|\cdot\|_{\mathcal{L}}}.$$

The famous open (until 1973) problem in Banach space theory is known as the "approximation problem" and asks whether every Banach space Y has the approximation property. It was settled in the negative by Enflo (1973), who found a separable, reflexive Banach space (necessarily infinite dimensional), which lacks the approximation property.

Let us also mention a few things about the spectrum of compact operators. The spaces in much of applied mathematics are actually real vector spaces, as was the case in our study so far. However, to work all the time in real spaces is mathematically inconvenient. Eigenvalue-eigenvector theory is such an instance. The theory is crippled if we insist on real vector spaces. For this reason in the next definition we consider a complex Banach space.

DEFINITION 3.1.28 *Let X be a complex Banach space and $L \in \mathcal{L}(X)$. The* ***resolvent set*** *$\varrho(L)$* ***of*** *L is defined by*

$$\varrho(L) \stackrel{df}{=} \big\{\lambda \in \mathbb{C} : \ (\lambda id_{_X} - L)^{-1} \text{ exists and belongs in } \mathcal{L}(X)\big\}.$$

The operator

$$R(\lambda) \stackrel{df}{=} (\lambda id_{_X} - L)^{-1}$$

is called the resolvent of L at λ. The points of $\varrho(L)$ are called ***regular values of*** *L. The set*

$$\sigma(L) \stackrel{df}{=} \mathbb{C} \setminus \varrho(L)$$

is called the ***spectrum of*** *L. The* ***point spectrum of*** *L is the subset $\sigma_p(L)$ of $\sigma(L)$ defined by*

$$\sigma(L) \stackrel{df}{=} \big\{\lambda \in \sigma(L) : \ \ker(\lambda id_{_X} - L) \neq \emptyset\big\}.$$

The elements of $\sigma_p(L)$ are called ***eigenvalues of*** *L and for each $\lambda \in \sigma_p(L)$ the closed subspace $\ker(\lambda id_{_X} - L)$ of X is the* ***eigenspace*** *corresponding to the eigenvalue λ, while the nonzero elements of $\ker(\lambda id_{_X} - L)$ are called* ***eigenvectors*** *of L.*

REMARK 3.1.29 If $\dim X = +\infty$ and $L \in \mathcal{L}_c(X)$, then $0 \in \sigma(L)$. Indeed, if L has a bounded inverse, then we could define an equivalent norm

$$\|\!|x|\!\|_{_X} \stackrel{df}{=} \big\|L(x)\big\|_{_X} \quad \text{on } X,$$

whose closed unit ball is $\overline{L\big(B_1^X\big)}$. But the latter set is compact (since $L \in \mathcal{L}_c(X)$) and so $\dim X < +\infty$, a contradiction. Also, if $\dim X < +\infty$, then from linear algebra we know that the operator $\lambda id_{_X} - L$ is invertible (and automatically $(\lambda id_{_X} - L)^{-1}$ is continuous) if and only if $\lambda id_{_X} - L$ is bijective. So in this case $\sigma(L) = \sigma_p(L)$. In general we can have that $\sigma_p(L) = \emptyset$ and $\sigma(L) \neq \emptyset$.

EXAMPLE 3.1.30 Consider the Hilbert space $L^2\big([0,1]\big)$ (over the complex scalars). Let $L \in \mathcal{L}\big(L^2\big([0,1]\big)\big)$ be defined by

$$L(x)(t) \stackrel{df}{=} tx(t) \qquad \forall\, t \in [0,1].$$

We claim that $\sigma_p(L) = \emptyset$. Indeed, if for some $\lambda \in \mathbb{C}$ we have

$$\lambda x(t) \;=\; tx(t) \qquad \forall\, t \in [0,1],$$

then

$$(\lambda - t)x(t) \;=\; 0$$

and so

$$x(t) \;=\; 0 \quad \text{for a.a. } t \in [0,1].$$

On the other hand

$$[0,1] \;\subseteq\; \sigma(L).$$

To this end let $\lambda \in [0,1]$ and take $\varepsilon > 0$, such that

$$[\lambda, \lambda + \varepsilon] \subseteq [0,1] \quad \text{or} \quad [\lambda - \varepsilon, \lambda] \subseteq [0,1].$$

To fix things, we assume that the first is true. We define

$$x_\varepsilon(t) \stackrel{df}{=} \begin{cases} \frac{1}{\sqrt{\varepsilon}} & \text{if} \quad t \in [\lambda, \lambda + \varepsilon]. \\ 0 & \text{if} \quad t \notin [\lambda, \lambda + \varepsilon]. \end{cases}$$

Then we have

$$\int\limits_0^1 x_\varepsilon(t)^2\, dt \;=\; \int\limits_\lambda^{\lambda+\varepsilon} \frac{1}{\varepsilon}\, dt \;=\; 1$$

and so we have

$$\|x_\varepsilon\|_2 \;=\; 1.$$

Also

$$\big(\lambda id_{_X} - L\big)(x_\varepsilon)(t) \;=\; (\lambda - t)x_\varepsilon(t) \qquad \forall\, t \in [0,1].$$

Therefore

$$\big\|(\lambda id_{_X} - L)(x_\varepsilon)\big\|_2^2 \;=\; \int\limits_\lambda^{\lambda+\varepsilon} \frac{1}{\varepsilon}(\lambda - t)^2\, dt \;=\; \frac{\varepsilon^2}{3}$$

and so

$$\big(\lambda id_{_X} - L\big)(x_\varepsilon) \;\longrightarrow\; 0 \quad \text{in } L^2\big([0,1]\big) \quad \text{as } \varepsilon \searrow 0.$$

If $\lambda id_{_X} - L$ has a bounded inverse, then

$$x_\varepsilon \;=\; \big(\lambda id_{_X} - L\big)^{-1}(\lambda id_{_X} - L)(x_\varepsilon) \;\longrightarrow\; 0 \quad \text{as } \varepsilon \searrow 0,$$

a contradiction to the fact that $\|x_\varepsilon\|_2 = 1$ for all $\varepsilon > 0$.

Using the theory of analytic functions one can show the following proposition.

PROPOSITION 3.1.31
If X is a complex Banach space and $L \in \mathcal{L}(X)$,
then $\sigma(L) \neq \emptyset$.

REMARK 3.1.32 The result is no longer valid if we consider a real Banach space.

PROPOSITION 3.1.33
If X is a complex Banach space and $\lambda \neq 0$ is an eigenvalue of $L \in \mathcal{L}_c(X)$,
then $\dim(\lambda id_X - L)^{-1}(0) < +\infty$ (i.e., the eigenspace corresponding to λ is finite dimensional).

PROOF Set

$$N_\lambda \stackrel{df}{=} (\lambda id_X - L)^{-1}(0)$$

and let B be a bounded subset of N_λ. For each $x \in B$, we have $L(x) = \lambda x$. Since $L \in \mathcal{L}_c(X)$, we have that $\overline{L(B)} \subseteq X$ is compact. Hence $\lambda \overline{B} \subseteq X$ is compact. Since all bounded sets in N_λ are relatively compact, it follows that $\dim N_\lambda < +\infty$. □

PROPOSITION 3.1.34
If X is a complex Banach space, $L \in \mathcal{L}_c(X)$ and $\varepsilon > 0$,
then L has only finite many linear independent eigenvectors corresponding to eigenvalues having absolute value larger than ε.

PROOF Let $\{x_n\}_{n \geqslant 1}$ be a sequence of distinct eigenvectors corresponding to eigenvalues λ_k satisfying $|\lambda_k| > \varepsilon$. Set

$$X_n \stackrel{df}{=} \text{span}\,\{x_k\}_{k=1}^n \qquad \forall\, n \geqslant 1.$$

Note that

$$L(X_n) \;=\; X_n$$

and use Riesz lemma (see Proposition A.3.15), to obtain

$$y_n \in X_n \quad \text{with } \|y_n\| = 1,$$

such that

$$d\big(y_n, X_{n-1}\big) \;\geqslant\; \frac{1}{2}.$$

Let

$$u_n \stackrel{df}{=} \frac{y_n}{\lambda_n}.$$

Then

$$\|u_n\|_X \;<\; \frac{1}{\varepsilon} \quad \text{and} \quad L(u_n) \in X_n.$$

Also if $y_n = \sum_{k=1}^{n} a_k x_k$, we have

$$L(u_n) - y_n \;=\; \sum_{k=1}^{n} \left(\frac{\lambda_k}{\lambda_n} - 1\right) a_k x_k \;=\; \sum_{k=1}^{n-1} \left(\frac{\lambda_k}{\lambda_n} - 1\right) a_k x_k \;\in\; X_{n-1}.$$

If $n > m$, then

$$L(u_m) \in X_m \subseteq X_{n-1} \quad \text{and} \quad L(u_n) - y_n \in X_{n-1}.$$

So we have

$$\begin{aligned} &\big\|L(u_n) - L(u_m)\big\| \;\geqslant\; d\big(L(u_n), X_{n-1}\big) \\ &= d\big(L(u_n) + y_n - L(u_n), X_{n-1}\big) \;=\; d\big(y_n, X_{n-1}\big) \;\geqslant\; \frac{1}{2}, \end{aligned}$$

so the sequence $\big\{L(u_n)\big\}_{n\geqslant 1}$ has no convergent subsequence, a contradiction to the fact that $L \in \mathcal{L}_c(X)$. □

PROPOSITION 3.1.35
If X is a Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then $R(\lambda id_X - L)$ is closed.

PROOF Without any loss of generality, we may assume that $\lambda = 1$. Set

$$V \;\stackrel{df}{=}\; id_X - L \quad \text{and} \quad N_1 \;\stackrel{df}{=}\; V^{-1}\big(\{0\}\big).$$

If $\lambda \notin \sigma_p(L)$, then

$$N_1 \;=\; \{0\}.$$

If $\lambda \in \sigma_p(L)$, then

$$\dim N_1 \;<\; +\infty$$

(see Proposition 3.1.33). So in both cases we see that $\dim N_1 < +\infty$. Thus we can write that

$$X \;=\; N_1 \oplus Em$$

with E being a closed subspace of X. Let

$$\widehat{V} \;\stackrel{df}{=}\; V|_E.$$

We have

$$V(X) \;=\; V(E) \;=\; \widehat{V}(E) \quad \text{and} \quad \widehat{V}^{-1}\big(\{0\}\big) \;=\; V^{-1}\big(\{0\}\big) \cap E \;=\; \{0\}.$$

This shows that $\widehat{V}$ is bijective form E into X. We claim that

$$\inf_{\substack{x \in E \\ \|x\|_X = 1}} \left\|\widehat{V}(x)\right\|_X > 0.$$

Suppose that the claim is not true. Then we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq E$ with $\|x_n\|_X = 1$, such that

$$\left\|\widehat{V}(x_n)\right\|_X \searrow 0.$$

Since $L \in \mathcal{L}_c(X)$, by passing to a suitable subsequence if necessary, we may assume that

$$L(x_n) \longrightarrow u \quad \text{in } X.$$

Then

$$x_n = \big(\widehat{V} + L\big)(x_n) \longrightarrow u \quad \text{in } X$$

and so $\|u\|_X = 1$. Moreover,

$$\widehat{V}(x_n) \longrightarrow \widehat{V}(u) \quad \text{in } X$$

and so

$$\widehat{V}(u) = 0,$$

a contradiction to the fact that $\widehat{V}\colon E \longrightarrow X$ is bijective. So the claim is true and therefore there exists $c > 0$, such that

$$\left\|\widehat{V}(x)\right\|_X \geqslant c\|x\|_X \qquad \forall\, x \in E.$$

This implies that $\widehat{V}(E)$ is closed. Indeed, let

$$u_n \in \widehat{V}(E) \qquad \forall\, n \geqslant 1$$

and assume that

$$u_n \longrightarrow u \quad \text{in } X.$$

Then

$$u_n = \widehat{V}(x_n) \quad \text{with } x_n \in E \qquad \forall\, n \geqslant 1.$$

We have

$$\begin{aligned} \|x_n - x_m\|_X &\leqslant \frac{1}{c}\left\|\widehat{V}(x_n - x_m)\right\|_X \\ &= \frac{1}{c}\|u_n - u_m\|_X \longrightarrow 0 \quad \text{as } n, m \to +\infty. \end{aligned}$$

So

$$x_n \longrightarrow x \quad \text{in } X$$

for some $x \in E$ and

$$\widehat{V}(x_n) \longrightarrow \widehat{V}(x) = u \in \widehat{V}(E) \quad \text{in } X.$$

Finally recall that $\widehat{V}(E) = V(E)$. ∎

To produce a characterization of the spectrum of a compact operator, we shall need that following straightforward auxiliary result.

LEMMA 3.1.36
If X is a Banach space, $L \in \mathcal{L}_c(X)$ and $E = (id_X - L)(X)$ is a proper subspace of X,
then for every $\varepsilon > 0$ we can find $x_\varepsilon \in \overline{B}_1^X$, such that

$$d_X\big(L(x_\varepsilon), L(E)\big) \;>\; 1 - \varepsilon.$$

PROOF By virtue of the Riesz lemma (see Proposition A.3.15), we can find $x_\varepsilon \in X$ with $\|x_\varepsilon\|_X = 1$, such that

$$d\big(x_\varepsilon, E\big) \;>\; 1 - \varepsilon.$$

Note that

$$(id_X - L)(x_\varepsilon) \quad \text{in } E$$

and $L(E) \subseteq E$. Therefore

$$d_X\big(L(x_\varepsilon), L(E)\big) \;\geqslant\; d_X\big(x_\varepsilon - (I - L)(x_\varepsilon), E\big) \;=\; d_X\big(x_\varepsilon, E\big) \;>\; 1 - \varepsilon.$$

□

Using this lemma we can have the following remarkable property of the spectrum of a compact operator.

THEOREM 3.1.37
If X is a complex Banach space, $L \in \mathcal{L}_c(X)$ and $\lambda \in \sigma(L) \setminus \{0\}$,
then $\lambda \in \sigma_p(L)$.

PROOF Without any loss of generality, we may assume that $\lambda = 1$. Let

$$V \;\stackrel{df}{=}\; id_X - L$$

and suppose that

$$V^{-1}\big(\{0\}\big) \;=\; \{0\}$$

(i.e., $\lambda = 1 \notin \sigma_p(L)$). We set

$$E_n \;\stackrel{df}{=}\; V^n(X)$$

and note that

$$E_n \;=\; V^n(X) \;=\; V^{n-1}\big(V(X)\big) \;\subseteq\; V^{n-1}(X) \;=\; E_{n-1} \qquad \forall\, n \geqslant 1.$$

From Proposition 3.1.35, we know that

$$E_n \text{ is closed} \qquad \forall\, n \geqslant 1.$$

Suppose that

$$E_{n+1} \;=\; E_n \qquad \forall\, n \geqslant 1.$$

Then according to Lemma 3.1.36, we can find $x_n \in \overline{B}_1^{E_n}$, such that

$$d_x\big(L(x_n), L(E_{n+1})\big) \;>\; \frac{1}{2} \qquad \forall\, n \geqslant 1,$$

so

$$\big\|L(x_n) - L(x_m)\big\|_X \;>\; \frac{1}{2} \qquad \forall\, n \neq m,$$

a contradiction to the compactness of L. So

$$E_{n+1} \;\neq\; E_n,$$

for some $n \geqslant 1$. We shall show that

$$X \;=\; E_0 \;=\; E_1.$$

Suppose that this is not true, i.e.,

$$E_0 \;\neq\; E_1.$$

Let $m \geqslant 1$ be the smallest positive integer, such that

$$E_{m-1} \;\neq\; E_m \;=\; E_{m+1}.$$

We choose

$$y \;\in\; E_{m-1} \setminus E_m.$$

Then

$$V(y) \;\in\; E_m \;=\; E_{m+1}.$$

Hence we can find $z \in E_m$, such that

$$V(y) \;=\; V(z) \quad \text{and} \quad y \;\neq\; z,$$

since $y \notin E_m = E_{m+1}$. Therefore

$$V(y - z) \;=\; 0$$

and so

$$y - z \;\in\; \ker V,$$

a contradiction to the hypothesis that $V^{-1}\big(\{0\}\big) = \{0\}$. So V is surjective and by Banach's theorem (see Theorem A.3.6), we have that V^{-1} exists and is bounded. Hence

$$\lambda \;=\; 1 \;\notin\; \sigma(L),$$

a contradiction. □

From Theorem 3.1.37, Proposition 3.1.34 and the well known fact from linear algebra, which says that eigenvectors corresponding to distinct eigenvalues are linear independent, we obtain the following characterization of the spectrum of a compact operator.

THEOREM 3.1.38
If X is an infinite dimensional complex Banach space and $L \in \mathcal{L}_c(X)$, then

(a) *$\sigma(L)$ is a countable compact set whose only possible limit point is 0;*

(b) *$\sigma(L) = \{0\} \cup \sigma_p(L)$;*

(c) *if $\lambda \in \sigma_p(L) \setminus \{0\}$, then the eigenspace of L corresponding to L is finite dimensional.*

REMARK 3.1.39 The above theorem does not say that $\sigma(L)$ is the disjoint union of $\{0\}$ and $\sigma_p(L)$. For example if $X = l^2$ and $L \in \mathcal{L}_c(l^2)$ is given by

$$L\big(\{x_n\}_{n\geqslant 1}\big) \stackrel{df}{=} \big(x_1, 0, 0, \dots\big),$$

then $\lambda = 0$ is an eigenvalue of L and the associated eigenspace is infinite dimensional (it has codimension equal to 1). On the other hand the operator $L \in \mathcal{L}_c(l^2)$ in Example 3.1.25 is bijective and so does not have 0 as an eigenvalue.

As we mentioned in the beginning of this section, compact operators generalize to infinite dimensions the properties of operators between finite dimensional spaces. One such property is that if $\dim X < +\infty$ and $L \in \mathcal{L}(X)$, then L is surjective if and only if L is injective. The result is no longer true if $\dim X = +\infty$. For example let $X = l^2$ and let

$$L\big(\{x_n\}_{n\geqslant 1}\big) \stackrel{df}{=} \big(0, x_1, x_2, \dots\big)$$

(the right shift operator). However, if $L \in \mathcal{L}_c(X)$, then as we show in the sequel, the result is true for $id_{_X} - L$. We start with a definition.

DEFINITION 3.1.40 *Let X be a Banach space, $A \subseteq X$ and $C \subseteq X^*$. We introduce the set $A^\perp \subseteq X^*$ (pronounced "A perp") and ${}^\perp C \subseteq X$ (pronounced "perp C"), defined by*

$$A^\perp \stackrel{df}{=} \Big\{x^* \in X^* :\ \langle x^*, a\rangle_{_X} = 0 \text{ for all } a \in A\Big\},$$

$${}^\perp C \stackrel{df}{=} \Big\{x \in X :\ \langle c^*, x\rangle_{_X} = 0 \text{ for all } c^* \in C\Big\}.$$

REMARK 3.1.41 The sets $A^{\perp}$ and ${}^{\perp}C$ are closed subsets of X^* and X respectively and

$$ {}^{\perp}\big(A^{\perp}\big) \;=\; \overline{\operatorname{span} A} \quad \text{and} \quad \big({}^{\perp}C\big)^{\perp} \;=\; \overline{\operatorname{span}}^{w^*} C. $$

Also if E is a closed subspace of X, then

$$ \big(X/_E\big)^* \;=\; E^{\perp} \quad \text{and} \quad X^*/_{E^{\perp}} \;=\; E^* $$

(see, e.g., Beauzamy (1982, pp. 41 and 43)).

LEMMA 3.1.42
If X, Y are two Banach spaces and $L \in \mathcal{L}(X;Y)$, then

$$ \ker L \;=\; {}^{\perp}\big(L^*(Y^*)\big) \quad \textit{and} \quad \ker L^* \;=\; L(X)^{\perp}. $$

PROOF Recall that Y^* is a separating family of functions on Y. So $x \in \ker L$ if and only if

$$ \big\langle L^*(y^*), x\big\rangle_X \;=\; \big\langle y^*, L(x)\big\rangle_Y \;=\; 0 \qquad \forall\, y^* \in Y^*, $$

hence $x \in {}^{\perp}\big(L^*(Y)\big)$.

In a similar fashion, we have that $y^* \in \ker L^*$ if and only if

$$ \big\langle y^*, L(x)\big\rangle_Y \;=\; \big\langle L^*(y^*), x\big\rangle_X \;=\; 0 \qquad \forall\, x \in X, $$

so $y^* \in L(X)^{\perp}$. ∎

LEMMA 3.1.43
If X is a Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then

$$ R(\lambda id_X - L) = X \textit{ implies that } \ker(\lambda id_X - L) = \{0\}, $$

i.e.,

$$ \textit{if } \lambda id_X - L \textit{ is surjective, then it is injective.} $$

PROOF Without any loss of generality we may assume that $\lambda = 1$. Recall that L commutes with $(id_X - L)^n$ (consider the polynomial expansion of $(id_X - L)^n$). So

$$ L\big(\ker(id_X - L)^n\big) \;\subseteq\; \ker(id_X - L)^n \qquad \forall\, n \geqslant 1. $$

Suppose that although $R(id_X - L)(X) = X$, the operator $id_X - L$ is not injective. Note that

$$ (id_X - L)^n(X) = X \qquad \forall\, n \geqslant 1 $$

and so $(id_X - L)^{n+1}$ maps some elements of X to 0 that $(id_X - L)^n$ does not. Hence

$$\ker (id_X - L)^n \subsetneq \ker (id_X - L)^{n+1}.$$

Using Riesz lemma (see Proposition A.3.15), we can find

$$x_n \in \ker (id_X - L)^{n+1} \text{ with } \|x_n\|_X = 1,$$

such that

$$\|x_n - y\|_X \geqslant \frac{1}{2} \qquad \forall\, n \geqslant 1,\ y \in \ker (id_X - L)^n.$$

If $n > m$, we have

$$(id_X - L)(x_n) + L(x_m) \in \ker (id_X - L)^n$$

and so

$$\big\|L(x_n) - L(x_m)\big\|_X = \big\|x_n - \big((id_X - L)(x_n) + L(x_m)\big)\big\|_X \geqslant \frac{1}{2},$$

a contradiction to the fact that $L \in \mathcal{L}_c(X)$. □

PROPOSITION 3.1.44
If X is Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then

$$\dim \ker (\lambda id_X - L) = \operatorname{codim} R(\lambda id_X - L)$$

(recall that $\operatorname{codim} R(\lambda id_X - L) = \dim \Big(X/_{R(\lambda id_X - L)}\Big)$*).*

PROOF Without any loss of generality, we may assume that $\lambda = 1$. From Remark 3.1.41 and Lemma 3.1.42, we have that

$$\Big(X/_{R(\lambda id_X - L)}\Big)^* = R(id_X - L)^\perp = \ker \big(id_X^* - L^*\big). \tag{3.2}$$

From Theorem 3.1.22, we know that $L^* \in \mathcal{L}_c(X^*)$ and so Proposition 3.1.33 implies that

$$\dim \ker \big(id_X^* - L^*\big) < +\infty.$$

A finite dimensional Banach space has the same dimension as its dual. So from (3.2), we have that

$$\operatorname{codim} R(id_X - L) = dim \ker (id_X^* - L^*). \tag{3.3}$$

Because $L^* \in \mathcal{L}(X^*)$, from Proposition 3.1.35, we have that $R(id_X^* - L^*)$ is closed, hence w^*-closed too (see Remark 3.1.21). So from Remark 3.1.41 and Lemma 3.1.42, we have

$$\ker (id_X - L)^\perp = \big({}^\perp R(id_X^* - L^*)\big)^\perp = R\big(id_X^* - L^*\big) = R\big(id_X^* - L^*\big),$$

so

$$X^*/_{R(id_X^* - L^*)} \;=\; X^*/_{\ker\,(id_X - L)^\perp} \;=\; \big[\ker\,(id_X - L)\big]^*.$$

Using as before the fact that a finite dimensional Banach space has the same dimension as its dual, we obtain

$$\operatorname{codim} R(id_X^* - L^*) \;=\; \dim\ker\,(id_X - L). \tag{3.4}$$

Suppose that

$$\dim\ker\,(id_X - L) \;>\; \operatorname{codim} R(id_X - L).$$

Then we can find a closed subspace E of X, such that

$$X \;=\; R(id_X - L) \oplus E.$$

Let P_E be the projection operator onto E. Then

$$\ker \mathrm{P}_E \;=\; R(id_X - L).$$

We have that

$$X/_{\ker \mathrm{P}_E} \;=\; X/_{R(id_X - L)} \;=\; E$$

and so

$$\operatorname{codim} R(id_X - L) \;=\; \dim E.$$

Therefore there is a bounded linear operator T which is not injective and maps $\ker\,(id_X - L)$ onto E. Then

$$T \;\in\; \mathcal{L}_c\big(\ker\,(id_X - L); X\big).$$

Let F be a closed subspace of X, such that

$$X \;=\; \ker\,(id_X - L) \oplus F$$

and P_0 the projection operator onto $\ker\,(id_X - L)$ and with kernel F. Set

$$G \;\stackrel{df}{=}\; L + T\mathrm{P}_0.$$

Evidently

$$G \;\in\; \mathcal{L}_c(X)$$

and we have

$$\begin{aligned}(id_X - G)(X) \;&=\; \big((id_X - L) - T\mathrm{P}_0\big)\big(\ker\,(id_X - L)\big)\\ &\quad+\; \big((id_X - L) - T\mathrm{P}_0\big)(F)\\ &=\; E + (id_X - L)(F) \;=\; E + (id_X - L)(X)\\ &=\; X,\end{aligned}$$

so, from Lemma 3.1.43, we have that

$$id_X - G \text{ is injective.}$$

But there is a nonzero

$$u \in \ker T \subseteq \ker (id_X - L),$$

such that

$$(id_X - G)(u) = (id_X - L)(u) - T\mathrm{P}_0(u) = 0,$$

a contradiction. So, it follows that

$$\dim \ker (id_X - L) \leqslant \operatorname{codim} R(id_X - L). \tag{3.5}$$

Moreover, because $L^* \in \mathcal{L}_c(X^*)$, we also have

$$\dim \ker (id_X^* - L^*) \leqslant \operatorname{codim} R(id_X^* - L^*). \tag{3.6}$$

From (3.3), (3.4), (3.5) and (3.6), we conclude that

$$\dim \ker (id_X - L) = \operatorname{codim} R(id_X - L).$$

□

A byproduct of the above proof is the following result.

COROLLARY 3.1.45
If X is a Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then

$$\dim \ker (\lambda id_X - L) = \dim \ker (\lambda id_X^* - L^*).$$

Clearly Proposition 3.1.44 permits the improvement of Lemma 3.1.43. This is done in the next theorem which summarizes all the above properties of a compact operator.

THEOREM 3.1.46
If X is a Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then

(a) $\ker (\lambda id_X - L)$ *is finite dimensional;*

(b) $R(\lambda id_X - L)$ *is closed and* $R(\lambda id_X - L) = \ker (\lambda id_X^* - L^*)^{\perp}$;

(c) $\ker (\lambda id_X - L) = \{0\}$ *if and only if* $R(\lambda id_X - L) = X$;

(d) $\dim \ker (\lambda id_X - L) = \dim \ker (\lambda id_X^* - L^*)$.

REMARK 3.1.47 Statement **(c)** expresses the fact that $\lambda id_X - L$ is injective if and only if $\lambda id_X - L$ is surjective, a well known property of linear operators between finite dimensional spaces.

This leads us to the ***Fredholm alternative theorem***, an important tool in the study of integral equations and boundary value problems.

THEOREM 3.1.48 (Fredholm Alternative Theorem)
If X is Banach space, $L \in \mathcal{L}_c(X)$ and λ is a nonzero scalar, then one and only one of the following two alternatives holds:

(a) *for every $u \in X$, the equation*

$$(\lambda id_X - L)(x) \;=\; u$$

has a unique solution $x \in X$; or

(b) *the homogeneous equation*

$$(\lambda id_X - L)(x) \;=\; 0$$

has N linear independent solutions with $N \geqslant 1$; in this case the nonhomogeneous equation

$$(\lambda id_X - L)(x) \;=\; u$$

has a solution if and only if u verifies N conditions of orthogonality, i.e.,

$$u \;\in\; \ker(\lambda id_X^* - L^*)^{\perp}.$$

Next let us say a few words about the spectrum of a self-adjoint, compact operator on a Hilbert space.

DEFINITION 3.1.49 *Let H be a Hilbert space and $L \in \mathcal{L}(H)$. We say that L is **self-adjoint** (or **hermitian**) if and only if $L^* = L$, i.e.,*

$$\big(L(x), y\big)_H \;=\; \big(x, L(y)\big)_H \qquad \forall\; x, y \in H.$$

REMARK 3.1.50 If H is a complex Hilbert space and $L \in \mathcal{L}(H)$ is a self-adjoint operator, then

$$\big(L(x), x\big)_H \;=\; \big(x, L(x)\big)_H \;=\; \overline{\big(L(x), x\big)_H},$$

hence

$$\big(L(x), x\big)_H \;\in\; \mathbb{R}.$$

Also one can check that

$$\|L^n\|_{\mathcal{L}} \;=\; \|L\|_{\mathcal{L}}^n \qquad \forall\; n \geqslant 1$$

and

$$\|L\|_{\mathcal{L}} \;=\; \sup_{\|x\|_H \leqslant 1} \big|\big(L(x), x\big)_H\big|.$$

PROPOSITION 3.1.51
If H is a Hilbert space and $L \in \mathcal{L}(H)$ is a self-adjoint operator, then all eigenvalues of L are real and eigenvectors corresponding to different eigenvalues are orthogonal.

PROOF Let λ be a eigenvalue with an eigenvector x. We have

$$\big(L(x), x\big)_H \;=\; \big(\lambda x, x\big)_H,$$

so, from Remark 3.1.50, we have

$$\lambda \;=\; \frac{(L(x), x)_H}{\|x\|_H^2} \;\in\; \mathbb{R}.$$

Also if μ is another eigenvalue with an eigenvector y, we have

$$\big(L(x), y\big)_H \;=\; \lambda\,(x, y)_H \quad \text{and} \quad \big(L(y), x\big)_H \;=\; \mu\big(x, y\big)_H.$$

Since L is self-adjoint, it follows that

$$(\lambda - \mu)\,(x, y)_H \;=\; 0.$$

Because $\lambda \neq \mu$, we conclude that $(x, y)_H = 0$. □

PROPOSITION 3.1.52
If H is Hilbert space and $L \in \mathcal{L}(H)$ is a self-adjoint operator, then $\lambda \in \sigma(L)$ if and only if

$$\inf_{\|x\|_H = 1} \big\|(\lambda id_X - L)(x)\big\|_H \;=\; 0.$$

PROOF "$\Longleftarrow$": If $\lambda \in \varrho(L)$, then

$$(\lambda id_X - L)^{-1} \;\in\; \mathcal{L}(H)$$

and so for $x \in H$ with $\|x\|_H = 1$, we have

$$\begin{aligned} 1 \;=\; \|x\|_H \;&=\; \big\|(\lambda id_X - L)^{-1}(\lambda id_X - L)(x)\big\|_H \\ &\leqslant\; \big\|\lambda id_X - L\big\|_{\mathcal{L}}^{-1} \big\|(\lambda id_X - L)(x)\big\|_H, \end{aligned}$$

so

$$\inf_{\|x\|_H = 1} \big\|(\lambda id_X - L)(x)\big\|_H \;\geqslant\; \big\|(\lambda id_X - L)^{-1}\big\|_{\mathcal{L}}^{-1} \;>\; 0.$$

"$\Longrightarrow$": We proceed by contradiction. So suppose that

$$\inf_{\|x\|_H = 1} \big\|(\lambda id_X - L)(x)\big\|_H \;=\; c \;>\; 0.$$

Then by positive homogeneity, we have

$$\big\|(\lambda id_{_X} - L)(x)\big\|_{_H} \;\geqslant\; c\,\|x\|_{_H} \qquad \forall\, x \in X.$$

Hence $\lambda id_{_X} - L$ is injective. If we show that $\lambda id_{_X} - L$ is also surjective, then by Banach's theorem (see Theorem A.3.6), we have that

$$(\lambda id_{_X} - L)^{-1} \;\in\; \mathcal{L}(H),$$

a contradiction to the fact that $\lambda \in \sigma(L)$. We establish the surjectivity of $\lambda id_{_X} - L$ in two steps. First we show that $(\lambda id_{_X} - L)(H)$ is dense in H and then that $(\lambda id_{_X} - L)(H)$ is closed in H.

Suppose that $(\lambda id_{_X} - L)(H)$ is not dense in H. Then we can find $u \in H\backslash\{0\}$, such that

$$\big(u, (\lambda id_{_X} - L)(x)\big)_{_H} \;=\; 0 \qquad \forall\, x \in H.$$

Since L is self-adjoint, we have that

$$0 \;=\; \big(u, (\lambda id_{_X} - L)(x)\big)_{_H} \;=\; \big((\overline{\lambda} id_{_X} - L)(u), x\big)_{_H} \qquad \forall\, x \in H,$$

hence

$$\big(\overline{\lambda} id_{_X} - L\big)(u) \;=\; 0.$$

This means that

$$\overline{\lambda} \;\in\; \sigma_p(L).$$

But from Proposition 3.1.51, we know that

$$\sigma_p(L) \;\subseteq\; \mathbb{R}.$$

Hence

$$\overline{\lambda} \;=\; \overline{\overline{\lambda}} = \lambda.$$

Therefore

$$\big(\lambda id_{_X} - L\big)(u) \;=\; 0, \quad u \neq 0,$$

a contradiction to the fact that

$$\big\|(\lambda id_{_X} - L)(u)\big\|_{_H} \;\geqslant\; c\,\|u\|_{_H} \;>\; 0.$$

This proves that $(\lambda id_{_X} - L)(H)$ is dense in H.

Now we show that $(\lambda id_{_X} - L)(H)$ is closed in H. To this end let

$$(\lambda id_{_X} - L)(x_n) \;\longrightarrow\; y \quad \text{in } H.$$

Then we have

$$\big\|(\lambda id_{_X} - L)(x_n - x_m)\big\|_{_H} \;\longrightarrow\; 0 \quad \text{as } n, m \to +\infty.$$

Since

$$\big\|(\lambda id_{_X} - L)(x_n - x_m)\big\|_{_H} \;\geqslant\; c\,\|x_n - x_m\|_{_H},$$

we have that

$$\|x_n - x_m\|_H \longrightarrow 0 \quad \text{as } n, m \to +\infty.$$

Therefore

$$x_n \longrightarrow x \in H$$

and so

$$(\lambda id_X - L)(x_n) \longrightarrow (\lambda id_X - L)(x),$$

hence

$$y = (\lambda id_X - L)(x),$$

which proves the closedness of $(\lambda id_X - L)(H)$.

We conclude that

$$\big(\lambda id_X - L\big)(H) = H$$

and so by Banach's theorem, we have

$$(\lambda id_X - L)^{-1} \in \mathcal{L}(H),$$

a contradiction to the fact that $\lambda \in \sigma(L)$. □

In Proposition 3.1.51, we saw that if $L \in \mathcal{L}(H)$ is a self-adjoint operator, then $\sigma_p(L) \subseteq \mathbb{R}$. Next we show that in fact the whole spectrum is real.

PROPOSITION 3.1.53
If H is a Hilbert space and $L \in \mathcal{L}(H)$ is a self-adjoint operator, then $\sigma(L) \subseteq \mathbb{R}$.

PROOF Let $\lambda = a + ic$ with $c \neq 0$. We show that $\lambda \in \varrho(L)$. For every $x \in H$, we have

$$\begin{aligned} & \big((\lambda id_X - L)(x), x\big)_H - \big(x, (\lambda id_X - L)(x)\big)_H \\ = \; & \lambda \|x\|_H^2 - \big(L(x), x\big)_H - \overline{\lambda} \|x\|_H^2 + \big(x, L(x)\big)_H \\ = \; & \big(\lambda - \overline{\lambda}\big) \|x\|_H^2 = 2ic \|x\|_H^2 . \end{aligned}$$

So for every $x \in H$, we have

$$\begin{aligned} 2|c| \|x\|_H^2 &= \big|\big((\lambda id_X - L)(x), x\big)_H - \big(x, (\lambda id_X - L)(x)\big)_H\big| \\ &\leqslant \big|\big((\lambda id_X - L)(x), x\big)_H\big| + \big|\big((x, (\lambda id_X - L)(x)\big)_H\big| \\ &\leqslant 2\big\|(\lambda id_X - L)(x)\big\|_H \|x\|_H , \end{aligned}$$

hence

$$c \|x\|_H \leqslant \big\|(\lambda id_X - L)(x)\big\|_H .$$

Invoking Proposition 3.1.52, we infer that

$$\lambda \in \varrho(L).$$

Therefore $\sigma(L) \subseteq \mathbb{R}$. □

We can say more about the position of $\sigma(L)$ in the real line $\mathbb{R}$ when $L \in \mathcal{L}(H)$ is self-adjoint.

PROPOSITION 3.1.54
If H is a Hilbert space and $L \in \mathcal{L}(H)$ is a self-adjoint operator, then $\sigma(L) \subseteq [m, M]$, where

$$m \stackrel{df}{=} \inf_{\|x\|_H=1} \big(L(x), x\big)_H \quad \text{and} \quad M \stackrel{df}{=} \sup_{\|x\|_H=1} \big(L(x), x\big)_H.$$

Moreover, $m, M \in \sigma(L)$.

PROOF From Proposition 3.1.53, we know that $\sigma(L) \subseteq \mathbb{R}$. Let $r > 0$ and $\lambda \stackrel{df}{=} M + r$. Then for every $x \in H$ with $\|x\|_H = 1$, we have

$$\begin{aligned} \big((\lambda id_X - L)(x), x\big)_H &= \lambda \|x\|_H^2 - \big(L(x), x\big)_H \\ \geqslant \lambda \|x\|_H^2 - M \|x\|_H^2 &= r \|x\|_H^2 = r, \end{aligned}$$

so

$$r \leqslant \big\|(\lambda id_X - L)(x)\big\|_H.$$

Invoking Proposition 3.1.52, we infer that $\lambda \in \varrho(L)$. Similarly if $\lambda = m - r$. So

$$\sigma(L) \subseteq [m, M].$$

Next we show that $M \in \sigma(L)$. Note that

$$\sigma(L - \mu id_X) = \sigma(L) - \mu \qquad \forall\, \mu \in \mathbb{R}.$$

So by replacing L with $L + \mu id_X$ with $\mu > 0$ sufficiently large, we may assume that $0 \leqslant m \leqslant M$. Then

$$M = \|L\|_{\mathcal{L}}$$

(see Remark 3.1.50). Let $\{x_n\}_{n \geqslant 1} \subseteq X$ be a sequence, such that

$$\|x_n\|_H = 1 \qquad \forall\, n \geqslant 1 \quad \text{and} \quad \big(L(x_n), x_n\big)_H \nearrow M = \|L\|_{\mathcal{L}}.$$

Then we have

$$\begin{aligned} \big\|(Mid_X - L)(x_n)\big\|_H^2 &= \big(Mx_n - L(x_n), Mx_n - L(x_n)\big)_H \\ = M^2 \|x_n\|_H^2 + \big\|L(x_n)\big\|_H^2 - 2M\big(L(x_n), x_n\big)_H \\ \leqslant M^2 + M^2 - 2M\big(L(x_n), x_n\big)_H, \end{aligned}$$

hence

$$\big\|(Mid_X - L)(x_n)\big\|_H \longrightarrow 0.$$

So Proposition 3.1.52 implies that $M \in \sigma(L)$.

The proof that $m \in \sigma(L)$ is similar. ∎

PROPOSITION 3.1.55
If H is a Hilbert space and $L \in \mathcal{L}_c(H)$ is a self-adjoint operator, then $\sigma_p(L) \neq \emptyset$.

PROOF If $L = 0$, then $\lambda = 0$ is an eigenvalue of L.

If $L \neq 0$, then by Proposition 3.1.54, at least one of m or M is a nonzero element of $\sigma(L)$. Invoking Theorem 3.1.38**(b)**, we conclude that

$$\sigma_p(L) \neq \emptyset.$$

□

PROPOSITION 3.1.56
If H is a Hilbert space and $L \in \mathcal{L}_c(H)$ is a self-adjoint operator, then there is a orthonormal basis of H consisting of eigenvectors of L.

PROOF For $\lambda \in \sigma_p(L)$ let

$$E(\lambda) \stackrel{df}{=} (\lambda id_{_X} - L)^{-1}(0)$$

(the eigenspace corresponding to the eigenvalue λ). Let $B(\lambda)$ be an orthonormal basis for each finite dimensional eigenspace $E(\lambda)$. By virtue of Proposition 3.1.51, we have that

$$\bigcup_{\lambda \in \sigma_p(L)} B(\lambda) \text{ is an orthonormal set in } H.$$

Suppose that

$$\overline{\text{span}} \bigcup_{\lambda \in \sigma_p(L)} B(\lambda) \neq H.$$

Then set

$$F \stackrel{df}{=} \left[\overline{\text{span}} \bigcup_{\lambda \in \sigma_p(L)} B(\lambda)\right]^{\perp}.$$

Clearly

$$L(F) \subseteq F.$$

So $L|_{_F}$ has an eigenvalue (see Proposition 3.1.55). Let $u \in F \setminus \{0\}$ be an eigenvector of $L|_{_F}$. Evidently u is an eigenvector of L and so

$$F \cap \overline{\text{span}} \bigcup_{\lambda \in \sigma_p(L)} B(\lambda) \supsetneq \{0\},$$

a contradiction. Therefore

$$\overline{\text{span}} \bigcup_{\lambda \in \sigma_p(L)} B(\lambda) = H.$$

□

Now we can state the so-called ***spectral theorem*** for compact self-adjoint operators on a separable Hilbert space.

THEOREM 3.1.57 (Spectral Theorem)
If H is an infinite dimensional separable Hilbert space and $L \in \mathcal{L}_c(H)$ is a self-adjoint operator,
then there exists an orthonormal basis $\{e_n\}_{n\geqslant 1}$ of H formed by eigenvectors of L, such that

$$L(x) \;=\; \sum_{n=1}^{\infty} \lambda_n \,(x, e_n)_H \, e_n \qquad \forall\, x \in H,$$

with $\{\lambda_n\}_{n\geqslant 1}$ being the eigenvalues corresponding to $\{e_n\}_{n\geqslant 1}$.

PROOF From Proposition 3.1.56, we know that there exists an orthonormal basis of H consisting of eigenvectors of L. This orthonormal basis is countable, because H is separable. Denote it by $\{e_n\}_{n\geqslant 1}$. For every $x \in H$ and $m > k \geqslant 1$ and since

$$|\lambda_n| \;\leqslant\; \sup_{\|x\|_H \leqslant 1} \left|\big(L(x), x\big)_H\right| \;=\; \|L\|_{\mathcal{L}}$$

(see Remark 3.1.50), we have

$$\begin{aligned} \left\| \sum_{n=k}^{\infty} \lambda_n \,(x, e_n)_H \, e_n \right\|_H^2 \;&=\; \sum_{n=k}^{m} \left|\lambda_n \,(x, e_n)_H\right|^2 \\ \leqslant\; \|L\|_{\mathcal{L}}^2 \sum_{n=k}^{m} \left|\,(x, e_n)_H\right|^2 \;&\longrightarrow\; 0 \quad \text{as } k, m \to +\infty, \end{aligned}$$

so $\sum\limits_{n=1}^{\infty} \lambda_n \,(x, e_n)_H \, e_n$ is convergent in H.

Moreover, if $x \in H$ with $\|x\|_H \leqslant 1$, then for every $m \geqslant 1$, we have

$$\begin{aligned} &\left\| \sum_{n=1}^{m} \lambda_n \,(x, e_n)_H \, e_n \right\|_H^2 \;=\; \sum_{n=1}^{m} \lambda_n^2 \left|\,(x, e_n)_H\right|^2 \;\leqslant\; \|L\|_{\mathcal{L}}^2 \sum_{n=1}^{m} \left|\,(x, e_n)_H\right|^2 \\ &\leqslant\; \|L\|_{\mathcal{L}}^2 \sum_{n=1}^{\infty} \left|\,(x, e_n)_H\right|^2 \;=\; \|L\|_{\mathcal{L}}^2 \,\|x\|_H^2 \,. \end{aligned} \tag{3.7}$$

Therefore, if we define

$$T(x) \;\stackrel{df}{=}\; \sum_{n=1}^{\infty} \lambda_n \,(x, e_n)_H \, e_n,$$

from (3.7), we see that $T \in \mathcal{L}(H)$. Note that $L(e_n) = T(e_n)$ for all $n \geqslant 1$. So by linearity and continuity, we conclude that $L = T$. ∎

Before passing to Fredholm operators, let us mention two more results on compact maps.

PROPOSITION 3.1.58
If X, Y are two Banach spaces, $U \subseteq X$ is an open set, $f \in K(U;Y)$ and it is Fréchet differentiable,
then

$$f'(x) \ \in \ \mathcal{L}_c(X;Y) \qquad \forall\, x \in U.$$

PROOF Suppose that $f'(x)$ is not compact. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ with

$$\|u_n\|_X \ \leqslant \ 1 \qquad \forall\, n \geqslant 1$$

and $\varepsilon > 0$, such that

$$\big\|f'(x)u_n - f'(x)u_m\big\|_Y \ \geqslant \ \varepsilon \qquad \forall\, n \neq m.$$

We have

$$f(x+h) - f(x) \ = \ f'(x)h + o_x(h),$$

where

$$\|o_x(h)\|_Y \ \leqslant \ \frac{\varepsilon}{3}\,\|h\|_X \qquad \forall\, h \in X,\ \|h\|_X \leqslant \delta,$$

for some $\delta = \delta(\varepsilon, x) > 0$. Therefore

$$\begin{aligned} &\big\|f(x+\delta u_n) - f(x+\delta u_m)\big\|_Y \\ \geqslant\ & \delta\big\|f'(x)(u_n - u_m)\big\|_Y - \big\|o_x(\delta u_n)\big\|_Y - \big\|o_x(\delta u_m)\big\|_Y \\ \geqslant\ & \varepsilon\delta - \frac{2\varepsilon}{3}\delta \ = \ \frac{\varepsilon}{3}\delta, \end{aligned}$$

a contradiction to the fact that $f \in K(U;Y)$. □

The converse of the above result is also true, provided that the map

$$x \ \longmapsto \ f'(x)$$

belongs in $K(U;\mathcal{L}(X;Y))$. For details see Vaĭnberg (1973, pp. 47 and 51).

PROPOSITION 3.1.59
Let X, Y be two Banach spaces.

(a) *If $f\colon X \longrightarrow Y$ is a Fréchet differentiable operator, $f'(x) \in \mathcal{L}_c(X;Y)$ for every $x \in X$ and $x \longmapsto f'(x)$ belongs in $K\big(X;\mathcal{L}(X;Y)\big)$,*
then $f \in K(X;Y)$.

(b) *If $f \in K(X;Y)$, f is Fréchet differentiable, and $x \longmapsto f'(x)$ belongs in $K\big(X;\mathcal{L}(X;Y)\big)$,*
then f is completely continuous.

The study of compact operators leads us to the following definition.

DEFINITION 3.1.60 *Let X, Y be two Banach spaces and $L \in \mathcal{L}(X;Y)$. We say that L is a* ***Fredholm operator****, if*

$$\alpha(L) \;=\; \dim\ker L \;<\; +\infty \quad \textit{and} \quad \beta(L) \;=\; \operatorname{codim} R(L) \;<\; +\infty.$$

The class of Fredholm operators is denoted by $\Phi(X;Y)$. The quantity $\alpha(L)$ is called the ***kernel index*** *of L and the quantity $\beta(L)$ is called the* ***deficiency index*** *of L. The* ***index*** *of L is defined by*

$$\operatorname{ind}(L) \;\stackrel{df}{=}\; \alpha(L) - \beta(L).$$

If we have only that $\alpha(L) < +\infty$ and that $R(L)$ is closed, then we say that L is a ***semi-Fredholm operator*** *and the class of all semi-Fredholm operators is denoted by $\Phi_+(X;Y)$. If $X = Y$ we write $\Phi(X)$ and $\Phi_+(X)$.*

REMARK 3.1.61 We have

$$\Phi(X;Y) \;\subseteq\; \Phi_+(X;Y),$$

since as we show in the sequel, the condition that $\beta(L) < +\infty$ implies that $R(L)$ is closed.

LEMMA 3.1.62
If X, Y are two Banach spaces, $L \in \mathcal{L}(X;Y)$ is injective and

$$L^{-1}\colon R(L) \;\longrightarrow\; X$$

is bounded,
then $R(L)$ is closed.

PROOF Let

$$L(x_n) \;\longrightarrow\; y \quad \text{in } Y.$$

Since by hypothesis L has a bounded inverse on $R(L)$, we must have that

$$\big\|L(x_n) - L(x_m)\big\|_Y \;\geqslant\; c\,\|x_n - x_m\|_X \qquad \forall\, n \neq m,$$

for some $c > 0$. Therefore $\{x_n\}_{n\geqslant 1} \subseteq X$ is a Cauchy sequence and we have that

$$x_n \;\longrightarrow\; x,$$

for some $x \in X$. Hence

$$L(x_n) \;\longrightarrow\; L(x) \quad \text{in } Y$$

and so

$$y \;=\; L(x).$$

Therefore $y \in R(L)$ and we conclude that $R(L)$ is closed in Y. ∎

The next definition formalizes an idea which was used in earlier proofs, when we wanted to get rid of the nontrivial kernel of an operator $L \in \mathcal{L}(X;Y)$.

DEFINITION 3.1.63 *Let X, Y be two Banach spaces and*

$$L \in \mathcal{L}_c(X;Y).$$

*The operator $\widehat{L}$ **induced** by L is the operator from $X/_{\ker L}$ into Y defined by*

$$\widehat{L}\big([x]\big) \stackrel{df}{=} L(x) \qquad \forall\, x \in X.$$

REMARK 3.1.64 Evidently $\widehat{L}$ is injective and

$$R\big(\widehat{L}\big) = R(L).$$

In fact it is straightforward to show that $\widehat{L}$ is continuous and

$$\|L\|_{\mathcal{L}} = \big\|\widehat{L}\big\|_{\mathcal{L}}.$$

PROPOSITION 3.1.65
If X, Y are two Banach spaces and $L \in \mathcal{L}(X;Y)$, then $R(L)$ is closed if and only if there exists $c > 0$, such that

$$\big\|L(x)\big\|_Y \geqslant c d_X(x, \ker L) \qquad \forall\, x \in X.$$

PROOF Consider the operator $\widehat{L}$ induced by L (see Definition 3.1.63). We know that $\widehat{L}$ is injective and

$$R\big(\widehat{L}\big) = R(L)$$

(see Remark 3.1.64). By virtue of Lemma 3.1.62, $R(L) = R\big(\widehat{L}\big)$ is closed if and only if $\widehat{L}^{-1}$ has a bounded inverse which in turn is equivalent to saying that

$$0 < c = \inf_{x \notin \ker L} \frac{\|\widehat{L}([x])\|_Y}{\|[x]\|} = \inf_{x \notin \ker L} \frac{\|L(x)\|_Y}{d_X(x, \ker L)}.$$

□

We can define the quantity

$$\gamma(L) \stackrel{df}{=} \inf_{x \notin \ker L} \frac{\|L(x)\|_Y}{d_X(x, \ker L)}.$$

This quantity is known as the ***minimum modulus*** of L. From the previous discussion, we have the following proposition.

PROPOSITION 3.1.66
If X, Y are two Banach spaces and $L\colon X \longrightarrow Y$ is linear, then any two of the following three properties imply the other:

(a) $L \in \mathcal{L}(X;Y)$;

(b) $R(L)$ *is closed in* Y;

(c) $\gamma(L) > 0$.

PROPOSITION 3.1.67
If X, Y are two Banach spaces, $L \in \mathcal{L}(X;Y)$ and suppose that E is a closed subspace of Y, such that $R(L) \oplus E$ is closed in Y, then $R(L)$ is closed.

PROOF Let $L_0 \in \mathcal{L}(X \times E, X \times Y)$ be defined by

$$L_0(x,u) \stackrel{df}{=} L(x) + u \qquad \forall\ (x,u) \in X \times E.$$

Since $R(L) \cap E = \{0\}$, we have that

$$\ker L_0 \ = \ \ker L \times \{0\}.$$

By hypothesis

$$R(L_0) \ = \ R(L) \oplus E$$

is closed. So according to Proposition 3.1.66, we have that

$$\gamma(L_0) \ > \ 0.$$

Then for all $x \in X$, we have

$$\begin{aligned} \big\|L(x)\big\|_Y \ &= \ \big\|L_0(x,0)\big\|_{X\times X} \\ \geqslant \ \gamma(L_0) d\big((x,0), \ker L_0\big) \ &= \ \gamma(L_0) d_{_X}(x, \ker L), \end{aligned}$$

so

$$\gamma(L) \ \geqslant \ \gamma(L_0) \ > \ 0.$$

Invoking Proposition 3.1.66, we conclude that $R(L) \subseteq Y$ is closed. □

COROLLARY 3.1.68
If X, Y are two Banach spaces, $L \in \mathcal{L}(X;Y)$ and $\beta(L) < +\infty$, then $R(L)$ is closed.

PROOF We have

$$Y \ = \ R(L) \oplus E$$

for some finite dimensional subspace E of Y. Apply Proposition 3.1.67. □

This corollary implies that every $L \in \Phi(X;Y)$ has closed range and so

$$\Phi(X;Y) \subseteq \Phi_+(X;Y)$$

(see Remark 3.1.61). The propositions that follow summarize some of the basic properties of Fredholm operators.

PROPOSITION 3.1.69
If X, Y are two Banach spaces and $L \in \Phi(X;Y)$, then

(a) *if $\operatorname{ind} L = 0$ and $\ker L = \{0\}$, then for every $y \in Y$ the equation $L(x) = y$ has a unique solution and L^{-1} exists and is bounded (i.e., $L^{-1} \in \mathcal{L}(X;Y)$);*

(b) *for given $y \in Y$, the equation $L(x) = y$ has a solution if and only if*

$$\langle y^*, y \rangle_X = 0 \qquad \forall\, y^* \in \ker L^*,$$

i.e., $y \in {}^{\perp}(\ker L^*)$;

(c) *$L^* \in \Phi(Y^*;X^*)$ and $\alpha(L^*) = \beta(L)$, $\beta(L^*) = \alpha(L)$, $\operatorname{ind} L^* = -\operatorname{ind} L$.*

PROOF **(a)** Since $\ker L = \{0\}$, L is injective. Also because $\operatorname{ind} L = 0$, we have $\beta(L) = 0$ and so $R(L) = Y$. Invoking Banach's Theorem, we conclude that $L^{-1} \in \mathcal{L}(X;Y)$ and $L(x) = y$ has a unique solution.

(b) From Corollary 3.1.68, we know that $R(L)$ is closed. Hence

$$R(L) = {}^{\perp}(\ker L^*).$$

(c) Since $R(L)$ is closed, we have

$$\overline{\ker L^*}^{w^*} = R(L)^{\perp} \quad \text{and} \quad {}^{\perp}\big(R(L^*)\big) = \overline{\ker L}.$$

So

$$\alpha(L^*) = \beta(L) \quad \text{and} \quad \beta(L^*) = \alpha(L).$$

Because $L \in \Phi(X;Y)$, it follows that

$$L^* \in \Phi(Y^*;X^*) \quad \text{and} \quad \operatorname{ind} L^* = -\operatorname{ind} L.$$

□

The next proposition gives a basic stability property of Fredholm operators.

PROPOSITION 3.1.70
If X, Y are two Banach spaces, $L \in \Phi(X;Y)$ and $T \in \mathcal{L}_c(X;Y)$, then $G \stackrel{df}{=} L + T \in \Phi(X;Y)$ and $\operatorname{ind} G = \operatorname{ind} L$.

REMARK 3.1.71 The result is also true if instead of $T \in \mathcal{L}_c(X;Y)$ we assume that $T \in \mathcal{L}(X;Y)$ with $\|T\|_{\mathcal{L}} \leqslant \delta$ for some $\delta = \delta(L) > 0$. Also note that in particular Proposition 3.1.70 implies that if $T \in \mathcal{L}_c(X;Y)$ then $id_X - T \in \Phi(X;Y)$.

PROPOSITION 3.1.72
If X is a Banach space and $L \in \mathcal{L}(X)$,
then $L \in \Phi_+(X)$ if and only if for every closed and bounded set $B \subseteq X$, $L|_B$ is proper.

PROOF "$\Longrightarrow$": Let $\{x_n\}_{n\geqslant 1} \subseteq B$ be such that

$$L(x_n) \longrightarrow u \quad \text{in } X.$$

We have that

$$X = \ker L \oplus E,$$

with a closed subspace $E \subseteq X$ (since $\dim \ker L < +\infty$). So $x_n = z_n + e_n$ with $z_n \in \ker L$, $e_n \in E$. We have

$$L(x_n) = L(e_n) \longrightarrow u \quad \text{in } X.$$

Then operator $L|_E = \widehat{L}$ (see Definition 3.1.63) is injective and so by Banach's Theorem,

$$L^{-1} \in \mathcal{L}\big(R(L);E\big).$$

Therefore, we have

$$e_n \longrightarrow e \quad \text{in } X,$$

for some $e \in E$. The sequence $\{z_n\}_{n\geqslant 1} \subseteq \ker L$ is bounded. So exploiting the finite dimensionality of $\ker L$, we have that the sequence $\{z_n\}_{n\geqslant 1}$ is relatively compact in X. Therefore we conclude that the sequence $\{x_n\}_{n\geqslant 1}$ is relatively compact in X, which proves that $L|_B$ is proper.

"$\Longrightarrow$": The set

$$\big\{x \in X : \ x \in \overline{\ker L}, \ \|x\|_X \leqslant 1\big\}$$

is compact by hypothesis. So it follows that $\ker L$ is finite dimensional. We can write

$$X = \ker L \oplus E,$$

with a closed subspace $E \subseteq X$. Then Proposition 3.1.67 implies that $R(L)$ is closed, hence $L \in \Phi_+(X)$. ∎

3.2 Operators of Monotone Type

Operators of monotone type were introduced to provide an analytic framework broader than compact operators in order to study nonlinear functional equations. Their systematic study of monotone operators starts in early 1960's and marks the advent of nonlinear functional analysis. Monotone operators are rooted in the theory of variational problems. Moreover, recalling that the Gâteaux derivative of a convex function is the prototypical example of a nonlinear monotone operator, it is no surprise that for a long period the theory of monotone operators and convex analysis developed in parallel and interacted heavily.

The mathematical framework of the analysis in this section is the following. Let X be a reflexive Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle_X$ we denote the duality pairing for the spaces X^* and X. Also

$$A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$$

is a generally multivalued operator. The domain $D(A)$ of A is defined by

$$D(A) \stackrel{df}{=} \big\{x \in X : \ A(x) \neq \emptyset\big\}$$

and the graph $\operatorname{Gr} A$ of A is defined by

$$\operatorname{Gr} A \stackrel{df}{=} \big\{(x, x^*) \in X \times X^* : \ x^* \in A(x)\big\}.$$

Also we can define $A^{-1}\colon X^* \supseteq D^* \longrightarrow 2^X$ b

$$\operatorname{Gr} A^* \stackrel{df}{=} \big\{(x^*, x) \in X^* \times X : \ (x, x^*) \in \operatorname{Gr} A\big\}.$$

Note that A^{-1} is always defined as a multivalued operator. Some of the results in this section are actually true in a more general setting. However, in order to have a uniform presentation, we have chosen to work in the above setting, which after all is what we encounter in most applications.

DEFINITION 3.2.1 *Let $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ be a multivalued operator.*

(a) *We say that A is **monotone**, if*

$$\langle x^* - y^*, x - y \rangle_X \ \geqslant \ 0,$$

for all $x, y, \in D(A)$ and all $x^* \in A(x)$, $y^* \in A(y)$.

(b) *We say that A is* ***strictly monotone****, if it is monotone and*

$$\langle x^* - y^*, x - y\rangle_X \ > \ 0,$$

for all $x, y \in D(A)$, $x \neq y$ and all $x^* \in A(x)$, $y^* \in A(y)$.

(c) *We say that A is* ***strongly monotone****, if there exists $c > 0$, such that*

$$\langle x^* - y^*, x - y\rangle_X \ \geqslant \ c\,\|x - y\|_X^2\,,$$

for all $x, y, \in D(A)$ and all $x^* \in A(x)$, $y^* \in A(y)$.

(d) *We say that A is* ***uniformly monotone****, if there exists a continuous function $c\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is strictly increasing, $c(0) = 0$, $c(r) \longrightarrow +\infty$ as $r \to +\infty$ and*

$$\langle x^* - y^*, x - y\rangle_X \ \geqslant \ c\big(\,\|x - y\|_X\,\big)\,\|x - y\|_X\,,$$

for all $x, y, \in D(A)$ and all $x^* \in A(x)$, $y^* \in A(y)$.

(e) *We say that A is* ***coercive****, if $D(A)$ is bounded or $D(A)$ is unbounded and*

$$\frac{\inf\{\langle x^*, x\rangle_X :\ x^* \in A(x)\}}{\|x\|_X} \ \longrightarrow \ +\infty \quad \text{as } \|x\|_X \to +\infty,\ x \in D(A).$$

We say that A is ***weakly coercive****, if $D(A)$ is bounded or $D(A)$ is unbounded and*

$$\inf_{x^* \in A(x)} \|x^*\|_{X^*} \ \longrightarrow \ +\infty \quad \text{as } \|x\|_X \to +\infty,\ x \in D.$$

REMARK 3.2.2 If A is strongly monotone, then A is uniformly monotone.

If A is uniformly monotone, then A is strictly monotone.

If A is strictly monotone, then A is monotone.

If A is uniformly monotone, then A is coercive.

If A is coercive, then A is weakly coercive.

Sometimes it is convenient to identify A with its graph. For this reason some authors speak of monotone sets in $X \times X^*$.

DEFINITION 3.2.3 *A monotone map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is said to be* ***maximal monotone****, if the inequality*

$$\langle x^* - y^*, x - y\rangle_X \ \geqslant 0 \qquad \forall\ (x, x^*) \in \operatorname{Gr} A$$

implies that $(y, y^*) \in \operatorname{Gr} A$.

REMARK 3.2.4 The above definition implies that the graph of a maximal monotone map is not properly included in the graph of another monotone map (i.e., it is maximal with respect to inclusion).

EXAMPLE 3.2.5 An increasing continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is maximal monotone. However, an increasing discontinuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is monotone but not maximal monotone, since it admits a monotone extension by filling in the jumps at the discontinuity points. This example underlines the necessity of multivalued operators in the study of maximal monotonicity.

The next result is an immediate consequence of Definition 3.2.1.

PROPOSITION 3.2.6
A map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is maximal monotone if and only if $A^{-1}\colon X^* \supseteq D(A^*) \longrightarrow 2^X$ is maximal monotone.

PROPOSITION 3.2.7
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map, then for every $x \in D(A)$, the set $A(x)$ is nonempty, convex and closed.

PROOF Since $x \in D(A)$, $A(x) \neq \emptyset$. Let $x^*, y^* \in A(x)$. Set

$$u_\lambda^* \stackrel{df}{=} \lambda x^* + (1-\lambda)y^* \qquad \forall\, \lambda \in [0,1].$$

For all $(z, z^*) \in \operatorname{Gr} A$, we have

$$\langle u_\lambda^* - z^*, x - z\rangle_X = \lambda \langle x^* - z^*, x - z\rangle_X + (1-\lambda)\langle y^* - z^*, x - z\rangle_X \geqslant 0,$$

hence $u_\lambda^* \in A(x)$ (see Definition 3.2.3). Therefore $A(x)$ is convex. Also suppose that $\{x_n^*\}_{n\geqslant 1} \subseteq A(x)$ is a sequence, such that

$$x_n^* \longrightarrow x^* \quad \text{in } X^*.$$

We have

$$\langle x_n^* - z^*, x - z\rangle_X \geqslant 0 \qquad \forall\, n \geqslant 1,\ (z, z^*) \in \operatorname{Gr} A.$$

In the limit as $n \to +\infty$, we have

$$\langle x^* - z^*, x - z\rangle_X \geqslant 0,$$

hence $(x, x^*) \in \operatorname{Gr} A$, i.e., $A(x)$ is closed in X^*. ∎

A fundamental property of monotone maps is local boundedness.

DEFINITION 3.2.8 *A monotone map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is said to be **locally bounded** at $x \in D(A)$, if there exists $M > 0$ and $r > 0$, such that*

$$\|y^*\|_{X^*} \leqslant M \qquad \forall\, y \in D(A) \cap \overline{B}_r(x),\ y^* \in A(y).$$

DEFINITION 3.2.9 *If $C \subseteq X$ is a nonempty set, a point $x \in C$ is an* ***absorbing point*** *of C, if the set $C - x$ is absorbing, i.e., $X = \bigcup_{\lambda>0} \lambda(C - x)$.*

REMARK 3.2.10 If $\operatorname{int} C \neq \emptyset$, then any $x \in \operatorname{int} C$ is an absorbing point of C. If $C = \partial B_1 \cup \{0\}$, then 0 is an absorbing point although $\operatorname{int} C = \emptyset$.

PROPOSITION 3.2.11
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is monotone and $x \in D(A)$ is an absorbing point of $D(A)$,
then A is locally bounded at x.

PROOF Without any loss of generality we may assume that $x = 0$ and $0 \in A(0)$ (i.e., $(0,0) \in \operatorname{Gr} A$). Indeed if this is not the case, we choose $x^* \in A(x)$ and consider the map

$$A_1(y) \stackrel{df}{=} A(y + x) - x^*.$$

Evidently A_1 is still monotone, $(0,0) \in \operatorname{Gr} A_1$ and $D(A_1) = D(A) - x$. So we can replace A with A_1. Therefore we need to show that A is locally bounded at 0.

For every $u \in X$, we define

$$\varphi(u) \stackrel{df}{=} \sup_{\substack{y \in D(A) \\ \|y\|_X \leqslant 1 \\ y^* \in A(y)}} \langle y^*, u - y\rangle_X .$$

Clearly φ is the supremum of affine continuous functions, hence φ is convex and lower semicontinuous and because $(0,0) \in \operatorname{Gr} A$, we have $\varphi \geqslant 0$. The set

$$C \stackrel{df}{=} \{u \in X :\ \varphi(u) \leqslant 1\}$$

is closed and convex. We claim that $0 \in C$. Indeed because $(0,0) \in \operatorname{Gr} A$, we have

$$0 \leqslant \langle y^*, y\rangle_X \qquad \forall\ (y, y^*) \in \operatorname{Gr} A$$

and so $\varphi(0) \leqslant 0$. Let

$$E \stackrel{df}{=} C \cap (-C).$$

This is a closed, convex and symmetric set. We claim that it is absorbing too. Let $u \in X$. Since by hypothesis $D(A)$ is absorbing, we can find $\lambda > 0$, such that $\lambda u \in D(A)$, i.e., $A(\lambda u) \neq \emptyset$. Choose $v^* \in A(\lambda u)$. If $(y, y^*) \in \operatorname{Gr} A$, from the monotonicity of A, we have

$$\langle y^*, \lambda u - y\rangle_X \leqslant \langle v^*, \lambda u - y\rangle_X ,$$

so

$$\varphi(\lambda u) \leqslant \sup_{\substack{y \in D(A) \\ \|y\|_X \leqslant 1}} \langle v^*, \lambda u - y\rangle_X \leqslant \langle v^*, \lambda u\rangle_X + \|v^*\|_{X^*} < +\infty.$$

Choose $t \in (0,1)$, such that $t\varphi(\lambda u) < 1$. Because φ is convex, we have

$$\varphi(t\lambda u) \leqslant t\varphi(\lambda u) + (1-t)\varphi(0) = t\varphi(\lambda u) < 1,$$

so $t\lambda u \in C$.

This shows that C is absorbing, hence E is absorbing too. Thus E is a neighbourhood of the origin and so we can find $\delta > 0$, such that

$$\varphi(u) \leqslant 1 \qquad \forall\ \|u\|_X \leqslant 2\delta.$$

This means that

$$\langle y^*, u\rangle_X \leqslant 1 + \langle y^*, y\rangle_X \qquad \forall\ y \in D(A),\ \|y\|_X \leqslant 1,\ y^* \in A(y),\ \|u\|_X \leqslant 2\delta.$$

Therefore, if $y \in D(A) \cap \overline{B}_\delta$ and $y^* \in A(y)$, we have

$$2\delta\,\|y^*\|_{X^*} = \sup_{\|u\|_X \leqslant 2\delta} \langle y^*, u\rangle_X \leqslant 1 + \|y^*\|_{X^*}\,\|y\|_X \leqslant 1 + \delta\,\|y^*\|_{X^*},$$

so $\|y^*\|_{X^*} \leqslant \frac{1}{\delta}$. □

Using this result, we can determine the continuity properties of maximal monotone maps. First let us recall the following notion from multivalued analysis.

DEFINITION 3.2.12 *Let Y, Z be two Hausdorff topological spaces. A multifunction (set-valued map) $F\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ is said to be **upper semicontinuous**, if for any closed set $C \subseteq Z$, the set*

$$F^-(C) \stackrel{df}{=} \{y \in Y :\ F(y) \cap C \neq \emptyset\}$$

is closed.

REMARK 3.2.13 It is easy to check that the above definition is equivalent to saying that for any open set $U \subseteq Z$, the set

$$F^+(U) \stackrel{df}{=} \{y \in Y :\ F(y) \subseteq U\}$$

is open. Moreover, if for all $y \in Y$, the set $F(y) \subseteq Z$ is closed and Z is regular, then upper semicontinuity of F implies that $\operatorname{Gr} F \subseteq Y \times Z$ is closed (see Hu & Papageorgiou (1997, p. 41)). The converse is true if F is locally compact, i.e., for every $y \in Y$, we can find a neighbourhood U of y, such that $\overline{F(U)}$ is compact in Z. Finally note that if F is single-valued, then the notion of upper semicontinuity coincides with that of continuity.

PROPOSITION 3.2.14
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is a maximal monotone map and* $\operatorname{int} D(A) \neq \emptyset$, *then* $A|_{\operatorname{int} D(A)}$ *is upper semicontinuous from* X *with the norm topology into* X^* *with the weak topology.*

PROOF Let $C \subseteq X^*$ be a weakly closed set. We need to show that the set

$$\left(A|_{\operatorname{int} D(A)}\right)^{-}(C) \;=\; \left\{x \in \operatorname{int} D(A) :\; A(x) \cap C \neq \emptyset\right\}$$

is closed in $\operatorname{int} D(A)$. To this end let

$$\{x_n\}_{n\geqslant 1} \;\subseteq\; \left(A|_{\operatorname{int} D(A)}\right)^{-1}(C)$$

be a sequence, such that

$$x_n \;\longrightarrow\; x \quad \text{in } X,$$

for some $x \in \operatorname{int} D(A)$. Let

$$x_n^* \;\in\; A(x_n) \cap C \qquad \forall\, n \geqslant 1.$$

Then Proposition 3.2.11 implies that the sequence $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ is bounded. By virtue of the reflexivity of X^* and the Eberlein-Smulian Theorem (see Theorem A.3.8), we may assume that

$$x_n^* \;\xrightarrow{w}\; x^* \quad \text{in } X^*.$$

Clearly $x^* \in C$. Also, we have

$$\langle x_n^* - y^*, x_n - y\rangle_X \;\geqslant\; 0 \qquad \forall\, n \geqslant 1,\ (y, y^*) \in \operatorname{Gr} A,$$

so

$$\lim_{n\to+\infty} \langle x_n^* - y^*, x_n - y\rangle_X \;=\; \langle x^* - y^*, x - y\rangle_X \;\geqslant\; 0.$$

Because A is maximal monotone, we infer that $x^* \in A(x)$.

Therefore

$$x \;\in\; \left(A|_{\operatorname{int} C}\right)^{-1}(C)$$

and this proves the claimed upper semicontinuity of $A|_{\operatorname{int} D(A)}$. □

A careful reading of the previous proof reveals that the following is also true.

PROPOSITION 3.2.15
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is a maximal monotone map,*
then $\operatorname{Gr} A \subseteq X \times X_w^*$ *and* $\operatorname{Gr} A \subseteq X_w \times X^*$ *are closed sets (here* Z_w *denotes the space* Z *furnished with the weak topology).*

DEFINITION 3.2.16 *Let Y, Z be two Banach spaces and let $V\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ be a multifunction.*

(a) *We say that V is **demicontinuous**, if it is upper semicontinuous from Y with the norm topology into Z with the weak topology.*

(b) *We say that V is **hemicontinuous**, if for all $x, y \in Y$, the multivalued map $\lambda \longmapsto V\big(\lambda x + (1-\lambda)y\big)$ is upper semicontinuous from $[0,1]$ into Z with the weak topology.*

(c) *We say that V is **bounded**, if it maps bounded sets in Y into bounded sets in Z.*

REMARK 3.2.17 Evidently demicontinuity implies hemicontinuity. For monotone maps $A\colon X \longrightarrow 2^{X^*}$ with $D(A) = X$, the converse is also true.

PROPOSITION 3.2.18
If $A\colon X \longrightarrow 2^{X^*}$ is a monotone hemicontinuous map with $D(A) = X$, then A is demicontinuous.

PROOF If $C \subseteq X^*$ is w-closed, we need to show that the set

$$A^-(C) \;=\; \big(x \in X :\; A(x) \cap C \neq \emptyset\big)$$

is norm closed in X. To this end let

$$\{x_n\}_{n \geqslant 1} \;\subseteq\; A^-(C)$$

be a sequence, such that

$$x_n \;\longrightarrow\; x \quad \text{in } X.$$

Let

$$x_n^* \;\in\; A(x_n) \cap C \qquad \forall\, n \geqslant 1.$$

Then Proposition 3.2.11 implies that the sequence $\{x_n^*\}_{n \geqslant 1} \subseteq X^*$ is bounded and so we may assume that

$$x_n^* \;\xrightarrow{w}\; x^* \quad \text{in } X^*.$$

Set

$$y_\lambda \;\stackrel{df}{=}\; x + \lambda y,$$

and let

$$y_\lambda^* \in A(y_\lambda) \qquad \forall\, \lambda > 0,\; y \in X.$$

From the monotonicity of A, we have

$$\big\langle x_n^* - y_\lambda^*, x_n - x\big\rangle_X - \lambda \,\langle x_n^* - y_\lambda^*, y\rangle_X \;\geqslant\; 0 \qquad \forall\, n \geqslant 1,$$

so

$$\langle x_n^* - y_\lambda^*, y\rangle_X \leqslant \frac{1}{\lambda}\langle x_n^* - y_\lambda^*, x_n - x\rangle_X \qquad \forall\, n \geqslant 1.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\langle x^* - y_\lambda^*, y\rangle_X \leqslant 0.$$

Next let

$$\lambda \searrow 0.$$

Due to the hemicontinuity of A, we may say that

$$y_\lambda^* \xrightarrow{w} y^* \quad \text{in } X^*,$$

for some $y^* \in A(x)$. So we obtain that

$$\langle x^* - y^*, y\rangle_X \leqslant 0.$$

Because $y \in X$ was arbitrary, it follows that

$$x^* = y^* \in A(x).$$

Therefore $x \in A^-(C)$ and we have proved the demicontinuity of A. □

Next we give a sufficient condition for maximality of a monotone map.

PROPOSITION 3.2.19
If $A\colon X \longrightarrow 2^{X^*}$ is a monotone map with $D(A) = X$, which is hemicontinuous and for every $x \in X$, the set $A(x) \subseteq X^*$ is closed and convex, then A is maximal monotone.

PROOF Suppose that $\widehat{A}$ is a monotone extension of A and $x_0^* \in \widehat{A}(x_0)$. We need to show that $x_0^* \in A(x_0)$. If this is not true, then from the strong separation theorem (see Theorem A.3.2), we can find $u \in X \setminus \{0\}$, such that

$$\langle x^*, u\rangle_X < \langle x_0^*, u\rangle \qquad \forall\, x^* \in A(x_0). \tag{3.8}$$

Let $\lambda > 0$ and $x_\lambda = x_0 + \lambda u$. By virtue of the monotonicity of $\widehat{A}$, we have

$$\lambda\langle x_\lambda^* - x_0^*, u\rangle_X \geqslant 0 \qquad \forall\, x_\lambda^* \in A(x_\lambda),$$

so

$$\langle x_\lambda^* - x_0^*, u\rangle_X \geqslant 0 \qquad \forall\, x_\lambda^* \in A(x_\lambda). \tag{3.9}$$

Because of the hemicontinuity of A, we can say that

$$x_\lambda^* \xrightarrow{w} x^* \quad \text{in } X^* \quad \text{as } \lambda \searrow 0,$$

for some $x^* \in A(x_0)$. So from (3.9), we have that

$$\langle x^* - x_0^*, u\rangle_X \geqslant 0,$$

which contradicts (3.8). □

At this point let us give some standard examples of maximal monotone maps.

EXAMPLE 3.2.20 **(a)** Let H be a Hilbert space and let $C \subseteq H$ be a closed, convex set. It is well known that for each $x \in H$, there exists a unique element in C, denoted by $\mathrm{proj}_C(x)$, such that

$$\big\|x - \mathrm{proj}_C(x)\big\|_X \;=\; \inf_{c\in C} \|x - c\|_X$$

(best approximation of x in C). The map $\mathrm{proj}_C : H \longrightarrow C$ is known as the ***metric projection*** on C. Then proj_C is a maximal monotone map. Indeed, recalling that

$$\big\langle x - \mathrm{proj}_C(x), c - \mathrm{proj}_C(x)\big\rangle_X \;\leqslant\; 0 \qquad \forall\, c \in C,$$

then we can easily check that

$$\big\|\mathrm{proj}_C(x) - \mathrm{proj}_C(y)\big\|_X^2 \;\leqslant\; \big\langle x - y, \mathrm{proj}_C(x) - \mathrm{proj}_C(y)\big\rangle_X$$

and so

$$\big\|\mathrm{proj}_C(x) - \mathrm{proj}_C(y)\big\|_X \;\leqslant\; \|x - y\|_X\,,$$

i.e., the map $x \longmapsto \mathrm{proj}_C(x)$ is nonexpansive and it is monotone. So by Proposition 3.2.19, the map $x \longmapsto \mathrm{proj}_C(x)$ is maximal monotone.

(b) If H is a Hilbert space and $A\colon H \longrightarrow H$ is nonexpansive (i.e., Lipschitz continuous with Lipschitz constant equal to 1), then it is easy to check that $id_X + A$ is maximal monotone.

(c) Let X be a reflexive Banach space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ a proper (i.e., not identically $+\infty$), convex, lower semicontinuous function. The subdifferential of φ is defined by

$$\partial\varphi(x) \;\stackrel{df}{=}\; \Big\{x^* \in X^* :\; \langle x^*, x - y\rangle_X \leqslant \varphi(y) - \varphi(x) \qquad \forall\, y \in X\Big\}.$$

Then $\partial\varphi\colon X \longrightarrow 2^{X^*}$ is a maximal monotone map. We shall prove this and more in Section 4.3, where we conduct a detailed study of the convex subdifferential. For the moment, we keep in mind the maximality of the subdifferential map, in order to better understand the next example.

(d) Let X be a reflexive Banach space and consider the map $\mathcal{F}\colon X \longrightarrow 2^{X^*}$ defined by

$$\mathcal{F}(x) \;\stackrel{df}{=}\; \Big\{x^* \in X^* :\; \langle x^*, x\rangle_X = \|x\|_X^2 = \|x^*\|_{X^*}^2 \Big\}.$$

According to the Hahn-Banach Theorem, we see that $\mathcal{F}(x) \neq \emptyset$ for all $x \in X$. Moreover, we have that

$$\mathcal{F}(x) \;=\; \partial\varphi(x), \quad \text{where} \quad \varphi(x) \;=\; \frac{1}{2}\,\|x\|_X^2\,.$$

Indeed, if $x^* \in \mathcal{F}(x)$, then

$$\begin{aligned} &\langle x^*, y-x\rangle_X \;\leqslant\; \|x^*\|_{X^*}\,\|y\|_X - \|x\|_X^2 \\ \leqslant\; &\frac{1}{2}\big(\,\|x\|_X^2 + \|y\|_X^2\,\big) - \|x\|_X^2 \;=\; \varphi(y) - \varphi(x), \end{aligned}$$

so $x^* \in \partial\varphi(x)$. Conversely, if $x^* \in \partial\varphi(x)$. Let $\psi(x) \stackrel{df}{=} \|x\|_X$ for all $x \in X$. By $\varphi'(x;\cdot)$ and $\psi'(x;\cdot)$ we denote the directional derivatives at $x \in X$ of the convex functions φ and ψ respectively, i.e., for all $h \in X$, we have

$$\begin{aligned} \varphi'(x;h) \;&\stackrel{df}{=}\; \lim_{\lambda\searrow 0} \frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda} \\ \psi'(x;h) \;&\stackrel{df}{=}\; \lim_{\lambda\searrow 0} \frac{\psi(x+\lambda h) - \psi(x)}{\lambda}. \end{aligned}$$

The limits exist since the difference quotients decrease as $\lambda \searrow 0$, because of the convexity of the functions. Then we have

$$\begin{aligned} \psi'(x;h)\,\|x\|_X \;&=\; \lim_{\lambda\searrow 0} \frac{\|x+\lambda h\|_X\,\|x\|_X - \|x\|_X^2}{\lambda} \\ &\leqslant\; \lim_{\lambda\searrow 0} \frac{1}{2}\,\frac{\|x+\lambda h\|_X^2 - \|x\|_X^2}{\lambda} \;=\; \varphi'(x;h) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \varphi'(x;h) \;&=\; \lim_{\lambda\searrow 0} \frac{1}{2}\,\frac{\|x+\lambda h\|_X^2 - \|x\|_X^2}{\lambda} \\ &=\; \lim_{\lambda\searrow 0} \left[\frac{1}{2}\,\frac{\|x+\lambda h\|_X - \|x\|_X}{\lambda}\big(\,\|x+\lambda h\|_X + \|x\|_X\,\big)\right] \\ &=\; \psi'(x;h)\,\|x\|_X\,. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we infer that

$$\varphi'(x;h) \;=\; \psi'(x;h)\,\|x\|_X\,.$$

Clearly from the definition of $\partial\varphi(x)$, we see that $x^* \in \partial\varphi(x)$ if and only if

$$\langle x^*, h\rangle_X \;\leqslant\; \varphi'(x;h) = \psi'(x;h)\,\|x\|_X\,.$$

So, we have

$$\left\langle \frac{x^*}{\|x\|_X}, h \right\rangle_X \;\leqslant\; \psi'(x;h) \;\leqslant\; \psi(x+h) - \psi(x) \;\leqslant\; \|h\|_X \qquad \forall\; h \in X,$$

so

$$\|x^*\|_{X^*} \leqslant \|x\|_X . \tag{3.12}$$

On the other hand since

$$\left\langle \frac{x^*}{\|x\|_X}, h \right\rangle_X \leqslant \psi'(x;h) \qquad \forall\, h \in X,$$

we have that $\frac{x^*}{\|x\|_X} \in \partial\psi(x)$ and so

$$\left\langle \frac{x^*}{\|x\|_X}, y - x \right\rangle_X \leqslant \psi(y) - \psi(x) \qquad \forall\, y \in X.$$

Let $y = 0$. We obtain

$$\left\langle \frac{x^*}{\|x\|_X}, x \right\rangle_X \geqslant \|x\|_X .$$

Therefore

$$\|x^*\|_{X^*} \geqslant \|x\|_X . \tag{3.13}$$

From (3.12) and (3.13), it follows that $\|x^*\|_{X^*} = \|x\|_X$, hence $x^* \in \mathcal{F}(x)$.

Note that if X is a Hilbert space, then $\mathcal{F}$ is the canonical isomorphism between X and X^*. So if $X = H$ is a pivot Hilbert space (i.e., $H = H^*$), then $\mathcal{F}$ is an identity operator.

REMARK 3.2.21 The duality map introduced in Example 3.2.20**(a)** actually can be defined on any Banach space (not necessarily reflexive) and is essentially dependent on the norm of the space. More precisely, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on X and $\mathcal{F}_1$ and $\mathcal{F}_2$ the corresponding duality maps, then we need not have $\mathcal{F}_1 = \mathcal{F}_2$. In fact in the proposition that follows, we show that the geometry of X and X^* is closely related to the properties of the duality map $\mathcal{F}$.

PROPOSITION 3.2.22
If X is a reflexive Banach space with X^* strictly convex,
then the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is single-valued, odd, demicontinuous, maximal monotone, coercive and bounded.

PROOF Let $x_1^*, x_2^* \in \mathcal{F}(x)$. Then we have

$$\langle x_k^*, x\rangle_X = \|x\|_X^2 = \|x_k^*\|_{X^*}^2 \quad \text{for } k \in \{1,2\}.$$

So, we have

$$2\,\|x_1^*\|_{X^*}\,\|x\|_X \leqslant \|x_1^*\|_{X^*}^2 + \|x_2^*\|_{X^*}^2 = \langle x_1^* + x_2^*, x\rangle_X \leqslant \|x_1^* + x_2^*\|_X\,\|x\|_X ,$$

thus

$$\|x_1^*\|_{X^*} \leqslant \frac{1}{2}\,\|x_1^* + x_2^*\|_{X^*}$$

and so $x_1^* = x_2^*$ due to the strict convexity of X^*.

Clearly $\mathcal{F}(-x) = -\mathcal{F}(x)$, i.e., $\mathcal{F}$ is odd.

To show the demicontinuity of $\mathcal{F}$, suppose that $\{x_n\}_{n\geqslant 1} \subseteq X$ is a sequence, such that

$$x_n \longrightarrow x \quad \text{in } X,$$

for some $x \in X$. Then

$$\left\|\mathcal{F}(x_n)\right\|_{X^*} = \|x_n\|_X \longrightarrow \|x\|_X$$

and so the sequence $\{\mathcal{F}(x_n)\}_{n\geqslant 1} \subseteq X^*$ is bounded. Because of the reflexivity of X^*, we may assume that

$$\mathcal{F}(x_n) \xrightarrow{w} x^* \quad \text{in } X^*.$$

We have

$$\begin{aligned} \left\langle x^*, h\right\rangle_X &= \lim_{n\to+\infty} \left\langle \mathcal{F}(x_n), h\right\rangle_X \\ &\leqslant \lim_{n\to+\infty} \|x_n\|_X \|h\|_X = \|x\|_X \|h\|_X \qquad \forall\, h \in X \end{aligned} \tag{3.14}$$

and

$$\left\langle x^*, x\right\rangle_X = \lim_{n\to+\infty} \left\langle \mathcal{F}(x_n), x\right\rangle_X = \lim_{n\to+\infty} \|x_n\|_X^2 = \|x\|_X^2 . \tag{3.15}$$

From (3.14), we have

$$\|x^*\|_{X^*} \leqslant \|x\|_X$$

and from (3.15), we have

$$\|x^*\|_{X^*} \geqslant \|x\|_X .$$

Therefore

$$\|x^*\|_{X^*} = \|x\|_X$$

and so $x^* \in \mathcal{F}(x)$, which proves the demicontinuity of $\mathcal{F}$.

Maximal monotonicity follows from Examples 3.2.20**(c)** and **(d)**. A more direct proof is the following. Let $x, y \in X$. Then we have

$$\begin{aligned} \left\langle \mathcal{F}(x) - \mathcal{F}(y), x - y\right\rangle_X &\geqslant \|x\|_X^2 + \|y\|_X^2 - 2\|x\|_X \|y\|_X \\ &= \left(\|x\|_X - \|y\|_X\right)^2 \geqslant 0, \end{aligned} \tag{3.16}$$

so $\mathcal{F}$ is monotone. Invoking Proposition 3.2.19, we conclude that $\mathcal{F}$ is maximal monotone. Also we have

$$\left\langle \mathcal{F}(x), x\right\rangle_X = \|x\|_X^2 ,$$

i.e., $\mathcal{F}$ is coercive.

Finally it is clear that $\mathcal{F}$ is bounded. □

REMARK 3.2.23 As we shall see later in this section (see Corollary 3.2.31), the maximal monotonicity and coercivity of $\mathcal{F}$ imply that $\mathcal{F}$ is surjective. Also we have

$$\varphi'(x;h) \;=\; \big\langle \mathcal{F}(x), h\big\rangle_X \qquad \forall\, h \in X,$$

which means that φ is Gâteaux differentiable at every $x \in X$ and

$$\varphi'(x) \;=\; \mathcal{F}(x).$$

Moreover, from Example 3.2.20**(d)**, we see that the map $x \longmapsto \psi(x) = \|x\|_X$ is Gâteaux differentiable at every $x \neq 0$ and

$$\psi'(x) \;=\; \frac{\mathcal{F}(x)}{\|x\|_X}.$$

It is a result of Banach space theory that the reflexive Banach space X is smooth (i.e., its norm is Gâteaux differentiable at every $x \neq 0$) if and only if X^* is strictly convex. Similarly the reflexive Banach space X is strictly convex if and only if X^* is smooth (see Day (1973, p. 144)).

PROPOSITION 3.2.24
If X is a reflexive Banach space and both X and X^* are strictly convex, then the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is strictly monotone and bijective and $\mathcal{F}^{-1}$ is the duality map of X^*.

PROOF Suppose that

$$\big\langle \mathcal{F}(x) - \mathcal{F}(y), x - y\big\rangle_X \;=\; 0.$$

From (3.16), we have

$$\begin{aligned} 0 \;&=\; \left\langle \mathcal{F}(x) - \mathcal{F}\left(\frac{x+y}{2}\right), \frac{x-y}{2}\right\rangle_X - \left\langle \mathcal{F}\left(\frac{x+y}{2}\right) - \mathcal{F}(y), \frac{x-y}{2}\right\rangle_X \\ &\geqslant \left(\|x\|_X - \left\|\frac{x+y}{2}\right\|_X\right)^2 + \left(\left\|\frac{x+y}{2}\right\|_X - \|y\|_X\right)^2, \end{aligned}$$

so

$$\|x\|_X \;=\; \left\|\frac{x+y}{2}\right\|_X \;=\; \|y\|_X\,.$$

Because X is strictly convex, it follows that $x = y$. So $\mathcal{F}$ is strictly monotone. Hence it is injective (see also Proposition 3.2.22). Moreover, we know that $\mathcal{F}$ is surjective (see Remark 3.2.23). Therefore $\mathcal{F}$ is bijective. Finally, it is clear that $\mathcal{F}^{-1}\colon X^* \longrightarrow X$ is the duality map of X^*. ∎

PROPOSITION 3.2.25
If X is a reflexive Banach space and X^* is a locally uniformly convex space (see Definition A.3.21),
then the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is continuous.

PROOF Let $\{x_n\}_{n\geqslant 1} \subseteq X$ be a sequence, such that

$$x_n \longrightarrow x \quad \text{in } X,$$

for some $x \in X$. Then

$$\|\mathcal{F}(x_n)\|_{X^*} \longrightarrow \|\mathcal{F}(x)\|_{X^*}.$$

Moreover, because $\mathcal{F}$ is demicontinuous (see Proposition 3.2.22), we have

$$\mathcal{F}(x_n) \xrightarrow{w} \mathcal{F}(x) \quad \text{in } X^*.$$

Since X^* is locally uniformly convex, it has the Kadec-Klee property (see Remark A.3.22) and so

$$\mathcal{F}(x_n) \longrightarrow \mathcal{F}(x) \quad \text{in } X^*.$$

Therefore $\mathcal{F}$ is continuous. ∎

REMARK 3.2.26 Under the hypotheses of Proposition 3.2.25, the map $x \longmapsto \psi(x) = \|x\|_X$ is Fréchet differentiable at every $x \neq 0$. Indeed, from Remark 3.2.23, we know that the map

$$x \longmapsto \psi(x)$$

is Gâteaux differentiable at every $x \neq 0$. Also Proposition 3.2.25 says that the map

$$x \longmapsto \psi'(x) = \frac{\mathcal{F}(x)}{\|x\|_X}$$

is continuous on $X \setminus \{0\}$. So ψ is Fréchet differentiable on $X \setminus \{0\}$.

Combining Propositions 3.2.24 and 3.2.25, we obtain the following proposition.

PROPOSITION 3.2.27
If X is a reflexive Banach space and both X and X^* are locally uniformly convex (see Definition A.3.21),
then the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is a homeomorphism.

PROPOSITION 3.2.28
If X is a reflexive Banach space and X^* is uniformly convex,
then the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is uniformly continuous on bounded subsets of X.

PROOF First we show that $\mathcal{F}$ is uniformly continuous on

$$\partial B_1 \stackrel{df}{=} \big\{x \in X : \ \|x\|_X = 1\big\}.$$

If this is not the case, then we can find $\varepsilon > 0$ and two sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subseteq \partial B_1$, such that

$$\|x_n - y_n\|_X \ \longrightarrow \ 0$$

and

$$\big\|\mathcal{F}(x_n) - \mathcal{F}(y_n)\big\|_{X^*} \ \geqslant \ \varepsilon \qquad \forall\, n \geqslant 1.$$

We have

$$\begin{aligned} &\big\|\mathcal{F}(x) + \mathcal{F}(y)\big\|_{X^*} \|x\|_X \ \geqslant \ \big\langle \mathcal{F}(x) + \mathcal{F}(y), x\big\rangle_X \\ = \ &\big\langle \mathcal{F}(x), x\big\rangle_X + \big\langle \mathcal{F}(y), y\big\rangle_X + \big\langle \mathcal{F}(y), x - y\big\rangle_X \\ \geqslant \ &\|x\|_X^2 + \|y\|_X^2 - \|y\|_X \|x - y\|_X \qquad \forall\, x, y \in X. \end{aligned}$$

Putting

$$x \ = \ x_n, \quad y \ = \ y_n \qquad \forall\, n \geqslant 1,$$

we obtain

$$\frac{1}{2}\big\|\mathcal{F}(x_n) + \mathcal{F}(y_n)\big\|_{X^*} \ = \ 1 - \frac{1}{2}\|x_n - y_n\|_X\,,$$

which contradicts the uniform convexity of X^*.

Recall that

$$\mathcal{F}(\lambda u) \ = \ \lambda \mathcal{F}(u) \qquad \forall\, \lambda > 0,\ u \in X.$$

For $x, y \in X \setminus \{0\}$, we have

$$\begin{aligned} \big\|\mathcal{F}(x) - \mathcal{F}(y)\big\|_{X^*} \ &= \ \left\| \|x\|_X \mathcal{F}\left(\frac{x}{\|x\|_X}\right) - \|y\|_X \mathcal{F}\left(\frac{y}{\|y\|_X}\right)\right\|_{X^*} \\ \leqslant \ \|x\|_X &\left\|\mathcal{F}\left(\frac{x}{\|x\|_X}\right) - \mathcal{F}\left(\frac{y}{\|y\|_X}\right)\right\|_{X^*} + \|x - y\|_X \left\|\mathcal{F}\left(\frac{y}{\|y\|_X}\right)\right\|_{X^*}. \end{aligned}$$

From the uniform continuity of $\mathcal{F}$ on ∂B_1, it follows that $\mathcal{F}$ is uniformly continuous on bounded sets located outside some neighbourhood of the origin. Since $\mathcal{F}$ is continuous at $x = 0$ and $\mathcal{F}(0) = 0$, we conclude that $\mathcal{F}$ is uniformly continuous on bounded sets. □

Using the duality map, we can have a necessary and sufficient condition for the maximality of a monotone operator A.

THEOREM 3.2.29

If both X and X^* are strictly convex and $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a monotone map,

then A is maximal monotone if and only if $R(A + \lambda\mathcal{F}) = X^*$ for all $\lambda > 0$ (equivalently for some $\lambda > 0$).

Theorem 3.2.29 is also a surjectivity result. One of the reasons that maximal monotone operators are important in applications is their remarkable surjectivity properties. We start with a necessary and sufficient condition in order to have surjectivity.

THEOREM 3.2.30
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is a monotone map,*
then $R(A) = X^*$ *if and only if* A^{-1} *is locally bounded.*

PROOF "$\Longrightarrow$": Since A is maximal monotone, so is A^{-1}. Because

$$D(A^{-1}) \;=\; R(A) \;=\; X^*,$$

from Proposition 3.2.11, we have that A^{-1} is locally bounded.

"$\Longleftarrow$": To show that $R(A) = X^*$, it suffices to show that $R(A)$ is both closed and open in X^*. First we show that $R(A)$ is closed. To this end let $\{x_n^*\}_{n\geqslant 1} \subseteq R(A)$ and suppose that

$$x_n^* \;\longrightarrow\; x^* \quad \text{in } X^*.$$

We have

$$x_n^* \;\in\; A(x_n)$$

and from the monotonicity of A, it follows that

$$\left\langle x_n^* - y^*, x_n - y\right\rangle_X \;\geqslant\; 0 \qquad \forall\, n \geqslant 1,\ (y, y^*) \in \operatorname{Gr} A. \tag{3.17}$$

Since by hypothesis A^{-1} is locally bounded, the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that

$$x_n \;\xrightarrow{w}\; x \quad \text{in } X.$$

Passing to the limit as $n \to +\infty$ in (3.17), we obtain

$$\left\langle x^* - y^*, x - y\right\rangle_X \;\geqslant\; 0 \qquad \forall\, (y, y^*) \in \operatorname{Gr} A,$$

so $x^* \in A(x)$ (since A is maximal monotone). Therefore $x^* \in R(A)$ and so we have proved that $R(A)$ is closed.

Next we show that $R(A)$ is open in X^*. Let $x^* \in R(A)$. We have

$$x^* \;\in\; A(x).$$

By considering if necessary

$$\widehat{A}(y) \;=\; A(y + x)$$

(maximal monotonicity is invariant under translation), we may assume that $x = 0$. Let $r > 0$ be such that $A^{-1}|_{B_r(x^*)}$ is bounded, where

$$B_r(x^*) \stackrel{df}{=} \{y^* \in X^* : \|y^* - x^*\|_X < r\}.$$

By Troyanski's renorming theorem (see Theorem A.3.23), we may assume without any loss of generality, that both X and X^* are locally uniformly convex. Let $y^* \in B_{\frac{r}{2}}(x^*)$. Then by Theorem 3.2.29, for every $\lambda > 0$, the operator equation

$$x^*_\lambda + \lambda \mathcal{F}(x_\lambda) = y^*, \quad x^*_\lambda \in A(x_\lambda) \tag{3.18}$$

has a solution x_λ. Because A is maximal monotone, we have

$$\langle y^* - \lambda \mathcal{F}(x_\lambda) - x^*, x_\lambda \rangle_X \geqslant 0 \qquad \forall\, \lambda > 0$$

(recall that $x = 0$), so

$$\|y^* - x^*\|_{X^*} \|x_\lambda\|_X \geqslant \lambda \|x_\lambda\|^2_X$$

and thus

$$\lambda \|x_\lambda\|_X < \frac{r}{2} \qquad \forall\, \lambda > 0.$$

From (3.18), we see that

$$\|y^* - x^*_\lambda\|_{X^*} = \lambda \|\mathcal{F}(x_\lambda)\|_{X^*} = \lambda \|x_\lambda\|_X < \frac{r}{2} \qquad \forall\, \lambda > 0, \tag{3.19}$$

so

$$\|x^*_\lambda - x^*\|_{X^*} < r \qquad \forall\, \lambda > 0.$$

Recall that $A^{-1}|_{B_r(x^*)}$ is bounded. So we have that $\{x_\lambda\}_{\lambda > 0} \subseteq X$ is bounded. Using this in (3.19), we see that

$$x^*_\lambda \longrightarrow y^* \quad \text{in } X^* \quad \text{as } \lambda \searrow 0.$$

Since from the first part of the proof of this implication, we have that $R(A)$ is closed, it follows that $y^* \in R(A)$ and so

$$B_{\frac{r}{2}} \subseteq R(A),$$

which proves that $R(A)$ is open in X^*. Thus we conclude that $R(A) = X^*$. □

COROLLARY 3.2.31
If $A: X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone and weakly coercive, then A is surjective (i.e., $R(A) = X^*$).

PROOF By Troyanski's renorming theorem (see Theorem A.3.23), we may assume that both X and X^* are locally uniformly convex. The weak coercivity condition is equivalent to saying that A^{-1} is locally bounded. So we can apply Theorem 3.2.30 and conclude that $R(A) = X^*$. □

COROLLARY 3.2.32
If $A\colon X \longrightarrow 2^{X^*}$ *is monotone, hemicontinuous with* $D(A) = X$ *and weakly coercive,*
then A *is surjective (i.e.,* $R(A) = X^*$*).*

In a finite dimensional context we can drop the monotonicity hypothesis from the above result, provided we assume coercivity. Namely, we have the following result.

PROPOSITION 3.2.33
If X *is a finite dimensional Banach space,* $F\colon X \longrightarrow 2^{X^*}$ *is an upper semicontinuous and coercive multifunction with nonempty, compact and convex values,*
then F *is surjective.*

PROOF For every $y^* \in X^*$, the multifunction

$$F_{y^*}(x) \stackrel{df}{=} F(x) - y^*$$

satisfies the same hypotheses as F. So it suffices to show that $0 \in R(F)$.

Suppose that

$$0 \notin R(F).$$

Then by the strong separation theorem (see Theorem A.3.2), for every $x \in X$ we can find $u(x) \in X \setminus \{0\}$, such that

$$0 < \inf_{x^* \in F(x)} \langle x^*, u(x) \rangle_X.$$

Since by hypothesis F is coercive, given $M > 0$ we can find $r = r(M) > 0$, such that

$$\frac{\langle x^*, x \rangle_X}{\|x\|_X} \geqslant M \qquad \forall\ \|x\|_X \geqslant r,\ x^* \in F(x),$$

so

$$\langle x^*, x \rangle_X \geqslant Mr \qquad \forall\ \|x\|_X = r,\ x^* \in F(x).$$

For such $x \in X$, we can take $u(x) = x$. Now let $x \in X \setminus \{0\}$. We define

$$U(x) \stackrel{df}{=} \left\{ y \in X : \inf_{y^* \in F(y)} \langle y^*, x \rangle_X > 0 \right\}.$$

Because of our hypotheses on F,

$$\text{the map } y \longmapsto \inf_{y^* \in F(y)} \langle y^*, x \rangle_X \text{ is lower semicontinuous}$$

(see Hu & Papageorgiou (1997, p. 83)) and so the set $U(x)$ is open. From the above, we have that $\{U(x)\}_{x \in X \setminus \{0\}}$ is an open cover of X.

We choose

$$\text{an open cover } \{V_k\}_{k=1}^m \text{ of } \overline{B}_r \stackrel{df}{=} \big(x \in X : \ \|x\|_X \leqslant r\big),$$

such that for each $k \in \{1, \dots, m\}$, we can find $x_k \in X$, such that

$$V_k \subseteq U(x_k)$$

and

$$\text{if } V_k \cap \partial B_r \neq \emptyset, \quad \text{then } x_k \in V_k \cap \partial B_r \quad \text{and} \quad \operatorname{diam} V_k \ < \ \frac{r}{2}.$$

Also we choose a continuous partition of unity $\{\vartheta_k\}_{k=1}^m$ subordinate to $\{V_k\}_{k=1}^m$. Then we define the function

$$f \colon \overline{B}_r \longrightarrow X,$$

by setting

$$f(x) \ \stackrel{df}{=} \ \sum_{k=1}^m \vartheta_k(x) x_k.$$

Evidently f is continuous and for each $k \geqslant 1$, such that $\vartheta_k(x) > 0$ and for each $x^* \in F(x)$, we have

$$\langle x^*, x_k \rangle_X \ > \ 0$$

(since $x \in V_k \subseteq U(x_k)$). So for each $x \in \overline{B}_r$ and any $x^* \in F(x)$, we have

$$\big\langle x^*, f(x) \big\rangle_X \ = \ \sum_{k=1}^m \vartheta_k(x) \, \langle x^*, x_k \rangle_X \ > \ 0,$$

hence

$$f(x) \ \neq \ 0 \qquad \forall \ x \in \overline{B}_r.$$

Therefore

$$d_B(f, B_r, 0) \ = \ 0,$$

where $d_B(f, B_r, 0)$ denotes the Brouwer degree of f on B_r with respect to 0.

On the other hand, if $x \in \partial B_r$, $f(x)$ is a convex combination of the points $\{x_k\}_{k=1}^m \subseteq \partial B_r$ and

$$\|x_k - x\|_X \ < \ \frac{r}{2} \qquad \forall \ k \in \{1, \dots, m\}.$$

So it follows that

$$\big\|f(x) - x\big\|_X \ \leqslant \ \frac{r}{2} \qquad \forall \ x \in \partial B_r.$$

Thus f is homotopic to the identity and so

$$d_B(f, B_r, 0) \ = \ 1,$$

a contradiction. This proves that $0 \in R(F)$ and so F is surjective. ∎

Also the solution set of the operator inclusion $y^* \in A(x)$, with

$$A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$$

being maximal monotone, has nice properties.

PROPOSITION 3.2.34
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone map, then for every $y^* \in X^*$, the set $A^{-1}(y^*)$ is closed and convex.

PROOF Since A is maximal monotone, then so is A^{-1}. Then from Proposition 3.2.7, we know that for all $y^* \in X^*$, the set $A^{-1}(y^*)$ is closed and convex and it is nonempty if $y^* \in R(A) = D(A^{-1})$. □

Now note that weak coercivity of A implies that A^{-1} is locally bounded. So as an immediate consequence of Proposition 3.2.34, we obtain the following structural result for the solution set of operator inclusions with maximal monotone and weakly coercive maps.

COROLLARY 3.2.35
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is a maximal monotone and weakly coercive map, then for every $y^* \in X^*$, we have that $A^{-1}(y^*)$ is convex and weakly compact.

Recall that when H is a pivot Hilbert space, then the duality map $\mathcal{F}$ equals the identity operator. Using this fact, for a given maximal monotone map $A\colon H \supseteq D(A) \longrightarrow 2^H$, we can define some single-valued regular approximation of A. More precisely, we make the following definition.

DEFINITION 3.2.36 *Let H be a pivot Hilbert space (i.e., $H = H^*$) and let*

$$A\colon H \supseteq D(A) \longrightarrow 2^H$$

*be a maximal monotone map. For every $\lambda > 0$, we define the **resolvent operator** of A, by*

$$J_\lambda \stackrel{df}{=} \left(id_{_H} + \lambda A\right)^{-1}$$

*and the **Yosida approximation** of A, by*

$$A_\lambda \stackrel{df}{=} \frac{1}{\lambda}\left(id_{_H} - J_\lambda\right).$$

REMARK 3.2.37 By virtue of Theorem 3.2.29, we have that

$$D(J_\lambda) \;=\; D(A_\lambda) \;=\; H \qquad \forall\, \lambda > 0.$$

Moreover, it is easy to check that J_λ is single-valued and then so is A_λ.

The next theorem summarizes the main properties of the resolvent and Yosida approximation of a maximal monotone operator A.

THEOREM 3.2.38
If H is a pivot Hilbert space and $A\colon H \supseteq D(A) \longrightarrow 2^H$ is a maximal monotone map,
then for every $\lambda > 0$, we have

(a) *J_λ is nonexpansive (i.e., Lipschitz continuous with Lipschitz constant 1);*

(b) *$A_\lambda(x) \in A\big(J_\lambda(x)\big)$ for every $x \in H$;*

(c) *A_λ is monotone and Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$;*

(d) *$\big\|A_\lambda(x)\big\|_H \leqslant \big\|A^0(x)\big\|_H$ for every $x \in D(A)$, where $A^0(x) = \operatorname{proj}_{A(x)}(0)$ (recall that $A(x)$ is closed and convex; see Proposition 3.2.7 and Example 3.2.20(a));*

(e) *$\lim\limits_{\lambda \searrow 0} A_\lambda(x) = A^0(x)$ for all $x \in D(A)$;*

(f) *$\overline{D(A)}$ is convex and $\lim\limits_{\lambda \searrow 0} J_\lambda(x) = \operatorname{proj}_{\overline{D}}(x)$ for every $x \in H$.*

PROOF **(a)** For $x, y \in H$, we have

$$x - y \;\in\; J_\lambda(x) - J_\lambda(y) + \lambda\big(A\big(J_\lambda(x)\big) - A\big(J_\lambda(y)\big)\big).$$

We take the inner product with $J_\lambda(x) - J_\lambda(y)$ and use the monotonicity of A. We have

$$\begin{aligned}\big\|J_\lambda(x) - J_\lambda(y)\big\|_H^2 &\leqslant \big\langle x - y, J_\lambda(x) - J_\lambda(y)\big\rangle_H \\ &\leqslant \|x - y\|_H \big\|J_\lambda(x) - J_\lambda(y)\big\|_H,\end{aligned}$$

so

$$\big\|J_\lambda(x) - J_\lambda(y)\big\|_H \;\leqslant\; \|x - y\|_H .$$

(b) This follows from the equivalence

$$\big(x, x^*\big) \in \operatorname{Gr} A_\lambda \quad\Longleftrightarrow\quad \big(x - \lambda x^*, x^*\big) \in \operatorname{Gr} A. \tag{3.20}$$

(c) Because J_λ is nonexpansive (see **(a)**), it follows that $id_H - J_\lambda$ is monotone (see Example 3.2.20**(b)**) and so A_λ is monotone too. We have

$$x - y \;=\; J_\lambda(x) - J_\lambda(y) + \lambda\big(A_\lambda(x) - A_\lambda(y)\big),$$

and

$$\big\langle x-y, A_\lambda(x)-A_\lambda(y)\big\rangle_H \;=\; \big\langle J_\lambda(x)-J_\lambda(y), A_\lambda(x)-A_\lambda(y)\big\rangle_H + \lambda\big\|A_\lambda(x)-A_\lambda(y)\big\|_H^2.$$

From the monotonicity of A and **(b)**, we have

$$\lambda\big\|A_\lambda(x)-A_\lambda(y)\big\|_H^2 \;\leqslant\; \|x-y\|_H\,\big\|A_\lambda(x)-A_\lambda(y)\big\|_H,$$

so

$$\big\|A_\lambda(x)-A_\lambda(y)\big\|_H \;\leqslant\; \frac{1}{\lambda}\,\|x-y\|_H\,.$$

Invoking Proposition 3.2.19, we conclude that A is maximal monotone.

(d) From **(b)**, we have that

$$\big\langle A^0(x)-A_\lambda(x), x-J_\lambda(x)\big\rangle_H \;\geqslant\; 0 \qquad \forall\, x\in D(A),\ \lambda>0,$$

so

$$\big\|A_\lambda(x)\big\|_H^2 \;\leqslant\; \big\langle A^0(x), A_\lambda(x)\big\rangle_H \;\leqslant\; \big\|A^0(x)\big\|_H\big\|A_\lambda(x)\big\|_H \tag{3.21}$$

and thus

$$\big\|A_\lambda(x)\big\|_H \leqslant\; \big\|A^0(x)\big\|_H.$$

(e) Using (3.20), we can easily verify that

$$(A_\lambda)_\mu \;=\; A_{\lambda+\mu} \qquad \forall\, \lambda,\mu>0.$$

So from **(d)** and (3.21), we see that

$$\big\|A_{\lambda+\mu}(x)\big\|_H \;\leqslant\; \big\|A_\lambda(x)\big\|_H \qquad \forall\, x\in H,\ \lambda,\mu>0 \tag{3.22}$$

and

$$\big\|A_{\lambda+\mu}(x)\big\|_H^2 \;\leqslant\; \big\langle A_\lambda(x), A_{\lambda+\mu}(x)\big\rangle_H \qquad \forall\, x\in H,\ \lambda,\mu>0. \tag{3.23}$$

From (3.22) and (3.23), it follows that

$$\big\|A_{\lambda+\mu}(x)-A_\lambda(x)\big\|_H^2 \;\leqslant\; \big\|A_\lambda(x)\big\|_H^2-\big\|A_{\lambda+\mu}(x)\big\|_H^2 \qquad \forall\, x\in H,\ \lambda,\mu>0.$$

Therefore, since from **(d)**, $\big\{\big\|A_\lambda(x)\big\|_H\big\}_{\lambda>0}$ is bounded for $\lambda>0$ small enough, we infer that $\big\{A_\lambda(x)\big\}_{\lambda>0}$ is Cauchy and so

$$A_\lambda(x) \;\longrightarrow\; y \quad \text{in } H \quad \text{as } \lambda\searrow 0.$$

By definition, we have

$$x-J_\lambda(x) \;=\; \lambda A_\lambda(x)$$

and so

$$J_\lambda(x) \longrightarrow x \quad \text{in } H \quad \text{as } \lambda \searrow 0.$$

Using **(b)** and the maximal monotonicity of A, we have that $y \in A(x)$, hence $y = A^0(x)$.

(f) Let

$$C \stackrel{df}{=} \overline{\operatorname{conv}} D(A)$$

and $x \in H$. We have

$$\left\langle A_\lambda(x) - u, J_\lambda(x) - z \right\rangle_H \geqslant 0 \qquad \forall\, (z,u) \in \operatorname{Gr} A,$$

so

$$\left\langle x - J_\lambda(x) - \lambda u, J_\lambda(x) - z \right\rangle_H \geqslant 0 \qquad \forall\, (z,u) \in \operatorname{Gr} A$$

and thus

$$\left\|J_\lambda(x)\right\|_H^2 \leqslant \left\langle x - \lambda u, J_\lambda(x) - z \right\rangle_H + \left\langle J_\lambda(x), z \right\rangle_H \qquad \forall\, (z,u) \in \operatorname{Gr} A. \tag{3.24}$$

From (3.24), it follows that $\left\{J_\lambda(x)\right\}_{\lambda>0} \subseteq H$ is bounded. We choose a sequence $\lambda_n \searrow 0$, such that

$$J_{\lambda_n}(x) \stackrel{w}{\longrightarrow} v \quad \text{in } H.$$

Then by passing to the limit in (3.24) (with $\lambda = \lambda_n$), we obtain

$$\|v\|_H^2 \leqslant \langle x, v - z \rangle_H + \langle v, z \rangle_H \qquad \forall\, z \in D(A),$$

so

$$\langle v - x, v - z \rangle_H \leqslant 0 \qquad \forall\, z \in D(A)$$

and so

$$\langle x - v, z - v \rangle_H \leqslant 0 \qquad \forall\, z \in C. \tag{3.25}$$

Since $v \in C$, from (3.25), it follows that $v = \operatorname{proj}_C(x)$. It remains to show that

$$C = \overline{D(A)}.$$

To this end, note that

$$J_\lambda(x) \in D(A) \qquad \forall\, x \in H$$

and

$$J_\lambda(z) \longrightarrow z \quad \text{as } \lambda \searrow 0 \qquad \forall\, z \in C.$$

Therefore, it follows that $C \subseteq \overline{D(A)}$, hence $C = \overline{D(A)}$. ∎

REMARK 3.2.39 From the proof of **(e)**, it follows that if $x \notin D$, then

$$\big\|A_\lambda(x)\big\|_H \nearrow +\infty \quad \text{as } \lambda \searrow 0.$$

Perturbation results for maximal monotone operators play an important role in applications. In this direction we have the following result.

THEOREM 3.2.40
If H is a pivot Hilbert space, $A\colon H \supseteq D(A) \longrightarrow 2^H$ and $B\colon H \supseteq D(B) \longrightarrow 2^H$ are two maximal monotone maps and $0 \in \operatorname{int}\big(D(A) \setminus D(B)\big)$, then

$$A + B\colon H \supseteq D(A) \cap D(B) \longrightarrow 2^H$$

is maximal monotone too.

PROOF We start by showing that if $S\colon H \longrightarrow H$ is a Lipschitz continuous map with Lipschitz constant $k_S > 0$, then the map

$$A + S\colon H \supseteq D(A) \longrightarrow 2^H$$

is maximal monotone. To this end, we choose $\mu > 0$, such that $\mu k_S < 1$. Then for a given $y \in H$, the equation

$$x + \mu A(x) + \mu S(x) \ni y \tag{3.26}$$

is equivalent to

$$x = \big(id_H + \mu A\big)^{-1}\big(y - \mu S(x)\big) = J_\mu\big(y - \mu S(x)\big). \tag{3.27}$$

Note that the map

$$z \longmapsto E_\mu(z) = J_\mu\big(y - \mu S(x)\big)$$

is Lipschitz continuous with Lipschitz constant $\mu k_S < 1$. So by the Banach fixed point theorem (see Theorem 7.1.2), equation (3.27) (hence inclusion (3.26) too) has a unique solution $x \in D(A)$. By virtue of Theorem 3.2.29, this means that $\mu(A + S)$ is maximal monotone (note that since H is a pivot Hilbert space, $\mathcal{F} = id_H$). So $A + S$ is maximal monotone too.

Using this general fact and Theorems 3.2.29 and 3.2.38**(c)**, we see that for every $\lambda > 0$ we can find $x_\lambda \in D(A)$, such that

$$x_\lambda + A(x_\lambda) + B_\lambda(x_\lambda) \ni y. \tag{3.28}$$

We take the inner product with $x_\lambda - z$, for some $z \in D(A) \cap D(B)$. Exploiting the monotonicity of A and B_λ, we obtain

$$\|x_\lambda - z\|_H \leqslant \big\|y - z - A^0(z) - B_\lambda(z)\big\|_H \qquad \forall\ \lambda > 0.$$

Using also Theorem 3.2.38**(d)**, we get

$$\|x_\lambda\|_H \leqslant 2\|z\|_H + \|y\|_H + \|A^0(z)\|_H + \|B^0(z)\|_H \qquad \forall\, \lambda > 0. \tag{3.29}$$

Because of our hypothesis concerning the domains $D(A)$ and $D(B)$, we can find $\varepsilon > 0$, such that

$$\overline{B}_\varepsilon \stackrel{df}{=} \{z \in H : \|z\|_H \leqslant \varepsilon\} \subseteq D(B) - D(A).$$

Let $z \in \overline{B}_\varepsilon$. Then

$$z = b - a \quad \text{with } b \in D(B) \quad \text{and} \quad a \in D(A).$$

Exploiting the monotonicity of B_λ, we have

$$\langle B_\lambda(x_\lambda), b)\rangle_H \leqslant \langle B_\lambda(x_\lambda), x_\lambda\rangle_H - \langle B_\lambda(b), x_\lambda - b\rangle_H.$$

From Theorem 3.2.38**(d)**, we have

$$\langle B_\lambda(x_\lambda), z\rangle_H \leqslant \langle B_\lambda(x_\lambda), x_\lambda - a\rangle_H + \|B^0(b)\|_H \|x_\lambda - b\|_H,$$

so

$$\langle B_\lambda(x_\lambda), z\rangle_H \leqslant \langle y - x_\lambda - u_\lambda, x_\lambda - a\rangle_H + \|B^0(b)\|_H \|x_\lambda - b\|_H,$$

with $u_\lambda \in A(x_\lambda)$. As A is monotone, we have

$$\begin{aligned}\langle B_\lambda(x_\lambda), z\rangle_H \leqslant\ & \|x_\lambda - a\|_H \left(\|y - x_\lambda\|_H + \|A^0(a)\|_H\right) \\ & + \|B^0(b)\|_H \|x_\lambda - b\|_H .\end{aligned}$$

From (3.29), we have that $\{x_\lambda\}_{\lambda>0}$ is bounded. Thus for every $z \in \overline{B}_\varepsilon$, we can find $c(z) > 0$, such that

$$\langle B_\lambda(x_\lambda), z\rangle_H \leqslant c(z).$$

Invoking the uniform boundedness principle (see Theorem A.3.4), we have

$$\sup_{\lambda>0} \|B_\lambda(x_\lambda)\|_H < +\infty. \tag{3.30}$$

From (3.28), we have

$$x_\lambda - x_\mu \in -A(x_\lambda) + A(x_\mu) - B_\lambda(x_\lambda) + B_\mu(x_\mu) \qquad \forall\, \lambda, \mu > 0,$$

so from the monotonicity of A, we have

$$\|x_\lambda - x_\mu\|_H^2 \leqslant -\langle B_\lambda(x_\lambda) - B_\mu(x_\mu), x_\lambda - x_\mu\rangle_H \qquad \forall\, \lambda, \mu > 0.$$

Invoking also Theorem 3.2.38**(b)**, we obtain

$$\|x_\lambda - x_\mu\|_H^2 \leqslant -\langle B_\lambda(x_\lambda) - B_\mu(x_\mu), \lambda B_\lambda(x_\lambda) - \mu B_\lambda(x_\mu)\rangle_H \qquad \forall\, \lambda, \mu > 0.$$

Using (3.30), we infer that $\{x_\lambda\}_{\lambda>0} \subseteq H$ is Cauchy and so

$$x_\lambda \longrightarrow x \quad \text{as } \lambda \searrow 0. \tag{3.31}$$

Also we can say that

$$B_\lambda(x_\lambda) \xrightarrow{w} v \quad \text{in } H, \quad \text{as } \lambda \searrow 0. \tag{3.32}$$

Note that

$$\|J_\lambda(x_\lambda) - x\|_H \leqslant \lambda \|B_\lambda(x_\lambda)\|_H + \|x_\lambda - x\|_H,$$

so, using (3.30) and (3.31), we have

$$J_\lambda(x_\lambda) \longrightarrow x \quad \text{in } H, \quad \text{as } \lambda \searrow 0.$$

Since B is maximal monotone and

$$B_\lambda(x_\lambda) \in B\big(J_\lambda(x_\lambda)\big)$$

(see Theorem 3.2.38**(b)**), in the limit as $\lambda \searrow 0$, we obtain $(x, v) \in \operatorname{Gr} B$ (see Proposition 3.2.15 and (3.32)). Passing to the limit as $\lambda \searrow 0$ in (3.28) and using the fact that A is maximal monotone, we obtain that $x \in D(A)$ and

$$x + A(x) + B(x) \ni y,$$

so

$$R\big(id_H + A + B\big) = H$$

and finally $A + B$ is maximal monotone (see Theorem 3.2.29). □

For operators from a reflexive Banach space into its dual, we have the following perturbation result.

THEOREM 3.2.41
If X is a reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ and $B\colon X \supseteq D(B) \longrightarrow 2^{X^*}$ are two maximal monotone maps and $D(A) \cap \operatorname{int} D(B) \neq \emptyset$ (or $D(B) \cap \operatorname{int} D(A) \neq \emptyset$),
then $A + B\colon X \supseteq D(A) \cap D(B) \longrightarrow 2^{X^*}$ is maximal monotone.

REMARK 3.2.42 Since $\operatorname{int} D(A) - D(B) \subseteq \operatorname{int}\big(D(A) - D(B)\big)$, we see that the hypothesis of Theorem 3.2.40 is weaker than that of Theorem 3.2.41. Note that the condition $0 \in \operatorname{int}\big(D(A) - D(B)\big)$ may hold even if $\operatorname{int} D(A) = \operatorname{int} D(B) = \emptyset$.

Another useful perturbation result is given in the next theorem.

THEOREM 3.2.43
If H is a pivot Hilbert space, $A\colon H \supseteq D(A) \longrightarrow 2^H$ and $B\colon H \supseteq D(B) \longrightarrow 2^H$ are maximal monotone maps, $D(A) \cap D(B) \neq \emptyset$ and

$$0 \leqslant \langle y, B_\lambda(x) \rangle_H \qquad \forall\ (x,y) \in \operatorname{Gr} A,\ \lambda > 0,$$

then $A + B$ is maximal monotone.

PROOF Let $y \in H$ and consider the inclusion

$$x + A(x) + B_\lambda(x) \ni y. \tag{3.33}$$

From the proof of Theorem 3.2.40, we know that (3.33) has a unique solution

$$x_\lambda \in D(A)$$

and $\{x_\lambda\}_{\lambda>0} \subseteq H$ is bounded. Take the inner product of (3.33) with $B_\lambda(x_\lambda)$ and use the hypothesis, to obtain that

$$\sup_{\lambda>0} \|B_\lambda(x_\lambda)\|_H < +\infty.$$

Then the remainder of the proof goes as the proof of Theorem 3.2.40. □

Next we introduce some generalizations of the concept of monotonicity, which are useful in the study of nonlinear partial differential equations. The mathematical setting remains the same. Namely, X is a reflexive Banach space, X^* is its topological dual and $A\colon X \longrightarrow 2^{X^*}$ is an operator.

DEFINITION 3.2.44 *The map $A\colon X \longrightarrow 2^{X^*}$ is said to be **pseudomonotone**, if*

(a) *the set $A(x)$ is nonempty, convex and weakly compact for all $x \in X$;*

(b) *A is upper semicontinuous from each finite dimensional subspace V of X, into X^* furnished with the weak topology;*

(c) *if $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ are sequences, such that*

$$x_n^* \in A(x_n),$$

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

for some $x \in X$ and

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x \rangle_X \leqslant 0,$$

then for each $y \in X$, we can find $v^*(y) \in A(x)$, such that

$$\langle v^*(y), x - y \rangle_X \leqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - y \rangle_X.$$

To be able to deal with problems in which the nonlinear operators are not everywhere defined and which are not continuous even in a mild sense, we introduce the following notion.

DEFINITION 3.2.45 *A map $A\colon X \longrightarrow 2^{X^*}$ is said to be **generalized pseudomonotone**, if for any sequences $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$, such that*

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

for some $x \in X$ and

$$x_n^* \xrightarrow{w} x^* \quad \text{in } X^*,$$

for some $x^* \in X^*$, with $x_n^* \in A(x_n)$ for $n \geqslant 1$ and

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle_X \leqslant 0,$$

we have that $x^* \in A(x)$ and

$$\langle x_n^*, x^*\rangle_X \longrightarrow \langle x^*, x\rangle_X.$$

An immediate consequence of this definition is the following result.

PROPOSITION 3.2.46
A map $A\colon X \longrightarrow 2^{X^*}$ is generalized pseudomonotone if and only if $A^{-1}\colon X^* \longrightarrow 2^X$ is generalized pseudomonotone.

The class of generalized pseudomonotone maps contains the maximal monotone ones.

PROPOSITION 3.2.47
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is maximal monotone, then A is generalized pseudomonotone.

PROOF Let $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ be two sequences, such that

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

for some $x \in X$ and

$$x_n^* \xrightarrow{w} x^* \quad \text{in } X^*,$$

for some $x^* \in X^*$, with $x_n^* \in A(x_n)$ for $n \geqslant 1$ and

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle_X \leqslant 0.$$

We need to show that $x^* \in A(x)$ and

$$\langle x_n^*, x\rangle_X \longrightarrow \langle x^*, x\rangle_X.$$

Let $(u, u^*) \in \operatorname{Gr} A$. Then since A is monotone, we have

$$0 \leqslant \langle x_n^* - u^*, x_n - u \rangle_X \qquad \forall\, n \geqslant 1. \tag{3.34}$$

Also we have

$$\langle x_n^*, x_n \rangle_X = \langle x_n^* - u^*, x_n - u \rangle_X + \langle x_n^*, u \rangle_X + \langle u^*, x_n \rangle_X - \langle u^*, u \rangle_X.$$

Note that

$$\langle x_n^*, u \rangle_X + \langle u^*, x_n \rangle_X - \langle u^*, u \rangle_X \longrightarrow \langle x^*, u \rangle_X + \langle u^*, x \rangle_X - \langle u^*, u \rangle_X,$$

so from (3.34), we have

$$\langle x^*, x \rangle_X \geqslant \limsup_{n \to +\infty} \langle x_n^*, x_n \rangle_X \geqslant \langle x^*, u \rangle_X + \langle u^*, x \rangle_X - \langle u^*, u \rangle_X$$

and thus

$$\langle x^* - u^*, x - u \rangle_X \geqslant 0.$$

Since $(u, u^*) \in \operatorname{Gr} A$ was arbitrary and A is maximal monotone, it follows that $x^* \in A(x)$. Therefore

$$\langle x_n^* - x^*, x_n - x \rangle_X \geqslant 0 \qquad \forall\, n \geqslant 1,$$

so

$$\liminf_{n \to +\infty} \langle x_n^*, x \rangle_X \geqslant \langle x^*, x \rangle_X$$

and thus

$$\langle x_n^*, x_n \rangle_X \longrightarrow \langle x^*, x \rangle_X,$$

i.e., A is generalized pseudomonotone. □

In fact every pseudomonotone map is generalized pseudomonotone.

THEOREM 3.2.48
If $A\colon X \longrightarrow 2^{X^*}$ is a pseudomonotone map, then A is generalized pseudomonotone

PROOF Suppose that $\{(x_n, x_n^*)\}_{n \geqslant 1} \subseteq \operatorname{Gr} A$ is a sequence, such that

$$(x_n, x_n^*) \overset{w \times w}{\longrightarrow} (x, x^*) \quad \text{in } X \times X^*,$$

for some $(x, x^*) \in X \times X^*$ and

$$\limsup_{n \to +\infty} \langle x_n^*, x_n - x \rangle_X \leqslant 0.$$

By virtue of pseudomonotonicity of A, for every $y \in X$, we can find $v^*(y) \in A(x)$, such that

$$\langle v^*(y), x - y\rangle_X \leqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - y\rangle_X.$$

We may assume that

$$\langle x_n^*, x_n\rangle_X \longrightarrow \xi,$$

for some $\xi \in \mathbb{R}$ and so

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle_X = \xi - \langle x^*, x\rangle_X \leqslant 0. \tag{3.35}$$

Also, we have

$$\xi - \langle x^*, y\rangle_X \geqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - y\rangle_X \geqslant \langle v^*(y), x - y\rangle_X,$$

so, from (3.35), we have

$$\langle x^*, x - y\rangle_X \geqslant \langle v^*(y), x - y\rangle_X \qquad \forall\, y \in X. \tag{3.36}$$

We claim that $x^* \in A(x)$. If this is not the case, then since $A(x)$ is convex and w-compact (see Definition 3.2.44), we can find $u \in X$, such that

$$\langle x^*, u\rangle_X < \inf_{z^*\in A(x)} \langle z^*, u\rangle_X. \tag{3.37}$$

Let $y = x - u$ in (3.36). Then

$$\langle x^*, u\rangle_X \geqslant \langle v^*(y), u\rangle_X, \tag{3.38}$$

with

$$v^*(y) \in A(x).$$

Comparing (3.37) and (3.38), we reach a contradiction. Therefore

$$x^* \in A(x).$$

Finally, if $y = x \in X$, then

$$\liminf_{n\to+\infty} \langle x_n^*, x_n - x\rangle_X \geqslant \langle v^*(x), x - x\rangle_X = 0,$$

so

$$\liminf_{n\to+\infty} \langle x_n^*, x_n\rangle_X \geqslant \langle x^*, x\rangle_X$$

and recalling the choice of the sequences $\{x_n\}_{n\geqslant 1}$ and $\{x_n^*\}_{n\geqslant 1}$, we get

$$\langle x_n^*, x_n\rangle_X \longrightarrow \langle x^*, x\rangle_X.$$

□

There is a converse to this proposition.

PROPOSITION 3.2.49
If $A\colon X \longrightarrow 2^{X^*}$ is a bounded generalized pseudomonotone map and for every $x \in X$, the set $A(x)$ is nonempty, convex and weakly compact, then A is pseudomonotone.

PROOF First we show that if $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ are two sequences, such that $x_n \xrightarrow{w} x$ in X, for some $x \in X$, $x_n^* \in A(x_n)$ for $n \geqslant 1$ and

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle_X \leqslant 0,$$

then for each $u \in X$, we can find $y^*(u) \in A(x)$, such that

$$\langle y^*(u), x - u\rangle_X \leqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - u\rangle_X.$$

Suppose that this is not true. Then we can find $u \in X$, such that

$$\liminf_{n\to+\infty} \langle x_n^*, x_n - u\rangle_X < \inf_{v^*\in A(x)} \langle v^*, x - u\rangle_X.$$

By passing to a suitable subsequence, we may say that

$$\lim_{n\to+\infty} \langle x_n^*, x_n - u\rangle_X < \inf_{v^*\in A(x)} \langle v^*, x - u\rangle_X.$$

Since A is bounded, we have that the sequence $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ is bounded. So by virtue of the Eberlein-Smulian theorem (see Theorem A.3.8), we may assume that

$$x_n^* \xrightarrow{w} x^* \quad \text{in } X^*.$$

Because A is generalized pseudomonotone, it follows that $x^* \in A(x)$ and

$$\langle x_n^*, x_n\rangle_X \longrightarrow \langle x^*, x\rangle_X$$

(see Definition 3.2.45). Therefore

$$\lim_{n\to+\infty} \langle x_n^*, x_n - u\rangle_X = \langle x^*, x - u\rangle_X < \inf_{v^*\in A(x)} \langle v^*, x - u\rangle_X,$$

a contradiction, since $x^* \in A(x)$.

Next we show that A is upper semicontinuous from X into X^* furnished with the weak topology. By virtue of the boundedness of A and of Remark 3.2.13, it suffices to show that if $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ are two sequences, such that

$$x_n \longrightarrow x \quad \text{in } X,$$

for some $x \in X$,

$$x_n^* \xrightarrow{w} x^* \quad \text{in } X^*,$$

for some $x^* \in X^*$ and $x_n^* \in A(x_n)$ for $n \geqslant 1$, then $x^* \in A(x)$. But this follows from the fact that A is generalized pseudomonotone. Thus we have shown that A is pseudomonotone (see Definition 3.2.44). ∎

Combining this result with Propositions 3.2.11 and 3.2.47, we obtain the following corollary.

COROLLARY 3.2.50
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is maximal monotone and* $D(A) = X$, *then* A *is pseudomonotone.*

The class of pseudomonotone maps is invariant under addition of operators.

PROPOSITION 3.2.51
If $A_1, A_2\colon X \longrightarrow 2^{X^*}$ *are two pseudomonotone maps, then* $A_1 + A_2$ *is pseudomonotone too.*

PROOF Evidently for each $x \in X$, the set

$$(A_1 + A_2)(x) \;=\; A_1(x) + A_2(x)$$

is nonempty, convex and weakly compact. Moreover, it is easy to see that the map

$$x \longmapsto (A_1 + A_2)(x)$$

is upper semicontinuous from every finite dimensional subspace of X into X^* equipped with the weak topology.

Next we show that if $\{x_n\}_{n\geqslant 1} \subseteq X$ and $\{x_n^*\}_{n\geqslant 1} \subseteq X^*$ are two sequences, such that

$$x_n \;\xrightarrow{w}\; x \quad \text{in } X,$$

for some $x \in X$, $x_n^* \in (A_1 + A_2)(x_n)$ for $n \geqslant 1$ and

$$\limsup_{n\to+\infty} \left\langle x_n^*, x_n - x \right\rangle_X \;\leqslant\; 0,$$

then for every $u \in X$, we can find

$$y^*(u) \;\in\; (A_1 + A_2)(x),$$

such that

$$\left\langle y^*(u), x - u \right\rangle_X \;\leqslant\; \liminf_{n\to+\infty} \left\langle x_n^*, x_n - u \right\rangle_X.$$

Let

$$x_n^* \;\stackrel{df}{=}\; y_n^* + z_n^* \quad \text{with} \quad y_n^* \in A_1(x_n), \quad z_n^* \in A_2(x_n) \qquad \forall\, n \geqslant 1.$$

Then, we have

$$\limsup_{n\to+\infty} \left[\left\langle y_n^*, x_n - x \right\rangle_X + \left\langle z_n^*, x_n - x \right\rangle_X \right] \;\leqslant\; 0. \tag{3.39}$$

We claim that (3.39) implies

$$\limsup_{n\to+\infty} \left\langle y_n^*, x_n - x \right\rangle_X \leqslant 0 \tag{3.40}$$

and

$$\limsup_{n\to+\infty} \left\langle z_n^*, x_n - x \right\rangle_X \leqslant 0.$$

Suppose that (3.40) is not true. Then at least one of the two lim sup is strictly bigger than zero. To fix things, suppose that

$$\limsup_{n\to+\infty} \left\langle y_n^*, x_n - x \right\rangle_X > 0.$$

Then we can find $c > 0$ and a suitable subsequence (denoted with the same index), such that

$$\lim_{n\to+\infty} \left\langle y_n^*, x_n - x \right\rangle_X > c = 0.$$

Then because of (3.39), we have that

$$\limsup_{n\to+\infty} \left\langle z_n^*, x_n - x \right\rangle_X \leqslant -c < 0. \tag{3.41}$$

By virtue of the pseudomonotonicity of A_2, for every $u \in X$, we can find

$$y_2^*(u) \in A_2(x),$$

such that

$$\left\langle y_2^*(u), x - u \right\rangle_X \leqslant \liminf_{n\to+\infty} \left\langle z_n^*, x_n - u \right\rangle_X.$$

Let $u = x$. Then we have

$$\liminf_{n\to+\infty} \left\langle z_n^*, x_n - x \right\rangle_X \geqslant 0. \tag{3.42}$$

Comparing (3.41) and (3.42), we reach a contradiction. This proves (3.40). Since both A_1 and A_2 are pseudomonotone, given $u \in X$, we can find

$$y_1^*(u) \in A_1(x) \quad \text{and} \quad y_2^*(u) \in A_2(x),$$

such that

$$\left\langle y_1^*(u), x - u \right\rangle_X \leqslant \liminf_{n\to+\infty} \left\langle y_n^*, x - u \right\rangle_X$$

and

$$\left\langle y_2^*(u), x - u \right\rangle_X \leqslant \liminf_{n\to+\infty} \left\langle z_n^*, x - u \right\rangle_X.$$

Let

$$y^*(u) \stackrel{df}{=} y_1^*(u) + y_2^*(u) \in (A_1 + A_2)(x).$$

We have

$$\begin{aligned} \left\langle y^*(u), x - u \right\rangle_X &\leqslant \liminf_{n\to+\infty} \left\langle y_n^*, x_n - u \right\rangle_X + \liminf_{n\to+\infty} \left\langle z_n^*, x_n - u \right\rangle_X \\ &\leqslant \liminf_{n\to+\infty} \left\langle x_n^*(u), x_n - u \right\rangle_X, \end{aligned}$$

which means that $A_1 + A_2$ is pseudomonotone. ∎

As was the case with maximal monotone operators, pseudomonotone operators exhibit remarkable surjectivity properties.

THEOREM 3.2.52
If $A\colon X \longrightarrow 2^{X^*}$ is pseudomonotone and coercive, then $R(A) = X^*$, i.e., A is surjective.

PROOF Let $\mathcal{T}$ be the family of all finite dimensional subspaces of X, equipped with the partial order defined by inclusion. Let $V \in \mathcal{T}$ and let $i_V\colon V \longrightarrow X$ denote the embedding operator. Then

$$i_V^*\colon X^* \longrightarrow V^*$$

is the corresponding projection operator onto V^*. Then

$$A_V \;=\; i_V^* A i_V\colon V \longrightarrow 2^{V^*}.$$

Clearly A_V has nonempty, convex and compact values and it is upper semicontinuous. Moreover, for every $x_V^* \in A_V(x)$, we have

$$x_V^* \;=\; i_V^* x^*$$

for some $x^* \in A(x)$ and so

$$\left\langle x_V^*, x\right\rangle_V \;=\; \left\langle i_V^* x^*, x\right\rangle_V \;=\; \left\langle x^*, i_V(x)\right\rangle_X,$$

so A_V is coercive too.

To prove the theorem, it suffices to show that

$$0 \;\in\; R(A).$$

Because of Proposition 3.2.33, for every $V \in \mathcal{T}$, we can find $x_V \in V$, such that

$$0 \;\in\; A_V(x_V),$$

hence

$$0 \;=\; i_V^* x_V^*,$$

for some $x_V^* \in A(x_V)$. By virtue of the coercivity of A, we have that $\{x_V\}_{V\in\mathcal{T}} \subseteq X$ is bounded.

For $V \in \mathcal{T}$, let

$$E_V \;\stackrel{df}{=}\; \bigcup_{\substack{V' \in \mathcal{T}\\ V' \supseteq V}} \{x_{V'}\}.$$

Then

$$E_V \;\subseteq\; \overline{B}_M$$

for some $M > 0$ large enough. Because X is reflexive (hence $\overline{B}_M$ is weakly compact), from the finite intersection property, we have that

$$\bigcap_{V \in \mathcal{T}} \overline{E}_V^w \neq \emptyset.$$

Let

$$x_0 \in \bigcap_{V \in \mathcal{T}} \overline{E}_V^w \quad \text{and} \quad y \in X.$$

Choose $V \in \mathcal{T}$, such that $\{x_0, y\} \subseteq V$. Let $\{x_{V_k}\}_{k \geqslant 1} \subseteq E_V$ be such that

$$x_{V_k} \xrightarrow{w} x_0 \quad \text{in } X \quad \text{as } k \to +\infty.$$

Recall that $0 = i_{V_k}^* x_{V_k}^*$ with $x_{V_k}^* \in A(x_{V_k})$. So we have

$$\langle x_{V_k}^*, x_{V_k} - x_0 \rangle_X = 0 \qquad \forall\, k \geqslant 1.$$

Since A is pseudomonotone, we can find $y^*(y) \in A(x_0)$, such that

$$\langle y^*(y), x_0 - y \rangle_X \leqslant \lim_{k \to +\infty} \langle x_{V_k}^*, x_{V_k} - y \rangle_X = 0 \qquad \forall\, y \in X. \tag{3.43}$$

Suppose that $0 \notin A(x_0)$. Then by the strong separation theorem (see Theorem A.3.2), we can find $y \in X$, such that

$$0 < \inf_{v^* \in A(x_0)} \langle v^*, x_0 - y \rangle_X. \tag{3.44}$$

Comparing (3.43) and (3.44), we obtain a contradiction. This proves the surjectivity of A. □

The following classes of operators are often useful in nonlinear operator equations.

DEFINITION 3.2.53 *Let*

$$A\colon X \supseteq D(A) \longrightarrow 2^{X^*} \quad \text{and} \quad B\colon X \supseteq D(B) \longrightarrow 2^{X^*}$$

be two maps.

(a) *We say that B is **smooth**, if $D(B) = X$ and B is bounded, coercive and maximal monotone.*

(b) *We say that A is **regular**, if it is generalized pseudomonotone and for every smooth operator $B\colon X \supseteq D(B) \longrightarrow 2^{X^*}$, we have*

$$R(A + B) = X^*.$$

The next proposition gives an important example of a regular generalized pseudomonotone map.

PROPOSITION 3.2.54
If $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *is a pseudomonotone operator and there exists* $c > 0$, *such that*

$$\langle x^*, x\rangle_X \geqslant -c\,\|x\|_X \qquad \forall\ (x, x^*) \in \operatorname{Gr} A,$$

then A *is regular (so also generalized pseudomonotone).*

PROOF First note that A is a generalized pseudomonotone operator (see Theorem 3.2.48). Also let $B\colon X \supseteq D(B) \longrightarrow 2^{X^*}$ be an arbitrary smooth map. Then from Corollary 3.2.50, B is pseudomonotone. Then Proposition 3.2.51 implies that $A+B$ is pseudomonotone. Moreover, $A+B$ is coercive. So Theorem 3.2.52 implies that

$$R(A+B) = X^*,$$

hence A is regular. □

We introduce two more classes of nonlinear operators of monotone type.

DEFINITION 3.2.55 *Let* $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ *be a map.*

(a) *We say that* A *is* ***of type*** (M), *if for every* $x \in X$, *the set* $A(x)$ *is nonempty, convex, weakly compact, it is upper semicontinuous from every finite dimensional subspace* V *of* X *into* X^* *furnished with the weak topology and if*

$$x_n \xrightarrow{w} x \quad \text{in } X,$$
$$x_n^* \xrightarrow{w} x^* \quad \text{in } X^*$$

with $(x_n, x_n^*) \in \operatorname{Gr} A$, *then*

$$(x, x^*) \in \operatorname{Gr} A.$$

(b) *We say that* A *is of* ***type*** $(S)_+$, *if* A *is single valued with* $D(A) = X$ *and for every sequence* $\{x_n\}_{n\geqslant 1} \subseteq X$, *such that*

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

for some $x \in X$ *and*

$$\limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle_X \leqslant 0,$$

we have that

$$x_n \longrightarrow x \quad \text{in } X.$$

REMARK 3.2.56 The prototype for operators of type (M) is a monotone, hemicontinuous map. Similarly the prototype for operators of type $(S)_+$ is a uniformly monotone operator defined everywhere. If A is of type (M) (respectively of type $(S)_+$) and $B\colon X \longrightarrow X^*$ is completely continuous, then $A+B$ is of type (M) (respectively of type $(S)_+$). In fact in the $(S)_+$ case, B can be compact.

We close this section by briefly discussing two important examples of maximal monotone maps.

PROPOSITION 3.2.57
If X is a separable, reflexive Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is maximal monotone with $0 \in A(0)$ and $\widehat{A}\colon L^p(T;X) \supseteq \widehat{D} \longrightarrow 2^{L^{p'}(T;X^*)}$, where $T=[0,b]$, $p \in (1,+\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$ is defined by

$$\widehat{A}(x) \stackrel{df}{=} \left\{ h \in L^{p'}\big(T;X^*\big) : \ h(t) \in A\big(x(t)\big) \quad \text{for a.a. } t \in T \right\} \qquad \forall\, x \in \widehat{D},$$

where

$$\widehat{D} \stackrel{df}{=} \Big\{ x \in L^p(T;X) : \ x(t) \in D(A) \text{ for a.a. } t \in T \text{ and there exists } h \in L^{p'}\big(T;X^*\big) \text{ such that } h(t) \in A\big(x(t)\big) \text{ for a.a. } t \in T \Big\},$$

then $\widehat{A}$ is maximal monotone too.

PROOF By Troyanski's renorming theorem (see Theorem A.3.23), we may assume without any loss of generality that both X and X^* are locally uniformly convex (see Definition A.3.21). Let

$$\mathcal{F}\colon X \longrightarrow X^*$$

be the duality map of X. From Proposition 3.2.27, we have that $\mathcal{F}$ is a homeomorphism. Let

$$J_0\colon L^p(T;X) \longrightarrow 2^{L^{p'}(T;X^*)}$$

be defined by

$$J_0(x)(\cdot) \stackrel{df}{=} \big\|\mathcal{F}\big(x(\cdot)\big)\big\|_{X^*}^{p-2}\mathcal{F}\big(x(\cdot)\big).$$

It is easy to see that J_0 is continuous, strictly monotone and so maximal monotone too (see Proposition 3.2.18). Clearly $\widehat{A}$ is a monotone map. We claim that

$$R\big(\widehat{A}+J_0\big) \ = \ L^{p'}\big(T;X^*\big).$$

To this end let $h \in L^{p'}\big(T; X^*\big)$ and consider the multifunction $S\colon T \longrightarrow 2^{X^*}$, defined by

$$S(t) \stackrel{df}{=} \big\{x \in X :\ A(x) + \varphi(x) \ni h(t)\big\},$$

where $\varphi\colon X \longrightarrow X^*$ is the monotone, continuous (hence maximal monotone) map, defined by

$$\varphi(x) \stackrel{df}{=} \big\|\mathcal{F}(x)\big\|_{X^*}^{p-2}\mathcal{F}(x) \qquad \forall\ x \in X.$$

We know that

$$A + \varphi\colon X \supseteq D(A) \longrightarrow 2^{X^*} \text{ is maximal monotone}$$

(see Theorem 3.2.41). Moreover, because $0 \in A(0)$, we have

$$\big\langle x^* + \varphi(x), x\big\rangle_X \geqslant \big\langle \varphi(x), x\big\rangle_X = \big\|\mathcal{F}(x)\big\|_{X^*}^p = \|x\|_X^p \qquad \forall\ (x, x^*) \in \operatorname{Gr} A,$$

hence $A + \varphi$ is coercive. Therefore

$$R(A + \varphi) = X^*$$

(see Corollary 3.2.31). It follows that

$$S(t) \neq \emptyset \qquad \forall\ t \in T.$$

Note that

$$\operatorname{Gr} S = \big\{(t, x) \in T \times X :\ \big(x, \varphi(x) - h(t)\big) \in \operatorname{Gr} A\big\}.$$

Let

$$\vartheta\colon T \times X \longrightarrow X \times X^*$$

be the function, defined by

$$\vartheta(t, x) \stackrel{df}{=} \big(x, \varphi(x) - \xi(t)\big).$$

Clearly ϑ is a Carathéodory function (i.e., $t \longmapsto \vartheta(t, x)$ is measurable and $x \longmapsto \vartheta(t, x)$ is continuous). Therefore ϑ is jointly measurable. Note that

$$\operatorname{Gr} S = \vartheta^{-1}\big(\operatorname{Gr} A\big)$$

and from Proposition 3.2.15, we know that

$$\operatorname{Gr} A \subseteq X \times X_w^* \text{ is closed}$$

(here by X_w^* we denote the space X^* furnished with the weak topology). Hence

$$\operatorname{Gr} S \in \mathcal{B}\big(X \times X_w^*\big)$$

(by $\mathcal{B}(Z)$ we denote the Borel σ-field of Z). Since X^*_w is a Souslin space (see Definition A.2.29**(b)**), then

$$\mathcal{B}\big(X \times X^*_w\big) \;=\; \mathcal{B}(X) \times \mathcal{B}\big(X^*_w\big)$$

(see Proposition A.2.34**(b)**). Moreover,

$$\mathcal{B}\big(X^*\big) \;=\; \mathcal{B}\big(X^*_w\big).$$

Therefore

$$\mathcal{B}(X) \times \mathcal{B}\big(X^*_w\big) \;=\; \mathcal{B}(X) \times \mathcal{B}\big(X^*\big) \;=\; \mathcal{B}\big(X \times X^*\big).$$

So we have that

$$\operatorname{Gr} S \;\in\; \mathcal{B}\big(X \times X^*\big)$$

and we can apply the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33) to obtain a measurable map $x\colon T \longrightarrow X$, such that

$$x(t) \;\in\; S(t) \qquad \forall\, t \in T.$$

We have

$$h(t) \;\in\; A\big(x(t)\big) + \varphi\big(x(t)\big) \quad \text{for a.a. } t \in T.$$

Taking duality brackets with $x(t)$, we obtain

$$\big\|x(t)\big\|^p_X \;\leqslant\; \big\langle h(t), x(t)\big\rangle_X \qquad \forall\, t \in T$$

(recall that $0 \in A(0)$) and so

$$\big\|x(t)\big\|^{p-1}_X \;\leqslant\; \big\|h(t)\big\|_{X^*},$$

from which it follows that $x \in L^p(T;X)$. This proves that

$$R\big(\widehat{A} + J_0\big) \;=\; L^{p'}\big(T;X^*\big).$$

Using this surjectivity property we shall establish the maximal monotonicity of the monotone operator $\widehat{A}$. To this end suppose that for some

$$(y,v) \;\in\; L^p(T;X) \times L^{p'}\big(T;X^*\big),$$

we have

$$\big\langle u - v, x - y\big\rangle_{pp'} \;\geqslant\; 0 \qquad \forall\, (x,u) \in \operatorname{Gr} \widehat{A}, \tag{3.45}$$

where by $\langle\cdot,\cdot\rangle_{pp'}$ we denote the duality brackets for the pair of spaces $\big(L^{p'}\big(T;X^*\big), L^p(T;X)\big)$, i.e.,

$$\langle u, v\rangle_{pp'} \;\stackrel{df}{=}\; \int_0^b \big\langle u(t), v(t)\big\rangle_X \, dt \qquad \forall\, (u,v) \in L^p(T;X) \times L^{p'}\big(T;X^*\big).$$

Since $\widehat{A} + J_0$ is surjective, we can find $x_1 \in \widehat{D}$, such that

$$u + J_0(x_1) \;=\; v + J_0(y),$$

for some $u \in \widehat{A}(x_1)$. Returning to (3.45) and setting $x = x_1$, we obtain

$$\big\langle J_0(y) - J_0(x_1), x_1 - y \big\rangle_{pp'} \;\geqslant\; 0. \tag{3.46}$$

But J_0 is strictly monotone. So from (3.46), it follows that

$$x_1 \;=\; y,$$

hence

$$y \in \widehat{D} \quad \text{and} \quad v \;=\; u \in \widehat{A}(x_1).$$

This proves the maximality of $\widehat{A}$. □

The second important class of maximal monotone operators that we would like to check closing this section are the linear ones. Note that for a linear operator

$$A\colon X \longrightarrow X^*,$$

monotonicity is equivalent to saying that

$$\big\langle A(x), x \big\rangle_X \;\geqslant\; 0 \qquad \forall\, x \in D(A).$$

For linear monotone operators we can characterize maximality using the adjoint operator or in terms of the density of its domain. This brings us to the "doorsteps" of the Hille-Yosida theorem and the theory of semigroups of operators, which form the subject of the next section.

THEOREM 3.2.58
If $A\colon X \supseteq D(A) \longrightarrow X^*$ is a linear monotone operator, then the following statements are equivalent:

(a) *A is maximal monotone;*

(b) *A^* is maximal monotone;*

(c) *A and A^* are both monotone and A is closed and densely defined.*

In particular if $X = H$ is a Hilbert space and

$$A\colon H \longrightarrow H^*$$

is linear maximal monotone, then A is symmetric if and only if A is self-adjoint. Maximal monotonicity is crucial here since it may happen that A is monotone symmetric, but A^* is not monotone.

REMARK 3.2.59 In Section 4.3 we shall return to the subject of monotone operators, by examining in more detail the subdifferential of a proper, convex and lower semicontinuous function (see Example 3.2.20**(c)**). This is a special kind of monotone map, known as cyclically monotone. As we shall see there, not every monotone map is of the subdifferential type.

3.3 Accretive Operators and Semigroups of Operators

In the previous section we studied operators (in general nonlinear) from a Banach space X into its dual X^*. In this section we deal with operators from X into X which still exhibit a "monotonicity" property. These are the so-called accretive operators. Of course the two classes of monotone and accretive operators coincide when $X = H$ is a Hilbert space. Accretive operators are intimately connected to the theory of generation of semigroups, which are a basic tool in the study of evolution equations. So the second half of this section is devoted to the presentation of the basics of the theory of semigroups of operators (linear and nonlinear).

Trying to extend the notion of monotonicity to maps from a Banach space X into itself, immediately we face the problem of finding a substitute for the duality brackets. There are two equivalent ways to do this. The first is to use the duality map, which essentially brings us back to the familiar setting of the dual pair (X, X^*). The second approach replaces the duality brackets by a so-called semi-inner product, which is a kind of inner product for the Banach space.

Let us start by giving the definition of accretivity based on the duality map and then proceed to introduce semi-inner products on a Banach space and show how they can be used.

DEFINITION 3.3.1 *Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow 2^X$ be an operator.*

(a) *We say that A is **accretive**, if for every $(x_1, u_1), (x_2, u_2) \in \operatorname{Gr} A$, there exists $x^* \in \mathcal{F}(x_1 - x_2)$, such that $\langle x^*, u_1 - u_2 \rangle_X \geqslant 0$.*

(b) *An accretive operator is said to be **maximal accretive**, if its graph is not properly included in the graph of another accretive operator.*

(c) *Finally an accretive operator is said to be m-**accretive**, if*

$$R(id_X + A) = X.$$

REMARK 3.3.2 When $X = H = H^*$ is a Hilbert pivot space, then $\mathcal{F} = id_X$ and so the notion of accretivity (respectively maximal accretivity) coincides with that of monotonicity (respectively maximal monotonicity). Moreover, in this case by virtue of Theorem 3.2.29, maximal accretivity and m-accretivity coincide. However, this is not in general true. We can find a maximal accretive operator which is not m-accretive (see Miyadera (1992, pp. 42–44)). If $-A$ is an accretive operator, then A is called a ***dissipative operator***. This terminology originates from mechanics, where dissipative forces are forces which do not increase the energy.

The next lemma leads to an alternative definition of an accretive operator.

LEMMA 3.3.3
If X is a Banach space and $x, y \in X$, then $\|x\|_X \leqslant \|x + \lambda y\|_X$ for all $\lambda > 0$ if and only if there exists $x^* \in \mathcal{F}(x)$, such that $\langle x^*, y\rangle_X \geqslant 0$.

PROOF "$\Longrightarrow$": Without any loss of generality we may assume that $x \neq 0$. Let $x^*_\lambda \in \mathcal{F}(x + \lambda y)$, $x^*_\lambda \neq 0$ for all $\lambda > 0$. Set

$$v^*_\lambda \stackrel{df}{=} \frac{x^*_\lambda}{\|x^*_\lambda\|_{X^*}} \qquad \forall\ \lambda > 0.$$

If $\lambda_n \searrow 0$, by Alaoglu's theorem (see Theorem A.3.9), we can find $v^* \in X^*$ with $\|v^*\|_{X^*} \leqslant 1$, such that

$$\langle v^*, v\rangle_X = \lim_{n \to +\infty} \langle v^*_{\lambda_n}, v\rangle_X \qquad \forall\ v \in X.$$

By hypothesis we have

$$\|x\|_X \leqslant \|x + \lambda_n y\|_X = \langle v^*_{\lambda_n}, x + \lambda_n y\rangle_X \leqslant \|x\|_X + \lambda_n \langle v^*_{\lambda_n}, y\rangle_X,$$

so

$$\langle v^*, y\rangle_X \geqslant 0. \tag{3.47}$$

Also, since

$$\|x + \lambda_n y\|_X = \langle v^*_{\lambda_n}, x + \lambda_n y\rangle \longrightarrow \langle v^*, x\rangle_X,$$

we have that

$$\|x\|_X \leqslant \langle v^*, x\rangle_X,$$

so

$$\|x\|_X = \langle v^*, x\rangle_X.$$

It follows that $x^* = \|x\|_X\, v^* \in \mathcal{F}(x)$. We have

$$\langle x^*, y\rangle_X \geqslant \|x\|_X \langle v^*, y\rangle_X \geqslant 0$$

(see (3.47)).

"$\Longleftarrow$": From the definition of the duality map $\mathcal{F}$ (see Example 3.2.20**(d)**) and the hypothesis that

$$\langle x^*, y\rangle_X \geqslant 0,$$

we have

$$\|x\|^2_X = \langle x^*, x\rangle_X \leqslant \langle x^*, x + \lambda y\rangle_X \leqslant \|x^*\|_{X^*} \|x + \lambda y\|_X.$$

So, because $x^* \in \mathcal{F}(x)$ and thus $\|x\|_X = \|x^*\|_{X^*}$, we have

$$\|x\|_X \leqslant \|x + \lambda y\|_X.$$

□

Using this lemma, we have the following alternative characterization of accretivity (known as ***Kato's criterion***).

PROPOSITION 3.3.4 (Kato's Criterion)
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$, then A is accretive if and only if for all $\lambda > 0$ and any $(x_1, u_1), (x_2, u_2) \in \operatorname{Gr} A$, we have

$$\|x_1 - x_2\|_X \leqslant \|x_1 - x_2 + \lambda(u_1 - u_2)\|_X.$$

Next we define the semi-inner products on X.

DEFINITION 3.3.5 *Let X be a Banach space and $x, y \in X$. We define the semi-inner products $(\cdot,\cdot)_\pm$ by the following:*

$$(y,x)_+ \stackrel{df}{=} \|x\|_X \lim_{\lambda \searrow 0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda} = \|x\|_X \inf_{\lambda>0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda}$$

$$(y,x)_- \stackrel{df}{=} \|x\|_X \lim_{\lambda \nearrow 0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda} = \|x\|_X \sup_{\lambda<0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda}.$$

REMARK 3.3.6 These definitions are rather natural since both limits exist due to the convexity of the function $x \longmapsto \psi(x) = \|x\|_X$ and they coincide with the inner product $(\cdot,\cdot)_X$ if $X = H$ is a Hilbert space. Moreover, if

$$\varphi(x) \stackrel{df}{=} \frac{1}{2}\|x\|_X^2,$$

then

$$\varphi'_+(x;y) = \|x\|_X \psi'_+(x;y) \qquad \forall\, x, y \in X,$$
$$\varphi'_-(x;y) = \|x\|_X \psi'_-(x;y) \qquad \forall\, x, y \in X,$$

where

$$\varphi'_+(x;y) \stackrel{df}{=} \lim_{\lambda \searrow 0} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda},$$
$$\psi'_+(x;y) \stackrel{df}{=} \lim_{\lambda \searrow 0} \frac{\psi(x+\lambda y) - \psi(x)}{\lambda},$$
$$\varphi'_-(x;y) \stackrel{df}{=} \lim_{\lambda \nearrow 0} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda},$$
$$\psi'_-(x;y) \stackrel{df}{=} \lim_{\lambda \nearrow 0} \frac{\psi(x+\lambda y) - \psi(x)}{\lambda}$$

(see Example 3.2.20**(d)**). Also note that $(y,x)_+ = -(-y,x)_-$ for all $x, y \in X$.

Since $\mathcal{F}(x) = \partial\varphi(x)$, the set $\partial\varphi(x)$ is w^*-compact, convex with support function $\varphi'_+(x;\cdot)$ and

$$\varphi'_+(x;y) = -\varphi'_-(x;-y) \qquad \forall\, x, y \in X,$$

from Definition 3.3.5, we deduce at once the following properties.

PROPOSITION 3.3.7
If X is a Banach space and $x, y \in \mathbb{R}$,
then

$$(y,x)_+ \;=\; \max_{x^* \in \mathcal{F}(x)} \langle x^*, y\rangle_X$$

and

$$(y,x)_- \;=\; \min_{x^* \in \mathcal{F}(x)} \langle x^*, y\rangle_X .$$

REMARK 3.3.8 If X^* is strict convex, then $\mathcal{F}$ is single-valued and so

$$(y,x)_+ \;=\; (y,x)_- \;=\; \big\langle \mathcal{F}(x), y\big\rangle_X \qquad \forall\, x, y \in X.$$

In particular a Hilbert space H is strictly convex and in that case $\mathcal{F} = id_X$. So

$$(y,x)_+ \;=\; (y,x)_- \;=\; (x,y)_H .$$

In the next proposition we have collected some basic properties of the semi-inner products.

PROPOSITION 3.3.9
If X is a Banach space, $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$,
then

(a) $(ax + y, x)_\pm = a\,\|x\|_X^2 + (y,x)_\pm$;

(b) $(\lambda y, \mu x)_\pm = \lambda\mu\,(y,x)_\pm$ *provided that* $\lambda, \mu \geqslant 0$;

(c) $(z+y, x)_\pm \leqslant \|z\|_X\,\|x\|_X + (y,x)_\pm$;

(d) $(\cdot,\cdot)_+ : X \times X \longrightarrow \mathbb{R}$ *is upper semicontinuous;*

(e) $(\cdot,\cdot)_- : X \times X \longrightarrow \mathbb{R}$ *is lower semicontinuous.*

PROOF **(a)** It follows easily from Proposition 3.3.7.

(b) Note that

$$\mathcal{F}(\mu x) \;=\; \mu \mathcal{F}(x)$$

and so

$$\begin{aligned}(\lambda y, \mu x)_+ \;&=\; \max_{x^* \in \mathcal{F}(x)} \big\langle \mu x^*, \lambda y\big\rangle_X \\ =\; \lambda\mu \max_{x^* \in \mathcal{F}(x)} \big\langle x^*, y\big\rangle_X \;&=\; \lambda\mu\,(y,x)_+ .\end{aligned}$$

Similarly for $(\cdot,\cdot)_-$.

(c) It follows easily from Proposition 3.3.7.

(d) From Definition 3.3.5, we know that

$$(y,x)_+ \;=\; \|x\|_X \inf_{\lambda>0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda}.$$

Note that the function $(y,x) \longmapsto \|x\|_X \dfrac{\|x+\lambda y\|_X - \|x\|_X}{\lambda}$ is continuous. Hence

$$\|x\|_X \inf_{\lambda>0} \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda} \;=\; \inf_{\lambda>0} \|x\|_X \frac{\|x+\lambda y\|_X - \|x\|_X}{\lambda}$$

is upper semicontinuous.

(e) Note that

$$(y,x)_- \;=\; (-y,x)_+$$

and use part **(d)**. □

The next proposition summarizes the different equivalent ways we can use to define accretive operators. It follows immediately from the previous discussion.

THEOREM 3.3.10
If X is a Banach space and

$$A\colon X \supseteq D(A) \;\longrightarrow\; 2^X$$

is an operator,
then the following two statements are equivalent:

(a) *A is an accretive operator;*

(b) *for every $(x_1,u_2),(x_2,u_2) \in \operatorname{Gr} A$, any one of the following three statements is true:*
$[A_1]$ $\|x_1 - x_2\|_X \leqslant \|x_1 - x_2 + \lambda(u_1 - u_2)\|_X$ *for all $\lambda > 0$;*
$[A_2]$ $\psi'_+\big(x_1 - x_2; u_1 - u_2\big) \geqslant 0$;
$[A_3]$ $\big(u_1 - u_2, x_1 - x_2\big)_+ \geqslant 0$.

Motivated from Proposition 3.3.4, we make the following definition.

DEFINITION 3.3.11 *Let X be a Banach space and let*

$$A\colon X \supseteq D(A) \;\longrightarrow\; 2^X$$

be an accretive operator. For every $\lambda > 0$ we introduce

$$J_\lambda \;\stackrel{df}{=}\; \big(id_X + \lambda A\big)^{-1} \quad \text{and} \quad A_\lambda \;\stackrel{df}{=}\; \frac{1}{\lambda}\big(id_X - J_\lambda\big),$$

both defined on $R\big(id_X + \lambda A\big)$.

In the next proposition we have collected some elementary properties of the operators J_λ and A_λ.

PROPOSITION 3.3.12
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an accretive operator, then

(a) *J_λ is nonexpansive on $R(id_X + \lambda A)$, i.e.,*

$$\big\|J_\lambda(x) - J_\lambda(y)\big\|_X \;\leqslant\; \|x-y\|_X \qquad \forall\, x,y \in R(id_X + \lambda A);$$

(b) *A_λ is accretive and Lipschitz continuous with constant $\frac{2}{\lambda}$ on $R(id_X + \lambda A)$;*

(c) *$A_\lambda(x) \in A\big(J_\lambda(x)\big)$ for every $x \in R(id_X + \lambda A)$;*

(d) *$\big\|A_\lambda(x)\big\|_X \leqslant \big|A(x)\big| \stackrel{df}{=} \inf\limits_{u\in A(x)} \|u\|_X$ for every $x \in D(A) \cap R(id_X + \lambda A)$;*

(e) *$\lim\limits_{\lambda \searrow 0} J_\lambda(x) = x$ for every $x \in \overline{D(A)} \cap \Big(\bigcap\limits_{\lambda>0} R(id_X + \lambda A)\Big)$.*

PROOF **(a)** This follows at once from Proposition 3.3.4.

(b) Let

$$y_k \;\stackrel{df}{=}\; \big(id_X + \mu A_k\big)(x_k) \qquad \forall\, k \in \{1,2\},\ \mu > 0.$$

Then

$$y_k \;=\; \Big(id_X + \frac{\mu}{\lambda}\big(id_X - J_\lambda\big)\Big)(x_k) \qquad \forall\, k \in \{1,2\}$$

and so

$$y_1 - y_2 + \frac{\mu}{\lambda}\big(J_\lambda(x_1) - J_\lambda(x_2)\big) \;=\; \Big(1 + \frac{\mu}{\lambda}\Big)(x_1 - x_2).$$

Hence

$$\Big(1 + \frac{\mu}{\lambda}\Big)\|x_1 - x_2\|_X \;\leqslant\; \|y_1 - y_2\|_X + \frac{\mu}{\lambda}\big\|J_\lambda(x_1) - J_\lambda(x_2)\big\|_X$$

and from part **(a)**, we have

$$\|x_1 - x_2\|_X \;\leqslant\; \|y_1 - y_2\|_X\,,$$

which proves the accretivity of A_λ. Also, from part **(a)**, we have

$$\big\|A_\lambda(x_1) - A_\lambda(x_2)\big\| \;=\; \frac{1}{\lambda}\,\|x_1 - x_2 + J_\lambda(x_1) - J_\lambda(x_2)\|_X \;\leqslant\; \frac{2}{\lambda}\,\|x_1 - x_2\|_X$$

and so A_λ is a Lipschitz continuous operator with constant $\frac{2}{\lambda}$.

(c) We have

$$A_\lambda(x) \in \frac{1}{\lambda}\big((id_X+\lambda A)\big(J_\lambda(x)\big)-J_\lambda(x)\big) = A\big(J_\lambda(x)\big) \qquad \forall\, x \in R(id_X+\lambda A).$$

(d) We have

$$\begin{aligned} A_\lambda(x) &= \frac{1}{\lambda}\big(J_\lambda\big((id_X+\lambda A)x\big)-J_\lambda(x)\big) \\ &= \frac{1}{\lambda}\big(J_\lambda\big(x+\lambda u\big)-J_\lambda(x)\big) \qquad \forall\, u \in A(x). \end{aligned} \tag{3.48}$$

Hence

$$\big\|A_\lambda(x)\big\|_X \leqslant \|u\|_X\,,$$

which implies

$$\big\|A_\lambda(x)\big\|_X \leqslant \big|A(x)\big|.$$

(e) From part **(d)**, we have

$$\big\|J_\lambda(x)-x\big\|_X = \lambda\big\|A_\lambda(x)\big\|_X \leqslant \lambda\big|A(x)\big| \qquad \forall\, x \in D(A)\cap R(id_X+\lambda A).$$

Hence

$$J_\lambda(x) \longrightarrow x \quad \text{as } \lambda \searrow 0, \qquad \forall\, x \in D(A)\cap R(id_X+\lambda A)$$

and by uniform continuity this extends to all $x \in \overline{D(A)}\cap R(id_X+\lambda A)$ (see part **(b)**). □

REMARK 3.3.13 When $X = H$ is a Hilbert space, then Proposition 3.3.12 coincides with Theorem 3.2.38. Also note that if $\lambda, \mu > 0$ and $x \in D(J_\lambda) = R(id_X+\lambda A)$, then

$$\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda(x) \in D(J_\mu) = R(id_X+\mu A)$$

and

$$J_\lambda(x) = J_\mu\left(\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda(x)\right)$$

(this last equality is usually known as ***Resolvent Identity***). To see this let $x = y+\lambda v$ with $(y,v) \in \operatorname{Gr} A$. So $J_\lambda(x) = y$. Hence we have

$$\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda(x) = \frac{\mu}{\lambda}(y+\lambda v)+\frac{\lambda-\mu}{\lambda}y = y+\mu v \in R(id_X+\mu A) = D(J_\mu)$$

and

$$J_\mu\left(\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda(x)\right) = J_\mu(y+\mu v) = y = J_\lambda(x).$$

Using the resolvent identity, for $\lambda \geqslant \mu > 0$, we have that

$$\begin{aligned}
\lambda\|A_\lambda(x)\|_X &= \|J_\lambda(x) - x\|_X \leqslant \|J_\lambda(x) - J_\mu(x)\|_X + \|J_\mu(x) - x\|_X \\
&= \left\|J_\mu\left(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda(x)\right) - J_\mu(x)\right\|_X + \|J_\mu(x) - x\|_X \\
&\leqslant \left\|\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda(x) - x\right\|_X + \|J_\mu(x) - x\|_X \\
&= (\lambda-\mu)\|A_\lambda(x)\|_X + \mu\|A_\mu(x)\|_X,
\end{aligned}$$

so

$$\|A_\lambda(x)\|_X \leqslant \|A_\mu(x)\|_X \qquad \forall\, \lambda \geqslant \mu$$

and thus $\{\|A_\lambda(x)\|_X\}_{\lambda>0}$ is increasing as λ decreases to 0^+.

PROPOSITION 3.3.14
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an accretive operator, then $R(id_X + \lambda A) = X$ for all $\lambda > 0$, if it holds for some $\lambda > 0$.

PROOF Suppose that

$$R(id_X + \lambda A) = X$$

for some $\lambda > 0$ and $\mu > \frac{\lambda}{2}$. Then for $u \in X$, we have

$$u \in \big(id_X + \mu A\big)(x)$$

or equivalently

$$\big(id_X + \lambda A\big)(x) \ni \frac{\lambda}{\mu}u + \left(1 - \frac{\lambda}{\mu}\right)x,$$

which in turn is equivalent to saying that $K(x) = x$ for the contraction

$$K(x) \stackrel{df}{=} J_\lambda\left(\frac{\lambda}{\mu}u + \left(1 - \frac{\lambda}{\mu}\right)x\right) \qquad \forall\, x \in X.$$

By Banach's fixed point theorem (see Theorem 7.1.2) $K(x) = x$ has a unique solution and so

$$R(id_X + \mu A) = X \qquad \forall\, \mu > \frac{\lambda}{2}.$$

Then by induction we conclude that

$$R(id_X + \mu A) = X \forall \mu > 0.$$

□

REMARK 3.3.15 Using Proposition 3.3.14, we can say that an operator $A\colon X \supseteq D(A) \longrightarrow 2^X$ is said to be ***m*-accretive** if and only if

$$R(id_X + \lambda A) = X \quad \text{for some } \lambda > 0$$

(equivalently for all $\lambda > 0$; see Definition 3.3.1).

PROPOSITION 3.3.16
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator, then

(a) *A is maximal accretive;*

(b) *A is closed;*

(c) *if $x_\lambda \longrightarrow x$ and $A_\lambda(x) \longrightarrow u$ in X as $\lambda \searrow 0$, then*

$$(x, u) \in \operatorname{Gr} A.$$

PROOF **(a)** Let $(x_0, u_0) \in X \times X$ and suppose that

$$\|x_0 - x\|_X \leqslant \|x_0 - x + \lambda(u_0 - u)\|_X \qquad \forall\, \lambda > 0,\ (x, u) \in \operatorname{Gr} A. \tag{3.49}$$

We need to show that

$$(x_0, u_0) \in \operatorname{Gr} A$$

(see Definition 3.3.1 and Proposition 3.3.4). Since A is m-accretive, we have that

$$X = R(id_X + A)$$

and so we can find $(x, u) \in \operatorname{Gr} A$, such that

$$x + u = x_0 + u_0.$$

Using this in (3.49), we obtain $x = x_0$, hence $(x_0, u_0) \in \operatorname{Gr} A$.

(b) Let $\{(x_n, u_n)\}_{n \geqslant 1} \subseteq \operatorname{Gr} A$ and assume that

$$(x_n, u_n) \longrightarrow (x, u) \quad \text{in } X \times X.$$

We need to show that

$$(x, u) \in \operatorname{Gr} A.$$

By virtue of the m-accretivity of A, we have

$$\|x_n - y\|_X \leqslant \|x_n - y + \lambda(u_n - v)\|_X \qquad \forall\, n \geqslant 1,\ \lambda > 0,\ (y, v) \in \operatorname{Gr} A,$$

so

$$\|x - y\|_X \leqslant \|x - y + \lambda(u - v)\|_X \qquad \forall\, \lambda > 0,\ (y, v) \in \operatorname{Gr} A. \tag{3.50}$$

But from part **(a)**, we know that A is maximal accretive. So from (3.50), it follows that $(x, u) \in \operatorname{Gr} A$.

(c) From Proposition 3.3.12**(a)** and **(e)**, we have that

$$J_\lambda(x_\lambda) \longrightarrow x \quad \text{in } X, \quad \text{as } \lambda \searrow 0,$$

while from Proposition 3.3.12**(c)**, we know that

$$A_\lambda(x_\lambda) \in A\big(J_\lambda(x_\lambda)\big) \qquad \forall\, \lambda > 0.$$

Then using part **(b)**, we infer that $(x, u) \in \operatorname{Gr} A$. □

We can improve conclusion **(b)** of Proposition 3.3.16, provided we strengthen the condition on the space X.

PROPOSITION 3.3.17
If X is a reflexive Banach space with X^* being locally uniformly convex and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator,
then $\operatorname{Gr} A$ *is sequentially closed in $X \times X_w$.*

PROOF Suppose that $\big\{(x_n, u_n)\big\}_{n \geqslant 1} \subseteq \operatorname{Gr} A$ is a sequence, such that

$$x_n \longrightarrow x \quad \text{and} \quad u_n \xrightarrow{w} u \quad \text{in } X.$$

We need to show that

$$(x, u) \in \operatorname{Gr} A.$$

Because A is m-accretive, from Theorem 3.3.10**(b)**[A_3], we have

$$\big\langle \mathcal{F}(x_n - y), u_n - v \big\rangle_X \geqslant 0 \qquad \forall\, n \geqslant 1,\ (y, v) \in \operatorname{Gr} A. \tag{3.51}$$

But from Proposition 3.2.25, we know that the duality map

$$\mathcal{F}\colon X \longrightarrow X^*$$

is continuous. So if we pass to the limit as $n \to +\infty$ in (3.51), we obtain

$$\big\langle \mathcal{F}(x - y), u - v \big\rangle_X \geqslant 0 \qquad \forall\, (y, v) \in \operatorname{Gr} A,$$

so

$$\big(u - v, x - y\big)_+ \geqslant 0 \qquad \forall\, (y, v) \in \operatorname{Gr} A$$

and thus, from Proposition 3.3.16**(a)**, we conclude that

$$(x, u) \in \operatorname{Gr} A.$$

□

Another useful result that can be proved by imposing extra conditions on the space X is the following one.

PROPOSITION 3.3.18
If X is a Banach space with X^* strictly convex and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is a maximal accretive operator,
then the set $A(x) \subseteq X$ is convex and closed for any $x \in D(A)$.

PROOF Because X^* is strictly convex, the duality map $\mathcal{F}\colon X \longrightarrow X^*$ is single-valued. First we show that for $x \in D(A)$, the set $A(x)$ is convex. So let $u, v \in A(x)$ and set

$$w \;=\; tu + (1-t)v \quad \text{with } t \in [0,1].$$

For all $(y,h) \in \operatorname{Gr} A$, we have that

$$\big\langle \mathcal{F}(x-y), w-h\big\rangle_X \;=\; t\big\langle \mathcal{F}(x-y), u-h\big\rangle_X + (1-t)\big\langle \mathcal{F}(x-y), v-h\big\rangle_X \;\geqslant\; 0,$$

so from the maximality of A, we have that $(x,w) \in \operatorname{Gr} A$.

Next we show that the set $A(x)$ is closed in X. To this end let $\{u_n\}_{n\geqslant 1} \subseteq A(x)$ be a sequence, such that

$$u_n \;\longrightarrow\; u \quad \text{in } X.$$

We have

$$\big\langle \mathcal{F}(x-y), u_n - v\big\rangle_X \;\geqslant\; 0 \qquad \forall\, n \geqslant 1,\ (y,v) \in \operatorname{Gr} A,$$

so

$$\big\langle \mathcal{F}(x-y), u - v\big\rangle_X \;\geqslant\; 0 \qquad \forall\, (y,v) \in \operatorname{Gr} A$$

and so, from the maximality of A, we have that $(x,u) \in \operatorname{Gr} A$. □

We continue with the properties of maximal accretive and m-accretive operators.

PROPOSITION 3.3.19
If X is an uniformly convex Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator,
then $\overline{D(A)}$ is convex.

PROOF Let

$$D_0 \;\stackrel{df}{=}\; \left\{ x \in \overline{\operatorname{conv}} D(A) :\ \lim_{\lambda \searrow 0} J_\lambda(x) = x \right\}.$$

Evidently $\overline{D(A)} = D_0$ and D_0 is closed. So it suffices to show that D_0 is convex. For $x, y \in D_0$, we have

$$\left\| J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda(x) \right\|_X \;\leqslant\; \left\| \frac{x-y}{2} \right\|_X \qquad \forall\, \lambda > 0 \tag{3.52}$$

and

$$\left\| J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda(y)\right\|_X \leqslant \left\|\frac{x-y}{2}\right\|_X \qquad \forall\ \lambda > 0 \tag{3.53}$$

(see Proposition 3.3.12(**a**)). From (3.52) and (3.53), it follows that

$$\left\{J_\lambda\left(\frac{x+y}{2}\right)\right\}_{\lambda\in(0,1)} \subseteq X \text{ is bounded.}$$

Since X is reflexive (being uniformly convex; see Remark A.3.22), we can find a sequence $\lambda_n \searrow 0$, such that

$$J_{\lambda_n}\left(\frac{x+y}{2}\right) \xrightarrow{w} h \quad \text{in } X.$$

So if we pass to the limit as $n \to +\infty$ in (3.52) and (3.53), we obtain

$$\|h-x\|_X \leqslant \left\|\frac{x-y}{2}\right\|_X \quad \text{and} \quad \|h-y\|_X \leqslant \left\|\frac{x-y}{2}\right\|_X. \tag{3.54}$$

We have

$$\|x-y\|_X \leqslant \|x-h\|_X + \|h-y\|_X \leqslant \|x-y\|_X. \tag{3.55}$$

From (3.54) and (3.55), it follows that

$$\|x-h\|_X = \|y-h\|_X = \left\|\frac{x-y}{2}\right\|_X$$

and this by virtue of the uniform convexity of X implies that $h = \frac{x+y}{2}$. Thus we have

$$J_{\lambda_n}\left(\frac{x+y}{2}\right) \xrightarrow{w} \frac{x+y}{2} \quad \text{in } X. \tag{3.56}$$

Moreover, we have

$$\begin{aligned} \left\|\frac{y-x}{2}\right\|_X &\leqslant \liminf_{n\to+\infty} \left\| J_{\lambda_n}\left(\frac{x+y}{2}\right) - J_{\lambda_n}(x)\right\|_X \\ &\leqslant \limsup_{n\to+\infty} \left\| J_{\lambda_n}\left(\frac{x+y}{2}\right) - J_{\lambda_n}(x)\right\|_X \leqslant \left\|\frac{y-x}{2}\right\|_X, \end{aligned}$$

so

$$\left\| J_{\lambda_n}\left(\frac{x+y}{2}\right) - J_{\lambda_n}(x)\right\|_X \longrightarrow \left\|\frac{y-x}{2}\right\|_X. \tag{3.57}$$

Since

$$J_{\lambda_n}(x) \longrightarrow x \quad \text{in } X,$$

from (3.56) and (3.57) and the Kadec-Klee property (see Remark A.3.22), we conclude that

$$J_{\lambda_n}\left(\frac{x+y}{2}\right) \longrightarrow \frac{x+y}{2}$$

and so $\frac{x+y}{2} \in D_0$, which proves the convexity of D_0, hence the convexity of $\overline{D(A)}$. ∎

Next we prove two perturbation results for m-accretive operators. To do this we shall need the following auxiliary result.

LEMMA 3.3.20
If X is a Banach space with X^* being uniformly convex (hence X is reflexive) and $A\colon X \supseteq D(A) \longrightarrow 2^X$, and $B\colon X \supseteq D(B) \longrightarrow 2^X$ are two m-accretive operators with $D(A) \cap D(B) \neq \emptyset$,
then for every $u \in X$ and every $\lambda > 0$, the operator inclusion

$$x + A(x) + B_\lambda(x) \ \ni \ u$$

has a unique solution $x_\lambda \in D(A)$ and $\{x_\lambda\}_{\lambda>0}$ is bounded.
Moreover, if $\left\{B_\lambda(x_\lambda)\right\}_{\lambda\in(0,1)}$ is bounded, then the operator inclusion

$$x + A(x) + B(x) \ \ni \ u$$

has a unique solution $x \in D(A) \cap D(B)$ and

$$x_\lambda \ \longrightarrow \ x \quad \text{in } X, \quad \text{as } \lambda \searrow 0.$$

PROOF The operator inclusion

$$x + A(x) + B_\lambda(x) \ \ni \ u$$

is equivalent to

$$x \ = \ \left(1 + \frac{\lambda}{\lambda+1}A\right)^{-1}\left(\frac{\lambda}{\lambda+1}u + \frac{1}{\lambda+1}(id_X + \lambda B)^{-1}(x)\right),$$

so

$$x = K_\lambda(x), \tag{3.58}$$

with

$$K_\lambda \ \stackrel{df}{=} \ J^A_{\frac{\lambda}{\lambda+1}}\left(\frac{\lambda}{\lambda+1}u + \frac{1}{\lambda+1}J^A_\lambda\right) \qquad \forall\ \lambda > 0.$$

Since the operators $J^A_{\frac{\lambda}{\lambda+1}}$ and J^B_λ are nonexpansive on X (see Proposition 3.3.12**(a)**), we can check that

$$\left\|K_\lambda(x) - K_\lambda(y)\right\|_X \ \leqslant \ \frac{1}{\lambda+1}\left\|x - y\right\|_X \qquad \forall\ x, y \in X,\ \lambda > 0.$$

Invoking Banach's fixed point theorem (see Theorem 7.1.2), we infer that (3.58) has a unique solution $x_\lambda \in D(A)$ for $\lambda > 0$. Let

$$z \in D(A) \cap D(B) \quad \text{and} \quad u_\lambda \in z + A(z) + B_\lambda(z).$$

From Proposition 3.3.12**(b)**, we know that B_λ is accretive and since the sum of accretive operators is clearly accretive, we have that the operator

$$A + B_\lambda \colon X \supseteq D(A) \longrightarrow 2^X$$

is accretive. So

$$\big\langle \mathcal{F}(x_\lambda - z), u - x_\lambda - (u_\lambda - z) \big\rangle_X \ \geqslant\ 0,$$

hence

$$\|x_\lambda - z\|_X^2 \ \leqslant\ \big\langle \mathcal{F}(x_\lambda - z), u - u_\lambda \big\rangle_X \ \leqslant\ \|x_\lambda - z\|_X \, \|u - u_\lambda\|_X \,,$$

from which it follows that

$$\|x_\lambda - z\|_X \ \leqslant\ \|u - u_\lambda\|_X \,. \tag{3.59}$$

Because

$$\big\|B_\lambda(z)\big\|_X \ \leqslant\ \big|B(z)\big|$$

(see Proposition 3.3.12**(d)**), from (3.59), we infer that $\{x_\lambda\}_{\lambda>0}$ is bounded.

Now suppose that $\big\{B_\lambda(x_\lambda)\big\}_{\lambda>0}$ is bounded. For $\lambda, \mu > 0$, we have

$$u - x_\lambda - B_\lambda(x_\lambda) \ \in\ A(x_\lambda) \quad \text{and} \quad u - x_\mu - B_\mu(x_\mu) \ \in\ A(x_\mu).$$

Exploiting the accretivity of the operator A, we obtain

$$\big\langle \mathcal{F}(x_\lambda - x_\mu), x_\mu - x_\lambda + B_\mu(x_\mu) - B_\lambda(x_\lambda) \big\rangle_X \ \geqslant\ 0,$$

so

$$\|x_\lambda - x_\mu\|_X^2 \ \leqslant\ \big\langle \mathcal{F}(x_\lambda - x_\mu), B_\mu(x_\mu) - B_\lambda(x_\lambda) \big\rangle_X .$$

Because

$$B_\lambda(x_\lambda) \ \in\ B\big(J_\lambda(x_\lambda)\big) \quad \text{and} \quad B_\mu(x_\mu) \ \in\ B\big(J_\mu(x_\mu)\big)$$

(see Proposition 3.3.12**(c)**) and B is accretive, we have

$$\big\langle \mathcal{F}\big(J_\mu(x_\mu) - J_\lambda(x_\lambda)\big), B_\mu(x_\mu) - B_\lambda(x_\lambda) \big\rangle_X \ \geqslant\ 0.$$

It follows that

$$\begin{aligned} &\|x_\lambda - x_\mu\|_X^2 \\ &\leqslant\ \big\langle \mathcal{F}(x_\lambda - x_\mu) + \mathcal{F}\big(J_\mu(x_\mu) - J_\lambda(x_\lambda)\big), B_\mu(x_\mu) - B_\lambda(x_\lambda) \big\rangle_X . \end{aligned} \tag{3.60}$$

Since

$$\lambda B_\lambda(x_\lambda) \ =\ x_\lambda - J_\lambda(x_\lambda)$$

and by hypothesis $\big\{B_\lambda(x_\lambda)\big\}_{\lambda\in(0,1)}$ is bounded, we have that

$$\big\|x_\lambda - J_\lambda(x_\lambda)\big\|_X \ \leqslant\ M_1 \lambda \qquad \forall\ \lambda \in (0,1),$$

for some $M_1 > 0$.

Because X^* is uniformly convex, we have that duality map is uniformly continuous on bounded sets of X (see Proposition 3.2.28). Therefore since the duality map is odd (see Proposition 3.2.22), we see that given $\varepsilon > 0$, for all $\lambda, \mu > 0$ small enough, we have

$$\begin{aligned} &\left\|\mathcal{F}\big(x_\lambda - x_\mu\big) + \mathcal{F}\big(J_\mu(x_\mu) - J_\lambda(x_\lambda)\big)\right\|_X \\ = \ &\left\|\mathcal{F}\big(J_\mu(x_\mu) - J_\lambda(x_\lambda)\big) - \mathcal{F}\big(x_\mu - x_\lambda\big)\right\|_X \ \leqslant \ \varepsilon. \end{aligned}$$

So from (3.60), we have

$$\|x_\lambda - x_\mu\|_X^2 \ \leqslant \ M_1\varepsilon \qquad \forall\ \lambda, \mu > 0 \text{ small enough.}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$x_\lambda \ \longrightarrow \ x \quad \text{in } X, \quad \text{as } \lambda \searrow 0.$$

Let $\lambda_n \searrow 0$ be such that

$$B_{\lambda_n}(x_{\lambda_n}) \ \xrightarrow{w} \ z \quad \text{in } X$$

(recall that X is reflexive and by hypothesis $\big\{B_\lambda(x_\lambda)\big\}_{\lambda\in(0,1)}$ is bounded). Set

$$v_n \ \stackrel{df}{=} \ u - x_{\lambda_n} - B_{\lambda_n}(x_{\lambda_n}) \qquad \forall\ n \geqslant 1.$$

Then

$$v_n \xrightarrow{w} v = u - x - z \quad \text{in } X$$

and

$$v_n \in A(x_{\lambda_n}).$$

Invoking Proposition 3.3.17, we have that $(x, v) \in \operatorname{Gr} A$. Also

$$\big(J_{\lambda_n}(x_{\lambda_n}), B_{\lambda_n}(x_{\lambda_n})\big) \ \in \ \operatorname{Gr} B$$

and

$$J_{\lambda_n}(x_{\lambda_n}) \ \longrightarrow \ x \quad \text{and} \quad B_{\lambda_n}(x_{\lambda_n}) \ \xrightarrow{w} \ x \quad \text{in } X.$$

So once again via Proposition 3.3.17, we have that $(x, z) \in \operatorname{Gr} B$. Thus finally

$$x \in D(A) \cap D(B) \quad \text{and} \quad u \ = \ x + v + z$$

with

$$v \in A(x) \quad \text{and} \quad z \in B(x).$$

∎

Using this lemma we can prove two perturbation theorems for m-accretive operators.

THEOREM 3.3.21
If X is a Banach space with X^* being uniformly convex (hence X is reflexive) and

$$A\colon X \supseteq D(A) \longrightarrow 2^X, \quad B\colon X \supseteq D(B) \longrightarrow 2^X$$

are two m-accretive operators, such that

(i) $D(A) \cap D(B) \neq \emptyset$;

(ii) $\langle \mathcal{F}(B_\lambda(x)), u \rangle_X \geqslant 0$ *for all* $\lambda > 0$ *and all* $(x,u) \in \operatorname{Gr} A$,

then $A + B$ *is m-accretive.*

PROOF Let $u \in X$. By virtue of Lemma 3.3.20, we can find a unique $x_\lambda \in D(A)$, such that

$$u \ \in \ x_\lambda + A(x_\lambda) + B_\lambda(x_\lambda) \qquad \forall\ \lambda > 0.$$

We take the duality brackets with $\mathcal{F}(B_\lambda(x_\lambda))$. Using (ii) and the fact that $\{x_\lambda\}_{\lambda>0}$ is bounded (see Lemma 3.3.20), we obtain that $\{B_\lambda(x_\lambda)\}_{\lambda\in(0,1)}$ is bounded. Then by virtue of Lemma 3.3.20, we have that

$$x_\lambda \ \longrightarrow \ x \quad \text{in } X \quad \text{as } \lambda \searrow 0$$

and

$$u \ \in \ x + A(x) + B(x),$$

i.e.,

$$R(id_X + A + B) \ = \ X,$$

which means that $A + B$ is m-accretive. ∎

THEOREM 3.3.22
If X is a Banach space with X^* being uniformly convex (hence X is reflexive) and

$$A\colon X \supseteq D(A) \longrightarrow 2^X, \quad B\colon X \supseteq D(B) \longrightarrow 2^X$$

are two m-accretive operators, such that

(i) $D(A) \subseteq D(B)$;

(ii) for each $r > 0$, *there are* $c < 1$ *and* $d > 0$, *such that*

$$|B(x)| \ \leqslant \ c|A(x)| + d \qquad \forall\ x \in D(A),\ \|x\|_X \leqslant r,$$

then $A + B$ *is m-accretive.*

PROOF Let $u \in X$ and let $x_\lambda \in D(A)$ be the unique solution of the operator inclusion

$$u \in x_\lambda + A(x_\lambda) + B_\lambda(x_\lambda).$$

Since $x_\lambda \in D(A) \subseteq D(B)$ (see (i)) and

$$\|B_\lambda(x_\lambda)\|_X \leqslant |B(x_\lambda)| \qquad \forall\, \lambda > 0$$

(see Proposition 3.3.12**(d)**), from condition (ii) and since $\{x_\lambda\}_{\lambda>0}$ is bounded (see Lemma 3.3.20), we obtain

$$|A(x_\lambda)| \leqslant c|A(x_\lambda)| + d' \qquad \forall\, \lambda > 0,$$

for some $d' > 0$, so $\{A(x_\lambda)\}_{\lambda>0}$ is bounded (since $c < 1$).

Using this fact in condition (ii), we infer that $\{B_\lambda(x_\lambda)\}_{\lambda\in(0,1)}$ is bounded. Then invoking Lemma 3.3.20, we finish the proof as before. □

REMARK 3.3.23 Condition (ii) of Theorem 3.3.22 can be replaced by the following local condition

(ii)' for every $x_0 \in D(A)$, we can find a neighbourhood U of x_0 and constants $c < 1$ and $d > 0$, such that

$$|B(x)| \leqslant c|A(x)| + d \qquad \forall\, x \in D(A) \cap U$$

(see Kato (1967)).

In applications in general Theorem 3.3.21 is more convenient than Theorem 3.3.22.

We present an application of the perturbation results in the study of elliptic boundary value problems. To this end let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We shall need the following existence, uniqueness and regularity result due to Agmon, Douglis & Nirenberg (1959).

THEOREM 3.3.24
If $\Omega \subseteq \mathbb{R}^N$ *is as above,* $p \in (1, +\infty)$ *and* $f \in L^p(\Omega)$, *then there exists unique* $x \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, *such that*

$$-\Delta x(z) + x(z) = f(z) \quad \text{for a.a. } z \in Z, \quad x|_{\partial Z} = 0.$$

Moreover, if $\partial\Omega$ *is a* C^{m+1}*-manifold for some* $m \geqslant 1$ *and* $f \in W^{m,p}(\Omega)$, *then* $x \in W^{m+2,p}(\Omega)$ *and*

$$\|x\|_{W^{m+2,p}(\Omega)} \leqslant c\,\|f\|_{W^{m,p}(\Omega)} \qquad \forall\, x \in W^{m+2,p}(\Omega),$$

for some $c = c(m, p, \Omega) > 0$.

Let

$$\xi\colon \mathbb{R} \supseteq D(\xi) \longrightarrow 2^{\mathbb{R}}$$

be a maximal monotone map with $0 \in \xi(0)$. We consider the ***realization*** (lifting) of ξ on $L^p(\Omega) \times L^p(\Omega)$ for $p \in (1, +\infty)$. So we define

$$\widehat{\xi}\colon L^p(\Omega) \supseteq D(\widehat{\xi}) \longrightarrow 2^{L^p(\Omega)}$$

by

$$\widehat{\xi}(x) \stackrel{df}{=} \left\{u \in L^p(\Omega) :\ u(z) \in \xi\big(x(z)\big) \text{ for a.a. } z \in Z\right\} \qquad \forall\, x \in D(\widehat{\xi}),$$

where

$$D(\widehat{\xi}) \stackrel{df}{=} \Big\{x \in L^p(\Omega) :\ \text{there exists } u \in L^p(\Omega),$$
$$\text{such that } u(z) \in \xi\big(x(z)\big) \text{ for a.a. } z \in Z\Big\}.$$

A simple measurable selection argument establishes that $\widehat{\xi}$ is m-accretive and we have

$$\big[(id_{_X} + \lambda\widehat{\xi})^{-1}x\big](z) \;=\; (1 + \lambda\xi)^{-1}x(z) \quad \text{for a.a. } z \in Z$$

with $X = L^p(\Omega)$ and

$$\widehat{\xi}_\lambda(x)(z) \;=\; \xi_\lambda\big(x(z)\big) \quad \text{for a.a. } z \in Z, \text{ all } \lambda > 0 \text{ and all } x \in L^p(\Omega).$$

We consider the operator $K\colon L^p(\Omega) \supseteq D(K) \longrightarrow 2^{L^p(\Omega)}$, defined by

$$K(x) \;=\; -\Delta x + \widehat{\xi}(x) \qquad \forall\, x \in D(K), \tag{3.61}$$

where

$$D(K) \stackrel{df}{=} W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \cap D(\widehat{\xi}).$$

PROPOSITION 3.3.25
If $K\colon L^p(\Omega) \supseteq D(K) \longrightarrow 2^{L^p(\Omega)}$ is the operator defined by (3.61), then K is m-accretive.

PROOF It is easy to check that the duality map on $L^p(\Omega)$,

$$\mathcal{F}\colon L^p(\Omega) \longrightarrow L^{p'}(\Omega)$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$), is defined by

$$\mathcal{F}(x)(\cdot) \stackrel{df}{=} x(\cdot)\frac{|x(\cdot)|^{p-2}}{\|x\|_p^{p-2}}.$$

Using this we can check that the operator

$$-\Delta\colon L^p(\Omega) \supseteq W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is accretive. Moreover, by virtue of Theorem 3.3.24, it follows that $-\Delta$ is m-accretive. We have

$$\big\langle \mathcal{F}\big(\widehat{\xi}_\lambda(x)\big), -\Delta x\big\rangle_{L^p(\Omega)} \;=\; \int_\Omega -\Delta x(z)\xi_\lambda\big(x(z)\big)\big|\xi_\lambda\big(x(z)\big)\big|^{p-2}\,dz. \tag{3.62}$$

If

$$\varphi_\lambda(r) \;\stackrel{df}{=}\; \xi_\lambda(r)\big|\xi_\lambda(r)\big|^{p-2},$$

then φ_λ is a Lipschitz continuous map and

$$\varphi'_\lambda(r) \;=\; (p-1)\big|\xi_\lambda(r)\big|^{p-2}\xi'_\lambda(r) \quad \text{for a.a. } r \in \mathbb{R}.$$

Also

$$\varphi\big(x(\cdot)\big) \in W_0^{1,p}(\Omega)$$

(see Proposition 2.4.25 and Remark 2.4.26) and

$$D\varphi_\lambda\big(x(z)\big) \;=\; \varphi'_\lambda\big(x(z)\big)Dx(z) \quad \text{for a.a. } z \in Z.$$

Performing an integration by parts on the right hand side integral of (3.62) and recalling that $\beta_\lambda(0) = 0$ (since $0 \in \beta(0)$), we obtain

$$\big\langle \mathcal{F}\big(\widehat{\xi}_\lambda(x)\big), -\Delta x\big\rangle_{L^p(\Omega)} \;=\; \int_\Omega \varphi'_\lambda\big(x(z)\big)\big\|Dx(z)\big\|^2_{\mathbb{R}^N}\,dz \;\geqslant\; 0,$$

since $\varphi'_\lambda\big(x(z)\big) \geqslant 0$, because φ_λ is monotone increasing. Applying Theorem 3.3.21, with data $A = -\Delta$, $B = \widehat{\xi}$ and $X = L^p(\Omega)$ (note that $X^* = L^{p'}(\Omega)$ is uniformly convex since $p' \in (1, +\infty)$), we obtain that $A + B = K$ is m-accretive. □

The main reason for studying accretive operators is the fact that they are closely related with the generation of semigroups (linear and nonlinear). The theory of semigroups is a valuable tool in the study of partial differential equations, of Volterra integral equations and of control problems. In the rest of this section, we will see how m-accretive operators lead to semigroups of operators, which in turn describe the time-evolution of a dynamical system monitored by a differential equation in a Banach space (evolution equation). So we start our discussion with an existence result for evolution equations in which the input and Cauchy data are regular (smooth). First two useful auxiliary results.

LEMMA 3.3.26
If X is a Banach space, $x\colon T=[0,b]\longrightarrow X$ is weakly differentiable at $t\in T$, i.e.,

$$\frac{d}{ds}\big\langle x^*, x(s)\big\rangle_X\Big|_{s=t} \;=\; \big\langle x^*, x'(t)\big\rangle_X \qquad \forall\ x^*\in X^*,$$

and $s\longmapsto \big\|x(s)\big\|_X$ is differentiable at $s=t$,
then

$$\big\|x(t)\big\|_X\left[\frac{d}{ds}\big\|x(s)\big\|_X\right]\Big|_{s=t} \;=\; \big\langle x^*, x(t)\big\rangle_X \qquad \forall\ x^*\in \mathcal{F}\big(x(t)\big).$$

PROOF For every $x^*\in \mathcal{F}\big(x(t)\big)$ and $r>0$, we have

$$\big\langle x^*, x(t+r)-x(t)\big\rangle_X \;\leqslant\; \|x^*\|_{X^*}\left(\big\|x(t+r)\big\|_X-\big\|x(t)\big\|_X\right).$$

Since by hypothesis x is weakly differentiable at t, dividing with $r>0$ and letting $r\searrow 0$, we obtain

$$\big\langle x^*, x(t)\big\rangle_X \;\leqslant\; \big\|x(t)\big\|_X\left[\frac{d}{ds}\big\|x(s)\big\|_X\right]\Big|_{s=t}. \tag{3.63}$$

On the other hand, since

$$\big\langle x^*, x(t)-x(t-r)\big\rangle_X \;\geqslant\; \|x^*\|_{X^*}\left(\big\|x(t)\big\|_X-\big\|x(t-r)\big\|_X\right),$$

arguing as above, we obtain

$$\big\|x(t)\big\|_X\left[\frac{d}{ds}\big\|x(s)\big\|_X\right]\Big|_{s=t} \;\leqslant\; \big\langle x^*, x'(t)\big\rangle_X. \tag{3.64}$$

From (3.63) and (3.64), we conclude the desired equality. □

The second auxiliary result is a Gronwall-type lemma which is used frequently in the study of evolution equations.

LEMMA 3.3.27
If $\varphi\in L^1(T)$, $\varphi(t)\geqslant 0$ for almost all $t\in T$, $\eta\in\mathbb{R}$, $u\in C(T)$ and

$$\frac{1}{2}u(t)^2 \;\leqslant\; \frac{1}{2}\eta^2+\int_0^t \varphi(s)u(s)\,ds \qquad \forall\ t\in T,$$

then

$$\big|u(t)\big| \;\leqslant\; |\eta|+\int_0^t \varphi(s)\,ds \qquad \forall\ t\in T.$$

PROOF Let

$$\xi_\varepsilon(t) \;=\; \frac{1}{2}(\eta+\varepsilon)^2 + \int_0^t \varphi(s)u(s)\,ds,$$

with $\varepsilon > 0$ and $t \in T$. Then

$$\xi'_\varepsilon(t) \;=\; \varphi(t)u(t) \quad \text{for a.a. } t \in T.$$

Moreover,

$$\frac{1}{2}u(t)^2 \;\leqslant\; \xi_0(t) \;\leqslant\; \xi_\varepsilon(t) \qquad \forall\, \varepsilon > 0,\; t \in T,$$

so

$$\xi'_\varepsilon(t) \;\leqslant\; \varphi(t)\sqrt{2}\sqrt{\xi_\varepsilon(t)} \qquad \forall\, \varepsilon > 0,\; t \in T.$$

Because $t \longmapsto \xi_\varepsilon(t)$ is absolutely continuous with values in $\mathbb{R}_+$, we have

$$\big(\sqrt{\xi_\varepsilon(t)}\big)' \;=\; \frac{1}{2\sqrt{\xi_\varepsilon(t)}}\xi'_\varepsilon(t) \quad \text{for a.a. } t \in T,$$

thus

$$\big(\sqrt{\xi_\varepsilon(t)}\big)' \;\leqslant\; \frac{1}{\sqrt{2}}\varphi(t) \quad \text{for a.a. } t \in T$$

and so

$$\sqrt{\xi_\varepsilon(t)} \;\leqslant\; \sqrt{\xi_\varepsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t \varphi(s)\,ds \qquad \forall\, t \in T.$$

Therefore, it follows that

$$\big|u(t)\big| \;\leqslant\; \sqrt{2}\sqrt{\xi_\varepsilon(t)} \;\leqslant\; \sqrt{2\xi_\varepsilon(0)} + \int_0^t \varphi(s)\,ds$$

$$=\; |\eta+\varepsilon| + \int_0^t \varphi(s)\,ds.$$

Let $\varepsilon \searrow 0$, to conclude that

$$\big|u(t)\big| \;\leqslant\; |\eta| + \int_0^t \varphi(s)\,ds \qquad \forall\, t \in T.$$

□

Using these results, we can prove the first existence theorem for evolution equations driven by m-accretive operators.

THEOREM 3.3.28

If X is a Banach space with X^* being uniformly convex (hence X is reflexive), $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator, $x_0 \in D$, $f \in W^{1,1}\big((0,b);X\big)$ and $\omega \in \mathbb{R}$,
then we can find a unique $x \in W^{1,\infty}\big((0,b);X\big)$, such that

$$\begin{cases} x'(t) + A\big(x(t)\big) \ni \omega x(t) + f(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x_0. \end{cases}$$

PROOF First we show that the solution if it exists is unique. So suppose that $x_1, x_2 \in W^{1,\infty}\big((0,b);X\big)$ are two solutions of the evolution Cauchy problem. We have

$$\big(x_1(t) - x_2(t)\big)' + A\big(x_1(t)\big) - A\big(x_2(t)\big) \ \ni \ \omega\big(x_1(t) - x_2(t)\big) \quad \text{for a.a. } t \in T.$$

Let

$$\varphi(t) \ = \ \big\|x_1(t) - x_2(t)\big\|.$$

Since $x_1, x_2 \in W^{1,\infty}\big((0,b);X\big)$, they are Lipschitz continuous functions (see Theorem 2.2.24) and so they are differentiable almost everywhere on T (see Theorem 2.2.17). Moreover, φ is a Lipschitz continuous function too, thus differentiable almost everywhere on T. So we can use Lemma 3.3.26 and obtain

$$\begin{aligned} & \big\|x_1(t) - x_2(t)\big\|_X \frac{d}{dt}\big\|x_1(t) - x_2(t)\big\|_X \\ = \ & \big\langle \mathcal{F}\big(x_1(t) - x_2(t)\big), x_1'(t) - x_2'(t)\big\rangle_X \qquad \forall\, t \in T. \end{aligned}$$

Since A is m-accretive, we also have

$$\big\|x_1(t) - x_2(t)\big\|_X \frac{d}{dt}\big\|x_1(t) - x_2(t)\big\|_X \ \leqslant \ \omega\big\|x_1(t) - x_2(t)\big\|_X^2 \qquad \forall\, t \in T,$$

so

$$\frac{d}{dt}\big\|x_1(t) - x_2(t)\big\|_X \ \leqslant \ \omega\big\|x_1(t) - x_2(t)\big\|_X \qquad \forall\, t \in T.$$

Because $x_1(0) - x_2(0) = 0$, by Gronwall's inequality (the differential form), we obtain

$$\|x_1(t) - x_2(t)\|_X \ = \ 0 \qquad \forall\, t \in T.$$

Therefore $x_1 = x_2$.

To establish the existence of a solution, first we consider the following approximate evolution equation.

$$\begin{cases} x'(t) + A_\lambda\big(x(t)\big) = \omega x(t) + f(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x_0, \end{cases} \tag{3.65}$$

with $\lambda > 0$. Because A_λ is a Lipschitz continuous operator (see Proposition 3.3.12**(b)**), problem (3.65) has a unique solution $x_\lambda \in C^1(T;X)$. Using Lemma 3.3.26, we see that for all $\lambda, \mu > 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 + \big\langle \mathcal{F}\big(x_\lambda(t) - x_\mu(t)\big), A_\lambda\big(x_\lambda(t)\big) - A_\mu\big(x_\mu(t)\big)\big\rangle_X \\ &= \omega\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 \qquad \forall\, t \in T, \end{aligned}$$

so by Gronwall's inequality (see Theorem A.4.7), we have

$$\begin{aligned} &\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 \\ &\leqslant -2\int_0^b e^{2\omega(t-s)}\big\langle \mathcal{F}\big(x_\lambda(s) - x_\mu(s)\big), A_\lambda\big(x_\lambda(s)\big) - A_\mu\big(x_\mu(s)\big)\big\rangle_X\, ds \quad t \in T. \end{aligned}$$

Exploiting the accretivity of A and the fact that

$$A_\alpha\big(x_\alpha(t)\big) \;\in\; A\big(J_\alpha\big(x_\alpha(t)\big)\big) \qquad \forall\, t \in T$$

(see Proposition 3.3.12**(c)**), we can write that

$$\begin{aligned} &\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 \\ &\leqslant -2\int_0^b e^{2\omega(t-s)}\big\langle \mathcal{F}\big(x_\lambda(s) - x_\mu(s)\big) - \mathcal{F}\big(J_\lambda\big(x_\lambda(s)\big) - J_\mu\big(x_\mu(s)\big)\big) \times \\ &\qquad \times A_\lambda\big(x_\lambda(s)\big) - A_\mu\big(x_\mu(s)\big)\big\rangle_X\, ds \qquad \forall\, t \in T, \end{aligned}$$

so

$$\begin{aligned} &\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 \\ &\leqslant 2\int_0^b e^{2\omega(t-s)}\big\|\mathcal{F}\big(x_\lambda(s) - x_\mu(s)\big) - \mathcal{F}\big(J_\lambda\big(x_\lambda(s)\big) - J_\mu\big(x_\mu(s)\big)\big)\big\|_{X^*} \times \\ &\qquad \times \big\|A_\lambda\big(x_\lambda(s)\big) - A_\mu\big(x_\mu(s)\big)\big\|_X\, ds \qquad \forall\, t \in T. \end{aligned} \tag{3.66}$$

We claim that

$$\big\|x'_\lambda(t)\big\|_X \;\leqslant\; \big|A(x_0)\big| + \omega\,\|x_0\|_X + \big\|f(0)\big\|_X + e^{2\omega b}\,\|f'\|_1\,. \tag{3.67}$$

Indeed, from (3.65), for every $r \in (0,b)$, we have

$$\begin{aligned} &\big(x_\lambda(t+r) - x_\lambda(t)\big)' + A_\lambda\big(x_\lambda(t+r)\big) - A_\lambda\big(x_\lambda(t)\big) \\ &= \omega\big(x_\lambda(t+r) - x_\lambda(t)\big) + f(t+r) - f(t) \quad \text{for a.a. } t \in T_r = [0, t-r]. \end{aligned}$$

Exploiting the accretivity of A_λ and Lemma 3.3.26, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\|x_\lambda(t+r)-x_\lambda(t)\big\|_X^2 \\ \leqslant\ &\omega\big\|x_\lambda(t+r)-x_\lambda(t)\big\|_X^2 \\ &+\big\langle \mathcal{F}\big(x_\lambda(t+r)-x_\lambda(t)\big), f(t+r)-f(t)\big\rangle_X \quad \text{for a.a. } t\in T_r, \end{aligned}$$

so

$$\begin{aligned} &\big\|x_\lambda(t+r)-x_\lambda(t)\big\|_X^2 \\ \leqslant\ &\big\|x_\lambda(r)-x_\lambda(0)\big\|_X^2 \\ &+2\int_0^t e^{2\omega(t-s)}\big\|x_\lambda(s+r)-x_\lambda(s)\big\|_X\big\|f(s+r)-f(s)\big\|_X\,ds. \end{aligned}$$

Thus by Lemma 3.3.27, we obtain

$$\big\|x_\lambda(t+r)-x_\lambda(t)\big\|_X \leqslant \big\|x_\lambda(r)-x_\lambda(0)\big\|_X+\int_0^t e^{2\omega(t-s)}\big\|f(s+r)-f(s)\big\|_X\,ds.$$

Dividing with $r>0$ and letting $r\searrow 0$, we obtain

$$\begin{aligned} \big\|x'_\lambda(t)\big\|_X &\leqslant \big\|\omega x_0+f(0)-A_\lambda(x_0)\big\|_X+\int_0^b e^{2\omega(t-s)}\big\|f'(s)\big\|_X\,ds \\ &\leqslant \big|A(x_0)\big|+\omega\,\|x_0\|_X+\big\|f(0)\big\|_X+e^{2\omega b}\,\|f'\|_1\,. \end{aligned}$$

This proves (3.67), from which we infer that there exists $M_1>0$, such that

$$\big\|x'_\lambda(t)\big\|_X \leqslant M_1 \qquad \forall\ \lambda>0,\ t\in T. \tag{3.68}$$

Since

$$x_\lambda(t) = x_0+\int_0^t x'_\lambda(s)\,ds \qquad \forall\ \lambda>0,\ t\in T,$$

it follows that there exists $M_2>0$, such that

$$\big\|x_\lambda(t)\big\|_X \leqslant M_2 \qquad \forall\ \lambda>0,\ t\in T. \tag{3.69}$$

Returning to (3.65) and using (3.68) and (3.69), we obtain $M_3>0$, such that

$$\big\|A_\lambda\big(x_\lambda(t)\big)\big\|_X \leqslant M_3 \qquad \forall\ \lambda>0,\ t\in T, \tag{3.70}$$

so

$$\big\|x_\lambda(t)-J_\lambda\big(x_\lambda(t)\big)\big\|_X = \lambda\big\|A_\lambda\big(x_\lambda(t)\big)\big\|_X \leqslant \lambda M_3 \qquad \forall\,\lambda>0,\ t\in T. \tag{3.71}$$

From (3.71), it follows that

$$\big\|x_\lambda(t) - J_\lambda\big(x_\lambda(t)\big)\big\|_X \;\longrightarrow\; 0 \quad \text{as } \lambda \searrow 0, \text{ uniformly on } T.$$

Because $\mathcal{F}$ is uniformly continuous on bounded sets, from (3.66) and (3.70), we see that for a given $\varepsilon > 0$, for all $\lambda, \mu > 0$ small enough, we have

$$\big\|x_\lambda(t) - x_\mu(t)\big\|_X^2 \;\leqslant\; M_4\varepsilon,$$

for some $M_4 > 0$, so

$$x_\lambda \;\longrightarrow\; x \quad \text{in } C(T;X) \quad \text{as } \lambda \searrow 0.$$

Moreover, from (3.67) we infer that x is a Lipschitz continuous function, i.e., $x \in W^{1,\infty}\big((0,b);X\big)$. We claim that this is the solution of the evolution equation. To this end let $(y,z) \in \operatorname{Gr} A$ and let us set

$$y_\lambda \;\stackrel{df}{=}\; y + \lambda z \qquad \forall\, \lambda > 0.$$

Hence $z = A_\lambda(y_\lambda)$. Using (3.65) and the accretivity of A_λ, we obtain

$$\begin{aligned}\big\|x_\lambda(t) - y_\lambda\big\|_X^2 \;&\leqslant\; \big\|x_\lambda(t_0) - y_\lambda\big\|_X^2 \\ &+ \int_{t_0}^{t} \big\langle \mathcal{F}\big(x_\lambda(s) - y\big), \omega x_\lambda(s) + f(s) - z\big\rangle_X \, ds \qquad \forall\, t, t_0 \in T.\end{aligned}$$

Letting $\lambda \searrow 0$, we get

$$\big\|x(t) - y\big\|_X^2 - \big\|x(t_0) - y\big\|_X^2 \;\leqslant\; 2\int_0^t \big\langle \mathcal{F}\big(x(s) - y\big), \omega x(s) + f(s) - z\big\rangle_X. \tag{3.72}$$

Note that for any $z, h \in X$, we have

$$\big\langle \mathcal{F}(h), z - h\big\rangle_X \;\leqslant\; \|h\|_X \, \|z\|_X - \|h\|_X^2 \;\leqslant\; \frac{1}{2}\big(\, \|z\|_X^2 - \|h\|_X^2 \,\big).$$

Using this in (3.72), we have

$$\begin{aligned}&\left\langle \mathcal{F}\big(x(t_0) - y\big), \frac{x(t) - x(t_0)}{t - t_0} \right\rangle_X \\ &\leqslant\; \frac{1}{t - t_0} \int_{t_0}^{t} \big\langle \mathcal{F}\big(x(s) - y\big), \omega x(s) + f(s) - z\big\rangle_X \, ds. \end{aligned} \tag{3.73}$$

Let $t_0 \in T$ be a point of differentiability of x. Passing to the limit as $t \to t_0$ in (3.73), we obtain

$$\big\langle \mathcal{F}\big(x(t_0) - y\big), x'(t_0)\big\rangle_X \;=\; \big\langle \mathcal{F}\big(x(t_0) - y\big), \omega x(t_0) + f(t_0) - z\big\rangle_X,$$

so

$$\big\langle \mathcal{F}\big(x(t_0)-y\big), u_0 - z\big\rangle_X \ \geqslant\ 0, \tag{3.74}$$

with

$$u_0 \ \stackrel{df}{=}\ -x'(t_0) + \omega x(t_0) + f(t_0) \ \in\ A\big(x(t_0)\big)$$

(see (3.65)). Since A is m-accretive, hence maximal accretive, from (3.74), we conclude that

$$-x'(t_0) + \omega x(t_0) + f(t_0) \ \in\ A\big(x(t_0)\big).$$

Because x is almost everywhere differentiable (recall that $x \in W^{1,\infty}\big((0,b);X\big)$), we conclude that x solves (3.65). □

Next we want to consider evolution equations with less regular data. This can be done with remarkable success using the theory of semigroups of operators. In what follows we present some basic aspects of this theory. We start with the linear theory and then pass the nonlinear one.

Let us motivate the definition of a semigroup of bounded linear operators (linear semigroup). So let X be a Banach space and $A \in \mathcal{L}(X)$. We consider the following Cauchy problem

$$\begin{cases} x'(t) = Ax(t) & \forall\ t \geqslant 0, \\ x(0) = x_0 \in X. \end{cases} \tag{3.75}$$

It is easy to check that the function $x(t) \stackrel{df}{=} e^{tA}x_0$ for $t \geqslant 0$, $x \in C^1(\mathbb{R}_+;X)$ is the unique solution of (3.75). Let us mention the basic properties of this solution. First, for every fixed $t \geqslant 0$, the map $x_0 \longmapsto x(t)$ is linear. Moreover, since

$$\big\|x(t)\big\|_X \ \leqslant\ e^{t\|A\|_{\mathcal{L}}}\, \|x_0\|_X\,,$$

it is also bounded. Second, as $t \searrow 0$, we have that

$$x(t) \ \longrightarrow\ x_0 \quad \text{in } X$$

and $x(0) = x_0$. Finally third, by virtue of the uniqueness of the solution of (3.75), if we start with initial condition $x(t_0)$, $t_0 \geqslant 0$ and move for time $t \geqslant 0$, we must reach the state $x(t+t_0)$ (recall that $e^{(t+t_0)A} = e^{tA}e^{t_0A}$). Generalizing these properties we obtain the notion of a C_0-semigroup of linear operators.

DEFINITION 3.3.29 *Let X be a Banach space and $\big\{S(t)\big\}_{t\geqslant 0} \subseteq \mathcal{L}(X)$. We call S a C_0-**semigroup** on X if the following conditions hold:*

(a) *$S(0) = id_X$;*

(b) *$S(t+s)=S(t)S(s)$ for all $s,t \geqslant 0$;*

(c) *$\lim\limits_{t\to 0} \big\|S(t)x - x\big\|_X = 0$ for all $x \in X$.*

REMARK 3.3.30 Property **(b)** is the ***semigroup property***, while property **(c)** implies that the function

$$t \longmapsto S(t)$$

is continuous from $\mathbb{R}_+$ into $\mathcal{L}(X)$ furnished with the strong operator topology. If $A \in \mathcal{L}(X)$, then

$$\left\{S(t) = e^{tA}\right\}_{t \geqslant 0} \text{ is a } C_0\text{-semigroup.}$$

Also if

$$X = C_b(\mathbb{R})$$

(the space of bounded continuous functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ equipped with the supremum norm) and

$$S(t)f(\cdot) = f(t+\cdot) \qquad \forall\ f \in C_b(\mathbb{R}),$$

then

$$\left\{S(t)\right\}_{t \geqslant 0} \text{ is a } C_0\text{-semigroup.}$$

PROPOSITION 3.3.31
If X is a Banach space and $\left\{S(t)\right\}_{t \geqslant 0}$ is a C_0-semigroup on X, then there exist $M \geqslant 1$ and $\omega \geqslant 0$, such that

$$\left\|S(t)\right\|_{\mathcal{L}} \leqslant Me^{\omega t} \qquad \forall\ t \geqslant 0.$$

PROOF By virtue of property **(c)** in Definition 3.3.29, we can find $M \geqslant 1$ and $\delta > 0$, such that

$$\left\|S(t)\right\|_{\mathcal{L}} \leqslant M \qquad \forall\ t \in [0, \delta].$$

Let

$$\omega \stackrel{df}{=} \frac{\ln M}{\delta} \geqslant 0.$$

For a given $t \geqslant 0$, we can find an integer $n \geqslant 0$ and $\vartheta \in [0, \delta)$, such that

$$t = n\delta + \vartheta.$$

Because of the semigroup property, we have

$$S(t) = S(\delta)^n S(\vartheta),$$

so

$$\begin{aligned} \left\|S(t)\right\|_{\mathcal{L}} &\leqslant \left\|S(\delta)\right\|_{\mathcal{L}}^n \left\|S(\vartheta)\right\|_{\mathcal{L}} \\ &\leqslant M^n M = Me^{\omega t}, \end{aligned}$$

since

$$\ln M^n = n \ln M = m\omega\delta \leqslant \omega t.$$

∎

Using this bound, we can improve **(c)** in Definition 3.3.29.

COROLLARY 3.3.32
If X is a Banach space and $\{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup on X, then for all $x \in X$, the map $t \longmapsto S(t)x$ is continuous from $\mathbb{R}_+$ into X.

PROOF Let $r > 0$. Then using the semigroup property and Proposition 3.3.31, we obtain

$$\begin{aligned} & \|S(t+r)x - S(t)x\|_X \\ = \; & \|S(t)\|_{\mathcal{L}} \|S(r)x - x\|_X \\ \leqslant \; & Me^{\omega t}\|S(r)x - x\|_X \longrightarrow 0 \quad \text{as } r \searrow 0. \end{aligned}$$

So the function

$$t \longmapsto S(t)x$$

is continuous on $\mathbb{R}_+$. □

DEFINITION 3.3.33 *Let X be a Banach space and let $\{S(t)\}_{t\geqslant 0}$ be a C_0-semigroup on X. From Proposition 3.3.31, we know that*

$$\|S(t)\|_{\mathcal{L}} \leqslant Me^{\omega t} \qquad \forall\, t \geqslant 0$$

for some $M \geqslant 1$ and $\omega \geqslant 0$. If $M = 1$ and $\omega = 0$, i.e.,

$$\|S(t)\|_{\mathcal{L}} \leqslant 1 \qquad \forall\, t \geqslant 0,$$

*then we say that we have a **contraction semigroup**.*

The following notion is central in the theory of linear semigroups and is the starting point for determining those operators which generate contraction semigroups (see the Hille-Yosida Theorem 3.3.46).

DEFINITION 3.3.34 *Let X be a Banach space and let $\{S(t)\}_{t\geqslant 0}$ be a C_0-semigroup on X. We introduce the **generator** (or **infinitesimal generator**) of the semigroup S as the linear operator $A\colon X \supseteq D(A) \longrightarrow X$, defined by*

$$Ax \stackrel{df}{=} \lim_{t\searrow 0} \frac{S(t)x - x}{t} \qquad \forall\, x \in D(A),$$

where

$$D(A) \stackrel{df}{=} \left\{ x \in X : \lim_{t\searrow 0} \frac{S(t)x - x}{t} \quad \textit{exists} \right\}.$$

In general the operator A is not bounded.

EXAMPLE 3.3.35 **(a)** Let $A \in \mathcal{L}(X)$ and $S(t) = e^{At}$ for $t \geqslant 1$. This is a C_0-semigroup. Then for every $x \in X$, we have

$$\frac{e^{tA}x - x}{t} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} A^k x = Ax + \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} A^k x$$

and

$$\begin{aligned} \left\| \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} A^k x \right\|_X &\leqslant \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} \|A\|_{\mathcal{L}}^k \|x\|_X \\ &\leqslant t \|A\|_{\mathcal{L}}^2 \|x\|_X \sum_{k=0}^{\infty} \frac{t^k}{k!} \|A\|_{\mathcal{L}}^k \\ &= |t| \|A\|_{\mathcal{L}}^2 \|x\|_X e^{t\|A\|_{\mathcal{L}}} \longrightarrow 0 \quad \text{as } t \searrow 0. \end{aligned}$$

Therefore the generator of S is A.

(b) Let

$$X = C(B; X) \quad \text{and} \quad S(t)f(\cdot) \stackrel{df}{=} f(t + \cdot)$$

(see Remark 3.3.30). We have

$$\lim_{t \searrow 0} \frac{(S(t)f - f)(s)}{t} = D^+ f(s) \quad \text{if it exists.}$$

So if $f \in D(A)$, then $D^+ f$ exists at all $s \geqslant 0$ and it is bounded and uniformly continuous. Also we have

$$\begin{aligned} \frac{f(s) - f(s-t)}{t} &= \frac{f(s-t+t) - f(s-t)}{t} \\ &= D^+ f(s-t) + \frac{o(t)}{t} \longrightarrow D^+ f(s) \quad \text{as } t \searrow 0 \end{aligned}$$

(since $D^+ f$ is continuous). Therefore if $f \in D(A)$, then $D^+ f = D^- f$, i.e., $f'(t)$ exists at all $t \in \mathbb{R}$ and $f' \in C_b(\mathbb{R})$. So

$$D(A) = \left\{ f \in C_b(\mathbb{R}) : \ f' \text{ exists everywhere and } f' \in C_b(\mathbb{R}) \right\}$$

and $Af = f'$ for all $f \in D(A)$.

More generally, if $X = H = L^2(0, b)$ and

$$S(t)x(s) \stackrel{df}{=} \begin{cases} x(t+s) & \text{if} \quad t + s \in (0, b), \\ 0 & \text{if} \quad t + s \notin (0, b), \end{cases}$$

then the generator of S is the operator $A \colon H \supseteq D(A) \longrightarrow H$, defined by

$$Ax(t) = \frac{d}{dt} x(t),$$

with

$$D(A) = \left\{ x \in W^{1,2}(0, b) : \ x(b) = 0 \right\}.$$

In the next proposition, we summarize the differential properties of C_0-semigroups.

PROPOSITION 3.3.36
If X is a Banach space and $\{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup on X with generator A,
then for all $x \in D(A)$ and all $t \geqslant 0$, we have

(a) $S(t)x \in D(A)$;

(b) $\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax$ *for* $t > 0$;

(c) $S(t)x - x = \int_0^t S(r)Ax\,dr$;

(d) $D(A)$ *is dense in* X *and the operator* A *is closed (i.e.,* $\operatorname{Gr} A \subseteq X \times X$ *is closed).*

PROOF **(a)** For $r > 0$ we have

$$\frac{S(t+r)x - S(t)x}{r} = S(t)\frac{S(r)x - x}{r} \longrightarrow S(t)Ax \quad \text{as } r \searrow 0.$$

Hence $S(t)x \in D(A)$.

(b) From **(a)**, we have

$$\frac{d^+}{dt}S(t)x = S(t)Ax.$$

Also note that

$$\frac{S(t+r)x - S(t)x}{r} = S(t)\frac{S(r)x - x}{r}$$
$$= \left(\frac{S(r) - id_X}{r}\right)S(t)x \longrightarrow AS(t) \quad \text{as } r \searrow 0,$$

so

$$\frac{d^+}{dt}S(t)x = S(t)Ax = AS(t)x.$$

On the other hand, for $t > r > 0$, we have

$$\frac{S(t)x - S(t-r)x}{r} = S(t-r)\frac{S(r)x - x}{x} \longrightarrow S(t)Ax \quad \text{as } r \searrow 0,$$

so

$$\frac{d^-}{dt}S(t)x = S(t)Ax.$$

Also since

$$S(t-r)\frac{S(r)x - x}{r} = \left(\frac{S(r) - id_X}{r}\right)S(t-r),$$

we have

$$\frac{d^-}{dt}S(t)x \;=\; S(t)Ax \;=\; AS(t)x.$$

So finally, we conclude that

$$\frac{d}{dt}S(t)x \;=\; S(t)Ax \;=\; AS(t)x.$$

(c) By part **(b)**, the function $t \longmapsto S(t)x$ is continuously differentiable. So for all $x^* \in X^*$, we have

$$\big\langle x^*, S(t)x - x\big\rangle_X \;=\; \int_0^t \frac{d}{dr}\big\langle x^*, S(r)x\big\rangle_X \, dr$$

$$= \int_0^t \left\langle x^*, \frac{d}{dt}S(r)x \right\rangle_X dr \;=\; \left\langle x^*, \int_0^t S(r)Ax\, dr \right\rangle_X,$$

hence

$$S(t)x - x \;=\; \int_0^t S(r)Ax\, dr \qquad \forall\; t \geqslant 0.$$

(d) For $t > r > 0$ and $x \in X$, we have

$$\frac{S(r) - id_X}{r}\left(\int_0^t S(\tau)x\, d\tau\right)$$

$$= \frac{1}{r}\int_0^t \big(S(\tau + r)x - S(\tau)x\big)\, d\tau$$

$$= \frac{1}{r}\left[\int_r^{t+r} S(\tau)x\, d\tau - \int_0^t S(\tau)x\, d\tau\right]$$

$$= \frac{1}{r}\left[\int_r^{t+r} S(\tau)x\, d\tau - \int_0^r S(\tau)x\, d\tau\right] \;\longrightarrow\; S(t)x - x \quad \text{as } r \searrow 0,$$

so

$$\int_0^t S(\tau)x\, d\tau \;\in\; D(A).$$

But note that

$$\lim_{t \searrow 0} \frac{1}{t}\int_0^t S(\tau)x\, d\tau \;=\; x$$

and since $x \in X$ was arbitrary, we conclude that $D(A)$ is dense in X.

Next let $\{x_n\}_{n\geqslant 1} \subseteq D(A)$ and assume that

$$x_n \longrightarrow x \quad \text{and} \quad A(x_n) \longrightarrow y \quad \text{in } X.$$

For every $r \geqslant 0$, we have

$$\big\|S(r)Ax_n - S(r)y\big\|_X \leqslant Me^{\omega r}\big\|Ax_n - y\big\|_X$$

(see Proposition 3.3.31) and thus

$$S(\cdot)Ax_n \longrightarrow S(\cdot)y \quad \text{in } X \text{ uniformly on } [0,t],\ t \geqslant 0.$$

From **(c)** we know that

$$S(t)x_n - x_n = \int_0^t S(r)Ax_n\, dr.$$

Passing to the limit as $n \to +\infty$, we obtain

$$S(t)x - x = \int_0^t S(r)y\, dr.$$

Hence

$$\lim_{t\searrow 0} \frac{S(t)x - x}{t} = y,$$

which implies that $x \in D(A)$ and $y = Ax$, i.e., A is closed. □

REMARK 3.3.37 Using part **(b)** of Proposition 3.3.36 and induction, we can show that for all $n \geqslant 1$, all $x \in D(A^n)$ and all $t > 0$, we have

$$\frac{d^n}{dt^n}S(t)x = S(t)A^n x = A^n S(t)x.$$

Moreover, it can be shown that

$$\text{the set } \bigcap_{n=1}^{\infty} D(A^n) \text{ is dense in } X.$$

For details we refer to Pazy (1983, p. 6). Also because of parts **(b)** and **(d)** of Proposition 3.3.36 and using Theorem 2.1.17, we can rewrite part **(c)** as follows

(c)' for all $t \geqslant 0$ and all $x \in X$, we have

$$S(t)x - x = A\int_0^t S(\tau)x\, d\tau.$$

COROLLARY 3.3.38
An operator $A\colon X \supseteq D(A) \longrightarrow X$ can be generator of at most one C_0-semigroup.

PROOF Suppose that S_1 and S_2 are two C_0-semigroups with generator A. Let $x \in D(A)$ and $t \geqslant 0$ and define the function

$$u(s) \stackrel{df}{=} S_1(t-s)S_2(s)x \qquad \forall\, s \in (0,t).$$

From Proposition 3.3.36**(b)**, we know that $u \in C^1\big((0,t);X\big)$ and

$$\begin{aligned} u'(s) &= -AS_1(t-s)S_2(s)x + S_1(t-s)AS_2(s)x \\ &= -S_1(t-s)AS_2(s)x + S_1(t-s)AS_2(s)x \;=\; 0, \end{aligned}$$

so

$$S_1(t)x - S_2(t)x \;=\; u(0)-u(t) \;=\; -\int_0^t u'(s)\,ds \;=\; 0 \qquad \forall\, x \in D(A). \tag{3.76}$$

Because $D(A)$ is dense in X (see Proposition 3.3.36**(d)**), from (3.76), it follows that

$$S_1(t) \;=\; S_2(t) \qquad \forall\, t \geqslant 0.$$

□

If $\big\{S(t)\big\}_{t\geqslant 0}$ is a C_0-semigroup on a Banach space X, then $\big\{S(t)^*\big\}_{t\geqslant 0}$ still has the semigroup property but need not be a C_0-semigroup. In fact in general we can only show that for every $x^* \in X^*$, $t \longmapsto S(t)^*x^*$ is weakly*-continuous at $t = 0$, i.e.,

$$w^*\text{-}\lim_{t\searrow 0} S(t)^*x \;=\; x.$$

So the map $S(t) \longmapsto S(t)^*$ does not preserve the strong continuity at $t = 0$.

EXAMPLE 3.3.39 Let $X = C_0(\mathbb{R})$ (see Section 2.3) and let $\big\{S(t)\big\}_{t\geqslant 0}$ be the C_0-semigroup of left translations, i.e.,

$$\big(S(t)f\big)(s) \;\stackrel{df}{=}\; f(t+s) \qquad \forall\, t,s \geqslant 0,\ f \in C_0(\mathbb{R}).$$

We know that $X^* = NBV(\mathbb{R})$, the space of all normalized functions of bounded variation with the total variation norm

$$\|\vartheta\|_{TV(\mathbb{R})} \;=\; (\operatorname{Var}\vartheta)(\mathbb{R}) \;=\; \int_{\mathbb{R}} \big|\,d\vartheta(t)\big|.$$

By saying that ϑ is normalized, we mean that

$$\vartheta(s) \;=\; \frac{\vartheta(s^+)+\vartheta(s^-)}{2} \qquad \forall\, s\in\mathbb{R}$$

and

$$\vartheta(-\infty) \;=\; \lim_{s\to-\infty}\vartheta(s) \;=\; 0 \quad\text{and}\quad \vartheta(+\infty) \;=\; \lim_{s\to+\infty}\vartheta(s) \;=\; 0$$

(see also Theorem 2.3.41). For all $f\in C_0(\mathbb{R})$ and all $\vartheta\in NBV(\mathbb{R})$, we have

$$\big\langle \vartheta, S(t)f\big\rangle \;=\; \int_{\mathbb{R}} f(t+s)\,d\vartheta(s) \;=\; \int_{\mathbb{R}} f(s)\,d_s\vartheta(s-t) \;=\; \big\langle S(t)^*\vartheta, f\big\rangle,$$

so

$$\big(S(t)^*\vartheta\big)(s) \;=\; \vartheta(s-t),$$

i.e., $\big\{S(t)^*\big\}_{t\geqslant 0}$ is the right translation of ϑ.

By a theorem of Plessner (1929), we know that

$$\Big[\big\|\vartheta(\cdot-t)-\vartheta(\cdot)\big\|_{TV(\mathbb{R})} \;\longrightarrow\; 0 \quad \text{as } t\searrow 0\Big] \;\Longleftrightarrow\; \Big[\vartheta\in AC(\mathbb{R})\Big].$$

So the function $t\longmapsto S(t)^*\vartheta$ is not in general strongly continuous at $t=0$ (unless of course $\vartheta\in AC(\mathbb{R})$).

Note that in the previous example $X=C_0(\mathbb{R})$ is not a reflexive Banach space. This is not an accident.

PROPOSITION 3.3.40
If H is a Hilbert space, $H^*=H$ (i.e., H is a pivot space) and $\big\{S(t)\big\}_{t\geqslant 0}$ is a C_0-semigroup on H,
then $\big\{S(t)^*\big\}_{t\geqslant 0}$ is a C_0-semigroup on H.

PROOF Clearly $S(\cdot)^*$ satisfies the semigroup property. Therefore we need to show that for all $h\in H$, the map $t\longmapsto S(t)^*h$ is strongly continuous at $t=0$. For every $x,h\in H$, the function

$$s \;\longmapsto\; \big(S(s)^*h,x\big)_H \;=\; \big(h,S(s)x\big)_H$$

is continuous on $\mathbb{R}$. Therefore for all $h\in H$, the function $s\longmapsto S(s)^*h$ is weakly continuous. So it follows from the uniform boundedness principle (see Theorem A.3.4) and the semigroup property that the function $s\longmapsto S(s)^*h$ is bounded on any compact interval of $\mathbb{R}_+$. Also it is weakly measurable. Moreover, if $\{s_n\}_{n\geqslant 1}$ is an enumeration of the rationals in $\mathbb{R}_+$ and we consider

$$L \;\stackrel{df}{=}\; \operatorname{span}_{\mathbb{Q}}\big\{S(s_n)^*h\big\}_{n\geqslant 1}$$

(i.e., finite linear combinations of $\{S(s_n)^*h\}_{n\geqslant 1}$ with rational coefficients), then L is countable and $H_0 \stackrel{df}{=} \overline{\text{span}}\, L$ is separable. Since the function $s \longmapsto S(s)^*h$ is weakly continuous, we see that $\{S(s)^*h\}_{s\geqslant 0} \subseteq H_0$. Therefore we have proved that the function $s \longmapsto S(s)^*h$ is weakly measurable and separably valued; hence by Theorem 2.1.3, it is strongly measurable. We infer that $S(\cdot)^*h \in L^1_{\text{loc}}(\mathbb{R}; H)$. If $t > 0$ and $\eta \in (0, t)$, we have

$$\begin{aligned}
\big\|S(t+r)^*h - S(t)^*h\big\|_H &= \left\|\frac{1}{\eta}\int_0^\eta \big(S(t+r)^*h - S(t)^*h\big)\,d\tau\right\|_H \\
&= \left\|\frac{1}{\eta}\int_0^\eta S(\tau)^*\big(S(t+r-\tau)^*h - S(t-\tau)^*h\big)\,d\tau\right\|_H \\
&\leqslant \frac{1}{\eta}M_1\int_0^\eta \big\|S(t+r-\tau)^*h - S(t-\tau)^*h\big\|_H\,d\tau, \qquad (3.77)
\end{aligned}$$

for some $M_1 > 0$, so by Corollary 2.3.8, we have

$$\begin{aligned}
&\lim_{r\searrow 0}\big\|S(t+r)^*h - S(t)^*h\big\|_H \\
&\leqslant \lim_{r\searrow 0}\frac{M_1}{\eta}\int_0^\eta \big\|S(t+r-\tau)^*h - S(t-\tau)^*h\big\|_H\,d\tau = 0. \qquad (3.78)
\end{aligned}$$

Finally let $t_n \searrow 0$ and

$$C \stackrel{df}{=} \overline{\text{conv}}\{S(t_n)^*h\}_{n\geqslant 1}.$$

Because

$$S(t_n)^*h \xrightarrow{w} h \quad \text{in } H,$$

we have that $h \in C$. So if

$$E \stackrel{df}{=} \text{span}\{S(t_n)^*h\}_{n\geqslant 1},$$

then from (3.77), it follows that

$$\lim_{n\to+\infty} S(t_n)^*h = h \qquad \forall\, h \in E.$$

Because

$$\sup_{n\geqslant 1}\big\|S(t_n)^*\big\|_{\mathcal{L}} \leqslant M_2,$$

for some $M_2 > 0$, we conclude that

$$\lim_{n\to+\infty} S(t_n)^*h = h \qquad \forall\, h \in H.$$

□

REMARK 3.3.41 In fact the result is true if H is replaced by a reflexive Banach space. This is a consequence of the fact that if $\{S(t)\}_{t\geqslant 0}$ is a semigroup of linear operators, such that for all $x \in X$, the function $t \longmapsto S(t)x$ is strongly measurable (this is the case if for all $x \in X$, the function $t \longmapsto S(t)x$ is weakly continuous), then S is a C_0-semigroup. For details we refer to Hille & Phillips (1957, pp. 305–306).

Of great importance in applications are theorems which give necessary and sufficient conditions for an operator A to be the infinitesimal generator of a C_0-semigroup. The basic result in this direction is the celebrated Hille-Yosida theorem. To state and prove this fundamental result we need some preparation.

DEFINITION 3.3.42 *Let X be a Banach space and let $A\colon X \supseteq D(A) \longrightarrow X$ be a closed, linear operator.*

(a) *The **resolvent set** $\varrho(A)$ of A is defined by*

$$\varrho(A) \stackrel{df}{=} \{\lambda \in \mathbb{R} :\ \lambda id_X - A\colon X \supseteq D(A) \longrightarrow X \text{ is bijective}\}.$$

(b) *If $\lambda \in \varrho(A)$, then the **resolvent operator** $R_\lambda\colon X \longrightarrow X$ is defined by*

$$R_\lambda x \stackrel{df}{=} (\lambda id_X - A)^{-1} \qquad \forall\ x \in X.$$

REMARK 3.3.43 It is easy to check that R_λ is closed. So by the closed graph theorem (see Theorem A.3.7), we have that $R_\lambda \in \mathcal{L}(X)$. Moreover, we have $AR_\lambda x = R_\lambda Ax$ for all $x \in D(A)$ (see the proof of Proposition 3.3.44**(b)**).

PROPOSITION 3.3.44
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow X$ is a closed, linear operator, then

(a) *for $\lambda, \mu \in \varrho(A)$, we have*

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$$

(resolvent identity) and $R_\lambda R_\mu = R_\mu R_\lambda$;

(b) *if A is the generator of a C_0-semigroup S and*

$$\|S(t)\|_{\mathcal{L}} \leqslant Me^{\omega t} \qquad \forall\ t \geqslant 0,$$

then

$$\lambda \in \varrho(A) \qquad \forall\ \lambda > \omega$$

and

$$R_\lambda x = \int_0^{+\infty} e^{-\lambda t} S(t)x\, dt \qquad \forall\ x \in X.$$

PROOF Let $x \in D(A)$. We have

$$\begin{aligned} & (\lambda id_X - A)\big[R_\lambda - R_\mu\big](\mu id_X - A)(x) \\ = \; & (\mu id_X - A)x - (\lambda id_X - A)x \;=\; (\mu - \lambda)x, \end{aligned}$$

so

$$R_\lambda - R_\mu \;=\; (\mu - \lambda) R_\lambda R_\mu. \tag{3.79}$$

The commutation of R_λ and R_μ follows by interchanging λ and μ in (3.79).

(b) Let us set

$$\widehat{R}_\lambda x \;\stackrel{df}{=}\; \int_0^{+\infty} e^{-\lambda t} S(t)x \, dt \qquad \forall\ \lambda > \omega,\ x \in X.$$

This operator is well defined, since

$$\big\| e^{-\lambda t} S(t)x \big\|_X \;\leqslant\; M e^{(\omega - \lambda)t} \, \|x\|_X$$

and the function

$$t \longmapsto e^{-\lambda t} S(t)x$$

is continuous, thus strongly measurable. We have

$$\big\| \widehat{R}_\lambda \big\|_{\mathcal{L}} \;\leqslant\; M \int_0^{+\infty} e^{-(\lambda - \omega)t} \, dt \;\leqslant\; \frac{M}{\lambda - \omega},$$

i.e., $\widehat{R}_\lambda \in \mathcal{L}(X)$.

We show that $\widehat{R}_\lambda x \in D(A)$ and

$$(\lambda id_X - A)\widehat{R}_\lambda x \;=\; x \qquad \forall\ x \in X.$$

We have

$$\begin{aligned} & \frac{S(s) - id_X}{s} \widehat{R}_\lambda x \;=\; \frac{1}{s} \int_0^{+\infty} e^{-\lambda t} \big(S(t+s) - S(t) \big) x \, dt \\ = \; & \frac{e^{\lambda s} - 1}{s} \int_s^{\infty} e^{-\lambda t} S(t)x \, dx - \frac{1}{s} \int_0^s e^{-\lambda t} S(t)x \, dt. \end{aligned}$$

Passing to the limit as $s \searrow 0$, we obtain

$$A\widehat{R}_\lambda x \;=\; \lambda \widehat{R}_\lambda x - x \qquad \forall\ x \in X. \tag{3.80}$$

Using Proposition 3.3.36**(b)** and Theorem 2.1.17, for all $x \in D(A)$, we have

$$\begin{aligned} \widehat{R}_\lambda Ax &= \int_0^{+\infty} e^{-\lambda t} S(t)Ax\,dt \\ &= \int_0^{+\infty} e^{-\lambda t} AS(t)x\,dt \\ &= A\int_0^{+\infty} e^{-\lambda t} S(t)x\,dt = A\widehat{R}_\lambda x. \end{aligned} \tag{3.81}$$

From (3.80) and (3.81), it follows that

$$\widehat{R}_\lambda(\lambda id_X - A)x = x$$

and

$$(\lambda id_X - A)\widehat{R}_\lambda x = x \qquad \forall\, x \in D(A),$$

so

$$R_\lambda = \widehat{R}_\lambda \quad \text{and} \quad \lambda \in \varrho(A).$$

□

REMARK 3.3.45 Because of Proposition 3.3.44**(b)**, we see that the resolvent operator is the Laplace transform of the C_0-semigroup generated by A. Hence the function

$$\lambda \longmapsto R(\lambda)$$

is analytic on $\varrho(A)$.

Now we are ready for the theorem characterizing the generators of C_0-semigroups.

THEOREM 3.3.46 (Hille-Yosida Theorem)
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow X$ is a closed, densely defined, linear operator,
then A is the generator of a C_0-semigroup if and only if there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$, such that

$$\left\|R_\lambda^k\right\|_{\mathcal{L}} \leqslant \frac{M}{(\lambda-\omega)^k} \qquad \forall\, \lambda \in \varrho(A),\ \lambda > \omega.$$

In this case we have

$$\left\|S(t)\right\|_{\mathcal{L}} \leqslant Me^{\omega t} \qquad \forall\, t \geqslant 0.$$

PROOF "$\Longrightarrow$": From Proposition 3.3.44, we know that if $\lambda > \omega$, then $\lambda \in \varrho(A)$ and we have

$$R_\lambda x \;=\; \int_0^{+\infty} e^{-\lambda t} S(t)x\, dt \qquad \forall\, x \in X,$$

so

$$\frac{d^{(k-1)}}{d\lambda^{k-1}} R_\lambda x \;=\; R_\lambda^{k-1} x \;=\; \int_0^{+\infty} (-t)^{k-1} e^{-\lambda t} S(t)x\, dt \qquad \forall\, k \geqslant 1$$

and thus

$$\left\| R_\lambda^{k-1} \right\|_{\mathcal{L}} \;\leqslant\; M \int_0^{+\infty} t^{k-1} e^{-(\lambda-\omega)t}\, dt \;=\; M(k-1)!(\lambda-\omega)^{-k}. \tag{3.82}$$

But the function $\lambda \longmapsto R_\lambda$ is analytic on $\varrho(A)$ (see Remark 3.3.45). Hence

$$R_\lambda^{(k-1)} \;=\; (-1)^{k-1}(k-1)! R_\lambda^k. \tag{3.83}$$

From (3.82) and (3.83), it follows that

$$\left\| R_\lambda^k \right\|_{\mathcal{L}} \;\leqslant\; \frac{M}{(\lambda-\omega)^k}.$$

"$\Longleftarrow$": For $\lambda > \omega$, let us set

$$A_\lambda \;\stackrel{df}{=}\; \lambda^2 R_\lambda - \lambda id_X.$$

We have that $R_\lambda \in \mathcal{L}(X)$ and so we obtain the C_0-semigroup

$$S_\lambda(t) \;=\; e^{A_\lambda t} \;=\; e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} (\lambda id_X - A)^{-n}$$

(see Remark 3.3.30). We shall show that as $\lambda \to +\infty$, then $S_\lambda(t)$ converges in the strong operator topology to $S(t)$, $t \geqslant 0$, which is the desired semigroup. Note that

$$x \;=\; R_\lambda(\lambda id_X - A)x \;=\; \lambda R_\lambda x - R_\lambda Ax \;=\; \lambda R_\lambda x - AR_\lambda x \qquad \forall\, x \in D(A)$$

(see Remark 3.3.43), so

$$A_\lambda x \;=\; \lambda R_\lambda Ax \;=\; \lambda AR_\lambda x. \tag{3.84}$$

Also, we have

$$\left\| \lambda R_\lambda x - x \right\|_X \;=\; \left\| \lambda R_\lambda Ax \right\|_X \;\leqslant\; \frac{M}{\lambda-\omega} \left\| Ax \right\|_X \qquad \forall\, x \in D(A),$$

so

$$\lambda R_\lambda x \longrightarrow x \quad \text{in } X \quad \text{as } \lambda \to +\infty, \qquad \forall\, x \in D(A).$$

But $D(A)$ is dense in X. So for a given $x \in X$, we can find a sequence $\{x_m\}_{m\geqslant 1} \subseteq D(A)$, such that

$$x_m \longrightarrow x \quad \text{in } X.$$

Then for $\lambda_n \longrightarrow +\infty$ as $n \to +\infty$, we have

$$\lambda_n R_{\lambda_n} x_m \longrightarrow x_m \quad \text{in } X \quad \text{as } n \to +\infty.$$

By the double limit lemma (see Proposition A.2.35), we can find an increasing sequence $\{m(n)\}_{n\geqslant 1}$ (not necessarily strictly) to $+\infty$, such that

$$\lambda_n R_{\lambda_n} x_{m(n)} \longrightarrow x \quad \text{in } X, \quad \text{as } n \to +\infty.$$

Then we have

$$\begin{aligned} \big\|\lambda_n R_{\lambda_n} x - x\big\|_X &\leqslant \big\|\lambda_n R_{\lambda_n} x - \lambda_n R_{\lambda_n} x_{m(n)}\big\|_X + \big\|\lambda_n R_{\lambda_n} x_{m(n)} - x\big\|_X \\ &\leqslant \frac{\lambda_n M}{\lambda_n - \omega}\big\|x - x_{m(n)}\big\|_X + \big\|\lambda_n R_{\lambda_n} x_{m(n)} - x\big\|_X \longrightarrow 0, \end{aligned}$$

so

$$\lambda R_\lambda x \longrightarrow x \quad \text{in } X \quad \text{as } \lambda \to +\infty \qquad \forall\, x \in X.$$

Then because of (3.84), we have

$$A_\lambda x \longrightarrow Ax \quad \text{in } X \quad \text{as } \lambda \to +\infty \qquad \forall\, x \in D(A).$$

For every $\lambda > \omega$ and $t \geqslant 0$, we have

$$\big\|S_\lambda(t)\big\|_{\mathcal{L}} \leqslant e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} \frac{M}{(\lambda - \omega)^n} = M e^{\frac{\lambda\omega}{\lambda-\omega} t}.$$

Also from Proposition 3.3.44**(a)**, we have that

$$A_\lambda A_\mu = A_\mu A_\lambda \qquad \forall\, \lambda, \mu > 0$$

and so

$$A_\lambda S_\mu(t) = S_\mu(t) A_\lambda \qquad \forall\, \lambda, \mu > 0,\ t > 0.$$

From Proposition 3.3.36**(b)**, it follows that

$$\begin{aligned} S_\lambda(t)x - S_\mu(t)x &= \int_0^t \frac{d}{dt}\big(S_\mu(t-s) S_\lambda(s) x\big)\, ds \\ &= \int_0^t S_\mu(t-s)(A_\lambda - A_\mu) S_\lambda(s) x\, ds \\ &= \int_0^t S_\mu(t-s) S_\lambda(s)(A_\lambda - A_\mu) x\, ds \qquad \forall\, x \in D(A), \end{aligned}$$

so

$$\begin{aligned} &\big\|S_\lambda(t)x - S_\mu(t)x\big\|_X \\ &\leqslant\ M^2 e^{\frac{\mu\omega}{\mu-\omega}t}\big\|(A_\lambda - A_\mu)x\big\|_X \int_0^t e^{-\frac{(\lambda-\mu)\omega^2 s}{(\mu-\omega)(\lambda-\omega)}}\,ds. \end{aligned}$$

Let $\lambda > \mu$. We have

$$\big\|S_\lambda(t)x - S_\mu(t)x\big\|_X \ \leqslant\ M^2 e^{\frac{\mu\omega}{\mu-\omega}t}\big\|(A_\lambda - A_\mu)x\big\|_X \ \longrightarrow\ 0 \quad \text{as } \lambda, \mu \to +\infty,$$

and thus $S_\lambda(t)x$ converges to some limit as $\lambda \to +\infty$ uniformly on compact intervals.

Denote this limit by $S(t)x$, $x \in D(A)$. As before exploiting the density of $D(A)$ in X, we have that

$$S_\lambda(t)x \ \longrightarrow\ S(t)x \quad \text{in } X \quad \text{as } \lambda \to +\infty \qquad \forall\ x \in X$$

and the convergence is uniform on compact intervals in $\mathbb{R}_+$. This means that the function $t \longmapsto S(t)x$ is continuous and since S clearly satisfies the semigroup property and $S(0) = id_X$, we conclude that S is a C_0-semigroup.

It remains to show that A is the generator of S. Let $\widehat{A}$ be the generator of S. From Proposition 3.3.36**(c)**, we know that

$$S_\lambda(t)x - x \ =\ \int_0^t S_\lambda(s)A_\lambda x\,ds \qquad \forall\ t \geqslant 0,\ \lambda > 0,\ x \in D(A). \tag{3.85}$$

Note that

$$\begin{aligned} &\big\|S_\lambda(s)A_\lambda x - S(s)Ax\big\|_X \\ &\leqslant\ \big\|S_\lambda(s)A_\lambda x - S_\lambda(s)Ax\big\|_X + \big\|S_\lambda(s)Ax - S(s)Ax\big\|_X \\ &\leqslant\ \big\|S_\lambda(s)\big\|_{\mathcal{L}}\big\|A_\lambda x - Ax\big\|_X + \big\|S_\lambda(s)Ax - S(s)Ax\big\|_X \\ &\qquad\qquad \forall\ s \in [0,t],\ \lambda > 0,\ x \in D(A), \end{aligned}$$

so

$$\big\|S_\lambda(s)A_\lambda x - S(s)Ax\big\|_X \ \longrightarrow\ 0 \quad \text{as } \lambda \to +\infty.$$

Thus if we pass to the limit as $\lambda \to +\infty$ in (3.85), we obtain

$$S(t)x - x \ =\ \int_0^t S(s)Ax\,ds \qquad \forall\ t \geqslant 0,\ x \in D(A),$$

so

$$D(A) \ \subseteq D(\widehat{A}) \quad \text{and} \quad Ax \ =\ \widehat{A}x \qquad \forall\ x \in D(A),$$

i.e., $\widehat{A}$ is an extension of A.

Now if $\lambda > \omega$, then

$$\lambda \in \varrho(A) \cap \varrho(\widehat{A})$$

and so

$$(\lambda id_X - \widehat{A})\big(D(A)\big) \;=\; (\lambda id_X - A)\big(D(A)\big) \;=\; X.$$

Thus $\big(\lambda id_X - A\big)|_{D(A)}$ is bijective and so $D(A) = D(\widehat{A})$, i.e., $A = \widehat{A}$. □

REMARK 3.3.47 The operator $A_\lambda \in \mathcal{L}(X)$ introduced in the above proof is known as the ***Yosida approximation*** of A. Note that if A is also dissipative (see Remark 3.3.2), then A_λ coincides with the notion introduced in Definition 3.3.11. However, there A was not necessarily linear.

The following generation theorem for perturbed operators is useful in applications and is known as Phillips theorem.

THEOREM 3.3.48 (Phillips Theorem)
If X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow X$ is the generator of a C_0-semigroup and $B \in \mathcal{L}(X)$,
then $A + B\colon X \supseteq D(A) \longrightarrow X$ is also the generator of a C_0-semigroup.

Another important generation result is the so-called ***Lumer-Phillips theorem***

THEOREM 3.3.49 (Lumer-Phillips Theorem)
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow X$ is a densely defined, linear, m-dissipative operator,
then A is the generator of a contraction semigroup.

PROOF By Lemma 3.3.3, we have that

$$\big\|\lambda x - Ax\big\|_X \;\geqslant\; \lambda \|x\|_X \qquad \forall\, \lambda > 0,\; x \in D(A).$$

Also

$$R(\lambda id_X - A) \;=\; X \qquad \forall\, \lambda > 0,$$

due to the m-dissipativity of A (see Definition 3.3.1 and Proposition 3.3.14). It follows that

$$\mathbb{R}_+ \;\subseteq\; \varrho(A)$$

and

$$\|R_\lambda\|_{\mathcal{L}} \;\leqslant\; \frac{1}{\lambda} \qquad \forall\, \lambda > 0.$$

Then by Theorem 3.3.46, A generates a contraction semigroup. □

Recall that if X is a Banach space and $A \in \mathcal{L}(X)$, then A is the generator of the semigroup

$$S(t) \stackrel{df}{=} e^{tA} \qquad \forall\, t \geqslant 0.$$

Moreover, from elementary analysis, we know that for the exponential function, we have

$$e^{-at} = \lim_{n \to +\infty} \left(1 + \frac{at}{n}\right)^{-n}.$$

In the next theorem, we show that even if A is unbounded, the limit expression is valid for the semigroup generated by A. The result is known as the ***exponential formula***. First a lemma.

LEMMA 3.3.50
If X is a Banach space and $B \in \mathcal{L}(X)$ with $\|B\|_{\mathcal{L}} \leqslant 1$, then

$$\left\|e^{n(B-id_X)}x - B^n x\right\|_X \leqslant \sqrt{n}\,\|x - Bx\|_X \qquad \forall\, n \geqslant 1,\ x \in X.$$

PROOF For $k > n$, we have

$$\begin{aligned}\left\|B^k x - B^n x\right\|_X &= \sum_{m=n}^{k-1} \left(B^{m+1}x - B^m x\right) \\ \leqslant \|x - Bx\|_X \sum_{m=n}^{k-1} \|B^m\|_{\mathcal{L}} &\leqslant |k-n| \left\|x - Bx\right\|_X.\end{aligned}$$

Then for any $t > 0$, we have

$$\begin{aligned}\left\|e^{t(B-id_X)}x - B^n x\right\|_X &= \left\|e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!}\left(B^k x - B^n x\right)\right\|_X \\ \leqslant\ & e^{-t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}|k-n|\right)\left\|x - Bx\right\|_X \\ \leqslant\ & e^{-t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(k-n)^2\right)^{\frac{1}{2}} \left\|x - Bx\right\|_X \\ =\ & e^{-\frac{t}{2}}\left(t^2 - (2n-1)t + n^2\right)^{\frac{1}{2}} e^{\frac{t}{2}} \left\|x - Bx\right\|_X.\end{aligned}$$

Let $t = n$. We conclude that

$$\left\|e^{n(B-id_X)}x - B^n x\right\|_X \leqslant \sqrt{n}\left\|x - Bx\right\|_X.$$

□

Using this auxiliary result we can prove the exponential formula.

THEOREM 3.3.51
If X is a Banach space and S is a contraction semigroup on X with generator A,
then

$$S(t)x \;=\; \lim_{n\to+\infty}\left(id_X - \frac{t}{n}A\right)^{-n}x \;=\; \lim_{n\to+\infty}\left(\frac{n}{t}R_{\frac{n}{t}}\right)^{n}x \qquad \forall\, x\in X.$$

PROOF For every $n \geqslant 1$ and $t>0$, we have

$$\frac{n}{t}R_{\frac{n}{t}} \;=\; \frac{n}{t}\left(\frac{n}{t}id_X - A\right)^{-1} \;=\; \left(id_X - \frac{t}{n}A\right)^{-1}.$$

Also we have

$$tA_{\frac{n}{t}} \;=\; t\left(\frac{n^2}{t^2}R_{\frac{n}{t}} - \frac{n}{t}id_X\right) \;=\; n\left(\frac{n}{t}R_{\frac{n}{t}} - id_X\right).$$

Note that

$$\left\|\frac{n}{t}R_{\frac{n}{t}}\right\|_{\mathcal{L}} \;\leqslant\; 1.$$

So we can apply Lemma 3.3.50 with $B=\frac{n}{t}R_{\frac{n}{t}}$. We obtain

$$\left\|\exp\left(n\left(\frac{n}{t}R_{\frac{n}{t}} - id_X\right)\right)x - \left(\frac{n}{t}R_{\frac{n}{t}}\right)^{n}x\right\|_X \;\leqslant\; \sqrt{n}\left\|\frac{n}{t}R_{\frac{n}{t}}x - x\right\|_X.$$

From the proof of Theorem 3.3.46, we know that

$$\left\|\frac{n}{t}R_{\frac{n}{t}}x - x\right\|_X \;\leqslant\; \frac{t}{n}\left\|Ax\right\|_X \qquad \forall\, x\in D(A).$$

Therefore it follows that

$$\left\|\exp\left(tA_{\frac{n}{t}}\right)x - \left(\frac{n}{t}R_{\frac{n}{t}}\right)^{n}x\right\|_X \;\leqslant\; \frac{t}{\sqrt{n}}\left\|Ax\right\|_X \qquad \forall\, x\in D(A).$$

Again from the proof of Theorem 3.3.46, we know that

$$S(t)x \;=\; \lim_{n\to+\infty} exp\left(tA_{\frac{n}{t}}\right)x \qquad \forall\, x\in X.$$

So we infer that for fixed $t>0$,

$$S(t)x \;=\; \lim_{n\to+\infty}\left(\frac{n}{t}R_{\frac{n}{t}}\right)^{n}x \qquad \forall\, x\in D(A). \tag{3.86}$$

But

$$\left\|e^{tA_{\frac{n}{t}}}\right\|_{\mathcal{L}} \;\leqslant\; 1 \quad\text{and}\quad \left\|\left(\frac{n}{t}R_{\frac{n}{t}}\right)^{n}\right\|_{\mathcal{L}} \;\leqslant\; 1$$

and $D(A)$ is dense in X. So (3.86) is valid for all $x\in X$. □

Before passing to the nonlinear semigroup theory, let us see how we can use semigroups to extend the notion of a solution for an inhomogeneous evolution equation. So let X be a Banach space and A the generator of a C_0-semigroup S and $T = [0, b]$. Let $f \in L^1(T; X)$ and consider the evolution equation

$$\begin{cases} x'(t) = Ax(t) + f(t) & \forall\, t \in T = [0, b], \\ x(0) = x_0. \end{cases} \tag{3.87}$$

DEFINITION 3.3.52 **(a)** *A function $x \in W^{1,1}\big((0, b); X\big)$ is a **strong solution** of (3.87), if $x(0) = x_0$ and it satisfies the equation almost everywhere (hence $x(t) \in D(A)$ for almost all $t \in T$).*

(b) *A function $x \in C(T; X)$ is a **weak solution** of (3.87), if for all $x^* \in X^*$, the function $t \longmapsto \big\langle x^*, x(t)\big\rangle_X$ is absolutely continuous and*

$$\big\langle x^*, x(t)\big\rangle_X \;=\; \langle x^*, x_0\rangle_X + \int_0^t \big\langle A^* x^*, x(s)\big\rangle_X \, ds + \int_0^t \big\langle x^*, f(s)\big\rangle \, ds \qquad \forall\, t \in T.$$

(c) *A function $x \in C(T; X)$ is a **mild solution** of (3.87), if*

$$x(t) \;=\; S(t)x_0 + \int_0^t S(t-s) f(s)\, ds \qquad \forall\, t \in T.$$

The following interesting result is due to Ball (1977), where the reader can find its proof.

THEOREM 3.3.53
If X is a Banach space, A is the generator of a C_0-semigroup and $f \in L^1(T; X)$,
then $x \in C(T; X)$ is a mild solution of (3.86) if and only if it is a weak solution.

REMARK 3.3.54 In contrast to the strong solution, the mild solution makes sense without having that

$$x(t) \in D(A) \quad \text{for a.a. } t \in T.$$

Also we need not have $x_0 \in D(A)$ (nonregular initial condition). Moreover, it is easy to check that the mild solution

$$(f, x_0) \longmapsto x(\cdot; f, x_0)$$

is Lipschitz continuous on $L^1(T; X) \times X$.

Now we move to nonlinear semigroups.

DEFINITION 3.3.55 *Let X be a Banach space and let $C \subseteq X$ be a nonempty set. A family of maps*

$$S(t)\colon C \longrightarrow C, \quad t \geqslant 0$$

is said to be a ***semigroup of nonexpansive maps*** *if*

(a) $S(0) = id_C$;

(b) $S(t+s) = S(t) \circ S(s)$ *for all* $t, s \geqslant 0$;

(c) $\|S(t)x - S(t)y\|_X \leqslant \|x - y\|_X$ *for all* $t \geqslant 0$ *and all* $x, y \in C$;

(d) $S(t)x \longrightarrow x$ *in* X *as* $t \searrow 0$ *for all* $x \in C$.

REMARK 3.3.56 Evidently a semigroup S on nonexpansive maps can be extended uniquely to a semigroup of nonexpansive maps on $\overline{C}$ and so in Definition 3.3.55 we may assume without any loss of generality that $C \subseteq X$ is closed. If $C = X$ and $S(t) \in \mathcal{L}(X)$, then we recover Definition 3.3.33. Moreover, it is straightforward to check that

$$\mathbb{R}_+ \times C \ni (t, x) \longrightarrow S(t)x \in C \text{ is continuous.}$$

We shall prove a basic generation theorem for nonlinear semigroups of nonexpansive maps. The result will be a nonlinear analog of Theorems 3.3.49 and 3.3.51. To do this we need some preparation. First we prove a combinatorial lemma.

LEMMA 3.3.57
If $n \geqslant m \geqslant 1$ are integers and $\alpha, \beta > 0$ are such that $\alpha + \beta = 1$, then

$$\sum_{k=0}^{m} \binom{n}{k} \alpha^k \beta^{n-k} (m-k) \leqslant \left[(na-m)^2 + na\beta\right]^{\frac{1}{2}}$$

and

$$\sum_{k=m}^{n} \binom{k-1}{m-1} \alpha^m \beta^{k-m} (n-k) \leqslant \left[\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{a} + m - n\right)^2\right]^{\frac{1}{2}}.$$

PROOF Since $n \geqslant m$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{k=0}^{m} \binom{n}{k} \alpha^k \beta^{n-k} (m-k) \\ \leqslant\ &\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} (m-k) \end{aligned}$$

$$\leqslant \left(\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}(m-k)^2\right)^{\frac{1}{2}}. \tag{3.88}$$

From the binomial theorem, we know that

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k} \;=\; (\alpha+\beta)^n, \tag{3.89}$$

$$\sum_{k=0}^{n}k\binom{n}{k}\alpha^k\beta^{n-k} \;=\; \alpha n(\alpha+\beta)^{n-1} \tag{3.90}$$

and

$$\sum_{k=0}^{n}k^2\binom{n}{k}\alpha^k\beta^{n-k} \;=\; \alpha^2n(n-1)(\alpha+\beta)^{n-2}+\alpha n(\alpha+\beta)^{n-1}. \tag{3.91}$$

Using the equations of (3.89), (3.90) and (3.91) in the right hand side of (3.88) and since

$$\alpha+\beta \;=\; 1,$$

we obtain

$$\sum_{k=0}^{m}\binom{n}{k}\alpha^k\beta^{n-k}(m-k) \;\leqslant\; \left[(n\alpha-m)^2+n\alpha\beta\right]^{\frac{1}{2}}.$$

Also using once more the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\sum_{k=m}^{n}\binom{k-1}{m-1}\alpha^m\beta^{k-m}(n-k)\\
&\leqslant \sum_{k=m}^{\infty}\binom{k-1}{m-1}\alpha^m\beta^{k-m}|n-k| \qquad (3.92)\\
&\leqslant \left(\sum_{k=m}^{\infty}\binom{k-1}{m-1}\alpha^m\beta^{k-m}\right)^{\frac{1}{2}}\left(\sum_{k=m}^{\infty}\binom{k-1}{m-1}\alpha^m\beta^{k-m}(n-k)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Recall that

$$\sum_{k=m}^{\infty}\binom{k-1}{m-1}\beta^{k-m} \;=\; \frac{1}{(1-\beta)^m} \qquad \forall\ \beta\in(0,1). \tag{3.93}$$

Using (3.93) and the identities obtained by differentiating it with respect to β, in the right hand side of (3.92), we obtain

$$\sum_{k=m}^{n}\binom{k-1}{m-1}\alpha^m\beta^{k-m}(n-k) \;\leqslant\; \left[\frac{m\beta}{\alpha^2}+\left(\frac{m\beta}{\alpha}+m-n\right)^2\right]^{\frac{1}{2}}.$$

□

Using this auxiliary result we can obtain some estimates for the family $\{J_\lambda^n\}_{n\geqslant 1,\lambda>0}$.

LEMMA 3.3.58
If X is a Banach space, $A\colon X\supseteq D(A)\longrightarrow X$ is an m-accretive operator, $\lambda\geqslant\mu>0$ and $n\geqslant m\geqslant 1$ are integers,
then

(a) $\|J_\lambda^n(x)-x\|_X\leqslant n\|J_\lambda(x)-x\|_X$ *for all $x\in X$;*

(b)

$$\|J_\mu^n(x)-J_\lambda^m(x)\|_X \leqslant \sum_{k=0}^{m}\alpha^k\beta^{n-k}\binom{n}{k}\|J_\lambda^{m-k}(x)-x\|_X + \sum_{k=m}^{n}\alpha^m\beta^{k-m}\binom{k-1}{m-1}\|J_\mu^{n-k}(x)-x\|_X,$$

where

$$a=\frac{\mu}{\lambda}\quad\text{and}\quad\beta=\frac{\lambda-\mu}{\lambda}.$$

PROOF **(a)** Using Proposition 3.3.12**(a)**, we have

$$\|J_\lambda^n x-x\|_X = \left\|\sum_{k=0}^{n-1}\left(J_\lambda^{n-k}(x)-J_\lambda^{n-(k+1)}(x)\right)\right\|_X \leqslant \sum_{k=0}^{n-1}\|J_\lambda(x)-x\|_X = n\|J_\lambda(x)-x\|_X.$$

(b) For integers $1\leqslant i\leqslant m$ and $1\leqslant k\leqslant m$, we set

$$a_{k,i}\overset{df}{=}\|J_\mu^i(x)-J_\lambda^k(x)\|_X.$$

Using the resolvent identity (see Remark 3.3.13), we obtain

$$\begin{aligned} a_{k,i} &= \left\|J_\mu^i(x)-J_\mu\left(\frac{\mu}{\lambda}J_\lambda^{k-1}(x)+\frac{\lambda-\mu}{\lambda}J_\lambda^k(x)\right)\right\|_X \\ &\leqslant \left\|J_\mu^{i-1}(x)-\frac{\mu}{\lambda}J_\lambda^{k-1}(x)-\frac{\lambda-\mu}{\lambda}J_\lambda^k(x)\right\|_X \\ &\leqslant \frac{\mu}{\lambda}\|J_\mu^{i-1}(x)-J_\lambda^{k-1}(x)\|_X+\frac{\lambda-\mu}{\lambda}\|J_\mu^{i-1}(x)-J_\lambda^k(x)\|_X \\ &= \alpha a_{k-1,i-1}+\beta a_{k,i-1}. \end{aligned} \tag{3.94}$$

Inequalities (3.94) can be solved to estimate $a_{m,n}$ in terms of $a_{k,0}$ and $a_{0,i}$. This way we obtain the inequality in part **(b)** of the lemma. ∎

Now we are ready for the generation theorem for nonlinear semigroups of nonexpansive maps.

THEOREM 3.3.59
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator, then

$$S(t)x \;=\; \lim_{n\to+\infty} \left(id_X + \frac{t}{n}A\right)^{-n} x$$

exists for each $x \in \overline{D(A)}$, uniformly in t on compact intervals in $\mathbb{R}_+$. Moreover, $\{S(t)\}_{t\geqslant 0}$ is a semigroup of nonexpansive maps on $\overline{D(A)}$ and for each $x \in D(A)$ and $t > 0$, we have

$$\big\|S(t)x - x\big\|_X \;\leqslant\; t\big|A(x)\big| \;=\; t \inf_{u\in A(x)} \|u\|_X\,.$$

PROOF Let $x \in D(A)$, $\lambda \geqslant \mu > 0$ and $n \geqslant m \geqslant 1$ positive integers. Using Lemmata 3.3.57 and 3.3.58, we obtain

$$\big\|J^n_\mu(x) - J^m_\lambda(x)\big\|_X \;\leqslant\; \left(\left[(n\mu - \lambda m)^2 + n\mu(\lambda-\mu)\right]^{\frac{1}{2}}\right. \tag{3.95}$$

$$\left. + \left[m\lambda(\lambda-\mu) + (m\lambda - n\mu)^2\right]^{\frac{1}{2}}\right)\big|A(x)\big|. \tag{3.96}$$

Taking $\mu = \frac{t}{n}$ and $\lambda = \frac{t}{m}$ in (3.96), we obtain

$$\big\|J^n_{\frac{t}{n}}(x) - J^m_{\frac{t}{m}}(x)\big\|_X \;\leqslant\; 2t\left(\frac{1}{m} - \frac{1}{n}\right)^{\frac{1}{2}} |Ax|, \tag{3.97}$$

so

$$\lim_{n\to+\infty} J^n_{\frac{t}{n}}(x) \;=\; S(t)x \tag{3.98}$$

exists uniformly in t on compact intervals of $\mathbb{R}_+$. Moreover, since

$$J^n_{\frac{t}{n}} \text{ is nonexpansive on } X,$$

we see that

$$\big\|S(t)x - S(t)y\big\|_X \;\leqslant\; \|x-y\|_X \qquad \forall\, t \geqslant 0,\; x,y \in D(A).$$

Therefore

$$S(t)x \;=\; \lim_{n\to+\infty} J^n_{\frac{t}{n}}(x) \quad \text{exists for } x \in \overline{D(A)}$$

and

$$S(t)(\cdot) \text{ is nonexpansive on } \overline{D(A)}.$$

Also if in (3.96), we let $n = m$, $\mu = \frac{t}{n}$ and $\lambda = \frac{s}{n}$ with $0 \leqslant t \leqslant s$ and then pass to the limit as $n \to +\infty$, we obtain

$$\big\|S(t)x - S(s)x\big\|_X \leqslant 2|t-s||Ax| \qquad \forall\, x \in D(A). \tag{3.99}$$

From (3.99), it follows that the function $t \longmapsto S(t)x$ is continuous for all $x \in D(A)$ and then by a use of the double limit lemma (see Proposition A.2.35) as in the proof of Theorem 3.3.46, we obtain that the function $t \longmapsto S(t)x$ is continuous on $\mathbb{R}_+$ for all $x \in \overline{D(A)}$. Finally we need to verify the semigroup property (see Definition 3.3.55**(b)**). From (3.97) and (3.98), we have

$$S(t)^m x = \lim_{n\to+\infty} \big(J^n_{\frac{t}{n}}\big)^m(x) = \lim_{n\to+\infty} \big(J^m_{\frac{t}{n}}\big)^n(x).$$

Therefore,

$$\begin{aligned} S(mt)x &= \lim_{n\to+\infty} J^n_{\frac{mt}{n}}(x) = \lim_{k\to+\infty} J^{mk}_{\frac{mt}{mk}}(x) \\ &= \lim_{k\to+\infty} \big(J^m_{\frac{t}{k}}\big)^k(x) = S(t)^m x. \end{aligned} \tag{3.100}$$

Then if $i, k, r, s \geqslant 0$ are integers, we have

$$\begin{aligned} S\left(\frac{i}{k} + \frac{r}{s}\right)x &= S\left(\frac{is+rk}{ks}\right)x = S\left(\frac{1}{ks}\right)^{is+rk} x \\ = S\left(\frac{1}{ks}\right)^{is} S\left(\frac{1}{ks}\right)^{rk} x &= S\left(\frac{i}{k}\right)S\left(\frac{r}{s}\right)x, \end{aligned}$$

so

$$S(t+\tau)x = S(t)\circ S(\tau)x \quad \text{for all rational } t, \tau \geqslant 0 \text{ and all } x \in \overline{D(A)}.$$

Exploiting the continuity in t and the nonexpansiveness in x, we conclude that

$$S(t+\tau) = S(t)\circ S(\tau) \qquad \forall\, t, \tau \geqslant 0.$$

□

In applications to evolution equations, it is very helpful to know if

$$S(t)\colon C \longrightarrow C, \quad t > 0$$

is a compact map (see Definition 3.1.1). So we make the following definition.

DEFINITION 3.3.60 *Let X be a Banach space and $C \subseteq X$ be a nonempty, closed set and $S(t)\colon C \longrightarrow C$, $t \geqslant 0$, a semigroup of nonexpansive maps. We say that S is **compact**, if for all $t > 0$, $S(t)$ is a compact map.*

REMARK 3.3.61 Since $S(0) = id_C$, then $S(0)$ is not in general compact unless $C \subseteq X$ is compact or X is finite dimensional. Next we present two simple but typical examples of (linear) semigroups which are compact and noncompact respectively.

EXAMPLE 3.3.62 **(a)** $X = H = L^2(0,\pi)$ and

$$A\colon H \supseteq D(A) \longrightarrow H$$

is defined by

$$Ax \stackrel{df}{=} -\frac{d^2x}{dt^2} \qquad \forall\, x \in W^{2,2}(0,\pi) \cap W_0^{1,2}(0,\pi).$$

By integration by parts, we can check that A is monotone (i.e., accretive). Also for every $f \in L^2(0,\pi)$ the boundary value problem

$$\begin{cases} -x''(t) + x(t) = f(t) & \text{for a.a. } t \in [0,\pi], \\ x(0) = x(\pi) = 0, \end{cases}$$

has a unique solution $x \in D(A)$. Hence A is maximal monotone (i.e., m-accretive). Note that

$$\overline{D(A)} \;=\; H$$

and then because of Theorem 3.3.49, $-A$ generates a contraction semigroup S on H. We know that

$$\left\{\lambda_k = k^2\right\}_{k\geqslant 1} \text{ is the spectrum of} \left(-\frac{d^2}{dt^2}, W_0^{1,2}(0,\pi)\right)$$

and the corresponding eigenfunctions

$$\left\{\sqrt{\frac{2}{\pi}}\sin kx\right\}_{k\geqslant 1}$$

form an orthonormal basis for H. Then using sine Fourier expansion, we can easily verify that

$$S(t)x(\tau) \;=\; \sum_{k=1}^{\infty} a_k e^{-k^2 t}\sqrt{\frac{2}{\pi}}\sin k\tau \qquad \forall\, x \in H,\ t \geqslant 0,\ \tau \in [0,\pi],$$

with a_k being the k-th Fourier coefficient, defined by

$$a_k \;\stackrel{df}{=}\; \sqrt{\frac{2}{\pi}}\int_0^{\pi} x(s)\sin ks\, ds.$$

If we set

$$S_n(t)x(\tau) \stackrel{df}{=} \sum_{k=1}^{n} a_k e^{-k^2 t}\sqrt{\frac{2}{\pi}}\sin k\tau \qquad \forall\, x \in H,\ t > 0,\ \tau \in [0,\pi],$$

then

$$S_n \in \mathcal{L}_f(H)$$

(i.e., S_n is of finite rank; see Definition 3.1.23) and for all $t > 0$,

$$S_n(t)x \longrightarrow S(t)x \quad \text{in } H, \text{ uniformly on bounded subsets of } H.$$

Therefore $S(t)$ is compact for $t > 0$ (see Proposition 3.1.18).

(b) Let $X = H = L^2(0, 2\pi)$ and let

$$A\colon X \supseteq D(A) \longrightarrow H$$

be defined by

$$Ax \stackrel{df}{=} \frac{dx}{dt} \qquad \forall\, x \in D(A),$$

with

$$D(A) \stackrel{df}{=} \left\{x \in W^{2,2}(0,2\pi) :\ x(0) = x(2\pi),\ x'(0) = x'(2\pi)\right\}.$$

A simple integration by parts reveals that A is monotone (i.e., accretive). Also for every $\lambda > 0$ and every $h \in L^2(0, 2\pi)$, the problem

$$\begin{cases} x(t) + \lambda\frac{dx}{dt}(t) = h(t) & \forall\, t \in [0,2\pi], \\ x(0) = x(2\pi), \\ x'(0) = x'(2\pi), \end{cases}$$

has a unique solution $x \in W^{2,2}(0, 2\pi)$ and so A is maximal monotone (i.e., A is m-accretive). Hence by Theorem 3.3.49, $-A$ generates a contraction semigroup S on H. This semigroup is defined by

$$S(t)x(\tau) = \begin{cases} x(\tau + t) & \text{if} \quad \tau + t \in [0, 2\pi], \\ x(\tau + t - 2\pi) & \text{if} \quad \tau + t > 2\pi, \end{cases}$$

for all $x \in H$, $\tau \in [0, 2\pi]$ and $t \geqslant 0$. Then for every $t \geqslant 0$, S is an isometry on H and so $S(t)$ is not compact for $t > 0$.

REMARK 3.3.63 Roughly speaking, we can say that compact semigroups of nonexpansive maps are generated by m-accretive operators acting in a finite dimensional Banach space or by m-accretive operators arising in the study of parabolic problems. In contrast hyperbolic problems (even very simple ones) generate noncompact semigroups.

The compactness of a semigroup S of nonexpansive maps is closely related to the compactness of the nonexpansive operators $J_\lambda = (id_x + \lambda A)^{-1}$ (see Definition 3.3.11 and Proposition 3.3.12**(a)**). For this reason we need to have a result determining the relationship between $S(t)$ and J_λ, $t, \lambda > 0$.

LEMMA 3.3.64
If X is a Banach space, $C \subseteq X$ is a nonempty, closed set and

$$S(t)\colon C \longrightarrow C, \quad t \geqslant 0$$

is a semigroup of nonexpansive maps,
then

$$\big\|S(t)x - x\big\|_X \leqslant \frac{2}{t}\int_0^t \big\|S(\tau)x - x\big\|_X \, d\tau \qquad \forall\ x \in C,\ t > 0.$$

PROOF From Definition 3.3.55**(b)** and **(c)**, we have

$$\left\|S(t)x - \frac{1}{t}\int_0^t S(\tau)\, d\tau\right\|_X \tag{3.101}$$

$$= \left\|\frac{1}{t}\int_0^t \big(S(t)x - S(\tau)x\big)\, d\tau\right\|_X \tag{3.102}$$

$$\leqslant \frac{1}{t}\int_0^t \big\|S(t-\tau)x - x\big\|_X \, d\tau$$

$$= \frac{1}{t}\int_0^t \big\|S(\tau)x - x\big\|_X \, d\tau \qquad \forall\ x \in C,\ t > 0. \tag{3.103}$$

Using this inequality, we obtain

$$\begin{aligned}
&\big\|S(t)x - x\big\|_X \\
&\leqslant \left\|S(t)x - \frac{1}{t}\int_0^t S(\tau)x\, d\tau\right\|_X + \frac{1}{t}\left\|\int_0^t \big(S(\tau)x - x\big)\, d\tau\right\|_X \\
&\leqslant \frac{1}{t}\int_0^t \big\|\big(S(\tau)x - x\big)\big\|_X \, d\tau + \frac{1}{t}\int_0^t \big\|\big(S(\tau)x - x\big)\big\|_X \, d\tau \\
&\leqslant \frac{2}{t}\int_0^t \big\|\big(S(\tau)x - x\big)\big\|_X \, d\tau \qquad \forall\ x \in X,\ t > 0.
\end{aligned}$$

□

In order to derive the desired relations between $S(t)$ and J_λ, with $t, \lambda > 0$, we need to return to nonlinear evolution equations and discuss their solvability when the data are nonregular.

So let $T = [0, b]$, X be a Banach space,

$$A\colon X \supseteq D(A) \longrightarrow 2^X$$

be an m-accretive operator and $f \in L^1(T; X)$. We consider the following nonlinear evolution inclusion:

$$\begin{cases} x'(t) + A\big(x(t)\big) \ni f(t) & \forall\, t \in T, \\ x(0) = x_0. \end{cases} \tag{3.104}$$

From Theorem 3.3.28, we know that if X^* is uniformly convex (hence X is reflexive), $x_0 \in D(A)$ and $f \in W^{1,1}\big((0,b); X\big)$, then there exists unique $x \in W^{1,\infty}\big((0,b); X\big)$ satisfying (3.104) for almost all $t \in T$. Such a solution is usually called ***strong solution***. However, if $x_0 \in \overline{D(A)} \setminus D(A)$ or $f \in L^1(T; X) \setminus W^{1,1}\big((0,b); X\big)$ or X is not reflexive, then there are examples showing that (3.104) need not have a strong solution (for details we refer to Crandall & Liggett (1971)). So if we want to develop a general theory concerning evolution equations of the form (3.104), we need to introduce a new broader solution concept.

Suppose that x is a strong solution. Then

$$-x'(t) + f(t) \in A\big(x(t)\big) \quad \text{for a.a. } t \in T.$$

Because A is accretive, using Theorem 3.3.10, we obtain

$$0 \leqslant \big(-x'(t) + f(t) - v, x(t) - y\big)_+ \quad \text{for a.a. } t \in T \text{ and all } (y, v) \in \operatorname{Gr} A,$$

so

$$\big(x'(t), x(t) - y\big)_+ \leqslant \big(f(t) - v, x(t) - y\big)_+ \quad \text{for a.a. } t \in T.$$

By virtue of Lemma 3.3.26, we have that

$$\big(x'(t), x(t) - y\big)_+ = \frac{d}{dt}\big\|x(t) - y\big\|_X^2 \quad \text{for a.a. } t \in T.$$

So we obtain

$$\frac{1}{2}\frac{d}{dt}\big\|x(t) - y\big\|_X^2 \leqslant \big(f(t) - v, x(t) - y\big)_+ \quad \text{for a.a. } t \in T.$$

Integrating both sides of this inequality over $[s, t] \subseteq [0, b]$, we obtain

$$\big\|x(t) - y\big\|_X^2 \leqslant \big\|x(s) - y\big\|_X^2 + 2\int_s^t \big(f(\tau) - v, x(\tau) - y\big)_+ \, d\tau$$
$$0 \leqslant s \leqslant t \leqslant b,\ (y, v) \in \operatorname{Gr} A.$$

This leads to the introduction of a new more general solution notion for problem (3.104).

DEFINITION 3.3.65 *Let $x \in X$ and $f \in L^1(T;X)$. A function $x\colon T \longrightarrow X$ is said to be an **integral solution** of the Cauchy problem (3.104), if*

(a) $x(0) = x$;

(b) $x \in C(T;X)$;

(c) *for all $0 \leqslant s \leqslant t \leqslant b$ and all $(y,v) \in \operatorname{Gr} A$, we have*

$$\frac{1}{2}\|x(t)-y\|_X^2 \leqslant \frac{1}{2}\|x(s)-y\|_X^2 + \int_s^t \big(f(\tau)-v, x(\tau)-y\big)_+ \, d\tau.$$

REMARK 3.3.66 The previous discussion shows that every strong solution is also an integral solution. Moreover, for every $x_0 \in \overline{D(A)}$, the function

$$x(t) \stackrel{df}{=} S(t)x_0 = \lim_{n\to+\infty} \left(id_X + \frac{t}{n}A\right)^{-n} x_0$$

is an integral solution of the autonomous Cauchy problem

$$\begin{cases} x'(t) + A\big(x(t)\big) \ni 0 & \forall\, t \geqslant 0, \\ x(0) = x_0. \end{cases}$$

For details we refer to Barbu (1976, p. 124).

More generally we have the following result due to Bénilan (1972) (see also Barbu (1976, p. 124) and Miyadera (1992, p. 160)).

THEOREM 3.3.67
If X is a Banach space, $A\colon X \supseteq 2^X$ is an m-accretive operator, $x_0 \in \overline{D(A)}$ and $f \in L^1(T;X)$,
then problem (3.104) has a unique integral solution $x(\cdot;f) \in C(T;X)$.
Moreover, if $f_1, f_2 \in L^1(T;X)$ and

$$x_1(\cdot) = x(\cdot;f_1), \quad x_2(\cdot) = x(\cdot;f_2),$$

we have

$$\|x_1(t)-x_2(t)\|_X^2 \leqslant \|x_1(s)-x_2(s)\|_X^2 + \int_s^t \big(f_1(\tau)-f_2(\tau), x_1(\tau)-x_2(\tau)\big)_+ \, d\tau$$

and

$$\|x_1(t)-x_2(t)\|_X \leqslant \|x_1(s)-x_2(s)\|_X + \int_s^t \|f_1(\tau)-f_2(\tau)\|_X \, d\tau \qquad \forall\, 0 \leqslant s \leqslant t \leqslant b.$$

Now we have all the necessary tools to establish the relation between $S(t)$ and J_λ for all $t, \lambda > 0$.

PROPOSITION 3.3.68
If X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator and

$$S(t)\colon \overline{D(A)} \longrightarrow \overline{D(A)}, \quad t \geqslant 1$$

is the semigroup of nonexpansive maps generated by A, then for all $x_0 \in \overline{D}$ and all $t, \lambda > 0$, we have

(a) $\|S(t)x_0 - x_0\|_X \leqslant \left(2 + \frac{t}{\lambda}\right) \|J_\lambda(x_0) - x_0\|_X$;

(b) $\|J_\lambda(x_0) - x_0\|_X \leqslant \frac{2}{t}\left(1 + \frac{\lambda}{t}\right) \int_0^t \|S(\tau)x_0 - x_0\|_X \, d\tau$.

PROOF **(a)** From Definition 3.3.55**(c)** and Theorem 3.3.59, for all $x_0 \in \overline{D(A)}$, $(y, v) \in \operatorname{Gr} A$ and $t > 0$, we have

$$\begin{aligned} & \|S(t)x_0 - x_0\|_X \\ \leqslant\ & \|S(t)x_0 - S(t)y\|_X + \|S(t)y - y\|_X + \|y - x_0\|_X \\ \leqslant\ & 2\|x_0 - y\|_X + \|S(t)y - y\|_X \\ \leqslant\ & 2\|x_0 - y\|_X + t\|v\|_X . \end{aligned} \tag{3.105}$$

In (3.105), let us set $y = J_\lambda(x)$ and $v = A_\lambda(x_0) \in A\big(J_\lambda(x_0)\big)$ (see Proposition 3.3.12**(c)**). We obtain

$$\begin{aligned} & \|S(t)x_0 - x_0\|_X \\ \leqslant\ & 2\|x_0 - J_\lambda(x_0)\|_X + t\|A_\lambda(x_0)\|_X \\ \leqslant\ & 2\|x_0 - J_\lambda(x_0)\|_X + \frac{t}{\lambda}\|x_0 - J_\lambda(x_0)\|_X \\ =\ & \left(2 + \frac{t}{\lambda}\right)\|x_0 - J_\lambda(x_0)\|_X . \end{aligned}$$

(b) We know that

$$x(t) \ = \ S(t)x_0$$

is the unique integral solution of the autonomous Cauchy problem:

$$\begin{cases} x'(t) + A\big(x(t)\big) \ni 0 & \forall\ t \in T, \\ x(0) = x_0, \end{cases}$$

and

$$\frac{1}{2}\|S(t)x_0 - y\|_X^2 \leqslant \frac{1}{2}\|x_0 - y\|_X^2 + \int_0^t \big(-v, S(\tau)x_0 - y\big)_+ \, d\tau \qquad \forall\ (y,v) \in \operatorname{Gr} A,\ t \geqslant 0$$

(see Definition 3.3.65 and Remark 3.3.66). Using Definition 3.3.5 and Lemma 3.3.27, we obtain

$$\|S(t)x_0 - y\|_X \leqslant \|x_0 - y\|_X + \frac{1}{\lambda}\int_0^t \big(\|S(\tau)x_0 - y - \lambda v\|_X - \|S(\tau)x_0 - y\|_X\big)\, d\tau \qquad \forall\ \lambda > 0.$$

Let

$$y = J_\lambda(x_0) \quad \text{and} \quad v = A_\lambda(x_0).$$

We have

$$\|S(t)x_0 - J_\lambda(x_0)\|_X \leqslant \|x_0 - J_\lambda(x_0)\|_X + \frac{1}{\lambda}\int_0^t \big(\|S(\tau)x_0 - x_0\|_X - \|S(\tau)x_0 - J_\lambda(x_0)\|\big)\, d\tau. \tag{3.106}$$

From the triangle inequality, we have

$$-\|S(t)x_0 - x_0\|_X \leqslant \|S(t)x_0 - J_\lambda(x_0)\|_X - \|J_\lambda(x_0) - x_0\|_X.$$

Using this in (3.106), we obtain

$$\begin{aligned} &-\|S(t)x_0 - x_0\|_X + \|J_\lambda(x_0) - x_0\|_X \\ &\leqslant \|J_\lambda(x_0) - x_0\|_X + \frac{1}{\lambda}\int_0^t \big(2\|S(\tau)x_0 - x_0\|_X - \|J_\lambda(x_0) - x_0\|_X\big)\, d\tau, \end{aligned}$$

so

$$\|J_\lambda(x_0) - x_0\|_X \leqslant \frac{\lambda}{t}\|S(t)x_0 - x_0\|_X + \frac{2}{t}\int_0^t \|S(\tau)x_0 - x_0\|_X \, d\tau$$

and from Lemma 3.3.64, we have

$$\|J_\lambda(x_0) - x_0\|_X \leqslant \frac{2}{t}\left(1 + \frac{\lambda}{t}\right)\int_0^t \|S(\tau)x_0 - x_0\|_X \, d\tau.$$

□

LEMMA 3.3.69
If X is a Banach space, $C \subseteq X$ is a nonempty, closed set,

$$f_n \colon C \longrightarrow X \text{ are compact maps for } n \geqslant 1$$

and

$$f_n(x) \longrightarrow f(x) \quad \text{in } X,$$

uniformly on bounded subsets of C,
then $f \colon C \longrightarrow X$ is compact.

PROOF Clearly $f \colon C \longrightarrow X$ is continuous. Next let $B \subseteq C$ be a bounded set. Then for a given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon, B) \geqslant 1$, such that

$$\big\|f_n(x) - f(x)\big\|_X < \frac{\varepsilon}{2} \qquad \forall\, n \geqslant n_0,\ x \in B. \tag{3.107}$$

For $n \geqslant n_0$,

$$\text{the set } \overline{f_n(B)} \text{ is compact in } X.$$

So we can find $\{x_k\}_{k=1}^{N_0}$, where $N_0 = N_0(n, \varepsilon) \geqslant 1$, such that

$$\overline{f_n(B)} \subseteq \bigcup_{k=1}^{N_0} B_{\frac{\varepsilon}{2}}(x_k). \tag{3.108}$$

Let $x \in B$. From (3.108), we see that there exists $k \in \{1, \ldots, N_0\}$, such that

$$\big\|f_n(x) - x_k\big\| < \frac{\varepsilon}{2}. \tag{3.109}$$

Therefore using (3.107) and (3.109), we have

$$\big\|f(x) - x_k\big\|_X \leqslant \big\|f(x) - f_n(x)\big\|_X + \big\|f_n(x) - x_k\big\|_X < \varepsilon,$$

so

$$f(x) \in B_\varepsilon(x_k)$$

and thus

$$f(B) \subseteq \bigcup_{k=1}^{N_0} B_\varepsilon(x_k),$$

i.e., $f(B)$ is totally bounded, thus relatively compact in X. □

DEFINITION 3.3.70 *Let X be a Banach space, $C \subseteq X$ nonempty, closed and let*

$$S(t) \colon C \longrightarrow C, \quad t \geqslant 0,$$

*be a semigroup of nonexpansive maps. We say that S is **equicontinuous** (respectively **weakly equicontinuous**) if for each bounded set $B \subseteq C$, the family of functions $\big\{S(\cdot)x\big\}_{x \in B}$ is equicontinuous (respectively weakly equicontinuous) at each $t > 0$.*

As expected compactness and equicontinuity of nonlinear semigroups are closely related.

PROPOSITION 3.3.71
If X is a Banach space, $C \subseteq X$ is a nonempty, closed set and

$$S(t)\colon C \longrightarrow C, \quad t \geqslant 0$$

is a compact semigroup for nonexpansive maps,
then S is equicontinuous.

PROOF Let $B \subseteq C$ be a bounded set, let $t > 0$ and choose $r \in (0, t)$. Because

$$\overline{S(t-r)B} \text{ is compact,}$$

we can find $\{x_k\}_{k=1}^{N(\varepsilon)} \subseteq B$, such that

$$\overline{S(t-r)B} \subseteq \bigcup_{k=1}^{N_\varepsilon} B_{\frac{\varepsilon}{3}}\big(S(t-r)x_k\big). \tag{3.110}$$

For each $k \in \big\{1, \dots, N(\varepsilon)\big\}$, the map

$$t \longmapsto S(t)x_k \text{ is continuous on } \mathbb{R}_+$$

and so we can find $\delta = \delta(\varepsilon, t) \in (0, r)$, such that

$$\big\|S(t+h)x_k - S(t)x_k\big\|_X \leqslant \frac{\varepsilon}{3} \qquad \forall\, k \in \big\{1, \dots, N(\varepsilon)\big\},\ h \in [-\delta, \delta]. \tag{3.111}$$

Then because of (3.110), for each $x \in B$, we can find $k \in \big\{1, \dots, N(\varepsilon)\big\}$, such that

$$\big\|S(t-r)x - S(t-r)x_k\big\|_X \leqslant \frac{\varepsilon}{3}. \tag{3.112}$$

So using the semigroup property and (3.111) and (3.112), we obtain

$$\begin{aligned}
&\big\|S(t+h)x - S(t)x\big\|_X \\
\leqslant\ &\big\|S(t+h)x - S(t+h)x_k\big\|_X + \big\|S(t+h)x_k - S(t)x_k\big\|_X \\
&+ \big\|S(t)x_k - S(t)x\big\|_X \\
\leqslant\ &2\big\|S(t-r)x - S(t-r)x_k\big\|_X + \big\|S(t+h)x_k - S(t)x_k\big\|_X \leqslant \varepsilon \\
&\qquad\qquad \forall\, x \in B,\ h \in [-\delta, \delta].
\end{aligned}$$

This proves that $\big\{S(\cdot)x\big\}_{x \in B}$ is equicontinuous at every $t > 0$. ∎

Now we are ready to present a characterization of compact nonlinear semigroups.

THEOREM 3.3.72
If X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator and

$$S(t)\colon \overline{D(A)} \longrightarrow \overline{D(A)}, \quad t \geqslant 0$$

is the semigroup of nonexpansive maps generated by A according to Theorem 3.3.59,
then the following statements are equivalent:

(a) *the semigroup S is compact;*

(b) *for each $\lambda > 0$, J_λ is a compact map and the semigroup S is equicontinuous.*

PROOF "**(a)**$\Longrightarrow$**(b)**": The equicontinuity of S follows from Proposition 3.3.71. So we need to show that for each $\lambda > 0$, J_λ is a compact map. From Theorem 3.3.59, we have

$$\begin{aligned} &\big\|S(t)\circ J_\lambda(x) - J_\lambda(x)\big\|_X \leqslant t\big\|A_\lambda(x)\big\|_X \\ = \ &\frac{t}{\lambda}\big\|x - J_\lambda(x)\big\|_X \qquad \forall\, x \in X,\ \lambda > 0 \end{aligned} \tag{3.113}$$

(recall that $A_\lambda(x) \in A\big(J_\lambda(x)\big)$; see Proposition 3.3.12**(c)**). Because J_λ is nonexpansive, it maps bounded sets to bounded sets. So from (3.113), it follows that

$$S(t)\circ J_\lambda \longrightarrow J_\lambda \quad \text{as } t \to 0^+,$$

uniformly on bounded sets of X. But $S(t)\circ J_\lambda$ is compact, since $S(t)$ is. Therefore Lemma 3.3.69 implies that J_λ is compact.

"**(b)**$\Longrightarrow$**(a)**": Using Proposition 3.3.68**(b)**, we have

$$\begin{aligned} &\big\|\big(J_\lambda\circ S(t)\big)(x) - S(t)x\big\|_X \\ \leqslant \ &\frac{4}{\lambda}\int_0^\lambda \big\|S(t+\tau)x - S(t)x\big\|_X\, d\tau \qquad \forall\, \lambda, t > 0,\ x \in \overline{D(A)}. \end{aligned} \tag{3.114}$$

Because S is equicontinuous, for every bounded set $B \subseteq \overline{D(A)}$ and for each $t > 0$, we can find $\omega\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, such that

$$\lim_{r \searrow 0} \omega(r) = 0$$

and

$$\big\|S(t+\tau)x - S(t)x\big\|_X \leqslant \omega(\tau) \qquad \forall\, \tau \geqslant 0,\ x \in B. \tag{3.115}$$

Using (3.115) in (3.114), we obtain

$$\big\|\big(J_\lambda\circ S(t)\big)(x) - S(t)x\big\|_X \leqslant 4 \sup_{\tau\in[0,\lambda]} \omega(\tau) \qquad \forall\, x \in B,$$

so

$$J_\lambda \circ S(t) \longrightarrow S(t) \quad \text{as } \lambda \searrow 0,$$

uniformly on bounded subsets of $\overline{D(A)}$.

Note that

$$J_\lambda \circ S \text{ is compact}$$

(see the first part of **(b)**). So $S(t)$ is compact for all $t > 0$. □

In the case of linear semigroups, the above theorem takes the following form.

COROLLARY 3.3.73
If X is a Banach space, $A\colon X \supseteq D(A) \longrightarrow X$ is densely defined, linear, m-accretive operator and $S(t)\colon X \longrightarrow X$, $t \geqslant 0$ is the contraction semigroup generated by $-A$ according to Theorem 3.3.49,
then the following statements are equivalent:

(a) *the semigroup S is compact;*

(b) *for each $\lambda > 0$, J_λ is a compact operator and the map $t \longmapsto S(t)$ is continuous from $\mathbb{R}_+$ into $\mathcal{L}(X)$ with the operator norm topology.*

REMARK 3.3.74 Using the resolvent identity (see Remark 3.3.13), we see that in Theorem 3.3.72 and in Corollary 3.3.73,

$$\text{the map } J_\lambda \text{ is compact for all } \lambda > 0$$

if and only if

$$\text{it is compact for some } \lambda > 0.$$

The next proposition gives an equivalent condition for the map J_λ to be compact.

PROPOSITION 3.3.75
If X is a Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^X$ is an m-accretive operator,
then the following statements are equivalent:

(a) *for each $\lambda > 0$, J_λ is a compact map;*

(b) *for every $m > 0$, the level set*

$$L_m \stackrel{df}{=} \left\{ x \in D(A) : \ \|x\|_X + |A(x)| \leqslant m \right\}$$

is relatively compact in X.

PROOF "**(a)**$\Longrightarrow$**(b)**": From Proposition 3.3.12**(d)**, we have

$$\big\|A_\lambda(x)\big\|_X \leqslant \big|A(x)\big| \leqslant m \qquad \forall\, x \in L_m,\ \lambda > 0,$$

so

$$\big\|x - J_\lambda(x)\big\|_X \leqslant m\lambda \qquad \forall\, x \in L_m,\ \lambda > 0$$

and thus

$$J_\lambda \longrightarrow id_{L_m} \quad \text{as } \lambda \searrow 0, \quad \text{uniformly on } L_m.$$

From Lemma 3.3.69, it follows that L_m (which is bounded) is relatively compact.

"**(b)**$\Longrightarrow$**(a)**": Let $B \subseteq X$ be bounded and $\lambda > 0$. Since J_λ is nonexpansive $J_\lambda(B)$ is bounded. Because

$$A_\lambda(x) \in A\big(J_\lambda(x)\big) \qquad \forall\, x \in X,$$

we have

$$\begin{aligned} & \big\|J_\lambda(x)\big\|_X + \big|A\big(J_\lambda(x)\big)\big| \\ \leqslant\ & \big\|J_\lambda(x)\big\|_X + \big\|A_\lambda(x)\big\|_X \\ =\ & \big\|J_\lambda(x)\big\|_X + \frac{1}{\lambda}\big\|x - J_\lambda(x)\big\|_X \qquad \forall\, x \in X. \end{aligned}$$

So there exists $m > 0$ large enough, such that

$$J_\lambda(B) \subseteq L_m,$$

hence

$$\overline{J_\lambda(B)} \text{ is compact}$$

and so J_λ is a compact map. □

In Section 4.3, we will return to semigroups, when we will examine the subdifferential of a convex function. Before concluding this section, we would like to make an interesting remark concerning accretive operators.

REMARK 3.3.76 In a Hilbert space maximal accretivity (i.e., maximal monotonicity) and m-accretivity coincide (see Theorem 3.2.29). In a general Banach space this is no longer true. For a counterexample we refer to Crandall & Liggett (1971) (see also Miyadera (1992, pp. 42–44)).

3.4 The Nemytskii Operator and Integral Functions

In this section first we examine the Nemytskii (or superposition) operator, which is an important nonlinear operator that arises in many applications and then we pass to the study of nonlinear integral functionals, which leads naturally to the topic of the next section, which is the theory of Young measures.

Consider a set Ω, which in most cases is a measure space or a metric space or both and let X, Y be two Hausdorff topological spaces, which in our analysis as well as in most applications are either Euclidean spaces or Banach spaces. Let $f\colon \Omega \times X \longrightarrow Y$ and consider the nonlinear operator

$$N_f(u)(z) \stackrel{df}{=} f\big(z, u(z)\big) \qquad \forall\, z \in \Omega,$$

which to each function $u\colon \Omega \longrightarrow X$ assigns the Y-valued $z \longmapsto f\big(z, u(z)\big)$. This operator is known in the literature as the ***Nemytskii operator*** corresponding to the function f (also known as the ***superposition operator*** of f, or the ***composition operator*** of f, or the ***substitution operator*** of f).

Since in many applications the Nemytskii operator N_f acts on a Lebesgue space L^p, it is important to know under what conditions N_f maps L^p into another Lebesgue space L^r. It turns out that this leads to a particular growth condition on f, namely

$$f(z, x) \;=\; O(|x|^{\frac{p}{r}}),$$

which is both a necessary and a sufficient condition for N_f to act between L^p and L^r. This is the well known ***Krasnoselskii's theorem***, which here we prove in a more general form, namely when N_f acts on Lebesgue-Bochner spaces. We start with a definition.

DEFINITION 3.4.1 *Let (Ω, Σ) be a measurable space and let X, Y be two Hausdorff spaces. A function $f\colon \Omega \times X \longrightarrow Y$ is said to be a* ***Carathéodory function****, if*

- **(a)** *for every $x \in X$, the function $z \longmapsto f(z, x)$ is $\big(\Sigma, \mathcal{B}(Y)\big)$-measurable, with $\mathcal{B}(Y)$ being the Borel σ-field of Y;*
- **(b)** *for every $z \in \Omega$, the function $x \longmapsto f(z, x)$ is continuous.*

REMARK 3.4.2 If X is a separable metric space and Y is a metric space, then the function $(z, x) \longmapsto f(z, x)$ is $\Sigma \times \mathcal{B}(X)$-measurable, with $\mathcal{B}(X)$ being the Borel σ-field of X (i.e., f is jointly measurable). Therefore f is sup-measurable (superpositionally measurable), meaning that for every measurable function $u\colon \Omega \longrightarrow X$, the function $z \longmapsto f\big(z, u(z)\big)$ is measurable, i.e., the Nemytskii operator N_f maps measurable functions to measurable ones (for details see Denkowski, Migórski & Papageorgiou (2003*a*, pp. 189–190)).

In what follows, to avoid repeating the same hypotheses, we fix (Ω, Σ, μ) to be a nonatomic, σ-finite, complete measure space (in applications usually Ω is a subset of $\mathbb{R}^N$, equipped with the Lebesgue measure) and X, Y are two separable Banach spaces.

LEMMA 3.4.3
If $h\colon \Omega \times X \longrightarrow \mathbb{R}_+$ is a Carathéodory function, such that $h(z,0) = 0$ for all $z \in \Omega$ and

$$\left\|N_h(u)\right\|_{L^r(\Omega)} \leqslant c^r \qquad \forall\, u \in L^p(\Omega; X),$$

for some $c > 0$,
then

$$\mu(E_k) = 0 \qquad \forall\, k \geqslant 1,$$

where

$$E_k \stackrel{df}{=} \left\{ z \in \Omega : \sup_{\|x\|_X \leqslant k} h(z,x) = +\infty \right\} \qquad \forall\, k \geqslant 1.$$

PROOF Suppose that for some $k \geqslant 1$, we have $\mu(E_k) \neq 0$. Because the measure space is nonatomic, σ-finite, we can find $B_k \in \Sigma$, such that

$$B_k \subseteq E_k \quad \text{and} \quad 0 < \mu(B_k) < +\infty.$$

For every $z \in B_k$, we have

$$S_k(z) \stackrel{df}{=} \left\{ x \in X : \|x\|_X \leqslant k,\ h(z,x) > \frac{2c^r}{\mu(B_k)} \right\}.$$

Evidently

$$S_k(z) \neq \emptyset \qquad \forall\, z \in B_k$$

and

$$\operatorname{Gr} S_k \in (\Sigma \cap B_k) \times \mathcal{B}(X),$$

with $\mathcal{B}(X)$ being the Borel σ-field of X. We apply the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33) and obtain a $\big(\Sigma, \mathcal{B}(X)\big)$-measurable map $u_k\colon B_k \longrightarrow X$ such that

$$u_k(z) \in S_k(z) \qquad \forall\, z \in B_k.$$

We extend u_k to all of Ω by setting $u_k(z) = 0$ if $z \in \Omega \setminus B_k$. Since

$$h(z,0) = 0 \qquad \forall\, z \in \Omega$$

and $u_k \in L^p(\Omega; X)$, we have

$$\int_\Omega h\big(z, u_k(z)\big)^r \, d\mu = \int_{B_k} h\big(z, u_k(z)\big)^r \, d\mu > 2c^r,$$

a contradiction. ∎

Using this lemma, we can prove the general version of Krasnoselskii's theorem for N_f.

THEOREM 3.4.4
If $f\colon \Omega \times X \longrightarrow Y$ is a Carathéodory function, $p, r \in [1, +\infty)$ and N_f maps $L^p(\Omega; X)$ into $L^r(\Omega; Y)$,
then N_f is continuous, bounded (i.e., maps bounded sets into bounded sets) and there exist $a \in L^r(\Omega)_+$ and $c > 0$, such that

$$\big\|f(z,x)\big\|_Y \;\leqslant\; a(z) + c\,\|x\|_X^{\frac{p}{r}} \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

PROOF Let $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega; X)$ be a sequence, such that

$$u_n \;\longrightarrow\; u \quad \text{in } L^p(\Omega; X),$$

for some $u \in L^p(\Omega; X)$. Let

$$g\colon \Omega \times X \;\longrightarrow\; \mathbb{R}$$

be defined by

$$g(z,x) \;\stackrel{df}{=}\; \big\|f(z, x + u(z)) - f(z, u(z))\big\|_Y^r.$$

We pick a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$, such that

$$\big\|u_{n_k} - u\big\|^p_{L^p(\Omega;X)} \;\leqslant\; \frac{1}{2^k} \qquad \forall\; k \geqslant 1$$

and

$$u_{n_k}(z) \;\longrightarrow\; u(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Let

$$v_k \;\stackrel{df}{=}\; u_{n_k} - u \qquad \forall\; k \geqslant 1.$$

We have

$$v_k(z) \;\longrightarrow\; 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and so

$$g\big(z, v_k(z)\big) \;\longrightarrow\; 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega \quad \text{as } k \to +\infty.$$

Because

$$g(z,x) \;\geqslant\; 0 \qquad \forall\; (z,x) \in \Omega \times X$$

and

$$v_k(z) \;\longrightarrow\; 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

we can find $k(z) \in \mathbb{N}$, such that

$$\xi(z) \;=\; \sup_{k\geqslant 1} g\big(z, v_k(z)\big) \;=\; g\big(z, v_{k(z)}(z)\big).$$

Let

$$\widehat{v}(z) \stackrel{df}{=} v_{k(z)}(z).$$

Since ξ is Σ-measurable, we see that the function

$$z \longmapsto \widehat{v}(z)$$

is Σ-measurable. Moreover, we have

$$\begin{aligned}\int_\Omega \|\widehat{v}(z)\|_X^p \, d\mu &\leqslant \int_\Omega \sup_{k\geqslant 1} \|v_k(z)\|_X^p \, d\mu \\ \leqslant \sum_{k=1}^\infty \|v_k\|^p_{L^p(\Omega;X)} &= \sum_{k=1}^\infty \|u_{n_k} - u\|^p_{L^p(\Omega;X)} < +\infty,\end{aligned}$$

so $\widehat{v} \in L^p(\Omega;X)$.

Then from the definition of g and the hypothesis that N_f maps $L^p(\Omega;X)$ into $L^r(\Omega;X)$, we infer that

$$g\big(\cdot, \widehat{v}(\cdot)\big) \in L^1(\Omega)_+.$$

Since

$$g\big(z, v_k(z)\big) \leqslant g\big(z, \widehat{v}(z)\big) \qquad \forall\, z \in \Omega,\ k \geqslant 1$$

and

$$g\big(z, v_k(z)\big) \longrightarrow 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

form the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that

$$\int_\Omega g\big(z, v_k(z)\big)\, d\mu \longrightarrow 0.$$

Therefore

$$N_f\big(x_{n_k}\big) \longrightarrow N_f(x) \quad \text{in } L^r(\Omega;Y).$$

Since every subsequence of $\big\{N_f(x_n)\big\}_{n\geqslant 1}$ has a further subsequence converging in $L^r(\Omega;Y)$ to $N_f(x)$, we conclude that

$$N_f(x_n) \longrightarrow N_f(x) \quad \text{in } L^r(\Omega;Y)$$

and so the map

$$N_f\colon L^p(\Omega;X) \longrightarrow L^r(\Omega;Y)$$

is continuous.

Next we prove the boundedness of N_f. For $u \in L^p(\Omega;X)$, let

$$\widehat{f}(z,x) \stackrel{df}{=} f\big(z, x + u(z)\big) - f\big(z, u(z)\big).$$

Evidently $\widehat{f}$ is a Carathéodory function, $N_{\widehat{f}}$ maps $L^p(\Omega;X)$ into $L^r(\Omega;Y)$ and in addition

$$\widehat{f}(z,0) \;=\; 0 \qquad \forall\, z \in \Omega.$$

So without any loss of generality, we may assume that

$$f(z,0) \;=\; 0 \qquad \forall\, z \in \Omega.$$

Since N_f is continuous at 0, we can find $\varrho > 0$, such that

$$\big\|N_f(u)\big\|_{L^p(\Omega;Y)} \;\leqslant\; 1 \qquad \forall\; \|u\|_{L^p(\Omega;X)} \leqslant \varrho.$$

Then take an arbitrary $u \in L^p(\Omega;X)$ and let $n \geqslant 1$ be an integer, such that

$$n\varrho^p \;\leqslant\; \|u\|^p_{L^p(\Omega;X)} \;\leqslant\; (n+1)\varrho^p.$$

We write

$$\Omega \;=\; \bigcup_{k=1}^{m+1} \Omega_k$$

as a disjoint union, such that

$$\|u\|^p_{L^p(\Omega_k;X)} \;\leqslant\; \varrho^p \qquad \forall\, k \in \big\{1,\dots,k+1\big\}.$$

Then we have

$$\begin{aligned}\int\limits_{\Omega} \big\|f\big(z,u(z)\big)\big\|^r_Y\, d\mu \;&=\; \sum_{k=1}^{n+1} \int\limits_{\Omega_k} \big\|f\big(z,u(z)\big)\big\|^r_Y\, d\mu \\ \leqslant\; n+1 \;&\leqslant\; \left(\frac{\|u\|_{L^p(\Omega;X)}}{\varrho}\right)^p + 1,\end{aligned}$$

which proves that N_f is bounded.

Finally we prove the growth condition. Since N_f is bounded, we can find $c > 0$, such that

$$\big\|N_f(u)\big\|_{L^p(\Omega;Y)} \;\leqslant\; c \qquad \forall\; \|u\|_{L^p(\Omega;X)} \leqslant 1. \tag{3.116}$$

Let

$$h\colon \Omega \times X \;\longrightarrow\; \mathbb{R}$$

be defined by

$$h(z,x) \;\stackrel{df}{=}\; \Big[\big\|f(z,x)\big\|_Y - c\big\|x\big\|^{\frac{p}{r}}_X\Big]^+.$$

Using the inequality which says that

$$(\xi_1 - \xi_2)^r \leqslant \xi_1^r - \xi_2^r \qquad \forall\, \xi_1 > \xi_2,$$

we have

$$h(z,x)^r \;\leqslant\; \big\|f(z,x)\big\|_Y^r - c^r\|x\|_X^p \quad \text{when } h(z,x)>0. \tag{3.117}$$

Let $u \in L^p(\Omega;X)$ and let

$$C \;\stackrel{df}{=}\; \big\{z\in\Omega : h\big(z,u(z)\big)>0\big\}.$$

Then we can find an integer $n \geqslant 1$ and $\varepsilon \in [0,1)$, such that

$$\int\limits_C \big\|u(z)\big\|_X^p\, d\mu \;=\; n+\varepsilon.$$

So we can write

$$C \;=\; \bigcup_{k=1}^{n+1} C_k,$$

a disjoint union, such that

$$\int\limits_{C_k} \big\|u(z)\big\|_X^p\, d\mu \leqslant 1 \qquad \forall\; k\in\big\{1,\ldots,n+1\big\}.$$

Then assuming as before without any loss of generality, that

$$f(z,0) \;=\; 0 \qquad \forall\; z\in\Omega$$

and using (3.116), we obtain

$$\int\limits_C \big\|f\big(z,u(z)\big)\big\|_Y^r\, d\mu \;=\; \sum_{k=1}^{n+1}\int\limits_{C_k} \big\|f\big(z,u(z)\big)\big\|_Y^r\, d\mu \;\leqslant\; (n+1)c^r. \tag{3.118}$$

Returning to (3.117) and using (3.118), we have

$$\int\limits_\Omega h\big(z,u(z)\big)^r\, d\mu \;\leqslant\; (n+1)c^r-(n+\varepsilon)c^r \;\leqslant\; c^r \qquad \forall\; u\in L^p(\Omega;X). \tag{3.119}$$

So by virtue of Lemma 3.4.3, we have

$$\mu\left(\left\{z\in\Omega : \sup_{\|x\|_X\leqslant k} h(z,x)=+\infty\right\}\right) \;=\; 0 \qquad \forall\; k\geqslant 1. \tag{3.120}$$

Since by hypothesis the measure space is σ-finite, we can find $\{D_k\}_{k\geqslant 1}\subseteq\Sigma$, such that

$$\Omega \;=\; \bigcup_{k=1}^{\infty} D_k \quad \text{and} \quad \mu(D_k) \;<\; +\infty \qquad \forall\; k\geqslant 1.$$

For $z \in D_k$, let

$$V_k(z) \stackrel{df}{=} \Big\{ x \in X : \|x\|_X \leqslant k, \sup_{\|x\|_X \leqslant k} h(z,x) < +\infty, \sup_{\|x\|_X \leqslant k} h(z,x) - \frac{1}{k} \leqslant h(z,x) \Big\}.$$

Because of (3.120),

$$V_k(z) \neq \emptyset \quad \text{for } \mu\text{-a.a. } z \in D_k.$$

Also we have

$$\operatorname{Gr} V_k \ \in \ (\Sigma \cap D_k) \times \mathcal{B}(X).$$

So the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33) gives a $\big(\Sigma, \mathcal{B}(X)\big)$-measurable map $v_k \colon D_k \longrightarrow X$, such that

$$v_k(z) \ \in \ V_k(z) \qquad \forall\, z \in D_k.$$

Extend v_k to all of Ω by setting $v_k|_{\Omega \setminus D_k} = 0$. Let

$$a(z) \stackrel{df}{=} \sup_{x \in X} h(z,x).$$

Because h is a Carathéodory function and X is separable, a is Σ-measurable. Also we have

$$\sup_{\|x\|_X \leqslant k} h(z,x) - \frac{1}{k} \ \leqslant \ h\big(z, v_k(z)\big) \ \leqslant \ a(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

so

$$h\big(z, v_k(z)\big) \ \longrightarrow \ a(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega, \quad \text{as } k \to +\infty.$$

Note that $v_k \in L^p(\Omega; X)$ and so from (3.119), we have

$$\int_\Omega h\big(z, v_k(z)\big)^r \, d\mu \ \leqslant \ c^r \qquad \forall\, k \geqslant 1.$$

As $h \geqslant 0$, we can apply Fatou's lemma (see Theorem A.2.1) and obtain

$$\int_\Omega a(z)^r \, d\mu \ \leqslant \ \liminf_{k \to +\infty} \int_\Omega h\big(z, v_k(z)\big)^r \, d\mu \ \leqslant \ c^r$$

and thus $a \in L^r(\Omega)$.

Recalling the definition of $h(z,x)$, we conclude that

$$\big\|f(z,x)\big\|_Y \ \leqslant \ a(z) + c\,\|x\|_X^{\frac{p}{r}} \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

□

REMARK 3.4.5 By virtue of Theorem 3.4.4, the growth condition

$$\big\|f(z,x)\big\|_Y \leqslant a(z) + c\,\|x\|_X^{\frac{p}{r}} \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

with $a \in L^r(\Omega)$, $c > 0$ is both necessary and sufficient condition for the continuity and boundedness of the Nemytskii operator

$$N_f\colon L^p(\Omega;X) \longrightarrow L^r(\Omega;Y).$$

If in Theorem 3.4.4 we drop the hypothesis that $f(z,x)$ is a Carathéodory function and we only assume that the function $(z,x) \longmapsto f(z,x)$ is $\big(\Sigma \times \mathcal{B}(X), \mathcal{B}(Y)\big)$-measurable and for all $z \in \Omega$, the function $x \longmapsto f(z,x)$ is lower semicontinuous, then we no longer have the continuity of the Nemytskii operator N_f, even if it maps $L^p(\Omega;X)$ into $L^r(\Omega;Y)$.

To see this let $\Omega = [0,1]$ equipped with the Lebesgue measure, let $X = Y = \mathbb{R}$ and consider the function

$$f(x) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then N_f maps $L^p(\Omega)$ to $L^r(\Omega)$ for every $r \in [1,+\infty)$. However, if we consider

$$x_n(z) \stackrel{df}{=} \frac{z}{n},$$

then $N_f(x_n)$ does not converge in measure to zero. Also if $r = +\infty$ and N_f maps $L^p(\Omega;X)$ into $L^\infty(\Omega;Y)$ (f is still a Carathéodory function), then N_f is again bounded and there exists $M > 0$, such that

$$\big\|f(z,x)\big\|_Y \leqslant M \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } x \in X.$$

The proof which is similar to that of Theorem 3.4.4 is left to the reader. However,

$$N_f\colon L^p(\Omega;X) \longrightarrow L^\infty(\Omega;Y)$$

is not in general continuous as the following example illustrates.

Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, let $X = Y = \mathbb{R}$ and consider

$$f(x) \stackrel{df}{=} \begin{cases} -1 & \text{if} \quad x < -1, \\ x & \text{if} \quad -1 \leqslant x \leqslant 1, \\ 1 & \text{if} \quad 1 < x. \end{cases}$$

If we take $u_n(z) = z^n$, then

$$x_n \longrightarrow 0 \quad \text{in } L^p[0,1],$$

for all $p \in [1,+\infty)$. But $N_f(x_n)$ does not converge to zero in $L^\infty[0,1]$.

PROPOSITION 3.4.6
If $f\colon \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is a Carathéodory function, for all* $z \in \Omega$, $f(z,\cdot)$ *is a monotone map and* N_f *maps* $L^p\big(\Omega;\mathbb{R}^N\big)$ *into* $L^{p'}(\Omega;\mathbb{R}^N)$, *where* $p \in [1,+\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$,
then $N_f\colon L^p\big(\Omega;\mathbb{R}^N\big) \longrightarrow L^{p'}(\Omega;\mathbb{R}^N)$ *is a maximal monotone operator.*

PROOF If by $\langle\cdot,\cdot\rangle_{pp'}$ we denote the duality brackets for the pair of spaces $\big(L^{p'}(\Omega;\mathbb{R}^N), L^p\big(\Omega;\mathbb{R}^N\big)\big)$, for all $u,v \in L^p\big(\Omega;\mathbb{R}^N\big)$, we have

$$\big\langle N_f(u)-N_f(v),u-v\big\rangle_{pp'} = \int_\Omega \big(f\big(z,u(z)\big)-f\big(z,v(z)\big),u(z)-v(z)\big)_{\mathbb{R}^N}\,d\mu \geqslant 0$$

due to the monotonicity of $f(z,\cdot)$. Hence N_f is monotone. Moreover, by Theorem 3.4.4, N_f is continuous. Therefore, Proposition 3.2.19 implies that N_f is maximal monotone. □

PROPOSITION 3.4.7
If $f\colon \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is a Carathéodory function, such that*

(i) $f(z,\cdot)$ *is a strictly monotone map for* μ*-almost all* $z \in \Omega$;

(ii) $\big(f(z,x),x\big)_{\mathbb{R}^N} \geqslant c_1\|x\|^p_{\mathbb{R}^N} - a_1(z)$ *for* μ*-almost all* $z \in \Omega$ *and all* $x \in \mathbb{R}^N$ *with* $a_1 \in L^1(\Omega)_+$, $c>0$;

(iii) N_f *maps* $L^p\big(\Omega;\mathbb{R}^N\big)$ *into* $L^{p'}(\Omega;\mathbb{R}^N)$, *with* $p \in [1,+\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$,

then $N_f\colon L^p\big(\Omega;\mathbb{R}^N\big) \longrightarrow L^{p'}(\Omega;\mathbb{R}^N)$ *is an operator of type* $(S)_+$ *(see Definition 3.2.55(b)).*

PROOF Suppose that $\{u_n\}_{n\geqslant 1} \subseteq L^p\big(\Omega;\mathbb{R}^N\big)$ is a sequence, such that

$$u_n \longrightarrow u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big)$$

and

$$\limsup_{n\to+\infty}\big\langle N_f(u_n)-N_f(u),u_n-u\big\rangle_{pp'} \leqslant 0.$$

We need to show that

$$u_n \longrightarrow u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big).$$

From the monotonicity of N_f (see Proposition 3.4.6), we have that

$$\big\langle N_f(u_n)-N_f(u),u_n-u\big\rangle_{pp'} \longrightarrow 0.$$

Note that

$$\begin{aligned}&\big\langle N_f(u_n) - N_f(u), u_n - u\big\rangle_{pp'}\\ &= \int_\Omega \big(f\big(z,u_n(z)\big) - f\big(z,u(z)\big), u_n(z) - u(z)\big)_{\mathbb{R}^N}\, d\mu.\end{aligned}$$

Because of the monotonicity of $f(z,\cdot)$, by passing to a subsequence, we may assume that

$$\beta_n(z) \stackrel{df}{=} \big(f\big(z,u_n(z)\big) - f\big(z,u(z)\big), u_n(z) - u(z)\big)_{\mathbb{R}^N} \longrightarrow 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and

$$\big|\beta_n(z)\big| \leqslant k(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } n \geqslant 1,$$

with $k \in L^1(Z)_+$. From Theorem 3.4.4, we know that for μ-almost all $z \in \Omega$ and all $x \in \mathbb{R}^N$, we have

$$\big\|f(z,x)\big\|_{\mathbb{R}^N} \leqslant a(z) + c\,\|x\|_{\mathbb{R}^N}^{p-1},$$

with $a \in L^{p'}(\Omega)_+$, $c > 0$ (recall that $\frac{p}{p'} = p-1$). So for all $z \in \Omega \setminus D$, $\mu(D) = 0$ and all $n \geqslant 1$, we have

$$\begin{aligned}k(z) \geqslant \beta_n(z) \geqslant\ & c_1\big(\big\|u_n(z)\big\|_{\mathbb{R}^N}^p + \big\|u(z)\big\|_{\mathbb{R}^N}^p\big)\\ &- \big\|u_n(z)\big\|_{\mathbb{R}^N}\big(a(z) + c\big\|u(z)\big\|_{\mathbb{R}^N}^{p-1}\big)\\ &- \big\|u(z)\big\|_{\mathbb{R}^N}\big(a(z) + c\big\|u(z)\big\|_{\mathbb{R}^N}^{p-1}\big) - 2a_1(z). \qquad (3.121)\end{aligned}$$

Using Young's inequality (see Proposition A.4.5) with $\varepsilon > 0$, we have

$$c\big\|u_n(z)\big\|_{\mathbb{R}^N}\big\|u(z)\big\|_{\mathbb{R}^N}^{p-1} \leqslant \frac{\varepsilon}{p}\big\|u_n(z)\big\|_{\mathbb{R}^N}^p + \frac{c^{p'}}{\varepsilon p'}\big\|u(z)\big\|_{\mathbb{R}^N}^p \qquad (3.122)$$

and

$$c\big\|u(z)\big\|_{\mathbb{R}^N}\big\|u_n(z)\big\|_{\mathbb{R}^N}^{p-1} \leqslant \frac{c^p}{\varepsilon p}\big\|u(z)\big\|_{\mathbb{R}^N}^p + \frac{\varepsilon}{p'}\big\|u_n(z)\big\|_{\mathbb{R}^N}^p. \qquad (3.123)$$

Using (3.122) and (3.123) in (3.121), we obtain

$$\begin{aligned}c_1\big\|u_n(z)\big\|_{\mathbb{R}^N}^p \leqslant\ & k(z) + c_1\big\|u(z)\big\|_{\mathbb{R}^N}^p + \frac{\varepsilon}{p}\big\|u_n(z)\big\|_{\mathbb{R}^N}^p + \frac{c^{p'}}{\varepsilon p'}\big\|u(z)\big\|_{\mathbb{R}^N}^p\\ &+ a(z)\left(\big\|u_n(z)\big\|_{\mathbb{R}^N} + \big\|u(z)\big\|_{\mathbb{R}^N}\right)\\ &+ \frac{c^p}{\varepsilon p}\big\|u(z)\big\|_{\mathbb{R}^N}^p + \frac{\varepsilon}{p'}\big\|u_n(z)\big\|_{\mathbb{R}^N}^p + 2a_1(z). \qquad (3.124)\end{aligned}$$

Recall that $\frac{1}{p} + \frac{1}{p'} = 1$ and choose $\varepsilon < c$. Then from (3.124), it follows that the sequence

$$\big\{\big\|u_n(\cdot)\big\|_{\mathbb{R}^N}^p\big\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)_+$$

is uniformly integrable. Also for all $z \in \Omega \setminus D$, $\mu(D) = 0$, the sequence $\{u_n(z)\}_{n\geqslant 1} \subseteq \mathbb{R}^N$ is bounded. So by passing to a suitable subsequence (depending in general on $z \in \Omega \setminus D$), we may assume that

$$u_n(z) \longrightarrow \widehat{u}(z) \quad \text{in } \mathbb{R}^N.$$

Recall that $f(z,\cdot)$ is continuous and that

$$\beta_n(z) \longrightarrow 0,$$

so in the limit we obtain

$$\big(f\big(z,\widehat{u}(z)\big) - f\big(z,u(z)\big), \widehat{u}(z) - u(z)\big)_{\mathbb{R}^N} = 0.$$

Since by hypothesis $f(z,\cdot)$ is strictly monotone, we infer that

$$\widehat{u}(z) = u(z) \qquad \forall\, z \in \Omega \setminus D$$

and so it follows that

$$u_n(z) \longrightarrow u(z) \quad \text{in } \mathbb{R}^N, \qquad \forall\, z \in \Omega \setminus D,\ \mu(D) = 0. \tag{3.125}$$

From (3.125), the uniform integrability of the sequence

$$\big\{\|u_n(\cdot)\|^p_{\mathbb{R}^N}\big\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)_+$$

and Vitali's theorem (the extended dominated convergence theorem; see Theorem A.2.9), we obtain that

$$\|u_n\|_{L^p\left(\Omega;\mathbb{R}^N\right)} \longrightarrow \|u\|_{L^p\left(\Omega;\mathbb{R}^N\right)}.$$

Since

$$u_n \xrightarrow{w} u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big)$$

and the latter space is uniformly convex, from the Kadec-Klee property (see Remark A.3.22), we know that

$$u_n \longrightarrow u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big),$$

hence N_f is of type $(S)_+$. □

Next we pass to the study of integral functionals defined on Lebesgue-Bochner space. So if (Ω,Σ,μ) is a nonatomic, complete σ-finite measure space, X a separable Banach space and $f\colon \Omega \times X \longrightarrow \overline{R} = \mathbb{R} \cup \{+\infty\}$ is a $\Sigma \times \mathcal{B}(X)$-measurable function (an integrand), we consider the integral functional

$$I_f(u) \overset{df}{=} \int_\Omega f\big(z,u(z)\big)\,dz \qquad \forall\, u \in L^p(\Omega;X),$$

with $p \in [1,+\infty]$.

We start with a definition which extends the notion of a Carathéodory function (see Definition 3.4.1).

DEFINITION 3.4.8 *Let (Ω, Σ, μ) be a complete σ-finite measure space and X a separable metric space. We say that $f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a* ***normal integrand****, if*

(a) *f is $\Sigma \times \mathcal{B}(X)$-measurable; and*

(b) *the function $x \longmapsto f(z,x)$ is lower semicontinuous for μ-almost all $z \in \Omega$.*

We show that normal integrands which are bounded below can be realized as the upper envelope of a sequence of Carathéodory integrands.

PROPOSITION 3.4.9
If (Ω, Σ, μ) is a complete σ-finite measure space, X is a separable metric space with metric d_X, $f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ is a normal integrand and there exists a function $h\colon \Omega \longrightarrow \mathbb{R}$ (not necessarily measurable), such that

$$h(z) \leqslant f(z,x) \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } x \in X,$$

then we can find a sequence of functions $f_n\colon \Omega \times X \longrightarrow \mathbb{R}$ for $n \geqslant 1$, such that for all $n \geqslant 1$, we have

(a) *$h(z) \leqslant f_n(z,x) \leqslant n$ for μ-almost all $z \in \Omega$ and all $x \in X$;*

(b) *the function $z \longmapsto f_n(z,x)$ is measurable for all $x \in X$;*

(c) *$\big|f_n(z,x) - f_n(z,v)\big| \leqslant n d_X(x,v)$ for all $z \in \Omega$ and all $x, v \in \Omega$;*

(d) *$f_n(z,x) \nearrow f(z,x)$ for μ-almost all $z \in \Omega$ and all $x \in X$.*

PROOF **(a)** For every $n \geqslant 1$, let

$$\widehat{f}_n(z,x) \stackrel{df}{=} \inf_{y \in X} \big[f(z,y) + n d_X(y,x)\big].$$

Evidently, for all $n \geqslant 1$, μ-almost all $z \in \Omega$ and all $x \in X$, we have

$$h(z) \leqslant \widehat{f}_1(z,x) \leqslant \ldots \leqslant \widehat{f}_n(z,x) \leqslant \widehat{f}_{n+1}(z,x) \leqslant \ldots \leqslant f(z,x).$$

If we fix $n \geqslant 1$, $x \in X$ and $\lambda \in \mathbb{R}$, we have

$$\Big\{z \in \Omega : \ \widehat{f}_n(z,x) < \lambda\Big\} \ = \ \mathrm{proj}_{\Omega}\Big\{(z,y) \in \Omega \times X : \ f(z,y) + n d_X(y,x) < \lambda\Big\}.$$

By virtue of the joint measurability of f (see Definition 3.4.8), we deduce that

$$C_\lambda \stackrel{df}{=} \Big\{(z,y) \in \Omega \times X : \ f(z,y) + n d_X(y,x) < \lambda\Big\} \ \in \ \Sigma \times \mathcal{B}(X).$$

Hence from the Yankov-von Neumann-Aumann projection theorem (see Theorem A.2.32) and since by hypothesis Σ is μ-complete, we have

$$\mathrm{proj}_{\Omega} C_{\lambda} \in \Sigma$$

and so

$$\text{the function } z \longmapsto \widehat{f}_n(z,x) \text{ is measurable for all } x \in X \text{ and all } n \geqslant 1.$$

From the definition of $\widehat{f}_n$, we have

$$\widehat{f}_n(z,x) \leqslant f(z,y) + n d_X(y,x) \qquad \forall\ (z,y) \in \Omega \times X,$$

so

$$\widehat{f}_n(z,x) \leqslant f(z,y) + n d_X(y,v) + n d_X(v,x) \qquad \forall\ v \in X$$

and thus

$$\widehat{f}_n(z,x) - \widehat{f}_n(z,v) \leqslant n d_X(x,v).$$

Interchanging the roles of x and v in the above argument, we conclude that

$$\left|\widehat{f}_n(z,x) - \widehat{f}_n(z,v)\right| \leqslant n d_X(x,v) \qquad \forall\ z \in \Omega,\ x,v \in X.$$

Therefore $\widehat{f}_n$ is $\Sigma \times \mathcal{B}(X)$-measurable (see Remark 3.4.2). Moreover, since for all $(z,x) \in \Omega \times X$, the sequence $\{\widehat{f}_n(z,x)\}_{n \geqslant 1}$ is increasing, we have

$$\lim_{n \to +\infty} \widehat{f}_n(z,x) \leqslant f(z,x) \qquad \forall\ (z,x) \in \Omega \times X. \tag{3.126}$$

Let $D \subseteq \Omega$ be the μ-null set, such that the function $f(z,\cdot)$ is lower semicontinuous for all $z \in \Omega \setminus D$. Then for all $z \in \Omega \setminus D$, $x \in X$ and $\varepsilon > 0$, let $\{y_n\}_{n \geqslant 1} \setminus X$ be a sequence, such that

$$f(z,y_n) + n d_X(y_n,x) \leqslant \widehat{f}_n(z,x) + \varepsilon.$$

As $n \to +\infty$, either $\widehat{f}_n(z,x) \nearrow +\infty$, in which case equality holds in (3.126) or otherwise we have $y_n \longrightarrow x$ in X. Then because of the lower semicontinuity of $f(z,\cdot)$, we have

$$f(z,x) \leqslant \liminf_{n \to +\infty} f(z,y_n) \leqslant \lim_{n \to +\infty} \widehat{f}_n(z,x) + \varepsilon. \tag{3.127}$$

Since $z \in \Omega \setminus D$, $x \in X$ and $\varepsilon > 0$ were arbitrary, from (3.126) and (3.127), we infer that

$$\widehat{f}_n(z,x) \nearrow f(z,x) \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } x \in X.$$

Finally set

$$f_n(z,x) \stackrel{df}{=} \min\left\{\widehat{f}_n(z,x), n\right\}.$$

Then the sequence $\{f_n\}_{n \geqslant 1}$ is the desired sequence. □

Using this approximation result, we have another characterization of normal integrands. First let us recall the Scorza-Dragoni theorem, which is a parametrized version of Lusin's theorem (see Theorem A.2.11).

THEOREM 3.4.10 (Scorza-Dragoni Theorem)
If Ω, X are two Polish spaces (see Definition A.2.29(a)), Y is a separable metric space, μ is a tight Borel measure on Ω (see Remark 3.4.11) and $f\colon \Omega \times X \longrightarrow Y$ is a Carathéodory function,
then for every $\varepsilon > 0$, we can find a compact set $\Omega_\varepsilon \subseteq \Omega$, with

$$\mu(\Omega \setminus \Omega_\varepsilon) \ < \ \varepsilon,$$

such that

$$f|_{\Omega_\varepsilon \times X} \text{ is continuous.}$$

REMARK 3.4.11 Recall that μ is a tight Borel measure on Ω, if μ is finite and for every $\varepsilon > 0$, we can find a compact subset K_ε of Ω, such that

$$\mu(\Omega \setminus K_\varepsilon) \ < \ \varepsilon.$$

On a Polish space every finite Borel measure is tight.

Combining Proposition 3.4.9 with Theorem 3.4.10, we obtain the following characterization of normal integrands.

PROPOSITION 3.4.12
If Ω, X are two Polish spaces, μ is a finite Borel measure on Ω and $f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$,
then f is a normal integrand if and only if for every $\varepsilon > 0$ we can find a compact set $K_\varepsilon \subseteq \Omega$, such that

$$\mu(\Omega \setminus K_\varepsilon) \ < \ \varepsilon$$

and

$$f|_{K_\varepsilon \times X} \text{ is lower semicontinuous.}$$

Now we pass to the study of the integral functional

$$I_f(u) \ \stackrel{df}{=} \ \int_\Omega f\big(z, u(z)\big)\, d\mu \qquad \forall\ u \in L^p(\Omega; X),$$

with $p \in [1, +\infty]$. In our analysis we shall use the computational convention

$$+\infty - \infty \ = \ +\infty,$$

which is useful when dealing with integral functionals of $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$-valued functions.

THEOREM 3.4.13
If (Ω, Σ, μ) *is a nonatomic,* σ*-finite measure space,* X *is separable Banach space,* $f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ *is a normal integrand, the integral functional* I_f *is not identically* $+\infty$ *and* $p \in [1, +\infty)$,
then the following properties are equivalent:

(a) I_f *is lower semicontinuous on* $L^p(\Omega; X)$ *and* $I_f(u) > -\infty$ *for all* $u \in L^p(\Omega; X)$.

(b) $I_f\colon L^p(\Omega; X) \longrightarrow \overline{\mathbb{R}}$.

(c) *There exist* $\beta_1 \in \mathbb{R}$ *and* $\beta_2 \geqslant 0$, *such that*

$$I_f(u) \;\geqslant\; \beta_1 - \beta_2 \,\|u\|^p_{L^p(\Omega;X)} \qquad \forall\; u \in L^p(\Omega; X).$$

(d) *There exist* $a \in L^1(\Omega)$ *and* $c \geqslant 0$, *such that*

$$f(z, x) \;\geqslant\; a(z) - c\,\|x\|^p_X \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } x \in X.$$

PROOF Clearly implications **(d)**$\Longrightarrow$**(c)**$\Longrightarrow$**(b)** and **(a)**$\Longrightarrow$**(b)** hold.

"**(b)**$\Longrightarrow$**(d)**": Let us set

$$g \;\stackrel{df}{=}\; \min\big\{f, 0\big\}.$$

We claim that

$$\int_\Omega g\big(z, u(z)\big)\, d\mu \;>\; -\infty \qquad \forall\; u \in L^p(\Omega; X). \tag{3.128}$$

Suppose that (3.128) does not hold. Then for some $u_0 \in L^p(\Omega; X)$, we have

$$\int_\Omega g\big(z, u_0(z)\big)\, d\mu \;=\; -\infty.$$

Let

$$C \;\stackrel{df}{=}\; \Big\{z \in \Omega :\; f\big(z, u_0(z)\big) = g\big(z, u_0(z)\big)\Big\} \;\in\; \Sigma.$$

For any given $v \in L^p(\Omega; X)$, we define

$$\widehat{u} \;\stackrel{df}{=}\; \chi_C u_0 + \chi_{C^c} v \;\in\; L^p(\Omega; X).$$

Then we have

$$-\infty \;<\; I_f(\widehat{u}) \;=\; \int_C g\big(z, u_0(z)\big)\, d\mu + \int_{C^c} f\big(z, v(z)\big)\, d\mu,$$

so

$$\int_{C^c} f\big(z, v(z)\big)\, d\mu \;=\; +\infty$$

and thus $I_f \equiv +\infty$, a contradiction.

From (3.128), it follows that N_g (the Nemytskii operator corresponding to g) maps $L^p(\Omega; X)$ into $L^1(\Omega)$. Invoking Theorem 3.4.4, we obtain **(d)**.

"**(d)**$\Longrightarrow$**(a)**": Let $\lambda \in \mathbb{R}$ and consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega; X)$, such that

$$u_n \;\longrightarrow\; u \quad \text{in } L^p(\Omega; X),$$

for some $u \in L^p(\Omega; X)$ and

$$I_f(u_n) \;\leqslant\; \lambda \qquad \forall\, n \geqslant 1.$$

By passing to a subsequence if necessary, we may also assume that

$$u_n(z) \;\longrightarrow\; u(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and

$$\big\|u_n(z)\big\|_X \;\leqslant\; k(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega,\; n \geqslant 1,$$

with $k \in L^1(\Omega)_+$. Then because of **(d)**, we can apply Fatou's lemma (see Theorem A.2.1) and obtain

$$\begin{aligned} I_f(u) \;&=\; \int_\Omega f\big(z, u(z)\big)\, d\mu \;\leqslant\; \int_\Omega \liminf_{n\to+\infty} f\big(z, u_n(z)\big)\, d\mu \\ &\leqslant\; \liminf_{n\to+\infty} \int_\Omega f\big(z, u_n(z)\big)\, d\mu \;\leqslant\; \lambda, \end{aligned}$$

so I_f is lower semicontinuous on $L^p(\Omega; X)$.

Moreover, it is clear that

$$I_f(u) \;>\; -\infty \qquad \forall\, u \in L^p(\Omega; X).$$

□

COROLLARY 3.4.14

If (Ω, Σ, μ) is a nonatomic, complete, σ-finite measure space, X is a separable Banach space,

$$f\colon \Omega \times X \longrightarrow \mathbb{R}$$

is a Carathéodory function, such that

$$\big|f(z,x)\big| \;\leqslant\; a(z) + c\,\|x\|_X^p \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } x \in X,$$

with $a \in L^1(\Omega)_+$, $c > 0$ and $p \in [1, +\infty)$,
then $I_f\colon L^p(\Omega; X) \longrightarrow \mathbb{R}$ is continuous.

For $p = +\infty$, we can state the following continuity result.

PROPOSITION 3.4.15
If (Ω, Σ, μ) is a nonatomic, complete, σ-finite measure space, X is a separable Banach space, $f\colon \Omega \times X \longrightarrow \mathbb{R}$ is a Carathéodory function and for all $r > 0$ we can find $a_r \in L^1(\Omega)_+$, such that

$$\big|f(z,x)\big| \leqslant a_r(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } \|x\|_X \leqslant r,$$

then $I_f\colon L^\infty(\Omega; X) \longrightarrow \mathbb{R}$ is continuous.

PROOF Suppose that $\{u_n\}_{n\geqslant 1} \subseteq L^\infty(\Omega; X)$ is a sequence, such that

$$u_n \longrightarrow u \quad \text{in } L^\infty(\Omega; X),$$

for some $u \in L^\infty(\Omega; X)$. Let

$$r \stackrel{df}{=} \sup_{n\geqslant 1} \|u_n\|_{L^\infty(\Omega;X)} < +\infty.$$

Then

$$\big|f\big(z, u_n(z)\big)\big| \leqslant a_r(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Since

$$f\big(z, u_n(z)\big) \longrightarrow f\big(z, u(z)\big) \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

from the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that

$$I_f(u_n) \longrightarrow I_f(u).$$

□

PROPOSITION 3.4.16
If (Ω, Σ, μ) is a nonatomic, complete, σ-finite measure space, X is a separable Banach space, $f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ is a normal integrand, such that $f(z, \cdot)$ is convex for μ-almost all $z \in \Omega$, $u_0 \in L^\infty(\Omega; X)$ and $I_f(u_0) \in \mathbb{R}$,
then the following conditions are equivalent:

(a) *I_f is continuous at u_0 with respect to norm topology on $L^\infty(\Omega; X)$.*

(b) *There exist $\varepsilon > 0$ and $a \in L^1(\Omega)$, such that*

$$\sup_{\|x\|_X \leqslant \varepsilon} f\big(z, x + u_0(z)\big) \leqslant a(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

PROOF "**(a)**$\Longrightarrow$**(b)**": Suppose that the implication is not true. Then for any $\varepsilon > 0$, the function

$$\xi_\varepsilon(z) \stackrel{df}{=} \sup_{\|x\|_X \leqslant \varepsilon} f\big(z, x + u_0(z)\big)$$

is not integrable (note that since f is a normal integrand, ξ_ε is measurable). Clearly

$$\int_\Omega \xi_\varepsilon(z)\,d\mu \;=\; +\infty \qquad \forall\, \varepsilon > 0.$$

So for every $\varepsilon > 0$ and $N \geqslant 1$, we can find a measurable function $\xi_{\varepsilon,N} : \Omega \longrightarrow \mathbb{R}$, such that

$$\int_\Omega \xi_{\varepsilon,N}(z)\,d\mu \;\geqslant\; N \quad \text{and} \quad \xi_{\varepsilon,N}(z) \;\leqslant\; \xi_\varepsilon(z) \qquad \forall\, z \in \Omega.$$

Then for every $z \in \Omega$, we define

$$S_{\varepsilon,N}(z) \stackrel{df}{=} \Big\{ x \in X :\; \big\|x - u_0(z)\big\|_X \leqslant \varepsilon,\; f(z,x) \geqslant \xi_{\varepsilon,N}(z) \Big\}.$$

Since $(z,x) \longmapsto \big\|x - u_0(z)\big\|_X$ is a Carathéodory function (hence jointly measurable; see Remark 3.4.2) and

$$(z,x) \longmapsto f(z,x) - \xi_{\varepsilon,N}(z)$$

is a $\Sigma \times \mathcal{B}(X)$-measurable function (since f is a normal integrand), we infer that

$$\operatorname{Gr} S_{\varepsilon,N} \;\in\; \Sigma \times \mathcal{B}(X).$$

Applying the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33), we obtain a measurable map $u_{\varepsilon,N} : \Omega \longrightarrow X$, such that

$$u_{\varepsilon,N}(z) \;\in\; S_{\varepsilon,N}(z) \qquad \forall\, z \in \Omega.$$

Clearly $u_{\varepsilon,N} \in L^\infty(\Omega; X)$, $\big\|u_{\varepsilon,N} - u_0\big\|_{L^\infty(\Omega;X)} \leqslant \varepsilon$ and

$$I_f(u_{\varepsilon,N}) \;\geqslant\; \int_\Omega \xi_{\varepsilon,N}(z)\,d\mu \;\geqslant\; N.$$

Since $\varepsilon > 0$ and $N \geqslant 1$ were arbitrary, it follows that the convex integral functional I_f is unbounded from above in every $L^\infty(\Omega; X)$-neighbourhood of u_0 and so it cannot be continuous at u_0, a contradiction.

"**(b)**$\Longrightarrow$**(a)**": For every

$$v \;\in\; \overline{B}_\varepsilon \stackrel{df}{=} \Big\{ v \in L^\infty(\Omega; X) :\; \|v\|_{L^\infty(\Omega;X)} \leqslant \varepsilon \Big\},$$

we have

$$I_f(u_0 \pm v) \;=\; \int_\Omega f\big(z, u_0(z) \pm v(z)\big)\,d\mu \;\leqslant\; \int_\Omega a(z)\,d\mu \;=\; \eta \;<\; +\infty.$$

Since I_f is convex and bounded above in a neighbourhood of u_0, it is continuous at u_0. $\square$

Thus far we have considered the norm topology on the Lebesgue-Bochner space. If we want to have weak lower semicontinuity, then, as we show, necessarily $f(z,\cdot)$ must be convex. For this purpose we need to use some tools from multivalued analysis, which for the convenience of the reader we recall here. Details can be found in Denkowski, Migórski & Papageorgiou (2003*a*, Chapter 4).

DEFINITION 3.4.17 *Let Y be a separable Banach space and*

$$G\colon \Omega \longrightarrow 2^Y \setminus \{\emptyset\}$$

is a multivalued (set-valued) map. We say that G is ***graph measurable****, if* $\operatorname{Gr} G \in \Sigma \times \mathcal{B}(Y)$*, where*

$$\operatorname{Gr} G \stackrel{df}{=} \big\{(z,y) \in \Omega \times Y :\ y \in G(z)\big\}.$$

Also, for any $p \in [1,+\infty]$, we set

$$S^p_G \stackrel{df}{=} \left\{ g \in L^p(\Omega;Y) :\ g(z) \in G(z) \text{ for } \mu\text{-a.a. } z \in \Omega \right\}$$

(the set of L^p-selections of the multifunction G)

The next result from multivalued analysis is the main tool in establishing the necessity of the convexity of $f(z,\cdot)$.

PROPOSITION 3.4.18
If (Ω,Σ,μ) is a nonatomic, complete, σ-finite measure space, Y is a separable Banach space and $G\colon \Omega \longrightarrow 2^Y \setminus \{\emptyset\}$ is graph measurable with $S^p_G \neq \emptyset$, $p \in [1,\infty)$,
then $\overline{S^p_G}^{\,w} = S^p_{\overline{\operatorname{conv}} G}$, where w stands for the weak topology on $L^p(\Omega;Y)$.

REMARK 3.4.19 There is an analogous result for $p = +\infty$. More precisely, let (Ω,Σ,μ) and Y be as in Proposition 3.4.18. We denote by $Y^*_{w^*}$ the dual space of Y furnished with the w^*-topology. From Theorem 2.2.12, we know that

$$L^\infty(\Omega;Y^*_{w^*}) \ = \ L^1(\Omega;Y)^*,$$

where $L^\infty(\Omega;Y^*_{w^*})$ is understood in the sense of Definition 2.2.10. Let

$$G\colon \Omega \longrightarrow 2^{Y^*} \setminus \{\emptyset\}$$

be a multifunction, such that

$$\operatorname{Gr} G \ \in \ \Sigma \times \mathcal{B}\big(Y^*_{w^*}\big) \quad \text{and} \quad S^\infty_G \neq \emptyset.$$

Then

$$\overline{S^\infty_G}^{\,w^*} \ = \ S^\infty_{\overline{\operatorname{conv}}^{\,w^*} G} \quad \text{in } L^\infty(\Omega;Y^*_{w^*}).$$

Using this general result about multifunctions, we can prove the following theorem about the integral functional I_f, according to which the weak lower semicontinuity of I_f on $L^1(\Omega; X)$ implies the convexity of $f(z, \cdot)$ for all $z \in \Omega$.

THEOREM 3.4.20
If (Ω, Σ, μ) is a nonatomic, complete, σ-finite measure space, X is a separable Banach space,

$$f\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$$

is a normal integrand, there exists $u_0 \in L^1(\Omega; X)$, such that

$$I_f(u_0) \ < \ +\infty$$

and I_f is weakly lower semicontinuous on $L^1(\Omega; X)$,
then

$$f(z, \cdot) \text{ is convex for all } z \in \Omega.$$

PROOF Without any loss of generality, we may assume that

$$f\big(z, u_0(z)\big) \ = \ 0 \qquad \forall\ z \in \Omega$$

(otherwise replace $f(z, x)$ by

$$\widehat{f}(z, x) = f(z, x) - f\big(z, u_0(z)\big)).$$

Consider the multifunction $E\colon \Omega \longrightarrow 2^{X \times \mathbb{R}}$, defined by

$$E(z) \ \stackrel{df}{=} \ \operatorname{epi} f(z, \cdot) \ = \ \Big\{(x, \lambda) \in X \times \mathbb{R} : f(z, x) \leqslant \lambda\Big\}$$

(the epigraph of $f(z, \cdot)$). Since

$$\big(u_0(z), 0\big) \ \in \ E(z) \qquad \forall\ z \in \Omega,$$

we see that E has nonempty values, which are also closed due to the lower semicontinuity of $f(z, \cdot)$. Moreover,

$$(u_0, 0) \ \in \ S^1_E.$$

We claim that

$$S^1_E \text{ is weakly closed in } L^1(\Omega; X).$$

To this end let $\big\{(u_\alpha, \lambda_\alpha)\big\}_{\alpha \in J}$ be a net in S^1_E and assume that

$$u_\alpha \ \stackrel{w}{\longrightarrow} \ u \quad \text{in } L^1(\Omega; X)$$

and

$$\lambda_\alpha \ \stackrel{w}{\longrightarrow} \ \lambda \quad \text{in } L^1(\Omega).$$

For every $C \in \Sigma$, we have that

$$\chi_C u_\alpha \xrightarrow{w} \chi_C u \quad \text{in } L^1(\Omega; X)$$

and so

$$\chi_C u_\alpha + \chi_{C^c} u_0 \xrightarrow{w} \chi_C u + \chi_{C^c} u_0 \quad \text{in } L^1(\Omega; X).$$

Also

$$\chi_C \lambda_\alpha \xrightarrow{w} \chi_C \lambda \quad \text{in } L^1(\Omega). \tag{3.129}$$

Note that

$$I_f\left(\chi_C u_\alpha + \chi_{C^c} u_0\right) = \int_C f\big(z, u_\alpha(z)\big)\, d\mu.$$

Since by hypothesis I_f is weakly lower semicontinuous on $L^1(\Omega; X)$, we have

$$\begin{aligned} \liminf_\alpha I_f\left(\chi_C u_\alpha + \chi_{C^c} u_0\right) &\geqslant I_f\left(\chi_C u + \chi_{C^c} u_0\right) \\ &= \int_C f\big(z, u(z)\big)\, d\mu, \end{aligned}$$

so

$$\liminf_\alpha \int_C f\big(z, u_\alpha(z)\big)\, d\mu \geqslant \int_C f\big(z, u(z)\big)\, d\mu.$$

Because

$$f\big(z, u_\alpha(z)\big) \leqslant \lambda_\alpha(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

we have

$$\int_C f\big(z, u_\alpha(z)\big)\, d\mu \leqslant \int_C \lambda_\alpha(z)\, d\mu,$$

so, from (3.129), we have

$$\int_C f\big(z, u(z)\big)\, d\mu \leqslant \int_C \lambda(z)\, d\mu.$$

The set $C \in \Sigma$ was arbitrary. Hence it follows that

$$f\big(z, u(z)\big) \leqslant \lambda(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and

$$(u, \lambda) \in S^1_E,$$

which proves that

$$S^1_E \text{ is weakly closed in } L^1(\Omega; X).$$

Then by virtue of Proposition 3.4.18, we have that

$$S^1_E = S^1_{\overline{\text{conv}}\, E}.$$

We claim that this implies that

$$E(z) \;=\; \overline{\operatorname{conv}} E(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

We proceed by contradiction. Suppose that

$$D(z) \;=\; \overline{\operatorname{conv}} E(z) \setminus E(z) \;\neq\; \emptyset \qquad \forall\, z \in C \in \Sigma,$$

with $\mu(C) > 0$. Clearly

$$\operatorname{Gr} D \;\in\; (\Sigma \cap C) \times \mathcal{B}(X)$$

and so by the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.33) we can find two $(\Sigma \cap C)$-measurable functions

$$\widehat{u}\colon \Omega \longrightarrow X \quad \text{and} \quad \widehat{\lambda}\colon C \longrightarrow \mathbb{R},$$

such that

$$\big(\widehat{u}(z), \widehat{\lambda}(z)\big) \;\in\; D(z) \qquad \forall\, z \in C.$$

Exploiting the σ-finiteness of the measure space, we can find

$$C' \;\subseteq\; C$$

with

$$C' \in \Sigma \quad \text{and} \quad 0 \;<\; \mu(C') < +\infty,$$

such that

$$\chi_{C'}\widehat{u} \;\in\; L^1(\Omega; X) \quad \text{and} \quad \chi_{C'}\widehat{\lambda} \;\in\; L^1(\Omega).$$

Then let

$$\overline{u} \;\stackrel{df}{=}\; \chi_{C'}\widehat{u} + \chi_{(C')^c} u_0 \quad \text{and} \quad \overline{\lambda} \;\stackrel{df}{=}\; \chi_{C'}\widehat{\lambda}.$$

Evidently $\big(\overline{u}, \overline{\lambda}\big) \in S^1_E$ and

$$\big(\overline{u}(z), \overline{\lambda}(z)\big) \;\in\; D(z) \qquad \forall\, z \in C',$$

a contradiction. Therefore

$$E(z) \;=\; \overline{\operatorname{conv}} E(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and by redefining E on a μ-null set, we may assume that

$$E(z) \;=\; \overline{\operatorname{conv}} E(z) \qquad \forall\, z \in \Omega.$$

This proves the convexity of $f(z, \cdot)$ for all $z \in \Omega$. □

In the next section we use the theory of Young measures to prove very general lower semicontinuity results for integral functionals. Moreover, in Section 4.3, we will focus on convex integral functionals.

3.5 Young Measures

According to Proposition 2.3.38, a sequence $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega)$, which converges weakly but not strongly in $L^1(\Omega)$, oscillates violently around its weak limit. However, in the limit all this information about the faster and faster oscillations is lost and only a mean value is recorded. Of course this is not satisfactory, because if for example on $\{u_n\}_{n\geqslant 1}$ we act with the Nemytskii operator N_f, we cannot say that

$$N_f(u_n) \xrightarrow{w} N_f(u) \quad \text{in } L^1(\Omega),$$

unless $f(z,\cdot)$ is affine. The idea is then to embed the sequence $\{u_n\}_{n\geqslant 1}$ into a larger space and consider the limit there. The appropriate space is that of probability-valued functions (parametrized measures). These are the Young measures.

In what follows let Ω and E be locally compact, σ-compact metric spaces, Σ a σ-field on Ω containing $\mathcal{B}(\Omega)$ (the Borel σ-field of Ω) and $\mu \in M(\Omega)_+$ (see Section 2.3), which is nonatomic and Σ is μ-complete. Also we set

$$M^1_+(E) \stackrel{df}{=} \left\{\lambda \in M(E)_+ :\ \lambda(E) = 1\right\} \tag{3.130}$$

(the probability measures on E) and

$$SM^1_+(E) \stackrel{df}{=} \left\{\lambda \in M(E)_+ :\ \lambda(E) \leqslant 1\right\}$$

(the subprobability measures on E).

DEFINITION 3.5.1 *A **transition probability** (respectively **transition subprobability**) on E is a function*

$$\lambda\colon \Omega \longrightarrow M^1_+(E) \ \textit{(respectively } \lambda\colon \Omega \longrightarrow SM^1_+(E)\textit{)},$$

such that for every $A \in B(E)$, we have that the function $z \longmapsto \lambda(z)(A)$ is Σ-measurable. By $\widehat{R}(\Omega, E)$ (respectively $\widehat{SR}(\Omega, E)$) we denote the space of transition probabilities (respectively subprobabilities) on Ω.

REMARK 3.5.2 On $M^1_+(E)$ we can consider the topology of narrow convergence (see Definition 2.3.42**(c)**). This is the relative topology on $M^1_+(E)$ induced by $w\big(M(E), C_b(E)\big)$. Recall that $M^1_+(E)$ furnished with this topology is a Polish space (see Definition A.2.29**(a)**). In the sequel by $M^1_+(E)_n$ we denote the space $M^1_+(E)$ equipped with the topology of narrow convergence.

PROPOSITION 3.5.3
If $\lambda\colon \Omega \longrightarrow M_+^1(E)$ *is a* $\big(\Sigma, \mathcal{B}\big(M_+^1(E)_n\big)\big)$*-measurable function, then* $\lambda \in \widehat{R}(\Omega, E)$.

PROOF Let U be an open set in E. The map $z \longmapsto \lambda(z)(U)$ is the composition of $\lambda\colon \Omega \longrightarrow M_+^1(E)$, which by hypothesis is $\big(\Sigma, \mathcal{B}\big(M_+^1(E)_n\big)\big)$-measurable and of the map $\xi\colon M_+^1(E) \longrightarrow [0,1]$, defined by

$$\xi(\nu) \;=\; \nu(U),$$

which is lower semicontinuous (by virtue of the Portmanteau theorem; see Theorem A.2.36). Therefore the map $z \longrightarrow \lambda(z)(U)$ is Σ-measurable. Since $\lambda(z) \in M_+^1(E)$ is regular for every $z \in \Omega$, we conclude that

$$\text{the map } z \longmapsto \lambda(z)(A) \text{ is } \Sigma\text{-measurable for all } A \in \Sigma.$$

□

PROPOSITION 3.5.4
If E *is compact and* $\lambda \in \widehat{R}(\Omega, E)$, *then* λ *is a* $\big(\Sigma, \mathcal{B}\big(M_+^1(E)_n\big)\big)$*-measurable function.*

PROOF Recall that $C(E)^* = M(E)$. Then for every $g \in C(E)$, we have that

$$\text{the map } z \longmapsto \big\langle \lambda(z), g \big\rangle_{C(E)} = \int_E g(x)\lambda(z)(dx) \text{ is } \Sigma\text{-measurable.}$$

Indeed, suppose that $s\colon E \longrightarrow \mathbb{R}$ is a simple function. Then since $\lambda \in R(\Omega; E)$, we have that

$$\text{the map } z \longmapsto \int_E s(x)\lambda(z)(dx) \text{ is } \Sigma\text{-measurable.}$$

We can find a sequence $\{s_n\}_{n\geqslant 1}$ of simple functions on E, such that

$$s_n(x) \;\longrightarrow\; g(x) \quad \text{uniformly on } E.$$

Then

$$\int_E s_n(x)\lambda(z)(dx) \;\longrightarrow\; \int_E g(x)\lambda(z)(dx),$$

which proves the Σ-measurability of the map $z \longmapsto \big\langle \lambda(z), g \big\rangle_{C(E)}$. So

$$\text{the map } z \longmapsto \lambda(z) \text{ is weakly}^*\text{-measurable.}$$

Because $M_+^1(E)_n$ is a compact metrizable space, we conclude that $z \longmapsto \lambda(z)$ is $\big(\Sigma, \mathcal{B}\big(M_+^1(E)_n\big)\big)$-measurable. □

Let us recall the notion of image measure.

DEFINITION 3.5.5 *Let $(S, \mathcal{T})$ be a measurable space, Y a Hausdorff topological space, $\xi\colon S \longrightarrow Y$ a $\big(\Sigma, \mathcal{B}(Y)\big)$-measurable function and $\lambda\colon \mathcal{T} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ a measure. The image of λ under ξ is the measure $\nu\colon \mathcal{B}(Y) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$, defined by*

$$\nu(A) \stackrel{df}{=} \lambda\big(\xi^{-1}(A)\big) \qquad \forall\ A \in \mathcal{B}(Y).$$

We often denote ν by $\lambda \circ \xi^{-1}$.

REMARK 3.5.6 If $g\colon Y \longrightarrow \mathbb{R}$ is a ν-integrable map or a measurable and positive map, then

$$\int_S g\big(\xi(s)\big)\, d\lambda(s) = \int_Y g(y)\, d\nu(y).$$

Now we can give the definition of Young measure.

DEFINITION 3.5.7 **(a)** *$\lambda \in M(\Omega \times E)_+$ is a **Young measure** with respect to μ, if*

$$\lambda(A \times E) = \mu(A) \qquad \forall\ A \in \Sigma.$$

By $\mathcal{Y}(\Omega, E, \mu)$, we denote the space of Young measures with respect to μ.

(b) *$\lambda \in M(\Omega \times E)_+$ is a **Young submeasure** with respect to μ, if*

$$\lambda(A \times E) \leqslant \mu(A) \qquad \forall\ A \in \Sigma.$$

By $S\mathcal{Y}(\Omega, E, \mu)$, we denote the space of Young submeasures with respect to μ.

(c) *If $u\colon \Omega \longrightarrow E$ is a measurable function, then the Young measure associated to u is the element $\nu \in \mathcal{Y}(\Omega, E, \mu)$, defined by*

$$\int_{\Omega \times E} h(z, x)\, d\nu = \int_\Omega h\big(z, u(z)\big)\, d\nu \qquad \forall\ h \in C_0(\Omega \times E).$$

REMARK 3.5.8 Let

$$\mathrm{proj}_\Omega\colon \Omega \times E \longrightarrow \Omega$$

be the projection map defined by

$$\mathrm{proj}_\Omega(z, x) \stackrel{df}{=} z \qquad \forall\ (z, x) \in \Omega \times E.$$

If $\nu \in \mathcal{Y}(\Omega, E, \mu)$, then

$$\mu = \nu \circ \mathrm{proj}_\Omega^{-1}$$

(see Definition 3.5.5). Moreover, we have

$$\nu \;=\; \mu \circ \eta^{-1} \quad \text{for all } \mu, \nu \text{ as in Definition 3.5.7(c)},$$

where $\eta\colon \Omega \longrightarrow \Omega \times E$ is defined by

$$\eta(z) \;\stackrel{df}{=}\; \big(z, u(z)\big) \qquad \forall\, z \in \Omega.$$

If $u_n\colon \Omega \longrightarrow E$ are measurable functions for $n \geqslant 1$,

$$u_n(z) \;\longrightarrow\; u(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and ν_n, ν are the Young measures associated to u_n and u respectively (see Definition 3.5.7(c)), then

$$\nu_n \;\stackrel{w}{\longrightarrow}\; \nu \quad \text{in } M^1_+(\Omega \times E)$$

(see Definition 2.3.42(b)). To see this let

$$h \in C_0(\Omega \times E).$$

Using Remark 3.5.6 and the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\lim_{n \to +\infty} \int\limits_{\Omega \times E} h(z,x)\, d\nu_n \;=\; \lim_{n \to +\infty} \int\limits_{\Omega} h\big(z, u_n(z)\big)\, d\mu$$

$$= \int\limits_{\Omega} h\big(z, u(z)\big)\, d\mu \;=\; \int\limits_{\Omega \times E} h(z,x)\, d\nu.$$

PROPOSITION 3.5.9
We have

$$S\mathcal{Y}(\Omega, E, \mu) \;=\; \Big\{ \nu \in M(\Omega \times E)_+ :\; \nu(\Omega \times E) \leqslant \mu(\Omega)$$
$$\text{and for all } \beta \in C_0(\Omega)_+ \text{ and all } \xi \in C_0(\Omega \times E)_+,$$
$$\text{such that } \xi(z,x) \leqslant \beta(z) \text{ for all } (z,x) \in \Omega \times E,$$
$$\text{we have } \int\limits_{\Omega \times E} \xi(z,x)\, d\nu \leqslant \int\limits_{\Omega} \beta(z)\, d\mu \Big\}.$$

Moreover, if E is compact, then

$$\mathcal{Y}(\Omega, E, \mu) \;=\; \Big\{ \nu \in S\mathcal{Y}(\Omega, E, \mu) :\; \text{for all } \beta \in C_0(\Omega)_+,$$
$$\text{we have } \int\limits_{\Omega} \beta(z)\, d\mu = \int\limits_{\Omega \times E} \beta(z)\, d\nu \Big\}.$$

PROOF Let $K \subseteq \Omega$ and $C \subseteq E$ be nonempty, compact sets, and $\varepsilon > 0$. Then by virtue of Urysohn's lemma (see Theorem A.1.13), we can find

$$\xi \in C_0(\Omega \times E)_+, \quad \beta \in C(\Omega)_+, \quad 0 \leqslant \xi, \beta \leqslant 1$$

and a compact set

$$K_1 \subseteq \Omega, \quad K_1 \supseteq K, \quad K_1 \neq K,$$

such that

$$\xi|_{K \times C} = 1, \quad \beta|_K = 1, \quad |\beta|_{K_1^c}| \leqslant \varepsilon, \quad \mu(K_1 \setminus K) \leqslant \varepsilon$$

and

$$\xi(z,x) \leqslant \beta(z) \qquad \forall\ (z,x) \in \Omega \times E.$$

Then we have

$$\begin{aligned} \nu(K \times C) &\leqslant \int_{\Omega \times E} \xi(z,x)\, d\nu \leqslant \int_{\Omega} \beta(z)\, d\mu \\ &\leqslant \mu(K) + \mu(K_1 \setminus K) + \varepsilon\mu(K_1^c) \leqslant \mu(K) + \varepsilon\big(1 + \mu(\Omega)\big). \end{aligned}$$

Let $\varepsilon \searrow 0$, to conclude that

$$\nu(K \times C) \leqslant \mu(K).$$

If $A \in \Sigma$, then we can find compact sets

$$K_n \subseteq A \quad \text{and} \quad C_n \subseteq E \qquad \forall\ n \geqslant 1,$$

such that

$$K_n \nearrow A \quad \text{and} \quad C_n \nearrow E.$$

Since

$$\nu(K_n \times C_n) \leqslant \mu(K_n) \qquad \forall\ n \geqslant 1,$$

passing to the limit as $n \to +\infty$, we obtain

$$\nu(A \times E) \leqslant \mu(A).$$

This proves the first equality.

If E is compact, simply note that if $\beta \in C_0(\Omega)$, then

$$\beta \in C_0(\Omega \times E).$$

So the second equality follows at once. □

Recall that

$$M(\Omega \times E) = C_0(\Omega \times E)^*$$

(see Theorem 2.3.41). So we can equip $\mathcal{Y}(\Omega, E, \mu)$ and $S\mathcal{Y}(\Omega, E, \mu)$ with the relative weak*-topology. If we do this we can have useful topological properties for the space of Young measures and of Young submeasures.

THEOREM 3.5.10
E is compact if and only if $\mathcal{Y}(\Omega, E, \mu)$ is compact.

PROOF "$\Longrightarrow$": First note that

$$\mathcal{Y}(\Omega, E, \mu) \subseteq S\mathcal{Y}(\Omega, E, \mu) \subseteq \overline{B}^*_{\mu(\Omega)},$$

where

$$\overline{B}^*_{\mu(\Omega)} \stackrel{df}{=} \left\{\lambda \in M(\Omega \times E) : |\lambda|(\Omega \times E) \leqslant \mu(\Omega)\right\}.$$

We know that the predual space $C_0(\Omega \times E)$ is separable. So on bounded subsets of $M(\Omega \times E)$, the relative weak*-topology is compact and metrizable. Therefore we have to show that $\mathcal{Y}(\Omega, E, \mu)$ is sequentially closed. Let $\{\nu_n\}_{n \geqslant 1} \subseteq \mathcal{Y}(\Omega, E, \mu)$ be a sequence, such that

$$\nu_n \stackrel{w^*}{\longrightarrow} \nu.$$

If $\beta \in C_0(\Omega)$, then from Proposition 3.5.9, we have

$$\int_\Omega \beta(z)\, d\mu = \int_{\Omega \times E} \beta(z)\, d\nu_n \qquad \forall\, n \geqslant 1.$$

Since $\beta \in C_0(\Omega \times E)$, in the limit as $n \to +\infty$, we obtain

$$\int_\Omega \beta(z)\, d\mu = \int_{\Omega \times E} \beta(z)\, d\nu$$

and thus due to Proposition 3.5.9, we have $\nu \in \mathcal{Y}(\Omega, E, \mu)$. This proves the compactness of $\mathcal{Y}(\Omega, E, \mu)$.

"$\Longleftarrow$": Suppose that E is not compact.

Then we can find a sequence $\{x_n\}_{n \geqslant 1} \subseteq E$ with no convergent subsequence. Let

$$\nu_n \in \mathcal{Y}(\Omega, E, \mu) \text{ be associated to the constant function } x_n \qquad \forall\, n \geqslant 1$$

(see Definition 3.5.7**(c)**). Then for each $\xi \in C_0(\Omega \times E)$, we have

$$\lim_{n \to +\infty} \int_{\Omega \times E} \xi(z, x)\, d\nu_n = \lim_{n \to +\infty} \int_\Omega \xi(z, x_n)\, d\mu = 0$$

and so

$$\nu_n \stackrel{w^*}{\longrightarrow} 0$$

and $0 \notin \mathcal{Y}(\Omega, E, \mu)$, a contradiction. ∎

THEOREM 3.5.11
$S\mathcal{Y}(\Omega, E, \mu)$ is compact and metrizable.

PROOF Recall that

$$S\mathcal{Y}(\Omega, E, \mu) \subseteq \overline{B}^*_{\mu(\Omega)}$$

(see the proof of Theorem 3.5.10) and $\overline{B}^*_{\mu(\Omega)}$ with the relative weak*-topology is compact metrizable. So we need to show that $S\mathcal{Y}(\Omega, E, \mu)$ is sequentially weak*-closed. To this end let $\{\nu_n\}_{n\geqslant 1} \subseteq S\mathcal{Y}(\Omega, E, \mu)$ be a sequence, such that

$$\nu_n \xrightarrow{w^*} \nu \quad \text{in } S\mathcal{Y}(\Omega, E, \mu),$$

for some $\nu \in M(\Omega \times E)_+$. Suppose that

$$\xi \in C_0(\Omega \times E)_+, \quad \beta \in C_0(\Omega)_+$$

and

$$\xi(z, x) \leqslant \beta(z) \qquad \forall\ (z, x) \in \Omega \times E.$$

Then from Theorem 3.5.10, we have

$$\int_{\Omega \times E} \xi(z, x)\, d\nu_n \leqslant \int_{\Omega} \beta(z)\, d\mu \qquad \forall\ n \geqslant 1,$$

so

$$\int_{\Omega \times E} \xi(z, x)\, d\nu \leqslant \int_{\Omega} \beta(z)\, d\mu$$

and from Proposition 3.5.9, we have that

$$\nu \in S\mathcal{Y}(\Omega, E, \mu).$$

□

Let us see how Young measures are related to transition probabilities (see Definition 3.5.1).

In what follows on $\widehat{SR}(\Omega, E)$ we consider the equivalence relation

$$\lambda_1 \sim \lambda_2 \iff \Big[\lambda_1(z) = \lambda_2(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega\Big].$$

Then we set

$$R(\Omega, E) \stackrel{df}{=} \widehat{R}(\Omega, E)/_\sim \quad \text{and} \quad SR(\Omega, E) \stackrel{df}{=} \widehat{SR}(\Omega, E)/_\sim.$$

THEOREM 3.5.12
There is a bijection $\psi\colon R(\Omega,E) \longrightarrow \mathcal{Y}(\Omega,E,\mu)$ *given by*

$$\psi(\lambda) \stackrel{df}{=} \nu \qquad \forall\ \lambda \in R(\Omega,E),$$

where

$$\nu(C) \stackrel{df}{=} \int_\Omega \int_E \chi_C(z,x)\lambda(z)(dx)\,d\mu \qquad \forall\ C \in \Sigma \times \mathcal{B}(E).$$

PROOF For any ν-integrable or positive and $\Sigma \times \mathcal{B}(E)$-measurable function $h\colon \Omega \times E \longrightarrow \mathbb{R}$, we have

$$\int_{\Omega\times E} h(z,x)\,d\nu = \int_\Omega \int_E h(z,x)\lambda(z)(dx)\,d\mu,$$

where $\nu = \psi(\lambda)$.

First we show that the map is injective. So let $\lambda_1, \lambda_2 \in R(\Omega,E)$ and suppose that

$$\psi(\lambda_1) = \psi(\lambda_2).$$

Then for $A \in \Sigma$ and $\eta \in C_0(E)$, we set

$$h(z,x) \stackrel{df}{=} \chi_A(z)\eta(x) \qquad \forall\ (z,x) \in \Omega \times E.$$

We have

$$\int_A \int_E \eta(x)\lambda_1(z)(dx)\,d\mu = \int_A \int_E \eta(x)\lambda_2(z)(dx)\,d\mu.$$

Because $A \in \Sigma$ was arbitrary, we obtain

$$\int_E \eta(x)\lambda_1(z)(dx) = \int_E \eta(x)\lambda_2(z)(dx) \quad \text{for } \mu\text{-a.a. } z \in \Omega \text{ and all } \eta \in C_0(E),$$

so

$$\lambda_1(z) = \lambda_2(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Therefore ψ is indeed injective.

Next we show that ψ is surjective. So let $\nu \in \mathcal{Y}(\Omega,E,\mu)$. Then for any $\varepsilon > 0$, we can find a sequence $\{D_k\}_{k\geqslant 1}$ of pairwise disjoint Borel subsets of E with

$$\operatorname{diam} D_k < \varepsilon \quad \text{and} \quad \nu\left(\Omega \times \bigcap_{k=1}^{\infty} D_k^c\right) = 0$$

and also a sequence $\{U_m\}_{m\geqslant 1}$ of pairwise disjoint open sets with

$$\operatorname{diam} U_m < \varepsilon, \quad \mu(\partial U_m) = 0 \quad \text{and} \quad \mu\left(\Omega \setminus \bigcup_{m=1}^{\infty} U_m\right) = 0.$$

For each $k \geqslant 1$, we pick $x_k \in D_k$ and then define

$$\lambda^\varepsilon(z) \stackrel{df}{=} \sum_{k=1}^{\infty} \frac{\nu(U_m \times D_k)}{\mu(D_k)} \delta_{x_k} \qquad \forall\, z \in U_m,$$

where δ_{x_k} is the Dirac measure concentrated on x_k. Then

$$\lambda^\varepsilon \in R(\Omega, E)$$

and let

$$\nu^\varepsilon = \psi(\lambda^\varepsilon).$$

For all $C \in \Sigma \times \mathcal{B}(E)$, we have

$$\nu\Big(\bigcup_{\substack{(m,k) \\ U_m \times D_k \subseteq C}} U_m \times D_k \Big) \leqslant \nu^\varepsilon(C) \leqslant \nu\Big(\bigcup_{\substack{(m,k) \\ U_m \times D_k \supseteq C}} U_m \times D_k \Big).$$

Therefore for any open set $V \subseteq \Omega \times E$ with $\nu(\partial V) = 0$, we have

$$\lim_{\varepsilon \searrow 0} \nu^\varepsilon(V) = \nu(V).$$

Then by the regularity of ν and the Portmanteau theorem (see Theorem A.2.36), we have that

$$\nu^\varepsilon \stackrel{w^*}{\longrightarrow} \nu.$$

On the other hand note that $\{\lambda^\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^\infty(\Omega; M(E))$ and so by Alaoglu's theorem (see Theorem A.3.9), we may assume that

$$\lambda^\varepsilon \stackrel{w^*}{\longrightarrow} \lambda \quad \text{in } L^\infty(\Omega; M(E)).$$

Then if $\xi \in C_0(\Omega \times E)$, we have

$$\xi \in L^1\big(\Omega; C_0(E)\big)$$

and so

$$\int_{\Omega \times E} \xi(z,x)\, d\nu = \lim_{\varepsilon \searrow 0} \int_{\Omega \times E} \xi(z,x)\, d\nu^\varepsilon = \lim_{\varepsilon \searrow 0} \int_\Omega \int_E \xi(z,x) \lambda^\varepsilon(z)(dx)\, d\mu$$
$$= \int_\Omega \int_E \xi(z,x) \lambda(z)(dx)\, d\mu.$$

Hence

$$\lambda \in R(\Omega, E)$$

and

$$\nu = \psi(\lambda).$$

∎

REMARK 3.5.13 Similarly we establish that ψ is a bijective from $SR(\Omega, E)$ onto $S\mathcal{Y}(\Omega, E, \mu)$. Moreover, if ν is the Young measure associated to a measurable function $u\colon \Omega \longrightarrow E$, then $\psi(\delta_u) = \nu$. Here δ_u is the Dirac transition probability associated to u, i.e.,

$$\delta_{u(z)}(A) \;=\; \begin{cases} 1 & \text{if} \quad u(z) \in A \\ 0 & \text{if} \quad u(z) \notin A \end{cases} \qquad \forall\, A \in \mathcal{B}(E).$$

Also the identification obtained in Theorem 3.5.12 implies that the weak*-topology on $\mathcal{Y}(\Omega, E, \mu)$ (resulting from the pair of spaces $\big(M(\Omega \times E), C_0(\Omega \times E)\big)$) is equivalent to the weak*-topology on $R(\Omega, E)$ (resulting from the pair $\big(L^\infty(\Omega; M(E)), L^1\big(\Omega; C_0(E)\big)\big)$). Finally we should mention that Theorem 3.5.12 is also known as the ***disintegration theorem***.

The next approximation result is important in many applications.

THEOREM 3.5.14
If $F\colon \Omega \longrightarrow 2^E \setminus \{\emptyset\}$ *is a multifunction, such that* $\operatorname{Gr} F \in \Sigma \times \mathcal{B}(E)$, *then*

$$R_F(\Omega, E) \tag{3.131}$$
$$= \overline{\big(\delta_u :\; u\colon \Omega \longrightarrow E \text{ measurable},\; u(z) \in F(z) \text{ for } \mu\text{-a.a. } z \in \Omega\big)}^{w^*},$$

where

$$R_F(\Omega, E) \stackrel{df}{=} \big\{\lambda \in R(\Omega, E) :\; \lambda(z)\big(F(z)\big) = 1 \quad \text{for a.a. } z \in \Omega\big\}$$

and the closure is taken with respect to the weak*-topology on $L^\infty(\Omega; M(E))$.

PROOF In what follows by $M^1_+(E)_{w^*}$ (respectively $M^1_+(E)_n$) we denote the space $M^1_+(E)$ furnished with the relative weak*-topology of $M(E)$ (respectively the narrow topology); see Definition 2.3.42 and Remark 2.3.43. We know that $M^1_+(E)_n$ is a Polish space and because the narrow topology is stronger than the weak*-topology, we infer that $M^1_+(E)_{w^*}$ is a Souslin space (see Definition A.2.29**(b)**). So the Borel σ-fields of $M^1_+(E)_n$ and $M^1_+(E)_{w^*}$ are equal and we simply write $\mathcal{B}\big((E)\big)$.

Now let

$$S(z) \stackrel{df}{=} \big\{\lambda \in M^1_+(E) :\; \lambda\big(F(z)\big) = 1\big\}.$$

We claim that

$$\operatorname{Gr} S \;\in\; \Sigma \times \mathcal{B}\big((E)\big).$$

So let

$$C \in \Sigma \times \mathcal{B}(E)$$

and consider the map $\varphi_C\colon \Omega \times M^1_+(E) \longrightarrow \mathbb{R}$, defined by

$$\varphi_C(z, \lambda) \;=\; \big(\delta_z \otimes \lambda\big)(C) \qquad \forall\, (z, \lambda) \in \Omega \times M^1_+(E).$$

Here by δ_z we denote the Dirac measure concentrated at $z \in \Omega$. Let $\mathcal{T}$ be the collection of all sets $C \in \Sigma \times \mathcal{B}(E)$, such that φ_C is $\Sigma \times \mathcal{B}\big((E)\big)$-measurable. For $A \in \Sigma$ and $B \in \mathcal{B}(E)$, we have

$$\varphi_{A\times B}(z,\lambda) \;=\; \chi_A(z) \times \lambda(B) \;=\; \chi_A(z) \times \vartheta_B(\lambda),$$

where

$$\vartheta_B(\lambda) \;=\; \lambda(B) \qquad \forall\, \lambda \in M^1_+(E).$$

We show that ϑ_B is a Borel map on $M^1_+(E)$. Let $\mathcal{B}_1$ be all those sets $G \in \mathcal{B}(E)$, such that ϑ_G is $\mathcal{B}\big((E)\big)$-measurable. If G is open, then from the Portmanteau theorem (see Theorem A.2.36), we know that the map

$$\lambda \longmapsto \vartheta_G(\lambda)$$

is lower semicontinuous on $M^1_+(E)_n$. If U is an open set E and K is a closed set in E, then

$$\vartheta_{U\cap K} \;=\; \vartheta_U - \vartheta_{U\cap K^c},$$

so the map $\lambda \longmapsto \vartheta_{U\cap K}(\lambda)$ is Borel.

Hence if $\{U_k\}_{k=1}^N$ are open sets in E and $\{K_k\}_{k=1}^N$ are closed sets in E, then

$$\bigcup_{k=1}^{N} \big(U_k \cap K_k\big) \;\in\; \mathcal{B}_1$$

and so $\mathcal{B}_1$ is a monotone class. From the monotone class theorem, it follows that

$$\mathcal{B}_1 \;=\; \mathcal{B}(E).$$

So φ_G is $\mathcal{B}\big((E)\big)$-measurable for all $G \in \mathcal{B}(E)$. Therefore the map

$$(z,\lambda) \longmapsto \varphi_{A\times B}(z,\lambda) = \chi_A(z) \times \vartheta_B(\lambda)$$

is $\Sigma \times \mathcal{B}\big((E)\big)$-measurable for all $A \in \Sigma$ and all $B \in \mathcal{B}(E)$, hence $A \times B \in \mathcal{T}$. Clearly $\mathcal{T}$ is a monotone class and so it follows that $\mathcal{T} = \Sigma \times \mathcal{B}(E)$.

We have

$$\begin{aligned} &\big\{(z,\lambda) \in \Omega \times M^1_+(E) :\; (\delta_z \otimes \lambda)(\operatorname{Gr} F) = 1\big\} \\ =\; &\varphi^{-1}_{\operatorname{Gr} F}\big(\{1\}\big) \;=\; \operatorname{Gr} S \;\subseteq\; \Sigma \times \mathcal{B}\big((E)\big). \end{aligned}$$

Let D be the subset of $M^1_+(E)$ consisting of the Dirac measures and introduce

$$S_e(z) \;\stackrel{df}{=}\; \big\{\lambda \in D :\; \lambda\big(F(z)\big) = 1\big\}.$$

Then

$$\operatorname{ext} S(z) \;=\; S_e(z) \quad \text{and} \quad \operatorname{Gr} S_e \;\in\; \Sigma \times \mathcal{B}\big((E)\big).$$

Using Remark 3.4.19, we have

$$\overline{S^\infty_{S_e}}^{w^*} \;=\; S^\infty_{\overline{\operatorname{conv}}^* S_e} \;=\; S^\infty_S,$$

so we obtain (3.131). □

Exploiting the identification of $R(\Omega, E)$ with $\mathcal{Y}(\Omega, E, \mu)$ we obtain the following corollary of Theorem 3.5.14.

COROLLARY 3.5.15
The Young measures associated to measurable functions are dense in the space $\mathcal{Y}(\Omega, E, \mu)$ for the weak-topology on $M(\Omega \times E)$ (or equivalently by Theorem 3.5.12 for the weak*-topology on $L^\infty(\Omega; M(E))$).*

In the previous section we introduced the following classes of $\Sigma \times \mathcal{B}(X)$-measurable functions $f\colon \Omega \times E \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ (called integrands): the Carathéodory integrands and the normal integrands. Next we introduce further specifications of these classes, which will be helpful in our analysis.

DEFINITION 3.5.16 **(a)**

$$\mathcal{N}(\Omega, \Sigma, E) \stackrel{df}{=} \Big\{ f\colon \Omega \times E \longrightarrow \overline{\mathbb{R}} : \ f \text{ is } \Sigma \times \mathcal{B}(E)\text{-measurable and } f(z, \cdot) \text{ is lower semicontinuous} \Big\},$$

i.e., $\mathcal{N}(\Omega, \Sigma, E)$ is the set of all normal integrands; see also Definition 3.4.8.

(b)

$$\mathcal{N}_+(\Omega, \Sigma, E) \stackrel{df}{=} \Big\{ f \in \mathcal{N}(\Omega, \Sigma, E) : \ f \geqslant 0 \Big\},$$

i.e., $\mathcal{N}_+(\Omega, \Sigma, E)$ is the set of positive normal integrands.

(c)

$$\widehat{\mathcal{K}}_b(\Omega, \Sigma, E) \stackrel{df}{=} \Big\{ f \in \mathcal{N}(\Omega, \Sigma, E) : \ f(z, \cdot) \in C_b(E) \text{ for } \mu\text{-a.a. } z \in \Omega \text{ and the map } z \longmapsto \big\| f(z, \cdot) \big\|_{L^\infty(E)} \text{ belongs in } L^1(\Omega) \Big\},$$

i.e., $\widehat{\mathcal{K}}_b(\Omega, \Sigma, E)$ is the set of all C_b-Carathéodory integrands.

(d)

$$\widehat{\mathcal{K}}_0(\Omega, \Sigma, E) \stackrel{df}{=} \Big\{ f \in \mathcal{N}(\Omega, \Sigma, E) : \ f(z, \cdot) \in C_0(E) \text{ for } \mu\text{-a.a. } z \in \Omega \text{ and the map } z \longmapsto \big\| f(z, \cdot) \big\|_{L^\infty(E)} \text{ belongs in } L^\infty(\Omega) \Big\},$$

i.e., $\widehat{\mathcal{K}}_0(\Omega, \Sigma, E)$ is the set of all C_0-Carathéodory integrands.

From Proposition 3.4.9, we obtain the following fact.

PROPOSITION 3.5.17
If $f \in \mathcal{N}_+(\Omega, \Sigma, E)$,
then there exists a sequence $\{f_n\}_{n\geqslant 1} \subseteq \widehat{\mathcal{K}}_0(\Omega, \Sigma, E)$, *such that*

$$f_n \nearrow f.$$

PROPOSITION 3.5.18
If $f \in \mathcal{N}_+(\Omega, \Sigma, E)$,
then the map

$$\nu \longmapsto \int_{\Omega\times E} f\, d\nu$$

is w^**-lower semicontinuous on* $S\mathcal{Y}(\Omega, E, \mu)$.

PROOF By virtue of Proposition 3.5.17, we can find a sequence

$$\{f_n\}_{n\geqslant 1} \subseteq \widehat{\mathcal{K}}_0(\Omega, \Sigma, E),$$

such that

$$f_n \nearrow f.$$

By the monotone convergence theorem (see Theorem A.2.10), we have

$$\int_{\Omega\times E} f\, d\nu = \lim_{n\to+\infty} \int_{\Omega\times E} f_n\, d\nu.$$

Since

$$f_n \in L^1\big(\Omega; C_0(E)\big) \qquad \forall\, n \geqslant 1,$$

the map

$$\nu \longmapsto \int_{\Omega\times E} f_n\, d\nu$$

is w^*-continuous on $S\mathcal{Y}(\Omega, E, \mu)$ (see Remark 3.5.13). Therefore the map

$$\nu \longmapsto \int_{\Omega\times E} f\, d\nu$$

is w^*-lower semicontinuous on $S\mathcal{Y}(\Omega, E, \mu)$. □

PROPOSITION 3.5.19
If $\{u_n\colon \Omega \longrightarrow E\}_{n\geqslant 1}$ *is a sequence of* Σ*-measurable functions,* $\{\nu_n\}_{n\geqslant 1}$ *is a sequence of Young measures associated with* $\{u_n\}_{n\geqslant 1}$ *(i.e.,* $\nu_n = \delta_{u_n}$ *for* $n \geqslant 1$*) and*

$$\nu_n \xrightarrow{w^*} \nu \quad \text{in } S\mathcal{Y}(\Omega, E, \mu),$$

for some $\nu \in S\mathcal{Y}(\Omega, E, \mu)$,
then for μ*-almost all* $z \in \Omega$*, the subprobability measure* $\nu(z)$ *is supported by*

$$\limsup_{n\to+\infty} \{u_n(z)\} = \bigcap_{n=1}^{\infty} \overline{\{u_k(z) :\ k \geqslant n\}}.$$

PROOF We define

$$F_k(z) \stackrel{df}{=} \overline{\{u_n(z)\}}_{n\geqslant k} \qquad \forall\ z \in \Omega,\ k \geqslant 1$$

and

$$f_k(z,x) = i_{F_k(z)} = \begin{cases} 0 & \text{if } x \in F_k(z), \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently

$$f_k \in \mathcal{N}_+(\Omega, \Sigma, E)$$

and by virtue of Proposition 3.5.18, we have

$$\begin{aligned} 0 &\leqslant \int_{\Omega\times E} f_k(z,x)\, d\nu \\ &\leqslant \liminf_{n\to+\infty} \int_{\Omega\times E} f_k(z,x)\, d\nu_n \\ &= \liminf_{n\to+\infty} \int_{\Omega} f_k\big(z, u_n(z)\big)\, d\mu = 0, \end{aligned}$$

so

$$\int_{\Omega\times E} f_k(z,x)\, d\nu = 0.$$

From Remark 3.5.13, we have

$$\int_{\Omega}\int_{E} f_k(z,x)\nu(z)(dx)\, d\mu = 0,$$

so

$$\int_{E} f_k(z,x)\nu(z)(dx) = 0 \quad \text{for } \mu\text{-a.a. } z \in \Omega$$

and thus $\nu(z)$ is supported by $F_k(z)$ for μ-almost all $z \in \Omega$.

Because $k \geqslant 1$ was arbitrary, we obtain the conclusion of the proposition. □

When we examined the space of measures $M(E) = C_0(E)^*$, in addition to the weak*-topology, we considered also a finer topology called the ***narrow topology***, namely the $w\big(M(E), C_b(E)\big)$-topology. Exploiting the identification of $\mathcal{Y}(\Omega, E, \mu)$ with $R(\Omega, E)$, we do the same thing for the space $\mathcal{Y}(\Omega, E, \mu)$ of Young measures. We introduce a finer topology, which will lead to some powerful compactness result.

DEFINITION 3.5.20 *The **narrow topology** on $\mathcal{Y}(\Omega, E, \mu)$ is the weakest topology which makes continuous the linear functionals of the form*

$$\nu \longmapsto \int\limits_{\Omega \times E} f \, d\nu \qquad \forall\ f \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, E).$$

We say that the sequence $\{\nu_n\}_{n \geqslant 1}$ converges narrowly to ν, if

$$\lim_{n \to +\infty} \int\limits_{\Omega \times E} f \, d\nu_n = \int\limits_{\Omega \times E} f \, d\nu \qquad \forall\ f \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, E)$$

and we write

$$\nu_n \overset{n}{\longrightarrow} \nu.$$

REMARK 3.5.21 If E is compact, then the narrow and weak*-topologies coincide since $\widehat{\mathcal{K}}_b(\Omega, \Sigma, E) = \widehat{\mathcal{K}}_0(\Omega, \Sigma, E)$.

PROPOSITION 3.5.22
If $\{u_n \colon \Omega \longrightarrow E\}_{n \geqslant 1}$ is a sequence of Σ-measurable functions, $u \colon \Omega \longrightarrow E$ is a Σ-measurable function, $\{\nu_n\}_{n \geqslant 1}$ and ν are the Young measures associated to the functions $\{u_n\}_{n \geqslant 1}$ and u respectively,
then

$$u_n \overset{\mu}{\longrightarrow} u \iff \nu_n \overset{n}{\longrightarrow} \nu.$$

PROOF "$\Longrightarrow$": Since

$$u \overset{\mu}{\longrightarrow} u,$$

we can extract a subsequence $\big\{u_{n_k}\big\}_{k \geqslant 1}$, such that

$$u_{n_k}(z) \longrightarrow u(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Then for all

$$f \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, E),$$

by the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\lim_{n \to +\infty} \int\limits_{\Omega} f\big(z, u_{n_k}(z)\big) \, d\mu = \int\limits_{\Omega} f\big(z, u(z)\big) \, d\mu,$$

so

$$\lim_{k\to+\infty} \int\limits_{\Omega\times E} f(z,x)\,d\nu_{n_k} \;=\; \int\limits_{\Omega\times E} f(z,x)\,d\nu \qquad \forall\ f\in \widehat{\mathcal{K}}_b(\Omega,\Sigma,E)$$

and thus

$$\nu_{n_k} \;\xrightarrow{\;n\;}\; \nu \quad \text{as } k\to+\infty, \quad \text{in } \mathcal{Y}(\Omega,E,\mu).$$

Since every subsequence of $\{\nu_n\}_{n\geqslant 1}$ has a further subsequence converging narrowly to ν, we conclude that

$$\nu_n \;\xrightarrow{\;n\;}\; \nu \quad \text{in } \mathcal{Y}(\Omega,E,\mu).$$

"$\Longleftarrow$": Let

$$f(z,x) \;\stackrel{df}{=}\; \min\big\{1, d_{_E}\big(x,u(z)\big)\big\},$$

where $d_{_E}$ denotes the metric on E. Clearly

$$f\in \widehat{\mathcal{K}}_b(\Omega,\Sigma,E).$$

So

$$\lim_{n\to+\infty} \int\limits_{\Omega\times E} f(z,x)\,d\nu_n \;=\; \int\limits_{\Omega\times E} f(z,x)\,d\nu,$$

hence

$$\lim_{n\to+\infty} \int\limits_{\Omega} f\big(z,u_n(z)\big)\,d\mu \;=\; \int\limits_{\Omega} f\big(z,u(z)\big)\,d\mu \;=\; 0. \tag{3.132}$$

For a given $\varepsilon>0$, let

$$M_{\varepsilon,n} \;\stackrel{df}{=}\; \big\{z\in\Omega:\ d_{_E}\big(u_n(z),u(z)\big)\geqslant\varepsilon\big\} \qquad \forall\ n\geqslant 1.$$

We have

$$\varepsilon\mu(M_{\varepsilon,n}) \;\leqslant\; \int\limits_{M_{\varepsilon,n}} f\big(z,u_n(z)\big)\,d\mu$$

$$\leqslant\; \int\limits_{\Omega} f\big(z,u_n(z)\big)\,d\mu,$$

so, from (3.132), we have

$$\lim_{n\to+\infty} \mu(M_{\varepsilon,n}) \;=\; 0$$

and thus

$$u_n \;\xrightarrow{\;\mu\;}\; u.$$

□

By the Alexandrov one-point compactification (see Theorem A.1.3 and Remark A.1.4), we can find a compact metric space $\widehat{E}$, such that E is a dense subset of $\widehat{E}$.

PROPOSITION 3.5.23
If $f \in \mathcal{N}_+(\Omega, \Sigma, E)$,
then

(a) *there exists a sequence* $\{\widehat{f}_n\}_{n\geqslant 1} \subseteq \widehat{\mathcal{K}}_0(\Omega, \Sigma, \widehat{E})$, *such that*

$$\widehat{f}_n \nearrow f \quad \text{on } \Omega \times E;$$

(b) *the function* $\nu \longmapsto \int\limits_{\Omega\times E} f\, d\nu$ *is narrowly lower semicontinuous.*

PROOF **(a)** Let $d_{\widehat{E}}$ be the metric of $\widehat{E}$ which extends d_E. Then

$$\widehat{f}_n(z,x) = \inf_{y\in E}\left[f(z,y) + n d_{\widehat{E}}(y,x)\right] \qquad \forall\, n \geqslant 1,\ (z,x) \in \Omega \times \widehat{E}$$

is the desired sequence.

(b) Follows from Proposition 3.5.18 since the narrow topology is finer than the weak*-topology. □

PROPOSITION 3.5.24
The narrow topology on $\mathcal{Y}(\Omega, \Sigma, E)$ *is the weakest topology* τ, *such that the map* $\nu \longrightarrow \int\limits_{\Omega\times E} \widehat{f}\, d\nu$ *is continuous for all* $\widehat{f} \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, \widehat{E})$.

PROOF Note that $\widehat{\mathcal{K}}_b(\Omega, \Sigma, \widehat{E}) \subseteq \widehat{\mathcal{K}}_b(\Omega, \Sigma, E)$. So *a priori* the τ-topology is weaker than the narrow topology. Next let $f \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, E)$ and introduce

$$\beta(z) \stackrel{df}{=} \left\|f(z,\cdot)\right\|_{L^\infty(E)} \quad \text{and} \quad g(z,x) \stackrel{df}{=} f(z,x) + a(z).$$

Evidently $g \in \mathcal{N}_+(\Omega, \Sigma, E)$ and so by Proposition 3.5.23, we have that the map $\nu \longmapsto \int\limits_{\Omega\times E} g\, d\nu$ is τ-lower semicontinuous.

Since

$$\int_{\Omega\times E} f\, d\nu = \int_{\Omega\times E} g\, d\nu - \int_{\Omega} \beta\, d\mu,$$

we infer that the map $\nu \longrightarrow \int\limits_{\Omega\times E} f\, d\nu$ is τ-lower semicontinuous.

If we repeat the above argument with f replaced by $-f$, we reach the desired conclusion. □

PROPOSITION 3.5.25
If on $\mathcal{K}_0(\Omega,\Sigma,E)$ *we introduce the equivalence relation*

$$f_1 \sim f_2 \iff \mu\big(\{z\in\Omega:\ f_1(z,\cdot)\neq f_2(z,\cdot)\}\big) = 0,$$

then $\mathcal{K}_0(\Omega,\Sigma,E) = \widehat{\mathcal{K}}_0(\Omega,\Sigma,E)/_\sim$ *is in bijection with* $L^1\big(\Omega;C_0(E)\big)$.

PROOF For each $f\in\mathcal{K}_0(\Omega,\Sigma,E)$, let

$$\psi(f)(z) \stackrel{df}{=} f(z,\cdot).$$

Then $\psi(f)(z)\in C_0(E)$. Also the map

$$\Omega\ni z\longmapsto \psi(f)(z)\in C_0(E)$$

is measurable. To see this let $\{x_n\}_{n\geqslant 1}$ be dense in E. Then for all $h\in C_0(E)$, we have

$$\big\|\psi(f)(z)-h\big\|_{C_0(E)} = \sup_{n\geqslant 1}\big|f(z,x_n)-h(x_n)\big|,$$

so the map

$$z\longmapsto \big\|\psi(f)(z)-h\big\|_{C_0(E)}$$

is Σ-measurable for all $h\in C_0(E)$.

Since the space $C_0(E)$ is separable, it follows that the map

$$\Omega\ni z\longmapsto\psi(f)(z)\in C_0(E)$$

is strongly measurable (see Corollary 2.1.4). Moreover,

$$\int_\Omega \big\|\psi(f)(z)\big\|_{C_0(E)}\,d\mu < +\infty,$$

i.e., $\psi(f)\in L^1\big(\Omega;C_0(E)\big)$.

Therefore

$$\psi\colon \mathcal{K}_0(\Omega,\Sigma,E)\longrightarrow L^1\big(\Omega;C_0(E)\big)$$

is injective. Moreover, if $g\in L^1\big(\Omega;C_0(E)\big)$, then $g=\psi(f)$ with

$$f(z,x)\stackrel{df}{=} g(z)(x),$$

so ψ is bijective. □

REMARK 3.5.26 The above proposition permits the identification of $\mathcal{K}_0(\Omega,\Sigma,E)$ with $L^1\big(\Omega;C_0(E)\big)$, which is very convenient because of the identification of $\mathcal{Y}(\Omega,E,\mu)$ with $R(\Omega,E)$.

We want to investigate the compact sets in $\mathcal{Y}(\Omega, E, \mu)$. For this purpose it is useful to recall the basic results characterizing the compact sets of the space $M(E)$ furnished with the narrow topology, i.e., the $w\big(M(E), C_b(E)\big)$-topology. If E is compact, this topology coincides with the w^*-topology.

THEOREM 3.5.27 (Prohorov Theorem)
If Y is a Polish space (see Definition A.2.29(a)) and $C \subseteq M(Y)_+$ is a bounded set,
then C is relatively compact in the narrow topology if and only if C is uniformly tight, i.e., for every $\varepsilon > 0$, we can find a compact set

$$K_\varepsilon \subseteq Y,$$

such that

$$\sup_{\lambda \in C} \lambda\big(K_\varepsilon^c\big) \leqslant \varepsilon.$$

We have the following characterization of uniformly tight sets in $M(Y)_+$.

PROPOSITION 3.5.28
If Y is a Polish space and $C \subseteq M(Y)_+$ is nonempty,
then C is uniformly tight if and only if there exists $\psi \colon Y \longrightarrow \overline{\mathbb{R}}_+$, such that

(a) *the set $\big\{y \in Y : \psi(y) \leqslant t\big\}$ is compact for every $t \geqslant 0$ (i.e., ψ is* inf*-compact); and*

(b) $\sup_{\lambda \in C} \int_Y \psi(y)\lambda(dy) < +\infty$.

PROOF "$\Longrightarrow$": Let $\{K_n\}_{n \geqslant 1}$ be an increasing sequence of compact sets of Y, such that

$$\lambda(Y \setminus K_n) \leqslant \frac{1}{2^n} \qquad \forall\, n \geqslant 1,\ \lambda \in C.$$

Let us set

$$\psi \stackrel{df}{=} \sum_{n=1}^{\infty} \chi_{Y \setminus K_n}.$$

We have

$$\int_Y \psi(y)\lambda(dy) = \sum_{n=1}^{\infty} \int_Y \chi_{Y \setminus K_n}(y)\lambda(dy) \leqslant 1 \qquad \forall\, \lambda \in C.$$

Note that ψ is $\mathbb{N} \cup \{+\infty\}$-valued. So

$$\big\{\psi \leqslant t\big\} = \big\{\psi \leqslant [t]\big\} = K_{[t]+1},$$

where $[t]$ denotes the integer part of $t \geqslant 1$. Therefore ψ is inf-compact. ∎

Motivated from Theorem 3.5.27, we introduce the following definition.

DEFINITION 3.5.29 *A set $S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is said to be* ***uniformly tight****, if and only if for every $\varepsilon > 0$, there exists a compact set*

$$K_\varepsilon \subseteq E,$$

such that

$$\sup_{\nu \in S} \nu\big(\Omega \times (E \setminus K_\varepsilon)\big) \leqslant \varepsilon.$$

REMARK 3.5.30 The uniform tightness of $S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is equivalent to saying that $C = \big\{\nu \circ \mathrm{proj}_E^{-1}\big\}_{\nu \in S}$ is uniformly tight in $M(E)_+$ (see Theorem 3.5.27).

Using the identification of $\mathcal{Y}(\Omega, E, \mu)$ with $R(\Omega, E)$ and Proposition 3.5.28, we are led to the following characterization of uniformly tight sets in $\mathcal{Y}(\Omega, E, \mu)$.

PROPOSITION 3.5.31
$S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is uniformly tight if and only if there exists an inf*-compact function $\psi\colon E \longrightarrow \overline{\mathbb{R}}_+ \overset{df}{=} \mathbb{R}_+ \cup \{+\infty\}$, such that*

$$\sup_{\nu \in S} \int_\Omega \int_E \psi(x)\nu(z)(dx)\, d\mu < +\infty.$$

In the literature there is another notion of uniform tightness for Young measures (or equivalently for transition probabilities).

DEFINITION 3.5.32 *A set $S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is said to be uniformly B-tight, if there exists a $\Sigma \times \mathcal{B}(E)$-measurable function $\varphi\colon \Omega \times E \longrightarrow \overline{\mathbb{R}}_+$, such that $\varphi(z, \cdot)$ is* inf*-compact for all $z \in \Omega$, i.e., the set*

$$\big\{x \in E :\ \varphi(z, x) \leqslant t\big\}$$

is compact for all $t \geqslant 0$ (hence φ is a normal integrand, i.e., $\varphi \in \mathcal{N}_+(\Omega, \Sigma, E)$), such that

$$\sup_{\nu \in S} \int_{\Omega \times E} \varphi(z, x)\, d\nu = \sup_{\nu \in S} \int_\Omega \int_E \varphi(z, x)\nu(z)(dx)\, d\mu < +\infty.$$

REMARK 3.5.33 Evidently uniform tightness implies uniform B-tightness. In fact since E is a Polish space, the two notions are equivalent (see Valadier (1975, p. 165)).

LEMMA 3.5.34
If $S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is uniformly tight, then so is $\overline{S}$ (the closure in the narrow topology; see Definition 3.5.20).

PROOF By virtue of Definition 3.5.29, for a given $\varepsilon > 0$, we can find a compact set $K_\varepsilon \subseteq E$, such that

$$\sup_{\nu \in S} \nu\big(\Omega \times (E \setminus K_\varepsilon)\big) \leqslant \varepsilon.$$

Let us set

$$\psi(x) \stackrel{df}{=} \chi_{E \setminus K_\varepsilon}(x).$$

Then $\psi \in \mathcal{N}_+(\Omega, \Sigma, E)$ and if $\nu \in \mathcal{Y}(\Omega, E, \mu)$, we have

$$\int_{\Omega \times E} \psi(x)\, d\nu \leqslant \varepsilon \iff \nu\big(\Omega \times (E \setminus K_\varepsilon)\big) \leqslant \varepsilon.$$

Therefore, if $\nu \in \overline{S}$, by Proposition 3.5.23**(b)**, it follows that

$$\nu\big(\Omega \times (E \setminus K_\varepsilon)\big) \leqslant \varepsilon,$$

hence $\overline{S}$ is uniformly tight too. □

The next theorem is the extension of Theorem 3.5.27 to Young measures.

THEOREM 3.5.35
$S \subseteq \mathcal{Y}(\Omega, E, \mu)$ is relatively compact for the narrow topology if and only if S is uniformly tight.

PROOF "$\Longrightarrow$": Let $\varphi\colon \mathcal{Y}(\Omega, E, \mu) \longrightarrow M(E)_+$ be defined by

$$\varphi(\nu) \stackrel{df}{=} \nu \circ \mathrm{proj}_E^{-1} \qquad \forall\, \nu \in \mathcal{Y}(\Omega, E, \mu).$$

If $h \in C_0(E)$, then $h \circ \mathrm{proj}_E \in \widehat{\mathcal{K}}_b(\Omega, \Sigma, E)$ and so from Remark 3.5.6, we have

$$\int_E h(x) d\big(\nu \circ \mathrm{proj}_E^{-1}\big) = \int_{\Omega \times E} \big(h \circ \mathrm{proj}_E\big)(z, x)\, d\nu$$

and thus φ is continuous for the narrow topology on $\mathcal{Y}(\Omega, \Sigma, E)$ and on $M(E)_+$.

It follows then that $\varphi(S)$ is compact in $M(E)_+$ and so from Theorem 3.5.27, we have that $\varphi(S)$ is uniformly tight in $M(E)_+$. This then implies the uniform tightness of S (see Remark 3.5.30).

"$\Longleftarrow$": Let $\{\nu_n\}_{n\geqslant 1} \subseteq S$ be a sequence and consider $\widehat{\nu}_n \in \mathcal{Y}(\Omega, \widehat{E}, \mu)$, defined by

$$\widehat{\nu}_n(C) \stackrel{df}{=} \nu_n\big(C \cap (\Omega \times E)\big) \qquad \forall\, C \in \Sigma \times \mathcal{B}(\widehat{E}),\ n \geqslant 1.$$

Because the set $\widehat{E}$ is compact, the weak* and narrow topologies on $\mathcal{Y}(\Omega, \widehat{E}, \mu)$ coincide (see Remark 3.5.21). So because of Theorem 3.5.10, we may assume that

$$\widehat{\nu}_n \longrightarrow \widehat{\nu} \quad \text{in } \mathcal{Y}(\Omega, E, \mu),$$

with the narrow topology.

Since S is uniformly tight, for every $m \geqslant 1$, we can find a compact set $K_m \subseteq E$, such that

$$\nu_n\big(\Omega \times (E \setminus K_m)\big) \leqslant \frac{1}{m} \qquad \forall\, n \geqslant 1,$$

so

$$\widehat{\nu}_n\big(\Omega \times (E \setminus K_m)\big) \leqslant \frac{1}{m} \qquad \forall\, n \geqslant 1$$

and thus

$$\widehat{\nu}\big(\Omega \times (E \setminus K_m)\big) \leqslant \liminf_{n \to +\infty} \widehat{\nu}_n\big(\Omega \times (E \setminus K_m)\big) \leqslant \frac{1}{m}.$$

So $\widehat{\nu}$ is supported by $\Omega \times \bigcup_{m=1}^{\infty} K_m \subseteq \Omega \times E$. □

Since in the sequel we shall focus on Young measures associated to measurable functions, let us give some examples in this direction.

EXAMPLE 3.5.36 In the examples that follow $\Omega = [0,1]$, μ is the Lebesgue measure and $E = \mathbb{R}$.

(a) Consider the Rademacher functions

$$u_n(z) \stackrel{df}{=} \operatorname{sgn}\big(\sin(2^n \pi z)\big) \qquad \forall\, n \geqslant 1,$$

where

$$\operatorname{sgn} z \stackrel{df}{=} \begin{cases} \dfrac{z}{|z|} & \text{if} \quad z \neq 0, \\ 1 & \text{if} \quad z \neq 1. \end{cases}$$

So

$$u_n(z) = \begin{cases} 1 & \text{if } z \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right),\ k \text{ is even}, \\ -1 & \text{otherwise.} \end{cases}$$

We have

$$u_n \stackrel{w^*}{\longrightarrow} 0 \quad \text{in } L^1[0,1].$$

Let ν_n be the Young measure associated to u_n for $n \geqslant 1$. We have that

$$\nu_n \xrightarrow{w^*} \frac{1}{2}(\mu \otimes \delta_1) + \frac{1}{2}(\mu \otimes \delta_{-1}).$$

So we see that w^*-convergence in $L^\infty(\Omega)$ does not imply the w^*-convergence in $M(\Omega \times E)$ of the associated Young measures. Note that the sequence $\{u_n\}_{n\geqslant 1}$ has no subsequences converging μ-almost everywhere on $\Omega = [0,1]$ (note that $\|u_n - u_m\|_{L^1[0,1]} = 1$ for $n \neq m$ and compare with Remark 3.5.8).

(b) Let $\{u_n\}_{n\geqslant 1}$ be the Rademacher functions introduced above. Then define

$$\widehat{u}_n(z) \stackrel{df}{=} \begin{cases} u_n(z) & \text{if} \quad n \text{ is even,} \\ \frac{1}{2}u_n(z) & \text{if} \quad n \text{ is odd.} \end{cases}$$

Clearly we still have

$$\widehat{u}_n \xrightarrow{w^*} 0 \quad \text{in } L^\infty[0,1].$$

However, the sequence $\{\widehat{\nu}_n\}_{n\geqslant 1}$ of associated Young measures is not convergent, because

$$\widehat{\nu}_{2n} \xrightarrow{w^*} \frac{1}{2}(\mu \otimes \delta_1) + \frac{1}{2}(\mu \otimes \delta_{-1}) \quad \text{in } M(\Omega \times E)$$

and

$$\widehat{\nu}_{2n+1} \xrightarrow{w^*} \frac{1}{2}(\mu \otimes \delta_{\frac{1}{2}}) + \frac{1}{2}(\mu \otimes \delta_{-\frac{1}{2}}) \quad \text{in } M(\Omega \times E).$$

(c) Let

$$u_n(z) \stackrel{df}{=} \sin(nz) \qquad \forall\ z \in [0,1],\ n \geqslant 1.$$

We know that

$$u_n \xrightarrow{w^*} 0 \quad \text{in } L^\infty[0,1].$$

On the other hand it can be shown (see Tartar (1979, p. 148)), that if $\{\nu_n\}_{n\geqslant 1}$ is the sequence of associated Young measures, then

$$\nu_n \xrightarrow{w^*} \nu \quad \text{in } M(\Omega \times E),$$

where

$$\nu(A) = \int_A \frac{1}{\pi} \frac{1}{\sqrt{1-z^2}}\, dz \quad \text{for all measurable } A \subseteq [0,1].$$

(d) Let

$$u_n(z) \stackrel{df}{=} \vartheta_n \chi_{[0,\frac{1}{n}]}(z) \qquad \forall\ n \geqslant 1,$$

with $\vartheta_n \in \mathbb{R}$ for $n \geqslant 1$. No matter which is the sequence $\{\vartheta_n\}_{n\geqslant 1} \subseteq \mathbb{R}$, if $\{\nu_n\}_{n\geqslant 1}$ is the sequence of associated Young measures, we have

$$\nu_n \xrightarrow{w^*} \mu \otimes \delta_0 \quad \text{in } M(\Omega \times E).$$

(e) Let

$$u_n(z) = n \qquad \forall\, z \in [0,1],\ n \geqslant 1.$$

Also let $\{\nu_n\}_{n\geqslant 1}$ be the sequence of associated Young measures. Clearly

$$\nu_n \xrightarrow{w^*} 0 \quad \text{in } M(\Omega \times E).$$

REMARK 3.5.37 In Examples 3.5.36**(a)**, **(b)** and **(c)**, the function u_n have values in $[-1,1]$, so we can take $E = [-1,1]$ and then from Theorem 3.5.10, it follows that the sequence $\{\nu_n\}_{n\geqslant 1}$ is relatively w^*-compact in $M(\Omega \times E)$. In Examples 3.5.36**(d)** and **(e)**, we cannot assume that E is compact. Nevertheless the sequence $\{\nu_n\}_{n\geqslant 1}$ is uniformly tight, thus relatively compact for the narrow topology (hence for the weak*-topology too; see Theorem 3.5.35).

As we already mentioned, in the remaining part of this section, we look at Young measures associated to measurable functions. So in what follows $u_n \colon \Omega \longrightarrow E$ are Σ-measurable functions and $\{\nu_n\}_{n\geqslant 1}$ is the sequence of corresponding Young measures. Following Definition 3.5.32, we introduce the following notion.

DEFINITION 3.5.38 *The sequence $\{u_n\}_{n\geqslant 1}$ is uniformly tight, if the sequence $\{\nu_n\}_{n\geqslant 1}$ is uniformly tight (in the sense of Definition 3.5.32).*

REMARK 3.5.39 Since $\nu_n(z) = \delta_{u_n(z)}$, according to the above definition, the sequence $\{u_n\}_{n\geqslant 1}$ is uniformly tight, if for every $\varepsilon > 0$, we can find a compact set $K_\varepsilon \subseteq E$, such that

$$\sup_{n\geqslant 1} \mu\big(\{z \in \Omega :\ u_n(z) \notin K_\varepsilon\}\big) \leqslant \varepsilon.$$

By Proposition 3.5.31, equivalently we can say that there exists an inf-compact function $\psi \colon E \longrightarrow \overline{\mathbb{R}}_+$, such that

$$\sup_{n\geqslant 1} \int_\Omega \psi\big(u_n(z)\big)\, d\mu < +\infty.$$

In particular then if $E = \mathbb{R}^N$ and we take $\psi(x) = \|x\|_{\mathbb{R}^N}$, then we see that every bounded sequence $\{u_n\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ is uniformly tight.

THEOREM 3.5.40
If

$$\nu_n \xrightarrow{n} \nu$$

and $f \in \mathcal{N}(\Omega, \Sigma, E)$ *is such that the sequence* $\{f^-(\cdot, u_n(\cdot))\}_{n\geqslant 1}$ *is uniformly integrable in* $L^1(\Omega)$,
then

(a) *if* $\liminf_{n\to+\infty} \int_\Omega f(z, u_n(z))\, d\mu < +\infty$, *then* $\int_{\Omega\times E} f^+(z,x)\, d\nu < +\infty$;

(b) $\int_{\Omega\times E} f(z,x)\, d\nu \leqslant \liminf_{n\to+\infty} \int_\Omega f(z, u_n(z))\, d\mu$.

PROOF **(a)** Fix $c \geqslant 0$ and let

$$f_c \stackrel{df}{=} \max\{-c, f\}.$$

From Proposition 3.5.23**(b)**, we have

$$\int_{\Omega\times E} (f_c(z,x) + c)\, d\nu \leqslant \liminf_{n\to+\infty} \int_\Omega (f_c(z, u_n(z)) + c)\, d\mu,$$

so

$$\int_{\Omega\times E} f_c(z,x)\, d\nu \leqslant \liminf_{n\to+\infty} \int_\Omega f_c(z, u_n(z))\, d\mu < +\infty. \tag{3.133}$$

If we let $c = 0$, then from (3.133) and the uniform integrability hypothesis, we conclude that

$$\text{the sequence} \{f^+(\cdot, u_n(\cdot))\}_{n\geqslant 1} \text{ is bounded in } L^1(\Omega).$$

This proves **(a)**.

(b) Let

$$A_{n,c} \stackrel{df}{=} \{z \in \Omega :\ f(z, u_n(z)) < -c\}.$$

Then we have

$$\int_{A_{n,c}} f(z, u_n(z))\, d\mu \leqslant 0 \qquad \forall\, n \geqslant 1,$$

so

$$\int_{A_{n,c}} f^+(z, u_n(z))\, d\mu - \int_{A_{n,c}} f^-(z, u_n(z))\, d\mu \leqslant 0 \qquad \forall\, n \geqslant 1.$$

Thus for a given $\varepsilon > 0$, we can find $c > 0$ large enough so that

$$-\varepsilon \leqslant \int\limits_{A_{n,c}} f\big(z, u_n(z)\big)\, d\mu \leqslant 0. \tag{3.134}$$

Note that

$$f_c\big(z, u_n(z)\big) = f\big(z, u_n(z)\big) \quad \text{on } \Omega \setminus A_{n,c}$$

and

$$f_c\big(z, u_n(z)\big) = -c \quad \text{on } A_{n,c}.$$

Hence

$$\begin{aligned}
\int\limits_{\Omega} f\big(z, u_n(z)\big)\, d\mu &= \int\limits_{A_{n,c}} f\big(z, u_n(z)\big)\, d\mu + \int\limits_{\Omega\setminus A_{n,c}} f\big(z, u_n(z)\big)\, d\mu \\
&= \int\limits_{A_{n,c}} f\big(z, u_n(z)\big)\, d\mu + \int\limits_{\Omega} f_c\big(z, u_n(z)\big)\, d\mu - \int\limits_{A_{n,c}} f_c\big(z, u_n(z)\big)\, d\mu \\
&\geqslant \int\limits_{A_{n,c}} f\big(z, u_n(z)\big)\, d\mu + \int\limits_{\Omega} f_c\big(z, u_n(z)\big)\, d\mu \geqslant \int\limits_{\Omega} f_c\big(z, u_n(z)\big)\, d\mu - \varepsilon
\end{aligned}$$

(see (3.134)). So using (3.133) and the fact that

$$f \leqslant f_c,$$

we have

$$\begin{aligned}
\liminf_{n\to+\infty} \int\limits_{\Omega} f\big(z, u_n(z)\big)\, d\mu &\geqslant \liminf_{n\to+\infty} \int\limits_{\Omega} f_c\big(z, u_n(z)\big)\, d\mu - \varepsilon \\
&\geqslant \int\limits_{\Omega} f_c(z, x)\, d\nu - \varepsilon \geqslant \int\limits_{\Omega} f(z, x)\, d\nu - \varepsilon.
\end{aligned}$$

Let $\varepsilon \searrow 0$ to finish the proof of the theorem. □

COROLLARY 3.5.41
If $\nu_n \xrightarrow{n} \nu$, $f_0 \colon \Omega \times E \longrightarrow \mathbb{R}$ *is* $\Sigma \times \mathcal{B}(E)$*-measurable,* $f_0(z, \cdot) \in C(E)$ *for all* $z \in \Omega$ *and* $\big\{f_0\big(\cdot, u_n(\cdot)\big)\big\}_{n\geqslant 1}$ *is a sequence of uniformly integrable functions, then* f_0 *is* ν*-integrable and*

$$\int\limits_{\Omega\times E} f_0(z, x)\, d\nu = \lim_{n\to+\infty} \int\limits_{\Omega} f_0\big(z, u_n(z)\big)\, d\mu.$$

PROOF Use Theorem 3.5.40 with $f = f_0$ and $f = -f_0$. □

COROLLARY 3.5.42
If $\nu_n \xrightarrow{n} \nu$, $h\colon E \longrightarrow \mathbb{R}$ *is a continuous function and* $\{h(u_n(\cdot))\}_{n\geqslant 1}$ *is a sequence of uniformly integrable functions,*
then

(a) *for* μ*-almost all* $z \in \Omega$, *the function* h *is* $\nu(z)$*-integrable and*

$$\int_\Omega \int_E |h(x)|\nu(z)(dx)\, d\mu \;<\; +\infty;$$

(b)

$$h(u_n) \xrightarrow{w} \widehat{h} \quad \text{in } L^1(\Omega),$$

where

$$\widehat{h}(z) \stackrel{df}{=} \int_E h(x)\nu(z)(dx).$$

PROOF **(a)** Follows from Corollary 3.5.41.

(b) Let $\vartheta \in L^\infty(\Omega)$ and let us set

$$f_0(z,x) \stackrel{df}{=} \vartheta(z)h(x).$$

We have

$$\int_A f_0\big(z, u_n(z)\big)\, d\mu \;\leqslant\; \|\vartheta\|_\infty \int_A \big|h\big(u_n(z)\big)\big|\, d\mu \qquad \forall\ A \in \Sigma,$$

so $\{f_0(\cdot, u_n(\cdot))\}_{n\geqslant 1}$ is a sequence of uniformly integrable functions. We can apply Corollary 3.5.41 and obtain

$$\begin{aligned}\int_{\Omega\times E} f_0(z,x)\, d\nu \;&=\; \lim_{n\to+\infty} \int_\Omega f_0\big(z, u_n(z)\big)\, d\mu \\ &=\; \lim_{n\to+\infty} \int_\Omega \vartheta(z)h\big(u_n(z)\big)\, d\mu. \qquad (3.135)\end{aligned}$$

Because $\{h(u_n(\cdot))\}_{n\geqslant 1}$ is a sequence of uniformly integrable functions, from the Dunford-Pettis theorem (see Theorem 2.3.24), we may assume that

$$h(u_n) \xrightarrow{w} \widehat{h} \quad \text{in } L^1(\Omega).$$

So from (3.135,) we have

$$\int_{\Omega\times E} f_0(z,x)\, d\nu \;=\; \int_\Omega \vartheta(z)\Big(\int_E h(x)\nu(z)(dx)\Big)\, d\mu \;=\; \int_\Omega \vartheta(z)\widehat{h}(z)\, d\mu.$$

Let $\vartheta = \chi_A$, $A \in \Sigma$. Then we obtain

$$\int_A \int_E h(x)\nu(z)(dx)\,d\mu \;=\; \int_A \widehat{h}(z)\,d\mu \qquad \forall\ A \in \Sigma,$$

so

$$\int_E h(x)\nu(z)(dx) \;=\; \widehat{h}(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

As every subsequence of $\{h(u_n(\cdot))\}_{n\geqslant 1}$ has a further subsequence converging weakly in $L^1(\Omega)$ to $\widehat{h}$, we conclude that the original sequence $\{h(u_n(\cdot))\}_{n\geqslant 1}$ converges. □

REMARK 3.5.43 If E is compact, then

$$h(u_n) \xrightarrow{w^*} \widehat{h} \quad \text{in } L^\infty(\Omega).$$

PROPOSITION 3.5.44
If $E = \mathbb{R}^N$ and $\{u_n\}_{n\geqslant 1} \subseteq L^\infty(\Omega;\mathbb{R}^N)$, $\{\nu_n\}_{n\geqslant 1}$ are sequences, such that

$$u_n \xrightarrow{w^*} u \quad \textit{in } L^\infty(\Omega;\mathbb{R}^N)$$

and

$$\nu_n \xrightarrow{w^*} \nu \quad \textit{in } M(\Omega \times \mathbb{R}^N),$$

then

$$u(z) \;=\; \int_{\mathbb{R}^N} x\nu(z)(dx) \quad \textit{for } \mu\textit{-a.a. } z \in \Omega.$$

PROOF Since the sequence $\{u_n\}_{n\geqslant 1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^N)$, we may replace $E = \mathbb{R}^N$ by a compact subset of it. Then the narrow and weak* topologies coincide. Therefore from Corollary 3.5.42 (see also Remark 3.5.43) with $h(x) = x$, we obtain the result. □

REMARK 3.5.45 For a given measurable function $u\colon \Omega \longrightarrow E$, we define the barycenter of u to be the set

$$Bar(u) \stackrel{df}{=} \left\{\lambda \in R(\Omega, E) :\ u(z) = \int_E x\lambda(z)(dx) \quad \text{for } \mu\text{-a.a. } z \in \Omega\right\}.$$

So the conclusion of Proposition 3.5.44 says that $\nu \in R(\Omega,\mathbb{R}^N)$ belongs in the barycenter of u.

In Proposition 2.3.38 we produced a criterion for strong convergence in $L^1(\Omega)$. In the next proposition, using the tools provided by the theory of Young measures, we obtain a criterion for strong convergence in $L^p\big(\Omega;\mathbb{R}^N\big)$, $p \in [1,+\infty)$.

PROPOSITION 3.5.46
If $E = \mathbb{R}^N$, $\{u_n\}_{n\geqslant 1} \subseteq L^\infty\big(\Omega;\mathbb{R}^N\big)$ *and* $\{\nu_n\}_{n\geqslant 1} \subseteq M(\Omega\times E)$ *are sequences, such that*

$$u_n \xrightarrow{w^*} u \quad \text{in } L^\infty\big(\Omega;\mathbb{R}^N\big)$$

and

$$\nu_n \xrightarrow{w^*} \nu \quad \text{in } M(\Omega\times E),$$

then

$$u_n \longrightarrow u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big),$$

for $p \in [1,+\infty)$ *if and only if*

$$\nu(z) = \delta_{u(z)} \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

PROOF "$\Longrightarrow$": Let $h \in C_0\big(\mathbb{R}^N\big)$. Then

$$h(u_n) \longrightarrow h(u) \quad \text{in } L^p(\Omega).$$

On the other hand it is easy to see that $\big\{h\big(u_n(\cdot)\big)\big\}_{n\geqslant 1}$ is the sequence of uniformly integrable functions. So by virtue of Corollary 3.5.42**(b)**, we have that

$$h(u_n) \xrightarrow{w} \widehat{h} \quad \text{in } L^1(\Omega),$$

with

$$\widehat{h}(z) \stackrel{df}{=} \int_{\mathbb{R}^N} h(x)\nu(z)(dx).$$

We have

$$\begin{aligned} h\big(u(z)\big) &= \big\langle h, \delta_{u(z)}\big\rangle_{C_0(\mathbb{R}^N)} \\ = \int_{\mathbb{R}^N} h(x)\nu(z)(dx) &= \big\langle h, \nu(z)\big\rangle_{C_0(\mathbb{R}^N)} \quad \text{for } \mu\text{-a.a. } z \in \Omega, \end{aligned}$$

where by $\langle\cdot,\cdot\rangle_{C_0(\mathbb{R}^N)}$ we denote the duality brackets for the pair of spaces $\big(C_0\big(\mathbb{R}^N\big), M\big(\mathbb{R}^N\big)\big)$. Since $h \in C_0\big(\mathbb{R}^N\big)$ was arbitrary, we obtain

$$\delta_{u(z)} = \mu(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

(b) It suffices to prove the implication for $p > 1$ (recall that the embedding $L^p\big(\Omega;\mathbb{R}^N\big) \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ is continuous). If

$$\nu(z) = \delta_{u(z)} \quad \text{for } \mu\text{-a.a. } z \in \Omega,$$

then from Corollary 3.5.42**(b)** with

$$h(x) \;=\; \|x\|^p_{\mathbb{R}^N}\,,$$

we have

$$\|u_n\|^p_{\mathbb{R}^N} \xrightarrow{w} \|u\|^p_{\mathbb{R}^N} \quad \text{in } L^1(\Omega)$$

and so

$$\|u_n\|_p \longrightarrow \|u\|_p\,.$$

Note that

$$u_n \xrightarrow{w} u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big).$$

So from the Kadec-Klee property (see Remark A.3.22), we have that

$$u_n \longrightarrow u \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big).$$

□

We have another result in this direction. First a lemma.

LEMMA 3.5.47
If $\{u_n\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ *is a sequence of uniformly integrable functions, then we can find a subsequence* $\{\nu_{n_k}\}_{k\geqslant 1}$ *of* $\{\nu_n\}_{n\geqslant 1}$*, such that*

(a) $\nu_{n_k} \xrightarrow{n} \nu$ *in* $\mathcal{Y}(\Omega,\mathbb{R}^N,\mu)$*;*

(b) $\int\limits_{\mathbb{R}^N} x\nu(z)(dx) < +\infty$*;*

(c) $u_{n_k} \xrightarrow{w} u$ *in* $L^1\big(\Omega;\mathbb{R}^N\big)$*, with* $u(z) = \int\limits_{\mathbb{R}^N} x\nu(z)(dx)$.

PROOF **(a)** From Proposition 3.5.31 with $\psi(x) = \|x\|_{\mathbb{R}^N}$, we see that the sequence $\{\nu_n\}_{n\geqslant 1}$ is uniformly tight. So by virtue of Theorem 3.5.35, we can find a subsequence $\{\nu_{n_k}\}_{k\geqslant 1}$ of $\{\nu_n\}_{n\geqslant 1}$, such that

$$\nu_{n_k} \xrightarrow{n} \nu \quad \text{as } k \to +\infty \quad \text{in } \mathcal{Y}(\Omega,\mathbb{R}^N,\mu).$$

(b) For the subsequence obtained in part **(a)**, we see that we can apply Corollary 3.5.42**(a)** with $\varphi(x) = \|x\|_{\mathbb{R}^N}$ and obtain the desired conclusion.

(c) For the subsequence obtained in part **(a)**, we see that we can apply Corollary 3.5.42**(b)** with $\varphi(x) = x_k$ for all $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$ and obtain the desired conclusion. □

PROPOSITION 3.5.48
If $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega;\mathbb{R}^N)$ *is a sequence, such that*

$$u_n \xrightarrow{w} u \quad \text{in } L^1(\Omega;\mathbb{R}^N)$$

and for every subsequence $\{\nu_{n_k}\}_{k\geqslant 1}$ *of the sequence* $\{\nu_n\}_{n\geqslant 1}$ *for which we have*

$$\nu_{n_k} \xrightarrow{n} \nu \quad \text{as } k \to +\infty \quad \text{in } \mathcal{Y}(\Omega,\mathbb{R}^N,\mu),$$

ν *is the Young measure associated to a* Σ*-measurable function* $w\colon \Omega \longrightarrow \mathbb{R}^N$, *then* $w = u$ *and*

$$u_n \longrightarrow u \quad \text{in } L^1(\Omega;\mathbb{R}^N).$$

PROOF We have

$$\int_\Omega \|w(z)\|_{\mathbb{R}^N}\, d\mu = \int_\Omega \left\| \int_{\mathbb{R}^N} x\nu(z)(dx) \right\|_{\mathbb{R}^N} d\mu$$
$$\leqslant \liminf_{n\to+\infty} \int_\Omega \|u_{n_k}(z)\|_{\mathbb{R}^N}\, d\mu < +\infty,$$

so $w \in L^1(\Omega;\mathbb{R}^N)$.

If

$$\nu_{n_k} \xrightarrow{n} \nu = \delta_{w(\cdot)},$$

then by Lemma 3.5.47**(c)**, we have

$$u_{n_k} \xrightarrow{w} w \quad \text{in } L^1(\Omega;\mathbb{R}^N).$$

Hence $w = u$.

From Proposition 3.5.22, we have that

$$u_{n_k} \xrightarrow{\mu} u$$

and by passing to a further subsequence if necessary, we may assume that

$$u_{n_k}(z) \longrightarrow u(z) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Then from the extended dominated convergence theorem (Vitali's theorem; see Theorem A.2.9), we have that

$$u_n \longrightarrow u \quad \text{in } L^1(\Omega;\mathbb{R}^N).$$

□

Now we can prove a lower semicontinuity result for integral functionals.

THEOREM 3.5.49
If $\{u_n\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ *is a sequence, such that*

$$u_n \xrightarrow{w} u \quad \text{in } L^1\big(\Omega;\mathbb{R}^N\big),$$

$\{w_n\colon \Omega \longrightarrow E\}_{n\geqslant 1}$ *is a sequence of* Σ*-measurable functions, such that*

$$w_n \xrightarrow{\mu} w,$$

for some Σ*-measurable function* $w\colon \Omega \longrightarrow E$, $f \in \mathcal{N}\big(\Omega,\Sigma,E\times\mathbb{R}^N\big)$ *and*

(i) $f\big(z,w(z),\cdot\big)$ *is convex for* μ*-almost all* $z\in\Omega$*;*

(ii) $\big\{f^-\big(\cdot,w_n(\cdot),u_n(\cdot)\big)\big\}_{n\geqslant 1}$ *is a sequence of uniformly integrable functions,*

then

(a) *if* $\liminf\limits_{n\to+\infty}\int\limits_\Omega f\big(z,w_n(z),u_n(z)\big)\,d\mu<+\infty$, *then*

$$\int_\Omega f^+\big(z,w(z),u(z)\big)\,d\mu<+\infty;$$

(b) $\displaystyle\int_\Omega f\big(z,w(z),u(z)\big)\,d\mu \leqslant \liminf_{n\to+\infty}\int_\Omega f\big(z,w_n(z),u_n(z)\big)\,d\mu.$

PROOF By passing to a suitable subsequence, we may assume that the limit

$$\lim_{n\to+\infty}\int_\Omega f\big(z,w_n(z),u_n(z)\big)\,d\mu \quad \text{exists},$$

while by Proposition 3.5.22, we have

$$\delta_{w_n} \xrightarrow{n} \delta_w \quad \text{in } \mathcal{Y}(\Omega,\Sigma,E)$$

and by Lemma 3.5.47, we have

$$\delta_{u_n} \xrightarrow{n} \nu \quad \text{in } \mathcal{Y}(\Omega,\Sigma,\mathbb{R}^N).$$

It is easy to see that

$$\delta_{w_n}\otimes\delta_{u_n} \xrightarrow{n} \delta_w\otimes\nu \quad \text{in } \mathcal{Y}(\Omega,\Sigma,E\times\mathbb{R}^N).$$

Invoking Theorem 3.5.40**(a)**, we obtain the implication

$$\begin{gathered}\liminf_{n\to+\infty}\int_\Omega f\big(z,w_n(z),u_n(z)\big)\,d\mu<+\infty\\ \Downarrow\\ \int_\Omega\left(\int_{\mathbb{R}^N} f^+\big(z,w(z),x\big)\nu(z)(dx)\right)d\mu<+\infty.\end{gathered}\tag{3.136}$$

From Theorem 3.5.40**(b)**, we have

$$\int_\Omega \left(\int_{\mathbb{R}^N} f\big(z, w(z), x\big)\nu(z)(dx) \right) d\mu \leqslant \liminf_{n\to+\infty} \int_\Omega f\big(z, w_n(z), u_n(z)\big)\, d\mu. \tag{3.137}$$

Since by hypothesis $f\big(z, w(z), \cdot\big)$ is convex for μ-almost all $z \in \Omega$, using Jensen's inequality (see Theorem A.2.26), we obtain

$$\int_\Omega f\big(z, w(z), u(z)\big)\, d\mu = \int_\Omega \left(f\big(z, w(z)\big) \int_{\mathbb{R}^N} x\nu(z)(dx) \right) d\mu \leqslant \int_\Omega \left(\int_{\mathbb{R}^N} f\big(z, w(z), x\big)\nu(z)(dx) \right) d\mu. \tag{3.138}$$

Then part **(a)** follows from (3.136) and (3.138), while part **(b)** follows from estimates (3.137) and (3.138). □

There is a version of this result for integrands defined on Banach spaces.

THEOREM 3.5.50
If (Ω, Σ, μ) is a finite measure space, X is a separable Banach space, Y is a separable reflexive Banach space and

$$f : \Omega \times X \times Y \longrightarrow \overline{\mathbb{R}}$$

is a $\Sigma \times \mathcal{B}(X) \times \mathcal{B}(Y)$-measurable function, such that

(i) the function $(x, y) \longmapsto f(\omega, x, y)$ is lower semicontinuous for μ-almost all $\omega \in \Omega$;

(ii) the function $y \longmapsto f(\omega, x, y)$ is convex for μ-almost all $\omega \in \Omega$ and all $x \in X$;

(iii) there exist $\beta \in L^1(\Omega)$ and $c > 0$, such that

$$f(\omega, x, y) \geqslant \beta(\omega) - c\big(\|x\|_X + \|y\|_Y\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } (x, y) \in X \times Y,$$

then the functional

$$(u, v) \longrightarrow I_f(u, v) \stackrel{df}{=} \int_\Omega f\big(\omega, u(\omega), v(\omega)\big)\, d\mu$$

is sequentially lower semicontinuous from $L^1(\Omega; X) \times L^1(\Omega; Y)_w$ into $\overline{\mathbb{R}}$.

In Proposition 2.3.39 using an extremality condition, we obtained a result concerning strong convergence in $L^1\big(\Omega;\mathbb{R}^N\big)$. There the result was stated without proof. Now that we have in our disposal the tools of the theory of Young measures, we can give a proof of it.

PROPOSITION 3.5.51
If $\{u_n\}_{n\geqslant 1}\subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ *is a sequence, such that*

$$u_n \xrightarrow{w} u \quad \text{in } L^1\big(\Omega;\mathbb{R}^N\big),$$

for some $u\in L^1\big(\Omega;\mathbb{R}^N\big)$ *and*

$$u(z) \in \operatorname{ext}\left(\overline{\operatorname{conv}}\limsup_{n\to+\infty}\{u_n(z)\}\right) \quad \text{for } \mu\text{-a.a. } z\in\Omega,$$

then

$$u_n \longrightarrow u \quad \text{in } L^1\big(\Omega;\mathbb{R}^N\big).$$

PROOF Let

$$S(z) \stackrel{df}{=} \overline{\operatorname{conv}}\limsup_{n\to+\infty}\big\{u_n(z)\big\} \qquad \forall\, z\in\Omega.$$

Since

$$u(z) \in \operatorname{ext} S(z) \quad \text{for } \mu\text{-a.a. } z\in\Omega,$$

we can find a sequence $\big\{C_n(z)\big\}_{n\geqslant 1}$ of closed, convex sets in $S(z)$, such that

$$\bigcup_{n=1}^{\infty} C_n(z) = S(z)\setminus\big\{u(z)\big\} \quad \text{for } \mu\text{-a.a. } z\in\Omega.$$

Let us fix $z\in\Omega\setminus N$ with $\mu(N)=0$. From Lemma 3.5.47, we have

$$u(z) = \int_{\mathbb{R}^N} x\nu(z)(dx),$$

with $\nu(z)$ being supported by $\limsup_{n\to+\infty}\big\{u_n(z)\big\}$. Suppose that

$$\nu(z)\big(C_n(z)\big) > 0.$$

If $\nu(z)\big(C_n(z)\big)=1$, then $u(z)\in C_n(z)$, a contradiction.

Therefore

$$0 < \nu(z)\big(C_n(z)\big) < 1$$

and we can define

$$\lambda_1 \stackrel{df}{=} \frac{\nu(z)|_{C_n(z)}}{\nu(z)(C_n(z))} \quad \text{and} \quad \lambda_2 \stackrel{df}{=} \frac{\nu(z)|_{\mathbb{R}^N\setminus C_n(z)}}{1-\nu(z)(C_n(z))}.$$

It follows that

$$\begin{aligned} u(z) &= \int_{\mathbb{R}^N} x\nu(z)(dx) \\ &= \nu(z)\big(C_n(z)\big)\int_{\mathbb{R}^N} x\lambda_1(dx) + \big(1-\nu(z)\big(C_n(z)\big)\big)\int_{\mathbb{R}^N} x\lambda_2(dx), \end{aligned}$$

with

$$u(z) \neq \int_{\mathbb{R}^N} x\lambda_1(dx),$$

since

$$\int_{\mathbb{R}^N} x\lambda_1(dx) \in C_n(z).$$

So we have a contradiction to the hypothesis that $u(z)$ is extremal in $C(z)$. This implies that

$$\nu(z)\big(C_n(z)\big) = 0 \qquad \forall\, n \geqslant 1,\ z \in \Omega \setminus N,$$

hence

$$\nu(z)\big(\{u(z)\}\big) = 1,$$

i.e.,

$$\nu(z) = \delta_{u(z)} \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

Then the conclusion of the proposition follows from Proposition 3.5.46. □

REMARK 3.5.52 The result is true if (Ω, Σ, μ) is any finite measure space (see Proposition 2.3.39). However, since the theory in this section was developed for a locally compact, σ-compact metric space Ω we have kept this assumption.

We know that a bounded sequence $\{u_n\}_{n\geqslant 1} \subseteq L^1\big(\Omega;\mathbb{R}^N\big)$ is uniformly tight (see Proposition 3.5.31 with $\psi(x) = \|x\|_{\mathbb{R}^N}$) and so we can extract a subsequence $\big\{u_{n_k}\big\}_{k\geqslant 1}$, such that

$$\delta_{u_{n_k}} \xrightarrow{n} \nu \quad \text{as } k \to +\infty \quad \text{in } \mathcal{Y}(\Omega, \Sigma, \mathbb{R}^N)$$

(see Theorem 3.5.35). It is natural to ask what is the relation between the sequence $\{u_n\}_{n\geqslant 1}$ and the function

$$u(z) = \int_{\mathbb{R}^N} x\nu(z)(dx).$$

In this respect the Chacon biting lemma (see Theorem 2.3.26) is helpful. An equivalent reformulation of this result (for $X = \mathbb{R}^N$) is the following theorem.

THEOREM 3.5.53
If $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega;\mathbb{R}^N)$ *is a bounded sequence, then we can extract a subsequence* $\{u_{n_k}\}_{k\geqslant 1}$ *of* $\{u_n\}_{n\geqslant 1}$, *such that for every* $\delta > 0$, *there exists* $A \in \Sigma$, *with*

$$\mu(A) \ < \ \delta$$

and

$$u_{n_k} \ \xrightarrow{w} \ u \quad \text{in } L^1(\Omega \setminus A;\mathbb{R}^N), \quad \text{as } k \to +\infty.$$

PROPOSITION 3.5.54
If $\{u_n\}_{n\geqslant 1} \subseteq L^1(\Omega;\mathbb{R}^N)$ *is a bounded sequence, then we can extract a subsequence* $\{u_{n_k}\}_{k\geqslant 1}$ *and a decreasing sequence of sets* $\{A_k\}_{k\geqslant 1} \subseteq \Sigma$, *with*

$$\mu(A_k) \ \searrow \ 0,$$

such that

$$\delta_{u_{n_k}} \ \xrightarrow{n} \ \nu \quad \text{in } \mathcal{Y}(\Omega,\Sigma,\mathbb{R}^N)$$

and

$$w_k \ \stackrel{df}{=} \ \chi_{\Omega\setminus A_k} u_{n_k} \ \xrightarrow{w} \ u \quad \text{in } L^1(\Omega;\mathbb{R}^N),$$

with

$$u(z) \ \stackrel{df}{=} \ \int\limits_{\mathbb{R}^N} x\nu(z)(dx) \quad \text{for } \mu\text{-a.a. } z \in \Omega.$$

PROOF By virtue of Theorem 2.3.26, we can find a subsequence $\{u_{n_k}\}_{k\geqslant 1}$ of $\{u_n\}_{n\geqslant 1}$ and a decreasing sequence of sets $\{A_k\}_{n\geqslant 1} \subseteq \Sigma$ with

$$\mu(A_k) \ \searrow \ 0,$$

such that $\{w_k = \chi_{\Omega\setminus A_k} u_{n_k}\}_{k\geqslant 1}$ is a sequence of uniformly integrable functions. Then by virtue of Lemma 3.5.47, we need to show that

$$\delta_{w_k} \ \xrightarrow{n} \ \nu \quad \text{in } \mathcal{Y}(\Omega,\Sigma,\mathbb{R}^N).$$

To this end let

$$f \ \in \ \widehat{\mathcal{K}}_b(\Omega,\Sigma,\mathbb{R}^N).$$

If $\eta_k = \psi(\delta_{w_k})$ (see Theorem 3.5.12), then we have

$$\begin{aligned}
&\left| \int\limits_{\Omega\times\mathbb{R}^N} f(z,x)\, d\eta_n - \int\limits_{\Omega\times\mathbb{R}^N} f(z,x)\, d\nu_{n_k} \right| \\
&= \ \left| \int\limits_{A_k} \big(f(z,0) - f\big(z,u_{n_k}(z)\big)\big)\, d\mu \right| \\
&\leqslant \ 2\,\|\psi\|_{L^\infty(\Omega\times\mathbb{R}^N)}\, \mu(A_k) \ \longrightarrow \ 0 \quad \text{as } k \to +\infty,
\end{aligned}$$

so $\eta_n \xrightarrow{n} \nu$. ∎

3.6 Remarks

3.1: Compact maps was the first class of operators used to study nonlinear equations in infinite dimensional spaces. Leray & Schauder (1934) used compact perturbations of the identity in order to extend the Brouwer degree to infinite dimensional spaces. However, for linear operators the notion was first introduced by Riesz (1918). Earlier Hilbert (1906) introduced the notion of completely continuous linear operator between Banach spaces. As a property for linear operators, complete continuity actually lies properly between compactness and boundedness. Moreover, when the domain space X is reflexive, then the two notions coincide (see Corollary 3.1.8). The basic approximation result for compact maps stated in Theorem 3.1.10 is due to Schauder (1930). Proper maps (see Definition 3.1.13) are discussed in Berger (1977). Theorem 3.1.22 is due to Schauder (1930). The proof of Proposition 3.1.31 can be found in Reed & Simon (1972, p. 191). Theorem 3.1.38 on the spectral properties of compact linear operators is due to Riesz (1918). Compact linear operators and their spectral properties are discussed in detail in Dunford & Schwartz (1958), Kato (1976) and Yosida (1978).

The Fredholm alternative (see Theorem 3.1.48) was obtained in the context of linear integral equations by Fredholm (1903). The spectral theory of self-adjoint, compact, linear operators can be found in the books Akhiezer & Glazman (1961, 1963), Gohberg & Goldberg (1981), Halmos (1998) and Kato (1976). For further results of Fredholm operators we refer to Goldberg (1966), Kato (1976) and Schechter (1971). The proof of Proposition 3.1.70 can be found in Schechter (1971, p. 114).

3.2: Monotone operators are rooted in the calculus of variations and were introduced in the early sixties, in order to provide an analytical framework for the study of nonlinear operator equations broader than the one provided by compact operators. The first mention of monotone operators in a Hilbert space can be traced in the work of Golomb (1935) on nonlinear Hammerstein integral equations. However, the systematic development of the theory of monotone operators, started with Kachurovski (1960), who established that the derivative of a convex function is a monotone map and also introduced the term "monotone operator." Then Minty (1962) obtained the first existence result for nonlinear functional equations in Hilbert spaces under monotonicity assumptions. Even simple one dimensional examples reveal that a complete theory of maximal monotone maps requires the use of multivalued maps. Proposition 3.2.11 is due to Rockafellar (1969) who proved that a monotone map is locally bounded at every point in the interior of its domain. Here we have stated a slightly more general version of this result. Concerning Proposition 3.2.14, Kenderov (1974) proved that if X is separable, reflexive Banach space and $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is maximal monotone with $\operatorname{int} D(A) \neq \emptyset$,

then there is a dense G_δ subset D_0 of int $D(A)$, such that $A|_{D_0}$ is single valued and upper semicontinuous for the norm topologies on X and X^*. For additional results in this direction, we refer to Phelps (1993). The duality map (see Example 3.2.20**(d)**) plays a basic role in the study of the geometry of Banach spaces and in the theory of evolution equations. It was first introduced by Beurling & Livingston (1962). Its properties are studied by Browder (1976), Cioranescu (1990) and Zeidler (1990*b*). Theorem 3.2.29 is due to Minty (1962) for Hilbert spaces and Rockafellar (1970*c*) for Banach spaces. Its proof can be found in Zeidler (1990*b*, p. 881). Theorem 3.2.30 is due to Browder (1968) and together with Corollary 3.2.31 explains why maximal monotone operators are a powerful tool in the study of nonlinear operator equations. Theorem 3.2.40 is due to Attouch (1981), while Theorem 3.2.41 is due to Rockafellar (1970*c*). The proof of Theorem 3.2.41 can be found in Zeidler (1990*b*, p. 888). The notion of pseudomonotonicity was introduced by Brézis (1968) (using nets) and Browder (1976) (using sequences). The basic works on pseudomonotonicity are those by Browder & Hess (1972) and Kenmochi (1974, 1975). Of course the most important result is Theorem 3.2.52, due to Browder & Hess (1972). Monotone operators and operators of monotone-type are discussed in the books of Brézis (1973), Barbu (1976), Deimling (1985), Hu & Papageorgiou (1997), Morosanu (1988), Pascali & Sburlan (1978) and Showalter (1997). The proof of Theorem 3.2.58 can be found in Hu & Papageorgiou (1997, pp. 311–312).

3.3: Accretive operators were introduced by Kato (1967, 1968), who gave the complete characterization in metric terms involved in Proposition 3.3.4. In the first part of the section, dealing with accretive operators, we have summarized the results of Brézis (1971), Brézis & Pazy (1970), Crandall & Pazy (1969, 1970), Kato (1967, 1968) and Kenmochi (1972, 1973). Lemma 3.3.26 is due to Kato (1968, 1970), while the Gronwall-type inequality obtained in Lemma 3.3.27 can be found in Brézis (1973, p. 157). Theorem 3.3.28 can be found in Crandall & Pazy (1969).

The linear semigroup theory started developing as soon as it was realized that the theory has immediate applications to partial differential equations, Markov processes and ergodic theory. It developed rapidly during the 1940's and 1950's thanks to the seminal contributions of Hille, Phillips and Yosida. The main result of this theory is of course Theorem 3.3.46 (the Hille-Yosida generation theorem), for contraction semigroups (i.e., $M = 1$, $\omega = 0$) was proved independently by Hille (1942) and Yosida (1948), while the general case (proved in Theorem 3.3.46) is independently due to Feller (1953), Miyadera (1952) and Phillips (1953). The proof of Phillips theorem (see Theorem 3.3.48) can be found in Hille & Phillips (1957, p. 389). Theorem 3.3.49 is due to Lumer & Phillips (1961), while an early Hilbert space version of it was proved by Phillips (1959).

The exponential formula in Theorem 3.3.51 is due to Hille (1942). In fact Hille's proof of the generation theorem was based on it. Another representa-

tion formula can be found in Pazy (1983, p. 21).

THEOREM 3.6.1
If A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on X, then

$$S(t)x = \lim_{\lambda\to+\infty} e^{tA_\lambda x}.$$

A complete list of representation formulae can be found in Hille & Phillips (1957, p. 354).

Theorem 3.3.59 (the generation theorem for nonlinear semigroups) for Hilbert spaces was proved by Komura (1967), while the general case is due to Crandall & Liggett (1971). The notion of integral solution (see Definition 3.3.65) is due to Bénilan (1972). In the linear case, Proposition 3.3.71 is due to Lax (see Hille & Phillips (1957, p. 304)). For nonlinear semigroups, Proposition 3.3.71 and Theorem 3.3.72 were proved by Brézis (1974), while Corollary 3.3.73 is due to Pazy (1968). The theory of linear semigroups can be found in the books of Butzer & Berens (1967), Fattorini (1999), Goldstein (1985), Hille & Phillips (1957) and Pazy (1983), while the theory of nonlinear semigroups can be found in the books of Barbu (1976), Miyadera (1992), Pavel (1987) and Vrabie (1987).

3.4: Theorem 3.4.4 for functions defined on $\Omega \times \mathbb{R}$ with values in $\mathbb{R}$ was proved by Krasnoselskii (1964*b*, 1964*a*). The general case is due to Lucchetti & Patrone (1980). Moreover, in addition to Theorem 3.4.4, we can also show the continuity in measure of the operator N_f, already known when X and Y are Euclidean spaces.

PROPOSITION 3.6.2
If $\mu(\Omega) < +\infty$, $f\colon \Omega \times X \longrightarrow Y$ is a Carathéodory function, $\{x_n\colon \Omega \longrightarrow X\}_{n\geqslant 1}$ is a sequence of Σ-measurable functions and

$$x_n \xrightarrow{\mu} x,$$

then

$$N_f(x_n) \xrightarrow{\mu} N_f(x).$$

The proof of Scorza-Dragoni theorem (see Theorem 3.4.10) can be found in Denkowski, Migórski & Papageorgiou (2003*a*, p. 188).

Normal integrands were introduced by Rockafellar (1968). The characterization of lower semicontinuity of the integral functional I_f obtained in Theorem 3.4.13 for Euclidean spaces X and Y is due to Poljak (1969), while the general case can be found in Lucchetti & Patrone (1980). Proposition 3.4.16 is due to Ioffe & Levin (1972). The proof of Proposition 3.4.18 can be found in Denkowski, Migórski & Papageorgiou (2003*a*, p. 460). For $X = \mathbb{R}$, Theorem 3.4.20 can be found in Ekeland & Temam (1976), but their proof is different based on results from convex analysis.

Further results on Nemytskii operators and integral functionals can be found in the books of Appell & Zabrejko (1990), Buttazzo (1989), Cesari (1983), Ekeland & Temam (1976), Hu & Papageorgiou (1997), Ioffe & Tihomirov (1979) and Vaĭnberg (1973).

3.5: The theory of Young measures has its roots in the so-called "generalized curves" of Young (1942*a*, 1942*b*, 1969) for the study of variational problems which are not inf-compact and consequently do not have a solution. The needs of control theory (relaxation) and of the calculus of variations led to further development of the original ideas of Young. We refer to Berliocchi & Lasry (1973), Ekeland (1972), Ekeland & Temam (1976), Gamkrelidze (1978), Warga (1972). Recently there was a revival of the theory (motivated also by the needs of problems in theoretical mechanics), which can be traced in the works of Alibert & Bouchitté (1997), Balder (1984, 1997), Ball (1989), Ball & Murat (1989), Ball & Zhang (1990), Di Perna (1985), Di Perna & Majda (1987) and Tartar (1979). Our presentation here follows the survey paper of Valadier (1975). Applications of Young measures to control theory and mechanics can be found in the books of Denkowski, Migórski & Papageorgiou (2003*a*, 2003*b*), Gamkrelidze (1978), Hu & Papageorgiou (1997, 2000), Pedregal (1997), Roubiček (1997) and Warga (1972). For the proof of Prohorov theorem (see Theorem 3.5.27) we refer to Parthasarathy (1967, p. 47). Theorem 3.5.49 was obtained by De Giorgi (1968–1969), with $f \geqslant 0$. A more general version similar to that of Theorem 3.5.49 was proved by Ioffe (1977*a*, 1977*b*). His proof, which is also reproduced by Buttazzo (1989, p. 46), does not use Young measures and instead it is based on the approximation of f by certain affine functions. Two different proofs of Theorem 3.5.50 can be found in Balder (1987) and Hu & Papageorgiou (2000, p. 31).

Chapter 4

Smooth and Nonsmooth Analysis and Variational Principles

The purpose of this chapter is to outline the basic aspects of the smooth and nonsmooth calculus in Banach spaces. Special emphasis is given on the nonsmooth theory, which started developing in the 1960's, in order to provide a uniform viewpoint for the treatment of large classes of nonlinear extremal problems. The resulting subdifferential theories found also in many other applications and today are part of the so-called nonsmooth analysis, which is one of the most robust and interesting research areas of nonlinear functional analysis.

In Section 4.1 we present the basics of the smooth calculus in Banach spaces. We limit ourselves to the discussion of the Gâteaux and Fréchet derivatives, which are the two most useful derivatives for vector valued functions.

In Section 4.2 we consider convex functions defined on Banach spaces. We discuss their continuity and differentiability properties. It turns out that a purely algebraic condition (convexity) has remarkable and powerful topological and differentiability implications. Also differentiability results bring us in contact with the Banach space theory and in particular with the so-called Asplund spaces which have the useful property that every separable subspace has a separable dual. We also show that every convex continuous function is locally Lipschitz. Locally Lipschitz functions between Banach spaces are the objects of investigation in Section 4.3. If the two Banach spaces are finite dimensional, then the locally Lipschitz function is differentiable almost everywhere (for the Lebesgue measure). Here we see how this can be generalized to the case where the two spaces are infinite dimensional. The main difficulty is to produce a suitable notion of negligible sets. This is done using the notion of Haar-null sets. We study them in detail and eventually prove an infinite dimensional version of the Rademacher theorem on the almost everywhere differentiability of locally Lipschitz functions.

In Section 4.4 we pass to the nonsmooth part of this chapter. We examine the duality and subdifferentiability properties of convex functions and the subdifferentiability properties of locally Lipschitz functions. At the end of the section, using the notion of bornology, we briefly consider some more subdifferentials of proper functions.

In Section 4.5 we investigate integral functionals defined by convex or nonconvex normal integrands. We determine their duality and subdifferentiability

properties.

Finally in Section 4.6 we present some variational principles and their applications. Prominent in our discussion is the so-called Ekeland variational principle, in which we show that it is equivalent to some other powerful results of nonlinear analysis. We also use it to prove some surjectivity results for nonlinear maps, which extend corresponding results from the linear operator theory. This chapter illustrates in a rather convincing manner how methods and results of nonlinear analysis cover a wide area from a theoretical starting point (Banach space theory) to an applied end (optimization theory).

4.1 Differential Calculus in Banach Spaces

In this section we develop the basics of the differential calculus in Banach spaces. The geometric character of the operation of differentiation becomes very apparent in this general setting and leads naturally to generalizations such as the subdifferentials of convex and of locally Lipschitz functions. Moreover, the needs of the infinite dimensional variational problems which dominate the present landscape of nonlinear analysis require a differential calculus in Banach spaces, along the lines of the one existing in $\mathbb{R}^N$. This section shows that such a theory is possible and the analogy with the finite dimensional calculus is indeed remarkable.

In what follows X and Y are two Banach spaces. Additional hypotheses will be introduced as needed.

DEFINITION 4.1.1 *A map $f\colon X \longrightarrow Y$ is said to be **Gâteaux differentiable** at $x \in X$, if and only if there exists $A(x) \in \mathcal{L}(X;Y)$, such that*

$$\lim_{\lambda \to 0} \frac{f(x+\lambda h) - f(x)}{\lambda} \;=\; A(x)h \qquad \forall\, h \in X.$$

*The operator $A(x)$ is said to be the **Gâteaux derivative** of f at x. It is usually denoted by $f'_G(x)$. We say that f is Gâteaux differentiable, if it is Gâteaux differentiable at every $x \in X$.*

REMARK 4.1.2 If we set

$$\varphi(\lambda) \;\stackrel{df}{=}\; f(x+\lambda h),$$

then

$$f'_G(x)h \;=\; \frac{d}{d\lambda}\varphi(\lambda)|_{\lambda=0} \qquad \forall\, x, h \in X.$$

So the Gâteaux derivative is essentially a one dimensional concept, since it considers the difference quotients along rays. Clearly then the Gâteaux derivative $f'_G(x)$, if it exists, is unique.

Let us see some examples that illustrate the notion of Gâteaux derivative.

EXAMPLE 4.1.3 **(a)** If $f = A \in \mathcal{L}(X;Y)$, then $f'_G(x) = A$ for all $x \in X$.

(b) Let $X = \mathbb{R}^N$, $Y = \mathbb{R}^M$ and $f = (f_1, \dots, f_M)\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$. Let $A = (a_{kj})$ be an $M \times N$-matrix and let $h = e_j = (0, \dots, 0, 1, 0, \dots, 0)$ be the j-th coordinate vector. Then

$$\lim_{\lambda \to 0} \left\| \frac{f(x + \lambda h) - f(x) - \lambda A h}{\lambda} \right\|_Y = 0,$$

so

$$\lim_{\lambda \to 0} \left| \frac{f_k(x + \lambda e_j) - f_k(x) - \lambda a_{kj}}{\lambda} \right| = 0 \qquad \forall\, k \in \{1, \dots, M\},\ j \in \{1, \dots, N\}$$

and so

$$\frac{\partial f_k}{\partial x_j}(x) = a_{kj} \qquad \forall\, k \in \{1, \dots, M\},\ j \in \{1, \dots, N\}.$$

Therefore $f'_G(x)$ has the matrix representation

$$f'_G(x) = \left(\frac{\partial f_k}{\partial x_j}(x) \right)_{\substack{k \in \{1, \dots, M\} \\ j \in \{1, \dots, N\}}}.$$

This matrix is called the ***Jacobian matrix*** of f at $x \in \mathbb{R}^N$. If $Y = \mathbb{R}$, then

$$f'_G(x) = \left(\frac{\partial f}{\partial x_j}(x) \right)_{j=1}^{N},$$

known as the ***gradient*** of f at $x \in \mathbb{R}^N$.

(c) Let $X = Y = C\big([0,1]\big)$ and consider the ***Hammerstain integral operator***, defined by

$$\widehat{f}(x)(t) \stackrel{df}{=} \int_0^1 k(t,s) f\big(s, x(s)\big)\, ds \qquad \forall\, t \in [0,1],$$

where

$$k \in C\big([0,1];[0,1]\big) \quad \text{and} \quad \frac{\partial f}{\partial x} \in C\big([0,1] \times \mathbb{R}\big).$$

An easy calculation reveals that $\widehat{f}$ is Gâteaux differentiable and

$$\big(\widehat{f}'_G(x)h\big)(t) = \int_0^1 k(t,s) \frac{\partial f}{\partial x}\big(s, x(s)\big) h(s)\, ds \qquad \forall\, h \in C\big([0,1]\big).$$

A stronger differentiability notion is given in the next definition.

DEFINITION 4.1.4 *A map $f\colon X \longrightarrow Y$ is said to be* ***Fréchet differentiable*** *at $x \in X$ if there exists*

$$A(x) \ \in \ \mathcal{L}(X;Y),$$

such that

$$f(x+h) - f(x) \ = \ A(x)h + u(x,h),$$

where

$$\frac{\|u(x,h)\|_Y}{\|h\|_X} \ \longrightarrow \ 0 \quad as \ \|h\|_X \to 0.$$

The operator $A(x)$ is said to be the ***Fréchet derivative*** *of f at $x \in X$ and it is usually denoted by $f'_F(x)$. We say that f is Fréchet differentiable, if it is Fréchet differentiable at every $x \in X$.*

REMARK 4.1.5 It is easy to see that the Fréchet derivative $f'_F(x)$, if it exists, is unique. It is clear that if f is Fréchet differentiable at x, it is also Gâteaux differentiable at x. The converse is not true as the following example shows.

EXAMPLE 4.1.6 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$ and consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by

$$f(x) \ \stackrel{df}{=} \ \begin{cases} \frac{x_1^3 x_2}{x_1^4 + x_2^2} & \text{if} \quad x = (x_1, x_2) \neq (0,0), \\ 0 & \text{if} \quad x = (x_1, x_2) = (0,0). \end{cases}$$

The function f is Gâteaux differentiable at $x = 0$ and

$$f'_G(0) \ = \ 0.$$

However, it is not Fréchet differentiable at $x = 0$, since on the curve

$$h_1^2 \ = \ h_2,$$

we have

$$\frac{|f(h)|}{\|h\|_{\mathbb{R}^2}} \ = \ \frac{|h_1^3 h_2|}{h_1^4 + h_2^2} \frac{1}{\sqrt{h_1^2 + h_2^2}} \ = \ \frac{1}{2} \frac{|h_1|}{\sqrt{h_1^2 + h_2^2}},$$

so

$$\lim_{\|h\|_{\mathbb{R}^2} \to 0} \frac{|f(h)|}{\|h\|_{\mathbb{R}^2}} \ = \ \frac{1}{2} \neq 0.$$

The next proposition establishes the exact relation between Gâteaux and Fréchet derivatives.

PROPOSITION 4.1.7
If $f\colon X \longrightarrow Y$ is a Gâteaux differentiable function at all points of some neighbourhood of $x \in X$ and $f'_G(\cdot)$ is continuous at $x \in X$,
then f is also Fréchet differentiable at $x \in X$.

PROOF Let us set

$$u(x,h) \stackrel{df}{=} f(x+h) - f(x) - f'_G(x)h.$$

Then for every $y^* \in Y^*$, we have

$$\big\langle y^*, u(x,h)\big\rangle_Y = \big\langle y^*, f(x+h) - f(x)\big\rangle_Y - \big\langle y^*, f'_G(x)h\big\rangle_Y.$$

By virtue of the mean value theorem, we can find $\lambda \in (0,1)$ (depending on y^*), such that

$$\big\langle y^*, u(x,h)\big\rangle_Y = \big\langle y^*, f'_G(x+\lambda h)h - f'_G(x)h\big\rangle_Y.$$

We can find $y^* \in Y^*$ with $\|y^*\|_{Y^*} = 1$, such that

$$\big\|u(x,h)\big\|_Y = \big|\big\langle y^*, u(x,h)\big\rangle_Y\big|.$$

Then we have

$$\big\|u(x,h)\big\|_Y \leqslant \big\|f'_G(x+\lambda h) - f'_G(x)\big\|_{\mathcal{L}} \|h\|_X,$$

so

$$\frac{\|u(x,h)\|_Y}{\|h\|_X} \leqslant \big\|f'_G(x+\lambda h) - f'_G(x)\big\|_{\mathcal{L}},$$

and so

$$\frac{\|u(x,h)\|_Y}{\|h\|_X} \longrightarrow 0 \quad \text{as } \|h\|_X \to 0$$

(since by hypothesis $f'_G(\cdot)$ is continuous at $x \in X$). □

Before proceeding further, let us give some examples of Fréchet differentiable maps.

EXAMPLE 4.1.8 **(a)** Let $X = H$ be a Hilbert space. Let $A \in \mathcal{L}(H)$ and let $f\colon H \longrightarrow \mathbb{R}$ be defined by

$$f(x) \stackrel{df}{=} (Ax, x)_H \qquad \forall\, x \in H.$$

Then

$$f(x+h) - f(x) - \big(Ax + A^*x, h\big)_H = o\big(\|h\|_H\big) \quad \text{as } h \to 0$$

and so

$$f'_F(x) \;=\; A + A^*.$$

If $A = id_{_H}$, then $f(x) = \|x\|^2_H$ and we have that

$$f'_F(x) \;=\; 2x \qquad \forall\, x \in H.$$

(b) Let $\Omega = \mathbb{R}^N$ be an open set and let $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function. Suppose that

$$\big|f(z,x)\big| \;\leqslant\; a(z) + c|x|^{p-1} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $p \in [1, +\infty)$, $a \in L^{p'}(\Omega)_+$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $c > 0$.

Let F be the potential function corresponding to f, i.e.,

$$F(z,x) \;\stackrel{df}{=}\; \int_0^x f(z,r)\,dr.$$

Using the mean value theorem, we can see that

$$\big|F(z,x)\big| \;\leqslant\; \widehat{a}(z) + \widehat{c}|x|^p \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $\widehat{a} \in L^1(\Omega)_+$ and $\widehat{c} > 0$.

Then consider the continuous functional $\varphi\colon L^p(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\varphi(u) \;\stackrel{df}{=}\; \int_\Omega F\big(z, u(z)\big)\,dz.$$

We claim that φ is continuously Fréchet differentiable and

$$\varphi'(u) \;=\; N_f(u) \qquad \forall\, u \in L^p(\Omega).$$

To this end let

$$\begin{aligned}\xi(h) \;&=\; \int_\Omega F\big(z,(u+h)(z)\big)\,dz - \int_\Omega F\big(z,u(z)\big)\,dz \\ &\quad + \int_\Omega f\big(z,u(z)\big)h(z)\,dz.\end{aligned}$$

Note that

$$\begin{aligned}F\big(z,(u+h)(z)\big) - F\big(z,u(z)\big) \;&=\; \int_0^1 \frac{d}{dt} F\big(z,(u+th)(z)\big)\,dt \\ &=\; \int_0^1 f\big(z,(u+th)(z)\big)h(z)\,dt.\end{aligned}$$

Therefore, using Fubini's theorem and Hölder's inequality (see, e.g., Theorem A.2.27), we have

$$\begin{aligned} \big|\xi(h)\big| &= \int_\Omega \int_0^1 \big|f\big(z,(u+th)(z)\big) - f\big(z,u(z)\big)\big|\big|h(z)\big|\,dt\,dz \\ &\leqslant \int_0^1 \int_\Omega \big|f\big(z,(u+th)(z)\big) - f\big(z,u(z)\big)\big|\big|h(z)\big|\,dz\,dt \\ &\leqslant \|h\|_p \int_0^1 \big\|N_f(u+th) - N_f(u)\big\|_{p'}\,dt. \end{aligned}$$

Because N_f is continuous (see Theorem 3.4.4), by the Lebesgue dominated convergence theorem (see Theorem A.2.2), we can conclude that

$$\frac{\xi(h)}{\|h\|_p} \longrightarrow 0 \quad \text{as } \|h\|_p \to 0.$$

This proves that

$$\varphi'_F(u) = N_f(u)$$

and so $\varphi \in C^1\big(L^p(\Omega)\big)$. This example is important in the variational methods for the study of boundary value problems.

PROPOSITION 4.1.9
If $f\colon X \longrightarrow Y$ is a function which is Fréchet differentiable at $x \in X$, then f is continuous at $x \in X$.

PROOF Since f is Fréchet differentiable at x, we can find $\delta > 0$, such that

$$\big\|f(x+h) - f(x) - f'_F(x)h\big\|_Y \leqslant \|h\|_X \qquad \forall\ \|h\|_X \leqslant \delta,$$

so

$$\big\|f(x+h) - f(x)\big\|_Y \leqslant \big(1 + \|f'_F(x)\|_{\mathcal{L}}\big)\,\|h\|_X \qquad \forall\ \|h\|_X \leqslant \delta.$$

This proves the continuity of f at $x \in X$. □

REMARK 4.1.10 The above proposition is no longer true if Fréchet differentiability is replaced by Gâteaux differentiability. To see this consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by

$$f(x_1,x_2) \stackrel{df}{=} \begin{cases} \frac{x_1^4 x_2}{x_1^6+x_2^3} & \text{if} \quad (x_1,x_2) \neq 0 \\ 0 & \text{if} \quad (x_1,x_2) = 0 \end{cases} \qquad \forall\ x = (x_1,x_2) \in \mathbb{R}^2.$$

Then $f'_G(x_1,x_2) = 0$ but f is not continuous at the origin.

In the case of Gâteaux differentiable maps we can conclude that they are continuous along rays.

PROPOSITION 4.1.11
If $f\colon X \longrightarrow Y$ is a function, which is Gâteaux differentiable at $x \in X$, then

$$\lim_{\lambda \to 0} \big\| f(x+\lambda h) - f(x) \big\|_Y \;=\; 0 \qquad \forall\, h \in X.$$

PROOF Let

$$\varphi(\lambda) \stackrel{df}{=} f(x+\lambda h) \qquad \forall\, \lambda \in \mathbb{R}.$$

Then φ is differentiable at 0, hence continuous there. So

$$\varphi(\lambda) \;\longrightarrow\; \varphi(0) \quad \text{as } \lambda \to 0,$$

which implies that

$$f(x+\lambda h) \;\longrightarrow\; f(x) \quad \text{in } Y, \quad \text{as } \lambda \to 0.$$

□

We have a chain rule for these derivatives.

PROPOSITION 4.1.12
If Z is a Banach space too, $f\colon X \longrightarrow Y$ is a function which is Gâteaux differentiable at $x \in X$ and $g\colon Y \longrightarrow Z$ is a function which is Fréchet differentiable at $f(x)$,
then the function $k \stackrel{df}{=} g \circ f\colon X \longrightarrow Z$ is Gâteaux differentiable at $x \in X$ and

$$k'_G(x) \;=\; g'_F\big(f(x)\big) f'_G(x).$$

Moreover, if f is Fréchet differentiable at $x \in X$, then k is Fréchet differentiable at x.

PROOF For $\lambda \neq 0$, we have

$$\begin{aligned}
&\frac{1}{|\lambda|} \big\| k(x+\lambda h) - k(x) - \lambda g'_F\big(f(x)\big) f'_G(x) h \big\|_Z \\
\leqslant\; &\frac{1}{|\lambda|} \big\| g\big(f(x+\lambda h)\big) - g\big(f(x)\big) - \lambda g'_F\big(f(x)\big)\big(f(x+\lambda h) - f(x)\big) \big\|_Z \\
&+ \frac{1}{|\lambda|} \big\| g'_F\big(f(x)\big)\big(f(x+\lambda h) - f(x) - \lambda f'_G(x) h\big) \big\|_Z. \qquad (4.1)
\end{aligned}$$

Since f is Gâteaux differentiable at $x \in X$, the second summand in the right hand side of (4.1) goes to zero as $\lambda \to 0$. Also suppose that $f(x+\lambda h) \neq f(x)$. Then since

$$f(x+\lambda h) \;\longrightarrow\; f(x) \quad \text{in } Y \quad \text{as } \lambda \to 0$$

(see Proposition 4.1.11) and because g is Fréchet differentiable at $f(x)$, we have that the first summand in the right hand side of (4.1) goes to zero as $\lambda \to 0$. This proves that

$$k'_G(x) \;=\; g'_F\big(f(x)\big) f'_G(x).$$

The proof is similar if f is Fréchet differentiable at $x \in X$. □

COROLLARY 4.1.13
If $f\colon X \longrightarrow Y$ is a function, which is Gâteaux differentiable at every point of the interval

$$[x, x+h] \;\stackrel{df}{=}\; \big\{u \in X :\; u = \lambda x + (1-\lambda)(x+h),\;\; \lambda \in [0,1]\big\},$$

then

$$f(x+h) - f(x) \;=\; \int_0^1 f'_G(x+th)h\, dt.$$

In the next proposition we show that compactness of a map is passed to its Fréchet derivative.

PROPOSITION 4.1.14
If $f\colon X \longrightarrow Y$ is a function which is compact and Fréchet differentiable at $x \in X$,
then $f'_F(x) \in \mathcal{L}_c(X;Y)$.

PROOF Suppose that the proposition is not true. Then we can find $\varepsilon > 0$ and $\{x_n\}_{n\geqslant 1} \subseteq X$ with

$$\|x_n\|_X \;\leqslant\; 1 \qquad \forall\; n \geqslant 1,$$

such that

$$\big\|f'_F(x)x_n - f'_F(x)x_m\big\|_X \;>\; 3\varepsilon \qquad \forall\; n \neq m. \tag{4.2}$$

Because f is Fréchet differentiable at x, we have

$$f(x+h) - f(x) \;=\; f'_F(x)h + u(x,h)$$

and we can find $\delta > 0$, such that

$$\big\|u(x,h)\big\|_Y \;\leqslant\; \varepsilon\, \|h\|_X \qquad \forall\;\; \|h\|_X \leqslant \delta.$$

Therefore, from (4.2), we have

$$\begin{aligned} &\big\|f(x+\delta x_n) - f(x+\delta x_m)\big\|_Y \\ &\geqslant\; \delta\big\|f'_F(x)(x_n - x_m)\big\|_Y - \big\|u(x,\delta x_n)\big\|_Y - \big\|u(x,\delta x_m)\big\|_Y \\ &\geqslant\; 3\varepsilon\delta - \delta\varepsilon - \delta\varepsilon \;=\; \delta\varepsilon, \end{aligned}$$

a contradiction to the fact that f is compact. □

For the next proposition, we need to introduce the following definition.

DEFINITION 4.1.15 **(a)** *A function $f\colon [a,b] \longrightarrow X$ is said to be* ***right differentiable*** *at $t \in [a,b)$, if the limit*

$$\lim_{h \to 0^+} \frac{1}{h}\big[f(t+h) - f(t)\big] \quad \textit{exists.}$$

We denote this limit by $f'_+(t)$ and we call it the ***right derivative*** *of f at t. Evidently $f'_+(t) \in X$.*

(b) *Similarly a function $f\colon [a,b] \longrightarrow X$ is said to be* ***left differentiable*** *at $t \in (a,b]$, if the limit*

$$\lim_{h \to 0^-} \frac{1}{h}\big[f(t+h) - f(t)\big] \quad \textit{exists.}$$

We denote this limit by $f'_-(t)$ and we call it the ***left derivative*** *of f at t. Evidently $f'_-(t) \in X$.*

REMARK 4.1.16 A function $f\colon [a,b] \longrightarrow X$ is Fréchet differentiable at $t \in (a,b)$ if and only if $f'_-(t) = f'_+(t)$.

PROPOSITION 4.1.17
If $f\colon [a,b] \longrightarrow X$, $g\colon [a,b] \longrightarrow \mathbb{R}$, are continuous functions, $f'_+(t)$, $g'_+(t)$ exist at all $t \in (a,b)$ and

$$\big\|f'_+(t)\big\|_X \leqslant g'_+(t) \qquad \forall\, t \in (a,b),$$

then

$$\big\|f(b) - f(a)\big\|_X \leqslant g(b) - g(a).$$

PROOF Let $\varepsilon > 0$ be given and consider the set

$$U \stackrel{df}{=} \Big\{ t \in [a,b] :\ \big\|f(t) - f(a)\big\|_X > g(t) - g(a) + \varepsilon(t-a) + \varepsilon \Big\}.$$

Clearly U is an open set. Suppose that U is nonempty and let $c \stackrel{df}{=} \inf U$. We can say the following:

(a) $c > a$. This follows from the continuity of f and g;

(b) $c \notin U$: since U is open;

(c) $c < b$: otherwise $U = \{b\}$ which is not open.

So we have that $a < c < b$. By hypothesis we have

$$\left\|f'_+(c)\right\|_X \leqslant g'_+(c).$$

Let $h > 0$ be such that if $t \in [c, c+h]$, we have

$$\left\|f'_+(c)\right\|_X \geqslant \frac{\|f(t)-f(c)\|_X}{t-c} - \frac{\varepsilon}{2} \quad \text{and} \quad g'_+(c) \leqslant \frac{g(t)-g(c)}{t-c} + \frac{\varepsilon}{2}.$$

It follows that

$$\left\|f(t)-f(c)\right\|_X \leqslant g(t)-g(c)+\varepsilon(t-c) \qquad \forall\, t \in [c, c+h]. \tag{4.3}$$

Also because $c \notin U$, we have

$$\left\|f(c)-f(a)\right\|_X \leqslant g(c)-g(a)+\varepsilon(c-a)+\varepsilon. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\left\|f(t)-f(a)\right\|_X \leqslant g(t)-g(a)+\varepsilon(t-a)+\varepsilon \qquad \forall\, t \in [c, c+h].$$

We infer that $\inf U \geqslant c+h$, a contradiction. So $U = \emptyset$ and we obtain

$$\left\|f(t)-f(a)\right\|_X \leqslant g(t)-g(a)+\varepsilon(t-a)+\varepsilon \qquad \forall\, t \in [a, b].$$

Let $t = b$ and $\varepsilon \searrow 0$ to obtain the desired inequality. □

REMARK 4.1.18 We have an analogous result if we replace the right derivatives by the left ones. Moreover, we can weaken the hypotheses of Proposition 4.1.17 and assume that there is a countable set $D \subseteq [a, b]$, such that $f'_+(t)$, $g'_+(t)$ exist for all $t \in [a, b] \setminus D$ and

$$\left\|f'_+(t)\right\|_X \leqslant g'_+(t) \qquad \forall\, t \in [a, b] \setminus D.$$

COROLLARY 4.1.19
If $f\colon [a, b] \longrightarrow X$ is continuous, right differentiable at every $t \in (a, b)$ and

$$\left\|f'_+(t)\right\|_X \leqslant k \qquad \forall\, t \in (a, b),$$

then

$$\left\|f(t)-f(s)\right\|_X \leqslant k|t-s| \qquad \forall\, t, s \in [a, b].$$

COROLLARY 4.1.20
If $g\colon [a, b] \longrightarrow \mathbb{R}$ is a continuous function, which is right differentiable at every $t \in (a, b)$,
then g is increasing if and only if

$$g'_+(t) \geqslant 0 \qquad \forall\, t \in (a, b).$$

We have the following mean value theorem.

PROPOSITION 4.1.21 (Mean Value Theorem) *If $f: X \longrightarrow \mathbb{R}$ is a Gâteaux differentiable function, then we can find $\lambda_0 \in (0,1)$, such that*

$$f(x+h) - f(x) \;=\; \left\langle f'_G(x+\lambda_0 h), h \right\rangle_X.$$

PROOF Let

$$\varphi(\lambda) \stackrel{df}{=} f(x+\lambda h).$$

Recall that

$$\left\langle f'_G(x+\lambda_0 h), h \right\rangle_X \;=\; \varphi'(\lambda_0)$$

(see Remark 4.1.2). Using the mean value theorem for scalar functions, we can find $\lambda_0 \in (0,1)$, such that

$$\varphi(1) - \varphi(0) \;=\; \varphi'(\lambda_0),$$

so

$$f(x+h) - f(x) \;=\; \left\langle f'_G(x+\lambda_0 h), h \right\rangle_X.$$

□

In general for vector valued functions the mean value theorem fails as the next example illustrates.

EXAMPLE 4.1.22 Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x) \stackrel{df}{=} (x_1^3, x_2^2) \qquad \forall\; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

We have

$$f'(x) \;=\; \begin{bmatrix} 3x_1^2 & 0 \\ 0 & 2x_2 \end{bmatrix}.$$

If $x = \binom{0}{0}$ and $y = \binom{1}{1}$, then it is clear that there is no $z \in [x,y] = \left\{\lambda x + (1-\lambda)y :\; \lambda \in [0,1]\right\}$, such that $f(y) - f(x) = f'(z)(y-x)$.

For vector valued functions the mean value theorem takes an inequality form.

PROPOSITION 4.1.23 (Mean Value Theorem) *If $f: X \longrightarrow Y$ is a Gâteaux differentiable function and $x, h \in X$, $y^* \in Y^*$, then we can find $\lambda_0 \in (0,1)$, such that*

$$\left\langle y^*, f(x+h) - f(x) \right\rangle_Y \;=\; \left\langle y^*, f'_G(x+\lambda_0 h)h \right\rangle_Y$$

and

$$\left\| f(x+h) - f(x) \right\|_Y \;\leqslant\; \left\| f'_G(x+\lambda_0 h) \right\|_{\mathcal{L}} \|h\|_X .$$

PROOF Let

$$g(x) \stackrel{df}{=} \langle y^*, f(x)\rangle_Y.$$

Then

$$\langle g'_G(x), h\rangle_X = \langle y^*, f'_G(x)h\rangle_Y.$$

From Proposition 4.1.21, we know that we can find $\lambda_0 \in (0,1)$, such that

$$g(x+h) - g(x) = \langle g'_G(x+\lambda_0 h), h\rangle_X,$$

hence

$$\langle y^*, f(x+h) - f(x)\rangle_Y = \langle y^*, f'_G(x+\lambda_0 h)h\rangle_Y.$$

Since $y^* \in Y^*$ is arbitrary, we choose $\|y^*\|_{Y^*} = 1$, such that

$$\langle y^*, f(x+h) - f(x)\rangle_Y = \|f(x+h) - f(x)\|_Y.$$

So we can find $\lambda_0 \in (0,1)$, such that

$$\|f(x+h) - f(x)\|_Y = \langle y^*, f'_G(x+\lambda_0 h)h\rangle_Y \leqslant \|f'_G(x+\lambda_0 h)\|_{\mathcal{L}} \|h\|_X.$$

□

COROLLARY 4.1.24
If $U \subseteq X$ is connected and open $f\colon U \longrightarrow Y$ is a Gâteaux differentiable function and

$$f'_G(x) = 0 \qquad \forall\, x \in U,$$

then f is constant on U.

Next we state and prove two major results of differential calculus. These are the implicit function theorem and the inverse function theorem.

The implicit function theorem deals with the following situation. Let $f(x,y)$ and suppose that

$$f(x_0, y_0) = c.$$

Can we find a function $x \longmapsto y = g(x)$, which at least locally satisfies

$$f\big(x, g(x)\big) = c\ ?$$

We want g to be differentiable provided f is differentiable. Moreover, in the neighbourhood, where

$$f\big(x, g(x)\big) = c$$

is valid, $g(x)$ should be the unique solution. To better motivate this consider the following simple example.

EXAMPLE 4.1.25 Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x,y) \stackrel{df}{=} x^2 + y^2 - 1.$$

We consider the 0-level set of f, namely the set of those $x, y \in \mathbb{R}$ that satisfy

$$f(x,y) \;=\; 0,$$

which in our case is of course the unit circle. We look for a function $g(x)$, such that $f\big(x, g(x)\big) = 0$ for all x in the domain of g. Evidently

$$g(x) \;=\; \pm\sqrt{1-x^2}$$

and so g need not be unique unless we restrict its domain. Also near $x_0 = \pm 1$, g could be either square root, so it is not uniquely determined. Note that at $x_0 = \pm 1$, g is not differentiable and

$$\frac{\partial f}{\partial y} \;=\; 0.$$

So to produce a unique differentiable function g, such that

$$f\big(x, g(x)\big) \;=\; 0,$$

we need to look locally and impose some condition like

$$\frac{\partial f}{\partial y} \;\neq\; 0.$$

The proof of the implicit function theorem uses the Banach fixed point theorem, which we state here in the form needed and postpone the proof of the general version until Section 7.1.

PROPOSITION 4.1.26 (Banach Fixed Point Theorem)
If V is a Banach space, C is a closed subset of V and $S\colon C \longrightarrow C$ satisfies

$$\big\|S(v_1) - S(v_2)\big\|_V \;\leqslant\; k\,\|v_1 - v_2\|_V \qquad \forall\; v_1, v_2 \in C,$$

for some $k \in [0,1)$,
then there exists unique $v \in C$, such that $v = S(v)$.
Moreover, if we have a parametrized family $\big\{S(x)\big\}_{x\in U}$ (with U being an open subset of a Banach space W) satisfying the above contraction condition with $k \in [0,1)$ independent of x, then the unique solution $v = v(x)$ of

$$v \;=\; S(x)v$$

depends continuously on x.

Using this proposition we can prove the implicit function theorem. In what follows for a function $f(x,y)$ by $D_1 f(x,y)$ (respectively $D_2 f(x,y)$) we denote the partial derivative of f with respect to x (respectively y).

THEOREM 4.1.27 (Implicit Function Theorem)
If X, Y, Z are three Banach spaces, $U \subseteq X \times Y$ is an open set, $(x_0, y_0) \in U$, $f\colon U \longrightarrow Z$ is a continuous differentiable function, $f(x_0, y_0) = 0$ and

$$D_2 f(x_0, y_0) \in \mathcal{L}(X;Y) \text{ is invertible with a continuous inverse,}$$

i.e., $D_2 f(x_0, y_0)$ is an isomorphism,
then there exist neighbourhoods U_1 of x_0 and U_2 of y_0, such that $U_1 \times U_2 \subseteq U$ and a unique continuously differentiable function $g\colon U_1 \longrightarrow U_2$, such that

$$f\big(x, g(x)\big) \;=\; 0 \qquad \forall\, x \in U_1$$

and

$$Dg(x) \;=\; -\big(D_2 f\big(x, g(x)\big)\big)^{-1} D_1 f\big(x, g(x)\big) \qquad \forall\, x \in U_1.$$

PROOF Let

$$L_0 \;\stackrel{df}{=}\; D_2 f(x_0, y_0) \in \mathcal{L}(Y;Z).$$

By hypothesis L_0 is an isomorphism. Then the equation $f(x,y) = 0$ can be equivalently rewritten as

$$y \;=\; y - L_0^{-1} f(x,y). \tag{4.5}$$

The advantage of passing to (4.5) is that we can apply Proposition 4.1.26. Namely for every x, we look for a fixed point of $y \longmapsto y - L_0^{-1} f(x,y)$ and to do this we employ Proposition 4.1.26. Let us set

$$h(x,y) \;\stackrel{df}{=}\; y - L_0^{-1} f(x,y).$$

Since $L_0^{-1} \circ L_0 = id_{_Y}$, we have

$$h(x,y_1) - h(x,y_2) \;=\; L_0^{-1}\big[L_0(y_1 - y_2) - \big(f(x,y_1) - f(x,y_2)\big)\big].$$

Because f is C^1 at (x_0, y_0) and L_0 is an isomorphism, we can find $\delta_1 > 0$ and $\vartheta > 0$, such that if $\|x - x_0\|_X \leqslant \delta_1$, $\|y_1 - y_0\|_Y \leqslant \vartheta$, $\|y_2 - y_0\|_Y \leqslant \vartheta$, then

$$\big\|h(x,y_1) - h(x,y_2)\big\|_Y \;\leqslant\; \frac{1}{2}\,\|y_1 - y_2\|_Y\,. \tag{4.6}$$

Also because of the continuity of $h(\cdot, y_0)$, we can find $\delta_2 > 0$, such that if $\|x - x_0\|_X \leqslant \delta_2$, then

$$\big\|h(x,y_0) - h(x_0,y_0)\big\|_Y \;<\; \frac{\vartheta}{2}. \tag{4.7}$$

Therefore, from (4.6) and (4.7), if $\delta = \min\{\delta_1, \delta_2\}$ and $\|x - x_0\|_X \leqslant \delta$, $\|y_1 - y_0\|_Y \leqslant \vartheta$, we have

$$\begin{aligned} \big\|h(x,y) - y_0\big\|_Y &= \big\|h(x,y) - h(x_0,y_0)\big\|_Y \\ &\leqslant \big\|h(x,y) - h(x,y_0)\big\|_Y + \big\|h(x,y_0) - h(x_0,y_0)\big\|_Y \\ &\leqslant \frac{1}{2}\|y - y_0\|_Y + \frac{\vartheta}{2} \leqslant \vartheta. \end{aligned} \tag{4.8}$$

So we see that $h(x,\cdot)$ maps $\overline{B}_\vartheta(y_0) = \{y \in Y : \|y - y_0\|_Y \leqslant \vartheta\}$ onto itself as well as $B_\vartheta(y_0) = \{y \in Y : \|y - y_0\|_Y < \vartheta\}$ onto itself (see (4.7) and (4.8)), for all $x \in \overline{B}_\delta(x_0) = \{x \in X : \|x - x_0\|_X \leqslant \delta\}$. We can apply Proposition 4.1.26 to obtain the parametric family $\{y \longmapsto h(x,y)\}_{x \in \overline{B}_\delta(x_0)}$. So for every $x \in \overline{B}_\delta(x_0)$, we can find unique $y = y(x) \in \overline{B}_\vartheta(y_0)$, such that

$$h(x,y) = y,$$

hence $f(x,y) = 0$ and the function $g(x) = y(x)$ is continuous. Let

$$U_1 \stackrel{df}{=} B_\delta(x_0) \quad \text{and} \quad U_2 \stackrel{df}{=} B_\vartheta(y_0).$$

Evidently by choosing $\delta > 0$ and $\vartheta > 0$ small enough we can have that $U_1 \times U_2 \subseteq U$. We claim that the function $g\colon B_\delta(x_0) \longrightarrow Y$ is continuously differentiable. To this end let $(x_1, y_1) \in U_1 \times U_2$, $y_1 = g(x_1)$ (recall that $G(x,\cdot)$ maps U_2 into itself). Exploiting the differentiability of f at (x_1, y_1), we have

$$f(x,y) = A(x - x_1) + B(y - y_1) + u(x,y) \qquad \forall\ (x,y) \in U,$$

with

$$A \stackrel{df}{=} D_1 f(x_1,y_1), \quad B \stackrel{df}{=} D_2 f(x_1,y_1)$$

and

$$\lim_{(x,y)\to(x_1,y_1)} \frac{\|u(x,y)\|_Z}{\|(x - x_1, y - y_1)\|_{X\times Y}} = 0.$$

Recall that

$$f\big(x, g(x)\big) = 0 \qquad \forall\ x \in U_1.$$

Hence

$$g(x) = -B^{-1}A(x - x_1) + y_1 - B^{-1}u\big(x, g(x)\big). \tag{4.9}$$

We can find $r_1, r_2 > 0$, such that if $\|x - x_1\|_X \leqslant r_1$, $\|y - y_1\|_Y \leqslant r_2$, then

$$\big\|u(x,y)\big\|_Z \leqslant \frac{1}{2\,\|B^{-1}\|_{\mathcal{L}}}\big(\,\|x - x_1\|_X + \|y - y_1\|_Y\,\big),$$

so

$$\big\|u(x,g(x))\big\|_Z \leqslant \frac{1}{2\,\|B^{-1}\|_{\mathcal{L}}}\big(\big\|x - x_1\big\|_X + \big\|g(x) - g(x_1)\big\|_Y\big). \tag{4.10}$$

From (4.9) and (4.10), it follows that

$$\big\|g(x)-g(x_1)\big\|_Y \;\leqslant\; \big\|B^{-1}A\big\|_{\mathcal{L}}\,\|x-x_1\|_X+\frac{1}{2}\,\|x-x_1\|_X+\frac{1}{2}\big\|g(x)-g(x_1)\big\|_Y,$$

so

$$\big\|g(x)-g(x_1)\big\|_Y \;\leqslant\; \eta\,\|x-x_1\|_X\,, \tag{4.11}$$

with $\eta \stackrel{df}{=} 2\big\|B^{-1}A\big\|_{\mathcal{L}}+1$.

Let

$$v(x) \;\stackrel{df}{=}\; -B^{-1}u\big(x,g(x)\big).$$

From (4.9), we have

$$g(x)-g(x_1) \;=\; B^{-1}A(x-x_1)+v(x) \tag{4.12}$$

and since

$$\big\|v(x)\big\|_Y \;\leqslant\; \big\|B^{-1}\big\|_{\mathcal{L}}\big\|u\big(x,g(x)\big)\big\|_Z$$

and g is continuous, we have

$$\lim_{x\to x_1}\frac{\|v(x)\|_Y}{\|x-x_1\|_X} \;=\; 0. \tag{4.13}$$

From (4.12) and (4.13), it follows that g is Fréchet differentiable at $x_1\in U_1$ and

$$g'_F(x_1) \;=\; -B^{-1}A \;=\; -\big(D_2f(x_1,y_1)\big)^{-1}D_1f(x_1,y_1),$$

which means that g is continuously differentiable. □

DEFINITION 4.1.28 *Let Z be a Banach space and let $V\subseteq Z$ be a closed subspace of Z. We say that V is **complemented**, if there is a closed subspace W of Z, such that*

$$Z \;=\; V\oplus W$$

(i.e., $Z=V+W$ and $V\cap W=\{0\}$).

REMARK 4.1.29 The subspace $V\subseteq Z$ is complemented if and only if there exists a bounded linear projection of Z onto V, i.e., there exists $\mathrm{P}_V\in\mathcal{L}(Z)$, such that

$$\mathrm{P}_V|_V \;=\; id_V \quad\text{and}\quad \mathrm{P}_V(Z) \;=\; V.$$

The closed subspace c_0 of l^∞ is not complemented. If every closed subspace of a Banach space Z is complemented, then Z is isomorphic to a Hilbert space. Every subspace of the Banach space Z, which is either finite dimensional or it has finite codimension, is complemented. Finally in a Hilbert space every closed subspace is complemented (take the orthogonal complement).

An interesting consequence of Theorem 4.1.27 is the following corollary.

COROLLARY 4.1.30
If $U \subseteq X$ is open, $f\colon U \longrightarrow Y$ is a continuously differentiable function, $f'_F(x_0)$ is surjective and $\ker f'_F(x_0)$ is complemented,
then $f(U)$ contains a neighbourhood of $f(x_0)$.

PROOF Let $V \stackrel{df}{=} \ker f'_F(x_0)$. Then $X = V \oplus W$ (see Definition 4.1.28) and so for all $x \in X$ we have $x = v + w$, with $v \in V$ and $w \in W$. We write

$$f(x) \;=\; f(v, w).$$

Evidently

$$D_2 f(x_0) \in \mathcal{L}(W; Y) \quad \text{is an isomorphism.}$$

So we can apply Theorem 4.1.27 and conclude that $f(U)$ contains the neighbourhood U_2 of $y_0 = f(x_0)$ postulated by Theorem 4.1.27. □

In the next example we use Corollary 4.1.30 to prove an existence theorem for differential equations.

EXAMPLE 4.1.31 Let

$$X \;=\; C^1\big([0,1]\big) \quad \text{and} \quad Y \;=\; C\big([0,1]\big).$$

Let

$$f\colon X \longrightarrow Y$$

be defined by

$$f(x) \;\stackrel{df}{=}\; \frac{dx}{dt} + x^3.$$

It is easy to see that f is a C^1-map and

$$f'_F(0) \;=\; \frac{d}{dt} \;\in\; \mathcal{L}(X; Y).$$

From the fundamental theorem of calculus, we have that $f'_F(0)$ is surjective. Also $\ker f'_F(0)$ is the space of constant functions, hence it is complemented (see Remark 4.1.29). In fact the complement of $\ker f'_F(0)$ is given by

$$W \;\stackrel{df}{=}\; \left\{ x \in X : \int_0^1 x(t)\, dt = 0 \right\}.$$

So we can apply Corollary 4.1.30 and conclude the following:

"We can find $\varepsilon > 0$, such that if $y \in Y$ with $\|y\|_Y < \varepsilon$, then the differential equation

$$\frac{dx(t)}{dt} + x(t)^3 \;=\; y(t) \qquad \forall\, t \in [0,1]$$

has a solution $x \in X$."

Using the implicit function theorem (see Theorem 4.1.27), we can prove the inverse function theorem.

THEOREM 4.1.32 (Inverse Function Theorem)
If $U \subseteq Y$ is an open set, $f\colon U \longrightarrow X$ is a continuously differentiable function, $y_0 \in U$ and $f'_F(y_0) \in \mathcal{L}(Y;X)$ is an isomorphism,
then there exists a neighbourhood U' of y_0, $U' \subseteq U$ and V' a neighbourhood of $x_0 = f(y_0)$, such that $f\colon U' \longrightarrow V'$ is a diffeomorphism and

$$(f^{-1})'_F(x_0) \;=\; f'_F(y_0)^{-1}.$$

PROOF Let

$$h(x,y) \;\stackrel{df}{=}\; f(y) - x.$$

Then

$$D_2 h(x_0, y_0) \;=\; f'_F(y_0),$$

which by hypothesis is an isomorphism. So by virtue of Theorem 4.1.27 we can find a neighbourhood V' of x_0 and a continuously differentiable map $g\colon V' \longrightarrow Y$, such that

$$g(V') \;\subseteq\; U_0$$

for a neighbourhood U_0 of y_0,

$$h\big(x, g(x)\big) \;=\; 0 \qquad \forall\, x \in V'$$

(i.e., $f\big(g(x)\big) = x$ for all $x \in V'$) and

$$g(x_0) \;=\; y_0.$$

In the sequel we consider f restricted to $g(V')$. Since

$$f\big(g(x)\big) \;=\; x,$$

we see that g is injective on V', hence a bijection from V' onto $g(V')$. In addition

$$g(V') \;=\; f^{-1}(V') \text{ is open}$$

because f is continuous. So we set

$$U' \;=\; g(V')$$

and we have that $f\colon U' \longrightarrow V'$ is a bijection. Finally since

$$g'_F(x_0) \;=\; -\big(D_2h\big(x_0, g(x_0)\big)\big)^{-1} D_1h\big(x_0, g(x_0)\big),$$

we have

$$f'_F(x_0) \circ g'_F(x_0) \;=\; id_{_X},$$

hence

$$g'_F(x_0) \;=\; (f^{-1})'_F(x_0) \;=\; f'_F(y_0)^{-1}.$$

□

In finite dimensions this theorem has the following useful consequences.

COROLLARY 4.1.33
If $V \subseteq \mathbb{R}^N$ is an open set, $x_0 \in V$, $f\colon V \longrightarrow \mathbb{R}^M$ is a continuously differentiable function and $y_0 = f(x_0)$,
then

(a) *if $N \leqslant M$ and $f'_F(x_0)$ is of maximal rank (i.e., of rank N), then we can find a neighbourhood U' of y_0, V' a neighbourhood of x_0 and a continuously differentiable function $g\colon U' \longrightarrow \mathbb{R}^N$, such that*

$$(g \circ f)(x) \;=\; i(x) \qquad \forall\; x \in V',$$

where $i\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is the canonical injection, i.e.,

$$i(x_1, \ldots, x_N) \;=\; (x_1, \ldots, x_N, 0, \ldots, 0);$$

(b) *if $N \geqslant M$ and $f'_F(x_0)$ is of maximal rank (i.e., of rank M), then we can find a neighbourhood $\widehat{V}$ of x_0 and a continuously differentiable map $\vartheta\colon \widehat{V} \longrightarrow V$, with $\vartheta(x_0) = x_0$ and*

$$(f \circ \vartheta)(x) \;=\; \mathrm{proj}_{_{\mathbb{R}^M}}(x) \qquad \forall\; x \in \widehat{V},$$

where $\mathrm{proj}_{_{\mathbb{R}^M}} : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is the canonical projection, i.e.,

$$\mathrm{proj}_{_{\mathbb{R}^M}}(x_1, \ldots, x_N) \;=\; (x_1, \ldots, x_M).$$

PROOF **(a)** By hypothesis

$$\mathrm{rank}\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{\substack{i\in\{1,\ldots,M\}\\ j\in\{1,\ldots,N\}}} \;=\; N.$$

By relabelling things if necessary, we may assume that

$$\det\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{\substack{i\in\{1,\ldots,N\}\\ j\in\{1,\ldots,N\}}} \;\neq\; 0.$$

Let $\xi\colon V \times \mathbb{R}^{M-N} \longrightarrow \mathbb{R}^M$ be defined by

$$\xi(x_1, \ldots, x_M) \stackrel{df}{=} f(x_1, \ldots, x_N) + (0, \ldots, 0, x_{N+1}, \ldots, x_M).$$

We have

$$\det\left(\frac{\partial \xi_i}{\partial x_j}(x_0, 0)\right)_{\substack{i\in\{1,\ldots,M\}\\ j\in\{1,\ldots,M\}}} \neq 0.$$

So Theorem 4.1.27 implies that locally there exists an inverse g of f, such that

$$i(x) = (g \circ \xi \circ i)(x) = (g \circ f)(x).$$

(b) Again we may assume without any loss of generality that

$$\det\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{\substack{i\in\{1,\ldots,M\}\\ j\in\{1,\ldots,M\}}} \neq 0.$$

We define $\eta\colon V \longrightarrow \mathbb{R}^N$, by

$$\eta(x_1, \ldots, x_N) \stackrel{df}{=} \big(f_1(x), \ldots, f_M(x), x_{M+1}, \ldots, x_n\big).$$

Hence

$$\det\left(\frac{\partial \eta_i}{\partial x_j}(x_0)\right)_{\substack{i\in\{1,\ldots,N\}\\ j\in\{1,\ldots,N\}}} = \det\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{\substack{i\in\{1,\ldots,M\}\\ j\in\{1,\ldots,M\}}} \neq 0.$$

So by Theorem 4.1.32, we can find a neighbourhood $\widehat{V}$ of x_0 and a continuously differentiable map $\vartheta\colon \widehat{V} \longrightarrow V$, such that $\vartheta(x_0) = x_0$ and $\vartheta = \eta^{-1}$. Then

$$\mathrm{proj}_{\mathbb{R}^M}(x) = \big(\mathrm{proj}_{\mathbb{R}^M} \circ \eta \circ \vartheta\big)(x) = \big(f \circ \vartheta\big)(x).$$

□

REMARK 4.1.34 In Corollary 4.1.33, part **(a)** tells us that f locally looks like the inclusion map:

$$\begin{array}{ccc} V' \subseteq \mathbb{R}^N & \xrightarrow{\ f\ } & U' \subseteq \mathbb{R}^M \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathbb{R}^M & \end{array}$$

On the other hand part **(b)** tells us that f locally looks like the projection map:

$$\begin{array}{ccccc} & & \widehat{V} \subseteq \mathbb{R}^N & & \\ & {\scriptstyle \vartheta}\swarrow & & \searrow{\scriptstyle p} & \\ \vartheta(\widehat{V}) \subseteq V \subseteq \mathbb{R}^N & & \xrightarrow{\ f\ } & & \mathbb{R}^M \end{array}$$

Both diagrams are commutative.

4.2 Convex Functions

In this section we focus on convex functions and their differentiability properties. We show that the algebraic property of convexity has important topological consequences such as continuity and differentiability. The situation is especially pleasant in the context of separable Banach spaces.

Convex functions play a certain role in modern variational analysis and the applications require that we consider extended real valued convex functions, that is functions with values in $\overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$.

DEFINITION 4.2.1 *Let X be a Hausdorff topological space and let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a function. The **effective domain** of φ is the set*

$$\operatorname{dom}\varphi \stackrel{df}{=} \big\{x \in X :\ \varphi(x) < +\infty\big\}.$$

*We say that φ is **proper**, if $\operatorname{dom}\varphi \neq \emptyset$. The **epigraph** of φ is the set*

$$\operatorname{epi}\varphi \stackrel{df}{=} \big\{(x,\lambda) \in X \times \mathbb{R} :\ \varphi(x) \leqslant \lambda\big\}.$$

*The function φ is **lower semicontinuous**, if for every $\lambda \in \mathbb{R}$, the sublevel set*

$$L_\lambda^\varphi \stackrel{df}{=} \big\{x \in X :\ \varphi(x) \leqslant \lambda\big\}$$

*is closed. If X is a Hausdorff linear topological space, we say that φ is **convex**, if for all $x_1, x_2 \in \operatorname{dom}\varphi$ and all $\lambda \in [0,1]$, we have*

$$\varphi\big(\lambda x_1 + (1-\lambda)x_2\big) \ \leqslant\ \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2).$$

*We say that φ is **strictly convex** if the above inequality is strict when $x_1 \neq x_2$ and $\lambda \in (0,1)$.*

The cone of proper, convex and lower semicontinuous functions is denoted by $\Gamma_0(X)$.

REMARK 4.2.2 It is well known that φ is lower semicontinuous if and only if $\operatorname{epi}\varphi \subseteq X \times \mathbb{R}$ is closed or equivalently if $\varphi(x) \leqslant \liminf_\alpha \varphi(x_\alpha)$ for every net (x_α) converging to x. Also φ is convex if and only if $\operatorname{epi}\varphi \subseteq X \times \mathbb{R}$ is a convex set. This means that certain properties of proper, convex and lower semicontinuous functions can be deduced from these (rather special) closed, convex sets in $X \times \mathbb{R}$. So one can argue that the study of proper, convex, lower semicontinuous functions is a special case of the study of closed, convex sets. On the other hand, if C is a nonempty subset of X, we can introduce the ***indicator function*** of C, by

$$i_C(x) \stackrel{df}{=} \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $i_C \in \Gamma_0(X)$ if and only if C is closed and convex. This example shows that it is possible to deduce certain properties of a closed, convex set from the properties of its indicator function which belongs in $\Gamma_0(X)$. So one can argue that the study of closed, convex sets is a special case of the study of proper, convex, lower semicontinuous functions. Both points of view are legitimate and it is a matter of which approach is more convenient, the geometric or the analytical.

The next theorem summarizes the continuity properties of a proper, convex function.

THEOREM 4.2.3
If X is a Hausdorff linear space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, convex function, then the following statements are equivalent:

(a) *φ is bounded from above on a neighbourhood of $x_0 \in X$;*

(b) *φ is continuous at $x_0 \in X$;*

(c) $\operatorname{int}\operatorname{epi}\varphi \neq \emptyset$*;*

(d) $\operatorname{int}\operatorname{dom}\varphi \neq \emptyset$ *and* $\varphi|_{\operatorname{int}\operatorname{dom}\varphi}$ *is continuous.*

Moreover, if the above statements hold, then

$$\operatorname{int}\operatorname{epi}\varphi \;=\; \big\{(x,\lambda) \in X \times \mathbb{R} :\; x \in \operatorname{int}\operatorname{dom}\varphi,\; \varphi(x) < \lambda\big\}.$$

PROOF "**(a)**$\Longrightarrow$**(b)**": Let U be a neighbourhood of x_0, such that $\varphi|_U$ is bounded from above, i.e., there exists $c > 0$, such that

$$\varphi(x) \leqslant c \qquad \forall\, x \in U.$$

Replacing, if necessary, U by $U - x_0$ and $\varphi(x)$ by $\varphi(x + x_0) - \varphi(x_0)$, we may assume without any loss of generality that $x_0 = 0$ and that $\varphi(0) = 0$. We will show that φ is continuous at $x_0 = 0$. Let $\varepsilon \in (0, c]$ and let us define

$$V_\varepsilon \;\stackrel{df}{=}\; \Big(\frac{\varepsilon}{c}U\Big) \cap \Big(-\frac{\varepsilon}{c}U\Big).$$

Evidently V_ε is a symmetric neighbourhood of the origin. We shall show that

$$\big|\varphi(x)\big| \;\leqslant\; \varepsilon \qquad \forall\, x \in V_\varepsilon \tag{4.14}$$

(which implies the continuity of φ at $x_0 = 0$). So let $x \in V_\varepsilon$. We have $\frac{c}{\varepsilon}x \in U$ and because φ is convex, we have

$$\varphi(x) \;\leqslant\; \frac{\varepsilon}{c}\varphi\Big(\frac{c}{\varepsilon}x\Big) + \Big(1 - \frac{\varepsilon}{c}\Big)\varphi(0) \;\leqslant\; \frac{\varepsilon}{c}c \;=\; \varepsilon.$$

Also $-\frac{c}{\varepsilon}x \in U$ and so

$$\begin{aligned} 0 \;=\; \varphi(0) \;&=\; \varphi\left(\frac{1}{1+\frac{\varepsilon}{c}}x + \frac{\frac{\varepsilon}{c}}{1+\frac{\varepsilon}{c}}\left(-\frac{c}{\varepsilon}x\right)\right) \\ &\leqslant\; \frac{1}{1+\frac{\varepsilon}{c}}\varphi(x) + \frac{\frac{\varepsilon}{c}}{1+\frac{\varepsilon}{c}}\varphi\left(-\frac{c}{\varepsilon}x\right) \;\leqslant\; \frac{1}{1+\frac{\varepsilon}{c}}\varphi(x) + \frac{\varepsilon}{1+\frac{\varepsilon}{c}}, \end{aligned}$$

hence $-\varepsilon \leqslant \varphi(x)$. So finally we obtain (4.14) and this proves the continuity of φ at the origin.

"**(b)**$\Longrightarrow$**(a)**": Since φ is continuous at x_0, then it is bounded on a neighbourhood of x_0.

"**(a)**$\Longrightarrow$**(c)**": By hypothesis, there exists a neighbourhood U of x_0, such that

$$\varphi(x) \;\leqslant\; c \qquad \forall\, x \in U.$$

So $U \subseteq \operatorname{int}\operatorname{dom}\varphi$ and

$$\{(x,\lambda) \in X \times \mathbb{R} :\; x \in U,\; c < \lambda\} \;\subseteq\; \operatorname{epi}\varphi,$$

which implies that $\operatorname{int}\operatorname{epi}\varphi \neq \emptyset$.

"**(c)**$\Longrightarrow$**(a)**": Let $(x,\lambda) \in \operatorname{int}\operatorname{epi}\varphi$. We can find a neighbourhood U of x and $r > 0$, such that

$$U \times [\lambda - r, \lambda + r] \;\subseteq\; \operatorname{epi}\varphi,$$

hence $U \times \{\lambda\} \subseteq \operatorname{epi}\varphi$ and so

$$\varphi(x) \;\leqslant\; \lambda \qquad \forall\, x \in U,$$

which means that φ is bounded from above in a neighbourhood of x.

"**(a)**$\Longrightarrow$**(d)**": As before we may assume that $x_0 = 0$. Let U be the neighbourhood of $x_0 = 0$ postulated by part **(a)**. Evidently $U \subseteq \operatorname{dom}\varphi$ and so $\operatorname{int}\operatorname{dom}\varphi \neq \emptyset$. Let $x \in \operatorname{int}\operatorname{dom}\varphi$. Note that $\operatorname{dom}\varphi$ is convex. So we can find $r > 1$, such that $\widehat{x} = rx \in \operatorname{dom}\varphi$. Let

$$V \;\stackrel{df}{=}\; x + \left(1 - \frac{1}{r}\right)U,$$

which is a neighbourhood of x. Exploiting the convexity of φ, for all $u \in V$, we have

$$u \;=\; x + \left(1 - \frac{1}{r}\right)z \quad \text{with} \quad z \in U$$

and

$$\begin{aligned}\varphi(u) &= \varphi\left(\frac{1}{r}\widehat{x}+\left(1-\frac{1}{r}\right)z\right) \leqslant \frac{1}{r}\varphi(\widehat{x})+\left(1-\frac{1}{r}\right)\varphi(z)\\ &\leqslant \frac{1}{r}\varphi(\widehat{x})+\left(1-\frac{1}{r}\right)c = \widehat{c}.\end{aligned}$$

So φ is bounded from above in a neighbourhood of x, hence continuous at $x \in \operatorname{int}\operatorname{dom}\varphi$ (recall that **(a)**$\Longleftrightarrow$**(b)**).

"**(d)**$\Longrightarrow$**(a)**": Obvious.

Finally let us show that

$$\operatorname{int}\operatorname{epi}\varphi = \{(x,\lambda)\in X\times\mathbb{R}:\ x\in\operatorname{int}\operatorname{dom}\varphi,\ \varphi(x)<\lambda\}.$$

Let us denote the right hand side set by W. Clearly

$$\operatorname{int}\operatorname{epi}\varphi \subseteq W.$$

On the other hand let

$$\overline{x} \in \operatorname{int}\operatorname{dom}\varphi \quad\text{and}\quad \varphi(\overline{x}) < \overline{\lambda}.$$

Let

$$\widehat{\lambda}\in\big(\varphi(\overline{x}),\overline{\lambda}\big).$$

Because $\varphi|_{\operatorname{int}\operatorname{dom}\varphi}$ is continuous, there exists a neighbourhood U of $\overline{x}$, such that $U\subseteq\operatorname{int}\operatorname{dom}\varphi$ and

$$\varphi(x) < \widehat{\lambda} \qquad \forall\, x\in U.$$

Hence $\big(\overline{x},\overline{\lambda}\big)\in U\times\Big(\widehat{\lambda},+\infty\Big)\subseteq\operatorname{int}\operatorname{epi}\varphi$, and so

$$W\subseteq\operatorname{int}\operatorname{epi}\varphi.$$

□

REMARK 4.2.4 From the above theorem it follows that

$$\operatorname{int}\operatorname{dom}\varphi = \big\{x\in X:\ \text{there exists } \lambda\in\mathbb{R} \text{ such that } (x,\lambda)\in\operatorname{int}\operatorname{epi}\varphi\big\}.$$

Also a convex function φ can be continuous at a boundary point x of $\operatorname{dom}\varphi$ where $\varphi(x)=+\infty$. To see this consider the convex function $\varphi\colon\mathbb{R}\longrightarrow\overline{\mathbb{R}}$, defined by

$$\varphi(x) \overset{df}{=} \begin{cases}\frac{1}{x} & \text{if} \quad x\in(0,+\infty),\\ +\infty & \text{if} \quad x\in(-\infty,0].\end{cases}$$

Recall that in $\overline{\mathbb{R}}$ the neighbourhoods of $+\infty$ are all the sets $(\lambda,+\infty]$ with $\lambda\in\mathbb{R}$. If C is nonempty, closed, convex set in X, then the indicator function $i_C\in\Gamma_0(X)$ is continuous at x if and only if $x\in\operatorname{int}C$. Therefore, if $\operatorname{int}C=\emptyset$, then i_C is not continuous at any point $C=\operatorname{dom}i_C$.

If X is finite dimensional, the situation is remarkably simple.

PROPOSITION 4.2.5
If $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is convex and X is finite dimensional, then φ is continuous on $\operatorname{int}\operatorname{dom}\varphi$.

PROOF Let $x \in \operatorname{int}\operatorname{dom}\varphi$. We can find

$$\{e_k\}_{k=0}^N \subseteq X \quad (N = \dim X)$$

and $r > 0$, such that

$$B_r(x) \;\subseteq\; \operatorname{conv}\{e_k\}_{k=0}^N \;\subseteq\; \operatorname{dom}\varphi.$$

So if $y \in B_r(x)$, we can find

$$\{\lambda_k\}_{k=0}^N \;\subseteq\; [0,1],$$

such that

$$\sum_{k=0}^N \lambda_k \;=\; 1 \quad \text{and} \quad x = \sum_{k=0}^N \lambda_k e_k.$$

Then because φ is convex, we have

$$\begin{aligned}\varphi(x) &\leqslant \sum_{k=0}^N \lambda_k \varphi(e_k)\\ &\leqslant \left(\sum_{k=0}^N \lambda_k\right) \max_{k\in\{0,\dots,N\}} \varphi(e_k)\\ &= c \;<\; +\infty.\end{aligned}$$

So by virtue of Theorem 4.2.3,

$$\varphi|_{\operatorname{int}\operatorname{dom}\varphi} \text{ is continuous.}$$

□

To have an infinite dimensional analog of the above theorem, we need an extra condition on the function φ.

THEOREM 4.2.6
If X is a Banach space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is convex and lower semicontinuous, then $\varphi|_{\operatorname{int}\operatorname{dom}\varphi}$ is continuous.

PROOF We have

$$\operatorname{dom}\varphi \;=\; \bigcup_{n=1}^{\infty} (\varphi \leqslant n).$$

Let

$$x \;\in\; \operatorname{int}\operatorname{dom}\varphi.$$

Since φ is lower semicontinuous, the sets

$$\{\varphi \leqslant n\} \text{ are closed.}$$

So by the Baire category theorem (see Theorem A.1.10), we can find $n \geqslant 1$, such that

$$\operatorname{int}\{\varphi < n\} \neq \emptyset \quad \text{and} \quad \varphi(x) \;<\; n.$$

Let

$$y \;\in\; \operatorname{int}\{\varphi < n\}$$

and set

$$h(\lambda) \;\stackrel{df}{=}\; \varphi\big(x + \lambda(y-x)\big) \qquad \forall\ \lambda > 0.$$

Since $x \in \operatorname{int}\operatorname{dom}\varphi$, we can find $r > 0$, such that

$$\overline{B}_{r\|y-x\|_X}(x) \subseteq \operatorname{dom}\varphi.$$

We have

$$[-r, r] \;\subseteq\; \operatorname{dom} h,$$

hence

$$0 \;\in\; \operatorname{int}\operatorname{dom} h$$

and so h is continuous at 0 (see Proposition 4.2.5).

Because $h(0) < n$, we can find $\vartheta > 0$, such that

$$h(\lambda) \;<\; n \qquad \forall\ \lambda \in [-\vartheta, 0].$$

Let

$$z \;\stackrel{df}{=}\; x - \vartheta(y-x).$$

We have

$$z \in \{\varphi < n\}$$

and

$$y \in \operatorname{int}\{\varphi < n\}.$$

Because

$$x \;\in\; [y, z) \;=\; \{(1-\lambda)y + \lambda z :\ \lambda \in [0,1)\},$$

it follows that

$$x \;\in\; \operatorname{int}\{\varphi < n\}$$

and so φ is continuous at x (see Theorem 4.2.3). □

In fact we can specify the kind of continuity of φ on $\operatorname{int}\operatorname{dom}\varphi$.

THEOREM 4.2.7
If X is a Banach space, $U \subseteq X$ is an open set and $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex function,
then φ is continuous on U if and only if φ is locally Lipschitz on U.

PROOF "$\Longrightarrow$": Because φ is continuous on U, for a given $x_0 \in U$, we can find $\delta > 0$ and $c > 0$, such that

$$\overline{B}_{2\delta}(x_0) \subseteq U \quad \text{and} \quad \big|\varphi(y)\big| \leqslant c \qquad \forall\, y \in \overline{B}_{2\delta}(x_0).$$

Let

$$x, y \in \overline{B}_{\delta}(x_0) \quad \text{with } x \neq y.$$

Let us set

$$r \stackrel{df}{=} \|x - y\|_X$$

and

$$z \stackrel{df}{=} y + \frac{\delta}{r}(y - x).$$

Then $z \in \overline{B}_{2\delta}(x_0)$. Also we have

$$y = \frac{r}{r+\delta} z + \frac{\delta}{1+\delta} x.$$

So from the convexity of φ, we obtain

$$\varphi(y) \leqslant \frac{r}{r+\delta}\varphi(z) + \frac{\delta}{r+\delta}\varphi(x),$$

so

$$\begin{aligned} \varphi(y) - \varphi(x) &\leqslant \frac{r}{r+\delta}\big(\varphi(z) - \varphi(x)\big) \\ &\leqslant \frac{r}{\delta} 2c = \frac{2c}{\delta}\|x - y\|_X . \end{aligned}$$

Interchanging the roles of x and y in the above argument, we conclude that

$$\big|\varphi(y) - \varphi(x)\big| \leqslant \frac{2c}{\delta}\|x - y\|_X \qquad \forall\, x, y \in \overline{B}_{\delta}(x_0).$$

"$\Longleftarrow$": Obvious. ∎

For convex, continuous functions, it is possible to characterize Gâteaux and Fréchet differentiability at $x \in X$ only in terms of φ, that is without using the linear functionals $\varphi'_F(x)$ and $\varphi'_G(x)$.

PROPOSITION 4.2.8
If $U \subseteq X$ is an open set and $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex, continuous function, then φ is Gâteaux differentiable at $x \in X$ if and only if

$$\lim_{\lambda \searrow 0} \frac{\varphi(x+\lambda h) + \varphi(x-\lambda h) - 2\varphi(x)}{\lambda} \;=\; 0 \qquad \forall\, h \in X. \tag{4.15}$$

PROOF "$\Longrightarrow$": Since φ is Gâteaux differentiable at $x \in X$, we have

$$\lim_{\lambda \searrow 0} \frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda} \;=\; \varphi'_G(x)h$$

and

$$\lim_{\lambda \searrow 0} \frac{\varphi(x-\lambda h) - \varphi(x)}{\lambda} \;=\; \varphi'_G(x)(-h).$$

From these limits, we obtain immediately (4.15).

"$\Longleftarrow$": Let

$$\psi(\lambda) \stackrel{df}{=} \varphi(x+\lambda h) \qquad \forall\, \lambda \in \mathbb{R}.$$

Then ψ is convex and by hypothesis we have

$$\psi'_+(0) \;=\; \psi'_-(0).$$

Therefore ψ is differentiable at $\lambda = 0$ and so φ is Gâteaux differentiable at x (see Remark 4.1.2). ∎

In the case of Fréchet differentiability at x, (4.15) holds uniformly with respect to h when $\|h\|_X = 1$.

PROPOSITION 4.2.9
If $U \subseteq X$ is an open set and $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex, continuous function, then φ is Fréchet differentiable at $x \in U$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\varphi(x+\lambda h) + \varphi(x-\lambda h) - 2\varphi(x) \;<\; \lambda\varepsilon \qquad \forall\; \|h\|_X = 1,\; \lambda \in (0,\delta). \tag{4.16}$$

PROOF "$\Longrightarrow$": Since φ is Fréchet differentiable at $x \in X$, for a given $\varepsilon > 0$, we can find $\delta > 0$, such that

$$\varphi(x+\lambda h) - \varphi(x) - \langle \varphi'_F(x), \lambda h \rangle_X \;<\; \frac{\varepsilon}{2}\|h\|_X \qquad \forall\; \|h\|_X = 1,\; \lambda \in (0,\delta). \tag{4.17}$$

Rewriting (4.17) with h replaced by $-h$ and adding we obtain (4.16).

"$\Longleftarrow$": From Proposition 4.2.8 we know that φ is Gâteaux differentiable at x. The convexity φ implies that

$$\frac{\varphi(x+\lambda h)-\varphi(x)}{\lambda} - \big\langle \varphi'_G(x), h\big\rangle_X \;\geqslant\; 0 \qquad \forall\ \lambda > 0 \tag{4.18}$$

and

$$\frac{\varphi(x)-\varphi(x-\lambda h)}{\lambda} - \big\langle \varphi'_G(x), h\big\rangle_X \;\leqslant\; 0 \qquad \forall\ \lambda > 0. \tag{4.19}$$

Therefore

$$\begin{aligned} \lambda\varepsilon \;&>\; \varphi(x+\lambda h)+\varphi(x-\lambda h)-2\varphi(x) \\ &=\; \big[\varphi(x+\lambda h)-\varphi(x)-\lambda\big\langle \varphi'_G(x), h\big\rangle_X\big] \\ &\quad -\; \big[\varphi(x)-\varphi(x-\lambda h)-\lambda\big\langle \varphi'_G(x), h\big\rangle_X\big] \\ &\qquad\qquad \forall\ \|h\|_X = 1,\ \lambda\in(0,\delta). \end{aligned} \tag{4.20}$$

From (4.18) and (4.19), we see that the right hand side of (4.20) is the sum of two positive quantities both of which have to be less than $\lambda\varepsilon$ for all $\|h\|_X = 1$ and all $\lambda \in (0,\delta)$. This means that φ is Fréchet differentiable at x. ∎

It is well known from elementary calculus that for a function $\varphi\colon U \longrightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^N$ being an open set, existence of all partial derivatives at $x \in U$ does not imply Fréchet differentiability. However, if φ is convex this is true.

PROPOSITION 4.2.10
If $U \subseteq \mathbb{R}^N$ is an open set, $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex function and all the partial derivatives of φ at $x \in U$ exist,
then φ is Fréchet differentiable at $x \in U$.

PROOF The obvious candidate for Fréchet derivative of φ at x is the linear transformation determined by the partial derivatives, that is

$$A(x) \;\in\; \mathcal{L}\big(\mathbb{R}^N;\mathbb{R}\big) \;=\; \mathbb{R}^N,$$

defined by

$$\big(A(x),h\big)_{\mathbb{R}^N} \;\stackrel{df}{=}\; \sum_{k=1}^{N} \frac{\partial\varphi}{\partial x_k}(x)h_k \qquad \forall\ h=(h_1,\ldots,h_N)\in\mathbb{R}^N.$$

Let $r > 0$ be such that

$$B_r(x) \;\subseteq\; U.$$

For each $h \in B_r(0)$, let

$$\psi(h) \;\stackrel{df}{=}\; \varphi(x+h)-\varphi(x)-\big(A(x),h\big)_{\mathbb{R}^N}.$$

Evidently ψ is convex on $B_r(0)$. For each $k \in \{1, \dots, N\}$, let us define a function $\xi_k \colon B_r(0) \longrightarrow \mathbb{R}$, by

$$\xi_k(h) \stackrel{df}{=} \begin{cases} \dfrac{\psi(h_k e_k)}{h_k} & \text{if} \quad h_k \neq 0 \\ 0 & \text{if} \quad h_k = 0 \end{cases} \qquad \forall\, h \in B_r(0),$$

where $\{e_k\}_{k=1}^N$ is the orthonormal basis of $\mathbb{R}^N$. We have

$$\xi_k(h) \longrightarrow 0 \quad \text{as} \ \|h\|_{\mathbb{R}^N} \to 0.$$

For each $h = (h_1, \dots, h_N)$ with $\|h\|_{\mathbb{R}^N} < \frac{r}{N}$, because of the convexity of ψ, we have

$$\begin{aligned} \psi(h) &= \psi\Big(\sum_{k=1}^N \frac{1}{N} N h_k e_k\Big) \leqslant \frac{1}{N} \sum_{k=1}^N \psi\big(N h_k e_k\big) \\ &= \sum_{k=1}^N h_k \xi_k(Nh) \leqslant \|h\|_{\mathbb{R}^N} \sum_{k=1}^N \big|\xi_k(Nh)\big|. \end{aligned}$$

Since

$$0 = \psi\left(\frac{1}{2}h + \frac{1}{2}(-h)\right) \leqslant \frac{1}{2}\psi(h) + \frac{1}{2}\psi(-h),$$

we have

$$-\psi(-h) \leqslant \psi(h).$$

Therefore, it follows that

$$-\|h\|_{\mathbb{R}^N} \sum_{k=1}^N \big|\xi_k(-Nh)\big| \leqslant \psi(h) \leqslant \|h\|_{\mathbb{R}^N} \sum_{k=1}^N \big|\xi_k(Nh)\big|,$$

so

$$\frac{\psi(h)}{\|h\|_{\mathbb{R}^N}} \longrightarrow 0 \quad \text{as } h \to 0$$

and thus φ is Fréchet differentiable at $x \in U$ and

$$\varphi'_F(x) = A(x).$$

□

The above proposition implies that for convex functions on a finite dimensional Banach space the situation is straightforward; namely Gâteaux and Fréchet differentiability are equivalent (compare with Proposition 4.1.7).

COROLLARY 4.2.11

If X is a finite dimensional Banach space, $U \subseteq X$ is an open set and $\varphi \colon U \longrightarrow \mathbb{R}$ is a convex function,
then φ is Gâteaux differentiable at $x \in U$ if and only if it is Fréchet differentiable at $x \in U$.

From elementary convex analysis we know that a convex function $\varphi\colon (a,b) \longrightarrow \mathbb{R}$ is differentiable at all except at most a countable number of points of (a,b). We would like to identify those Banach spaces X where the convex continuous functions defined on open convex sets in X have similar Gâteaux differentiability properties.

The basic result in this direction is the so-called Mazur's theorem, which implies the automatic generic (i.e., in a dense G_δ-subset) Gâteaux differentiability of convex, continuous functions in separable Banach spaces.

THEOREM 4.2.12 (Mazur Theorem)
If X is a separable Banach space, $U \subseteq X$ is an open, convex set and $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex, continuous function,
then φ is Gâteaux differentiable on a dense G_δ subset of U.

PROOF For each $h \in X$ and $m \in \mathbb{N}$, let

$$S\left(h, \frac{1}{m}\right) \stackrel{df}{=} \left\{x \in U : \text{ there exists } \delta = \delta(x,m) > 0, \text{ such that } \sup_{\lambda \in (0,\delta)} \frac{\varphi(x+\lambda h) + \varphi(x-\lambda h) - 2\varphi(x)}{\lambda} < \frac{1}{m}\right\}.$$

Since by hypothesis φ is continuous, for a given $m \geqslant 1$ and every $k \geqslant 1$, the set

$$T_k(h) \stackrel{df}{=} \left\{x \in U : \frac{\varphi(x+\frac{1}{k}h) + \varphi(x-\frac{1}{k}h) - 2\varphi(x)}{\frac{1}{k}} < \frac{1}{m}\right\}$$

is open in U. It follows then that

$$S\left(h, \frac{1}{m}\right) = \bigcup_{k=1}^{\infty} T_k(h)$$

is open in U. We claim that it is also dense in U. Suppose that this is not the case. Then we can find $x_0 \in U$ and $r > 0$, such that

$$S\left(h, \frac{1}{m}\right) \cap B_r(x_0) = \emptyset.$$

If $\psi(\lambda) = \varphi(x_0 + \lambda h)$, it follows that the convex function ψ is not differentiable on $(-r, r)$, a contradiction. So $S\left(h, \frac{1}{m}\right)$ is dense in U.

Because X is separable, we can find a sequence $\{h_n\}_{n\geqslant 1}$ which is dense in $\partial B_1(0) = \left\{h \in X : \|h\|_X = 1\right\}$. By virtue of Proposition 4.2.8, φ is Gâteaux differentiable at x if the directional derivative exists in the directions $\{h_n\}_{n\geqslant 1}$. So φ is Gâteaux differentiable on the set $\bigcap_{n,m=1}^{\infty} S\left(h_n, \frac{1}{m}\right)$, which is dense in U (Baire's theorem) and of course G_δ. ∎

There exist nonseparable Banach spaces where the above theorem fails.

EXAMPLE 4.2.13 Let $X = l^\infty$ and for $x = \{x_n\}_{n\geqslant 1}$, let

$$\varphi(x) \stackrel{df}{=} \limsup_{n\to+\infty} |x_n|.$$

Then φ is a seminorm (hence it is convex) and it is also continuous since $\varphi(x) \leqslant \|x\|_\infty$. If $\varphi(x) = 0$, then

$$x_n \longrightarrow 0$$

and so taking $h = (1, 1, \ldots)$, we have

$$\frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda} = \frac{|\lambda|}{\lambda},$$

which shows that φ is not Gâteaux differentiable at all $x \in \varphi^{-1}\big(\{0\}\big)$. If $\varphi(x) > 0$, exploiting the positive homogeneity of φ, we may assume without any loss of generality that

$$\varphi(x) = 1.$$

Let $\big\{x_{n_k}\big\}_{k\geqslant 1}$ be a subsequence of $\{x_n\}_{n\geqslant 1}$, such that

$$|x_{n_k}| \longrightarrow 1 \quad \text{as } k \to +\infty.$$

By passing to a further subsequence if necessary, we may assume that all the elements of the $\big\{x_{n_k}\big\}_{k\geqslant 1}$ have the same sign. Moreover, since $\varphi(x) = \varphi(-x)$, we can say that

$$x_{n_k} > 0 \qquad \forall\, k \geqslant 1.$$

Let

$$h_n \stackrel{df}{=} \begin{cases} 0 & \text{if either } n \neq n_k \text{ for all } k \geqslant 1, \text{ or } n = n_k \text{ with } k \text{ odd}, \\ 1 & \text{if } n = n_k \text{ with } k \text{ even}. \end{cases}$$

Let us set $h = \{h_n\}_{n\geqslant 1} \in l^\infty$. We have

$$\frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda} = \begin{cases} 1 & \text{if} \quad \lambda > 0, \\ 0 & \text{if} \quad \lambda < 0. \end{cases}$$

So φ is nowhere Gâteaux differentiable.

REMARK 4.2.14 The above example should not lead to the conclusion that Theorem 4.2.12 fails in every nonseparable Banach space. There are nonseparable Banach spaces in which Theorem 4.2.12 remains valid, for example the class of weakly compactly generated Banach spaces. A Banach space is ***weakly compactly generated***, if there exists a weakly compact set C (which we can always take to be convex), whose linear span is dense in X

(i.e., $X = \overline{\text{span}}\, C$). Separable Banach spaces are weakly compactly generated. To see this let $\{x_n\}_{n\geqslant 1}$ be dense in $\partial B_1(0)$ and take

$$C \stackrel{df}{=} \left\{\frac{1}{n}x_n\right\} \cup \{0\}.$$

Note that C is actually compact. Also reflexive Banach spaces are weakly compact generated. In this case let $C \stackrel{df}{=} \overline{B}_1(0)$.

DEFINITION 4.2.15 *A Banach space X is said to be a* ***weak Asplund space****, if every convex, continuous function defined on an open, convex set $U \subseteq X$ is Gâteaux differentiable at each point of a dense G_δ subset U.*

REMARK 4.2.16 In the above definition the term weak has nothing to do with the weak topology. It is used because sometimes Gâteaux differentiability is called ***weak differentiability*** in contrast to the Fréchet differentiability, which is called ***strong differentiability***. Theorem 4.2.12 says that every separable Banach space is a weak Asplund space.

What about Fréchet differentiability? In this direction we have the following result due to Asplund (1968) and Lindenstrauss (1963) independently.

THEOREM 4.2.17
If X is a Banach space with separable dual, $U \subseteq X$ is an open, convex set and $\varphi\colon U \longrightarrow \mathbb{R}$ is a convex, continuous function,
then φ is Fréchet differentiable on a dense G_δ subset of U.

Then in analogy to Definition 4.2.15, we make the following definition.

DEFINITION 4.2.18 *A Banach space X is said to be an* ***Asplund space****, if every convex, continuous function defined on an open, convex set $U \subseteq X$ is Fréchet differentiable on a dense G_δ subset of U.*

REMARK 4.2.19 Theorem 4.2.17 implies that every Banach space X with a separable dual X^* is an Asplund space. Note that X is separable too. More generally, we can say that a separable Banach space X is an Asplund space if and only if it has a separable dual. If $X = C\big([0,1]\big)$ (whose dual $M\big([0,1]\big)$, the space of Radon measures is not separable) and $\varphi\colon X \longrightarrow \mathbb{R}_+$ is defined by

$$\varphi(x) \stackrel{df}{=} \|x\|_\infty,$$

then it can be shown that φ is not Fréchet differentiable at any point.

4.3 Haar Null Sets and Locally Lipschitz Functions

From real analysis we know that every Lipschitz continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable almost everywhere and is the integral of its derivative (a consequence of the fundamental theorem of the Lebesgue calculus). We saw that this theorem can be extended to vector valued functions $f\colon \mathbb{R} \longrightarrow X$, when X is a Banach space with the RNP (see Theorem 2.2.17 and Remark 2.2.18). We also proved another generalization of Lebesgue's original result, which is due to Rademacher and which says that a locally Lipschitz function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is differentiable almost everywhere (see Theorem 1.5.8 and Corollary 1.5.9). The purpose of this section is to combine these two generalizations; that is we want to prove a Rademacher's theorem for functions between Banach spaces.

The problem that we face when dealing with such a generalization is that we do not have a natural measure μ (such as the Lebesgue measure in $\mathbb{R}^N$), which produces a useful class of μ-null sets. So we need to devise new ways to come up with negligible sets. Our experience from the real line suggests that the approach cannot be purely topological and choose, say, the sets of first Baire category. There are several distinct ways to define negligible sets. Our goal is not to present all of them. Instead we will focus on the so-called Haar-null sets, which historically produced the first generalization of Rademacher's theorem.

As the name suggests Haar-null sets are defined on topological groups. So let us start with a brief discussion of them.

DEFINITION 4.3.1 *A **topological group** is a group G endowed with a Hausdorff topology, which is compatible with the group structure; that is the two maps*

$$G \times G \ni (x, y) \longmapsto xy \in G \quad \text{and} \quad G \ni x \longmapsto x^{-1} \in G$$

*are continuous ($G \times G$ is furnished with the product topology). An **isomorphism** of a topological group G onto a topological group $\widehat{G}$ is a group isomorphism of G onto $\widehat{G}$ which is bicontinuous. We say that G is an **Abelian topological group**, if G is Abelian.*

REMARK 4.3.2 The map $x \longmapsto x^{-1}$ is the ***inverse map***, the map $x \longmapsto ax$ is ***left translation*** by a and the map $x \longmapsto xb$ is ***right translation*** by b. All three maps are homeomorphisms of G into itself. The fact that translations are homeomorphisms implies that a topological group is ***topologically homogeneous***. Namely for any $a, b \in G$, the map $x \longmapsto ba^{-1}x$ is a homeomorphism of G which sends a to b. Therefore the topological structure at a is reflected at b. In particular then the topology is completely determined by the system of neighbourhoods of the neutral element e.

DEFINITION 4.3.3 *A metric d_G on a topological group G is said to be **left-invariant**, if*

$$d_G(ax, ay) = d_G(x, y) \qquad \forall\ a, x, y \in G$$

*and **right-invariant**, if*

$$d_G(xa, ya) = d_G(x, y) \qquad \forall\ a, x, y \in G.$$

*The metric d_G is **invariant**, if it is both left-invariant and right-invariant.*

REMARK 4.3.4 A celebrated theorem of Birkhoff-Kakutani (see, e.g., Hewitt & Ross (1963, pp. 68–70)) says that a topological group G is metrizable if and only if the neutral element e has a countable fundamental system of neighbourhoods. A metrizable topological group admits a left-invariant (or right-invariant) compatible metric. However, a metrizable topological group need not admit an invariant metric. For Abelian groups clearly left and right invariance are equivalent.

We shall consider separable Abelian topological groups and separable Banach spaces or otherwise we face serious measurability problems. In addition the Abelian topological group will be Polish. So let G be an Abelian Polish topological group. If G is locally compact, then it is well known that there is a unique (up to scalar multiplication) translation invariant measure μ on G called the Haar measure (see Dieudonné (1969, p. 244)). For G which is not locally compact, no invariant measure exists. Nevertheless, it is still possible to define the notion of Haar-null sets. Since G is Abelian, its operation will be denoted by "+." Also all measures considered in the sequel will be Borel without any explicit mention.

DEFINITION 4.3.5 *A Borel set $A \subseteq G$ is said to be Haar-null, if there is a probability measure μ on G, such that $\chi_A \star \mu = 0$, i.e.,*

$$\int_G \chi_A(x + y)\, \mu(dx) = 0 \qquad \forall\ y \in G$$

(the convolution of the characteristic function χ_A and the measure μ).

REMARK 4.3.6 So according to this definition the Borel set A is Haar-null if and only if there is a probability measure μ, such that all translates of A are μ-null, i.e.,

$$\mu(A + y) = 0 \qquad \forall\ y \in G.$$

The measure μ is called ***test measure*** for A.

The next proposition shows that when G is locally compact, then the notion of Haar-null set coincides with that of a set which is negligible with respect to the Haar measure on G.

PROPOSITION 4.3.7
If G is a locally compact, Abelian, Polish topological group and $A \subseteq G$ is a Borel set,
then the following two properties are equivalent:

(a) *A is Haar-null on G;*

(b) *A is negligible for the Haar measure on G.*

PROOF "**(a)**$\Longrightarrow$**(b)**": Let μ be a test measure for A and let h be a Haar measure on G. We know that h is σ-finite. Then by virtue of Fubini's theorem, we have

$$\int\limits_G \left[\int\limits_G \chi_A(x+y)\, h(dx) \right] \mu(dy) \;=\; \int\limits_G \left[\int\limits_G \chi_A(x+y)\, \mu(dy) \right] h(dx) \;=\; 0.$$

So for μ-almost all $y \in G$, we have

$$\int\limits_G \chi_A(x+y)\, h(dx) \;=\; 0,$$

hence

$$h(A) \;=\; \int\limits_G \chi_A(x)\, h(dx) \;=\; 0.$$

"**(b)**$\Longrightarrow$**(a)**": Again let h be a Haar measure. Let $f \in C_c(G)$, such that

$$\int\limits_G f(x) h(dx) \;=\; 1.$$

Let us set

$$\mu(B) \;\stackrel{df}{=}\; \int\limits_B f(x) h(dx) \qquad \forall\; B \in \mathcal{B}(G).$$

Evidently μ is a probability measure on G and we have

$$\begin{aligned} \int\limits_G \chi_A(x+y) \mu(dx) \;&=\; \int\limits_G \chi_A(x+y) f(x) h(dx) \\ \leqslant\; c \int\limits_G \chi_A(x+y) h(dx) \;&=\; c \int\limits_G \chi_A(x) h(dx) \;=\; 0, \end{aligned}$$

for some $c > 0$ (since $f \in C_c(G)$). ∎

REMARK 4.3.8 Since on $\mathbb{R}^N$, the Haar measures are multiples of the Lebesgue measure, it follows that the Haar-null sets are the Lebesgue-null sets.

PROPOSITION 4.3.9
If G is an Abelian, Polish topological group and $\{A_n\}_{n\geqslant 1}$ is a sequence of Haar-null sets in G,
then $\widehat{A} \stackrel{df}{=} \bigcup_{n=1}^{\infty} A_n$ is a Haar-null set in G.

PROOF Let $M_+^1(G)$ be the space of probability measures on G. Furnished with the narrow topology $w\big(M_+^1(G), C_b(G)\big)$, $M_+^1(G)$ becomes a Polish space (see, e.g., Denkowski, Migórski & Papageorgiou (2003*a*, p. 199)). Let ϱ be a complete metric on $M_+^1(G)$ generating the above narrow topology. By the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that if

$$\mu_n \longrightarrow \mu \quad \text{in } M_+^1(G),$$

then

$$\mu_n \star \nu \longrightarrow \mu \star \nu \quad \text{in } M_+^1(G).$$

Note that if a probability measure μ vanishes on a set A and all its translates, then the same is true for every translate of μ and every probability measure which is absolutely continuous with respect to μ (see Definition A.2.22). In particular this is the case for measures of the form

$$\mu_B(A) \stackrel{df}{=} \frac{\mu(A \cap B)}{\mu(B)},$$

with $B \in \mathcal{B}(G)$, such that $\mu(B) > 0$. So let μ' be a translation of μ, such that every neighbourhood of the neutral element e of G has strictly positive μ'-measure.

Then for $r > 0$, we set

$$u_r(C) \stackrel{df}{=} \frac{\mu'(C \cap B_r(e))}{\mu'(B_r(e))},$$

where

$$B_r(e) \stackrel{df}{=} \big\{x \in G : \ d_{_G}(x, e) < r\big\},$$

with $d_{_G}$ being a complete metric on G. Evidently

$$u_r \longrightarrow \delta_e \quad \text{in } M_+^1(G), \quad \text{as } r \searrow 0,$$

where δ_e is the Dirac measure with its mass at e. Therefore for every $\varepsilon > 0$, we can find $\nu \in M_+^1(G)$, such that $\chi_{_A} \star \nu = 0$ (i.e., ν is a test measure on A) and $\varrho(\nu, \delta_e) < \varepsilon$.

These observations imply that by induction we can generate a sequence $\{\nu_n\}_{n\geqslant 1} \subseteq M_+^1(G)$, such that

$$\chi_{A_n} \star \nu_n \;=\; 0 \qquad \forall\, n \geqslant 1$$

(i.e., ν_n is a test measure on A_n) and

$$\varrho\big(u, u \star \mu_n\big) \;<\; \frac{1}{2^n} \qquad \forall\, n \geqslant 1,$$

where u is any convolution of different μ_k's with $1 \leqslant k \leqslant n-1$. Since the metric ϱ on $M_+^1(G)$ is complete, we see that the measure

$$\mu \;\stackrel{df}{=}\; \prod_{k=1}^{\infty} \star \mu_k$$

is well defined. Because $\mu = \mu_n \star \nu_n$ for all $n \geqslant 1$, where

$$\nu_n \;=\; \prod_{k \neq n} \star \mu_k,$$

it follows that

$$\chi_{A_n} \star \mu \;=\; 0 \qquad \forall\, n \geqslant 1,$$

hence

$$\chi_{\bigcup_{n=1}^{\infty} A_n} \star \mu \;=\; 0.$$

This proves that $\bigcup_{n=1}^{\infty} A_n$ is a Haar-null set. □

COROLLARY 4.3.10
If G is an Abelian, Polish topological group and $A \subseteq G$ is Haar-null, then $A^c = G \setminus A$ is dense in G.

PROOF Let $\{x_n\}_{n\geqslant 1}$ be a sequence which is dense in G and let μ be a test measure for A. It suffices to show that $\operatorname{int} A = \emptyset$. Suppose that $\operatorname{int} A \neq \emptyset$. Then we can find a neighbourhood U of the neutral element e of G and $a \in A$, such that $a + U \subseteq G$. We have

$$\mu(U + b) \;=\; \mu(U + a) \;=\; \mu(U) \;=\; 0 \qquad \forall\, b \in G,$$

so

$$\mu(U + x_n) \;=\; 0 \qquad \forall\, n \geqslant 1.$$

Then because of Proposition 4.3.9, we have that

$$G \;=\; \bigcup_{n=1}^{\infty} (U + x_n) \quad \text{is Haar-null, with test measure } \mu,$$

a contradiction. □

EXAMPLE 4.3.11 Let G be a separable Banach space and let $A \subseteq X$ be a Borel set which intersects all the translates of a fixed line L in sets whose one-dimensional Lebesgue measure is zero. For example A can be a proper, Borel linear subspace of X. Then A is a Haar-null set. Any probability measure on the line L which is equivalent to the Lebesgue measure on L can be used as a test measure of A. Sets like A above are called ***directionally-null sets***.

Recall that a set $A \subseteq \mathbb{R}$ can have positive measure, and, in fact, its complement can be Lebesgue-null, without A including any interval of positive length (consider, e.g., the set of irrational numbers). However, if we take all differences between elements in A (i.e., $A - A$), then this set contains a nontrivial interval around zero. This fact is used in the construction of a nonmeasurable set (see, e.g., Halmos (1974, p. 69)). The same property can be proved if $\mathbb{R}$ is replaced by an Abelian, Polish topological group G and we consider a Borel set $A \subseteq G$ which is not Haar-null.

PROPOSITION 4.3.12
If G is an Abelian, Polish topological group and $A \subseteq G$ is a Borel set which is not Haar-null,
then $A - A$ is a neighbourhood of the neutral element e of G.

PROOF Let

$$S(A) \stackrel{df}{=} \{x \in G : \ (A + x) \cap A \text{ is not Haar-null}\}.$$

We claim that $S(A)$ is a neighbourhood of e. If we can show this then the proposition follows since

$$S(A) \ \subseteq \ A - A.$$

Suppose that the claim is not true. Then we can find a sequence $\{x_n\}_{n \geqslant 1} \subseteq G$, such that

$$d_{_G}(x_n, e) \ < \ \frac{1}{2^n} \qquad \forall \ n \geqslant 1$$

($d_{_G}$ being a complete metric on G) and

$$(A + x_n) \cap A \text{ is Haar-null} \qquad \forall \ n \geqslant 1.$$

Because of Proposition 4.3.9, the set

$$\bigcup_{n=1}^{\infty} [(A + x_n) \cap A]$$

is Haar-null and so its complement

$$\widehat{A} \stackrel{df}{=} A \setminus \bigcup_{n=1}^{\infty} [(A + x_n) \cap A]$$

is a Borel set which is not Haar-null (see Corollary 4.3.10). Note that

$$\big(\widehat{A}+x_n\big)\cap\widehat{A} \;=\; \emptyset \qquad \forall\, n\geqslant 1.$$

Let C be the Cantor group (i.e., $C=\{0,1\}^{\mathbb{N}}$). The elements of C are denoted by $\xi=\{\xi_n\}_{n\geqslant 1}$ with

$$\xi_n\in\{0,1\} \qquad \forall\, n\geqslant 1.$$

Let

$$\xi^{(n)} \;=\; \big\{\xi_k^{(n)}\big\}_{k\geqslant 1}$$

be the element of C, defined by

$$\xi_k^{(n)} \;\stackrel{df}{=}\; \begin{cases} 0 & \text{if} \quad k\neq n, \\ 1 & \text{if} \quad k=n. \end{cases}$$

Consider the map $\vartheta\colon C\longrightarrow G$, defined by

$$\vartheta(\xi) \;\stackrel{df}{=}\; \sum_{n=1}^{\infty}\xi_n x_n.$$

Note that

$$\vartheta\big(\xi^{(n)}\big) \;=\; x_n$$

and because d_G is complete and invariant, we have that ϑ is continuous.

If we consider the Haar probability measure h on C and we consider the image measure

$$h\circ\vartheta^{-1} \;=\; \mu \quad \text{on } G,$$

then because $\widehat{A}$ is not Haar-null, μ does not vanish on some translate $\widehat{A}$; that is we can find $y\in G$, such that

$$\vartheta^{-1}\big(\widehat{A}+y\big) \text{ has positive } h\text{-measure.}$$

So we must have that

$$\vartheta^{-1}\big(\widehat{A}+y\big)-\vartheta^{-1}\big(\widehat{A}+y\big) \text{ is a neighbourhood of } 0\in C.$$

This means that for all $n\geqslant 1$ large enough,

$$\xi^{(n)} \;\in\; \vartheta^{-1}\big(\widehat{A}+y\big)-\vartheta^{-1}\big(\widehat{A}+y\big).$$

Thus there are $\sigma,\tau\in\vartheta^{-1}\big(\widehat{A}+y\big)$ which differ only in the n-th coordinate and $x_n\in\widehat{A}-\widehat{A}$ for all these n. But this contradicts the fact that

$$\big(\widehat{A}+x_n\big)\cap\widehat{A} \;=\; \emptyset \qquad \forall\, n\geqslant 1.$$

□

REMARK 4.3.13 In the above proof, we have shown something stronger. Namely that $S(A)$ is an open set containing e. In fact this result can be generalized as follows: "If A, B are two Borel sets in G and

$$S(A,B) \stackrel{df}{=} \big\{x \in G : \ (A+x) \cap B \text{ is not Haar-null}\big\},$$

then $S(A,B)$ is an open (possibly empty) subset of G" (see Christensen (1974, p. 118)).

COROLLARY 4.3.14
(a) *If G is an Abelian, Polish topological group which is not locally compact, then every compact subset of G is Haar-null.*

(b) *If $G = X$ is a nonreflexive, separable Banach space, then every weak compact subset of X is Haar-null.*

Using the notion of the Haar-null sets we can extend Rademacher's theorem to Lipschitz continuous functions between certain Banach spaces.

LEMMA 4.3.15
If X and Y are two Banach spaces, $U \subseteq X$ is an open set, $f : U \longrightarrow Y$ is a Lipschitz continuous function, $G \subseteq X$ is a dense additive subgroup and for some $x_0 \in U$ and all $h \in G$,

$$\lim_{\lambda \to 0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda} \ = \ f'(x_0; h)$$

exists and $f'(x_0; \cdot)$ is additive, then f is Gâteaux differentiable at x_0.

PROOF Consider the following family of functions:

$$u_\lambda(h) \stackrel{df}{=} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda} \qquad \forall\ \lambda \neq 0,\ h \in X.$$

The family $\{u_\lambda\}_{\lambda \neq 0}$ is equicontinuous (since f is Lipschitz continuous on U) and since $\lim_{\lambda \to 0} u_\lambda(h)$ exists for all $h \in G$, it also exists for all $h \in \overline{G} = X$. Moreover, from the additivity of $f'(x_0; \cdot)$ on G follows the additivity on $\overline{G} = X$. In addition note that

$$\lim_{\lambda \to 0} u_\lambda(th) \ = \ t \lim_{\lambda \to 0} u_\lambda(h) \qquad \forall\ t \in \mathbb{R}.$$

So $f'(x_0; \cdot)$ is a linear operator which is bounded by the Lipschitz constant of f, i.e., $f'(x_0; \cdot) \in \mathcal{L}(X; Y)$. Therefore f is Gâteaux differentiable at x_0. ∎

PROPOSITION 4.3.16
If $U \subseteq \mathbb{R}^N$ is an open set, Y is a Banach space with the RNP and $f\colon U \longrightarrow Y$ is a Lipschitz continuous function,
then f is Gâteaux differentiable almost everywhere on U (on U we consider the N-dimensional Lebesgue measure).

PROOF Without any loss of generality, we may assume that $U = \mathbb{R}^N$. Let $\{x_n\}_{n\geqslant 1} \subseteq \mathbb{R}^N$ be dense and let

$$G \stackrel{df}{=} \operatorname{span}_{\mathbb{Q}} \{x_n\}_{n\geqslant 1}$$

(i.e., the linear combinations of the x_n's with rational coefficients). Clearly G is a countable dense additive subgroup of $\mathbb{R}^N$. From Theorem 2.2.17, we know that the directional derivatives

$$f'(x;h) \;=\; \lim_{\lambda\to 0} \frac{f(x+\lambda h)-f(x)}{\lambda}$$

exist in all directions $h \in G$ for almost all $x \in \mathbb{R}^N$. Then in the light of Lemma 4.3.15, if we can show that these directional derivatives are additive on G, then we will have the desired almost everywhere Gâteaux differentiability. To this end let $\varphi \in C_c^1\big(\mathbb{R}^N\big)$ be such that

$$\int\limits_{\mathbb{R}^N} \varphi(x)\,dx \;=\; 1.$$

For example a standard function with these properties is the function

$$\varphi(x) \;=\; \begin{cases} c\exp\left(\dfrac{1}{\|x\|_{\mathbb{R}^N}^2-1}\right) & \text{if} \quad \|x\|_{\mathbb{R}^N} \leqslant 1, \\ 0 & \text{if} \quad \|x\|_{\mathbb{R}^N} > 1, \end{cases}$$

with $c \in \mathbb{R}$ chosen so that we have the normalization condition

$$\int\limits_{\mathbb{R}^N} \varphi(x)\,dx \;=\; 1.$$

Let

$$g(x) \;\stackrel{df}{=}\; (f\star\varphi)(x) \;=\; \int\limits_{\mathbb{R}^N} f(y)\varphi(x-y)\,dy.$$

We know that

$$g \in C^1\big(\mathbb{R}^N;Y\big)$$

and so we have that

$$g'_G(x)h \;=\; \big(f\star\varphi'_G(x)\big)h$$

is linear in $h \in \mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

Since

$$\int_{\mathbb{R}^N} f(y)\varphi(x-y)\,dy \;=\; \int_{\mathbb{R}^N} f(x-y)\varphi(y)\,dy \qquad \forall\, x \in \mathbb{R}^N,\ h \in G,$$

using the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have

$$\begin{aligned} g'_G(x)h &= \lim_{\lambda \to 0} \frac{g(x+\lambda h) - g(x)}{\lambda} \\ &= \int_{\mathbb{R}^N} \varphi(y) \lim_{\lambda \to 0} \frac{f(x+\lambda h - y) - f(x-y)}{\lambda}\,dy \end{aligned}$$

and it follows that

$$g'_G(x)h \;=\; \big(\varphi \star \xi_h\big)(x),$$

where

$$\xi_h(x) \;\stackrel{df}{=}\; \lim_{\lambda \to 0} \frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda}$$

is a bounded measurable function. Then we have

$$\varphi \star \big(\xi_{h_1+h_2} - \xi_{h_1} - \xi_{h_2}\big) \;=\; 0 \qquad \forall\, h_1, h_2 \in G.$$

The same is true if φ is replaced by

$$\varphi_m(x) \;=\; m^N \varphi(mx).$$

Recall that for every bounded, measurable function

$$\widehat{g}\colon \mathbb{R}^N \longrightarrow Y,$$

we have that

$$\big(\varphi_m \star \widehat{g}\big)(x) \;\longrightarrow\; \widehat{g}(x) \quad \text{for a.a. } x \in \mathbb{R}^N$$

(see Proposition 2.4.12**(c)**; the result there is stated for $\mathbb{R}$-valued functions, but it can be extended to Y-valued functions by scalarization using elements of Y^* and recalling that by virtue of Theorem 2.1.3, we may assume without any loss of generality that Y is separable). So in the limit, we obtain that

$$\xi_{h_1+h_2}(x) \;=\; \xi_{h_1}(x) + \xi_{h_2}(x) \quad \text{for a.a. } x \in \mathbb{R}^N \text{ and all } h_1, h_2 \in G.$$

Because G is countable, the exceptional Lebesgue-null set is independent of $h_1, h_2 \in G$. □

Now we are ready for the infinite dimensional generalization of Rademacher's theorem (see Theorem 1.5.8 and Corollary 1.5.9).

THEOREM 4.3.17
If X is a separable Banach space, $U \subseteq X$ is an open set, Y is a Banach space with the RNP and $f : U \longrightarrow Y$ is a Lipschitz continuous function, then f is a Gâteaux differentiable function on a set D_f with $X \setminus D_f$ being Haar-null in X.

PROOF Without any loss of generality we may assume that $U = X$.

First we show that the set D_f of points of Gâteaux differentiability of f is a Borel subset of X. Indeed let $\{x_n\}_{n\geqslant 1}$ be dense in X and set

$$G \stackrel{df}{=} \operatorname{span}_{\mathbb{Q}} \{x_n\}_{n\geqslant 1}.$$

Then by virtue of Lemma 4.3.15, we have

$$D_f = \left\{ x \in X : \left| \frac{f(x+\lambda h) - f(x)}{\lambda} - \frac{f(x+rh) - f(x)}{r} \right| < \frac{1}{m}, \right.$$
$$\left. \lambda, r \in \mathbb{Q},\ |\lambda|, |r| \leqslant \frac{1}{n},\ h \in G,\ m, n \geqslant 1 \right\}.$$

So we see that $D_f \subseteq X$ is a Borel set.

Now let $\{y_n\}_{n\geqslant 1}$ be a sequence of linearly independent vectors in X, such that

$$\overline{\operatorname{span}}\, \{y_n\}_{n\geqslant 1} = X.$$

Let

$$V_m \stackrel{df}{=} \operatorname{span} \{y_n\}_{n=1}^{m}.$$

Then

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_m \subseteq \ldots \subseteq X$$

and

$$\overline{\bigcup_{m=1}^{\infty} V_m} = X.$$

Using Proposition 4.3.16, we can find $D_n \subseteq V_n$, such that f is Gâteaux differentiable on D_n and $V_n \setminus D_n$ is Lebesgue-null. By virtue of Lemma 4.3.15,

$$D_f = \bigcap_{n=1}^{\infty} D_n.$$

So D_f is a Haar-null set in X. ∎

COROLLARY 4.3.18
If X is a separable Banach space, Y is a Banach space with the RNP and $f : X \longrightarrow Y$ is a locally Lipschitz continuous function, then f is a Gâteaux differentiable function on a set D_f with $X \setminus D_f$ being Haar-null in X.

4.4 Duality and Subdifferentials

A basic theme that runs through the whole theory of convex analysis is that of "duality." Namely almost every mathematical notion is paired with another one, which is in some sense dual to it. So convex cones are associated to their polars (generalizing this way the pairing of tangent and normal spaces in differential geometry), closed convex sets are associated to their support functions (a pairing which permits an interchange between geometric and analytical reasoning), minimization problems in linear programming are associated to maximization problems, known as the dual problems, which provide valuable information about the solvability and the value of the original problem. In general duality permits us to establish close relations between otherwise disparate properties. The starting point of all these correspondence is a deep duality principle between certain pairs of convex functions (known as conjugate functions), which we study in the first half of this section.

The mathematical framework of our analysis is a Hausdorff locally convex vector space X and its dual X^* (i.e., the set of continuous, linear functionals on X). Additional hypotheses will be introduced as needed. We supply X with the $w(X, X^*)$-topology and X^* with the $w(X^*, X)$-topology. So (X^*, X) is a dual pair (or dual system) and we denote by $\langle \cdot, \cdot \rangle_X$ their pairing.

DEFINITION 4.4.1 *Let $\varphi\colon X \longrightarrow \mathbb{R}^* \stackrel{df}{=} \mathbb{R} \cup \{\pm\infty\}$. Then **Legendre-Fenchel transform** (or the **conjugate**) of φ is the function $\varphi^*\colon X^* \longrightarrow \mathbb{R}^*$, defined by*

$$\varphi^*(x^*) \stackrel{df}{=} \sup_{x \in X} \big[\langle x^*, x \rangle_X - \varphi(x) \big].$$

The function $(\varphi^*)^* = \varphi^{**}\colon X \longrightarrow \mathbb{R}^*$, defined by

$$\varphi^{**}(x) \stackrel{df}{=} \sup_{x^* \in X^*} \big[\langle x^*, x \rangle_X - \varphi^*(x^*) \big],$$

*is the **second conjugate** (or **biconjugate**) of φ.*

REMARK 4.4.2 If φ takes the value $-\infty$, then $\varphi^* \equiv +\infty$. Also if the effective domain $\operatorname{dom}\varphi$ is empty, then $\varphi^* \equiv +\infty$. For this reason of interest is the case where $\varphi\colon X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ and $\operatorname{dom}\varphi \neq \emptyset$ (i.e., φ is a proper function; see Definition 4.2.1). In this case $\varphi^*\colon X \longrightarrow \overline{\mathbb{R}}$ and it is proper too. The significance of φ^* is better understood using epigraphs (see Definition 4.2.1). So we have

$$(x^*, \mu) \in \operatorname{epi}\varphi^* \iff \Big[\langle x^*, x \rangle_X - \lambda \leqslant \mu \qquad \forall\ (x, \lambda) \in \operatorname{epi}\varphi \Big].$$

If we write the last inequality as

$$\langle x^*, x\rangle_X - \mu \leqslant \lambda \qquad \forall\ (x,\lambda) \in \operatorname{epi}\varphi,$$

we see that

$$(x^*,\mu) \in \operatorname{epi}\varphi^* \iff \Big[\langle x^*, x\rangle_X - \mu \ \leqslant\ \varphi(x) \qquad \forall\ x \in X\Big].$$

So

$$l_{(x^*,\mu)}(x) \stackrel{df}{=} \langle x^*, x\rangle_X - \mu$$

is a continuous affine minorant of φ. Therefore φ^* is proper if and only if φ admits a continuous affine minorant. Moreover, φ^* describes the family of all continuous affine minorants of φ. On the other hand also note that

$$\varphi^*(x^*) \leqslant \mu \iff \Big[l_{(x,\lambda)}(x^*) \ =\ \langle x^*, x\rangle_X - \lambda \ \leqslant\ \mu \qquad \forall\ (x,\lambda) \in \operatorname{epi}\varphi\Big].$$

So we see that φ^* is the pointwise supremum of all continuous affine functions $\{l_{(x,\lambda)}\}_{(x,\lambda)\in\operatorname{epi}\varphi}$. Therefore φ^{**} is the pointwise supremum of all continuous affine functions majorized by φ.

From these observations and recalling that the supremum of continuous affine functions on X is convex and lower semicontinuous, we obtain the following result.

PROPOSITION 4.4.3
If $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ *is a proper function,*
then $\varphi^* \in \Gamma_0(X^*)$.

Also directly from the definition of φ^*, we obtain the following two results.

PROPOSITION 4.4.4 (Young-Fenchel Inequality)
If $\varphi\colon X \longrightarrow \mathbb{R}^*$ *is a function,*
then

$$\varphi(x) + \varphi^*(x^*) \ \geqslant\ \langle x^*, x\rangle_X \qquad \forall\ x \in X,\ x^* \in X^*.$$

PROPOSITION 4.4.5
If $\varphi, \psi\colon \longrightarrow \mathbb{R}^*$ *and*

$$\varphi(x) \ \leqslant\ \psi(x) \qquad \forall\ x \in X,$$

then

$$\varphi^*(x^*) \ \geqslant\ \psi^*(x^*) \qquad \forall\ x^* \in X^*.$$

DEFINITION 4.4.6 **(a)** *Let $C \subseteq X$. The* ***support function*** *of the set C is the function*

$$\sigma_C : X^* \longrightarrow \mathbb{R}^*,$$

defined by

$$\sigma_C(x^*) \stackrel{df}{=} \sup_{c \in C} \langle x^*, c \rangle_X$$

(recall that $\sup_\emptyset = -\infty$). If $C \neq \emptyset$, then σ_C takes values in $\overline{\mathbb{R}}$.

(b) *The infimal convolution of functions $\varphi, \psi : X \longrightarrow \overline{\mathbb{R}}$ is the function*

$$\varphi \oplus \psi : X \longrightarrow \mathbb{R}^*,$$

defined by

$$\big(\varphi \oplus \psi\big)(x) \stackrel{df}{=} \inf_{y \in X} \big(\varphi(y) + \psi(x-y)\big) = \inf_{z+y=x} \big(\varphi(z) + \psi(y)\big).$$

We say that $\varphi \oplus \psi$ is ***exact at*** *x, if*

$$\big(\varphi \oplus \psi\big)(x) = \min_{y \in X} \big(\varphi(y) + \psi(x-y)\big)$$

(i.e., the infimum is attained).

We say that $\varphi \oplus \psi$ is ***exact****, if it is exact at every x.*

REMARK 4.4.7 Evidently if $C \subseteq X$ is nonempty, then $\sigma_C \in \Gamma_0(X^*)$ and $\sigma_C(0) = 0$. In fact σ_C is sublinear (i.e., subadditive and positively homogeneous). Moreover,

$$\sigma_C = \big(i_C\big)^*,$$

where i_C is the indicator function of the set $C \subseteq X$, i.e.,

$$i_C(x) \stackrel{df}{=} \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise} \end{cases}$$

(see Remark 4.2.2). If

$$\operatorname{sepi} \varphi \stackrel{df}{=} \big\{(x, \lambda) \in X \times \mathbb{R} : \ \varphi(x) < \lambda\big\}$$

(the ***strict epigraph*** of φ), then it is easy to check that

$$\operatorname{sepi}\big(\varphi \oplus \psi\big) = \operatorname{sepi} \varphi + \operatorname{sepi} \psi.$$

Also since

$$\big(\varphi \oplus \psi\big)(x) = \inf_{\substack{(z,\lambda) \in \operatorname{epi} \varphi \\ (y,\mu) \in \operatorname{epi} \varphi \\ z+y=x}} (\lambda + \mu) = \inf_{(x,\lambda) \in (\operatorname{epi} \varphi + \operatorname{epi} \psi)} \lambda,$$

we see that the infimal convolution of proper, convex functions φ and ψ is convex but not necessarily proper. For example, if $\varphi = i_C$ and $\psi = i_D$ and $C, D \subseteq X$ are two nonempty, convex, disjoint sets, then

$$i_C + i_D \equiv +\infty.$$

On the other hand, if φ and ψ are linear functionals which are not identical, then

$$\varphi \oplus \psi = -\infty.$$

In addition note that, if X is a normed space and $C \subseteq X$ is a nonempty set, then for all $x \in X$, we have

$$\begin{aligned} d_X(x, C) &= \inf_{c \in C} \|x - c\|_X \\ &= \inf_{y \in X} \big(\|x - y\|_X + i_C(y) \big) \\ &= \big(\|\cdot\|_X + i_C \big)(x). \end{aligned}$$

Hence for all nonempty, convex sets $C \subseteq X$, the distance function $d_X(\cdot, C)$ is convex. Moreover, it is easy to see that

$$\big| d_X(x, C) - d_X(y, C) \big| \leqslant \|x - y\|_X,$$

i.e., $d_X(\cdot, C)$ is nonexpansive. Finally for any index set I, we have

$$\left(\inf_{i \in I} \varphi_i \right)^* = \sup_{i \in I} \varphi_i^*.$$

PROPOSITION 4.4.8
If $\varphi, \psi \colon X \longrightarrow \overline{\mathbb{R}}$ are proper, convex functions, then

$$\big(\varphi \oplus \psi\big)^* = \varphi^* + \psi^*.$$

PROOF According to Definitions 4.4.1 and 4.4.6**(b)**, we have

$$\begin{aligned} \big(\varphi \oplus \psi\big)^*(x^*) &= \sup_{x \in X} \big(\langle x^*, x \rangle_X - \big(\varphi \oplus \psi\big)(x) \big) \\ &= \sup_{x \in X} \left(\langle x^*, x \rangle_X - \inf_{y \in X} \big(\varphi(y) + \psi(x - y) \big) \right) \\ &= \sup_{y, z \in X} \big(\langle x^*, y \rangle_X - \varphi(y) + \langle x^*, z \rangle_X - \psi(z) \big) \\ &= \varphi^*(x^*) + \psi^*(x^*). \end{aligned}$$

∎

PROPOSITION 4.4.9

If X and Y are two Hausdorff, locally convex spaces, $A \in \mathcal{L}(X;Y)$ is an isomorphism, $g\colon Y \longrightarrow \overline{\mathbb{R}}$ is a proper function and for $y_0 \in Y$, $x_0^* \in X^*$, $\xi_0 \in \mathbb{R}$ and $\lambda_0 > 0$, we set

$$\varphi(x) \stackrel{df}{=} \lambda_0 g(Ax + y_0) + \langle x_0^*, x\rangle_X + \xi_0 \qquad \forall\, x \in X,$$

then

$$\varphi^*(x^*) = \lambda_0 g^*\left(\frac{1}{\lambda_0}(A^{-1})^*(x^* - x_0^*)\right) - \left\langle x^* - x_0^*, A^{-1}y_0\right\rangle_X - \xi_0 \quad \forall x^* \in X^*.$$

PROOF We have

$$\begin{aligned}
\varphi^*(x^*) &= \sup_{x\in X}\left\{\langle x^*, x\rangle_X - \lambda_0 g(Ax + y_0) - \langle x_0^*, x\rangle_X - \xi_0\right\} \\
&= \lambda_0 \sup_{y\in Y}\left\{\frac{1}{\lambda_0}\left\langle (A^{-1})^*(x^* - x_0^*), y\right\rangle_Y - g(y)\right\} \\
&\quad - \left\langle x^* - x_0^*, A^{-1}y_0\right\rangle_X - \xi_0 \\
&= \lambda_0 g^*\left(\frac{1}{\lambda_0}(A^{-1})^*(x^* - x_0^*)\right) - \left\langle x^* - x_0^*, A^{-1}y_0\right\rangle_X - \xi_0.
\end{aligned}$$

□

COROLLARY 4.4.10

If $g\colon X \longrightarrow \overline{\mathbb{R}}$ a is proper function and $x_0 \in X$, $x_0^* \in X^*$, $\lambda_0 > 0$, $\vartheta_0 \in \mathbb{R}$, then

(a) *for* $\varphi(x) = g(x + x_0)$, $\varphi^*(x^*) = g^*(x^*) - \langle x^*, x_0\rangle_X$;
(b) *for* $\varphi(x) = g(x) + \langle x_0^*, x\rangle_X$, $\varphi^*(x^*) = g^*(x^* - x_0^*)$;
(c) *for* $\varphi(x) = \lambda_0 g(x)$, $\varphi^*(x^*) = \lambda_0 g^*\left(\frac{1}{\lambda_0}x^*\right)$;
(d) *for* $\varphi(x) = \lambda_0 g\left(\frac{1}{\lambda_0}x\right)$, $\varphi^*(x^*) = \lambda_0 g^*(x^*)$;
(e) *for* $\varphi(x) = g(\lambda_0 x)$, $\varphi^*(x^*) = g^*\left(\frac{1}{\lambda_0}x^*\right)$;
(f) *for* $\varphi(x) = \lambda_0 g(\frac{1}{\lambda_0}x + x_0) + \vartheta_0$, $\varphi^*(x^*) = \lambda_0 g^*(x^*) - \lambda_0\langle x^*, x_0\rangle_X - \vartheta_0$.

Let us give some examples of conjugate functions.

EXAMPLE 4.4.11 **(a)** Let X be a normed space and

$$C = \overline{B}_1 = \left\{x \in X : \|x\|_X \leqslant 1\right\}.$$

Then

$$\sigma_C(x^*) = i_C^*(x^*) = \sup_{c\in B_1}\langle x^*, c\rangle_X = \|x^*\|_{X^*}.$$

(b) If $K \subseteq X$ is a cone (i.e., $\lambda K \subseteq K$ for all $\lambda \geqslant 0$), then

$$\sigma_K = i_{-K^*},$$

where K^* is the dual cone, i.e.,

$$K^* \stackrel{df}{=} \{x^* \in X^* : \langle x^*, x\rangle_X \geqslant 0 \text{ for all } x \in K\}.$$

If K is a linear subspace of X, then

$$K^* = K^\perp \stackrel{df}{=} \{x^* \in X^* : \langle x^*, x\rangle_X = 0 \text{ for all } x \in K\}.$$

(c) Let X be a normed space and

$$\varphi(x) = \|x\|_X .$$

Then we have

$$\varphi^*(x^*) = \sup_{x\in X} [\langle x^*, x\rangle_X - \|x\|_X].$$

If $\|x^*\|_{X^*} \leqslant 1$, we have

$$\langle x^*, x\rangle_X \leqslant \|x\|_X$$

and so $\varphi^*(x^*) = 0$. On the other hand if $\|x^*\|_{X^*} > 1$, we can find $x \in X$, such that

$$\|x\|_X < \langle x^*, x\rangle_X$$

and so

$$\langle x^*, \lambda x\rangle_X - \|\lambda x\|_X = \lambda[\langle x^*, x\rangle_X - \|x\|_X] > 0 \qquad \forall\, \lambda > 0.$$

Therefore

$$\varphi^*(x^*) = +\infty.$$

So we conclude that

$$\varphi^* = i_{\overline{B}^*},$$

where $\overline{B}^* = \{x^* \in X^* : \|x^*\|_{X^*} \leqslant 1\}$.

(d) Let X be a normed space, $C \subseteq X$ nonempty set and

$$\varphi(x) \stackrel{df}{=} d_x(x, C) \qquad \forall\, x \in X.$$

Then from Remark 4.4.7, we know that

$$\varphi = \|\cdot\|_X \oplus i_C .$$

Then by virtue of Proposition 4.4.8, we have that

$$\varphi^* = \|\cdot\|_X^* + i_C^* = i_{\overline{B}^*} + \sigma_C$$

(see **(c)** and **(a)** above).

(e) If

$$\varphi(x) \stackrel{df}{=} \langle x_0^*, x\rangle_X + \xi_0 \qquad \forall\, x \in X,$$

with $x_0^* \in X$ and $\xi_0 \in \mathbb{R}$ (i.e., φ is a continuous, affine functions), then

$$\varphi^*(x^*) = \sup_{x\in X}\left[\langle x^* - x_0^*, x\rangle_X - \xi_0\right] = \begin{cases} -\xi_0 & \text{if} \quad x^* = x_0^*, \\ +\infty & \text{if} \quad x^* \neq x_0^*. \end{cases}$$

(f) If $\varphi\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is defined by

$$\varphi(x) \stackrel{df}{=} \frac{1}{p}\,\|x\|_{\mathbb{R}^N}^p\,,$$

with $p \in (1, +\infty)$, then

$$\varphi^*(x^*) = \frac{1}{p'}\,\|x^*\|_{\mathbb{R}^N}^{p'}\,,$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Indeed, let

$$g_{x^*}(x) \stackrel{df}{=} (x^*, x)_{\mathbb{R}^N} - \frac{1}{p}\,\|x\|_{\mathbb{R}^N}^p\,.$$

Then g_{x^*} is concave and

$$g'_{x^*}(x) = x^* - \|x\|_{\mathbb{R}^N}^{p-2}\,x.$$

We have that $g'_{x^*}(\widehat{x}) = 0$ at the unique point $\widehat{x}$, such that

$$(x^*, \widehat{x})_{\mathbb{R}^N} = \left\|\widehat{x}\right\|_{\mathbb{R}^N}^p = \left\|x^*\right\|_{\mathbb{R}^N}^{\frac{p}{p-1}}.$$

So if $p' = \frac{p}{p-1}$ and since

$$\varphi^*(x^*) = \sup_{x\in\mathbb{R}^N} g_{x^*}(x),$$

we have that

$$\varphi^*(x^*) = \frac{1}{p'}\,\|x^*\|_{\mathbb{R}^N}^{p'}\,.$$

More generally, let X be a normed space and let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be an even, convex function. If

$$\varphi(x) \stackrel{df}{=} g\big(\,\|x\|_X\,\big) \qquad \forall\, x \in X,$$

then

$$\varphi^*(x^*) = g^*\big(\,\|x^*\|_{X^*}\,\big) \qquad \forall\, x^* \in X^*.$$

PROPOSITION 4.4.12
If $\varphi: X \longrightarrow \overline{\mathbb{R}}$ is a convex and lower semicontinuous function, then φ admits a continuous affine minorant, i.e.,

$$\langle x_0^*, x\rangle_X - \xi_0 \leqslant \varphi(x) \qquad \forall\, x \in X,$$

for some $(x_0, \xi_0) \in X^* \times \mathbb{R}$.

PROOF Clearly we may assume that φ is proper, i.e.,

$$\varphi \in \Gamma_0(X).$$

Let $x_0 \in X$ and $\eta \in \mathbb{R}$ be such that

$$\eta < \varphi(x_0).$$

Then

$$(x_0, \eta) \notin \operatorname{epi}\varphi$$

and so by the strong separation theorem (see Theorem A.3.2), we can find $(x_0^*, \vartheta_0) \in X^* \times \mathbb{R}$, $(x_0^*, \vartheta_0) \neq (0,0)$ and $\xi \in \mathbb{R}$, such that

$$\langle x_0^*, x\rangle_X + \vartheta_0\lambda < \xi < \langle x_0^*, x_0\rangle_X + \vartheta_0\eta \qquad \forall\, (x,\lambda) \in \operatorname{epi}\varphi.$$

Let

$$(x, \lambda) = \big(x, \varphi(x)\big).$$

We have

$$\langle x_0^*, x\rangle_X + \vartheta_0\varphi(x) < \xi < \langle x_0^*, x_0\rangle_X + \vartheta_0\eta, \tag{4.21}$$

so $\vartheta_0 < 0$.

Without any loss of generality, we may assume that $\vartheta_0 = -1$. Then from (4.21), we have

$$\langle x_0^*, x\rangle_X - \varphi(x) < \xi_0 \qquad \forall\, x \in X,$$

so

$$\langle x_0^*, x\rangle_X - \xi_0 < \varphi(x) \qquad \forall\, x \in X.$$

□

PROPOSITION 4.4.13
For any function $\varphi: X \longrightarrow \mathbb{R}^*$, we have

$$\varphi^{**} \leqslant \varphi.$$

PROOF From the Young-Fenchel inequality (see Proposition 4.4.4), we have

$$\varphi^{**}(x) = \sup_{x^* \in X^*} \big[\langle x^*, x\rangle_X - \varphi^*(x^*)\big] \leqslant \varphi(x) \qquad \forall\, x \in X.$$

□

The next theorem is very important and determines when we have equality in Proposition 4.4.13.

THEOREM 4.4.14
If $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ *is a function,*
then $\varphi^{**} = \varphi$ *if and only if* φ *is convex and lower semicontinuous.*

PROOF "$\Longrightarrow$": Follows from Remark 4.4.2.

"$\Longleftarrow$": If $\varphi \equiv +\infty$, then

$$\varphi^* \equiv -\infty$$

and so

$$v = \varphi^{**} \equiv +\infty.$$

Therefore we may assume that φ is proper.

We know that we have $\varphi^{**} \leqslant \varphi$. So we need to show that the opposite inequality also holds. To this end let $x \in X$ and $\mu \in \mathbb{R}$ be such that

$$\mu < \varphi(x).$$

Then

$$(x, \mu) \notin \operatorname{epi} \varphi$$

and so we can apply the strong separation theorem (see Theorem A.3.2) and find $(x^*, \beta) \in X^* \times \mathbb{R}$, $(x^*, \beta) \neq (0, 0)$ and $\delta > 0$, such that

$$\langle x^*, y\rangle_X + \beta\lambda \leqslant \langle x^*, x\rangle_X + \beta\mu - \delta \qquad \forall\, (y, \lambda) \in \operatorname{epi} \varphi.$$

Since λ can increase to $+\infty$, from this inequality it follows that $\beta \leqslant 0$.

First suppose that $\beta < 0$. We have

$$\langle x^*, y\rangle_X + \beta\varphi(y) < \langle x^*, x\rangle_X + \beta\mu \qquad \forall\, y \in X,$$

from which it follows that

$$(-\beta\varphi)^*(x^*) \leqslant \langle x^*, x\rangle_X + \beta\mu.$$

Using Corollary 4.4.10**(c)**, we obtain

$$-\beta\varphi^*\left(\frac{x^*}{-\beta}\right) \leqslant \langle x^*, x\rangle_X + \beta\mu$$

and thus

$$\mu \leqslant \left\langle -\frac{x^*}{\beta}, x\right\rangle_X - \varphi^*\left(-\frac{x^*}{\beta}\right) \leqslant \varphi^{**}(x).$$

Because $\mu < \varphi(x)$ was arbitrary, we infer that

$$\varphi(x) \leqslant \varphi^{**}(x)$$

as desired.

Next assume that $\beta = 0$. We have

$$\langle x^*, y\rangle_X \leqslant \langle x^*, x\rangle_X - \delta \qquad \forall\, y \in \operatorname{dom}\varphi$$

and so, we see that

$$x \notin \operatorname{dom}\varphi \quad \text{and} \quad \varphi(x) = +\infty.$$

It is enough to show that also

$$\varphi^{**}(x) = +\infty.$$

Let $\eta \in \mathbb{R}$ be such that

$$\langle x^*, y\rangle_X < \eta < \langle x^*, x\rangle_X \qquad \forall\, y \in \operatorname{dom}\varphi.$$

As φ is bounded below by an affine function (see Proposition 4.4.12), we have that there exist $y^* \in X^*$ and $\vartheta \in \mathbb{R}$, such that

$$\langle y^*, y\rangle_X - \vartheta \leqslant \varphi(y) \qquad \forall\, y \in X.$$

So for all $\gamma > 0$, we have

$$\langle y^*, y\rangle_X - \vartheta + \gamma\big(\langle x^*, y\rangle_X - \eta\big) \leqslant \varphi(y) \qquad \forall\, y \in X.$$

Then

$$\langle y^* + \gamma x^*, y\rangle_X - \varphi(y) \leqslant \vartheta + \gamma\eta \qquad \forall\, y \in X$$

and so

$$\varphi^*(y^* + \gamma x^*) \leqslant \vartheta + \gamma\eta.$$

Therefore

$$\begin{aligned}\langle y^*, x\rangle_X - \vartheta + \gamma\big(\langle x^*, x\rangle_X - \eta\big) &\leqslant \langle y^* + \gamma x^*, x\rangle_X - \varphi^*(y^* + \gamma x^*)\\ &\leqslant \varphi^{**}(x).\end{aligned}$$

Since $\eta < \langle x^*, x\rangle_X$ and $\gamma > 0$ was arbitrary, we see that the left hand side is arbitrarily large and so

$$\varphi^{**}(x) = +\infty.$$

Thus

$$\varphi(x) \leqslant \varphi^{**}(x).$$

□

COROLLARY 4.4.15
If $C \subseteq X$ is a nonempty, closed and convex set, then $x \in C$ if and only if

$$\langle x^*, x \rangle_X \leqslant \sigma_C(x^*) \qquad \forall\, x^* \in X^*.$$

In Proposition 4.4.8, we saw that addition is the dual operation to infimal convolution. The next proposition shows that under some additional conditions, the converse is also true.

PROPOSITION 4.4.16
If $\varphi, \psi \colon X \longrightarrow \overline{\mathbb{R}}$ are proper, convex functions and there exists a point $x \in \operatorname{dom} \varphi$, such that ψ is continuous at x, then

$$\big(\varphi + \psi\big)^* = \varphi^* \oplus \psi^*.$$

Now we pass to the study of the subdifferential starting with convex subdifferentials. The convex subdifferential characterizes the local behaviour of convex functions, in a way which is analogous to that in which derivatives determine the local behaviour of smooth functions (see Section 4.1). In fact we can develop a subdifferential calculus which to a high degree parallels the differential calculus of smooth functions.

The mathematical setting remains as before. Namely X is a Hausdorff, locally convex vector space, X^* is its topological dual. The spaces X and X^* are supplied with the $w(X, X^*)$ and $w(X^*, X)$ topologies respectively. Let $\varphi \colon X \longrightarrow \overline{\mathbb{R}}$ be a proper, convex function and $x \in \operatorname{dom} \varphi$, $h \in X$. The function

$$u_x(\lambda) \stackrel{df}{=} \frac{\varphi(x + \lambda h) - \varphi(x)}{\lambda}$$

is increasing on $(0, +\infty)$. So we can make the following definition.

DEFINITION 4.4.17 *Let $\varphi \colon X \longrightarrow \overline{\mathbb{R}}$ be a proper, convex function and $x_0 \in \operatorname{dom} \varphi$. The **directional derivative** of φ at x_0 in the direction $h \in X$ is defined by*

$$\varphi'(x_0; h) \stackrel{df}{=} \inf_{\lambda > 0} \frac{\varphi(x_0 + \lambda h) - \varphi(x_0)}{\lambda} = \lim_{\lambda \searrow 0} \frac{\varphi(x_0 + \lambda h) - \varphi(x_0)}{\lambda}.$$

REMARK 4.4.18 Note that $\varphi'(x_0; h) \in \mathbb{R}^*$ and it is easy to see that $\varphi'(x_0; \cdot)$ is sublinear. Moreover, if X is a Banach space and $\varphi'(x_0; \cdot) \in X^*$, then φ is Gâteaux differentiable at x_0 and

$$\varphi'(x_0; \cdot) = \varphi'_G(x_0).$$

DEFINITION 4.4.19 *Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper function and $x_0 \in \operatorname{dom}\varphi$. The **subdifferential** of φ at x_0 is the subset $\partial\varphi(x_0)$ (possibly empty) of X^*, defined by*

$$\partial\varphi(x_0) \stackrel{df}{=} \Big\{x^* \in X^* : \ \langle x^*, y - x_0\rangle \leqslant \varphi(y) - \varphi(x_0) \ \ \textit{for all } y \in X\Big\}.$$

REMARK 4.4.20 From this definition we see that

$$x^* \in \partial\varphi(x_0) \quad \text{if and only if} \quad x_0 \in \operatorname{argmin}(\varphi - x^*),$$

where

$$\operatorname{argmin}\psi \stackrel{df}{=} \big\{x \in X : \ \psi(x) = \inf_X \psi\big\}.$$

The set $\partial\varphi(x)$ is always a closed and convex subset of X^* and it can be empty (consider for example the subdifferential $\partial\varphi(x)$ when $x \notin \operatorname{dom}\varphi$). The domain of the subdifferential multifunction $\partial\varphi$ is the set

$$D(\partial\varphi) \ = \ \big\{x \in X : \partial\varphi(x) \neq \emptyset\big\}.$$

The function φ is said to be subdifferentiable at $x \in X$, if $x \in D(\partial\varphi)$. The elements of $\partial\varphi(x)$ are called ***subgradients*** of φ at x. Using the epigraph of φ we can better understand the geometric meaning of the subdifferential. So φ is subdifferentiable at $x \in X$ and $x^* \in X^*$ is a subgradient of φ at x if and only if the graph of the continuous function

$$y \ \longmapsto \ \langle x^*, y - x\rangle_X + \varphi(x)$$

is a nonvertical supporting hyperplane to the set $\operatorname{epi}\varphi$ at $\big(x, \varphi(x)\big)$, that is the continuous affine function

$$l(y) \stackrel{df}{=} \langle x^*, y - x\rangle_X + \varphi(x)$$

is a minorant of φ which is exact at x, i.e.,

$$l \ \leqslant \ \varphi \quad \text{and} \quad l(x) \ = \ \varphi(x).$$

Since

$$l(x) \ \leqslant \ \varphi^{**}(x) \ \leqslant \ \varphi$$

(see Remark 4.4.2 and Proposition 4.4.13), we infer that

$$\text{if } \partial\varphi(x) \neq \emptyset, \ \text{ then } \varphi(x) = \varphi^{**}(x).$$

Consequently,

$$\text{if } \varphi(x) = \varphi^{**}(x), \ \text{ then } \partial\varphi(x) = \partial\varphi^{**}(x).$$

PROPOSITION 4.4.21
If $\varphi: X \longrightarrow \overline{\mathbb{R}}$ *is a function,*
then

$$x^* \in \partial\varphi(x) \iff \varphi(x) + \varphi^*(x^*) = \langle x^*, x\rangle_X .$$

PROOF "$\Longrightarrow$": From the definition of the subdifferential, we have

$$\langle x^*, y\rangle_X - \varphi(y) \leqslant \langle x^*, x\rangle_X - \varphi(x) \qquad \forall\, y \in X,$$

so

$$\varphi^*(x^*) + \varphi(x) \leqslant \langle x^*, x\rangle_X .$$

Since the opposite inequality is always true (see the Young-Fenchel inequality; Proposition 4.4.4), we conclude that

$$\varphi(x) + \varphi^*(x^*) = \langle x^*, x\rangle_X .$$

"$\Longleftarrow$": We have

$$\langle x^*, x\rangle_X - \varphi(x) = \varphi^*(x^*) \geqslant \langle x^*, y\rangle_X - \varphi(y) \qquad \forall\, y \in X.$$

Therefore

$$\langle x^*, y - x\rangle_X \leqslant \varphi(y) - \varphi(x) \qquad \forall\, y \in X,$$

hence

$$x^* \in \partial\varphi(x).$$

□

COROLLARY 4.4.22
If $\varphi: X \longrightarrow \overline{\mathbb{R}}$ *and* $x^* \in \partial\varphi(x)$,
then $x \in \partial\varphi^*(x^*)$.

PROOF Since $x^* \in \partial\varphi(x)$, we have

$$\varphi^*(x^*) + \varphi(x) = \langle x^*, x\rangle_X$$

(see Proposition 4.4.21). Then since $\varphi^{**} \leqslant \varphi$ (see Proposition 4.4.13), we obtain

$$\varphi^*(x^*) + \varphi^{**}(x) \leqslant \langle x^*, x\rangle_X .$$

A new appeal to Proposition 4.4.21 gives that

$$x \in \partial\varphi^*(x^*).$$

□

COROLLARY 4.4.23
If $\varphi \in \Gamma_0(X)$, *then*

$$x^* \in \partial\varphi(x) \iff x \in \partial\varphi^*(x^*).$$

PROOF Since $\varphi \in \Gamma_0(X)$, we have

$$\varphi = \varphi^{**}$$

(see Theorem 4.4.14). So from Corollary 4.4.22 we conclude the desired equivalence. □

Before continuing with the investigation of the subdifferentials in the context of convex functions, let us give some examples of subdifferentials.

EXAMPLE 4.4.24 **(a)** Let $\varphi\colon \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a proper, convex function and $x \in \operatorname{int}\operatorname{dom}\varphi$. Then it is easily seen that

$$\partial\varphi(x) = \left[f'_-(x), f'_+(x)\right].$$

(b) Let X be a Banach space and $\varphi(x) \stackrel{df}{=} \|x\|_X$. If $x \neq 0$, then

$$\partial\varphi(x) = \left\{x^* \in X^* : \|x^*\|_{X^*} = 1,\ \langle x^*, x\rangle_X = \|x\|_X\right\}.$$

Indeed, let $x^* \in X^*$ be such that

$$\|x^*\|_{X^*} = 1 \quad \text{and} \quad \langle x^*, x\rangle_X = \|x\|_X.$$

Then

$$\langle x^*, y\rangle_X \leqslant \|y\|_X \qquad \forall\ y \in X$$

and so

$$\langle x^*, y - x\rangle_X \leqslant \|y\|_X - \|x\|_X,$$

hence $x^* \in \partial\varphi(x)$.

On the other hand, let $x^* \in \partial\varphi(x)$. Then

$$-\|x\|_X \geqslant -\langle x^*, x\rangle_X$$

and

$$\|x\|_X = 2\|x\|_X - \|x\|_X \geqslant \langle x^*, x\rangle_X,$$

from which we infer that

$$\langle x^*, x\rangle_X = \|x\|_X.$$

Also

$$\langle x^*, \lambda y\rangle_X \leqslant \|x + \lambda y\|_X - \|x\|_X \qquad \forall\ y \in X,\ \lambda > 0,$$

hence

$$\langle x^*, y\rangle_X \;\leqslant\; \left\|\frac{1}{\lambda}x + y\right\|_X - \frac{1}{\lambda}\,\|x\|_X\,.$$

Let $\lambda \to +\infty$, to obtain

$$\langle x^*, y\rangle_X \;\leqslant\; \|y\|_X\,,$$

from which it follows that

$$\|x^*\|_{X^*} \;\leqslant\; 1.$$

But since $\langle x^*, x\rangle_X = \|x\|_X$, we conclude that

$$\|x^*\|_{X^*} \;=\; 1.$$

If $x = 0$, then

$$\partial\varphi(0) \;=\; \overline{B}_1^* \;=\; \big\{x^* \in X^* : \|x^*\|_{X^*} \leqslant 1\big\}.$$

Indeed note that

$$x^* \in \partial\varphi(0) \iff \Big[\langle x^*, x\rangle_X \leqslant \|x\|_X \qquad \forall\; x \in X\Big]$$

and the last inequality is equivalent to saying that $\|x^*\|_{X^*} \leqslant 1$.

(c) Let C be a closed, convex set in X and $\varphi(x) \stackrel{df}{=} i_C(x)$. Then

$$\begin{aligned}\partial\varphi(x) \;=\; N_C(x) &\stackrel{df}{=} \big\{x^* \in X^* :\; \langle x^*, c - x\rangle_X \leqslant 0 \;\text{ for all } c \in C\big\}\\ &= \big\{x^* \in X^* :\; \langle x^*, x\rangle_X = \sigma_C(x^*)\big\}.\end{aligned}$$

The set

$$\partial\varphi(x) \;=\; N_C(x)$$

is a nonempty (because $0 \in \partial\varphi(x) = N_C(x)$), closed and convex cone in X^*, known as the ***normal cone*** to C at x. It generalizes the notion of normal space (see Definition A.1.12**(b)**) in differential geometry.

If $x \notin C$, then

$$\partial\varphi(x) \;=\; N_C(x) \;=\; \emptyset.$$

So

$$D(\partial\varphi) \;=\; C$$

and

$$\partial\varphi(x) \;=\; N_C(x) \;=\; \{0\} \qquad \forall\; x \in \operatorname{int} C.$$

If $C = V$ is a linear subspace of X, then

$$\begin{aligned}\partial\varphi(x) \;=\; N_V(x) \;&=\; V^\perp\\ = \big\{x^* \in X^* :\; \langle x^*, v\rangle_X = 0 \;\text{ for all } v \in V\big\} &\qquad \forall\; x \in V.\end{aligned}$$

For convex functions we have an easy criterion for subdifferentiability at $x \in X$.

PROPOSITION 4.4.25
If X is a Banach space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a convex function which is continuous at $x \in X$,
then $\partial\varphi(x) \neq \emptyset$ and $\partial\varphi(x)$ is w^*-compact and convex in X^*.

PROOF From Theorem 4.2.3, we know that

$$\operatorname{int} \operatorname{epi} \varphi \;\neq\; \emptyset.$$

Since $\big(x, \varphi(x)\big)$ belongs to the boundary of $\operatorname{epi}\varphi$, we can apply the weak separation theorem (see Theorem A.3.1) and find $(x^*, \eta) \in X^* \times \mathbb{R}$, with $(x^*, \eta) \neq (0, 0)$, such that

$$\eta\big(\varphi(x) - \lambda\big) \;\leqslant\; \langle -x^*, x - y\rangle_X \qquad \forall\, (y, \lambda) \in \operatorname{epi}\varphi. \tag{4.22}$$

Since for fixed $y \in \operatorname{dom}\varphi$, λ can increase up to $+\infty$, from (4.22), we infer that $\eta \geqslant 0$. If $\eta = 0$, then

$$\langle -x^*, x - y\rangle_X \;\geqslant\; 0 \qquad \forall\, y \in \operatorname{dom}\varphi.$$

But $x \in \operatorname{int}\operatorname{dom}\varphi$ (see Theorem 4.2.3). So $x^* = 0$, a contradiction. So $\eta > 0$ and we take $\eta = 1$. Then from (4.22) with $\lambda = \varphi(y)$, we have

$$\varphi(x) - \varphi(y) \;\leqslant\; \langle -x^*, x - y\rangle_X\,,$$

so

$$-x^* \;\in\; \partial\varphi(x) \neq \emptyset.$$

From Theorems 4.2.3 and 4.2.7, we know that there exists $r > 0$, such that $\varphi|_{\overline{B}_r(x)}$ is Lipschitz continuous. So we have

$$\langle x^*, u\rangle_X \;\leqslant\; \varphi(x + u) - \varphi(x) \;\leqslant\; k\,\|u\|_X \qquad \forall\, u \in \overline{B}_r(0),$$

for some $k > 0$ and so

$$\|x^*\|_{X^*} \;\leqslant\; k.$$

By Alaoglu's theorem (see Theorem A.3.9) and since $\partial\varphi(x)$ is clearly w^*-closed, we conclude that it is w^*-compact and convex. □

REMARK 4.4.26 The result is actually true in the more general context of dual pairs of locally convex spaces. However, since the material of Section 4.2 was developed in the context of Banach spaces and to avoid introducing additional functional analytic material, we have stated the result in Banach spaces.

In fact for a continuous, convex function φ we can describe the subdifferential completely.

PROPOSITION 4.4.27
If X is a Banach space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a convex function which is continuous at $x \in X$,
then

$$\sigma_{\partial\varphi(x)}(h) \;=\; f'(x;h) \qquad \forall\, h \in X.$$

PROOF Let

$$\psi(h) \stackrel{df}{=} \varphi'(x;h) \qquad \forall\, h \in X.$$

Since φ is continuous at $x \in X$, we have

$$\partial\varphi(x) \;\neq\; \emptyset$$

(see Proposition 4.4.25). So we have

$$\langle x^*, h\rangle_X \;\leqslant\; \psi(h) \;\leqslant\; \varphi(x+h) - \varphi(x) \qquad \forall\, h \in X,\; x^* \in \partial\varphi(x),$$

so ψ is finite everywhere, hence continuous on X.

Also using Proposition 4.4.9, we see that the conjugate of the function

$$\psi_\lambda(h) \stackrel{df}{=} \frac{1}{\lambda}\big[\varphi(x+\lambda h) - \varphi(x)\big] \qquad \forall\, \lambda > 0$$

is the function

$$\psi_\lambda^*(x^*) \;=\; \frac{1}{\lambda}\big[\varphi^*(\lambda x^*) + \varphi(x) - \lambda\,\langle x^*, x\rangle_X\big] \qquad \forall\, \lambda > 0.$$

Since $\psi = \inf\limits_{\lambda>0} \psi_\lambda$, we have that

$$\psi^* \;=\; \sup_{\lambda>0} \psi_\lambda^*$$

(see Remark 4.4.2). Therefore

$$\psi^*(x^*) \;=\; \sup_{\lambda>0} \frac{1}{\lambda}\big[\varphi^*(\lambda x^*) + \varphi(x) - \langle \lambda x^*, x\rangle_X\big].$$

Then by virtue of Propositions 4.4.4 and 4.4.21, we have

$$\psi^*(x^*) \;=\; \begin{cases} 0 & \text{if } x^* \in \partial\varphi(x), \\ +\infty & \text{otherwise,} \end{cases}$$

i.e., $\psi^* = i_{\partial\varphi(x)}$ and so $\psi^{**} = \psi = \sigma_{\partial\varphi(x)}$ (see Theorem 4.4.14). ∎

REMARK 4.4.28 Again the result remains valid in the framework of dual pairs of locally convex spaces.

Next we show that for convex functions the case of Gâteaux differentiability is essentially the same as that of uniqueness of the subgradient.

PROPOSITION 4.4.29
Let X be a Banach space and let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper, convex function.

(a) *If φ is Gâteaux differentiable at x, then*

$$x \in D(\partial\varphi) \quad \text{and} \quad \partial\varphi(x) = \{\varphi'_G(x)\}.$$

(b) *If φ is continuous at x and $\partial\varphi(x)$ is a singleton, then φ is Gâteaux differentiable at x and*

$$\partial\varphi(x) \;=\; \{\varphi'_G(x)\}.$$

PROOF **(a)** Due to the convexity and Gâteaux differentiability of φ at x, we have

$$\begin{aligned} \langle \varphi'_G(x), h\rangle_X &\leqslant \frac{1}{\lambda}\big[\varphi(x+\lambda h) - \varphi(x)\big] \\ &\leqslant \varphi(x+h) - \varphi(x) \qquad \forall\ \lambda \in (0,1),\ h \in X, \end{aligned}$$

so $\varphi'_G(x) \in \partial\varphi(x)$.

Let $x^* \in X^*$ be any element of $\partial\varphi(x)$. We have

$$\langle x^*, h\rangle_X \;\leqslant\; \frac{1}{\lambda}\big[\varphi(x+\lambda h) - \varphi(x)\big] \qquad \forall\ \lambda > 0,\ h \in X,$$

so

$$\langle x^*, h\rangle_X \;\leqslant\; \langle \varphi'_G(x), h\rangle_X \qquad \forall\ h \in X$$

and thus $x^* = \varphi'_G(x)$, i.e.,

$$\partial\varphi(x) \;=\; \{\varphi'_G(x)\}.$$

(b) Since φ is convex, we have

$$\varphi(x) + \lambda\varphi'(x;h) \;\leqslant\; \varphi(x+\lambda h) \qquad \forall\ \lambda \in \mathbb{R},\ h \in X.$$

So the straight line

$$L \;\stackrel{df}{=}\; \big\{\big(x+\lambda h, \varphi(x) + \lambda\varphi'(x;h)\big) :\ \lambda \in \mathbb{R}\big\}$$

does not intersect $\operatorname{int}\operatorname{epi}\varphi \neq \emptyset$ (see Theorem 4.2.3). Then by the weak separation theorem (see Theorem A.3.1), we can find a closed hyperplane H containing line L, such that

$$H \cap \operatorname{int}\operatorname{epi}\varphi \;=\; \emptyset.$$

The hyperplane H is the graph of a continuous affine function l on X, such that

$$l(x) \;=\; \varphi(x).$$

Since by hypothesis

$$\partial\varphi(x) \;=\; \{x^*\},$$

the slope of l is x^* and because $L \subseteq H$, we have

$$\varphi'(x;h) \;=\; \langle x^*, h\rangle_X \qquad \forall\, h \in X.$$

Thus φ is Gâteaux differentiable at x and

$$\partial\varphi(x) \;=\; \{\varphi'_G(x)\}.$$

□

The next proposition explains the central role of the subdifferential in optimization theory. It is a direct consequence of Definition 4.4.19.

PROPOSITION 4.4.30
If $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper function,
then φ attains its minimum at $x \in \operatorname{dom}\varphi$ if and only if $0 \in \partial\varphi(x)$.

Next we will establish some basic rules of the subdifferential calculus. We start with two straightforward observations. Here $\varphi, \psi\colon X \longrightarrow \overline{\mathbb{R}}$ are proper functions. We have

$$\partial(\lambda\varphi)(x) \;=\; \lambda\partial\varphi(x) \qquad \forall\, \lambda > 0,\; x \in X \tag{4.23}$$

and

$$\partial\varphi(x) + \partial\psi(x) \;\subseteq\; \partial(\varphi+\psi)(x) \qquad \forall\, x \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi. \tag{4.24}$$

The next proposition provides a simple situation where equality in (4.24) is realized.

PROPOSITION 4.4.31
If $\varphi, \psi\colon X \longrightarrow \overline{\mathbb{R}}$ are proper, convex functions and there exists $\widehat{x} \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$ where φ is continuous,
then

$$\partial\varphi(x) + \partial\psi(x) \;=\; \partial(\varphi+\psi)(x) \qquad \forall\, x \in X.$$

PROOF Because of (4.24), we need to show that

$$\partial\varphi(x) + \partial\psi(x) \;\supseteq\; \partial(\varphi+\psi)(x) \qquad \forall\, x \in X. \tag{4.25}$$

To this end let $x^* \in \partial(\varphi + \psi)(x)$. Then $x \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$ and

$$\psi(x) - \psi(y) \;\leqslant\; \varphi(y) - \varphi(x) - \langle x^*, y - x\rangle_X \;=\; g(y) \qquad \forall\, y \in X.$$

We introduce the following two sets

$$C_1 \stackrel{df}{=} \operatorname{epi} g \quad \text{and} \quad C_2 \stackrel{df}{=} \big\{(y,\mu) \in X \times \mathbb{R} :\; \mu \leqslant \psi(x) - \psi(y)\big\}.$$

Both sets are convex and by virtue of Theorem 4.2.3, $\operatorname{int} C_1 \neq \emptyset$. Also $\operatorname{int} C_1 \cap C_2 = \emptyset$. Indeed,

$$g(y) \;\leqslant\; \mu \;\leqslant\; \psi(x) - \psi(y) \qquad \forall\, (y,\mu) \in \operatorname{int} C_1 \cap C_2$$

and so $g(y) = \mu$. Because $(y,\mu) \in \operatorname{int} C_1$, we have that

$$(y, \mu - \varepsilon) \in C_1$$

for $\varepsilon > 0$ small and so $g(y) \leqslant \mu - \varepsilon$, a contradiction. Since $\operatorname{int} C_1 \cap C_2 = \emptyset$, we can apply the weak separation theorem (see Theorem A.3.1) and produce $(z^*, \eta) \in X^* \times \mathbb{R}$, $(z^*, \eta) \neq (0,0)$, such that

$$\langle z^*, z\rangle_X + \eta\lambda \;\leqslant\; \langle z^*, y\rangle_X + \eta\mu \qquad \forall\, (z,\lambda) \in C_1,\; (y,\mu) \in C_2 \tag{4.26}$$

and the inequality is strict if $(z,\lambda) \in \operatorname{int} C_1$. Note that $(x,0) \in C_2$. Then from (4.26) and since λ can increase to $+\infty$, we obtain $\eta \leqslant 0$.

If $\eta = 0$, then

$$\langle z^*, z\rangle_X \;\leqslant\; \langle z^*, x\rangle_X \qquad \forall\, z \in \operatorname{dom} g.$$

But since g is continuous at x, $\operatorname{dom} g$ is a neighbourhood of x, hence $z^* = 0$, a contradiction to the fact that $(z^*, \eta) \neq (0,0)$.

So $\eta < 0$ and we may assume that $\eta = -1$. Then from (4.26), we have

$$\langle z^*, z\rangle_X - g(z) \;\leqslant\; \langle z^*, x\rangle_X \;\leqslant\; \langle z^*, y\rangle_X - \big(\psi(x) - \psi(y)\big) \qquad \forall\, z \in \operatorname{dom} g,\; y \in \operatorname{dom}\psi.$$

From the second inequality we have that $-z^* \in \partial\psi(x)$, while from the first we have that $x^* + z^* \in \partial\varphi(x)$. Then

$$x^* \;=\; x^* + z^* + (-z^*) \;\in\; \partial\varphi(x) + \partial\psi(x)$$

and we have proved (4.25). □

REMARK 4.4.32 Of course the result is also true for any family $\{\varphi_i\}_{i=1}^n$ of proper, convex functions on X, such that there exists $x \in \bigcap_{i=1}^n \operatorname{dom}\varphi_i$, where all but one of the functions are continuous.

PROPOSITION 4.4.33
If $A \in \mathcal{L}(X;Y)$ *and* $\varphi\colon Y \longrightarrow \overline{\mathbb{R}}$ *is a proper function, then*

$$A^*\partial\varphi(Ax) \;\subseteq\; \partial(\varphi \circ A)(x) \qquad \forall\, x \in X$$

and equality holds if in addition φ *is convex and continuous at a point in the range of* A.

PROOF The inclusion follows at once from the definitions. Let us prove that equality holds when φ is convex and continuous at the range of A. So let $x^* \in \partial(\varphi \circ A)(x)$. We have

$$\langle x^*, z - x\rangle_X + (\varphi \circ A)(x) \;\leqslant\; (\varphi \circ A)(z) \qquad \forall\, z \in X. \tag{4.27}$$

Let

$$L \stackrel{df}{=} \big\{\big(Az, \langle x^*, z - x\rangle_X + (\varphi \circ A)(x)\big) \in Y \times \mathbb{R} :\; z \in X\big\}.$$

This is an affine subspace of $Y \times \mathbb{R}$ and because of (4.27), L and $\operatorname{epi}\varphi$ have only boundary points in common, that is

$$L \cap \operatorname{int}\operatorname{epi}\varphi \;\neq\; \emptyset$$

(note that by Theorem 4.2.3, $\operatorname{int}\operatorname{epi}\varphi \neq \emptyset$). So we can apply the weak separation theorem (see Theorem A.3.1) and find a close hyperplane H containing L, such that

$$H \cap \operatorname{int}\operatorname{epi}\varphi \;=\; \emptyset.$$

The hyperplane H is the graph of a continuous affine function

$$l(y) \stackrel{df}{=} \langle y^*, y\rangle_Y + \mu \qquad \forall\, y \in Y,$$

with $(y^*, \mu) \in Y^* \times \mathbb{R}$. Since $H \supseteq L$, we have

$$\langle y^*, Az\rangle_Y + \mu \;=\; \langle x^*, z - x\rangle_X + (\varphi \circ A)(x) \qquad \forall\, z \in X,$$

so taking $z = 0$, we have

$$\mu \;=\; (\varphi \circ A)(x) - \langle x^*, x\rangle_X$$

and

$$\langle y^*, Az\rangle_Y \;=\; \langle x^*, z\rangle_X \qquad \forall\, z \in X.$$

From the second equality, we infer that

$$x^* \;=\; A^*y^*.$$

Also since

$$H \cap \operatorname{int}\operatorname{epi}\varphi \;=\; \emptyset,$$

we have

$$\langle y^*, y\rangle_Y + (\varphi \circ A)(x) - \langle A^* y^*, x\rangle_X \;\leqslant\; \varphi(y) \qquad \forall\, y \in Y,$$

so

$$\langle y^*, y - Ax\rangle_Y + (\varphi \circ A)(x) \;\leqslant\; \varphi(y) \qquad \forall\, y \in Y$$

and thus $y^* \in \partial\varphi\big(A(x)\big)$. We infer that

$$\partial(\varphi \circ A)(x) \;\subseteq\; A^* \partial\varphi(Ax) \qquad \forall\, x \in X.$$

Therefore equality must hold. ∎

Next we study the multifunction $\partial\varphi\colon X \longrightarrow 2^{X^*}$. The first result explains the connection between subdifferentials and maximal monotone maps and it generalizes the elementary fact that if $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, convex function, then φ' is increasing.

THEOREM 4.4.34
If X is a reflexive Banach space and $\varphi \in \Gamma_0(X)$, then $\partial\varphi\colon X \longrightarrow 2^{X^*}$ is a maximal monotone map.

PROOF Using Troyanski's renorming theorem (see Theorem A.3.23), we may assume that both X and X^* are locally uniformly convex (see Definition A.3.21). Let

$$\mathcal{F}\colon X \;\longrightarrow\; X^*$$

be the duality map of X. By Proposition 3.2.27, $\mathcal{F}$ is a homeomorphism.

Now, directly from the definition, we see that $\partial\varphi$ is monotone. So by virtue of Theorem 3.2.29 to prove the maximal monotonicity of $\partial\varphi$, it suffices to show that

$$R\big(\partial\varphi + \mathcal{F}\big) \;=\; X^*. \tag{4.28}$$

To this end let $x^* \in X^*$ and consider the function

$$\psi\colon X \;\longrightarrow\; \overline{\mathbb{R}},$$

defined by

$$\psi(x) \;\stackrel{df}{=}\; \frac{1}{2}\,\|x\|_X^2 + \varphi(x) - \langle x^*, x\rangle_X \qquad \forall\, x \in X.$$

Evidently

$$\psi \;\in\; \Gamma_0(X)$$

and

$$\psi(x) \;\longrightarrow\; +\infty \quad \text{as } \|x\|_X \to +\infty$$

(recall that φ is bounded below by a continuous affine function; see Proposition 4.4.12). So by the Weierstrass theorem, we can find

$$x_0 \in \operatorname{dom}\psi,$$

such that

$$\psi(x_0) = \inf_X \psi.$$

Then from Proposition 4.4.30, we have that

$$0 \in \partial\psi(x_0).$$

Using Proposition 4.4.31 (see also Remark 4.4.32), we have

$$\partial\varphi(x_0) = \mathcal{F}(x_0) + \partial\varphi(x_0) - x^*$$

(recall that $\partial\big(\frac{1}{2}\|\cdot\|_X^2\big)(x_0) = \mathcal{F}(x_0)$; see Example 3.2.20**(d)**). Hence

$$0 \in \partial\varphi(x_0) + \mathcal{F}(x_0) - x^*$$

and so

$$x^* \in \partial\varphi(x_0) + \mathcal{F}(x_0).$$

Because $x^* \in X^*$ was arbitrary, we conclude that (4.28) holds and thus $\partial\varphi$ is maximal monotone. □

REMARK 4.4.35 The result is actually true for X being any Banach space. For a proof of the result in this general case we refer to Rockafellar (1970*b*) (see also Phelps (1993, p. 59)).

Now we obtain some additional properties which characterize the subdifferentials within the class of maximal monotone maps.

DEFINITION 4.4.36 *Let X be a Banach space and $A\colon X \longrightarrow 2^{X^*}$. We say that A is n-**cyclically monotone** provided that*

$$\sum_{k=0}^{n} \langle x_k^*, x_k - x_{k+1}\rangle_X \geqslant 0,$$

whenever $n \geqslant 1$ and

$$x_0, x_2, \ldots, x_n \in X, \quad x_{n+1} = x_0$$

and

$$x_k^* \in A(x_k) \qquad \forall\, k \in \{0, 1, \ldots, n\}.$$

*We say that A is **cyclically monotone**, if it is n-cyclically monotone for every $n \geqslant 2$.*

*The map A is **maximal cyclically monotone**, if its graph is not properly included in the graph of a cyclically monotone map.*

REMARK 4.4.37 Clearly a 2-cyclically monotone map is monotone. So every cyclically monotone map is monotone.

PROPOSITION 4.4.38
Every monotone map $f\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is cyclically monotone.

PROOF Let

$$x_1, x_1, \ldots, x_n \ \in \ D(f)$$

and

$$x_k^* \ \in \ f(x_k) \qquad \forall\ k \in \{0, 1, \ldots, n\}.$$

We may assume that $x_k \leqslant x_{k+1}$ for all $k \in \{0, 1, \ldots, n-1\}$. Then $x_k^* \leqslant x_{k+1}^*$ for all $k \in \{0, 1, \ldots, n-1\}$ and we have

$$\begin{aligned} \sum_{k=0}^{n} x_k^*(x_k - x_{k+1}) \ &= \ \sum_{k=0}^{n-1} x_k^*(x_k - x_{k+1}) + x_n^*(x_n - x_0) \\ &= \ \sum_{k=0}^{n-1} (x_k^* - x_0^*)(x_k - x_{k+1}) \ \geqslant \ 0 \end{aligned}$$

(recall that $x_{n+1} = x_0$). ∎

Directly from the definition, we see that if $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, convex function, then $\partial\varphi$ is cyclically monotone. Moreover, if $\varphi \in \Gamma_0(X)$, then by virtue of Theorem 4.4.34 and Remark 4.4.35, we see that $\partial\varphi$ is maximal cyclically monotone. It turns out that subdifferentials are the only maximal cyclically monotone maps.

THEOREM 4.4.39
If X is a Banach space,
then a map $A\colon X \longrightarrow 2^{X^*}$ is maximal cyclically monotone if and only if there exists $\varphi \in \Gamma_0(X)$, such that $A = \partial\varphi$.

PROOF "$\Longrightarrow$": Let us fix $(x_0, x_0^*) \in \operatorname{Gr} A$ and for every $x \in X$ we define

$$\varphi(x) \ \stackrel{df}{=} \ \sup_{\substack{(x_k, x_k^*) \in \operatorname{Gr} A \\ k \in \{1, \ldots, n\} \\ n \geqslant 1}} \ \sum_{k=0}^{n} \left\langle x_k^*, x_{k+1} - x_k \right\rangle_X + \left\langle x_n^*, x - x_n \right\rangle_X.$$

Because φ is the supremum of continuous affine functions, it follows that φ is convex and lower semicontinuous. Moreover, since

$$\sum_{k=0}^{n} \left\langle x_k^*, x_{k+1} - x_k \right\rangle_X \ \leqslant \ 0$$

(due to the cyclical monotonicity of A), it follows that φ is proper, that is $\varphi \in \Gamma_0(X)$. Let $(x, x^*) \in \operatorname{Gr} A$ and $y \in X$. Since in the definition of φ, $n \geqslant 2$ is arbitrary, we have

$$\varphi(y) \;\geqslant\; \sum_{k=0}^{n} \langle x_k^*, x_{k+1} - x_k \rangle_X + \langle x_n^*, x - x_n \rangle_X + \langle x^*, y - x \rangle_X$$

(i.e., we have added the point $(x, x^*) \in \operatorname{Gr} A$ in the definition of φ). Hence we obtain

$$\varphi(y) \;\geqslant\; \varphi(x) + \langle x^*, y - x \rangle_X \qquad \forall\, y \in X,$$

so $x^* \in \partial\varphi(x)$.

Since $(x, x^*) \in \operatorname{Gr} A$ was arbitrary, we infer that $\operatorname{Gr} A \subseteq \operatorname{Gr} \partial\varphi$. Due to the maximality of φ, we conclude that $\operatorname{Gr} A = \operatorname{Gr} \partial\varphi$, hence $A = \partial\varphi$.

"$\Longleftarrow$": See the remark before the statement of the theorem. □

REMARK 4.4.40 In fact it can be shown that $\varphi \in \Gamma_0(X)$ is unique up to an additive constant; see Rockafellar (1970*b*).

COROLLARY 4.4.41
Any maximal monotone map $f\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ *has the form*

$$f(x) \;=\; \big[g'_-(x), g'_+(x)\big],$$

with $g \in \Gamma_0(\mathbb{R})$.

We can use Theorem 4.4.39 to characterize self-adjoint positive operators in a Hilbert space.

PROPOSITION 4.4.42
If H is a Hilbert space and $A\colon H \supseteq D(A) \longrightarrow H$ is a linear maximal monotone operator,
then A is maximal cyclically monotone if and only if A is self-adjoint.

PROOF "$\Longrightarrow$": By virtue of Theorem 4.4.39, we can find $\varphi \in \Gamma_0(H)$, such that $A = \partial\varphi$. Because $A(0) = 0$ and using Remark 4.4.40, we may assume that

$$\varphi(0) \;=\; 0.$$

For $x \in D(A)$, let

$$g(t) \;\stackrel{df}{=}\; \varphi(tx) \qquad \forall\, t \in [0, 1].$$

From Proposition 4.4.33, we have that

$$\partial g(t) \;=\; (\partial\varphi(tx), x)_H\,.$$

Using the definition of subdifferential, we infer that

$$\big|g(t)-g(s)\big| \;\leqslant\; \big(A(x),x\big)_H|t-s| \qquad \forall\, t,s\in[0,1]$$

and so g is differentiable for almost all $t\in[0,1]$ and

$$\frac{d}{dt}g(t) \;=\; t\big(A(x),x\big)_H.$$

Then we have

$$g(1)-g(0) \;=\; \int_0^1 g'(t)\,dt \;=\; \left(\int_0^1 t\,dt\right)\big(A(x),x\big)_H \;=\; \frac{1}{2}\big(A(x),x\big)_H,$$

so

$$\varphi(x) \;=\; \frac{1}{2}\big(A(x),x\big)_H \qquad \forall\, x\in D(A).$$

Then via an easy calculation, we obtain that

$$\big(\partial\varphi(x),y\big)_H \;=\; \frac{1}{2}\left[\big(A(x),y\big)_H+\big(x,A(y)\big)_H\right] \qquad \forall\, x,y\in D(A).$$

Since

$$\big(\partial\varphi(x),y\big)_H \;=\; \big(A(x),y\big)_H,$$

we obtain

$$\big(A(x),y\big)_H \;=\; \big(x,A(y)\big)_H \qquad \forall\, x,y\in D(A)$$

and so $A\subseteq A^*$. But A^* is monotone (see Theorem 3.2.58) and A is maximal. Therefore

$$A \;=\; A^*$$

and we conclude that A is self-adjoint.

"$\Longrightarrow$": Since A is self-adjoint, maximal monotone, there exists a square root of A with the same properties (see Kato (1976, p. 281)). So $A^{\frac{1}{2}}$ is closed (see Theorem 3.2.58) and if we set

$$\varphi(x) \;\stackrel{df}{=}\; \begin{cases} \frac{1}{2}\big\|A^{\frac{1}{2}}x\big\|_H^2 & \text{if } \; x\in D(A^{\frac{1}{2}}) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, x\in H,$$

then $\varphi\in\Gamma_0(H)$. Since

$$\big(A(x),y\big)_H \;=\; \big(A^{\frac{1}{2}}(x),A^{\frac{1}{2}}(y)\big)_H \qquad \forall\,(x,y)\in D(A)\times D\big(A^{\frac{1}{2}}\big),$$

we have

$$\big(A(x),y-x\big)_H \;\leqslant\; \frac{1}{2}\big\|A^{\frac{1}{2}}(y)\big\|_H^2-\frac{1}{2}\big\|A^{\frac{1}{2}}(x)\big\|_H^2 \qquad \forall\,(x,y)\in D(A)\times D\big(A^{\frac{1}{2}}\big),$$

so $A(x)\in\partial\varphi(x)$, i.e., $A\subseteq\partial\varphi$.

Because A is maximal, we conclude that

$$A \;=\; \partial\varphi.$$

□

Using Proposition 3.2.14, we obtain the following result.

PROPOSITION 4.4.43
If X is a reflexive Banach space and $\varphi: X \longrightarrow \mathbb{R}$ is a continuous, convex function,
then

$$\partial\varphi: X \longrightarrow 2^{X^*} \setminus \{\emptyset\}$$

is upper semicontinuous from X with norm topology into X^* with the weak topology.

REMARK 4.4.44 The result is true if X is any Banach space. In this case X^* is supplied with the w^*-topology. The proof remains the same.

DEFINITION 4.4.45 *Let Y and Z be Hausdorff topological spaces and let $S: Y \supseteq D(S) \longrightarrow 2^Z$ be a multifunction. A **selection** f of S is a single valued map*

$$f: Y \longrightarrow Z,$$

such that

$$f(y) \in S(y) \qquad \forall\, y \in D(S).$$

In the next proposition using selections of the subdifferential map, we characterize the Gâteaux and Fréchet differentiability of the convex function.

PROPOSITION 4.4.46
If X is a Banach space, $U \subseteq X$ is a nonempty, open convex set and

$$\varphi: U \longrightarrow \mathbb{R}$$

is a continuous, convex function,
then φ is Gâteaux (respectively Fréchet) differentiable at $x \in U$ if and only if there is a selection f of the subdifferential map $\partial\varphi$ which is norm-to-weak* (respectively norm-to-norm) continuous at x.

An interesting consequence of Proposition 4.4.46 is that Fréchet differentiable, convex functions are necessarily C^1-functions.

COROLLARY 4.4.47
If X is a Banach space, $U \subseteq X$ is a nonempty, open, convex set and $\varphi: U \longrightarrow \mathbb{R}$ is a convex and Fréchet differentiable function,
then the function

$$x \longmapsto \varphi'_F(x)$$

is norm-to-norm continuous from U into X^*, i.e.,

$$\varphi \in C^1(X).$$

Another such result is given in the next proposition.

PROPOSITION 4.4.48
If $\varphi \in \Gamma_0\big(\mathbb{R}^N\big)$, φ *is strictly convex and*

$$\frac{\varphi(x)}{\|x\|_{\mathbb{R}^N}} \longrightarrow +\infty \quad \text{as } \|x\|_{\mathbb{R}^N} \to +\infty,$$

then $\varphi^* \in C^1\big(\mathbb{R}^N\big)$.

PROOF Without any loss of generality, we may assume that $0 \in \operatorname{dom}\varphi$ and $\varphi(0) = 0$. Fix $x^* \in \mathbb{R}^N$ and consider the function

$$\psi_{x^*}(x) \stackrel{df}{=} (x^*, x)_{\mathbb{R}^N} - \varphi(x).$$

Evidently $-\psi_{x^*} \in \Gamma_0\big(\mathbb{R}^N\big)$, it is strictly convex and

$$\psi_{x^*}(x) \longrightarrow -\infty \quad \text{as } \|x\|_{\mathbb{R}^N} \to +\infty.$$

So by the Weierstrass theorem ψ_{x^*} attains its maximum on $\mathbb{R}^N$ and the maximizer x is unique. By Propositions 4.4.21 and Corollary 4.4.23, we have

$$\partial\varphi^*(x^*) = \{x\},$$

i.e., $\partial\varphi$ is single-valued. The map $x^* \longmapsto \partial\varphi^*(x^*)$ is closed. We show that it maps bounded sets to bounded sets, hence it is continuous. To this end let

$$\|x^*\|_{\mathbb{R}^N} \leqslant r \quad \text{and} \quad x = \partial\varphi^*(x^*).$$

We have $x^* \in \partial\varphi(x)$ and so

$$\frac{\varphi(x)}{\|x\|_{\mathbb{R}^N}} = \frac{\varphi(x) - \varphi(0)}{\|x\|_{\mathbb{R}^N}} \leqslant \frac{(x^*, x)_{\mathbb{R}^N}}{\|x\|_{\mathbb{R}^N}} \leqslant \|x^*\|_{\mathbb{R}^N} \leqslant r;$$

thus the set $\big\{\partial\varphi^*(x^*) : \|x^*\|_{\mathbb{R}^N} \leqslant r\big\}$ is bounded, i.e., $\partial\varphi^*$ is continuous.

Finally let $x = \partial\varphi^*(x^*)$ and

$$x_{h^*} = \partial\varphi^*(x^* + h^*),$$

for some $x^* \in \mathbb{R}^N$, $h^* \in \mathbb{R}^N \setminus \{0\}$. From the definition of the subdifferential, we have

$$\begin{aligned} 0 &\leqslant \frac{\varphi^*(x^* + h^*) - \varphi^*(x^*) - (h^*, x)_{\mathbb{R}^N}}{\|h^*\|_{\mathbb{R}^N}} \\ &\leqslant \frac{(h^*, x_{h^*} - x)_{\mathbb{R}^N}}{\|h^*\|_{\mathbb{R}^N}} \leqslant \|x_{h^*} - x\|_{\mathbb{R}^N}. \end{aligned}$$

From the continuity of $\partial\varphi^*$, we have

$$\|x_{h^*} - x\|_{\mathbb{R}^N} \longrightarrow 0 \quad \text{as } h^* \to 0.$$

So φ^* is differentiable at $x^* \in \mathbb{R}^N$ and the derivative is continuous, i.e., $\varphi^* \in C^1\big(\mathbb{R}^N\big)$. ∎

Before passing to the nonconvex subdifferentials, let us mention a few things about the ε-subdifferential (or approximate subdifferential), which is a useful tool in convex analysis. Its definition results from an innocent looking perturbation of the original subdifferential (see Definition 4.4.19), which however leads to some remarkable properties, that are different in nature from those of the "exact" subdifferential.

The mathematical setting remains unchanged with (X, X^*) being a dual system of Hausdorff locally convex spaces.

DEFINITION 4.4.49 *Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper function, $\varepsilon > 0$ and $x \in \operatorname{dom}\varphi$. The* ***$\varepsilon$-subdifferential*** *of φ at x is the set $\partial_\varepsilon\varphi(x)$ (possibly empty), defined by*

$$\partial_\varepsilon\varphi(x) \stackrel{df}{=} \Big\{x^* \in X^* :\ \langle x^*, y-x\rangle_X - \varepsilon \leqslant \varphi(y) - \varphi(x) \quad \textit{for all } y \in X\Big\}.$$

REMARK 4.4.50 Equivalently we can say that

$$x^* \ \in\ \partial_\varepsilon\varphi(x)$$

if and only if

$$\inf_X\big(\varphi - x^*\big) > -\infty \quad \text{and} \quad x \in \varepsilon - \operatorname{argmin}\big(\varphi - x^*\big),$$

with

$$\varepsilon\text{-}\operatorname{argmin}\big(\varphi - x^*\big) \ =\ \Big\{y \in X :\ \varphi(y) - \langle x^*, y\rangle_X \leqslant \inf_X\big(\varphi - x^*\big) + \varepsilon\Big\}.$$

Also

$$x^* \ \in\ \partial_\varepsilon\varphi(x)$$

if and only if

$$\varphi(x) + \varphi^*(x^*) - \langle x^*, x\rangle_X \ \leqslant\ \varepsilon.$$

Geometrically the definition of $\partial_\varepsilon\varphi(x)$ says that the epigraph of the continuous affine function with slope $x^* \in \partial_\varepsilon\varphi(x)$ and passing through $\big(x, \varphi(x) - \varepsilon\big)$ contains the epigraph of φ. So for $\varepsilon > 0$, $\partial_\varepsilon\varphi(x)$ is a global notion (in contrast to $\partial\varphi(x)$ which is local), i.e., it may be sensitive to variations of φ far away from x. When $\varepsilon = 0$, we recover Definition 4.4.19.

The next proposition establishes the main difference between approximate and exact subdifferentials.

PROPOSITION 4.4.51
If $\varphi \in \Gamma_0(X)$, $\varepsilon > 0$ and $x \in \operatorname{dom}\varphi$,
then

$$\partial_\varepsilon\varphi(x) \neq \emptyset$$

and it is w^*-closed, convex.

PROOF From Theorem 4.4.14, we have

$$-\varphi(x) \;=\; \inf_{x^* \in X^*} \left[\varphi^*(x^*) - \langle x^*, x\rangle_X\right].$$

Let

$$\psi(x^*) \stackrel{df}{=} \varphi^*(x^*) - \langle x^*, x\rangle_X \qquad \forall\, x^* \in X^*.$$

Let $x^* \in \varepsilon - \operatorname{argmin}\psi \neq \emptyset$. We have

$$\varphi^*(x^*) - \langle x^*, x\rangle_X \;\leqslant\; \inf_{X^*} \psi + \varepsilon \;=\; -\varphi(x) + \varepsilon,$$

hence

$$\varphi(x) + \varphi^*(x^*) - \langle x^*, x\rangle_X \;\leqslant\; \varepsilon,$$

which means that $x^* \in \partial_\varepsilon \varphi(x)$ (see Remark 4.4.7). So $\partial_\varepsilon \varphi(x) \neq \emptyset$ and clearly it is w^*-closed and convex. □

In the study of the ε-subdifferentials (with $\varepsilon > 0$), the directional derivative (see Definition 4.4.17) is replaced by the following quantity.

DEFINITION 4.4.52 *Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper function, $\varepsilon > 0$ and $x \in \operatorname{dom}\varphi$. For every $h \in X$, we define*

$$\varphi'_\varepsilon(x;h) \stackrel{df}{=} \inf_{\lambda > 0} \frac{\varphi(x+\lambda h) - \varphi(x) + \varepsilon}{\lambda}.$$

The next result is analogous to Proposition 4.4.27.

PROPOSITION 4.4.53
If X is a Banach space, $\varphi \in \Gamma_0(X)$, $\varepsilon > 0$ and $x \in \operatorname{dom}\varphi$,
then

$$\varphi'_\varepsilon(x;\cdot) \;=\; \sigma_{\partial_\varepsilon \varphi(x)}(\cdot).$$

Let us mention some basic calculus rules for the ε-subdifferential.

PROPOSITION 4.4.54
If X and Y are two Banach spaces, $A \in \mathcal{L}(X;Y)$, $\varphi \in \Gamma_0(Y)$, $\varepsilon > 0$ and $x \in A^{-1}(\operatorname{dom}\varphi)$,
then

$$\partial_\varepsilon(\varphi \circ A)(x) \;=\; \overline{A^*\big(\partial_\varepsilon \varphi\big(A(x)\big)\big)}^{w^*}.$$

PROOF Since $\varphi \circ A \in \Gamma_0(X)$, from Proposition 4.4.53, we have that

$$(\varphi \circ A)'_\varepsilon(x;h) \;=\; \sigma_{\partial_\varepsilon(\varphi\circ A)(x)}(h) \qquad \forall\, h \in X.$$

We have

$$(\varphi \circ A)'_\varepsilon(x;h) \;=\; \inf_{\lambda > 0} \frac{\varphi(A(x) + \lambda A(h)) - \varphi(A(x)) + \varepsilon}{\lambda} \;=\; \varphi'_\varepsilon\big(A(x);A(h)\big).$$

On the other hand, using Proposition 4.4.53, for all $h \in X$, we have

$$\begin{aligned}\sigma_{A^*(\partial_\varepsilon\varphi(A(x)))}(h) &= \sup_{y*\in\partial_\varepsilon\varphi(A(x))} \big\langle A^*(y^*), h\big\rangle_X \\ = \sup_{y*\in\partial_\varepsilon\varphi(A(x))} \big\langle y^*, A(h)\big\rangle_X &= \varphi'_\varepsilon\big(A(x); A(h)\big).\end{aligned}$$

So we conclude that

$$\sigma_{\partial_\varepsilon(\varphi\circ A)(x)}(h) = \sigma_{A^*(\partial_\varepsilon\varphi(A(x)))}(h) \qquad \forall\, h \in X,$$

hence we obtain the conclusion of the proposition. □

COROLLARY 4.4.55
If X and Y are two Banach spaces, $A \in \mathcal{L}(X;Y)$, $\varphi \in \Gamma_0(Y)$ and $x \in A^{-1}(\operatorname{dom}\varphi)$,
then

$$\partial(\varphi\circ A)(x) = \bigcap_{\varepsilon>0} \overline{A^*\big(\partial_\varepsilon\varphi\big(A(x)\big)\big)}^{w^*}.$$

Moreover, if X is reflexive, then we have

$$\partial(\varphi\circ A)(x) = \bigcap_{\varepsilon>0} \overline{A^*\big(\partial_\varepsilon\varphi\big(A(x)\big)\big)}.$$

PROOF Clearly

$$\partial(\varphi\circ A)(x) = \bigcap_{\varepsilon>0} \partial_\varepsilon(\varphi\circ A)(x).$$

Applying Proposition 4.4.54, we obtain the first equality.

For the second equality just note that if X is reflexive, then

$$\overline{A^*\big(\partial_\varepsilon\varphi\big(A(x)\big)\big)} = \overline{A^*\big(\partial_\varepsilon\varphi\big(A(x)\big)\big)}^{w} = \overline{A^*\big(\partial_\varepsilon\varphi\big(A(x)\big)\big)}^{w^*}.$$

□

COROLLARY 4.4.56
If X is a Banach space, $\varphi, \psi \in \Gamma_0(X)$ and $x \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$,
then

$$\partial(\varphi+\psi)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon\varphi(x) + \partial_\varepsilon\psi(x)}^{w^*}.$$

Moreover, if X is reflexive, then

$$\partial(\varphi+\psi)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon\varphi(x) + \partial_\varepsilon\psi(x)}.$$

PROOF Let

$$u(x,y) \stackrel{df}{=} \varphi(x)+\varphi(y) \qquad \forall\ (x,y)\in X\times X$$

and let $A\in\mathcal{L}(X;X\times X)$ be defined by

$$A(x) \stackrel{df}{=} \big(x,x\big) \qquad \forall\ x\in X.$$

Then we see that

$$u\circ A \ = \ \varphi+\psi \quad \text{on } X.$$

We have

$$A^*(x^*,y^*) \ = \ x^*+y^* \qquad \forall\ (x^*,y^*)\in X^*\times Y^*$$

and

$$u^*(x^*,y^*) \ = \ \varphi^*(x^*)+\psi^*(y^*) \qquad \forall\ (x^*,y^*)\in X^*\times Y^*.$$

Let $(x,y)\in \operatorname{dom} u$, $\varepsilon>0$ and $(x^*,y^*)\in\partial_\varepsilon u(x,y)$. We have

$$\varphi(x)+\psi(y)+\varphi^*(x^*)+\psi^*(y^*)-\langle x^*,x\rangle_X-\langle y^*,y\rangle_X \ \leqslant \ \varepsilon.$$

So there exists $\varepsilon_1,\varepsilon_2>0$, such that

$$\varepsilon_1+\varepsilon_2 \ = \ \varepsilon$$

and

$$x^* \ \in \ \partial_{\varepsilon_1}\varphi(x) \ \subseteq \ \partial_\varepsilon\varphi(x) \quad \text{and} \quad y^* \ \in \ \partial_{\varepsilon_2}\psi(y) \ \subseteq \partial_\varepsilon\psi(y).$$

Therefore, we infer that

$$\partial_\varepsilon u(x,y) \ \subseteq \ \partial_{\varepsilon_1}\varphi(x)\times\partial_{\varepsilon_2}\psi(y).$$

Using Corollary 4.4.55, we obtain

$$\partial(\varphi+\psi)(x) \ = \ \bigcap_{\varepsilon>0}\overline{A^*\big(\partial_\varepsilon u(x,x)\big)}^{w^*} \ \subseteq \ \bigcap_{\varepsilon>0}\overline{\partial_\varepsilon\varphi(x)+\partial_\varepsilon\psi(x)}^{w^*}. \tag{4.29}$$

On the other hand note that

$$\partial_\varepsilon\varphi(x)+\partial_\varepsilon\psi(x) \ \subseteq \ \partial_{2\varepsilon}(\varphi+\psi)(x)$$

and so

$$\overline{\partial_\varepsilon\varphi(x)+\partial_\varepsilon\psi(x)}^{w^*} \ \subseteq \ \partial_{2\varepsilon}(\varphi+\psi)(x).$$

From this inclusion it follows that

$$\bigcap_{\varepsilon>0}\overline{\partial_\varepsilon\varphi(x)+\partial_\varepsilon\psi(x)}^{w^*} \ \subseteq \ \bigcap_{\varepsilon>0}\partial_{2\varepsilon}(\varphi+\psi)(x) \ = \ \partial(\varphi+\psi)(x). \tag{4.30}$$

From (4.29) and (4.30), we conclude that

$$\partial(\varphi+\psi)(x) \;=\; \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon\varphi(x)+\partial_\varepsilon\psi(x)}^{w^*}.$$

Again if X is reflexive, then the norm closure and the weak closure coincide (due to the convexity of the set). □

Using the second characterization of the ε-subdifferential, we can easily prove the following rule. We leave the details to the reader.

PROPOSITION 4.4.57
If $\varphi,\psi\in\Gamma_0(X)$, $\varepsilon\geqslant 0$ *and there exist* $x_0\in\operatorname{dom}\varphi\cap\operatorname{dom}\psi$, *such that* φ *is continuous at* x_0,
then

$$\partial_\varepsilon(\varphi+\psi)(x) \;=\; \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2>0\\ \varepsilon_2+\varepsilon_2=\varepsilon}} [\partial_{\varepsilon_1}\varphi(x)+\partial_{\varepsilon_2}\psi(x)] \qquad \forall\; x\in\operatorname{dom}\varphi\cap\operatorname{dom}\psi.$$

Now we pass to a brief discussion of some nonconvex subdifferentials. Historically the first subdifferential defined for nonconvex functions is that for locally Lipschitz functions. The starting point for the introduction of such a subdifferential was Theorem 4.2.7 (that is that a continuous, convex function is locally Lipschitz) and when the underlying space is finite dimensional Theorem 1.5.8 and Corollary 1.5.9 (Rademacher's theorem).

The mathematical framework is a Banach space X with X^* its topological dual. Let us start by recalling the definition of a locally Lipschitz function, which is central in what follows.

DEFINITION 4.4.58 *A function* $\varphi\colon X\longrightarrow\mathbb{R}$ *is* ***locally Lipschitz****, if every point* $x\in X$ *admits a neighbourhood* $U\subseteq X$ *and a constant* k_U *(depending on* U*), such that*

$$\big|\varphi(y)-\varphi(z)\big| \;\leqslant\; k_U\,\|y-z\|_X \qquad \forall\; y,z\in U.$$

A locally Lipschitz function need not have directional derivatives in the sense of Definition 4.4.17. However, exploiting the local Lipschitz structure, we can define a generalized directional derivative as follows.

DEFINITION 4.4.59 *Let* $\varphi\colon X\longrightarrow\mathbb{R}$ *be a locally Lipschitz function. Then the* ***generalized directional derivative*** *of* φ *at* $x\in X$ *in the direction* $h\in X$ *is defined by*

$$\varphi^0(x;h) \;\stackrel{df}{=}\; \limsup_{\substack{x'\to x\\ \lambda\searrow 0}} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda}.$$

The utility of φ^0 follows from some useful properties that it exhibits.

PROPOSITION 4.4.60
If $\varphi\colon X \longrightarrow \mathbb{R}$ is a locally Lipschitz function, then

(a) *the function $h \longmapsto \varphi^0(x;h)$ is sublinear and Lipschitz continuous for all $x \in X$;*

(b) *the function $(x,h) \longmapsto \varphi^0(x;h)$ is upper semicontinuous on $X \times X$;*

(c) $\varphi^0(x;-h) = (-\varphi)^0(x;h)$.

PROOF **(a)** Clearly $\varphi(x;\cdot)$ is positively homogeneous. Also let $h_1, h_2 \in X$. We have

$$\begin{aligned}
\varphi^0(x;h_1+h_2) &= \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x'+\lambda(h_1+h_2))-\varphi(x')}{\lambda} \\
&= \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x'+\lambda h_1+\lambda h_2)-\varphi(x'+\lambda h_2)+\varphi(x'+\lambda h_2)-\varphi(x')}{\lambda} \\
&\leqslant \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x'+\lambda h_1+\lambda h_2)-\varphi(x'+\lambda h_2)}{\lambda} + \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x'+\lambda h_2)-\varphi(x')}{\lambda} \\
&= \varphi^0(x;h_1)+\varphi^0(x;h_2).
\end{aligned}$$

So we have proved that $\varphi^0(x;\cdot)$ is sublinear. Exploiting the local Lipschitzness of φ, we see that for all $x' \in X$ near $x \in X$ and for all $\lambda > 0$ near zero, we have

$$\frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda} \leqslant k\,\|h\|_X \qquad \forall\, h \in X,$$

so

$$\varphi^0(x;h) \leqslant k\,\|h\|_X \qquad \forall\, h \in X$$

and due to the sublinearity of $\varphi^0(x;\cdot)$, we have

$$\left|\varphi^0(x;h)\right| \leqslant k\,\|h\|_X \qquad \forall\, h \in X,$$

so finally we deduce that $\varphi^0(x;\cdot)$ is Lipschitz continuous.

(b) Let

$$(x_n,h_n) \longrightarrow (x,h) \quad \text{in } X \times X.$$

From the definition of $\varphi^0(x;h)$, we know that for every $n \geqslant 1$, we can find $v_n \in X$ and $\lambda_n \in (0,1)$, such that

$$\|v_n\|_X + \lambda_n \leqslant \frac{1}{n}$$

and

$$\varphi^0(x_n; h_n) \leqslant \frac{\varphi(x_n + v_n + \lambda_n h_n) - \varphi(x_n + v_n)}{\lambda_n} + \frac{1}{n},$$

so

$$\limsup_{n \to +\infty} \varphi^0(x_n, h_n) \leqslant \varphi^0(x; h),$$

i.e., the function $(x, h) \longmapsto \varphi^0(x, h)$ is upper semicontinuous.

(c) By definition, we have

$$\varphi^0(x; -h) = \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x' - \lambda h) - \varphi(x')}{\lambda}$$

$$= \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{(-\varphi)(y + \lambda h) - (-\varphi)(y)}{\lambda} = (-\varphi)^0(x; h)$$

(with $y = x' - \lambda h$). □

These properties lead to the following definition.

DEFINITION 4.4.61 *Let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The* ***generalized subdifferential*** *(or* ***Clarke subdifferential****) of φ at x is defined by*

$$\partial\varphi(x) \stackrel{df}{=} \left\{ x^* \in X^* : \langle x^*, x \rangle_X \leqslant \varphi^0(x; h) \quad \text{for all } h \in X \right\}.$$

The elements of $\partial\varphi(x)$ are called ***generalized gradients****.*

PROPOSITION 4.4.62
If $\varphi\colon X \longrightarrow \mathbb{R}$ is a locally Lipschitz function, then for every $x \in X$, the set $\partial\varphi(x) \subseteq X^*$ is nonempty, convex and w^*-compact, the multifunction $\partial\varphi\colon X \longrightarrow 2^{X^*} \setminus \{\emptyset\}$ is upper semicontinuous from X with the norm topology into X^* with the w^*-topology (see Definition 3.2.12) and

$$\varphi^0(x; h) = \sigma_{\partial\varphi(x)}(h) \qquad \forall\, (x, h) \in X \times X.$$

PROOF Since by Proposition 4.4.60**(a)**, $\varphi^0(x; \cdot)$ is sublinear, the Hahn-Banach theorem implies that it has a continuous linear minorant. Therefore $\partial\varphi(x) \neq \emptyset$. Clearly the set is convex and by virtue of Proposition 4.4.60**(a)**, it is also closed and bounded, hence w^*-compact (by Alaoglu's theorem; see Theorem A.3.9). To show the upper semicontinuity, let $C \subseteq X^*$ be a w^*-closed set and let

$$\{x_n\}_{n \geqslant 1} \subseteq \partial\varphi^-(C) = \left\{ x \in X : \partial\varphi(x) \cap C \neq \emptyset \right\}$$

be a sequence, such that

$$x_n \longrightarrow x \quad \text{in } X.$$

Let us take

$$x_n^* \in \partial\varphi(x_n) \cap C \qquad \forall\, n \geqslant 1.$$

Because Proposition 4.4.60**(a)**, the sequence $\{x_n^*\}_{n\geqslant 1}$ is bounded in X^*. So by Alaoglu's theorem (see Theorem A.3.9), we can find a subnet $\{x_\alpha^*\}_{\alpha\in J}$ of $\{x_n^*\}_{n\geqslant 1}$, such that

$$x_\alpha^* \xrightarrow{w^*} x^*.$$

We have

$$\langle x_\alpha^*, h\rangle_X \leqslant \varphi^0(x_\alpha; h) \qquad \forall\, h \in X.$$

Taking the limit with respect to $\alpha \in J$ and using Proposition 4.4.60**(b)**, we obtain

$$\langle x^*, h\rangle_X \leqslant \varphi^0(x; h) \qquad \forall\, h \in X,$$

so $x^* \in \partial\varphi(x)$.

Also $x^* \in C$, since $C \subseteq X^*$ is w^*-closed. Therefore

$$x^* \in \partial\varphi(x) \cap C,$$

hence $x \in \partial\varphi^-(C)$ which proves the upper semicontinuity of the multifunction.

Finally, using once more the Hahn-Banach theorem, for every $h_0 \in X$, we can find $x_0^* \in \partial\varphi(x)$, such that

$$\langle x_0^*, h\rangle_X \leqslant \varphi^0(x_0; h) \qquad \forall\, h \in X$$

and

$$\langle x_0^*, h_0\rangle_X = \varphi^0(x_0; h_0).$$

Therefore $\varphi^0(x;\cdot) = \sigma_{\partial\varphi(x)}(\cdot)$. □

PROPOSITION 4.4.63
Let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz function.

(a) *If φ is Gâteaux differentiable at $x \in X$, then $\varphi'_G(x) \in \partial\varphi(x)$.*

(b) *If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'_F(x)\}$ for all $x \in X$.*

(c) *If φ is also convex, then the convex and generalized subdifferentials of φ coincide.*

PROOF **(a)** From the definition of $\varphi^0(x;\cdot)$, we have

$$\langle \varphi'_G(x), h\rangle_X \leqslant \varphi^0(x;h) \qquad \forall\, h \in X.$$

So $\varphi'_G(x) \in \partial\varphi(x)$.

(b) If $\varphi \in C^1(X)$, then

$$\varphi^0(x;h) = \langle \varphi'_F(x), h\rangle_X \qquad \forall\, h \in X,$$

hence $\partial\varphi(x) = \{\varphi'_F(x)\}$.

(c) From the definition of $\varphi^0(x;h)$, we have

$$\varphi^0(x;h) = \lim_{\varepsilon \searrow 0} \sup_{\|x'-x\|_X \leqslant \varepsilon\delta} \sup_{0<\lambda\leqslant\varepsilon} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda},$$

with $\delta > 0$ arbitrary.

Because φ is convex,

$$\text{the map } \lambda \longmapsto \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda} \text{ is increasing on } (0,+\infty).$$

So we obtain

$$\varphi^0(x;h) = \lim_{\varepsilon \searrow 0} \sup_{\|x'-x\|_X \leqslant \varepsilon\delta} \frac{\varphi(x'+\varepsilon h)-\varphi(x')}{\varepsilon}.$$

From the local Lipschitz property of φ, we have

$$\left|\frac{\varphi(x'+\varepsilon h)-\varphi(x')}{\varepsilon} - \frac{\varphi(x+\varepsilon h)-\varphi(x)}{\varepsilon}\right| \leqslant 2\delta k \qquad \forall\, x' \in x+\varepsilon\delta\overline{B}_1,$$

with $k > 0$, so

$$\varphi^0(x;h) \leqslant \lim_{\varepsilon \searrow 0} \frac{\varphi(x+\varepsilon h)-\varphi(x)}{\varepsilon} + 2\delta k.$$

Since $\delta >$ was arbitrary, we obtain

$$\varphi^0(x;h) \leqslant \varphi'(x;h) \qquad \forall\, h \in X.$$

Because the opposite inequality is always true, we conclude that $\varphi^0(x;\cdot) = \varphi'(x;\cdot)$. Then by virtue of Proposition 4.4.27 and Definition 4.4.61, we conclude that the two subdifferentials coincide. □

EXAMPLE 4.4.64 If φ is differentiable but not C^1, then $\partial\varphi(x)$ need not be a singleton. To see this consider the function $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\varphi(x) \stackrel{df}{=} \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then φ is Lipschitz continuous on $[-1,1]$, $\varphi'(0) = 0$ but φ' is not continuous at $x = 0$. A straightforward calculation shows that $\varphi^0(0;h) = |h|$, hence $\partial\varphi(0) = [-1,1]$ and so it is not a singleton.

When $X = \mathbb{R}^N$, then we can use Corollary 1.5.9 (Rademacher's theorem) to give a definition which is less abstract and formal than Definition 4.4.61 in terms of the generalized directional derivative. The new definition is more geometric.

THEOREM 4.4.65
If $\varphi\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is a locally Lipschitz function and E is any Lebesgue-null set in $\mathbb{R}^N$,
then

$$\partial\varphi(x) \;=\; \operatorname{conv}\left\{ \lim_{n\to+\infty} \nabla\varphi(x_n) :\; x_n \longrightarrow x,\; x_n \notin E \cup D_\varphi^c \right\},$$

where $D_\varphi^c \subseteq \mathbb{R}^N$ is the Lebesgue-null set where φ fails to be differentiable (due to Rademacher's theorem).

PROOF Since, by Proposition 4.4.63**(a)**, we have

$$\nabla\varphi(x_n) \;\in\; \partial\varphi(x_n) \qquad \forall\; n \geqslant 1$$

and $\partial\varphi$ is locally bounded (see Proposition 4.4.60**(a)**), we see that the sequence $\{\nabla\varphi(x_n)\}_{n\geqslant 1}$ has a convergent subsequence. Then Proposition 4.4.62 implies that the limit of the subsequence belongs in $\partial\varphi(x)$. Therefore we have

$$\operatorname{conv}\left\{ \lim_{n\to+\infty} \nabla\varphi(x_n) :\; x_n \longrightarrow x,\; x_n \notin E \cup D_\varphi^c \right\} \;\subseteq\; \partial\varphi(x). \tag{4.31}$$

On the other hand let

$$\xi_h \;=\; \limsup_{\substack{y \to x \\ y \notin E\cup D_\varphi^c}} \big(\nabla\varphi(y), h\big)_{\mathbb{R}^N},$$

with $h \neq 0$. For a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that

$$\big(\nabla\varphi(y), h\big)_{\mathbb{R}^N} \;\leqslant\; \xi_h + \varepsilon \qquad \forall\; y \in x + \delta\overline{B}_1(0),\; y \notin E \cup D_\varphi^c.$$

For $t \in \left(0, \frac{\delta}{2\|h\|_{\mathbb{R}^N}}\right)$, we have

$$\varphi(y+th) - \varphi(y) \;\leqslant\; t(\xi_h + \varepsilon) \quad \text{for a.a. } y \in x + \frac{\delta}{2}\overline{B}_1(0),$$

so

$$\varphi^0(x;h) \;\leqslant\; \xi_h + \varepsilon.$$

Therefore the support function of the set

$$\operatorname{conv}\left\{ \lim_{n\to+\infty} \nabla\varphi(x_n) :\; x_n \longrightarrow x,\; x_n \notin E \cup D_\varphi^c \right\}$$

majorizes $\varphi^0(x;\cdot)$. This combined with (4.31) finishes the proof of the theorem. ∎

COROLLARY 4.4.66
If $\varphi\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ *is a locally Lipschitz function,*
then

$$\varphi^0(x;h) \;=\; \limsup_{\substack{x' \to x \\ y \in E \cup D_\varphi^c}} \big(\nabla\varphi(x'), h\big)_{\mathbb{R}^N}.$$

Finally let us state a few basic calculus rules for the generalized subdifferential.

PROPOSITION 4.4.67
If $\varphi\colon X \longrightarrow \mathbb{R}$ *is a locally Lipschitz function and* $\lambda \in \mathbb{R}$,
then

$$\partial(\lambda\varphi) \;=\; \lambda\partial\varphi.$$

PROOF Evidently the result is true if $\lambda \geqslant 0$, since

$$(\lambda\varphi)^0 \;=\; \lambda\varphi^0.$$

So we need to consider the case $\lambda < 0$. We may assume that $\lambda = -1$. Then

$$x^* \;\in\; \partial(-\varphi)(x)$$

if and only if

$$\langle x^*, h\rangle_X \;\leqslant\; (-\varphi)^0(x;h) \qquad \forall\, h \in X.$$

By Proposition 4.4.60**(c)**, we have

$$(-\varphi)^0(x;h) \;=\; \varphi^0(x;-h).$$

So

$$x^* \;\in\; \partial(-\varphi)(x)$$

if and only if

$$\langle x^*, h\rangle_X \;\leqslant\; \varphi^0(x;-h) \qquad \forall\, h \in X,$$

hence

$$-x^* \;\in\; \partial\varphi(x).$$

Thus finally we have that

$$x^* \in \partial(-\varphi)(x) \text{ if and only if } x^* \in -\partial\varphi(x).$$

□

An interesting consequence of this proposition is the following extension of Fermat's equation for local extrema.

PROPOSITION 4.4.68
If $\varphi\colon X \longrightarrow \mathbb{R}$ *is a locally Lipschitz function and has a local maximum or minimum at* $x \in X$,
then $0 \in \partial\varphi(x)$.

PROOF Since $\partial(-\varphi) = -\partial\varphi$, it suffices to prove the proposition for the case of a local minimum at $x \in X$. Then clearly

$$\varphi^0(x;h) \;\geqslant\; 0 \qquad \forall\, h \in X.$$

Hence $0 \in \partial\varphi(x)$. □

PROPOSITION 4.4.69
If $\varphi_k\colon X \longrightarrow \mathbb{R}$ *for* $k \in \{1,\ldots,n\}$ *are locally Lipschitz functions,*
then

$$\partial\Big(\sum_{k=1}^{n}\varphi_k\Big) \;\subseteq\; \sum_{k=1}^{n}\partial\varphi_k,$$

i.e., the generalized subdifferential is subadditive.

PROOF It suffices to prove the result for $n = 2$. The general case follows by induction. The support function of $\partial(\varphi_1+\varphi_2)$ is $(\varphi_1+\varphi_2)^0$ and the support function of $\partial\varphi_1 + \partial\varphi_2$ is $\varphi_1^0 + \varphi_2^0$. Also note that $\partial\varphi_1(x) + \partial\varphi_2(x)$ is convex and w^*-compact. Since

$$(\varphi_1 + \varphi_2)^0(x;\cdot) \;\leqslant\; \varphi_1^0(x;\cdot) + \varphi_2^0(x;\cdot),$$

we conclude that

$$\partial(\varphi_1 + \varphi_2)(x) \;\subseteq\; \partial\varphi_1(x) + \partial\varphi_2(x) \qquad \forall\, x \in X.$$

□

COROLLARY 4.4.70
If $\varphi_k\colon X \longrightarrow \mathbb{R}$ *for* $k \in \{1,\ldots,n\}$ *are locally Lipschitz functions and all but one are* C^1*-functions,*
then

$$\partial\Big(\sum_{k=1}^{n}\varphi_k\Big) \;=\; \sum_{k=1}^{n}\partial\varphi_k.$$

COROLLARY 4.4.71
If $\varphi_k\colon X \longrightarrow \mathbb{R}$ *are locally Lipschitz functions and* $\lambda_k \in \mathbb{R}$ *for* $k \in \{1,\ldots,n\}$,
then

$$\partial\Big(\sum_{k=1}^{n}\lambda_k\varphi_k\Big) \;\subseteq\; \sum_{k=1}^{n}\lambda_k\partial\varphi_k$$

and equality holds if all but one of the functions are C^1*-functions.*

The following mean value theorem is a useful tool in many applications.

THEOREM 4.4.72 (Mean Value Theorem)
If $U \subseteq X$ is an open set, $x, y \in X$, $[x,y] \subseteq U$, with

$$[x,y] \stackrel{df}{=} \big\{\lambda x + (1-\lambda)y : \ \lambda \in [0,1]\big\}$$

and $\varphi\colon U \longrightarrow \mathbb{R}$ is locally Lipschitz,
then there exists $u \in (x,y)$, with

$$(x,y) \stackrel{df}{=} \big\{\lambda x + (1-\lambda)y : \ \lambda \in (0,1)\big\}$$

and $u^* \in \partial\varphi(u)$, such that

$$\varphi(y) - \varphi(x) \ = \ \langle u^*, y-x\rangle_X \, .$$

PROOF Let

$$x_\lambda \stackrel{df}{=} x + \lambda(y-x)$$

and consider the function $f\colon [0,1] \longrightarrow \mathbb{R}$, defined by

$$f(\lambda) \stackrel{df}{=} \varphi(x_\lambda) \qquad \forall\ \lambda \in [0,1].$$

Clearly f is Lipschitz continuous on $[0,1]$. We claim that

$$\partial f(\lambda) \ \subseteq \ \big\{\, \langle u^*, y-x\rangle_X : \ u^* \in \partial\varphi(x_\lambda)\big\} \qquad \forall\ \lambda \in (0,1).$$

Since both sets are closed, convex, it suffices to show that

$$\sigma_{\partial f(\lambda)}(\pm 1) \ \leqslant \ \max_{u^* \in \partial\varphi(x_\lambda)} \pm \langle u^*, y-x\rangle_X \, . \tag{4.32}$$

To this end, for $h = \pm 1$, we have

$$\begin{aligned}
&\limsup_{\substack{\lambda' \to \lambda \\ t \searrow 0}} \frac{f(\lambda'+th) - f(\lambda')}{t} \\
&= \limsup_{\substack{\lambda' \to \lambda \\ t \searrow 0}} \frac{\varphi(x + (\lambda'+th)(y-x)) - \varphi(x+\lambda'(y-x))}{t} \\
&\leqslant \limsup_{\substack{z \to x_\lambda \\ t \searrow 0}} \frac{\varphi(z+th(y-x)) - \varphi(z)}{t} \\
&= \varphi^0\big(x_\lambda; h(y-x)\big) \ = \ \sup_{u^* \in \partial\varphi(x_\lambda)} \big\langle u^*, h(y-x)\big\rangle.
\end{aligned} \tag{4.33}$$

From (4.33), we obtain (4.32).

Now let

$$\xi(\lambda) \stackrel{df}{=} \varphi(x_\lambda) + \lambda\big(\varphi(x) - \varphi(y)\big) \qquad \forall\ \lambda \in [0,1].$$

We have

$$\xi(0) \;=\; \xi(1) \;=\; \varphi(x)$$

and so we can find $\lambda \in (0,1)$ at which ξ attains a local extremum. Then by Proposition 4.4.68, we have that

$$0 \;\in\; \partial\xi(\lambda),$$

which via Propositions 4.4.67 and 4.4.69 and (4.32) implies that

$$\varphi(y) - \varphi(x) \;\in\; \big\langle \partial\varphi(u), y - x \big\rangle_X,$$

with $u = x_\lambda$. □

The generalized subdifferential has a remarkable calculus which makes it very useful in applications. We mention only two rules which arise in applications. For their proofs and additional results in this direction we refer to Clarke (1983).

PROPOSITION 4.4.73 (Chain Rule)
If X and Y are two Banach spaces, $h \in C^1(X;Y)$ and $\varphi\colon Y \longrightarrow \mathbb{R}$ is a locally Lipschitz function,
then

$$\partial(\varphi \circ h)(x) \;\subseteq\; \partial\varphi\big(h(x)\big) \circ h'_F(x) \qquad \forall\ x \in X.$$

REMARK 4.4.74 Note that $h'_F(x) \in \mathcal{L}(X;Y)$. Using its adjoint

$$\big(h'_F(x)\big)^* \;\in\; \mathcal{L}(Y^*;X^*)$$

we can equivalently rewrite the above chain rule as

$$\partial(\varphi \circ h)(x) \;\subseteq\; \big(h'_F(x)\big)^* \partial\varphi\big(h(x)\big) \qquad \forall\ x \in X.$$

A useful consequence of the above chain rule is the following result.

COROLLARY 4.4.75
If X and Y are two Banach spaces, X is embedded continuously and densely in Y, $\varphi\colon Y \longrightarrow \mathbb{R}$ is a locally Lipschitz function and $\widehat{\varphi} \stackrel{df}{=} \varphi|_X$,
then

$$\partial\widehat{\varphi}(x) \;=\; \partial\varphi(x) \qquad \forall\ x \in X,$$

which means that every element in $\partial\widehat{\varphi}(x)$ admits a unique extension to an element of $\partial\varphi(x)$.

PROPOSITION 4.4.76 (Nonsmooth Lagrange Rule) *If X is a Banach space, $\varphi, f\colon X \longrightarrow \mathbb{R}$ are two locally Lipschitz functions and x is a local solution of the problem*

$$\inf_{f(x)\leqslant 0} \varphi(x),$$

then there exist $\lambda_0, \lambda_1 \geqslant 0$, not both zero, such that

$$0 \ \in \ \lambda_0 \partial\varphi(x) + \lambda_1 \partial f(x) \quad \textit{and} \quad 0 = \lambda_1 f(x).$$

REMARK 4.4.77 If

$$m(b) \stackrel{df}{=} \inf_{f(x)\leqslant b} \varphi(x),$$

$m(0)$ is finite and

$$\liminf_{b\to 0} \frac{m(b) - m(0)}{|b|} \ > \ -\infty$$

(calmness condition), then $\lambda_0 > 0$ and so by normalization we can assume that $\lambda_0 = 1$.

Let us conclude this section by mentioning some more nonconvex subdifferentials. This can be done using the following parent notion.

DEFINITION 4.4.78 *Let X be a Banach space. A **bornology** is a collection $\mathfrak{B}$ of bounded, symmetric (with respect to the origin) subsets of X, whose union is X.*

REMARK 4.4.79 By taking the collection of all finite symmetric sets, we have the so-called ***Gâteaux bornology*** denoted by $\mathfrak{B}_G$. Similarly if the collection consists of all bounded, symmetric sets, then we have the ***Fréchet bornology*** denoted by $\mathfrak{B}_F$. Finally if we consider all symmetric, compact sets, then the resulting bornology is the so-called ***Hadamard bornology*** denoted by $\mathfrak{B}_H$.

DEFINITION 4.4.80 *Let X be a Banach space and let $\mathfrak{B}$ be a bornology in X.*

(a) *The norm of X is said to be $\mathfrak{B}$-**smooth**, if it is Gâteaux differentiable at every $x \in X \setminus \{0\}$ and the defining limit exists uniformly on members of $\mathfrak{B}$.*

(b) *A function $\varphi\colon X \longrightarrow \mathbb{R}$ is said to be $\mathfrak{B}$-**differentiable** at $x \in X$ with $\mathfrak{B}$-**derivative** $\varphi'_{\mathfrak{B}}(x)$, if for every $C \in \mathfrak{B}$, we have*

$$\lim_{\lambda\to 0} \sup_{h\in C} \frac{\varphi(x+\lambda h) - \varphi(x) - \lambda \langle \varphi'_{\mathfrak{B}}(x), h\rangle_X}{\lambda} \ = \ 0.$$

(c) *Let $\varphi\colon X \longrightarrow \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ be a lower semicontinuous function and suppose that $\varphi(x) \in \mathbb{R}$. We say that φ is* ***$\mathfrak{B}$-subdifferentiable*** *at x, if there exists $x^* \in X^*$, such that for every $\varepsilon > 0$ and every $C \in \mathfrak{B}$, there exists $\delta > 0$ for which we have*

$$\langle x^*, h\rangle_X \leqslant \frac{\varphi(x+\lambda h) - \varphi(x)}{\lambda} + \varepsilon \qquad \forall\ \lambda \in (0,\delta),\ h \in C.$$

The elements $x^* \in X^*$ are called $\mathfrak{B}$-subderivatives of φ at x and the set of all $\mathfrak{B}$-subderivatives is the $\mathfrak{B}$-subdifferential of φ at x and it is denoted by $\partial_{\mathfrak{B}}\varphi(x)$.

REMARK 4.4.81 If in Definition 4.4.80**(b)**, $\mathfrak{B} = \mathfrak{B}_G$ (respectively $\mathfrak{B} = \mathfrak{B}_F$), then $\varphi'_{\mathfrak{B}}(x) = \varphi'_G(x)$ (respectively $\varphi'_{\mathfrak{B}}(x) = \varphi'_F(x)$).

If in Definition 4.4.80**(c)**, φ is $\overline{\mathbb{R}}$-valued, proper, convex, then

$$\partial_{\mathfrak{B}}\varphi(x) \ = \ \partial\varphi(x)$$

(the convex subdifferential; see Definition 4.4.19).

If in Definition 4.4.80**(c)**, φ is $\mathbb{R}$-valued, locally Lipschitz, then

$$\partial_{\mathfrak{B}}\varphi(x) \ = \ \partial\varphi(x)$$

(the generalized subdifferential; see Definition 4.4.61).

Finally if in Definition 4.4.80**(c)**, we reverse the inequality and replace ε be $-\varepsilon$, we obtain the $\mathfrak{B}$-superdifferential of φ at x denoted by $\partial^{\mathfrak{B}}\varphi(x)$. Note that

$$\partial^{\mathfrak{B}}(-\varphi)(x) \ = \ -\partial_{\mathfrak{B}}\varphi(x).$$

The elements of $\partial^{\mathfrak{B}}\varphi(x)$ are called ***$\mathfrak{B}$-superderivatives*** of φ at x.

PROPOSITION 4.4.82

Let X be a Banach space, $\mathfrak{B}$ a bornology in X and $\varphi\colon X \longrightarrow \mathbb{R}^*$.

(a) *If φ is a lower semicontinuous function, $\varphi(x) \in \mathbb{R}$ and $\partial_{\mathfrak{B}}\varphi(x)$ and $\partial^{\mathfrak{B}}\varphi(x)$ are nonempty sets,*
then φ is $\mathfrak{B}$-differentiable at x and $\partial\varphi_{\mathfrak{B}}(x) = \varphi'_G(x)$.

(b) *If φ is a concave function which is continuous in a neighbourhood of x and $\partial_{\mathfrak{B}}\varphi(x) \neq \emptyset$,*
then φ is $\mathfrak{B}$-differentiable at x and $\partial\varphi_{\mathfrak{B}}(x) = \{\varphi'_{\mathfrak{B}}(x)\}$.

(c) *If the norm of X is $\mathfrak{B}$-smooth and*

$$\mathcal{T} \stackrel{df}{=} \Big\{ s\colon X \longrightarrow \mathbb{R} : s(x) = \frac{1}{2}\sum_{n=1}^{\infty} \mu_n \|x - z_n\|_X^2 ,$$
$$\text{where } \mu_n \geqslant 0,\ \sum_{n=1}^{\infty} \mu_n = 1,\ \text{and } z_n \longrightarrow z \in X \Big\},$$

then the elements of $\mathcal{T}$ are everywhere $\mathfrak{B}$-differentiable.

PROOF **(a)** Let

$$x^* \in \partial_{\mathfrak{B}}\varphi(x) \quad \text{and} \quad y^* \in \partial^{\mathfrak{B}}\varphi(x).$$

Then for every $\varepsilon > 0$ and every $h \in X$, we have

$$\langle y^*, h\rangle_X - \langle x^*, h\rangle_X \leqslant 2\varepsilon,$$

hence

$$x^* = y^*.$$

Denote the common value by v^*. Then clearly

$$v^* = \varphi'_{\mathfrak{B}}(x) = \varphi'_G(x).$$

(b) The function $-\varphi$ is convex and continuous at $x \in X$. Then by Proposition 4.4.25 and Remark 4.4.81, we have

$$\partial(-\varphi)(x) = \partial_{\mathfrak{B}}(-\varphi)(x) \neq \emptyset.$$

Since

$$\partial^{\mathfrak{B}}\varphi(x) = -\partial_{\mathfrak{B}}(-\varphi)(x),$$

we have that

$$\partial^{\mathfrak{B}}\varphi(x) \neq \emptyset.$$

Also

$$\partial_{\mathfrak{B}}\varphi(x) = -\partial(-\varphi)(x) \neq \emptyset.$$

So we can apply part **(a)** and obtain the claim.

(c) Let $x \in X$. The sequence

$$\{x - z_n\}_{n\geqslant 1} \text{ is bounded in } X.$$

By hypothesis the norm of X is $\mathfrak{B}_G$-smooth and so X^* is strictly convex. Hence by Proposition 3.2.22, the duality map $\mathcal{F}$ is single valued. Then from the definition of $s \in \mathcal{T}$, the series

$$\sum_{n=1}^{\infty} \mu_n \mathcal{F}(x - z_n) \text{ is norm convergent in } X^*$$

to some $x^* \in X^*$. It is straightforward to check that

$$x^* = s'_{\mathfrak{B}}(x).$$

□

Finally let us give some particular subdifferentials associated with bornologies.

EXAMPLE 4.4.83 Let X be a Banach space and $\mathfrak{B}$ a bornology on X.

(a) Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper, lower semicontinuous function and $x \in \operatorname{dom}\varphi$. The ***viscosity $\mathfrak{B}$-subdifferential*** at x is the set $\partial_{\mathfrak{B}}\varphi(x)$ of all $x^* \in X^*$, such that there is a $\mathfrak{B}$-differentiable function f, such that $\varphi - f$ attains a local minimum at x and

$$x^* \;=\; f'_{\mathfrak{B}}(x).$$

(b) If in the above case we specialize the perturbation function f, we obtain the so-called ***proximal subdifferential*** $\partial_p\varphi(x)$ of φ at x. Assuming that the norm of X is Fréchet differentiable (off the origin), if

$$f(y) \;\overset{df}{=}\; \langle x^*, y-x\rangle_X - k\,\|y-x\|_X^2\,,$$

for some $k>0$ in **(a)**, then we obtain the proximal subdifferential $\partial_p\varphi(x)$ for a proper, lower semicontinuous function $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$. So $x^* \in \partial_p\varphi(x)$ if and only if for some $k>0$ and all y in the neighbourhood of x, we have

$$\varphi(y) - \varphi(x) + k\,\|y-x\|_X^2 \;\geqslant\; \langle x^*, y-x\rangle_X\,.$$

This subdifferential is useful if X is finite dimensional or if X is a Hilbert space.

(c) Let $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ be a proper, lower semicontinuous function and $x \in \operatorname{dom}\varphi$. The ***canonical $\mathfrak{B}$-subdifferential*** of φ at x, denoted by $\partial^-_{\mathfrak{B}}\varphi(x)$, is the set of all $x^* \in X^*$, such that

$$\liminf_{\lambda\to 0}\,\inf_{h\in C}\,\frac{1}{\lambda}\left[\varphi(x+\lambda h) - \varphi(x) - \lambda\,\langle x^*, h\rangle_X\right] \;\geqslant\; 0 \qquad \forall\, C \in \mathfrak{B}.$$

REMARK 4.4.84 Of all the subdifferentials the proximal is the smallest. The viscosity $\mathfrak{B}$-subdifferential is not greater than the corresponding canonical subdifferential. The Fréchet viscosity and canonical subdifferentials coincide, if there exists a Lipschitz continuous, Fréchet differentiable bump function (see Deville, Godefroy & Zizler (1993); recall that $b\colon X \longrightarrow \mathbb{R}$ is a ***bump function*** if it has a nonempty and bounded support). If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two bornologies in X and $\mathfrak{B}_2$ is finer than $\mathfrak{B}_1$ (i.e., for every $C_1 \in \mathfrak{B}_1$, we can find $C_2 \in \mathfrak{B}_2$, such that $C_1 \subseteq C_2$), then any $\mathfrak{B}_2$-subdifferential is not greater than the corresponding $\mathfrak{B}_1$-subdifferential.

4.5 Integral Functionals and Subdifferentials

Let (Ω, Σ, μ) be a σ-finite measure space and let X be a separable Banach space. In this section we describe the subdifferential theory of the integral functionals

$$I_\varphi(u) \;=\; \int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu,$$

where $\varphi\colon \Omega \times X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is a normal integrand (see Definition 3.4.8) and $u\colon \Omega \longrightarrow X$ belongs in some vector space of functions.

We adopt the convention that

$$+\infty + (-\infty) \;=\; +\infty$$

(i.e., we let $+\infty$ dominate over $-\infty$). Then for a normal integrand $\varphi\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$, we define the integral functional $I_\varphi\colon L^1(\Omega; X) \longrightarrow \mathbb{R}^*$, by

$$I_\varphi(u) \;\stackrel{df}{=}\; \begin{cases} \displaystyle\int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu & \text{if} \quad \displaystyle\int_\Omega \varphi\big(\omega, u(\omega)\big)^+ d\mu < +\infty, \\ +\infty & \text{if} \quad \displaystyle\int_\Omega \varphi\big(\omega, u(\omega)\big)^+ d\mu = +\infty. \end{cases}$$

Similarly, if φ^* is the conjugate of $\varphi(\omega, \cdot)$, we define the integral functional $I_{\varphi^*}\colon L^\infty\big(\Omega; X^*_{w^*}\big) \longrightarrow \mathbb{R}^*$, by

$$I_{\varphi^*}(v) \;\stackrel{df}{=}\; \begin{cases} \displaystyle\int_\Omega \varphi^*\big(\omega, v(\omega)\big)\, d\mu & \text{if} \quad \displaystyle\int_\Omega \varphi^*\big(\omega, v(\omega)\big)^+ d\mu < +\infty, \\ +\infty & \text{if} \quad \displaystyle\int_\Omega \varphi^*\big(\omega, v(\omega)\big)^+ d\mu = +\infty. \end{cases}$$

Recall that

$$\big(L^1(\Omega; X)\big)^* \;=\; L^\infty\big(\Omega; X^*_{w^*}\big)$$

(see Theorem 2.2.12; for the Banach space $L^\infty\big(\Omega; X^*_{w^*}\big)$ see Definition 2.2.10; note that Theorem 2.2.12 was stated with μ being a finite measure, but the result is true for μ being σ-finite, see Ionescu-Tulcea & Ionescu-Tulcea (1969, p. 95)). The duality brackets for the pair $\big(L^1(\Omega; X), L^\infty\big(\Omega; X^*_{w^*}\big)\big)$ are defined by

$$\langle u, v\rangle_{L^1(\Omega;X)} \;\stackrel{df}{=}\; \int_\Omega \big\langle v(\omega), u(\omega)\big\rangle_X\, d\mu \qquad \forall\; u \in L^1(\Omega; X),\; v \in L^\infty\big(\Omega; X^*_{w^*}\big).$$

We mention that the theory can be developed for integral functionals I_φ defined on $L^p(\Omega; X)$ with $p \in (1, +\infty)$. In this case

$$\big(L^p(\Omega; X)\big)^* \;=\; L^{p'}\big(\Omega; X^*_{w^*}\big)$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and if X^* is separable, then

$$\left(L^p(\Omega; X)\right)^* \;=\; L^{p'}\left(\Omega; X^*\right)$$

(see Theorem 2.2.9).

First we need to know that the integral in the definition of I_{φ^*} makes sense.

PROPOSITION 4.5.1
If $\varphi\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ *is a normal integrand,*
then $\varphi^*\colon \Omega \times X^* \longrightarrow \overline{\mathbb{R}}$ *defined by*

$$\varphi^*(\omega, x^*) \stackrel{df}{=} \sup_{x \in X} \left(\langle x^*, x\rangle_X - \varphi(\omega, x)\right)$$

is a convex normal integrand on $\Omega \times X^*_{w^*}$.

PROOF Consider the multifunction $E\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$, defined by

$$E(\omega) \stackrel{df}{=} \operatorname{epi} \varphi(\omega, \cdot) \;=\; \left\{(x, \lambda) \in X \times \mathbb{R} :\; \varphi(\omega, x) \leqslant \lambda\right\}.$$

Evidently E has nonempty and closed values (by virtue of the normality of $\varphi(\omega, \cdot)$). Moreover, we have

$$\begin{aligned} \operatorname{Gr} E \;&=\; \left\{(\omega, x, \lambda) \in \Omega \times X \times X :\; \varphi(\omega, x) \leqslant \lambda\right\} \\ &\in\; \Sigma \times \mathcal{B}(X) \times \mathcal{B}(\mathbb{R}) \;=\; \Sigma \times \mathcal{B}(X \times \mathbb{R}), \end{aligned}$$

where $\mathcal{B}(Z)$ is the Borel σ-field of Z. So we can find two sequence

$$\left\{u_n\colon \Omega \longrightarrow X\right\}_{n \geqslant 1} \quad \text{and} \quad \left\{\lambda_n\colon \Omega \longrightarrow \mathbb{R}\right\}_{n \geqslant 1}$$

of Σ-measurable functions, such that

$$E(\omega) \;=\; \overline{\left\{\left(u_n(\omega), \lambda_n(\omega)\right)\right\}}_{n \geqslant 1} \qquad \forall\, \omega \in \Omega$$

(see Denkowski, Migórski & Papageorgiou (2003*a*, p. 433)). Note that

$$\varphi^*(\omega, x^*) \;=\; \sup_{n \geqslant 1} \left[\langle x^*, u_n(\omega)\rangle_X - \lambda_n(\omega)\right].$$

Thus we conclude that the function $(\omega, x^*) \longmapsto \varphi^*(\omega, x^*)$ is a convex normal integrand on $\Omega \times X^*_{w^*}$. □

THEOREM 4.5.2
(a) *If* $I_\varphi\colon L^1(\Omega; X) \longrightarrow \mathbb{R}^*$ *is finite at* $u_0 \in L^1(\Omega; X)$,
then $\overline{(I_\varphi)}^* = I_{\varphi^*}$.

(b) *If* φ *is a convex normal integrand (i.e.,* $\varphi(\omega, \cdot)$ *is convex for* μ*-almost all* $\omega \in \Omega$*) and* I_φ *and* I_{φ^*} *are finite at* $u_0 \in L^1(\Omega; X)$ *and* $v_0 \in L^\infty\left(\Omega; X^*_{w^*}\right)$ *respectively,*
then I_φ *and* I_{φ^*} *are proper, convex and lower semicontinuous functionals which are conjugate to each other.*

PROOF **(a)** Evidently it suffices to show that for all $v \in L^{\infty}\big(\Omega; X^*_{w^*}\big)$, we have

$$\int_{\Omega} \varphi^*\big(\omega, v(\omega)\big)\, d\mu \;\leqslant\; \sup_{u \in L^1(\Omega;X)} \Big(\langle v, u \rangle_{L^1(\Omega;X)} - I_{\varphi}(u) \Big) \tag{4.34}$$

(see Proposition 4.4.4). Let $\xi \in \mathbb{R}$ be such that $\xi < I_{\varphi^*}(v)$. We can obtain (4.34) if we show that there exists $u \in L^1(\Omega; X)$, such that

$$\langle v, u \rangle_{L^1(\Omega;X)} - I_{\varphi}(u) \;\geqslant\; \xi.$$

Since by hypothesis $I_{\varphi}(u_0)$ is finite, we can find $\vartheta_0 \in L^1(\Omega)$, such that

$$\big\langle v(\omega), u_0(\omega) \big\rangle_X - \varphi\big(\omega, u_0(\omega)\big) \;\geqslant\; \vartheta_0(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega. \tag{4.35}$$

We obtain

$$\varphi^*\big(\omega, v(\omega)\big) \;\geqslant\; \vartheta_0(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

We claim that there exists a function $\beta \in L^1(\Omega)$, such that

$$\xi \;<\; \int_{\Omega} \beta(\omega)\, d\mu$$

and

$$\beta(\omega) \;<\; \varphi^*\big(\omega, v(\omega)\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Let $g \in L^1(\Omega)$, $g(\omega) > 0$ for μ-almost all $\omega \in \Omega$. If $I_{\varphi^*}(v)$ is finite, then let

$$\beta(\omega) \;\stackrel{df}{=}\; \varphi^*\big(\omega, v(\omega)\big) - \varepsilon g(\omega),$$

with $\varepsilon > 0$ sufficiently small (so that $\xi < \int_{\Omega} \beta(\omega)\, d\mu$). If $I_{\varphi^*}(v) = +\infty$, then we define

$$f_n(\omega) \;\stackrel{df}{=}\; \begin{cases} \min\Big\{ ng(\omega), \dfrac{1}{2}\varphi^*\big(\omega, v(\omega)\big) \Big\} & \text{if} \quad \varphi^*\big(\omega, v(\omega)\big) > 0, \\ \varphi^*\big(\omega, v(\omega)\big) - g(\omega) & \text{if} \quad \varphi^*\big(\omega, v(\omega)\big) \leqslant 0. \end{cases}$$

Note that

$$f_n(\omega) \;\longrightarrow\; \frac{1}{2}\varphi^*\big(\omega, v(\omega)\big) \qquad \forall\, \omega \in \big\{ \omega \in \Omega :\; \varphi^*\big(\omega, v(\omega)\big) > 0 \big\}.$$

Hence by the monotone convergence theorem (see Theorem A.2.10), we have that

$$\lim_{n \to +\infty} \int_{\Omega} f_n(\omega)\, d\mu \;=\; +\infty$$

and so we can find $n_0 \geqslant 1$ large enough so that

$$\xi \ < \ \int_\Omega f_{n_0}(\omega)\, d\mu.$$

Therefore if we take $\beta = f_{n_0}$, we have

$$\beta(\omega) \ < \ \varphi^*\big(\omega, v(\omega)\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Consider the multifunction $S\colon \Omega \longrightarrow 2^X$, defined by

$$S(\omega) \ \stackrel{df}{=} \ \big\{x \in S : \ \big\langle v(\omega), x\big\rangle_X - \varphi(\omega, x) \geqslant \beta(\omega)\big\} \qquad \forall\, \omega \in \Omega.$$

Evidently S has nonempty, closed values and

$$\operatorname{Gr} S \ = \ \big\{(\omega, x) \in \Omega \times X : \ x \in S(\omega)\big\} \ \in \ \Sigma \times \mathcal{B}(X).$$

By the Yankov-von Neumann-Aumann selection theorem (see Theorem A.2.32), we can find a Σ-measurable function $s\colon \Omega \longrightarrow X$, such that

$$s(\omega) \ \in \ S(\omega) \qquad \forall\, \omega \in \Omega.$$

Since μ is σ-finite, we can find $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) < +\infty$, such that $s|_{\Omega_0}$ is bounded and

$$\int_{\Omega_0} \beta(\omega)\, d\mu + \int_{\Omega \setminus \Omega_0} \vartheta_0(\omega)\, d\mu \ > \ \xi.$$

We set

$$u(\omega) \ \stackrel{df}{=} \ \begin{cases} s(\omega) & \text{if} \quad \omega \in \Omega_0, \\ u_0(\omega) & \text{if} \quad \omega \in \Omega \setminus \Omega_0. \end{cases}$$

Evidently $u \in L^1(\Omega; X)$ and we have

$$\big\langle v(\omega), u(\omega)\big\rangle_X - \varphi\big(\omega, u(\omega)\big) \ \geqslant \ \beta(\omega) \qquad \forall\, \omega \in \Omega_0$$

and

$$\big\langle v(\omega), u(\omega)\big\rangle_X - \varphi\big(\omega, u(\omega)\big) \ \geqslant \ \vartheta_0(\omega) \qquad \forall\, \omega \in \Omega \setminus \Omega_0$$

(see (4.35)). Therefore

$$\int_\Omega \big\langle v(\omega), u(\omega)\big\rangle_X\, d\mu - \int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu \ \geqslant \ \int_{\Omega_0} \beta(\omega)\, d\mu + \int_{\Omega \setminus \Omega_0} \vartheta_0(\omega)\, d\mu \ > \ \xi,$$

so $I_{\varphi^*}(v) = (I_\varphi)^*(v)$.

(b) Since

$$\varphi \ = \ \varphi^{**}$$

(see Theorem 4.4.14), this follows at once from part **(a)**. □

Next we consider integral functionals defined on the Lebesgue-Bochner space $L^\infty(\Omega; X^*)$. So as before (Ω, Σ, μ) is a σ-finite measure space, but now X is a separable Banach space with a separable dual X^*. Recall that $X^*_{w^*}$ (i.e., the Banach space X^* supplied with the w^*-topology) is a Souslin space (see Definition A.2.29**(b)**). It follows that $\mathcal{B}(X^*) = \mathcal{B}(X^*_{w^*})$ (see Denkowski, Migórski & Papageorgiou (2003*a*, p. 211). Also let $\varphi\colon \Omega \times X^*_{w^*} \longrightarrow \overline{\mathbb{R}}$ be a convex normal integrand. We consider the following integral functionals

$$I_\varphi(u) \stackrel{df}{=} \int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu \qquad \forall\, u \in L^1(\Omega; X^*) = L^1(\Omega; X^*_{w^*})$$

and

$$I_{\varphi^*}(v) \stackrel{df}{=} \int_\Omega \varphi^*\big(\omega, v(\omega)\big)\, d\mu \qquad \forall\, v \in L^\infty(\Omega; X).$$

From Theorem 4.5.2**(b)**, we know that if $\operatorname{dom} I_\varphi \neq \emptyset$ and $\operatorname{dom} I_{\varphi^*} \neq \emptyset$, then the functionals I_φ and I_{φ^*} are conjugate to each other. So we have

$$I_{\varphi^*}(v) = \sup_{u \in L^1(\Omega; X^*)} \big[\langle v, u\rangle_{L^1(\Omega; X^*)} - I_\varphi(u)\big]$$

and

$$I_\varphi(u) = \sup_{v \in L^\infty(\Omega; X^*)} \big[\langle u, v\rangle_{L^1(\Omega; X^*)} - I_{\varphi^*}(v)\big].$$

What about $\big(I_{\varphi^*}\big)^*$ defined on $(L^\infty(\Omega; X))^*$? To get an expression for this conjugate, first we need to introduce the structure of $(L^\infty(\Omega; X))^*$.

DEFINITION 4.5.3 *Let (Ω, Σ, μ) be a σ-finite measure space and let Y be a Banach space.*

(a) *A function $l \in \big(L^\infty(\Omega; Y)\big)^*$ is said to be* ***absolutely continuous with respect to*** *μ, if*

$$l(v) = \int_\Omega \big\langle u(\omega), v(\omega)\big\rangle_Y \qquad \forall\, v \in L^\infty(\Omega; Y),$$

with $u \in L^1(\Omega; Y^*_{w^*})$. The function u is said to be the ***density*** *of l with respect to μ. We have*

$$\|l\| = \|u\|_{L^1(\Omega; Y^*_{w^*})} = \int_\Omega \big\|u(\omega)\big\|_{Y^*}\, d\mu.$$

So we can identify an absolute continuous functional $l \in \big(L^\infty(\Omega; Y)\big)^*$, with its density with respect to μ.

(b) *A functional* $l \in \big(L^\infty(\Omega;Y)\big)^*$ *is said to be singular with respect to* μ*, if there exists a decreasing sequence* $\{C_n\}_{n\geqslant 1} \subseteq \Sigma$*, such that*

$$\mu(C_n) \searrow 0$$

and l *is supported by* C_n *for* $n \geqslant 1$*, that is if* $v \in L^\infty(\Omega;Y)$ *and* v *vanishes on some* C_n*, then* $l(v) = 0$*.*

REMARK 4.5.4 If μ is finite, then $l \in \big(L^\infty(\Omega;Y)\big)^*$ is singular if and only if for every $\varepsilon > 0$, we can find $A \in \Sigma$, such that $\mu(A) \leqslant \varepsilon$ and l is supported by A (i.e., if $v \in L^\infty(\Omega;Y)$, $v|_A = 0$, then $l(v) = 0$). For a given $l \in \big(L^\infty(\Omega;Y)\big)^*$ and $A \in \Sigma$, we define $l^A \in \big(L^\infty(\Omega;Y)\big)^*$, by

$$l^A(v) \stackrel{df}{=} l\big(\chi_A v\big) \qquad \forall\, v \in L^\infty(\Omega;Y).$$

It is easy to see that if $A, B \in \Sigma$ and $A \cap B = \emptyset$, then

$$\big\|l^{A\cup B}\big\|_{(L^\infty(\Omega;Y))^*} = \big\|l^A\big\|_{(L^\infty(\Omega;Y))^*} + \big\|l^B\big\|_{(L^\infty(\Omega;Y))^*}.$$

PROPOSITION 4.5.5
If (Ω, Σ, μ) *is a* σ*-finite measure space,* Y *is a Banach space,*

$$l \in \big(L^\infty(\Omega;Y)\big)^*$$

and for every $\varepsilon > 0$*, there exists* $A \in \Sigma$*, with*

$$\mu(A) \leqslant \varepsilon \quad \text{and} \quad \|l\|_{(L^\infty(\Omega;Y))^*} - \varepsilon \leqslant \big\|l^A\big\|_{(L^\infty(\Omega;Y))^*},$$

then $l \in \big(L^\infty(\Omega;Y)\big)^*$ *is singular with respect to* μ*.*

PROOF Let $\{A_n\}_{n\geqslant 1} \subseteq \Sigma$ be such that

$$\mu(A_n) \leqslant \frac{1}{2^n} \quad \text{and} \quad \|l\|_{(L^\infty(\Omega;Y))^*} - \frac{1}{2^n} \leqslant \big\|l^{A_n}\big\|_{(L^\infty(\Omega;Y))^*} \qquad \forall\, n \geqslant 1.$$

Since $\|l\|_{(L^\infty(\Omega;Y))^*} = \big\|l^{A_n}\big\|_{(L^\infty(\Omega;Y))^*} + \big\|l^{A_n^c}\big\|_{(L^\infty(\Omega;Y))^*}$, we have

$$\big\|l^{A_n^c}\big\|_{(L^\infty(\Omega;Y))^*} \leqslant \frac{1}{2^n} \qquad \forall\, n \geqslant 1.$$

Let

$$C_n \stackrel{df}{=} \bigcup_{k=n+1}^{\infty} A_k \qquad \forall\, n \geqslant 1.$$

We have

$$\mu(C_n) \leqslant \frac{1}{2^n} \qquad \forall\, n \geqslant 1.$$

We claim that

$$l \in \left(L^\infty(\Omega;Y)\right)^* \text{ is supported by } C_n \qquad \forall\, n \geqslant 1.$$

To this end let

$$v \in L^\infty(\Omega;Y) \quad \text{with } v|_{C_n} = 0.$$

So for all $k \geqslant n+1$, we have that

$$v \;=\; \chi_{A_k^c} v$$

and so

$$\left|l(v)\right| \;=\; \left|l^{A_k^c}(v)\right| \;\leqslant\; \left\|l^{A_k^c}\right\|_{(L^\infty(\Omega;Y))^*} \|v\|_{L^\infty(\Omega;Y)} \;\leqslant\; \frac{1}{2^k}\,\|v\|_{L^\infty(\Omega;Y)}\,.$$

Let $k \to +\infty$ to obtain that $l(v) = 0$ and so $l \in \left(L^\infty(\Omega;Y)\right)^*$ is singular. □

PROPOSITION 4.5.6
If (Ω, Σ, μ) is a finite measure space, Y is a Banach space,

$$L^\infty(\Omega) \otimes Y \;=\; \text{span}\left(\left\{gy :\; g \in L^\infty(\Omega),\; y \in Y\right\}\right),$$

$l \in \left(L^\infty(\Omega;Y)\right)^*$ *and*

$$l(w) \;=\; 0 \qquad \forall\, w \in L^\infty(\Omega) \otimes Y,$$

then l is singular.

PROOF Let Z be the subspace of $L^\infty(\Omega;Y)$ consisting of the equivalence classes of countably valued functions from Ω into Y. From Corollary 2.1.4, we know that Z is dense in $L^\infty(\Omega;Y)$. So for a given $\varepsilon > 0$, we can find $z \in Z$ with

$$\|z\|_{L^\infty(\Omega;Y)} \;\leqslant\; 1,$$

such that

$$\|l\|_{(L^\infty(\Omega;Y))^*} - \varepsilon \;\leqslant\; l(z).$$

So there exist a sequence $\{A_m\}_{m\geqslant 1} \subseteq \Sigma$ of pairwise disjoint sets with

$$\Omega \;=\; \bigcup_{m=1}^{\infty} A_m$$

and a sequence $\{x_m\}_{m\geqslant 1} \subseteq X$, such that

$$z(\omega) \;=\; x_m \qquad \forall\, \omega \in A_m.$$

Let $n \geqslant 1$ be large enough, such that

$$\mu\Big(\bigcup_{m=n}^{\infty} A_m\Big) \leqslant \varepsilon.$$

Since

$$w(\cdot) = \sum_{m=1}^{n-1} \chi_{A_m}(\cdot) x_m \in L^\infty(\Omega) \otimes Y,$$

we have

$$l(z) = l\Big(\chi_{\bigcup_{m=n}^{\infty} A_m} z\Big) \geqslant \|l\|_{(L^\infty(\Omega;Y))^*} - \varepsilon.$$

Applying Proposition 4.5.5, we obtain that $l \in \big(L^\infty(\Omega;Y)\big)^*$ is singular with respect to μ. □

Let us state the theorem which characterizes the dual space $(L^\infty(\Omega;Y))^*$. The decomposition produced by this theorem is analogous to the Lebesgue decomposition of measures. For a proof of the theorem we refer to Levin (1974).

THEOREM 4.5.7

If (Ω, Σ, μ) is a σ-finite measure space, Y is a Banach space and L_s is the space of singular continuous linear functionals on $L^\infty(\Omega;Y)$, then $\big(L^\infty(\Omega;Y)\big)^*$ is isometrically isomorphic to $L^1(\Omega;Y^*_{w^*}) \oplus L_s$ and

$$\|l\|_{(L^\infty(\Omega;Y))^*} = \|u\|_{L^1(\Omega;Y^*_{w^*})} + \|l_s\|_{L^\infty(\Omega;Y)^*},$$

where $l \in \big(L^\infty(\Omega;Y)\big)^*$, $u \in L^1(\Omega;Y^*_{w^*})$ and $l_s \in L_s$.

Now that we have a complete description of the dual space $\big(L^\infty(\Omega;Y)\big)^*$, we can return to our initial problem, namely the formula for $\big(I_{\varphi^*}\big)^*$, defined on $\big(L^\infty(\Omega;X)\big)^*$.

Recall that the mathematical setting is the following: (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space with a separable dual X^* and $\varphi\colon \Omega \times X^*_{w^*} \longrightarrow \overline{\mathbb{R}}$ is a convex normal integrand. As we already mentioned

$$\mathcal{B}(X^*) = \mathcal{B}(X^*_{w^*}).$$

We want to derive a formula for $\big(I_{\varphi^*}\big)^*\colon \big(L^\infty(\Omega;X)\big)^* \longrightarrow \overline{\mathbb{R}}$, defined by

$$\big(I_{\varphi^*}\big)^*(l) \stackrel{df}{=} \sup_{v \in L^\infty(\Omega;X)} \big[l(v) - I_{\varphi^*}(v)\big] \qquad \forall\, l \in \big(L^\infty(\Omega;X)\big)^*.$$

THEOREM 4.5.8
If the above hypotheses hold and $\operatorname{dom} I_\varphi$, $\operatorname{dom} I_{\varphi^*}$ *are nonempty, then*

$$\big(I_{\varphi^*}\big)^*(l) \;=\; I_\varphi(u) + \sigma_{\operatorname{dom} I_{\varphi^*}}(l_s),$$

where $l \in \big(L^\infty(\Omega;X)\big)^*$, $u \in L^1(\Omega;X^*)$ *is the density of* l *with respect to* μ *and* $l \in L_s$ *is the singular part of* l *with respect to* μ *(so we have that* $l = u + l_s$ *and* $\|l\|_{(L^\infty(\Omega;X))^*} = \|u\|_{L^1(\Omega;X^*)} + \|l_s\|_{(L^\infty(\Omega;X))^*}$*; see Theorem 4.5.7).*

PROOF By definition, we have

$$\begin{aligned}
\big(I_{\varphi^*}\big)^*(l) \;&=\; \sup_{v \in \operatorname{dom} I_{\varphi^*}} \big[\, \langle u, v\rangle_{L^\infty(\Omega;X)} + l_s(v) - I_{\varphi^*}(v)\big] \\
&\leqslant \sup_{v \in \operatorname{dom} I_{\varphi^*}} \big[\, \langle u, v\rangle_{L^\infty(\Omega;X)} - I_{\varphi^*}(v)\big] + \sup_{\widehat{v} \in \operatorname{dom} I_{\varphi^*}} l_s(\widehat{v}) \\
&=\; I_\varphi(u) + \sigma_{\operatorname{dom} I_{\varphi^*}}(l_s). \qquad (4.36)
\end{aligned}$$

Let $\{C_n\}_{n \geqslant 1} \subseteq \Sigma$ be a decreasing sequence, such that

$$\mu(C_n) \;\searrow\; 0$$

and l_s is supported by C_n for every $n \geqslant 1$. Also let

$$v_0 \in \operatorname{dom} I_{\varphi^*}, \quad \varepsilon \;>\; 0 \quad \text{and} \quad \xi \in \mathbb{R}$$

be such that

$$\xi \;<\; I_\varphi(u).$$

Note that

$$\varphi\big(\omega, u(\omega)\big) \;\geqslant\; \big\langle u(\omega), v_0(\omega)\big\rangle_X - \varphi^*\big(\omega, v_0(\omega)\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and so

$$I_\varphi(u) \;>\; -\infty.$$

Then for $n \geqslant 1$ large enough, we have

$$\int_{C_n} \big[\big\langle u(\omega), v_0(\omega)\big\rangle_X - \varphi^*\big(\omega, v_0(\omega)\big)\big]\, d\mu \;>\; -\varepsilon \qquad (4.37)$$

and

$$\int_{C_n^c} \varphi\big(\omega, u(\omega)\big)\, d\mu \;>\; \xi. \qquad (4.38)$$

We have

$$\begin{aligned}
\big(I_{\varphi^*}\big)^*(l) &= \sup_{v\in L^\infty(\Omega;X)}\left[l(v)-\int\limits_{C_n}\varphi^*\big(\omega,v(\omega)\big)\,d\mu-\int\limits_{C_n^c}\varphi^*\big(\omega,v(\omega)\big)\,d\mu\right]\\
&= \sup_{v\in L^\infty(\Omega\setminus C_n;X)}\left[\langle u,v\rangle_{L^\infty(\Omega;X)}-\int\limits_{C_n^c}\varphi^*\big(\omega,v(\omega)\big)\,d\mu\right]\\
&\quad+\sup_{\widehat{v}\in L^\infty(C_n;X)}\left[\langle u,\widehat{v}\rangle_{L^\infty(\Omega;X)}+l_s(\widehat{v})-\int\limits_{C_n}\varphi^*\big(\omega,\widehat{v}(\omega)\big)\,d\mu\right]\\
&= \int\limits_{C_n^c}\varphi\big(\omega,u(\omega)\big)\,d\mu\\
&\quad+\sup_{\widehat{v}\in L^\infty(C_n;X)}\left[l_s(\widehat{v})+\int\limits_{C_n}\left[\big\langle u(\omega),\widehat{v}(\omega)\big\rangle_X-\varphi^*\big(\omega,\widehat{v}(\omega)\big)\right]\,d\mu\right].
\end{aligned}$$

Let

$$\widehat{v}\ \stackrel{df}{=}\ \chi_{C_n}v_0.$$

Since

$$l_s\big(\chi_{C_n^c}v_0\big)\ =\ 0,$$

we have

$$l_s(v_0)\ =\ l_s\big(\chi_{C_n}v_0\big)$$

and so, using also (4.37) and (4.38), we obtain

$$\begin{aligned}
\big(I_{\varphi^*}\big)^*(l) &\geqslant \int\limits_{C_n^c}\varphi\big(\omega,u(\omega)\big)\,d\mu+l_s(v_0)\\
&\qquad+\int\limits_{C_n}\left[\,\big\langle u(\omega),v_0(\omega)\big\rangle_X-\varphi^*\big(\omega,v_0(\omega)\big)\right]d\mu\\
&\geqslant\ \xi+l_s(v_0)-\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we let $\varepsilon\searrow 0$ and have

$$\big(I_{\varphi^*}\big)^*(l)\ \geqslant\ \xi+l_s(v_0)\qquad\forall\ v_0\in\operatorname{dom}I_{\varphi^*},$$

so

$$\big(I_{\varphi^*}\big)^*(l)\ \geqslant\ I_\varphi(u)+\sigma_{\operatorname{dom}I_{\varphi^*}}(l_s). \tag{4.39}$$

From (4.36) and (4.39), we conclude that

$$\big(I_{\varphi^*}\big)^*(l)\ =\ I_\varphi(u)+\sigma_{\operatorname{dom}I_{\varphi^*}}(l_s).$$

□

REMARK 4.5.9 If $\operatorname{dom} I_{\varphi^*} = L^\infty(\Omega; X)$, then I_{φ^*} is continuous, in fact locally Lipschitz on $L^\infty(\Omega; X)$ (see Theorem 4.2.6 and 4.2.7). We will show that we can say more about the continuity of I_{φ^*}, when $\operatorname{dom} I_{\varphi^*} = L^\infty(\Omega; X)$ (see Proposition 4.5.14).

DEFINITION 4.5.10 *Let V and W be two linear spaces.*

(a) *We say that (V, W) form a **dual pair** (or **dual system**), if there exists a bilinear functional $b\colon V \times W \longrightarrow \mathbb{R}$, written $b(v, w) = \langle v, w\rangle$, such that*

(i) if $\langle v, w\rangle = 0$ for all $w \in W$, then $v = 0$;

(ii) if $\langle v, w\rangle = 0$ for all $v \in V$, then $w = 0$.

(b) *For a given dual pair (V, W) and a topology τ_V on V, we say that τ_V is compatible with the dual pair (V, W), if τ_V is a locally convex vector topology and $\left(V_{\tau_V}\right)^* = W$ (that is, if $v^* \in \left(V_{\tau_V}\right)^*$, then $v^*(v) = \langle v, w\rangle$ for some $w \in W$ and all $v \in V$ and conversely). So we view W as a subspace of the algebraic dual of V. Dually, a compatible topology τ_W on W is one such that $\left(W_{\tau_W}\right)^* = V$.*

(c) *The smallest topology on V compatible with the dual pair (V, W) is the weak topology denoted by $w(V, W)$. The largest topology on V compatible with the dual pair is the **Mackey topology** denoted by $m(V, W)$.*

REMARK 4.5.11 From the properties of the bilinear form $\langle \cdot, \cdot \rangle$, we see easily that a compatible topology is always Hausdorff. All compatible topologies have the same closed, convex sets and the same bounded sets (i.e., these properties are duality invariant). The Mackey topology $m(V, W)$ is the topology of uniform convergence on all balanced, convex, w-compact sets in W. Recall that $A \subseteq W$ is balanced if $\lambda A \subseteq A$ for all $|\lambda| \leqslant 1$. If $V = X$ is a Banach space and $W = X^*$, then $w(V, W)$ is the usual weak topology on X and $m(V, W)$ is the norm topology on X. On the other hand, if $V = X^*$ and $W = X$, then $w(V, W)$ is the weak*-topology on X^* and $m(V, W)$ is strictly smaller than the norm topology, unless X is reflexive.

DEFINITION 4.5.12 *Let (V, W) be a dual pair, let τ_W be a compatible topology on W and let $\varphi\colon W \longrightarrow \mathbb{R}^*$ be a function.*

*We say that φ is τ_W-inf-**compact** for the slope $v_0 \in V$, if for all $\lambda \in \mathbb{R}$, the set*

$$\{w \in W : \ \varphi(w) - \langle v_0, w\rangle \leqslant \lambda\}$$

is τ_W-compact.

*If $v_0 = 0$, then we simply say that φ is τ_W-inf-**compact**.*

The next proposition establishes an interesting connection between continuity of φ and τ_W-inf-compactness of φ^*.

PROPOSITION 4.5.13
If (V, W) *is a dual pair,* $w = w(V, W)$ *and* $\varphi \in \Gamma_0(V_w)$,
then φ *is finite and* $m(V, W)$*-continuous at* $v_0 \in V$ *if and only if* φ^* *is* $w(W, V)$-inf*-compact for the slope* $v_0 \in V$.

Using Theorem 4.5.8 and Proposition 4.5.13, we infer the following result for the integral functional I_{φ^*}.

PROPOSITION 4.5.14
If the hypotheses of Theorem 4.5.8 hold with $\operatorname{dom} I_{\varphi^*} = L^\infty(\Omega; X)$,
then

(a) I_{φ^*} *is a* $m\big(L^\infty(\Omega; X), L^1(\Omega; X^*)\big)$*-continuous function on* $L^\infty(\Omega; X)$;

(b) I_φ *is a* $w\big(L^1(\Omega; X^*), L^\infty(\Omega; X)\big)$-inf*-compact function for every slope in* $L^\infty(\Omega; X)$;

(c) *we have*

$$\big(I_{\varphi^*}\big)^*(l) \;=\; \begin{cases} I_\varphi(u) & \text{if } l = u \in L^1(\Omega; X^*), \\ +\infty & \text{otherwise.} \end{cases}$$

Another useful continuity result for the integral functional I_φ is the following.

PROPOSITION 4.5.15
If (Ω, Σ, μ) *is a nonatomic finite measure space,* X *is a separable Banach space,* $\varphi\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ *is a convex normal integrand,* $p \in [1, +\infty)$ *and* $I_\varphi\colon L^p(\Omega; X) \longrightarrow \overline{\mathbb{R}}$ *is continuous at a point,*
then I_φ *is continuous everywhere.*

PROOF Let $u_0 \in L^p(\Omega; X)$ be the point of continuity of I_φ. By considering if necessary the functional

$$x \;\longmapsto\; \widehat{I}_\varphi(x) \;\stackrel{df}{=}\; I_\varphi(u_0 + x) - I_\varphi(u_0),$$

we may assume without any loss of generality that $u_0 = 0$ and $I_\varphi(0) = 0$. So we can find $\delta > 0$, such that

$$I_\varphi(u) \;\leqslant\; 1 \qquad \forall\ \|u\|_{L^p(\Omega;X)} \leqslant \delta.$$

Let $x \in L^p(\Omega; X)$ be arbitrary. Exploiting the nonatomicity of μ and the absolute continuity of the Lebesgue integral, we can find $\delta_1 > 0$ and pairwise disjoint sets $\{A_k\}_{k=1}^N \subseteq \Sigma$, such that

$$\mu(A_k) \;\leqslant\; \delta_1 \quad \text{and} \quad \Big\|\chi_{A_k} u\Big\|_{L^p(\Omega;X)} \;\leqslant\; \delta \qquad \forall\ k \in \big\{1, \ldots, N\big\}.$$

We have

$$\int_{A_k} \varphi\big(\omega, u(\omega)\big)\, d\mu \;=\; \int_{\Omega} \varphi\big(\omega, \chi_{A_k}(\omega)u(\omega)\big)\, d\mu - \int_{A_k^c} \varphi(\omega, 0)\, d\mu \;\leqslant\; 1 + \xi,$$

for some $\xi > 0$ independent of $k \in \{1, \ldots, N\}$. Therefore, we conclude that

$$\sum_{k=1}^{N} \int_{A_k} \varphi\big(\omega, u(\omega)\big)\, d\mu \;=\; \int_{\Omega} \varphi\big(\omega, u(\omega)\big)\, d\mu \;=\; I_\varphi(u) \;<\; +\infty,$$

so I_φ is continuous everywhere on $L^p(\Omega; X)$ (see Theorem 4.2.3). □

Next we describe the subdifferential of the integral functional I_φ.

THEOREM 4.5.16
If (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space, $\varphi\colon \Omega \times X \longrightarrow \overline{\mathbb{R}}$ is a convex normal integrand and $I_\varphi\colon L^p(\Omega; X) \longrightarrow \overline{\mathbb{R}}$, $p \in [1, +\infty)$ is finite for at least one $u_0 \in L^p(\Omega; X)$,
then for every $u \in L^p(\Omega; X)$, we have that

$$u^* \in \partial I_\varphi(u) \subseteq L^{p'}\big(\Omega; X^*_{w^*}\big)$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$) if and only if

$$u^*(\omega) \;\in\; \partial\varphi\big(\omega, u(\omega)\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

PROOF According to Proposition 4.4.21, we have that $u^* \in \partial I_\varphi(u)$ if and only if

$$I_\varphi(u) + \big(I_\varphi\big)^*(u^*) \;=\; \langle u^*, v\rangle_{L^p(\Omega;X)}$$

(see Remark 2.2.13). From Theorem 4.5.2**(a)**, we know that

$$\big(I_\varphi\big)^* \;=\; I_{\varphi^*}.$$

So we have that $u^* \in \partial I_\varphi(u)$ if and only if

$$\int_{\Omega} \big[\varphi\big(\omega, u(\omega)\big) + \varphi^*\big(\omega, u^*(\omega)\big)\big]\, d\mu \;=\; \int_{\Omega} \big\langle u^*(\omega), u(\omega)\big\rangle_X\, d\mu.$$

The result now follows from the fact that

$$\varphi\big(\omega, u(\omega)\big) + \varphi^*\big(\omega, u^*(\omega)\big) \;\geqslant\; \big\langle u^*(\omega), u(\omega)\big\rangle_X \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

□

Before passing to the study of the subdifferentials of nonconvex integral functionals, let us prove a last result for convex subdifferentials. It concerns functionals defined on the space of continuous functions defined on a compact metric space K.

So let K be a compact metric space and consider the Banach space $C(K)$ (with the supremum norm). The Riesz-Markov representation theorem (see Theorem 2.3.41) says that

$$C(K)^* \;=\; M(K),$$

where $M(K)$ is the Banach space of Radon measures, i.e., the space of all signed Borel measures which are of bounded variation with the norm given by the total variation, namely

$$\|\mu\| \;=\; \sup\left\{\sum_{k=1}^{N}\big|\mu(A_k)\big| : \bigcup_{k=1}^{N} A_k \subseteq A,\ A_k \cap A_i = \emptyset \text{ for } k \neq i,\ N \geqslant 1\right\}.$$

In what follows by $\langle\cdot,\cdot\rangle_{C(K)}$ we denote the duality brackets for the pair $\big(C(K), M(K)\big)$, i.e.,

$$\langle \mu, u\rangle_{C(K)} \;\stackrel{df}{=}\; \int_K u(x)\, d\mu(x) \qquad \forall\ u \in C(K),\ \mu \in M(K).$$

We say that $\mu \in M(K)$ is ***positive***, denoted by $\mu \geqslant 0$, if

$$\langle \mu, u\rangle_{C(K)} \;\geqslant\; 0 \qquad \forall\ u \in C(K),\ u \geqslant 0.$$

If $e \in C(K)$ is the function, such that

$$e(x) = 1 \qquad \forall\ x \in K \quad \text{and} \quad \langle \mu, e\rangle_{C(K)} = 1,$$

then we say that the Radon measure $\mu \in M(K)$ has ***total mass one***.

A Radon measure $\mu \in M(K)$ vanishes in an open set $U \subseteq K$, if

$$\langle \mu, u\rangle_{C(K)} \;=\; 0 \qquad \forall\ u \in C(K),\ \operatorname{supp} u \subseteq U;$$

recall that

$$\operatorname{supp} u \;\stackrel{df}{=}\; \overline{\{x \in K :\ u(x) \neq 0\}}.$$

By using partition of unity, we can show that if μ vanishes in a collection of open sets U_r, then μ also vanishes on $\bigcup U_r$. Hence it follows that there exists a largest open set $\widehat{U}$ where μ vanishes. Then the set $K \setminus \widehat{U}$ is called the support of μ and is denoted by $\operatorname{supp}\mu$.

LEMMA 4.5.17
If $u \in C(K)$, $\mu \in M(K)$ *and* $u|_{\operatorname{supp}\mu} = 0$,
then $\langle \mu, u \rangle_{C(K)} = 0$.

PROOF If $d_{_K}$ is a metric on K, for each $\varepsilon > 0$, let

$$\big(\operatorname{supp}\mu\big)_{\varepsilon} \stackrel{df}{=} \big\{x \in K : \ d_{_K}\big(x, \operatorname{supp}\mu\big) < \varepsilon\big\}.$$

Using Urysohn's lemma (see Theorem A.1.13), for every $n \geqslant 1$, we can find $\vartheta_n \in C(K)$, such that

$$\vartheta_n|_{\overline{(\operatorname{supp}\mu)_{\frac{1}{n}}}} \ \equiv \ 0$$

and

$$\vartheta_n|_{(\operatorname{supp}\mu)^c_{\frac{2}{n}}} \ \equiv \ 1.$$

Then

$$\vartheta_n u \ \longrightarrow \ u \quad \text{in } C(K)$$

and so

$$\langle \mu, \vartheta_n u \rangle_{C(K)} \ \longrightarrow \ \langle \mu, u \rangle_{C(K)}.$$

Note that

$$\operatorname{supp}\vartheta_n u \ \subseteq \ \widehat{U} \ = \ K \setminus \operatorname{supp}\mu \qquad \forall\, n \geqslant 1.$$

Hence

$$\langle \mu, \vartheta_n u \rangle_{C(K)} \ = \ 0$$

and so we conclude that

$$\langle \mu, u \rangle_{C(K)} \ = \ 0.$$

□

PROPOSITION 4.5.18
If $\xi \colon C(K) \longrightarrow \mathbb{R}$ *is defined by*

$$\xi(u) \stackrel{df}{=} \max_{x \in K} u(x),$$

then ξ *is continuous, convex and for each* $u \in C(K)$, *we have that*

$$\mu \ \in \ \partial\xi(u)$$
if and only if
$$\mu \geqslant 0, \quad \langle \mu, e \rangle_{C(K)} = 1 \quad \textit{and} \quad \operatorname{supp}\mu \subseteq \big\{x \in K : \ \xi(u) = u(x)\big\}.$$

PROOF The convexity of ξ is clear. To establish the continuity of ξ, we argue as follows. Let

$$u, v \in C(K) \quad \text{and} \quad x_0 \in K$$

be such that

$$\xi(u) = \max_{x \in K} u(x) = u(x_0).$$

We have

$$\xi(u) - \xi(v) \leqslant u(x_0) - v(x_0) \leqslant \|u - v\|_\infty .$$

Reversing the roles of u and v in the above argument, we conclude that

$$\big|\xi(u) - \xi(v)\big| \leqslant \|u - v\|_\infty ,$$

i.e., ξ is Lipschitz continuous.

Now let us prove the description of $\partial\xi$.

(a) Let $\mu \in \partial\xi(u)$. Then we have

$$\langle \mu, v - u \rangle_{C(K)} \leqslant \xi(v) - \xi(u) \qquad \forall\, v \in C(K). \tag{4.40}$$

Let

$$g \in C(K), \quad g \geqslant 0$$

and let us set

$$v \stackrel{df}{=} u - g.$$

From (4.40), we have

$$-\langle \mu, g \rangle_{C(K)} \leqslant \max_{x \in K} \big(u - g\big)(x) - \max_{x \in K} u(x) \leqslant 0,$$

so

$$\langle \mu, g \rangle_{C(K)} \geqslant 0.$$

Also let $c \in \mathbb{R}$ and let us set

$$v \stackrel{df}{=} u + ce.$$

From (4.40), we have

$$c\,\langle \mu, e \rangle_{C(K)} \leqslant \max_{x \in K} \big(u + ce\big)(x) - \max_{x \in K} u(x) = c$$

(recall that $e \equiv 1$). Since $c \in \mathbb{R}$ was arbitrary, we obtain

$$\langle \mu, e \rangle_{C(K)} = 1.$$

Next we show that

$$\operatorname{supp} \mu \subseteq C \stackrel{df}{=} \big\{x \in K :\ \xi(u) = u(x)\big\}.$$

It suffices to show that μ vanishes in any open set $U \subseteq K \setminus C$. To this end let $g \in C(K)$ be such that $\operatorname{supp} g \subseteq U$. Also let

$$\eta \stackrel{df}{=} \xi(u) - \max_{x \in \operatorname{supp} g} u(x) > 0.$$

We choose $\varepsilon > 0$ so that

$$\pm\varepsilon g(x) < \eta \qquad \forall\, x \in K.$$

Then

$$u(x) \pm \varepsilon g(x) < \xi(u) \qquad \forall\, x \in \operatorname{supp} g$$

and

$$\xi(u \pm \varepsilon g) = \xi(u).$$

So if in (4.40), we set $v \stackrel{df}{=} u \pm \varepsilon g$, we obtain

$$\pm\varepsilon \langle \mu, g \rangle_{C(K)} \leqslant 0,$$

i.e., $\langle \mu, g \rangle_{C(K)} = 0$, so

$$\operatorname{supp} \mu \subseteq \big\{ x \in K : \xi(u) = u(x) \big\}.$$

(b) Note that

$$g = u - \xi(u)e$$

and

$$g(x) = 0 \qquad \forall\, x \in \operatorname{supp} \mu.$$

Using Lemma 4.5.17, we obtain

$$\langle \mu, g \rangle_{C(K)} = 0$$

and so

$$\langle \mu, u \rangle_{C(K)} = \xi(u).$$

Hence if $v \in C(K)$, we have

$$\xi(v) - \xi(u) = \xi(v) - \langle \mu, u \rangle_{C(K)}. \tag{4.41}$$

Let

$$g \stackrel{df}{=} \xi(v)e - v.$$

Evidently

$$g \geqslant 0$$

and so

$$\langle \mu, g \rangle_{C(K)} \geqslant 0,$$

hence

$$\xi(v) \;\geqslant\; \langle \mu, u \rangle_{C(K)}\,.$$

Using this in (4.41), we obtain

$$\xi(v) - \xi(u) \;\geqslant\; \langle \mu, v-u \rangle_{C(K)} \qquad \forall\, v \in C(K),$$

so $\mu \in \partial\xi(u)$. □

Now we consider nonconvex locally Lipschitz integral functionals. The mathematical setting is the following: (Ω, Σ, μ) is a finite measure space, X is a separable Banach space and

$$\varphi\colon \Omega \times X \longrightarrow \mathbb{R}$$

is a measurable function. We consider the integral functional

$$I_\varphi\colon L^p(\Omega; X) \longrightarrow \overline{\mathbb{R}},$$

$p \in [1, +\infty)$, defined by

$$I_\varphi(u) \;\stackrel{df}{=}\; \int_\Omega \varphi\big(\omega, u(\omega)\big)\, d\mu \qquad \forall\, u \in L^p(\Omega; X).$$

Our goal is to describe $\partial I_\varphi(u)$ under one the following two hypotheses:

$\underline{H(\varphi)_1}$ We have

$$\big|\varphi(\omega, x) - \varphi(\omega, y)\big| \;\leqslant\; k(\omega)\, \|x-y\|_X\,,$$

for μ-almost all $\omega \in \Omega$ and all $x, y \in X$, with $k \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

$\underline{H(\varphi)_2}$ For μ-almost all $\omega \in \Omega$, the function $\varphi(\omega, \cdot)$ is locally Lipschitz and

$$\|x^*\|_{X^*} \;\leqslant\; a(z)\left(1 + \|x\|_X^{p-1}\right),$$

for μ-almost all $\omega \in \Omega$, all $x \in X$ and all $x^* \in \partial\varphi(\omega, x)$, with $a \in L^\infty(\Omega)$.

THEOREM 4.5.19
If $\varphi\colon \Omega \times X \longrightarrow \mathbb{R}$ *is a measurable function and satisfies either hypotheses* $H(\varphi)_1$ *or* $H(\varphi)_2$,
then I_φ *is Lipschitz continuous on bounded sets of* $L^p(\Omega; X)$ *and if* $u^* \in \partial I_\varphi(u)$, *then*

$$u^*(\omega) \;\in \partial\varphi\big(\omega, u(\omega)\big) \quad \textit{for } \mu\textit{-a.a. } \omega \in \Omega.$$

PROOF *Case 1.* First suppose that hypothesis $H(\varphi)_1$ holds.

Then for $u, v \in L^p(\Omega; X)$, we have

$$\begin{aligned} & \big|I_\varphi(u) - I_\varphi(v)\big| \\ \leqslant & \int_\Omega \big|\varphi\big(\omega, u(\omega)\big) - \varphi\big(\omega, v(\omega)\big)\big|\, d\mu \\ \leqslant & \int_\Omega k(\omega)\big\|u(\omega) - v(\omega)\big\|_X\, d\mu \\ \leqslant & \|k\|_{p'}\, \|u - v\|_{L^p(\Omega;X)}\,, \end{aligned}$$

so I_φ is Lipschitz continuous (globally) on $L^p(\Omega; X)$.

Case 2. Next suppose that hypothesis $H(\varphi)_2$ holds.

We show that

$$I_\varphi \text{ is Lipschitz continuous on bounded sets of } L^p(\Omega; X).$$

Let $u, v \in L^p(\Omega; X)$ be such that

$$\|u\|_{L^p(\Omega;X)} \leqslant r \quad \text{and} \quad \|v\|_{L^p(\Omega;X)} \leqslant r.$$

Using Theorem 4.4.72, we can find

$$w(\omega) \in \big[u(\omega), v(\omega)\big] \stackrel{df}{=} \big\{(1-\lambda)u(\omega) + \lambda v(\omega) : \lambda \in [0,1]\big\}$$

and

$$w^*(\omega) \in \partial\varphi\big(\omega, w(\omega)\big),$$

such that

$$\begin{aligned} & \varphi\big(\omega, v(\omega)\big) - \varphi\big(\omega, u(\omega)\big) \\ = & \big\langle w^*(\omega), v(\omega) - u(\omega)\big\rangle_X \qquad \text{for } \mu\text{-a.a. } \omega \in \Omega. \end{aligned} \tag{4.42}$$

By virtue of hypothesis $H(\varphi)_2$, we have that

$$\begin{aligned} & \big\|w^*(\omega)\big\|_X \leqslant a(\omega)\left(1 + \big\|w(\omega)\big\|_X^{p-1}\right) \\ \leqslant & \widehat{a}(\omega)\left(1 + \big\|u(\omega)\big\|_X^{p-1} + \big\|v(\omega)\big\|_X^{p-1}\right) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega, \end{aligned} \tag{4.43}$$

with $\widehat{a} \in L^\infty(\Omega)$. Let

$$\eta(\omega) \stackrel{df}{=} \widehat{a}(\omega)\left(1 + \big\|u(\omega)\big\|_X^{p-1} + \big\|v(\omega)\big\|_X^{p-1}\right).$$

Then

$$\eta \in L^{p'}(\Omega)_+ \quad \text{and} \quad \|\eta\|_{p'} \leqslant c,$$

where $c > 0$ depends on $\|\widehat{a}\|_\infty$ and on $r > 0$. From (4.42) and (4.43), it follows that

$$\left|I_\varphi(v) - I_\varphi(u)\right| \leqslant c\,\|v-u\|_{L^p(\Omega;X)},$$

so I_φ is Lipschitz continuous on bounded sets.

Now let

$$u^* \in \partial I_\varphi(u) \subseteq L^{p'}\big(\Omega; X^*_{w^*}\big).$$

Then using Fatou's lemma (see Theorem A.2.1), we have

$$\begin{aligned}\langle u^*, h\rangle_{L^p(\Omega;X)} &\leqslant \big(I_\varphi\big)^0(u;h)\\ &\leqslant \int_0^b \varphi^0\big(\omega, u(\omega); h(\omega)\big)\,d\mu \qquad \forall\, h \in L^p(\Omega;X),\end{aligned}$$

so

$$\int_0^b \left[\varphi^0\big(\omega, u(\omega); h(\omega)\big) - \langle u^*(\omega), h(\omega)\rangle_X\right] d\mu \geqslant 0 \qquad \forall\, h \in L^p(\Omega;X).$$

Let

$$h \stackrel{df}{=} \chi_A z,$$

with $A \in \Sigma$, $z \in X$. We obtain

$$\int_A \left[\varphi^0\big(\omega, u(\omega); z\big) - \langle u^*(\omega), z\rangle_X\right] d\mu \geqslant 0.$$

Since $A \in \Sigma$ is arbitrary, we infer that

$$\big\langle u^*(\omega), z\big\rangle_X \leqslant \varphi^0\big(\omega, u(\omega); z\big) \quad \text{for a.a. } \omega \in \Omega \tag{4.44}$$

and the exceptional μ-null set is independent of $z \in X$ since X is separable. Since $z \in X$ is arbitrary, we conclude that

$$u^*(\omega) \in \partial\varphi\big(\omega, u(\omega)\big) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

□

REMARK 4.5.20 Note that under hypothesis $H(\varphi)_1$ we in fact proved that I_φ is Lipschitz continuous (globally).

4.6 Variational Principles

Suppose that X is a Banach space, $C \subseteq X$ is a nonempty, noncompact set and

$$\varphi\colon X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$$

is a proper, lower semicontinuous function, which is bounded below. Then the problem

$$\inf_{x \in C} \varphi(x)$$

need not have a solution. If $X = \mathbb{R}^N$, the situation can be remedied by considering a suitable small perturbation of φ. More specifically, for simplicity let

$$C \;=\; \mathbb{R}^N, \quad m \;=\; \inf_{x \in X} \varphi(x), \quad \varepsilon \;>\; 0$$

and take $x_0 \in \mathbb{R}^N$ to be such that

$$\varphi(x_0) \;\leqslant\; m + \varepsilon.$$

Consider the function

$$\varphi_\varepsilon(x) \;\stackrel{df}{=}\; \varphi(x) + \varepsilon \left\| x - x_0 \right\|_X .$$

Evidently $\varphi_\varepsilon\colon \mathbb{R}^N \longrightarrow \overline{\mathbb{R}}$ is proper, lower semicontinuous and in addition φ_ε is weakly coercive, i.e.,

$$\varphi_\varepsilon(x) \;\longrightarrow\; +\infty \quad \text{as } \left\| x \right\|_{\mathbb{R}^N} \to +\infty.$$

So invoking the Weierstrass theorem, we infer that φ_ε attains its infimum at a point $y \in \mathbb{R}^N$. Note that

$$\left\| y - x_0 \right\|_{\mathbb{R}^N} \;\leqslant\; 1.$$

Indeed, if

$$\left\| y - x_0 \right\|_{\mathbb{R}^N} \;>\; 1,$$

we have

$$\begin{aligned} \varphi_\varepsilon(y) \;&=\; \varphi(y) + \varepsilon \left\| y - x_0 \right\|_{\mathbb{R}^N} \;>\; \varphi(y) + \varepsilon \;\geqslant\; m + \varepsilon \\ &\geqslant\; \varphi(x_0) \;=\; \varphi_\varepsilon(x_0) \;\geqslant\; \varphi_\varepsilon(y), \end{aligned}$$

a contradiction. Also

$$\varphi_\varepsilon(y) \;\leqslant\; \varphi_\varepsilon(x) \qquad \forall\, x \in \mathbb{R}^N,$$

hence

$$\varphi(y) + \varepsilon \left\| y - x_0 \right\|_{\mathbb{R}^N} \;\leqslant\; \varphi(x) + \varepsilon \left\| x - x_0 \right\|_{\mathbb{R}^N},$$

so

$$\varphi(y) \leqslant \varphi(x) + \varepsilon \|x - y\|_{\mathbb{R}^N}.$$

So this argument shows that for a given $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$ satisfying

$$\varphi(x_0) \leqslant \inf_{x \in X} \varphi(x) + \varepsilon$$

(i.e., $x_0 \in \mathbb{R}^N$ is an ε-minimizer), we can find $y \in \mathbb{R}^N$, such that

$$\|y - x_0\|_{\mathbb{R}^N} \leqslant 1$$

and the function

$$x \longmapsto \varphi(x) + \varepsilon \|x - y\|_{\mathbb{R}^N}$$

attains its infimum at $y \in \mathbb{R}^N$. The main analytical tool in this argument was the theorem of Weierstrass, which guarantees a minimizer for a proper, lower semicontinuous, bounded from below function with at least one bounded sublevel set (this is the case if for example the function is weakly coercive). Evidently this is an essentially finite dimensional situation.

In an infinite dimensional space for this to work we need extra conditions, such as the reflexivity of X and the weak lower semicontinuity of the function. In general the argument fails. Nevertheless, we can salvage the principle formulated above. Namely if $x_0 \in X$ is an ε-minimizer of $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$, which is proper, lower semicontinuous, bounded from below, then a small Lipschitz perturbation of φ attains a strict minimum at a point $y \in X$, which is relatively close to x_0 (i.e., we can find a Lipschitz continuous function $h\colon X \longrightarrow \mathbb{R}$ with a small Lipschitz constant, such that $\varphi + h$ attains a strict minimum at $y \in X$). In fact this principle can be formulated in any complete metric space. This is the essence of the so-called Ekeland variational principle and its extensions. This result turned out to be an essential tool in many different areas of nonlinear analysis.

THEOREM 4.6.1

If (X, d_X) is a complete metric space, $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semicontinuous, bounded from below function, $\varepsilon > 0$ and $x_0 \in X$ satisfies

$$\varphi(x_0) \leqslant \inf_{x \in X} \varphi(x) + \varepsilon,$$

then for a given $\lambda > 0$, we can find $y_\lambda \in X$, such that

(a) $\varphi(y_\lambda) \leqslant \varphi(x_0)$;

(b) $d_X(y_\lambda, x_0) \leqslant \lambda$;

(c) $\varphi(y_\lambda) < \varphi(x) + \frac{\varepsilon}{\lambda} d_X(x, y_\lambda)$ *for all* $x \neq y_\lambda$.

PROOF By replacing φ with

$$\widehat{\varphi}(x) \stackrel{df}{=} \frac{1}{\varepsilon}\varphi(x)$$

and $d_{_X}(\cdot,\cdot)$ with

$$d_{_\lambda}(\cdot,\cdot) \stackrel{df}{=} \frac{1}{\lambda}d_{_X}(\cdot,\cdot),$$

without any loss of generality, we may assume that $\varepsilon = \lambda = 1$.

On X we define a relation by

$$\big[x \leqslant z\big] \quad \stackrel{df}{\Longleftrightarrow} \quad \big[\varphi(x) \leqslant \varphi(z) - d_{_X}(x,z)\big].$$

Evidently $x \leqslant x$ (i.e., the relation $\leqslant$ is reflexive). Also, if $x \leqslant z$, we have

$$\varphi(x) \;\leqslant\; \varphi(z) - d_{_X}(x,z)$$

and if $z \leqslant v$, we have

$$\varphi(z) \;\leqslant\; \varphi(v) - d_{_X}(z,v).$$

Thus, by the triangle inequality, it follows that

$$\varphi(x) \;\leqslant\; \varphi(v) - \big[d_{_X}(x,z) + d_{_X}(z,v)\big] \;\leqslant\; \varphi(v) - d_{_X}(x,v).$$

Therefore $x \leqslant v$ (i.e., the relation $\leqslant$ is transitive). Finally if $x \leqslant z$ and $z \leqslant x$, we obtain

$$d_{_X}(x,z) \;=\; 0,$$

hence $x = z$ (i.e., the relation $\leqslant$ is antisymmetric). So we conclude that the relation $\leqslant$ is a partial order.

Inductively we define a sequence $\{S_n\}_{n\geqslant 1}$ of subsets of X as follows. Let $x_1 = x_0$ and

$$S_1 \stackrel{df}{=} \big\{z \in X :\; z \leqslant x_1\big\},$$

$x_2 \in S_1$ is such that

$$\varphi(x_2) \;\leqslant\; \inf_{x\in S_1} \varphi(x) + \frac{1}{2^2}$$

and for the induction step, let

$$S_n \stackrel{df}{=} \big\{z \in X :\; z \leqslant x_n\big\},$$

$x_{n+1} \in S_n$ is such that

$$\varphi(x_{n+1}) \;\leqslant\; \inf_{x\in S_n} \varphi(x) + \frac{1}{2^{n+1}}.$$

Since $x_{n+1} \leqslant x_n$, we have that $S_{n+1} \subseteq S_n$ for $n \geqslant 1$ and by virtue of lower semicontinuity of φ, we have that S_n is closed. If $z \in S_{n+1}$, we have that $z \leqslant x_{n+1} \leqslant x_n$ and so

$$\begin{aligned} d_X(z, x_{n+1}) &\leqslant \varphi(x_{n+1}) - \varphi(z) \leqslant \inf_{x \in S_n} \varphi(x) + \frac{1}{2^{n+1}} - \varphi(z) \\ &\leqslant \varphi(z) + \frac{1}{2^{n+1}} - \varphi(z) = \frac{1}{2^{n+1}}, \end{aligned}$$

so

$$\operatorname{diam} S_{n+1} \leqslant \frac{1}{2^n} \qquad \forall\, n \geqslant 1,$$

i.e.,

$$\operatorname{diam} S_n \longrightarrow 0.$$

Because (X, d_X) is complete, by Cantor's theorem (see Theorem A.1.11), we have that

$$\bigcap_{n=1}^{\infty} S_n = \{y\}.$$

Since $y \in S_1$, we have $y \leqslant x_1 = x_0$ and so

$$\varphi(y) \leqslant \varphi(x_0) - d_X(y, x_0) \leqslant \varphi(x_0),$$

i.e., **(a)** holds.

Also we have (recall $\varepsilon = \lambda = 1$)

$$d_X(y, x_0) \leqslant \varphi(x_0) - \varphi(y) \leqslant \inf_{x \in X} \varphi(x) + 1 - \inf_{x \in X} \varphi(x) = 1,$$

i.e., **(b)** holds.

Finally to prove **(c)**, we need to show that $z \leqslant y$ implies $z = y$. Indeed, if $z \leqslant y$, then $z \leqslant x_n$ for all $n \geqslant 1$, hence

$$z \in \bigcap_{n=1}^{\infty} S_n,$$

which implies that $z = y$. □

REMARK 4.6.2 Note that in the above proposition conclusions **(b)** and **(c)** are somehow complementary and the choice of $\lambda > 0$ allows us to strike a balance between them depending on the application we have in mind. If $\lambda > 0$ is large then **(b)** provides little information on the whereabouts of y_λ while **(c)** tells us that y_λ is close to being a global minimizer of φ. The opposite situation occurs when $\lambda > 0$ is small. Then **(b)** implies that y_λ is close to x_0, but the inequality in **(c)** gives us little information. Two particular cases are of interest. The first corresponds to $\lambda = 1$, $\varepsilon > 0$ and the second to $\lambda = \sqrt{\varepsilon}$, $\varepsilon > 0$. In the first case we are not interested in conclusion **(b)** (i.e., we are not interested on how y_λ is located with respect to x_0). In the second case we are interested in both **(b)** and **(c)**. We state these two particular cases as corollaries.

COROLLARY 4.6.3
If (X, d_X) *is a complete metric space and* $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ *is a proper, lower semicontinuous, bounded from below function,*
then for every $\varepsilon > 0$, *we can find* $y_\varepsilon \in X$, *such that*

(a) $\varphi(y_\varepsilon) \leqslant \inf_{x\in X} \varphi(x) + \varepsilon$;

(b) $\varphi(y_\varepsilon) < \varphi(x) + \varepsilon d_X(x, y_\varepsilon)$ *for all* $x \neq y_\varepsilon$.

COROLLARY 4.6.4
If (X, d_X) *is a complete metric space,* $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ *is a proper, lower semicontinuous, bounded from below function,* $\varepsilon > 0$ *and* $x_\varepsilon \in X$ *satisfies*

$$\varphi(x_\varepsilon) \leqslant \inf_{x\in X} \varphi(x) + \varepsilon,$$

then we can find $y_\varepsilon \in X$, *such that*

(a) $\varphi(y_\varepsilon) \leqslant \varphi(x_\varepsilon)$;

(b) $d_X(y_\varepsilon, x_\varepsilon) \leqslant \sqrt{\varepsilon}$;

(c) $\varphi(y_\varepsilon) < \varphi(x) + \sqrt{\varepsilon} d_X(x, y_\varepsilon)$ *for all* $x \neq y_\varepsilon$.

If we put more structure on the space X, we can strengthen the conclusion of Theorem 4.6.1.

THEOREM 4.6.5
If X *is a Banach space and* $\varphi\colon X \longrightarrow \mathbb{R}$ *is a lower semicontinuous, bounded from below function which is Gâteaux differentiable,*
then for every $\varepsilon > 0$, *we can find* $x_\varepsilon \in X$, *such that*

$$\varphi(x_\varepsilon) \leqslant \inf_{x\in X} \varphi(x) + \varepsilon \quad \text{and} \quad \left\|\varphi'_G(x_\varepsilon)\right\|_{X^*} \leqslant \varepsilon.$$

PROOF By virtue of Corollary 4.6.3, we can find $x_\varepsilon \in X$, such that

$$\varphi(x_\varepsilon) \leqslant \inf_{x\in X} \varphi(x) + \varepsilon \quad \text{and} \quad \varphi(x_\varepsilon) \leqslant \varphi(x) + \varepsilon \left\|x - x_\varepsilon\right\|_X \qquad \forall\, x \in X.$$

Let $h \in X$ and $\lambda > 0$ be arbitrary. Let us set $x \stackrel{df}{=} x_\varepsilon + \lambda h$. We obtain

$$\frac{\varphi(x_\varepsilon) - \varphi(x_\varepsilon + \lambda h)}{\lambda} \leqslant \varepsilon \left\|h\right\|_X.$$

Passing to the limit as $\lambda \searrow 0$, we obtain

$$-\left\langle \varphi'_G(x), h\right\rangle_X \leqslant \varepsilon \left\|h\right\|_X \qquad \forall\, h \in X,$$

so $\left|\left\langle \varphi'_G(x), h\right\rangle_X\right| \leqslant \varepsilon \left\|h\right\|_X$ and thus $\left\|\varphi'_G(x)\right\|_{X^*} \leqslant \varepsilon$. □

An immediate consequence of this theorem is the following corollary.

COROLLARY 4.6.6
If X is a Banach space and $\varphi\colon X \longrightarrow \mathbb{R}$ is a lower semicontinuous, bounded from below function which is Gâteaux differentiable,
then there exists a sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$\varphi(x_n) \searrow \inf_{x\in X} \varphi(x) \quad \text{and} \quad \varphi'_G(x_n) \longrightarrow 0.$$

REMARK 4.6.7 The above corollary asserts the existence of a minimizing sequence, whose elements satisfy the first order necessary conditions, up to any desired approximation.

COROLLARY 4.6.8
If X is a Banach space and $\varphi\colon X \longrightarrow \mathbb{R}$ is a lower semicontinuous, bounded from below function which is Gâteaux differentiable,
then for each minimizing sequence $\{y_n\}_{n\geqslant 1}$ of φ (i.e., $\varphi(y_n) \searrow \inf_{x\in X} \varphi(x)$), we can find another minimizing sequence $\{x_n\}_{n\geqslant 1}$ of φ, such that:

(a) $\varphi(x_n) \leqslant \varphi(y_n)$;

(b) $\|x_n - y_n\|_X \longrightarrow 0$;

(c) $\left\|\varphi'_G(x_n)\right\|_{X^*} \longrightarrow 0$.

As we already said the Ekeland variational principle is a very powerful tool of nonlinear analysis. Below we show how the well known Caristi's fixed point theorem can be derived from the Ekeland variational principle. In fact we show the two results are equivalent, in the sense that the Ekeland variational principle can also be derived from Caristi's fixed point theorem. First we state and prove Caristi's fixed point theorem.

THEOREM 4.6.9 (Caristi Fixed Point Theorem)
If (X, d_X) is a complete metric space, $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semicontinuous, bounded from below function and $F\colon X \longrightarrow 2^X \setminus \{\emptyset\}$ is a multifunction, such that

$$\varphi(y) \leqslant \varphi(x) - d_X(y, x) \quad \text{for some } y \in F(x) \text{ and all } x \in X, \tag{4.45}$$

then there exists $x_0 \in X$, such that $x_0 \in F(x_0)$.

PROOF By virtue of Corollary 4.6.3 with $\varepsilon = 1$, we can find $x_0 \in X$, such that

$$\varphi(x_0) < \varphi(x) + d_X(x, x_0) \qquad \forall\, x \neq x_0. \tag{4.46}$$

We claim that $x_0 \in F(x_0)$. Suppose that this is not true. Then for all $y \in F(x_0)$, we have that $y \neq x_0$. Let $y \in F(x_0)$ be as in (4.45). We have

$$\varphi(y) \leqslant \varphi(x_0) - d_X(y, x_0) \quad \text{and} \quad \varphi(x_0) < \varphi(y) + d_X(y, x_0)$$

(see (4.46)), a contradiction. ∎

REMARK 4.6.10 We emphasize that on the multifunction F no regularity conditions were imposed except for (4.45), which is a mild restriction. Suppose that F has compact values and

$$h\big(F(x),F(y)\big) \leqslant kd_{_X}(x,y) \qquad \forall\, x,y\in X$$

and with $k\in(0,1)$. Here $h(\cdot,\cdot)$ stands for the Hausdorff metric on the nonempty and closed subsets of X. Then we can apply Theorem 4.6.9 with

$$\varphi(x) = \frac{1}{1-k}d_{_X}\big(x,F(x)\big).$$

Indeed let $y\in F(x)$ be such that $d_{_X}\big(x,F(x)\big)=d_{_X}(x,y)$. Such an element exists since $F(x)$ is compact. Then we have

$$\begin{aligned}(1-k)d_{_X}(x,y) &= d_{_X}\big(x,F(x)\big)-kd_{_X}(x,y)\\ \leqslant\ d_{_X}\big(x,F(x)\big)-h\big(F(x),F(y)\big) &\leqslant d_{_X}\big(x,F(x)\big)-d_{_X}\big(y,F(y)\big),\end{aligned}$$

so

$$\varphi(y) \leqslant \varphi(x)-d_{_X}(x,y).$$

So condition (4.45) is satisfied. The resulting fixed point theorem is a particular case of Nadler's fixed point theorem (see Theorem 7.4.3). Of course, if F is single valued, we recover the well known Banach's contraction principle (see Theorem 7.1.2). We should point out that Banach's fixed point theorem contains much more information.

PROPOSITION 4.6.11
The Caristi fixed point theorem (see Theorem 4.6.9) implies the Ekeland variational principle in the form of Corollary 4.6.3.

PROOF Let $\widehat{d}_{_X}=\varepsilon d_{_X}$. This is an equivalent metric on X. Proceeding by contradiction, suppose that there is no $x_\varepsilon\in X$ satisfying inequality **(b)** in Corollary 4.6.3. Then for every $x\in X$, we have

$$F(x)=\big\{y\in X:\ \varphi(x)\geqslant\varphi(y)+\widehat{d}_{_X}(x,y),\ y\neq x\big\}\neq\emptyset.$$

The multifunction F satisfies (4.45). So by Theorem 4.6.9, we can find $x_0\in X$, such that $x_0\in F(x_0)$. But this cannot happen since from the definition of F, $x\notin F(x)$ for all $x\in X$. □

There is another geometrical result of nonlinear analysis which is equivalent to some form of the Ekeland variational principle.

DEFINITION 4.6.12 *Let X be a normed space, $C\subseteq X$ a nonempty, convex set and $x\in X$. The **drop** associated with the pair (x,C), denoted by $D(x,C)$, is the convex hull of $\{x\}\cup C$, i.e.,*

$$D(x,C) = \big\{x+\lambda(c-x):\ c\in C,\ \lambda\in[0,1]\big\}.$$

REMARK 4.6.13 The set $D(x, C)$ is called a "drop," a suitable name given its geometry.

The next result is known as the drop theorem.

THEOREM 4.6.14 (Drop Theorem)
If X is a normed space, $A \subseteq X$ is a complete set, $y \in X \setminus A$, $R = d_X(y, A)$ and $0 < r < R < \varrho$,
then there exists $u \in A$, such that

$$u \in \overline{B}_\varrho(y) \quad \text{and} \quad D\big(u, \overline{B}_r(y)\big) \cap A \;=\; \{u\}.$$

PROOF By translating things, if necessary, we may assume that $y = 0$. Let

$$E \stackrel{df}{=} \overline{B}_\varrho \cap A.$$

This is a closed subset of A, hence it is a complete metric space (the metric induced by the norm of X). We introduce the continuous function

$$\varphi\colon E \longrightarrow \mathbb{R}_+,$$

defined by

$$\varphi(x) \stackrel{df}{=} \frac{\varrho + r}{R - r} \|x\|_X .$$

We apply Corollary 4.6.3 with $\varepsilon = 1$ to obtain $u \in E$, such that

$$\varphi(u) \;<\; \varphi(x) + \|u - x\|_X \qquad \forall\, x \in E,\ x \neq u. \tag{4.47}$$

We need to show that

$$D\big(u, \overline{B}_\varrho(0)\big) \cap A \;=\; \{u\}.$$

Suppose that this is not true and let $v \in D\big(u, \overline{B}_\varrho(0)\big) \cap A$. We have $v \in A$ and

$$v \;=\; (1 - \lambda)u + \lambda z$$

for some $z \in \overline{B}_r(0)$ and some $\lambda \in [0, 1]$. Since $v \neq u$ and $r < R$, it follows that $\lambda \in (0, 1)$. We have

$$\|v\|_X \;\leqslant\; (1 - \lambda)\|u\|_X + \lambda \|z\|_X .$$

Because $u \in A$, we have $\|u\|_X \geqslant R$ and so it follows that

$$\lambda(R - r) \;\leqslant\; \lambda\big(\|u\|_X - \|z\|_X\big) \;\leqslant\; \|u\|_X - \|v\|_X . \tag{4.48}$$

From (4.47) with $x = v$, we have

$$\frac{\varrho + r}{R - r}\|u\|_X \;<\; \frac{\varrho + r}{R - r}\|v\|_X + \|v - u\|_X \;=\; \frac{\varrho + r}{R - r}\|v\|_X + \lambda \|u - z\|_X ,$$

so

$$\frac{\varrho + r}{R - r} \left(\|u\|_X - \|v\|_X \right) \ < \ \lambda \|u - z\|_X$$

and thus using also (4.48), we have

$$\varrho + r \ < \ \|u - z\|_X .$$

But $\|u\|_X \leqslant \varrho$ (recall that $y = 0$) and $z \in \overline{B}_r(0)$. Hence

$$\|u - z\|_X \ \leqslant \ \varrho + r,$$

a contradiction. □

This geometrical result is in fact equivalent to the Ekeland variational principle stated in the following form which can be easily deduced from Corollary 4.6.3.

PROPOSITION 4.6.15
If (X, d_X) is a complete metric space and $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semicontinuous function which is bounded from below,
then for any $\beta > 0$ and any $x_0 \in X$, there exists $y \in X$, such that:

(a) $\varphi(y) < \varphi(x) + \beta d_X(y, x)$ *for all* $x \neq y$;

(b) $\varphi(y) \leqslant \varphi(x_0) - \beta d_X(y, x_0)$.

In this form the Ekeland variational principle is equivalent to the drop theorem. For the proof of this, see Penot (1986).

PROPOSITION 4.6.16
The drop theorem (see Theorem 4.6.14) is equivalent to the Ekeland variational principle in the form of Proposition 4.6.15.

We continue with the applications of the Ekeland variational principle.

PROPOSITION 4.6.17
If X is a Banach space, $\varphi\colon X \longrightarrow X$ is a lower semicontinuous, bounded from below function which is Gâteaux differentiable and there exist $a, c > 0$, such that

$$a \|x\|_X - c \ \leqslant \ \varphi(x) \qquad \forall\, x \in X, \tag{4.49}$$

then $\varphi'_G(X)$ is dense in $a\overline{B}_1^{X^*}$, where

$$\overline{B}_1^{X^*} \ = \ \left\{ x^* \in X^* : \ \|x^*\|_{X^*} \leqslant 1 \right\}.$$

PROOF Let $x^* \in a\overline{B}_1^{X^*}$ and consider the function

$$\psi(x) \stackrel{df}{=} \varphi(x) - \langle x^*, x\rangle_X .$$

Evidently ψ is lower semicontinuous, bounded from below (see (4.49)) and Gâteaux differentiable. Applying Theorem 4.6.5, we obtain $y_\varepsilon \in X$, such that

$$\left\|\psi'_G(y_\varepsilon)\right\|_{X^*} \leqslant \varepsilon.$$

But

$$\psi'_G(y_\varepsilon) = \varphi'_G(y_\varepsilon) - x^*.$$

Hence

$$\left\|\varphi'_G(y_\varepsilon) - x^*\right\|_{X^*} \leqslant \varepsilon.$$

Since $x^* \in a\overline{B}_1^{X^*}$ was arbitrary, we conclude that $\varphi'_G(X)$ is dense in $a\overline{B}_1^{X^*}$. □

COROLLARY 4.6.18
If X is a Banach space, $\varphi\colon X \longrightarrow \mathbb{R}$ is a lower semicontinuous, bounded from below function which is Gâteaux differentiable and there exists a continuous function $\vartheta\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$, such that

$$\frac{\vartheta(s)}{s} \longrightarrow +\infty \quad \text{as } s \to +\infty$$

and

$$\varphi(x) \geqslant \vartheta\big(\|x\|_X\big) \qquad \forall\, x \in X,$$

then $\varphi'_G(X)$ is dense in X^*.

PROOF Let $a > 0$. We can find $s_a > 0$, such that

$$\vartheta(s) \geqslant as \qquad \forall\, s \geqslant s_a.$$

Hence we have that

$$\varphi(x) \geqslant a\|x\|_X \qquad \forall\, x \in X,\ \|x\|_X \geqslant s_a.$$

On the other hand, if $\|x\|_X < s_a$, then $\varphi(x) \geqslant m_a$, where

$$m_a \stackrel{df}{=} \min_{s\in[0,s_a]} \vartheta(s).$$

Therefore, finally we have

$$\varphi(x) \geqslant a\|x\|_X - m_a.$$

Apply Proposition 4.6.17 to obtain that $\varphi'_G(X)$ is dense in $a\overline{B}_1^{X^*}$. Since $a > 0$ was arbitrary, we conclude that $\varphi'_G(X)$ is dense in X^*. □

Recall that if X is a Banach space and $\varphi \in \Gamma_0(X)$, then $D(\partial\varphi) \subseteq \operatorname{dom}\varphi$ (recall that $D(\partial\varphi) = \{x \in X : \partial\varphi(x) \neq \emptyset\}$ and $\operatorname{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}$). We would like to have a more precise relation between these two sets.

PROPOSITION 4.6.19
If X is a Banach space, $\varphi \in \Gamma_0(X)$, $x_0 \in \operatorname{dom}\varphi$, $\varepsilon > 0$ and $x_0^* \in \partial_\varepsilon\varphi(x_0)$, then there exist $x \in X$ and $x^* \in \partial\varphi(x)$, such that

(a) $\|x - x_0\|_X \leqslant \sqrt{\varepsilon}$;

(b) $\big|\varphi(x) - \varphi(x_0)\big| \leqslant \varepsilon + \sqrt{\varepsilon}$;

(c) $\|x^* - x_0^*\|_{X^*} \leqslant \sqrt{\varepsilon}\big(1 + \|x_0\|_X\big)$.

PROOF Let

$$\psi(x) \stackrel{df}{=} \varphi(x) - \langle x_0^*, x\rangle_X .$$

We have $\psi \in \Gamma_0(X)$ and

$$\psi(x) \geqslant -\varphi^*(x_0^*) \geqslant \varphi(x_0) - \langle x_0^*, x_0\rangle_X - \varepsilon$$

(see Remark 4.4.50). Moreover, from Remark 4.4.50, we have

$$\psi(x_0) \leqslant \inf_{x\in X} \psi(x) + \varepsilon.$$

On X we use the norm

$$|\!|\!| x |\!|\!|_X = \|x\|_X + \big|\langle x_0^*, x\rangle_X\big|,$$

which is equivalent to the original norm $\|\cdot\|_X$. Invoking Corollary 4.6.4, we obtain $x \in X$, such that

$$\psi(x) \leqslant \psi(x_0)$$

and

$$|\!|\!| x - x_0 |\!|\!|_X = \|x - x_0\|_X + \big|\langle x_0^*, x - x_0\rangle_X\big| \leqslant \sqrt{\varepsilon} \tag{4.50}$$

and

$$\varphi(x) - \langle x_0^*, x\rangle_X \leqslant \varphi(y) - \langle x_0^*, y\rangle_X + \sqrt{\varepsilon}\,|\!|\!| x - y |\!|\!|_X \qquad \forall\, y \in X. \tag{4.51}$$

From (4.51) we obtain that

$$\varphi(x) - \langle x_0^*, x\rangle_X = \inf_{y\in X} \psi_1(y),$$

where

$$\psi_1(y) \stackrel{df}{=} \big\{\varphi(y) - \langle x_0^*, y\rangle_X + \sqrt{\varepsilon}\,|\!|\!| x - y |\!|\!|_X\big\},$$

so, from Proposition 4.4.30, we have

$$0 \in \partial\psi_1(x).$$

Invoking Proposition 4.4.31, we have

$$\partial\psi_1(x) \;=\; \partial\varphi(x) - x_0^* + \sqrt{\varepsilon}\,\partial\big(\,\|\cdot\|_X\,\big)(0).$$

Note that

$$\partial\big(\,\|\cdot\|_X\,\big)(0) \;=\; u^* + \lambda x_0^*,$$

with $u^* \in \overline{B}_1^{X^*}$ and $\lambda \in [-1,1]$ (see Example 4.4.24**(b)**). Therefore, we have

$$0 \;=\; x^* - x_0^* + \sqrt{\varepsilon}\big(u^* + \lambda x_0^*\big),$$

with $x^* \in \partial\varphi(x)$, so

$$\|x^* - x_0^*\|_{X^*} \;\leqslant\; \sqrt{\varepsilon}\big(1 + \|x_0^*\|_{X^*}\big),$$

which proves **(c)**.

Also from (4.50) and since

$$\psi(x_0) \;\leqslant\; \inf_{x\in X}\psi(x) + \varepsilon,$$

we have

$$\big|\varphi(x) - \varphi(x_0)\big| \;\leqslant\; \psi(x_0) - \psi(x) + \big|\,\langle x_0^*, x - x_0\rangle_X\,\big| \;\leqslant\; \varepsilon + \sqrt{\varepsilon},$$

which proves **(b)**.

Finally again from (4.50), we have $\|x - x_0\|_X \leqslant \sqrt{\varepsilon}$, which proves **(a)**. □

THEOREM 4.6.20

If X is a Banach space and $\varphi \in \Gamma_0(X)$, then $D(\partial\varphi)$ is dense in $\operatorname{dom}\varphi$.

PROOF Let $x_0 \in \operatorname{dom}\varphi$, $n \geqslant 1$ and $x_{0n}^* \in \partial_{\frac{1}{n}}\varphi(x_0)$ (see Proposition 4.4.51). Invoking Proposition 4.6.19, we obtain $x_n \in D(\partial\varphi)$, such that

$$\|x_n - x_0\|_X \;\leqslant\; \frac{1}{\sqrt{n}} \quad\text{and}\quad \big|\varphi(x_n) - \varphi(x_0)\big| \;\leqslant\; \frac{1}{n} + \frac{1}{\sqrt{n}}.$$

□

REMARK 4.6.21 Actually in the proof of Theorem 4.6.20, we obtained something stronger. Namely if $x_0 \in \operatorname{dom}\varphi$, we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq D(\partial\varphi)$, such that

$$x_n \;\longrightarrow\; x_0 \quad \text{in } X$$

and

$$\varphi(x_n) \;\longrightarrow\; \varphi(x_0) \quad \text{in } \mathbb{R}.$$

Recall the following theorem from the theory of bounded linear operators between Banach spaces.

THEOREM 4.6.22
If X and Y are two Banach spaces and $A \in \mathcal{L}(X;Y)$, then the following statements are equivalent:

(a) *A is surjective;*

(b) *there exists $c > 0$, such that*

$$\|y^*\|_{Y^*} \leqslant c\,\|A^*y^*\|_{X^*} \qquad \forall\ y^* \in Y^*;$$

(c) *$N(A^*) = \{0\}$ and $R(A^*)$ is closed.*

REMARK 4.6.23 According to this theorem, if A is surjective, then A^* is injective. If one of the spaces X or Y is finite dimensional, then the converse is also true.

We want to produce a nonlinear analog of Theorem 4.6.22.

THEOREM 4.6.24
If X and Y are two Banach spaces, $\varphi\colon X \longrightarrow Y$ is a Gâteaux differentiable map, $\varphi(X)$ is closed in Y, $y \in Y$ and there exist $\varrho > 0$ and $k \in [0,1)$, such that

$$\varphi^{-1}\big(B_\varrho(y)\big) \neq \emptyset \tag{4.52}$$

and

$$\inf_{z \in \overline{R(\varphi'_G(x))}} \|y - \varphi(x) - z\|_Y \leqslant k\,\|y - \varphi(x)\|_Y \qquad \forall\ x \in \varphi^{-1}\big(B_\varrho(y)\big), \tag{4.53}$$

then $y \in \varphi(X)$.

PROOF Let

$$A = \varphi(X).$$

By hypothesis $A \subseteq Y$ is closed. We proceed by contradiction. So suppose that $y \notin \varphi(X)$. Let

$$R \stackrel{df}{=} d_Y(y, A)$$

and choose $\varrho, r > 0$, such that

$$r < R < \varrho \quad \text{and} \quad k\varrho < r.$$

Note that if (4.52) and (4.53) hold for some $\varrho_0 > 0$, then it also holds for any $\varrho \in (R, \varrho_0)$. According to Theorem 4.6.14, we can find $u_0 \in B_\varrho(y)$, such that

$$D(u_0, C) \cap A = \{u_0\},$$

where

$$C \stackrel{df}{=} \overline{B}_r(y).$$

Let $x_0 \in X$ be such that $\varphi(x_0) = u_0$. From (4.53) and recalling that $k\varrho < r$, we have

$$\inf_{z \in \overline{R(\varphi'_G(x_0))}} \|y - \varphi(x_0) - z\|_Y \leqslant k\|y - \varphi(x_0)\|_Y < r.$$

So we can find $h \in X$, such that

$$\|y - \varphi(x_0) - \varphi'_G(x_0)h\|_Y < r. \tag{4.54}$$

From this inequality, it follows that for $\lambda > 0$ small, we have

$$\left\| y - \varphi(x_0) - \frac{\varphi(x_0 + \lambda h) - \varphi(x_0)}{\lambda} \right\|_Y < r.$$

Let

$$v_\lambda \stackrel{df}{=} y - \varphi(x_0) - \frac{\varphi(x_0 + \lambda h) - \varphi(x_0)}{\lambda} \in Y.$$

From Definition 4.6.12, we see that

$$y - v_\lambda \in D(u_0, C),$$

where $C = \overline{B}_r(y)$, so

$$(1 - \lambda)u_0 + \lambda(y - v_\lambda) \in D(u_0, C) \qquad \forall\ \lambda \in (0, 1)$$

and thus

$$\varphi(x_0 + \lambda h) \in D(u_0, C) \qquad \forall\ \lambda > 0 \text{ small enough.}$$

Because $D(u_0, C) \cap A = \{u_0\}$, it follows that

$$\varphi(x_0 + \lambda h) = u_0 \qquad \forall\ \lambda > 0 \text{ small enough,}$$

so $\varphi'_G(x_0) = 0$.

Using this in (4.54), we obtain

$$\|y - \varphi(x_0)\|_X < r < R,$$

a contradiction. □

REMARK 4.6.25 If in the above theorem conditions (4.52) and (4.53) hold for all $y \in Y$, then we conclude that φ is surjective. Note that conditions (4.52) and (4.53) are in a sense complementary. Namely the larger $\varrho > 0$, the more difficult it is to verify (4.53).

COROLLARY 4.6.26
If X and Y are two Banach spaces, $\varphi\colon X \longrightarrow Y$ is a Gâteaux differentiable map, $\varphi(X)$ is closed in Y and

$$N\big(\big(\varphi'_G(x)\big)^*\big) \;=\; \{0\} \qquad \forall\, x \in X,$$

then φ is surjective.

PROOF Recall that

$$\overline{R\big(\varphi'_G(x)\big)} \;=\; {}^{\perp}N\big(\big(\varphi'_G(x)\big)^*\big)$$

(see, e.g., Denkowski, Migórski & Papageorgiou (2003*a*, p. 320)). Since by hypothesis $N\big(\big(\varphi'_G(x)\big)^*\big) = \{0\}$, it follows that (4.53) is true with $k = 0$. So for a given $y \in Y$, let $\varrho > 0$ be such that

$$d_{_Y}\big(y, \varphi(X)\big) \;<\; \varrho$$

and let $k = 0$. Then we can apply Theorem 4.6.24 and conclude that $y \in \varphi(X)$. This proves that φ is surjective. □

REMARK 4.6.27 It is clear from the proof of the above theorem that the hypothesis

$$N\big(\big(\varphi'_G(x)\big)^*\big) \;=\; \{0\} \qquad \forall\, x \in X$$

can be replaced by the hypothesis that

$$R\big(\varphi'_G(x)\big) \text{ is dense in } Y \qquad \forall\, x \in X.$$

Moreover, if $\varphi'_G(x) \in \Phi(X;Y)$ and $\operatorname{ind}\big(\varphi'_G(x)\big) = 0$ for all $x \in X$ (see Definition 3.1.60), then the hypothesis

$$N\big(\big(\varphi'_G(x)\big)^*\big) \;=\; \{0\} \qquad \forall\, x \in X$$

can be replaced by the hypothesis that

$$N\big(\varphi'_G(x)\big) \;=\; \{0\} \qquad \forall\, x \in X.$$

The discussion so far has illustrated the power of the Ekeland variational principle. The only difficulty that we encounter when using this principle is that the perturbation function $\varepsilon\,\|x - x_\varepsilon\|$ is not differentiable at the origin. So it is natural to ask whether it is possible to formulate a similar variational principle but for a different class of perturbations, which would include functions differentiable at points of interest. This was achieved by Borwein & Preiss (1987), who obtained the following theorem.

THEOREM 4.6.28
If X is a Banach space, $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semicontinuous, bounded from below function, $\varepsilon > 0$ and $x_0 \in X$ is such that

$$\varphi(x_0) \leqslant \inf_{x\in X} \varphi(x) + \varepsilon,$$

then for any $\lambda > 0$ and any $p \in [1, +\infty)$, we can find $y_\lambda \in X$, a sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$x_n \longrightarrow y_\lambda \quad \text{in } X$$

and a sequence $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ satisfying $\sum_{n=1}^{\infty} t_n = 1$, such that:

(a) $\varphi(y_\lambda) \leqslant \varphi(x_0)$;

(b) $\|x_n - x_0\|_X \leqslant \lambda$ *for all* $n \geqslant 1$;

(c) *the function* $\psi(x) = \varphi(x) + \frac{\varepsilon}{\lambda^p} \sum_{n=1}^{\infty} t_n \|x - x_n\|_X^p$ *attains a strict minimum at* y_λ.

The next step in the development of variational principles was made by Deville, Godefroy & Zizler (1993). Their starting point was the argument in the Borwein-Preiss variational principle. In that argument important is the function $\vartheta(x) \overset{df}{=} 1 - \|x\|_X$ and in particular what matters is the behaviour of ϑ within the unit ball. That is the behaviour of ϑ outside the domain where ϑ is nonnegative plays no role in the argument. So we may as well replace ϑ by the function

$$\widehat{\vartheta}(x) \overset{df}{=} \vartheta^+(x) = \max\{0, 1 - \|x\|_X\}.$$

Note that $\widehat{\vartheta}$ is a continuous bump function (i.e., a continuous function on X which has a nonempty and bounded support; see Remark 4.4.84). This observation is interesting because there are Banach spaces in which smooth bump functions can be found, but they do not have an equivalent differentiable norm (see Fabian (1997)).

THEOREM 4.6.29
If X is a Banach space which admits a Lipschitz continuous bump function that is Fréchet (respectively Gâteaux) differentiable, $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semicontinuous, bounded from below function and $\varepsilon > 0$,
then there exists a Lipschitz continuous function $g\colon X \longrightarrow \mathbb{R}$ which is Fréchet (respectively Gâteaux) differentiable, such that

$$\|g\|_\infty = \sup_{x\in X} |g(x)| \leqslant \varepsilon,$$
$$\|g'\|_\infty = \sup_{x\in X} \|g'(x)\|_{X^*} \leqslant \varepsilon$$

and $\varphi - g$ attains its minimum on X.

PROOF We do the proof for the Fréchet differentiable case. The proof of the Gâteaux differentiable case is done similarly. Let V be the linear space of all functions $g\colon X \longrightarrow \mathbb{R}$ which are Lipschitz continuous and Fréchet differentiable. Evidently

$$\big\|g'(x)\big\|_{X^*} \leqslant \mathrm{Lip}(g) \qquad \forall\ g \in V,\ x \in X$$

(by $\mathrm{Lip}(g)$ we denote the Lipschitz constant of g). So the function $x \longmapsto g'(x)$ is bounded and then by the mean value theorem (see Proposition 4.1.21), we have that $x \longmapsto g(x)$ is bounded too. It is easy to see that V supplied with the norm

$$\|g\|_V \ = \ \|g\|_\infty + \|g'\|_\infty$$

becomes a Banach space. For every $n \geqslant 1$, let

$$A_n \stackrel{df}{=} \Big\{ g \in V : \text{ there exists } x_0 \in X, \text{ such that } (\varphi - g)(x_0) < \inf_{x \in X \setminus B_{\frac{1}{n}}(x_0)} (\varphi - g)(x) \Big\},$$

where

$$B_{\frac{1}{n}}(x_0) \ = \ \big\{ x \in X : \ \|x - x_0\|_X < \frac{1}{n} \big\}.$$

We claim that for every $n \geqslant 1$, the set A_n is open and dense in V. To do this note that $\|\cdot\|_\infty \leqslant \|\cdot\|_V$. So it follows that A_n is open. To show the density of A_n, let $g \in A_n$ and $\varepsilon > 0$. We need to find $h \in V$ with $\|h\|_V < \varepsilon$ and $x_0 \in X$, such that

$$(\varphi - g - h)(x_0) \ < \ \inf_{x \in X \setminus B_{\frac{1}{n}}(x_0)} (\varphi - g - h)(x).$$

Using our assumption on the space X, we can find $b \in V$, such that

$$0 \ < \ b(0), \quad \|b\|_V \ < \ \varepsilon \quad \text{and} \quad b(x) \ = \ 0 \qquad \forall\ \|x\|_X \geqslant \frac{1}{n}.$$

The function $\varphi - g$ is bounded from below. So we can find $x_0 \in X$, such that

$$(\varphi - g)(x_0) \ < \ \inf_{x \in X} (\varphi - g)(x) + b(0).$$

Let

$$h(x) \stackrel{df}{=} b(x - x_0).$$

Evidently $h \in V$ with $\|h\|_V < \varepsilon$, and we have

$$\big(\varphi - g - h\big)(x_0) \ = \ \big(\varphi - g\big)(x_0) - b(0) \ < \ \inf_{x \in X} \big(\varphi - g\big)(x).$$

Since $h_{X \setminus B_{\frac{1}{n}}(x_0)} \equiv 0$, we have

$$\big(\varphi - g - h\big)(x) \ = \ \big(\varphi - g\big)(x) \ \geqslant \ \inf_{x \in X} \big(\varphi - g\big)(x) \qquad \forall\ x \in X \setminus B_{\frac{1}{n}}(x_0).$$

Hence $g+h \in A_n$ and this proves the density of A_n in V. Then by the Baire category theorem (see Theorem A.1.10), we have that

$$D \;=\; \bigcap_{n=1}^{\infty} A_n \subseteq Y \text{ is a dense } G_\delta \text{ set.}$$

Next we show that if $g \in D$, then $\varphi - g$ attains its minimum on X. From the definition of A_n, we can find $x_n \in X$, such that

$$\big(\varphi - g\big)(x_n) \;<\; \inf_{x \in X \setminus B_{\frac{1}{n}}(x_n)} \big(\varphi - g\big)(x).$$

For $m \geqslant n$, we have

$$x_m \;\in\; B_{\frac{1}{n}}(x_n),$$

or otherwise we would have

$$\big(\varphi - g\big)(x_n) \;<\; \big(\varphi - g\big)(x_m) \quad \text{and} \quad \|x_n - x_m\|_X \;\geqslant\; \frac{1}{n} \;>\; \frac{1}{m}. \tag{4.55}$$

By virtue of the second inequality and the choice of x_m, we have

$$\big(\varphi - g\big)(x_m) \;<\; \big(\varphi - g\big)(x_n),$$

which contradicts the first inequality in (4.55). Therefore we infer that

$$\{x_n\}_{n \geqslant 1} \subseteq X \text{ is a Cauchy sequence in } X$$

and

$$x_n \;\longrightarrow\; \widehat{x} \quad \text{in } X.$$

We claim that $\widehat{x}$ is a minimizer of $\varphi - g$.

Because $\varphi - g$ is lower semicontinuous, we have

$$\begin{aligned} \big(\varphi - g\big)(\widehat{x}) \;&\leqslant\; \liminf_{n \to +\infty} \big(\varphi - g\big)(x_n) \\ &\leqslant\; \liminf_{n \to +\infty} \left(\inf_{x \in X \setminus B_{\frac{1}{n}}(x_n)} \big(\varphi - g\big)(x) \right). \end{aligned} \tag{4.56}$$

If $u \in X$, $u \neq \widehat{x}$, then

$$\|x_n - u\|_X \;>\; \frac{1}{n} \qquad \forall\, n \geqslant 1 \text{ large enough}$$

and so for $n \geqslant 1$ large enough, we have

$$\inf_{x \in X \setminus B_{\frac{1}{n}}(x_n)} \big(\varphi - g\big)(x) \;\leqslant\; \big(\varphi - g\big)(u).$$

Using this in (4.56), we conclude that

$$\big(\varphi - g\big)(\widehat{x}) \;=\; \inf_{x \in X} \big(\varphi - g\big)(x).$$

∎

REMARK 4.6.30 If the norm of X is Fréchet differentiable away from the origin, then the function

$$x \longmapsto \|x\|_X^2$$

is a C^1-function on X. If $\xi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a C^1-function, such that

$$\xi(0) \;=\; 1 \quad \text{and} \quad \xi(s) \;=\; 0 \qquad \forall\, s \geqslant 1,$$

then

$$b(x) \stackrel{df}{=} \xi\big(\,\|x\|_X^2\,\big)$$

is a C^1-function on X, such that

$$b(0) \;=\; 1 \quad \text{and} \quad b(x) \;=\; 0 \qquad \forall\, x \notin B_1(0).$$

This is a C^1-bump function. Note that if X^* is separable, then X admits a C^1-bump function. Indeed in this case X admits an equivalent Fréchet differentiable norm and so the C^1-bump function is constructed as indicated above. Every separable Banach space admits an equivalent Gâteaux differentiable norm and so it admits a Lipschitz continuous, Gâteaux differentiable bump function.

The following result is an interesting consequence of Theorem 4.6.29.

PROPOSITION 4.6.31
If the Banach space X admits a Lipschitz continuous and Fréchet (respectively Gâteaux) differentiable bump function,
then every continuous convex function defined on X is Fréchet (respectively Gâteaux) differentiable on a dense subset of X. In particular, if the Banach space admits a Lipschitz continuous and Fréchet differentiable bump function, then X is an Asplund space (see Definition 4.2.18).

PROOF Again we do the proof for the Fréchet differentiable case. The proof for the Gâteaux differentiable case is similar.

Let φ be a continuous concave function on X and b be a Lipschitz continuous and Fréchet differentiable function on X with

$$b(0) \;\neq\; 0 \quad \text{and} \quad b(x) \;=\; 0 \qquad \forall\, x \notin B_1(0).$$

Let $x_0 \in X$ and choose $\delta > 0$, such that

$$\varphi(x_0) - 1 \;<\; \varphi(x) \qquad \forall\, x \in B_\delta(x_0).$$

Choose $m > \frac{1}{\delta}$. If $\|x - x_0\|_X \geqslant \delta$, we have

$$\big\|m(x - x_0)\big\|_X \;>\; 1$$

and so

$$b\big(m(x-x_0)\big) \;=\; 0.$$

We define the function

$$f(x) \;\stackrel{df}{=}\; \begin{cases} \dfrac{1}{b(m(x-x_0))^2} & \text{if} \quad b\big(m(x-x_0)\big) \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The function

$$\varphi + f \colon X \longrightarrow \overline{\mathbb{R}}$$

is proper, lower semicontinuous and bounded from below (by $\varphi(x_0) - 1$). So we can apply Theorem 4.6.29 and obtain a Lipschitz continuous and Fréchet differentiable function $g \colon X \longrightarrow \mathbb{R}$, such that $\varphi + f - g$ attains its minimum at some point $y_0 \in B_\delta(x_0)$ (since $f \equiv +\infty$ outside $B_\delta(x_0)$). Let U be a neighbourhood of y_0, such that the function

$$x \longmapsto b\big(m(x-x_0)\big)$$

is nonzero. Then the function f is Fréchet differentiable on U. We have

$$\varphi(y_0) + f(y_0) - g(y_0) \;\leqslant\; \varphi(y) + f(y) - g(y) \qquad \forall\, y \in U,$$

so

$$-\varphi(y) \;\leqslant\; -\varphi(y_0) - f(y_0) + g(y_0) + f(y) - g(y) \qquad \forall\, y \in U.$$

Let

$$v(y) \;\stackrel{df}{=}\; -\varphi(y_0) - f(y_0) + g(y_0) + f(y) - g(y) \qquad \forall\, y \in U.$$

We have

$$-\varphi(y) \;\leqslant\; v(y) \qquad \forall\, y \in U \quad \text{and} \quad -\varphi(y_0) \;=\; v(y_0).$$

If $\|h\|_X$ is small, from the convexity of $-\varphi$, we have

$$\begin{aligned} 0 \;&\leqslant\; (-\varphi)(y_0 + h) + (-\varphi)(y_0 - h) - (-2\varphi)(y_0) \\ &\leqslant\; v(y_0 + h) + v(y_0 - h) - 2v(y_0). \end{aligned} \tag{4.57}$$

Since v is Fréchet differentiable, we have

$$\begin{aligned} &v(y_0 + h) + v(y_0 - h) - 2v(y_0) \\ =\; &\big\langle v'_F(y_0), h \big\rangle_X - \big\langle v'_F(y_0), h \big\rangle_X + o\big(\|h\|_X\big). \end{aligned} \tag{4.58}$$

Hence from (4.57) and (4.58), it follows that

$$(-\varphi)(y_0 + h) + (-\varphi)(y_0 - h) - 2(-\varphi)(y_0) \;=\; o\big(\|h\|_X\big)$$

and this by virtue of Proposition 4.2.9 implies that $-\varphi$ is Fréchet differentiable at y_0. Since $x_0 \in X$ and $\delta > 0$ were arbitrary, we conclude that $-\varphi$ is Fréchet differentiable on a dense subset of X.

Finally for the last part of the proposition, recall that the set of points of differentiability of $-\varphi$ is a G_δ set (see the proof of Theorem 4.2.12). □

REMARK 4.6.32 It is not known whether every Asplund space admits a Fréchet differentiable bump function.

We conclude this section with a generalization of Theorem 4.6.1 which is useful when we study boundary value problems using variational methods. This generalization is due to Zhong (1997), where the interested reader can find its proof.

THEOREM 4.6.33
If $h\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a continuous, nondecreasing function, such that*

$$\int_0^{+\infty} \frac{1}{1+h(r)}\, dr \;=\; +\infty,$$

(X, d_X) *is a complete metric space,* $x_0 \in X$ *is fixed,* $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ *is a proper, lower semicontinuous and bounded below function,* $\varepsilon > 0$,

$$\varphi(y) \;\leqslant\; \inf_{x \in X} \varphi(x) + \varepsilon$$

and $\lambda > 0$,
then there exists $x_\lambda \in X$ *such that*

$$\varphi(x_\lambda) \;\leqslant\; \varphi(y), \quad d_X(x_\lambda, x_0) \;\leqslant\; r_0 + \overline{r}$$

and

$$\varphi(x_\lambda) \;\leqslant\; \varphi(x) + \frac{\varepsilon}{\lambda(1+h(d_X(x_0, x_\lambda)))} d_X(x_\lambda, x) \qquad \forall\, x \in X,$$

where

$$r_0 \;\stackrel{df}{=}\; d_X(x_0, y)$$

and $\overline{r} > 0$ *is such that*

$$\int_{r_0}^{r_0+\overline{r}} \frac{1}{1+h(r)}\, dr \;\geqslant\; \lambda.$$

REMARK 4.6.34 If $h \equiv 0$ and $x_0 = y$, then Theorem 4.6.33 reduces to Theorem 4.6.1.

4.7 Remarks

4.1: Gâteaux (1913) gave the definition of directional differentiability when X is simply a linear space and $Y = \mathbb{R}$. Afterwards, Lévy (1920) imposed the requirement that $f'(x;\cdot)$ must be linear. The Fréchet derivative was introduced by Fréchet (1920). Various parts of the calculus in Banach spaces can be found in the books of Abraham & Marsden (1978), Cartan (1967), Denkowski, Migórski & Papageorgiou (2003*a*, 2003*b*), Dieudonné (1969), Ioffe & Tihomirov (1979), Vaĭnberg (1973), Zeidler (1985*b*) and in the survey papers of Averbukh & Smolyanov (1967, 1968) and Nashed (1971). We should also mention Lusternik's theorem, which is useful in variational analysis. For a proof of it see Zeidler (1985*b*, pp. 287–289). First a definition.

DEFINITION 4.7.1 *Let X be a locally convex space, $C \subseteq X$ and $x_0 \in C$.*

(a) *An **admissible curve** in C through x_0 is a map $u\colon (-\varepsilon,\varepsilon) \longrightarrow C$ for some $\varepsilon > 0$, such that*

$$u(t) \in C \qquad \forall\, t \in (-\varepsilon,\varepsilon),$$

$$u(0) = x_0 \quad \text{and} \quad u'(0) \text{ exists.}$$

(b) *$h \in X$ is said to be a **tangent vector** to C at x_0 if and only if there exists an admissible curve in C through x_0, such that*

$$u'(0) = h,$$

that is if there exist an $\varepsilon > 0$ and a map

$$(-\varepsilon,\varepsilon) \ni \lambda \longmapsto r(\lambda) \in X,$$

such that

$$x_0 + \lambda h + r(\lambda) \in C \qquad \forall\, \lambda \in (-\varepsilon,\varepsilon)$$

and

$$\frac{\|r(\lambda)\|_X}{\lambda} \longrightarrow 0 \quad \text{as } \lambda \to 0.$$

(c) *The set of all vectors tangent to C at x_0 is a closed cone, which is nonempty since the origin belongs to it. This cone is usually called the **tangent cone** to C at x_0 and it is denoted by $T_C(x_0)$. If this cone is a subspace of X, then it is called the **tangent space** to C at x_0.*

THEOREM 4.7.2

(a) *If X and Y are two Banach spaces, U is a neighbourhood of $x_0 \in X$, $\varphi\colon U \longrightarrow Y$ is a Fréchet differentiable function and*

$$R\big(\varphi'_F(x_0)\big) \;=\; Y$$

(i.e., $x_0 \in U$ is a regular point of φ),
then the tangent space to the set

$$C \;\stackrel{df}{=}\; \big\{x \in X :\; \varphi(x) = \varphi(x_0)\big\}$$

coincides with the kernel of $\varphi'_F(x_0)$, i.e.,

$$T_C(x_0) \;=\; N\big(\varphi'_F(x_0)\big).$$

(b) *If X and Y are two Banach spaces, U is a neighbourhood of $x_0 \in X$, $\varphi \in C^1(U;Y)$ and for all $x \in U$, such that*

$$\varphi(x) \;=\; \varphi(x_0),$$

we have

$$R\big(\varphi'_F(x)\big) \;=\; Y$$

and $N\big(\varphi'_F(x)\big)$ is complemented in X (see Definition 4.1.28),
then the set

$$C \;\stackrel{df}{=}\; \big\{x \in U :\; \varphi(x) = \varphi(x_0)\big\}$$

is a C^1-manifold in X. Moreover, if $\varphi \in C^r(U;Y)$ with $r \geqslant 1$, then C is a C^r-manifold in X.

4.2: Convex functions play a central role in many applications. There are several books dealing with the continuity and differentiability properties of convex functions on finite or infinite dimensional Banach spaces. We mention the books of Rockafellar (1970*a*), Webster (1994) (convex functions defined on $\mathbb{R}^N$) and Barbu & Precupanu (1986), Ekeland & Temam (1976), Giles (1982), Ioffe & Tihomirov (1979), Laurent (1972), Phelps (1993) and Roberts & Varberg (1973) (convex functions defined on Banach and locally convex spaces). Note that the books of Giles (1982) and Phelps (1993) approach convex functions from the point of view of functional analysis and place special emphasis on the relations with Banach space theory. Theorem 4.2.12 is a classical result of Mazur (1933). For convex, continuous functions which are everywhere Gâteaux differentiable, there are some stronger results on Fréchet differentiability. In particular, Deville, Godefroy, Hare & Zizler (1987) characterize the separable Banach spaces X so that every continuous convex function

$\varphi\colon X \longrightarrow \mathbb{R}$, which is everywhere Gâteaux differentiable, must be Fréchet differentiable on a dense set. It turns out that X^* can be nonseparable, but X cannot contain a subspace isomorphic to l^1.

4.3: Haar-null sets (see Definition 4.3.5) were introduced by Christensen (1972), who also obtained all the results up until Corollary 4.3.14. Theorem 4.3.17 is also due to Christensen (1974). In the paper of Hunt, Sauer & Yorke (1992, 1993) (see also their addendum), we find relations between dynamical systems and Haar-null sets. There are other ways to define negligible sets in an infinite dimensional Banach space (such as Gauss-null sets, Aronszajn-null sets and cube-null sets). A detailed discussion of them and their use in the study of the differentiability properties of Lipschitz continuous functions can be found in the book of Benyamini & Lindenstrauss (1997).

4.4: Duality is in the core of convex analysis. The Legendre-Fenchel transform (see Definition 4.4.1) was first used for convex functions on $\mathbb{R}$ by Mandelbrojt (1939). This motivated Fenchel (1951) to introduce an important and more general definition for convex functions in $\mathbb{R}^N$. The transform introduced by Fenchel is an extension of the Legendre transform (see Legendre (1786)). This is why the transform is called Legendre-Fenchel transform. This notion was extended to dual pairs of locally convex spaces by Brondsted (1964), Moreau (1966–1967) and Rockafellar (1974). A special case of the inequality in Proposition 4.4.3**(a)** can be found in Young (1912) and for this reason the inequality is called Young-Fenchel inequality. We should point out that some authors (see Ioffe & Tichomirov (1968, 1979)) prefer to name Young-Fenchel transform for what we call here Legendre-Fenchel transform. The finite-dimensional duality theory can be found in the books of Fenchel (1951), Rockafellar (1970*a*), while the infinite dimensional duality theory can be found in the books of Barbu & Precupanu (1986), Ekeland & Temam (1976), Ioffe & Tihomirov (1979), Laurent (1972). Theorem 4.4.14 is the main result in the duality theory for convex functions and sometimes it is called the Fenchel-Moreau theorem. First Fenchel (1951) observed that $\varphi \in \Gamma_0\big(\mathbb{R}^N\big)$ if and only if it is supremum of all affine continuous functions majorized by φ. Soon thereafter Hörmander (1955) established the following result.

PROPOSITION 4.7.3

If X is a locally convex space,
then there is a bijective correspondence between nonempty, closed, convex sets and sublinear, $w(X^*, X)$-lower semicontinuous functions on X^* with values in $\overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$, which maps C into σ_C.

Theorem 4.4.14 in conjunction with Proposition 4.4.13 says that φ^{**} is the biggest convex and lower semicontinuous function majorized by φ (sometimes this is denoted by writing that $\varphi^{**} = \overline{\operatorname{conv}}\,\varphi$, which is a suggestive notation expressing the fact that $\operatorname{epi}\varphi^{**} = \overline{\operatorname{conv}}\operatorname{epi}\varphi$). This fact is important in control

theory in connection with the relaxation method. If the ambient space is $\mathbb{R}^N$, then using Carathéodory's theorem for convex sets in $\mathbb{R}^N$, we can have the following useful expression for φ^{**} (see Ioffe & Tihomirov (1979, p. 189)).

PROPOSITION 4.7.4
If $\varphi\colon \mathbb{R}^N \longrightarrow \overline{\mathbb{R}}$ *is a proper, lower semicontinuous function and* $\operatorname{dom}\varphi^{**} \subseteq \mathbb{R}^N$ *is closed,*
then

$$\varphi^{**}(x) \;=\; \inf\left\{ \sum_{k=1}^{N+1} \lambda_k \varphi(x_k) :\; x_k \in \mathbb{R}^N, \lambda_k \geqslant 0, \sum_{k=1}^{N+1} \lambda_k = 1, \sum_{k=1}^{N+1} \lambda_k x_k = x \right\}.$$

The operation of infimal convolution (see Definition 4.4.6**(b)**) was introduced by Moreau (1965, 1966–1967) and its duality properties were studied by Ioffe & Tichomirov (1968, 1979). The proof of Proposition 4.4.16 can be found in Ioffe & Tihomirov (1979, p. 178).

Although affine continuous supports for convex functions were considered earlier, the first systematic study of the subdifferential multifunction started with the works of Moreau (1965, 1966–1967) and Rockafellar (1966, 1970*b*). Moreau (1965) limits himself in the framework of Hilbert spaces, while Rockafellar (1970*b*) passes to general Banach spaces. We should also mention the related work of Pshenichnyi (1971) on quasi-differentiable functions. One of the main results of the convex subdifferential theory is Theorem 4.4.34 (see also Remark 4.4.35). This was first proved by Rockafellar (1966), but it was found that his proof had a gap. This was remedied by Rockafellar (1970*b*), where we find also the proof of Proposition 4.4.31. The notion of cyclically monotone operators is due to Rockafellar (1966), who proved Theorem 4.4.39 (see also Rockafellar (1970*b*, Theorem B, p. 210). Proposition 4.4.42 is due to Brézis (1973). For the proof of Proposition 4.4.46, we refer to Phelps (1993, p. 19).

The ε-subdifferential (see Definition 4.4.49) was investigated systematically by Hiriart-Urruty (1980, 1982), and Hiriart-Urruty & Phelps (1993). Convex subdifferentials found widespread applications in optimization, control theory and evolution equations, as seen in the books of Barbu (1976, 1994), Barbu & Precupanu (1986), Dontchev & Zolezzi (1993), Ekeland & Temam (1976), Hiriart-Urruty & Lemaréchal (1993), Hu & Papageorgiou (1997, 2000), Ioffe & Tihomirov (1979), Rockafellar (1970*a*, 1974), Rockafellar & Wets (1998) and Tiba (1990). The proof of Proposition 4.4.53 can be found in Rockafellar (1970*a*, pp. 219–220).

The subdifferential theory for locally Lipschitz functionals is due to Clarke (1975, 1981, 1983). Only Theorem 4.4.72 is due to Lebourg (1975). Applications of the generalized subdifferential can be found in the books of Clarke (1983, 1989), Clarke, Ledyaev, Stern & Wolenski (1998) and in Naniewicz & Panagiotopoulos (1995) and Gasiński & Papageorgiou (2005) (which deal with hemivariational inequalities). Of the other subdifferentials, the viscosity

subdifferential was explicitly defined by Deville, Godefroy & Zizler (1993) and studied by Borwein & Zhu (1996). The proximal subdifferential is discussed in Clarke (1989), Clarke, Ledyaev, Stern & Wolenski (1998) and Rockafellar & Wets (1998) and the canonical subdifferential was introduced by Penot (1978).

4.5: Integral functionals determined by (convex) normal integrands were first studied by Rockafellar (1968). Several results for integrands defined on $\Omega \times \mathbb{R}^N$ can be found in Rockafellar (1976) with additional results and extensions in Rockafellar (1971*a*, 1971*c*, 1971*b*). The work was extended by Levin (1973, 1974, 1975, 1980), who removed some restrictive finite dimensionality or reflexivity hypotheses.

Theorem 4.5.7, known as the Yosida-Hewitt decomposition theorem, was first proved by Yosida & Hewitt (1952) for $X = \mathbb{R}$ and μ being a finite measure. Another, more direct proof can be found in Dubovitskii & Miljutin (1968). Ioffe & Levin (1972) extended the result to a separable Banach space X and a finite measure μ. Their proof does not extend to μ being σ-finite. The general form (see Theorem 4.5.7) is due to Levin (1974). Theorem 4.5.2 is due to Rockafellar (1971*a*) (for a reflexive, separable Banach space X) and Levin (1975) (for a separable Banach space X). Similarly Theorem 4.5.8 is due to Rockafellar (1971*b*) (for $X = \mathbb{R}^N$), Rockafellar (1971*a*) (for a reflexive, separable Banach space X) and Levin (1975) (for a separable Banach space X). Proposition 4.5.13 is due to Moreau (1966–1967) and its proof can be also found in Laurent (1972, p. 348), while Theorem 4.5.16 is due to Rockafellar (1971*a*) Additional results on convex integral functionals can be found in Bismut (1973), Castaing & Valadier (1977), Papageorgiou (1986) and Valadier (1975). There is a continuous analog of the operation of infimal convolution (see Definition 4.4.6**(b)**).

DEFINITION 4.7.5 *Let (Ω, Σ, μ) be a finite, complete, nonatomic measure space and let*

$$\varphi\colon \Omega \times \mathbb{R}^N \longrightarrow \overline{\mathbb{R}}$$

be a $\Sigma \times \mathcal{B}(\mathbb{R}^N)$-measurable integrand. The inf***-convolution integral*** *of φ with respect to μ is the function*

$$\oint_{\Omega} \varphi_\omega \, d\mu\colon \mathbb{R}^N \longrightarrow \mathbb{R}^*,$$

defined by

$$\left(\oint_{\Omega} \varphi_\omega \, d\mu\right)(x) \stackrel{df}{=} \inf\left\{\lambda \in \mathbb{R} : \ (x,\lambda) \in \int_{\Omega} \operatorname{epi} \varphi(\omega,\cdot)\, d\mu\right\}.$$

REMARK 4.7.6 In the above definition

$$\int_\Omega \operatorname{epi}\varphi(\omega,\cdot)\,d\mu \;=\; \left\{\int_\Omega \big(u(\omega),\lambda(\omega)\big)\,d\mu :\; (u,\lambda)\in S^1_{\operatorname{epi}\varphi(\omega,\cdot)}\right\}.$$

If for all $x\in L^1\big(\Omega;\mathbb{R}^N\big)$, $I_\varphi(x)$ exists (possibly infinite), then

$$\left(\oint_\Omega \varphi_\omega\,d\mu\right)(x) \;=\; \inf\left\{I_\varphi(u) :\; u\in L^1\big(\Omega;\mathbb{R}^N\big),\; \int_\Omega u(\omega)\,d\mu = x\right\}.$$

In this form this operation arises in mathematical economics (see Aumann & Shapley (1974)).

The next result is due to Ioffe & Tichomirov (1968). It is the continuous analog of Proposition 4.4.8

PROPOSITION 4.7.7
If $\operatorname{dom}\left(\oint_\Omega \varphi_\omega\,d\mu\right)\neq\emptyset$,
then

$$\left(\oint_\Omega \varphi_\omega\,d\mu\right)^* \;=\; \int_\Omega \varphi^*_\omega\,d\mu.$$

4.6: Theorem 4.6.1 is due to Ekeland (1974). A detailed discussion with various applications can be found in Ekeland (1979, 1989). Theorem 4.6.9 with F being single valued was proved by Caristi (1976) using a different proof based on transfinite induction (see also Caristi & Kirk (1975)). Theorem 4.6.14 is due to Daneš (1972), but his proof used a result of Krasnoselskii & Zabreiko (1984). The proof given here is due to Brondsted (1974). Relations between these and other geometric theorems of nonlinear analysis were proved by Brézis & Browder (1976), Daneš (1972) and Penot (1986). Proposition 4.6.19 and Theorem 4.6.20 are due to Brondsted & Rockafellar (1965). For the proof of Theorem 4.6.22, we refer to Denkowski, Migórski & Papageorgiou (2003*a*, p. 384). Theorem 4.6.24 and Corollary 4.6.26 are due to Browder (1971*a*, 1971*b*). Another nonlinear surjectivity result due to Bates & Ekeland (1980) is the following.

PROPOSITION 4.7.8
If X and Y are two Banach spaces, $\varphi\colon X\longrightarrow Y$ is continuous, Gâteaux differentiable,

$$R\big(\varphi'_G(x)\big) \;=\; Y \qquad \forall\, x\in X$$

and there exists $k>0$, such that for all $x\in X$ and all $y\in Y$, there exists $z\in\big(\varphi'_G(x)\big)^{-1}(y)$ satisfying $\|z\|_X\leqslant k\,\|y\|_X$,
then $f(X)=Y$, i.e., f is surjective.

Theorem 4.6.28 is due to Borwein & Preiss (1987) and it is known as the Borwein-Preiss smooth variational principle. Theorem 4.6.29 is due to Deville, Godefroy & Zizler (1993). More applications of the Ekeland variational principle can be found in Barbu (1994), Denkowski, Migórski & Papageorgiou (2003*b*), Fattorini (1999), Li & Yong (1995) and Willem (1996).

Chapter 5

Critical Point Theory

Variational methods are a valuable tool in the analysis of nonlinear problems. According to these methods, we are trying to find solutions of a given nonlinear equation, by looking for critical (stationary) points of a functional defined on the function space in which we want the solution of our problem to lie. The Euler-Lagrange equation satisfied by a critical point is the nonlinear equation that we are trying to solve. The functional, whose critical points we are trying to determine, in many cases is unbounded (both from above and below – indefinite functional) and so we cannot expect to have global maxima or minima. Instead we look for local extrema or for saddle points using minimax arguments. So critical point theory is the main ingredient in the variational methods. In this chapter we present some aspects of critical point theory which are useful in the study of nonlinear boundary value problems.

A fruitful technique in obtaining critical points of a C^1-functional is based on deformation arguments along the gradient flow or a substitute of it when due to the geometry of the space or the lack of regularity of the functional it is impossible to use the gradient flow. For this reason in Section 5.1, we introduce the so-called pseudogradient vector field and the compactness-type conditions that this vector field must satisfy and we derive various deformation results, which describe the deformations of the sublevel sets of the functionals near a critical point where topologically interesting things may occur.

In Section 5.2 we use the deformation results and the geometric notion of linking sets in order to obtain minimax expressions for the critical values of the functionals. We prove a general minimax principle which generates as special cases the classical mountain pass theorem, saddle point theorem and generalized mountain pass theorem.

In Section 5.3 we prove strong forms of the mountain pass theorem, which, besides an existence statement for critical points, give in addition information about the fine structure of the functional near them.

In Section 5.4 we prove results establishing the existence of multiple critical points. For this purpose we introduce the notion of local linking, we impose symmetry conditions on the functional and we use the Krasnoselskii's genus of a set. The Krasnoselskii's genus is an example of a topological index. Historically the first topological index was introduced by Lusternik-Schnirelman in order to extend to nonlinear eigenvalue problems the theory of eigenvalues of quadratic forms developed by Courant, Weyl and others. It turns out that

the Lusternik-Schnirelman category is the maximal topological index which is invariant under homeomorphisms and satisfies certain properties (see Proposition 5.5.5).

In Section 5.6 we introduce the Lusternik-Schnirelman category and prove a basic multiplicity result for critical points of a C^1-functional. Also we develop the method of Lagrange multipliers for infinite dimensional constrained optimization problems, which provides the analytical framework for the study of nonlinear eigenvalue problems.

5.1 Deformation Results

Let X be a Hausdorff topological space and let $K \subseteq X$ be a closed subset. By a ***homotopy*** on X we mean a continuous function

$$h\colon [0,1] \times X \longrightarrow X,$$

such that

$$h(0,\cdot) \;=\; id_{_X},$$

i.e.,

$$h(0,x) \;=\; x \qquad \forall\, x \in X.$$

A homotopy h is sometimes called a ***deformation*** of X.

Let $\varphi\colon X \longrightarrow X$ be a continuous function and let $\mathcal{T}$ a family of subsets of X on which φ is bounded. We set

$$c \;=\; c(\varphi,\mathcal{T}) \;\stackrel{df}{=}\; \inf_{C\in\mathcal{T}} \sup_{x\in C} \varphi(x). \tag{5.1}$$

Our aim is to determine conditions which will guarantee that $c \in \varphi(K)$. The idea is to use deformations of the sublevels

$$\varphi^a \;\stackrel{df}{=}\; \varphi^{-1}\big((-\infty,a]\big) \;=\; \big\{x \in X :\; \varphi(x) \leqslant a\big\} \qquad \forall\, a \in \mathbb{R}.$$

To do this we shall use the following hypothesis.

$\underline{H(\varphi)}$ There is a homotopy $h\colon [0,1] \times X \longrightarrow X$, such that for all $a,b \in \mathbb{R}$, $a < b$ with $\varphi^{-1}\big([a,b]\big) \cap K = \emptyset$, there is $t > 0$ for which we have $h(t,\varphi^b) \subseteq \varphi^a$.

REMARK 5.1.1 Hypothesis $H(\varphi)$ tells us that nothing can happen topologically in between the levels a and b, if $[a,b]$ does not contain any elements of $\varphi(K)$.

Using this hypothesis, we can prove the following generalized minimax principle.

THEOREM 5.1.2 (Generalized Minimax Principle)
If X, K, φ, $\mathcal{T}$ and h are as above, hypothesis $H(\varphi)$ holds,

$$h(t, C) \in \mathcal{T} \qquad \forall\ C \in \mathcal{T},\ t > 0,$$

(i.e., $\mathcal{T}$ is h-invariant), $\varphi(K) \subseteq \mathbb{R}$ is closed and $c(\varphi, \mathcal{T})$ is finite, then $c(\varphi, \mathcal{T}) \in \varphi(K)$.

PROOF We proceed by contradiction. Suppose that $c(\varphi, \mathcal{T}) \in \mathbb{R} \setminus \varphi(K)$. Then because $\varphi(K)$ is closed, we can find $a < c(\varphi, \mathcal{T}) < b$, such that

$$\varphi^{-1}\big([a, b]\big) \cap K \ = \ \emptyset.$$

From (5.1) we see that we can find $C \in \mathcal{T}$, such that

$$\sup_{x \in C} \varphi(x) \ \leqslant \ b,$$

hence $C \subseteq \varphi^b$. By virtue of hypothesis $H(\varphi)$, we can find $t > 0$, such that

$$h(t, \varphi^b) \ \subseteq \ \varphi^a.$$

Hence $h(t, C) \subseteq \varphi^a$ and due to the invariance hypothesis $h(t, C) \in \mathcal{T}$. So we have

$$c(\varphi, \mathcal{T}) \ \leqslant \ \sup_{x \in h(t, C)} \varphi(x) \ \leqslant \ \sup_{x \in \varphi^a} \varphi(x) \ \leqslant \ a,$$

a contradiction to the choice of $a \in \mathbb{R}$. This proves that

$$c(\varphi, \mathcal{T}) \ \in \ \varphi(K).$$

□

REMARK 5.1.3 A careful reading of the above proof reveals that hypothesis $H(\varphi)$ can be weakened as follows.

$H(\varphi)'$ There is a homotopy $h\colon [0, 1] \times X \longrightarrow X$, such that for all $a, b \in \mathbb{R}$, $a < b$ with $\varphi^{-1}\big([a, b]\big) \cap K = \emptyset$ and for all $E \in \mathcal{T}$, there is $t = t(E, a, b) > 0$ for which we have $h(t, E \cap \varphi^b) \subseteq \varphi^a$.

EXAMPLE 5.1.4 **(a)** Let $\mathcal{T} = \{X\}$. Then

$$c(\varphi, \mathcal{T}) \ = \ \sup_{x \in X} \varphi(x).$$

(b) Let $\mathcal{T} = \big\{\{x\} : \ x \in X\big\}$. Then

$$c(\varphi, \mathcal{T}) \ = \ \inf_{x \in X} \varphi(x).$$

(c) Let $S^M \stackrel{df}{=} \big\{x \in \mathbb{R}^M : \|x\|_{\mathbb{R}^M} = 1\big\}$ and let $\mathcal{H}$ be a homotopy class of maps $g\colon S^M \longrightarrow X$. We set

$$\mathcal{T} \stackrel{df}{=} \big\{C \subseteq X : \ C = g(S^M), \ g \in \mathcal{H}\big\}.$$

Then $\mathcal{T}$ is invariant under all deformations of X.

In most applications of interest, X is a Banach space, $\varphi \in C^1(X)$ and K is the set of critical points of φ, i.e.,

$$K \ = \ K^{\varphi} \ = \ \big\{x \in X : \ \varphi'(x) = 0\big\}.$$

This set is closed. Then we can use Theorem 5.1.2 to determine critical points of φ. Two are the main ingredients of Theorem 5.1.2.

(a) The homotopies (deformations) h, which satisfy property $H(\varphi)$ with respect to K.

(b) The h-invariant families $\mathcal{T}$, which provide a measure for the number of critical points of φ.

In Section 5.6, following the idea of degree theory, we shall construct $\mathbb{Z}$-valued functions χ on certain subsets of 2^X, such that the properties of χ and assumptions like $H(\varphi)$ yield lower bounds for the number of critical points of φ (i.e., the number of elements of K). Recall that the degree

$$\big|d(\varphi, U, 0)\big|$$

provides a lower bound for the number of zeros of φ on U in the regular case.

In this section we focus on how to construct homotopies exhibiting property $H(\varphi)$. The idea is to consider homotopies of the underlying space X along the lines of steepest descent associated to φ, i.e., along the trajectories of

$$\dot{x}(t) \ = \ -\nabla\varphi\big(x(t)\big).$$

However, in many cases this differential system or its trajectory are not defined either due to the nature of X or due to the lack of regularity of φ. For example if X is a Banach space which is not a Hilbert space, then

$$\nabla\varphi(x) \in X^* \qquad \forall \ x \in X,$$

and so the differential system cannot be defined. For this reason we introduce a substitute of the gradient vector field, the so-called pseudogradient vector

field. Then we examine the deformation flow generated by this field. To determine the basic properties of this flow, we need a compactness-type condition. So we start our discussion with a presentation and study of this compactness condition. Before proceeding with this discussion, let us mention that the whole theory can be developed in the more general setting of C^1-functionals on complete, regular $C^{1,1}$-Banach manifolds with Finsler structure. However, in many applications the setting used here is sufficient.

DEFINITION 5.1.5 *Let X be a Banach space with X^* being its topological dual and let $\varphi \in C^1(X)$.*

(a) *We say that φ satisfies the* ***Palais-Smale condition at level*** *$c \in \mathbb{R}$ (PS_c-**condition** for short), if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that*

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad \varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*,$$

has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we simply say that φ satisfies the ***Palais-Smale condition*** *(PS-**condition** for short).*

(b) *We say that φ satisfies the* ***Cerami condition at level*** *$c \in \mathbb{R}$ (C_c-**condition** for short), if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that*

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad \big(1 + \|x_n\|_X\big)\varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*,$$

has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we simply say that φ satisfies the ***Cerami condition*** *(C-**condition** for short).*

REMARK 5.1.6 Evidently the Cerami condition is weaker than the Palais-Smale condition. However, it can be shown that if $\varphi \in C^1(X)$ is bounded below, then the two conditions are in fact equivalent (see Gasiński & Papageorgiou (2005, p. 127)). These conditions are compactness-type conditions which are rather strong. For example the constant functions and the functions $\cos x$ and $\sin x$ do not satisfy them. In what follows we prove a general result explaining this.

In what follows X is a Banach space. Additional hypotheses will be introduced as needed.

DEFINITION 5.1.7 *A functional $\varphi\colon X \longrightarrow \mathbb{R}$ is said to be* ***weakly coercive****, if*

$$\varphi(x) \longrightarrow +\infty \quad \text{as } \|x\|_X \to +\infty.$$

REMARK 5.1.8 Clearly weak coercivity of φ is equivalent to saying that for every $\lambda \in \mathbb{R}$ the set φ^λ is bounded, where

$$\varphi^\lambda \stackrel{df}{=} \big\{x \in X :\ \varphi(x) \leqslant \lambda\big\}.$$

PROPOSITION 5.1.9
If $\varphi \in C^1(X)$ and $c \in \mathbb{R}$ is such that φ^λ is unbounded for $\lambda > c$ and bounded for $\lambda < c$,
then we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$\varphi(x_n) \longrightarrow c, \quad \varphi'(x_n) \longrightarrow 0 \quad \text{in } X^* \quad \text{and} \quad \|x_n\|_X \longrightarrow +\infty.$$

PROOF For every $n \geqslant 1$, we can find $r_n \geqslant n$, such that

$$\varphi^{c-\frac{1}{n}} \subseteq \overline{B}_{r_n}(0) = \big\{x \in X : \|x\|_X \leqslant r_n\big\}.$$

Let us set

$$E_n \stackrel{df}{=} X \setminus \overline{B}_{r_n}(0) \quad \text{and} \quad \psi_n \stackrel{df}{=} \varphi|_{E_n}.$$

Then

$$c - \frac{1}{n} \leqslant \inf_{x\in E_n} \varphi(x) = c_n.$$

By hypothesis the set $\varphi^{c+\frac{1}{n}}$ is unbounded. So we can find $y_n \in X$, such that

$$\varphi(y_n) \leqslant c + \frac{1}{n} \quad \text{and} \quad \|y_n\|_X \geqslant r_n + 1 + \frac{1}{\sqrt{n}}.$$

This means that $y_n \in E_n$ and so we have

$$\varphi(y_n) \leqslant c + \frac{1}{n} \leqslant c_n + \frac{2}{n}.$$

Applying Theorem 4.6.1 (with $\varepsilon = \frac{2}{n}$ and $\lambda = \frac{1}{\sqrt{n}}$), we obtain $x_n \in E_n$, such that

$$\begin{aligned} &c - \frac{1}{n} \leqslant c_n \leqslant \varphi(x_n) \leqslant \varphi(y_n) \leqslant c + \frac{1}{n} \leqslant c + \frac{2}{n}, \\ &\varphi(x_n) \leqslant \varphi(x) + \frac{2}{\sqrt{n}} \|x - x_n\|_X \qquad \forall\, x \in E_n, \\ &\|x_n - y_n\|_X \leqslant \frac{1}{\sqrt{n}}. \end{aligned} \tag{5.2}$$

From the last inequality and since

$$\|y_n\|_X \geqslant r_n + 1 + \frac{1}{\sqrt{n}},$$

we have that

$$\|x_n\|_X \geqslant r_n + 1.$$

Hence $x_n \in \operatorname{int} E_n$ and so from (5.2), we obtain

$$\big\|\varphi'(x_n)\big\|_X \leqslant \frac{2}{\sqrt{n}}.$$

Thus finally for the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, we have that

$$\varphi(x_n) \longrightarrow c, \quad \varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*$$

and

$$\|x_n\|_X \longrightarrow +\infty.$$

□

COROLLARY 5.1.10
If $\varphi \in C^1(X)$ satisfies the PS_c-condition and φ^λ is bounded for all $\lambda < c$, then for some $\lambda_0 > 0$ the set $\varphi^{c+\lambda_0}$ is bounded.

REMARK 5.1.11 Clearly in the above corollary, we can assume that instead of the PS_c-condition, φ satisfies a weaker condition, namely if $\{x_n\}_{n\geqslant 1} \subseteq X$ is such that

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad \varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*,$$

then the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ has a bounded subsequence.

Another important consequence of Proposition 5.1.9 is the following result.

PROPOSITION 5.1.12
If $\varphi \in C^1(X)$ is bounded from below and it is not weakly coercive, then φ does not satisfy the PS_{c_0}-condition where

$$c_0 \stackrel{df}{=} \sup\{\lambda \in \mathbb{R} : \ \varphi^\lambda \text{ is bounded}\}.$$

PROOF Let

$$C \stackrel{df}{=} \{\lambda \in \mathbb{R} : \ \varphi^\lambda \text{ is bounded}\}$$

(recall that by definition the empty set is bounded) and

$$m \stackrel{df}{=} \inf_{x\in X} \varphi(x).$$

Since φ is bounded from below m is finite and

$$(-\infty, m] \subseteq C,$$

i.e.,

$$C \neq \emptyset.$$

Because φ is not coercive, we have that

$$c_0 = \sup C < +\infty$$

(see Remark 5.1.8). Then for all $\lambda > c_0$, φ^λ is unbounded and so by virtue of Proposition 5.1.9, φ does not satisfy the PS_{c_0}-condition. □

This proposition leads to the result relating the PS-condition with coercivity.

THEOREM 5.1.13
If $\varphi \in C^1(X)$ is bounded from below and satisfies the PS*-condition, then φ is weakly coercive.*

REMARK 5.1.14 For any functional $\varphi\colon X \longrightarrow \mathbb{R}$, the set

$$C \stackrel{df}{=} \big\{\lambda \in \mathbb{R} :\ \varphi^{\lambda} \text{ is bounded}\big\}$$

is a left half-line which is either open or closed. It can happen that $C = \emptyset$ or $C = \mathbb{R}$, the latter case corresponding to φ being weakly coercive. On the other hand, if

$$D \stackrel{df}{=} \big\{c \in \mathbb{R} :\ \varphi \text{ satisfies the } \mathrm{PS}_c\text{-condition}\big\},$$

then D is not necessarily a half-line, even if φ is bounded from below.

EXAMPLE 5.1.15 The converse of Theorem 5.1.13 is not in general true. Namely if $\varphi \in C^1(X)$, it is bounded from below and it is weakly coercive, then it need not satisfy the PS-condition. To see this let $X = H$ be a Hilbert space and

$$\varphi(x) \stackrel{df}{=} \begin{cases} \|x\|_H^2 \ln \|x\|_H & \text{if } x \in H \setminus \{0\}, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $\varphi \in C^1(H)$ and it is bounded from below and weakly coercive. However, if $\{x_n\}_{n\geqslant 1} \subseteq H$ is such that $\|x_n\|_H = \frac{1}{\sqrt{e}}$ and has no strongly convergent subsequence, then

$$\varphi(x_n) \ = \ -\frac{1}{2e}, \quad \varphi'(x_n) \ = \ 0 \qquad \forall\, n \geqslant 1$$

and so φ does not satisfy the $\mathrm{PS}_{-\frac{1}{2e}}$-condition.

Now we can start discussing how we can generate the deformation flow. As we already mentioned in order to guarantee that the differential system which we use to generate the flow is defined in X and has (at least locally) solutions, instead of the gradient vector field, we use the following substitute.

DEFINITION 5.1.16 *Let $\varphi \in C^1(X)$. A vector $v \in X$ is a* ***pseudogradient vector*** *for φ at x, if*

$$\|v\|_X \ \leqslant \ 2\big\|\varphi'(x)\big\|_{X^*} \quad \text{and} \quad \big\|\varphi'(x)\big\|_{X^*}^2 \ \leqslant \ \big\langle \varphi'(x), v\big\rangle_X.$$

We say that $V\colon \big\{x \in X :\ \varphi'(x) \neq 0\big\} \longrightarrow X$ is a ***pseudogradient vector field*** *for φ, if V is locally Lipschitz and for every $x \in \big\{x \in X :\ \varphi'(x) \neq 0\big\}$ (i.e., in the set of regular points of φ) and $V(x)$ is a pseudogradient vector for φ at x.*

REMARK 5.1.17 If $v \in X$ is a pseudogradient vector of φ at x, we have that

$$\left\|\varphi'(x)\right\|_{X^*}^2 \leqslant \left\langle \varphi'(x), v\right\rangle_X \leqslant \left\|\varphi'(x)\right\|_{X^*} \|v\|_X,$$

hence $\left\|\varphi'(x)\right\|_{X^*} \leqslant \|v\|_X$.

LEMMA 5.1.18
If Y is a metric space, V is a normed space, for every $y \in Y$, $F(y)$ is a nonempty, convex subset of V and for every $y \in Y$ we can find a neighbourhood U of y, such that

$$\bigcap_{y' \in U} F(y') \neq \emptyset,$$

then there exists a locally Lipschitz map $f\colon Y \longrightarrow V$, such that

$$f(y) \in F(y) \quad \text{for a.a. } y \in Y.$$

PROOF For every $y \in Y$, let $U(y)$ be a neighbourhood of y, such that

$$\bigcap_{y' \in U(y)} F(y') \neq \emptyset.$$

The collection $\left\{U(y)\right\}_{y \in Y}$ is an open cover of Y and Y being metric space is paracompact (see Definition A.1.7 and Remark A.1.8). So we can find a locally finite refinement $\{V_i\}_{i \in I}$ of $\left\{U(y)\right\}_{y \in Y}$ (see Definition A.1.5). First suppose that

$$V_i \neq Y \qquad \forall\, i \in I$$

and set

$$\vartheta_i(y) \stackrel{df}{=} d_{_Y}(y, Y \setminus V_i) \quad \text{and} \quad \xi(y) \stackrel{df}{=} \sum_{i \in I} \vartheta_i(y) \qquad \forall\, y \in Y.$$

Clearly ϑ_i is Lipschitz continuous and because $\{V_i\}_{i \in I}$ is a locally finite open cover, ξ is locally Lipschitz and

$$\xi(y) \neq 0 \qquad \forall\, y \in Y.$$

Setting

$$\psi_i(y) \stackrel{df}{=} \frac{\vartheta_i(y)}{\xi(y)} \qquad \forall\, i \in I,$$

we see that $\{\psi_i\}_{i \in I}$ is a locally Lipschitz partition of unity subordinate to $\{V_i\}_{i \in I}$.

Now suppose that

$$V_{i_0} = Y \text{ for some } i_0 \in I.$$

Let us set

$$\psi_{i_0} \equiv 1 \quad \text{and} \quad \psi_i \equiv 0 \qquad \forall\, i \neq i_0.$$

Again $\{\psi_i\}_{i\in I}$ is a locally Lipschitz partition of unity subordinate to $\{V_i\}_{i\in I}$.

Since the cover $\{V_i\}_{i\in I}$ refines the cover $\{U(y)\}_{y\in Y}$ (see Definition A.1.5), it follows that

$$\bigcap_{y'\in V_i} F(y') \neq \emptyset \qquad \forall\, i \in I.$$

Let

$$v_i \in \bigcap_{y'\in V_i} F(y) \qquad \forall\, i \in I$$

and define $f\colon Y \longrightarrow V$ by

$$f(y) \stackrel{df}{=} \sum_{i\in I} \psi_i(y) v_i.$$

The function f is locally Lipschitz. Also for every $y \in Y$, there are only finitely many $V_{i_1}, \ldots, V_{i_k}$ elements of the cover $\{V_i\}_{i\in I}$, such that

$$y \in V_{i_m} \qquad \forall\, m \in \{1, \ldots, k\}.$$

Then

$$f(y) = \sum_{m=1}^{k} \psi_{i_m}(y) v_{i_m} \quad \text{and} \quad \sum_{m=1}^{k} \psi_{i_m}(y) = 1.$$

Since

$$v_{i_m} \in F(y) \qquad \forall\, m \in \{1, \ldots, k\}$$

and $F(y)$ is convex, it follows that

$$f(y) \in F(y) \qquad \forall\, y \in Y.$$

□

Using this lemma, we can establish the existence of pseudogradient vector field.

THEOREM 5.1.19
If $\varphi \in C^1(X)$,
then there exists a pseudogradient vector field for φ.

PROOF Let

$$Y \stackrel{df}{=} \{x \in X : \varphi'(x) \neq 0\}.$$

For every $x \in Y$, let $G(x)$ be the set of all pseudogradient vectors for φ at $x \in Y$. Clearly

$$G(x) \text{ is a convex subset of } X.$$

Also for a given $x \in Y$, we can find $u \in X$, such that

$$\|u\|_X \leqslant 1 \quad \text{and} \quad \frac{4}{5}\|\varphi'(x)\|_{X^*} \leqslant \langle \varphi'(x), u\rangle_X.$$

Let us set

$$v \stackrel{df}{=} \frac{5}{3}\|\varphi'(x)\|_{X^*} u.$$

We have

$$\|v\|_X \leqslant \frac{5}{3}\|\varphi'(x)\|_{X^*} \quad \text{and} \quad \langle \varphi'(x), v\rangle_X \geqslant \frac{4}{3}\|\varphi'(x)\|^2_{X^*}.$$

Because $\varphi \in C^1(X)$, we can find a neighbourhood U of x, such that

$$\|v\|_X < 2\|\varphi'(y)\|_{X^*} \quad \text{and} \quad \langle \varphi'(y), v\rangle_X > \|\varphi'(y)\|^2_{X^*} \qquad \forall\, y \in U,$$

hence

$$v \in \bigcap_{y \in U} G(y).$$

Thus we can apply Lemma 5.1.18 and obtain a locally Lipschitz map

$$V : Y \longrightarrow X,$$

such that

$$V(x) \in G(x) \qquad \forall\, x \in Y.$$

Clearly V is the pseudogradient vector field for φ. □

REMARK 5.1.20 If $X = H$ is a Hilbert space and $\varphi \in C^2(X)$, then we can take

$$V(x) \stackrel{df}{=} \varphi'(x).$$

Also if $0 < \alpha < \beta$ and $\varphi \in C^1(X)$, then by setting

$$V_1(x) \stackrel{df}{=} \frac{\alpha+\beta}{2}\frac{V(x)}{\|\varphi'(x)\|^2_{X^*}} \qquad \forall\, x \in Y,$$

with V being the pseudogradient vector field obtained in Theorem 5.1.19 and

$$Y \stackrel{df}{=} \{x \in X : \varphi'(x) \neq 0\},$$

we have that V_1 is locally Lipschitz from Y into X and

$$\alpha \leqslant \langle \varphi'(x), V_1(x)\rangle_X \leqslant \|\varphi'(x)\|_{X^*}\|V_1(x)\|_X \leqslant \beta \qquad \forall\, x \in Y.$$

The deformation flow will be generated by solving a Cauchy problem in a Banach space. So let us recall the basic existence result concerning such problems.

THEOREM 5.1.21
If $U \subseteq X$ is an open subset, $f\colon U \longrightarrow X$ is a locally Lipschitz vector field and $x_0 \in U$,
then the Cauchy problem

$$\begin{cases} \dot{u}(t) = f\big(u(t)\big), \\ u(0) = x, \end{cases} \tag{5.3}$$

has a unique C^1-local solution

$$u(x)(\cdot) \;=\; u(x,\cdot)$$

defined on a maximal interval $\big(l_-(x), l_+(x)\big)$ containing 0. The set

$$W \;\stackrel{df}{=}\; \big\{(x,t):\; x \in U,\; t \in \big(l_-(x), l_+(x)\big)\big\}$$

is open and the map $(x,t) \longmapsto u(x,t)$ is locally Lipschitz from W into X. Moreover, if for some $x \in X$, $u\big(x, \big(l_-(x), l_+(x)\big)\big)$ lies in a complete subset of U, then

$$l_+(x) \;<\; +\infty \;\Longrightarrow\; \int_0^{l_+(x)} \big\|f\big(u(t)\big)\big\|_X \, dt \;=\; +\infty. \tag{5.4}$$

By imposing a sublinear growth condition on the vector field f, we can have a global solution.

THEOREM 5.1.22
If $U \subseteq X$ is an open subset, $f\colon U \longrightarrow X$ is a locally Lipschitz function,

$$\big\|f(x)\big\|_X \leqslant\; a + c\,\|x\|_X \qquad \forall\; x \in U,$$

for some $a, c > 0$ and the unique C^1-solution $(x,t) \longmapsto u(t,x)$ of the Cauchy problem (5.3) lies in complete subsets of U,
then

$$l_-(x) \;=\; -\infty \quad \textit{and} \quad l_+(x) \;=\; +\infty \qquad \forall\; x \in U,$$

for every $t \in \mathbb{R}$ the function $x \longmapsto u(x,t)$ is a homeomorphism and the function $X \times \mathbb{R} \ni (x,t) \longmapsto u(x,t) \in X$ is locally Lipschitz and bounded (i.e., maps bounded sets into bounded sets).

PROOF Suppose that for some $x \in U$, we have $l_+(x) < +\infty$. Then integrating, we have

$$\big\|u(x,t)\big\|_X \;\leqslant\; \|x\|_X + \int_0^t \big\|\dot{u}(x,s)\big\|_X \, ds$$

$$\leqslant \|x\|_X + al_+(x) + c\int_0^t \|u(x,s)\|_X\, ds \qquad \forall\, t \in [0, l_+(x)). \tag{5.5}$$

Invoking Gronwall's inequality (see Theorem A.4.7), we can find $\beta_1 = \beta_1(x) > 0$, such that

$$\|u(x,t)\|_X \leqslant \beta_1 \qquad \forall\, t \in [0, l_+(x)).$$

Then by virtue of the sublinear growth condition of f we see that there exists $\beta_2 = \beta_2(x) > 0$, such that

$$\|f(u(x,t))\|_X \leqslant \beta_2 \qquad \forall\, t \in [0, l_+(x)),$$

hence

$$\int_0^{l_+(x)} \|f(u(x,t))\|_X\, dt \leqslant \beta_2 l_+(x) < +\infty,$$

a contradiction to (5.4). This proves that $l_+(x) = +\infty$.

By reversing the time, in a similar fashion, we show that $l_-(x) = -\infty$. The uniqueness of the solution implies that

$$u^{-1}(x,t) = u(x,-t) \qquad \forall\, t \geqslant 0,\ x \in U.$$

So for every $t \geqslant 0$, the function $u(\cdot,t)$ is a homeomorphism. Finally from (5.5) and Gronwall's inequality (see Theorem A.4.7), we have

$$\|u(x,t)\|_X \leqslant \big(\|x\|_X + at_0\big)e^{ct_0} \qquad \forall\, t \in [0,t_0],$$

hence we conclude that the map $(x,t) \longmapsto u(t,x)$ is bounded and of course locally Lipschitz (see Theorem 5.1.21). □

REMARK 5.1.23 The above global existence theorem implies that for every compact set $T \subseteq \mathbb{R}$ and every closed set $C \subseteq U$, the set $u(C,T)$ is closed in U. To see this suppose that

$$u(x_n,t_n) \longrightarrow v \in U,$$

for some sequence $\{(x_n,t_n)\}_{n\geqslant 1} \subseteq C \times T$. Since T is compact, we may assume that

$$t_n \longrightarrow t \in T.$$

By virtue Theorem 5.1.22, we have

$$x_n = u^{-1}\big(u(x_n,t_n),t_n\big) \longrightarrow u^{-1}(v,t) \in C$$

and so

$$v = u\big(u(v,-t),t\big) \in u(C \times T).$$

Now we can start examining the deformation flow. First a definition to fix our terminology.

DEFINITION 5.1.24 *Recall that a homotopy on X is a continuous map $h\colon [0,1]\times X \longrightarrow X$, such that*

$$h(0,x) \;=\; x \qquad \forall\, x\in X.$$

We write h_t for the map

$$X \ni x \;\longmapsto\; h(t,x)\in X.$$

We say that h is a ***homotopy of homeomorphisms****, if for every $t\in[0,1]$, h_t is a homeomorphism on X. If $\varphi\in C^1(X)$, then we say that the homotopy h is φ-decreasing, if*

$$\varphi\big(h(t,x)\big) \;\leqslant\; \varphi\big(h(s,x)\big) \qquad \forall\, t,s\in[0,1],\ s\leqslant t,\ x\in X.$$

First we establish a quantitative deformation result.

PROPOSITION 5.1.25
If $\varphi\in C^1(X)$, $a,b\in\mathbb{R}$, $a<b$, $\delta>0$, $S\subseteq X$ is a closed subset and we have

$$\big\|\varphi'(x)\big\|_{X^*} \;\geqslant\; \frac{2(b-a)}{\delta} \qquad \forall\, x\in S\cap\varphi^{-1}\big([a,b]\big),$$

then for each $\varepsilon>0$ and for each closed subset $S_0\subseteq X$ with $S\cap S_0=\emptyset$, there is a φ-decreasing and locally Lipschitz homotopy of homeomorphisms h_t, such that:

(a) *If $x\in\varphi^b$ and $h(t,x)\in S$ for all $t\in[0,1]$,*
then $h(1,x)\in\varphi^a$.
Moreover, if $x\in\varphi^b$ and $h(t,x)\in S\cap\varphi_a$ for all $t\in[0,s]$, where

$$\varphi_a \;\stackrel{df}{=}\; \big\{\varphi\geqslant a\big\},$$

then

$$\varphi\big(h(s,x)\big) \;\leqslant\; \varphi(x)-(b-a)s.$$

(b) *If*

$$C \;\stackrel{df}{=}\; \varphi^{a-\varepsilon}\cup\varphi_{b+\varepsilon}\cup\left\{x\in X:\ \big\|\varphi'(x)\big\|_{X^*}\leqslant\frac{b-a}{\delta}\right\}\cup S_0,$$

then

$$h(t,x) \;=\; x \qquad \forall\,(t,x)\in[0,1]\times C.$$

(c) $\big\|h(t,x)-x\big\|_X\leqslant\delta t$ *for all $(t,x)\in[0,1]\times X$.*

PROOF Recall that

$$K = K^{\varphi} \stackrel{df}{=} \{x \in X : \varphi'(x) = 0\}$$

(the critical set of φ) and let

$$E \stackrel{df}{=} S \cap \varphi^{-1}([a,b]).$$

Also let

$$V_1 : X \setminus K \longrightarrow X$$

be the normalized pseudomonotone vector field given in Remark 5.1.20 with $\alpha = 1$, $\beta = 2$. The sets C and E are disjoint and closed. Let

$$\widehat{C} \stackrel{df}{=} \{x \in X : d_X(x,C) \leqslant d_X(x,E)\}.$$

Then $\widehat{C}$ is closed and

$$C \subseteq \operatorname{int} \widehat{C} \subseteq \widehat{C} \subseteq X \setminus E.$$

Invoking Urysohn's lemma (see Theorem A.1.13), we can find a locally Lipschitz map

$$\vartheta : X \longrightarrow [0,1],$$

such that

$$\vartheta|_{\widehat{C}} \equiv 0 \quad \text{and} \quad \vartheta|_E \equiv 1.$$

Then let

$$f : X \longrightarrow X$$

be the locally Lipschitz vector field defined by

$$f(x) \stackrel{df}{=} \begin{cases} \vartheta(x) V_1(x) & \text{if} \quad x \in X \setminus \widehat{C}, \\ 0 & \text{if} \quad \widehat{C}. \end{cases}$$

We consider the following Cauchy problem.

$$\begin{cases} \dot{u}(t) = -f(u(t)), \\ u(0) = x \in X. \end{cases} \tag{5.6}$$

Note that from the definition of E and ϑ, we have

$$\frac{2(b-a)}{\delta} \|V_1(x)\|_X \leqslant \|\varphi'(x)\|_{X^*} \|V_1(x)\|_X \leqslant 2,$$

hence

$$\|V_1(x)\|_X \leqslant \frac{\delta}{b-a} \qquad \forall\, x \in X$$

(see Remark 5.1.20).

So we can apply Theorem 5.1.22 and obtain a locally Lipschitz, bounded map

$$u\colon X \times \mathbb{R}_+ \longrightarrow X,$$

such that

$$\text{the map } u(\cdot,t) \text{ is a homeomorphism on } X \qquad \forall\, t \geqslant 1.$$

We have

$$\begin{aligned} \frac{d}{dt}\varphi\big(u(x,t)\big) &= \big\langle \varphi'\big(u(x,t)\big), \dot{u}(x,t)\big\rangle_X \\ &\leqslant -\vartheta\big(u(x,t)\big) \qquad \forall\, x \in X \end{aligned} \tag{5.7}$$

(see Remark 5.1.20 and recall the definition of ϑ). From (5.7) it follows that

$$\text{the function } t \longmapsto \varphi\big(u(x,t)\big) \text{ is decreasing on } \mathbb{R}_+.$$

Note that because of the uniqueness of the solution of the Cauchy problem (5.6) and the definition of the vector field f, we have that $x \in \widehat{C}$ if and only if

$$u(x,t) = x \qquad \forall\, t \geqslant 0.$$

So from (5.7), we infer that

$$\text{the function } t \longmapsto \varphi\big(u(x,t)\big) \text{ is strictly decreasing} \qquad \forall\, x \in X \setminus \widehat{C}.$$

If

$$u(x,t) \in E \qquad \forall\, t \in [0,s],$$

then integrating (5.7) on $[0,s]$, we obtain

$$\varphi\big(u(x,s)\big) \leqslant \varphi(x) - s. \tag{5.8}$$

Finally, we have

$$\begin{aligned} \big\|u(x,t) - x\big\|_X &\leqslant \int_0^t \big\|\dot{u}(x,s)\big\|_X\, ds \\ &= \int_0^t \big\|f\big(u(x,s)\big)\big\|_X\, ds \\ &\leqslant \frac{\delta}{b-a}t. \end{aligned} \tag{5.9}$$

So if we set

$$h(t,x) \overset{df}{=} u\big(x,(b-a)t\big),$$

then from (5.8) and (5.9) and the observation that $\widehat{C}$ is invariant with respect to the flow of (5.6), we see that **(a)**, **(b)** and **(c)** of the proposition are satisfied. ∎

Next we present some interesting consequences of the quantitative deformation result established in Proposition 5.1.25.

COROLLARY 5.1.26
If $\varphi \in C^1(X)$, $c \in \mathbb{R}$ *and* $\varepsilon > 0$,
then there exists a φ*-decreasing and locally Lipschitz homotopy of homeomorphisms* h_t, *such that:*

(a) *If* $x \in \varphi^{c+\varepsilon}$ *and*

$$\big\|\varphi'\big(h(t,x)\big)\big\|_{X^*} \geqslant 4\sqrt{\varepsilon} \qquad \forall\ t \in [0,1],$$

then

$$h(1,x) \in \varphi^{c-\varepsilon}.$$

Moreover, if

$$c-\varepsilon \ \leqslant\ \varphi\big(h(t,x)\big) \ \leqslant\ c+\varepsilon \qquad \forall\ t \in [0,s]$$

and

$$\big\|\varphi'\big(h(t,x)\big)\big\|_{X^*} \ \geqslant\ 4\sqrt{\varepsilon} \qquad \forall\ t \in [0,s],$$

then

$$\varphi\big(h(s,x)\big) \ \leqslant\ \varphi(x) - 2\varepsilon s.$$

(b) *If*

$$\big\|\varphi'(x)\big\|_{X^*} \ \leqslant\ 2\sqrt{\varepsilon} \quad \text{or} \quad x \notin \varphi^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big),$$

then

$$h(t,x) \ =\ x \qquad \forall\ t \in [0,1].$$

(c) $\big\|h(t,x)-x\big\|_X \leqslant \sqrt{\varepsilon}t$ *for all* $(t,x) \in [0,1] \times X$.

PROOF Apply Proposition 5.1.25, with

$$a \ =\ c-\varepsilon, \quad b \ =\ c+\varepsilon, \quad \delta \ =\ \sqrt{\varepsilon}$$

and

$$S \ =\ \big\{x \in X :\ \big\|\varphi'(x)\big\|_X \geqslant 4\sqrt{\varepsilon}\big\}$$

to obtain the corollary. □

We can improve the speed of decrease of the function

$$t \longmapsto \varphi\big(h(t,x)\big)$$

(see Corollary 5.1.26**(a)**) at the expense of the estimate in Corollary 5.1.26**(c)**.

COROLLARY 5.1.27
If $\varphi \in C^1(X)$, $c \in \mathbb{R}$ *and* $\varepsilon \in \big(0, \frac{1}{2}\big)$,
then there exists a φ*-decreasing and locally Lipschitz homotopy of homeomorphisms* h_t*, such that:*

(a) *If* $x \in \varphi^{c+\varepsilon}$ *and*

$$\big\|\varphi'\big(h(t,x)\big)\big\|_{X^*} \geqslant 4\sqrt{\varepsilon} \qquad \forall\, t \in [0,1],$$

then

$$h(1,x) \in \varphi^{c-\varepsilon}.$$

Moreover, if

$$c-\varepsilon \leqslant \varphi\big(h(t,x)\big) \leqslant c+\varepsilon \qquad \forall\, t \in [0,s]$$

and

$$\big\|\varphi'\big(h(t,x)\big)\big\|_{X^*} \geqslant 4\sqrt{\varepsilon} \qquad \forall\, t \in [0,s],$$

then

$$\varphi\big(h(s,x)\big) \leqslant \varphi(x) - s.$$

(b) *If*

$$\big\|\varphi'(x)\big\|_{X^*} \leqslant 2\sqrt{\varepsilon} \quad \text{or} \quad x \notin \varphi^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big),$$

then

$$h(t,x) = x \qquad \forall\, t \in [0,1].$$

(c) $\big\|h(t,x)-x\big\|_X \leqslant \min\Big\{\frac{t}{2\sqrt{\varepsilon}}, 4\sqrt{\varepsilon}\Big\}$ *for all* $(t,x) \in [0,1] \times X$.

PROOF Let $u(x,t)$ be the flow obtained in the proof of Proposition 5.1.25, when $a = c-\varepsilon$, $b = c+\varepsilon$ and $\delta = \sqrt{\varepsilon}$. From (5.9), we have

$$\big\|u(x,t)-x\big\|_X \leqslant \frac{t}{2\sqrt{\varepsilon}}.$$

Also from the proof of Proposition 5.1.25 (see (5.8)), we have that

$$\big\|u(x,t)-x\big\|_X \leqslant \frac{1}{\sqrt{\varepsilon}} \int_0^t \vartheta\big(u(x,s)\big)\, ds \leqslant \frac{1}{\sqrt{\varepsilon}}\big(\varphi(x) - \varphi\big(u(x,t)\big)\big) \leqslant 4\sqrt{\varepsilon}.$$

Let us set

$$h(t,x) \stackrel{df}{=} u(x,t) \qquad \forall\, (t,x) \in [0,1] \times X.$$

□

COROLLARY 5.1.28
If $\varphi \in C^1(X)$, $a, b \in \mathbb{R}$, $a < b$, $\delta > 0$, $C, E \subseteq X$ are two closed subsets, such that

$$\overline{C}_\delta \cap E \;=\; \emptyset,$$

where $\overline{C}_\delta = \{x \in X :\; d_{_X}(x, C) \leqslant \delta\}$ and

$$\big\|\varphi'(x)\big\|_{X^*} \;\geqslant\; \frac{4(b-a)}{\delta} \qquad \forall\; x \in \overline{C}_\delta \cap \varphi^{-1}\big([a,b]\big),$$

then for each $\varepsilon > 0$, there is a φ-decreasing and locally Lipschitz homotopy of homeomorphisms h_t, such that

(a) $h_1(C \cap \varphi^b) \subseteq \varphi^a$.

(b) *If $x \in E$ or $x \notin \varphi^{-1}\big([a-\varepsilon, b+\varepsilon]\big)$, then*

$$h(t,x) \;=\; x \qquad \forall\; t \in [0,1].$$

(c) $\big\|h(t,x) - x\big\|_{_X} \leqslant \delta t$ *for all* $(t,x) \in [0,1] \times X$.

PROOF Apply Proposition 5.1.25 with $S = \overline{C}_\delta$ and $S_0 = E$. Then it is clear that we obtain **(b)** and **(c)**.

Moreover, if $x \in C \cap \varphi^b$, then from Proposition 5.1.25**(c)**, we have that

$$h(t,x) \;\in\; S \qquad \forall\; t \in [0,1].$$

Therefore

$$h(1, C \cap \varphi^b) \;\subseteq\; \varphi^a.$$

□

Now we enter into the picture the PS-condition (see Definition 5.1.5**(a)**). This condition combined with the previous results will give us a complete description of the deformation flow.

Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. As before

$$K \;=\; K^\varphi \;=\; \big\{x \in X :\; \varphi'(x) = 0\big\}$$

(the set of critical points of φ) and

$$K_c^\varphi \;\stackrel{df}{=}\; \big\{x \in K^\varphi :\; \varphi(x) = c\big\}$$

(the set of critical points of φ with energy level c). If φ satisfies the PS-condition, then clearly K_c^φ is compact.

THEOREM 5.1.29
If $\varphi \in C^1(X)$, $\varphi(K^\varphi) \cap [a,b] = \emptyset$ *and* φ *satisfies the* PS_c*-condition for every* $c \in [a,b]$,
then there exist $\varepsilon > 0$ *and a* φ*-decreasing, locally Lipschitz homotopy of homeomorphisms* h_t, *such that*

$$h(1, \varphi^b) \ \subseteq \ \varphi^a$$

and

$$h(t,x) \ = \ x \qquad \forall\ (t,x) \in [0,1] \times \big(X \setminus \varphi^{-1}\big([a-\varepsilon, b+\varepsilon]\big)\big).$$

PROOF Since by hypothesis $\varphi(K^\varphi) \cap [a,b] = \emptyset$ and φ satisfies the PS_c-condition for all $c \in [a,b]$, we can find $\varepsilon > 0$, such that

$$\big\|\varphi'(x)\big\|_X \ \geqslant \ 2\varepsilon(b-a) \qquad \forall\ x \in \varphi^{-1}\big([a,b]\big).$$

Then apply Proposition 5.1.25 with $S = X$ and $\delta = \frac{1}{\varepsilon}$, to finish the proof. □

THEOREM 5.1.30
If $\varphi \in C^1(X)$, $c \in \mathbb{R}$, φ *satisfies the* PS_c*-condition and* U *is a neighbourhood of* K_c^φ *(we take* $U = \emptyset$ *if* $K_c^\varphi = \emptyset$),
then there exist $\varepsilon > 0$ *and a* φ*-decreasing, locally Lipschitz homotopy of homeomorphisms* h_t, *such that*

(a) $h\big(1, \varphi^{c+\varepsilon} \setminus U\big) \subseteq \varphi^{c-\varepsilon}$;

(b) $h(t,x) = x$ *for all* $(t,x) \in [0,1] \times \big(X \setminus \varphi^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big)\big)$;

(c) $\big\|h(t,x) - x\big\|_X \leqslant \sqrt{\varepsilon} t$ *for all* $(t,x) \in [0,1] \times X$.

PROOF Let $C = X \setminus U$. Since φ satisfies the PS_c-condition, we can find $\varepsilon > 0$, such that

$$\overline{C}_{\sqrt{\varepsilon}} \cap K_c^\varphi \ = \ \emptyset$$

and

$$\big\|\varphi'(x)\big\|_X \ \geqslant \ 8\sqrt{\varepsilon} \qquad \forall\ x \in C_{\sqrt{\varepsilon}} \cap \varphi^{-1}\big([c-\varepsilon, c+\varepsilon]\big).$$

Apply Corollary 5.1.28, with

$$a \ = \ c-\varepsilon, \quad b \ = \ c+\varepsilon, \quad \delta \ = \ \sqrt{\varepsilon}, \quad C \ = \ X \setminus U \quad \text{and} \quad E \ = \ \emptyset.$$

□

REMARK 5.1.31 Part **(a)** of the above theorem says that starting a little bit above a critical level c, we will either bypass the critical neighbourhood U and reach a harmless level $a < c$ or we will land on U, the only place near K_c^φ where topologically interesting things may occur.

THEOREM 5.1.32
If $\widehat{\varphi} \in C^1(X)$, $\widehat{c} \in \mathbb{R}$, $\widehat{\varphi}$ *satisfies the* $\mathrm{PS}_{\widehat{c}}$*-condition,* $A, B \subseteq X$ *are two disjoint closed subsets,* $A \cap K_{\widehat{c}}^{\widehat{\varphi}} = \emptyset$ *and*

$$\sup_{x \in A} \widehat{\varphi}(x) \ \leqslant \ \widehat{c} \leqslant \inf_{x \in B} \widehat{\varphi}(x),$$

then there exist $\varepsilon > 0$ *and a* $\widehat{\varphi}$*-decreasing, locally Lipschitz homotopy of homeomorphisms* h_t*, such that*

$$h(1, A) \ \subseteq \ \widehat{\varphi}^{c-\varepsilon}$$

and

$$h(t, x) \ = \ x \qquad \forall \ (t, x) \in [0, 1] \times \big(B \cup \big(X \setminus \widehat{\varphi}^{-1}\big([\widehat{c} - 2\varepsilon, \widehat{c} + 2\varepsilon]\big)\big)\big).$$

PROOF By virtue of Theorem 5.1.30, we can find $\varepsilon > 0$ and a $\widehat{\varphi}$-decreasing, locally Lipschitz homotopy of homeomorphisms $\widehat{h}_t$, such that

$$\widehat{h}(1, \widehat{C}) \ \subseteq \ \widehat{\varphi}^{\widehat{c}-\varepsilon}$$

and

$$\widehat{h}(t, x) \ = \ x \qquad \forall \ (t, x) \in [0, 1] \times \big(X \setminus \widehat{\varphi}^{-1}\big([\widehat{c} - 2\varepsilon, \widehat{c} + 2\varepsilon]\big)\big).$$

We know that for every $x \in X$, the map

$$\widehat{h}_x(t) \ \stackrel{df}{=} \ \widehat{h}(t, x)$$

solves a Cauchy problem:

$$\begin{cases} \dot{\widehat{h}}_x(t) = -f\big(\widehat{h}_x(t)\big), \\ \widehat{h}_x(0) = x, \end{cases}$$

where f is a locally Lipschitz, bounded vector field, such that

$$f|_{X \setminus \widehat{\varphi}^{-1}([\widehat{c}-2\varepsilon, \widehat{c}+2\varepsilon])} \ \equiv \ 0.$$

From the proof of Proposition 5.1.25, we know that for every $x \in A$,

$$\text{the map } t \longmapsto f\big(\widehat{h}(t, x)\big) \text{ is strictly decreasing.}$$

Hence

$$\widehat{h}\big([0, 1] \times A\big) \cap B \ = \ \emptyset.$$

From Remark 5.1.23, we know that

$$\widehat{h}\big([0, 1] \times A\big) \text{ is closed in } X.$$

So by Urysohn's lemma (see Theorem A.1.13) we can find a locally Lipschitz function

$$\vartheta_1 \colon X \longrightarrow [0,1],$$

such that

$$\vartheta_1|_{\widehat{h}([0,1]\times A)} \equiv 1 \quad \text{and} \quad \vartheta_1|_B \equiv 0.$$

The map

$$x \longmapsto \big(\vartheta_1 \circ f\big)(x)$$

is still locally Lipschitz and bounded. Therefore the Cauchy problem:

$$\begin{cases} \dot{u}(t) = -\vartheta_1\big(u(t)\big) f\big(u(t)\big), \\ u(0) = x, \end{cases}$$

generates a $\widehat{\varphi}$-decreasing, locally Lipschitz homotopy of homeomorphisms η_t, such that

$$\eta(t,x) = x \qquad \forall\ (t,x) \in [0,1] \times \big(B \cup \big(X \setminus \widehat{\varphi}^{-1}\big([\widehat{c}-2\varepsilon, \widehat{c}+2\varepsilon]\big)\big)\big).$$

From the definition of ϑ_1 and the uniqueness of the solution of the Cauchy problem, we have that

$$\eta(t,x) = \widehat{h}(t,x) \qquad \forall\ (t,x) \in [0,1] \times A,$$

hence

$$\eta(1,A) = \widehat{h}(1,A) \subseteq \widehat{\varphi}^{\widehat{c}-\varepsilon}.$$

So finally let us set

$$h(t,x) \stackrel{df}{=} \eta(t,x) \qquad \forall\ (t,x) \in [0,1] \times X.$$

□

Note that in the next result we allow $b = +\infty$, in which case $\varphi^b \setminus K_b^\varphi = X$.

THEOREM 5.1.33 (Second Deformation Theorem)
If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leqslant +\infty$, φ satisfies the PS_c-condition for every $c \in [a,b)$, φ has no critical values in (a,b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ,
then there exists a φ-decreasing homotopy $h \colon [0,1] \times \big(\varphi^b \setminus K_b^\varphi\big) \longrightarrow \varphi^b$, such that

$$h\big(1, \varphi^b \setminus K_b^\varphi\big) \subseteq \varphi^a$$

and

$$h(t,x) = x \qquad \forall\ (t,x) \in [0,1] \times \varphi^a.$$

PROOF Let

$$V_1 \colon X \setminus K \longrightarrow X$$

be the normalized pseudogradient vector field determined in Remark 5.1.20 with $\alpha = 1$, $\beta = 2$. For $x = \varphi^{-1}\big([a,b]\big) \setminus K_b^{\varphi}$, we consider the Cauchy problem

$$\begin{cases} \dot{u}(t) = -V_1\big(u(t)\big), \\ u(0) = x. \end{cases} \tag{5.10}$$

By virtue of Theorem 5.1.21, problem (5.10) has a unique solution $u(x,\cdot)$ defined on a maximal interval $\big[0, l_+(x)\big)$. On this interval, we have

$$\frac{d}{dt}\varphi\big(u(t,x)\big) \;\leqslant\; -1 \tag{5.11}$$

(recall that $\alpha = 1$).

Claim 1. If $\varphi\big(u\big(x,t(x)\big)\big) = a$ for some $t(x) < l_+(x)$,
then $t(x)$ is unique and the map $x \longmapsto t(x)$ is continuous.

The uniqueness of $t(x)$ is immediate from (5.11). Note that $t(x)$ is characterized by

$$\varphi\big(u(x,t)\big) \;<\; a \;<\; \varphi\big(u(x,s)\big) \quad \text{for } s < t(x) < t < l_+(x). \tag{5.12}$$

Suppose that

$$x_n \;\longrightarrow\; x \quad \text{in } X.$$

For a given $\varepsilon > 0$ small enough, from (5.12), we have

$$\varphi\big(u\big(x_n, t(x_n) + \varepsilon\big)\big) \;<\; a \;<\; \varphi\big(u\big(x_n, t(x_n) - \varepsilon\big)\big).$$

Exploiting the continuity of the map $y \longmapsto \varphi\big(u(y,t)\big)$, we can find $n_0 = n_0(\varepsilon) \geqslant 1$, such that

$$\varphi\big(u\big(x, t(x_n) + \varepsilon\big)\big) \;<\; a \;<\; \varphi\big(u\big(x, t(x_n) - \varepsilon\big)\big) \qquad \forall\, n \geqslant n_0.$$

From the intermediate value theorem and the uniqueness of $t(x)$, we obtain that

$$\big|t(x_n) - t(x)\big| \;\leqslant\; \varepsilon \qquad \forall\, n \geqslant n_0,$$

hence

$$t(x_n) \;\longrightarrow\; t(x),$$

which proves Claim 1.

For $x \in \varphi^{-1}\big([a,b]\big) \setminus K_b^{\varphi}$, we put $t(x) = l_+(x)$ if $\varphi\big(u(t,x)\big) > a$ for all $t < l_+(x)$.

Claim 2. If $\{x_n\}_{n\geqslant 1} \subseteq \varphi^{-1}\big((a,b]\setminus K_b^{\varphi}\big)$, $v \in \varphi^{-1}(a)$ and $v = \lim_{n\to+\infty} u(x_n, s_n)$ for some sequence $0 \leqslant s_n < t(x_n)$,
then for every sequence $\{t_n\}_{n\geqslant 1}$ with $s_n \leqslant t_n < t(x_n)$, we have $v = \lim_{n\to+\infty} u(x_n, t_n)$.

Let $\varepsilon > 0$ be such that

$$K \cap \overline{B}_{\varepsilon}(v) \cap \varphi^{-1}\big([a,b]\big) \subseteq \{v\},$$

where

$$\overline{B}_{\varepsilon}(v) = \big\{v' \in X : \|v' - v\|_X \leqslant \varepsilon\big\}$$

and

$$b_1 \stackrel{df}{=} \sup \varphi\big(\overline{B}_{\varepsilon}(v)\big) < b.$$

We show that there exists $n_0 \geqslant 1$

$$u(x_n, t_n) \in \overline{B}_{\varepsilon}(v) \qquad \forall\, n \geqslant n_0.$$

Indeed, if this is not the case, we can find a subsequence $\big\{(x_{n_k}, t_{n_k})\big\}_{k\geqslant 1}$ of $\big\{(x_n, t_n)\big\}_{n\geqslant 1}$, such that

$$\big\|u\big(x_{n_k}, t_{n_k}\big) - v\big\|_X > \varepsilon \qquad \forall\, k \geqslant 1.$$

By hypothesis

$$\big\|u\big(x_{n_k}, s_{n_k}\big) - v\big\|_X < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant k_0.$$

So exploiting the continuity of u, we can find $r_{n_k}, \lambda_{n_k} \in \big[s_{n_k}, t_{n_k}\big]$, $r_{n_k} < \lambda_{n_k}$, such that

$$\big\|u\big(x_{n_k}, r_{n_k}\big) - v\big\|_X = \frac{\varepsilon}{2},$$

$$\big\|u\big(x_{n_k}, \lambda_{n_k}\big) - v\big\|_X = \varepsilon$$

and

$$u\big(x_{n_k}, t\big) \in A \qquad \forall\, t \in \big[r_{n_k}, \lambda_{n_k}\big],$$

where A is the annulus, defined by

$$A \stackrel{df}{=} \left\{y \in X : \frac{\varepsilon}{2} \leqslant \|y - v\|_X \leqslant \varepsilon\right\}.$$

Since φ satisfies the PS_c-condition for every $c \in [a,b)$, we have that

$$\delta = \inf_{x\in A\cap\varphi^{-1}([a,b_1])} \big\|\varphi'(x)\big\|_{X^*} > 0.$$

But from (5.10), we have

$$\frac{\varepsilon}{2}\big\|u\big(x_{n_k},\lambda_{n_k}\big)-u\big(x_{n_k},r_{n_k}\big)\big\|_X \;\leqslant\; \int\limits_{r_{n_k}}^{\lambda_{n_k}} \big\|\dot{u}\big(x_{n_k},\tau\big)\big\|_X\,d\tau$$

$$\leqslant\; 2\int\limits_{r_{n_k}}^{\lambda_{n_k}} \frac{d\tau}{\|\varphi'(u(x_{n_k},\tau))\|_{X^*}} \;\leqslant\; 2\frac{\lambda_{n_k}-r_{n_k}}{\delta}. \tag{5.13}$$

So we deduce that

$$\varphi\big(u\big(x_{n_k},\lambda_{n_k}\big)\big)-\varphi\big(u\big(x_{n_k},r_{n_k}\big)\big) \;=\; \int\limits_{r_{n_k}}^{\lambda_{n_k}} \frac{d}{dt}\varphi\big(u\big(x_{n_k},t\big)\big)\,dt$$

$$\leqslant\; \int\limits_{r_{n_k}}^{\lambda_{n_k}} \big\langle\varphi'\big(u\big(x_{n_k},t\big)\big),\dot{u}\big(x_{n_k},t\big)\big\rangle_X\,dt \;\leqslant\; -\big(\lambda_{n_k}-r_{n_k}\big);$$

hence, using also (5.13), we obtain

$$a \;\leqslant\; \varphi\big(u\big(x_{n_k},\lambda_{n_k}\big)\big) \;\leqslant\; \varphi\big(u\big(x_{n_k},r_{n_k}\big)\big)-\big(\lambda_{n_k}-r_{n_k}\big) \;\leqslant\; \varphi\big(u\big(x_{n_k},s_{n_k}\big)\big)-\frac{\delta\varepsilon}{4}.$$

By hypothesis

$$\varphi\big(u\big(x_{n_k},s_{n_k}\big)\big) \;\longrightarrow\; \varphi(v) \;=\; a.$$

Thus in the limit we have

$$a \;\leqslant\; a-\frac{\delta\varepsilon}{4},$$

a contradiction. This proves Claim 2.

Claim 3. <u>If</u> $x\in\varphi^{-1}\big((a,b]\big)\setminus K_b^\varphi$ is such that $t(x)=l_+(x)$, <u>then</u> $v=\lim\limits_{t\to l_+(x)} u(t,x)$ exists and $v\in K_a^\varphi$.

Suppose that the claim is not true. Because K_a^φ is compact, by virtue of Claim 2 (with $x_n=x$ for all $n\geqslant 1$), we cannot find a sequence $\{s_n\}_{n\geqslant 1}\subseteq\big[0,l_+(x)\big)$, such that

$$d_X\big(u(x,s_n),K_a^\varphi\big) \;\longrightarrow\; 0.$$

Therefore we can find $\varepsilon>0$ and $\delta\in\big(0,l_+(x)\big)$, such that

$$d_X\big(u(x,t),K_a^\varphi\big) \;>\; \varepsilon \qquad \forall\, t\in\big[\delta,l_+(x)\big).$$

Note that the set $u\big(x,[0,\delta]\big)$ is compact and

$$u\big(x,[0,\delta]\big)\cap K_a^\varphi \;=\; \emptyset.$$

So by making $\varepsilon > 0$ even smaller if necessary, we can say that

$$u(x,t) \ \in \ \varphi^{-1}\big([a,\varphi(x)]\big) \cap \big\{y \in X : \ d_{_X}(y, K_a^{\varphi}) \geqslant \varepsilon\big\} \qquad \forall\, t \in \big[0, l_+(x)\big).$$

This set is complete and

$$0 \ < \ \varphi\big(u(t,x)\big) \ \leqslant \ \varphi(x) - t \qquad \forall\, t \in \big[0, l_+(x)\big).$$

So it follows that

$$l_+(x) \ \leqslant \ \varphi(x) - a \ < \ +\infty$$

and then Theorem 5.1.22 implies that

$$+\infty \ = \ \int_0^{l_+(x)} \big\|V\big(u(x,t)\big)\big\|_{_X}\, dt \ \leqslant \ 2 \int_0^{l_+(x)} \frac{dt}{\|\varphi'(u(x,t))\|_{_{X^*}}}.$$

This means that we can find a sequence $t_n \longrightarrow l_+(x)$, such that

$$\big\|\varphi'\big(u(x,t_n)\big)\big\|_{_{X^*}} \ \longrightarrow \ 0.$$

Note that

$$\varphi\big(u(x,t_n)\big) \ \leqslant \ b_1,$$

for some $b_1 < b$. So the PS-condition is valid and we can find a subsequence $\{s_n\}_{n\geqslant 1}$ of $\{t_n\}_{n\geqslant 1}$, such that

$$u(x,s_n) \ \longrightarrow \ v,$$

for some $v \in \varphi^{-1}\big([a,b_1]\big) \cap K^{\varphi}$. Because of the hypotheses on φ, we have $\varphi(v) = a$, i.e., $v \in K_a^{\varphi}$, a contradiction.

Because of Claims 1 and 2, the limit $\lim\limits_{t\to t(x)} u(t,x) = u\big(x,t(x)\big)$ exists for all $x \in \varphi^{-1}\big((a,b]\big) \setminus K_b^{\varphi}$.

Claim 4 If $\{x_n\}_{n\geqslant 1} \subseteq \varphi^{-1}\big((a,b]\big) \setminus K_b^{\varphi}$, $x \in \varphi^{-1}(a)$ and $x_n \longrightarrow x$ in X, then $x = \lim\limits_{n\to+\infty} u(x_n,s_n)$ for every sequence $\{s_n\}_{n\geqslant 1}$, such that $0 \leqslant s_n \leqslant l_+(x)$.

Because of Claim 2, it suffices to consider the case $s_n = l_+(x_n)$. We can find $t_n < l_+(x_n)$, such that

$$\big\|u\big(x_n,t_n\big) - u\big(x_n,l_+(x_n)\big)\big\|_{_X} \ \leqslant \ \frac{1}{n} \qquad \forall\, n \geqslant 1.$$

Since $x_n \longrightarrow x$, by virtue of Claim 2, we have that

$$u(x_n,t_n) \ \longrightarrow \ x$$

and so we conclude that

$$u\big(x_n, l_+(x_n)\big) \longrightarrow x \quad \text{in } X.$$

Claim 5. If $\{x_n\}_{n\geqslant 1} \subseteq \varphi^{-1}\big((a,b]\big) \setminus K_b^\varphi$, $x_n \longrightarrow x \in \varphi^{-1}\big((a,b]\big) \setminus K_b^\varphi$ and $l_+(x) = t(x)$,
then for every sequence $\{t_n\}_{n\geqslant 1}$ with $0 < t_n < t(x_n)$ and $l_+(x) \leqslant \liminf_{n\to+\infty} t_n$, we have

$$u\big(x, t(x)\big) = \lim_{n\to+\infty} u\big(x_n, t(x_n)\big) = \lim_{n\to+\infty} u(x_n, t_n).$$

Let

$$v \stackrel{df}{=} u\big(x, t(x)\big).$$

Let $s_1 \in \big(0, l_+(x)\big)$ be such that

$$u(x, s_1) \in \overline{B}_{\frac{1}{2}}(v).$$

For $n \geqslant 1$ large enough, we have that

$$u(x_n, s_1) \in \overline{B}_1(v).$$

Since

$$l_+(x) \leqslant \liminf_{n\to+\infty} t_n,$$

we can find $n_1 \geqslant 1$ large enough so that $s_1 < t_{n_1}$. Continuing this way we generate $s_k < t_{n_k}$, such that

$$u\big(x_{n_k}, s_k\big) \in \overline{B}_{\frac{1}{k}}(v) \qquad \forall\ k \geqslant 1.$$

As before, because of Claim 2, we have that

$$u\big(x_{n_k}, s_k\big) \longrightarrow v$$

and so, it follows that

$$u\big(x_{n_k}, t_k\big) \longrightarrow v.$$

Now let $t_n < t(x_n)$ be such that

$$\big\|u\big(x_n, t(x_n)\big) - u\big(x_n, t_n\big)\big\|_X \longrightarrow 0$$

and

$$\varphi\big(u(x_n, t_n)\big) \longrightarrow a.$$

It cannot happen that

$$\liminf_{n\to+\infty} t_n < l_+(x),$$

because then we can have

$$t_{n_k} \longrightarrow \tau < l_+(x)$$

and hence

$$\varphi\big(u(x,\tau)\big) \;=\; a,$$

which contradicts the assumption that $t(x) = l_+(x)$. Therefore

$$l_+(x) \;\leqslant\; \liminf_{n\to+\infty} t_n$$

and from the first part of the proof of this claim, we have

$$u(x_n, t_n) \;\longrightarrow\; v$$

and so

$$u\big(x_n, t(x_n)\big) \;\longrightarrow\; v.$$

This proves Claim 5.

For each $x \in \varphi^a$, we put

$$t(x) \;=\; 0.$$

Let

$$\xi\colon \mathbb{R}_+ \times \left(\varphi^b \setminus K_b^{\varphi}\right) \longrightarrow \varphi^b$$

be defined by

$$\xi(t,x) \;\stackrel{df}{=}\; \begin{cases} x & \text{if} \quad t(x) = 0, \\ u(x,t) & \text{if} \quad 0 \leqslant t < t(x), \\ u\big(x, t(x)\big) & \text{if} \quad 0 < t(x) \leqslant t. \end{cases} \tag{5.14}$$

Claim 6. ξ is continuous.

Let $(t_n, x_n) \longrightarrow (t,x)$ and assume that $a \leqslant \varphi(x)$.
First suppose that $t(x) = 0$. Then since

$$\xi(t_n, x_n) \;=\; u(x_n, s_n),$$

with $s_n \leqslant t(x_n)$, from Claim 4, we have that

$$\xi(t_n, x_n) \;\longrightarrow\; x = \xi(t,x).$$

Next suppose that $t(x) > 0$. If $t < t(x)$, we have

$$\varphi\big(u(x,t)\big) \;>\; a$$

and so

$$\varphi\big(u(x_n, t_n)\big) \;>\; a \qquad \forall\; n \geqslant n_0,$$

for some $n_0 \geqslant 1$. Hence $t_n < t(x_n)$ and so from (5.14), we have

$$\xi(t_n, x_n) \;=\; u(x_n, t_n) \;\longrightarrow\; u(x,t) \;=\; \xi(t,x).$$

Finally let $0 < t(x) \leqslant t$. If $t(x) < l_+(x)$, then Claim 1 implies that

$$t(x_n) \;\longrightarrow\; t(x)$$

and so from the continuity of u, we have

$$\xi(t_n, x_n) \;\longrightarrow\; \xi(t,x).$$

If $t(x) = l_+$, we invoke Claim 5. This proves Claim 6.

By virtue of Claim 3, for every $x \in \varphi^b \setminus K_b^\varphi$,

$$\text{the limit } \lim_{t \to +\infty} \xi(t,x) = \widehat{\xi}(x) \text{ exists.}$$

Let

$$h\colon [0,1] \times \big(\varphi^b \setminus K_b^\varphi\big) \longrightarrow \varphi^b$$

be defined by

$$h(t,x) \;\stackrel{df}{=}\; \begin{cases} \xi\left(\frac{t}{1-t}, x\right) & \text{if} \quad t \in [0,1), \\ \widehat{\xi}(x) & \text{if} \quad t = 1. \end{cases}$$

Evidently h is φ-decreasing,

$$h_0 \;=\; id_X, \quad h_t|_{\varphi^a} \;=\; id_X|_{\varphi^a} \quad \text{and} \quad h\big(1, \varphi^b \setminus K_b^\varphi\big) \subseteq \varphi^a.$$

So if we show that h is continuous, we are done.

Claim 7. If $x_n \longrightarrow x$ and $t_n \longrightarrow +\infty$, then $\xi(t_n, x_n) \longrightarrow \widehat{\xi}(x)$.

If $t(x) = 0$, we argue as in the corresponding situation in the proof of Claim 6.

If $0 < t(x) < l_+(x)$, then

$$0 \;<\; t(x_n) \;<\; 2t(x) \;<\; +\infty,$$

for $n \geqslant 1$ large enough and so

$$\xi(t_n, x_n) \;=\; u\big(x_n, t(x_n)\big) \;\longrightarrow\; u\big(x, t(x)\big) \;=\; \widehat{\xi}(x).$$

If $t(x) = l_+(x)$, then since

$$\liminf_{n \to +\infty} t_n \;=\; +\infty \geqslant l_+(x),$$

from Claim 5, we have

$$\xi(x_n, t_n) \;\longrightarrow\; u\big(x, t(x)\big) = \widehat{\xi}(x).$$

This proves Claim 7.

From Claims 6 and 7 and the definition of h, we conclude that h is continuous. This completes the proof of the theorem. □

Next we prove a deformation result under the weaker compactness condition of Definition 5.1.5**(b)**.

THEOREM 5.1.34
If $\varphi \in C^1(X)$, $c \in \mathbb{R}$, φ satisfies the C_c-condition, $\varepsilon_0 > 0$, U is a neighbourhood of K_c^φ (if $K_c^\varphi = \emptyset$, then we take $U = \emptyset$) and $\lambda > 0$,
then there exists $\varepsilon \in (0, \varepsilon_0)$ and a φ-decreasing, locally Lipschitz homotopy of homeomorphisms $h\colon [0,1] \times X \longrightarrow X$, such that for all $(t,x) \in [0,1] \times X$, we have

(a) $\big\|h(t,x) - x\big\|_X \leqslant \lambda\big(1 + \|x\|_X\big)t$;

(b) *if $h(t,x) \neq x$, then $\varphi\big(h(t,x)\big) < x$;*

(c) *if $\big|\varphi(x) - c\big| \geqslant \varepsilon_0$, then $h(t,x) = x$;*

(d) $h\big(\{1\} \times \varphi^{c+\varepsilon}\big) \subseteq \varphi^{c-\varepsilon} \cup U$.

PROOF Due to the C_c-condition, K_c^φ is compact. So we can find $\varrho > 0$, such that

$$\operatorname{int} B_{3\varrho}(K_c^\varphi) \ \subseteq \ U.$$

We claim that we can find $\widehat{\varepsilon} \in \big(1, \frac{1}{2}\varepsilon_0\big)$ and $\mu > 0$, such that

$$\big(1 + \|x\|_X\big)\varphi'(x) \ \geqslant \ \mu \qquad \forall\, x \in \varphi^{-1}\big([c - 2\widehat{\varepsilon}, c + 2\widehat{\varepsilon}]\big) \cap \big(\operatorname{int} B_\varrho(K_c^\varphi)\big)^c.$$

Indeed, if this is not the case, we can find $x_n \notin \operatorname{int} B_\varrho(K_c^\varphi)$, such that

$$\varphi(x_n) \ \longrightarrow \ c \quad \text{and} \quad \big(1 + \|x\|_X\big)\varphi'(x_n) \ \longrightarrow \ 0.$$

Because of the C_c-condition, we can find a subsequence $\{x_{n_k}\}_{k \geqslant 1}$ of $\{x_n\}_{n \geqslant 1}$, such that

$$x_{n_k} \ \longrightarrow \ x \quad \text{in } X \quad \text{as } k \to +\infty.$$

Then

$$\varphi(x) \ = \ c, \quad x \notin \operatorname{int} B_\varrho(K_c^\varphi) \quad \text{and} \quad \varphi'(x) = 0,$$

a contradiction.

We consider the sets

$$C \ \stackrel{df}{=} \ \varphi^{-1}\big([c - \widehat{\varepsilon}, c + \widehat{\varepsilon}]\big) \cap \operatorname{int} B_{2\varrho}(K_c^\varphi)$$

and

$$E \ \stackrel{df}{=} \ \big\{x \in X : \ \big|\varphi(x) - c\big| \geqslant 2\widehat{\varepsilon}\big\} \cup \overline{B_\varrho(K_c^\varphi)}$$

and let

$$\vartheta\colon X \longrightarrow [0,1]$$

be a locally Lipschitz function, such that

$$\vartheta\big|_C \ \equiv \ 0 \quad \text{and} \quad \vartheta\big|_E \ \equiv \ 1.$$

Let $\sigma > 0$ be such that $e^{\sigma} - 1 \leqslant \lambda$ and let V_1 be the normalized pseudogradient vector field of Remark 5.1.20 with $\alpha = 1$, $\beta = 2$. We set

$$f(x) \stackrel{df}{=} \begin{cases} -\mu\sigma\vartheta(x)V_1(x) & \text{if } x \in \varphi^{-1}\big([c - 2\widehat{\varepsilon}, c + 2\widehat{\varepsilon}]\big) \cap \big(\operatorname{int} B_{\varrho}(K_c^{\varphi})\big)^c, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $f\colon X \longrightarrow X$ is locally Lipschitz and

$$\big\|f(x)\big\|_X \leqslant \sigma\big(1 + \|x\|_X\big) \quad \text{and} \quad \big\langle \varphi'(x), f(x)\big\rangle_X \leqslant -\frac{1}{4}\mu\sigma\vartheta(x) \qquad \forall\, x \in X. \tag{5.15}$$

Consider the Cauchy problem

$$\begin{cases} \dot{u}(t) = f\big(u(t)\big), \\ u(0) = x. \end{cases} \tag{5.16}$$

We know that (5.16) has a unique solution $u(x)(\cdot)$ and if we set

$$h(t, x) \stackrel{df}{=} u(x)(t),$$

then h_t is a φ-decreasing, locally Lipschitz, bounded homotopy of homeomorphisms (see Theorem 5.1.22). Moreover,

$$\text{if } h(t, x) \neq x, \quad \text{then } \varphi\big(h(t, x)\big) < \varphi(x)$$

(see the proof of Proposition 5.1.25). Also from (5.15), we have

$$\begin{aligned} \big\|h(t, x) - x\big\|_X &\leqslant \sigma \int_0^t \big(1 + \big\|h(s, x)\big\|_X\big)\, ds \\ &\leqslant \sigma \int_0^t \big\|h(s, x) - x\big\|_X\, ds + \sigma\big(1 + \|x\|_X\big)t. \end{aligned}$$

Hence by Gronwall's inequality (see Theorem A.4.7) and the choice of $\sigma > 0$, we have

$$\big\|h(t, x) - x\big\|_X \leqslant \lambda\big(1 + \|x\|_X\big)t.$$

Finally let $R > 0$ be such that

$$\overline{B_{2\vartheta}(K_c^{\varphi})} \subseteq \operatorname{int} B_R(0)$$

and $\varepsilon \in (0, \widehat{\varepsilon}]$ satisfying

$$8\varepsilon \leqslant \mu\sigma, \quad 8\lambda(1 + R)\varepsilon \leqslant \mu\sigma\varrho.$$

Let $x \in \varphi^{c+\varepsilon}$ and suppose by contradiction that

$$\varphi\big(h(1, x)\big) > c - \varepsilon \quad \text{and} \quad h(1, x) \notin U$$

(see **(d)**). Then

$$c - \varepsilon \;<\; \varphi\big(h(t,x)\big) \;\leqslant\; c + \varepsilon \qquad \forall\, t \in [0,1].$$

Also

$$h\big([0,1] \times \{x\}\big) \cap \operatorname{int} B_{2\varrho}(K_c^\varphi) \;=\; \emptyset,$$

or otherwise from the second inequality in (5.15), we would have

$$\frac{1}{4}\mu\sigma \;\leqslant\; \varphi(x) - \varphi\big(h(1,x)\big) \;<\; 2\varepsilon,$$

a contradiction to the choice of $\varepsilon \in (0, \widehat{\varepsilon})$. So we can find $0 \leqslant t_1 < t_2 \leqslant 1$, such that

$$d_{_X}\big(h(t_1,x), K_c^\varphi\big) \;=\; 2\varrho, \quad d_{_X}\big(h(t_2,x), K_c^\varphi\big) \;=\; 3\varrho$$

and

$$2\varrho \;<\; d_{_X}\big(h(t,x), K_c^\varphi\big) \;<\; 3\varrho \qquad \forall\, t \in (t_1, t_2).$$

We have

$$\frac{1}{4}\mu\sigma(t_2 - t_1) \;\leqslant\; \varphi\big(h(t_1,x)\big) - \varphi\big(h(t_2,x)\big) \;<\; 2\varepsilon.$$

Because $h(t_2,x) = h\big(t_2 - t_1, h(t_1,x)\big)$, we have

$$\begin{aligned} \varrho \;&\leqslant\; \big\|h(t_2,x) - h(t_1,x)\big\|_{_X} \\ &\leqslant\; \lambda\left(1 + \big\|h(t_1,x)\big\|_{_X}\right)(t_2 - t_1) \;<\; \lambda(1+R)\frac{8\varepsilon}{\mu\sigma}, \end{aligned}$$

a contradiction to the choice of $R > 0$ and $\varepsilon \in \big(0, \widehat{\varepsilon}\big)$. This completes the proof. □

Thus far we have assumed that $\varphi \in C^1(X)$. There is an extension of the theory to the case where

$$\varphi = \varphi_1 + \varphi_2 \quad \text{with} \quad \varphi_1 \in C^1(X) \quad \text{and} \quad \varphi_2 \in \Gamma_0(X) \tag{5.17}$$

(see Definition 4.2.1).

DEFINITION 5.1.35 *If φ is as in (5.17), then we say that $x \in X$ is a **critical point** of φ, if $x \in \operatorname{dom}\varphi_2 = \big\{x \in X :\ \varphi_2(x) < +\infty\big\}$ and it satisfies the inequality*

$$0 \;\leqslant\; \big\langle \varphi_1'(x), h - x\big\rangle_{_X} + \varphi_2(h) - \varphi_2(x) \qquad \forall\, h \in X. \tag{5.18}$$

REMARK 5.1.36 Evidently (5.17) is equivalent to saying that

$$-\varphi_1'(x) \;\in\; \partial\varphi_2(x).$$

It is easy to see that every local minimizer of φ is a critical point of φ.

In this general situation the Palais-Smale compactness condition takes the following form which is motivated by the Ekeland variational principle (see Section 4.6).

DEFINITION 5.1.37 *Let φ be as in (5.17). We say that φ satisfies the* ***generalized Palais-Smale condition*** *(*G-PS-***condition** *for short), if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that*

$$\text{the sequence } \{\varphi(x_n)\}_{n\geqslant 1} \text{ is bounded}$$

and

$$-\varepsilon_n \|h - x_n\|_X \leqslant \langle \varphi_1'(x_n), h - x_n\rangle_X + \varphi_2(h) - \varphi_2(x_n) \qquad \forall\, h \in X,$$

with $\varepsilon_n \to 0$, has a strongly convergent subsequence.

The next lemma permits an equivalent reformulation of Definition 5.1.37, which in many situations is more appropriate.

LEMMA 5.1.38
If $\psi \in \Gamma_0(X)$, $\psi(0) = 0$ and

$$\psi(x) \geqslant -\|x\|_X \qquad \forall\, x \in X,$$

then there exists $x^* \in X^*$, such that $\|x^*\|_{X^*} \leqslant 1$ and

$$\langle x^*, x\rangle_X \leqslant \psi(x) \qquad \forall\, x \in X.$$

PROOF Let

$$\psi_1(x) \stackrel{df}{=} \psi(x) + \|x\|_X \qquad \forall\, x \in X.$$

By hypothesis $\psi_1 \in \Gamma_0(X)$, $\psi_1 \geqslant 0$ and

$$0 = \psi_1(0) = \inf_{x\in X} \psi_1(x).$$

So we have $0 \in \partial\psi_1(0)$ (see Proposition 4.4.30). By virtue of Proposition 4.4.31, we have

$$\partial\psi_1(0) = \partial\psi(0) + \partial\,\|\cdot\|_X\,(0).$$

But we know that

$$\partial\,\|\cdot\|_X\,(0) = \overline{B}_1^{X^*} = \{x^* \in X^* :\ \|x^*\|_{X^*} \leqslant 1\}$$

(see Example 4.4.24**(b)**). So

$$0 \in \partial\psi(0) + \overline{B}_1^{X^*}$$

and this means that we can find $x^* \in X^*$ with $\|x^*\|_{X^*} \leqslant 1$, such that

$$\langle x^*, x\rangle_X \leqslant \psi(x) \qquad \forall\, x \in X.$$

□

Using the above lemma, we can make the following definition.

DEFINITION 5.1.39 *Let φ be as in (5.17). We say that φ satisfies the* G-PS′-***condition**, if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1}$ is bounded and*

$$\langle x_n^*, h - x_n\rangle_X \;\leqslant\; \langle \varphi_1'(x_n), h - x_n\rangle_X + \varphi_2(h) - \varphi_2(x_n) \qquad \forall\, h \in X,$$

with

$$x_n^* \to 0 \quad \text{in } X^*,$$

has a strongly convergent subsequence.

REMARK 5.1.40 The above condition means that

$$x_n^* \;\in\; \varphi_1'(x_n) + \partial\varphi_2(x_n)$$

and in this form the similarity with Definition 5.1.5**(a)** becomes apparent. Indeed, if $\varphi_2 \equiv 0$, then we recover Definition 5.1.5**(a)**.

PROPOSITION 5.1.41
If φ is as in (5.17),
then G-PS-*condition and* G-PS′-*condition are equivalent.*

PROOF Clearly G-PS′-condition implies G-PS-condition. So we need to show the opposite implication. Suppose that $\{x_n\}_{n\geqslant 1} \subseteq X$ is a sequence, such that the sequence $\{\varphi(x_n)\}_{n\geqslant 1}$ is bounded and

$$-\varepsilon_n \|h - x_n\|_X \;\leqslant\; \langle \varphi_1'(x_n), h - x_n\rangle_X + \varphi_2(h) - \varphi_2(x_n) \qquad \forall\, h \in X,$$

with $\varepsilon_n \to 0$. If $\varepsilon_n \leqslant 0$, take

$$x_n^* \;=\; 0.$$

If $\varepsilon_n > 0$, let

$$\psi(v) \;\stackrel{df}{=}\; \frac{1}{\varepsilon_n}\left[\langle \varphi_1'(x_n), v\rangle_X + \varphi_2(v + x_n) - \varphi_2(x_n)\right] \qquad \forall\, v \in X.$$

Evidently

$$\psi \in \Gamma_0(X),$$
$$\psi(0) \;=\; 0$$

and

$$-\|v\|_X \;\leqslant\; \psi(v) \qquad \forall\, v \in X.$$

So we can apply Lemma 5.1.38 and obtain $v_n^* \in X^*$ with $\|v_n^*\|_{X^*} \leqslant 1$, such that

$$\langle v_n^*, v\rangle_X \;\leqslant\; \psi(v) \qquad \forall\, v \in X.$$

Setting

$$x_n^* \stackrel{df}{=} \varepsilon_n v_n^*,$$

we have

$$\langle x_n^*, h - x_n \rangle_X \leqslant \langle \varphi_1'(x_n), h - x_n \rangle_X + \varphi_2(h) - \varphi_2(x_n) \qquad \forall\, h \in X,$$

with

$$x_n^* \to 0 \quad \text{in } X^*.$$

□

Let φ be as in (5.17). If $c \in \mathbb{R}$, as before we set

$$\begin{aligned} K^\varphi &\stackrel{df}{=} \big\{x \in X :\ x \text{ is a critical point of } \varphi\big\}, \\ K_c^\varphi &\stackrel{df}{=} \big\{x \in X :\ x \in K^\varphi,\ \varphi(x) = c\big\}, \\ \varphi^c &\stackrel{df}{=} \big\{x \in X :\ \varphi(x) \leqslant c\big\}. \end{aligned}$$

Using the Ekeland variational principle, we can have the following deformation result.

THEOREM 5.1.42
If φ is as in (5.17), φ satisfies the G-PS-*condition, $c \in \mathbb{R}$, $\varepsilon_0, \delta > 0$ and U is a neighbourhood of K_c^φ,*
then there exists $\varepsilon \in (0, \varepsilon_0)$, such that for each compact $G \subseteq \big(\varphi^{c+\varepsilon} \setminus \varphi^{c-\varepsilon}\big) \setminus U$, we can find a closed set F, such that $G \subseteq \operatorname{int} F$, $t_0 > 0$ and a homotopy $h_t \colon X \longrightarrow X$, $0 \leqslant t \leqslant t_0$, such that

(a) $\big\|h(t,x) - x\big\|_X \leqslant t$ *for all* $x \in X$;

(b) $\varphi\big(h(t,x)\big) \leqslant \varphi(x) - 2\varepsilon t$ *for all* $x \in F$;

(c) $\varphi\big(h(t,x)\big) \leqslant \varphi(x)$ *for all* $x \in X$;

(d) $\sup\limits_{x \in F} \varphi\big(h(t,x)\big) - \sup\limits_{x \in F} \varphi(x) \leqslant -2\varepsilon t$.

REMARK 5.1.43 If φ_1 and φ_2 are both even and G is symmetric, i.e.,

$$G = -G,$$

then h_t may be chosen to be odd.

5.2 Minimax Theorems

In this section we use the deformation results to derive minimax expressions characterizing the critical values of a C^1-functional.

We start with a geometric notion which plays a central role in what follows.

DEFINITION 5.2.1 *Let Y be a Hausdorff topological space and C_0, E two distinct subsets of Y, such that E is closed and $C_0 \cap E = \emptyset$. We say that the sets C_0 and E are **linking** in Y, if there exists a set $C \supseteq C_0$, such that for any $\gamma \in C(C;Y)$ satisfying $\gamma|_{C_0} = id_{C_0}$, we have $\gamma(C) \cap E \neq \emptyset$.*

REMARK 5.2.2 Suppose that C_0 and E are linking sets and $\psi\colon Y \longrightarrow Y$ is a homeomorphism. Then it is easy to see that $\psi(C_0)$ and $\psi(E)$ are linking too. If Y is locally convex and C_0 is the relative boundary of a nonempty, bounded and convex set C, then above definition is equivalent to saying that $C_0 \cap E = \emptyset$ and C_0 is not contractible in $Y \setminus E$ (see Gasiński & Papageorgiou (2005, p. 136)).

EXAMPLE 5.2.3 **(a)** Let X be a Banach space, $x_0, x_1 \in X$ with $\|x_0 - x_1\|_X > r$,

$$C_0 = \{x_0, x_1\} \quad \text{and} \quad E = \partial B_r(x_0) = \big\{x \in X : \|x - x_0\|_X = r\big\}.$$

The sets C_0 and E are linking in X. Indeed let

$$C \stackrel{df}{=} \big\{x \in X : (1-t)x_0 + tx_1,\ t \in [0,1]\big\}$$

and consider $\gamma \in C(C;X)$, such that $\gamma(x_0) = x_0$ and $\gamma(x_1) = x_1$. Evidently the set $\gamma(C)$ is connected. If $\gamma(C) \cap E = \emptyset$, then we have $\gamma(C) = U_1 \cup U_2$, with

$$U_1 \stackrel{df}{=} \gamma(C) \cap B_r(x_0) \quad \text{and} \quad U_2 \stackrel{df}{=} \gamma(C) \cap \overline{B}_r(x_0)^c.$$

Since both sets U_1 and U_2 are open, we contradict the connectedness of $\gamma(C)$.

(b) Let X be a Banach space and suppose that $X = Y \oplus V$, with $\dim Y < +\infty$. Let

$$C_0 \stackrel{df}{=} \partial B_r(0) \cap Y = \big\{x \in Y : \|x\|_X = r\big\} \quad \text{and} \quad E \stackrel{df}{=} V.$$

Then the sets C_0 and E are linking in X. To see this, let $\text{proj}_Y \in \mathcal{L}(X)$ be the projection onto Y (it exists since Y is finite dimensional) and let

$$C \stackrel{df}{=} \overline{B}_r(0) \cap Y = \big\{x \in Y : \|x\|_X \leqslant r\big\}.$$

For a given $\gamma \in C(C;X)$, such that $\gamma|_{C_0} = id_{C_0}$, we claim that $\gamma(C) \cap E \neq \emptyset$. To this end it suffices to show that $0 \in \mathrm{proj}_{_Y}\big(\gamma(C)\big)$. So let $h\colon [0,1]\times Y \longrightarrow Y$ be defined by

$$h(t,y) \stackrel{df}{=} t\mathrm{proj}_{_Y}\big(\gamma(y)\big) + (1-t)y.$$

Evidently h_t is a homotopy and $h_1 = \mathrm{proj}_{_Y} \circ \gamma$. Moreover, for every $t \in [0,1]$, we have $h_t|_{C_0} = id_{C_0}$. Then from the homotopy invariance and normalization of Brouwer's degree, we have

$$D_B\big(\mathrm{proj}_{_Y} \circ \gamma, B_r(0) \cap Y, 0\big) \;=\; D_B\big(id_{_Y}, \mathrm{int}\, B_r(0) \cap Y, 0\big) \;=\; 1,$$

which means that $0 \in \mathrm{proj}_{_Y}(C)$.

(c) Let X be a Banach space and suppose that $X = Y \oplus V$, with $\dim Y < +\infty$. Let $e \in V$ with $\|e\|_{_X} = 1$ and $0 < \varrho < r_1$, $0 < r_2$ be given. We set

$$C_0 \stackrel{df}{=} \big\{\lambda e + y :\; y \in Y \text{ and } \lambda \in \{0, r_1\} \text{ or } \|y\|_{_X} = r_2\big\} \quad \text{and} \quad E \stackrel{df}{=} \partial B_\varrho(0) \cap V.$$

Then we claim that the sets C_0 and E are linking in X. To show this as before, let $\mathrm{proj}_{_Y} \in \mathcal{L}(X)$ be the projection on Y,

$$C \;\stackrel{df}{=}\; \big\{\lambda e + y :\; y \in Y,\; \lambda \in [0, r_1],\; \|y\|_{_X} \leqslant r_2\big\}$$

(the cylinder whose boundary is C_0) and suppose that $\gamma \in C(C;X)$ with $\gamma_{C_0} = id_{C_0}$. We will show that $\gamma(C) \cap E \neq \emptyset$. To have this, we must show that there exists $x \in C$ which satisfies

$$\big\|\gamma(x)\big\|_{_X} \;=\; \varrho \quad \text{and} \quad \mathrm{proj}_{_Y}\big(\gamma(x)\big) \;=\; 0.$$

So let $h\colon [0,1] \times \mathbb{R} \times Y \longrightarrow Y$ be defined by

$$\begin{aligned} &h\big(t,(\lambda,y)\big) \\ &\stackrel{df}{=}\; \big(t\big\|\gamma(x) - \mathrm{proj}_{_Y}\big(\gamma(x)\big)\big\|_{_X} + (1-t)\lambda - \varrho,\; t\mathrm{proj}_{_Y}\big(\gamma(x)\big) + (1-t)y\big), \end{aligned}$$

where $x = \lambda + ey$. Clearly h is continuous and

$$h\big(0,(\lambda,y)\big) \;\stackrel{df}{=}\; (\lambda - \varrho, y).$$

Moreover, if $x = \lambda e + y \in C_0$, we have

$$h\big(t,(\lambda,y)\big) \;=\; \big(t\,\|x-y\|_{_X} + (1-t)\lambda - \varrho, y\big) \;=\; (\lambda - \varrho, y) \;\neq\; 0.$$

Identifying C with a subset of $\mathbb{R}\times Y$ by virtue of the decomposition $x = \lambda e + y$ and exploiting the homotopy invariance of Brouwer's degree, we have

$$D_B\big(h_1, \mathrm{int}\, C, 0\big) \;=\; D_B\big(h_0, \mathrm{int}\, C, 0\big) \;=\; 1.$$

So there exists $x = \lambda e + y \in C$, such that $h(1,x) = 0$. This is equivalent to

$$\big\|\gamma(x)\big\|_{_X} \;=\; \varrho \quad \text{and} \quad \mathrm{proj}_{_Y}\big(\gamma(x)\big) \;=\; 0.$$

Using the notion of linking sets, we can prove the following general minimax principle. In what follows X is a Banach space.

THEOREM 5.2.4
If C_0 and E are closed sets which are linking in X, $\varphi \in C^1(X)$,

$$\sup_{x\in C_0} \varphi(x) \leqslant \inf_{x\in E} \varphi(x),$$

φ satisfies the PS_c*-condition, where*

$$c \stackrel{df}{=} \inf_{\gamma\in\Gamma} \sup_{x\in C} \varphi\big(\gamma(x)\big),$$

with $C \supseteq C_0$ as in Definition 5.2.1 and

$$\Gamma \stackrel{df}{=} \big\{\gamma \in C(C;X) : \ \gamma|_{C_0} = id_{C_0}\big\},$$

then $c \geqslant \inf_{x\in E} \varphi(x)$ and c is a critical value of φ. Moreover, if

$$c = \inf_{x\in E} \varphi(x),$$

then $K_c^\varphi \cap E \neq \emptyset$.

PROOF Since the sets C_0 and E are linking, for a given $\gamma \in \Gamma$, we have

$$\gamma(C) \cap E \neq \emptyset.$$

Therefore

$$\sup_{x\in C} \varphi\big(\gamma(x)\big) \geqslant \inf_{x\in E} \varphi(x) \qquad \forall\ \gamma \in \Gamma,$$

hence

$$c \geqslant \inf_{x\in E} \varphi(x).$$

First suppose that

$$c > \inf_{x\in E} \varphi(x).$$

Let us set

$$\varepsilon_0 \stackrel{df}{=} c - \inf_{x\in E} \varphi(x) > 0$$

and assume that $K_c^\varphi = \emptyset$. Then with $U = \emptyset$, let $\varepsilon > 0$ and $h\colon [0,1]\times X \longrightarrow X$ be as postulated by Theorem 5.1.34. The choice of $\varepsilon_0 > 0$ implies that

$$h_t|_{C_0} = id_{C_0} \qquad \forall\ t \in [0,1].$$

Choose $\gamma \in \Gamma$, such that

$$\varphi\big(\gamma(x)\big) \leqslant c + \varepsilon \qquad \forall\ x \in C$$

and let us set

$$\xi \;=\; h_1 \circ \gamma \;\in\; C(C;X).$$

If $x \in C_0$, then

$$\xi(x) \;=\; h_1\big(\gamma(x)\big) \;=\; h_1(x) \;=\; x,$$

hence $\xi \in \Gamma$. Moreover, note that

$$\gamma(x) \;\in\; \varphi^{c+\varepsilon} \qquad \forall\, x \in C.$$

So by virtue of Theorem 5.1.34**(d)**, we have that

$$\varphi\big(\xi(x)\big) \;\leqslant\; c-\varepsilon,$$

which contradicts the definition of c (recall that $\xi \in \Gamma$).

Next suppose that

$$c \;=\; \inf_{x\in E} \varphi(x).$$

We will show that

$$K_c^{\varphi} \cap E \neq \emptyset.$$

Suppose that the opposite holds, namely that $K_c^{\varphi} \cap E = \emptyset$. We apply Theorem 5.1.32, with

$$\widehat{\varphi} \;=\; -\varphi, \quad \widehat{c} \;=\; c, \quad A \;=\; E \quad \text{and} \quad B \;=\; C_0.$$

So we obtain $\varepsilon > 0$ and a continuous map

$$h\colon [0,1] \times X \longrightarrow X$$

as postulated by that theorem. Again we choose $\gamma \in \Gamma$, such that

$$\varphi\big(\gamma(x)\big) \;<\; c+\varepsilon \qquad \forall\, x \in C.$$

Let

$$\xi_1 \;\stackrel{df}{=}\; h_1^{-1} \circ \gamma \in C(C;X).$$

We have

$$\xi_1(x) \;=\; h_1^{-1}\big(\gamma(x)\big) \;=\; h_1^{-1}(x) \;=\; x \qquad \forall\, x \in C_0,$$

i.e., $\xi_1 \in \Gamma$. Since C_0 and E are linking in X, we have

$$\xi_1(C) \cap E \;\neq\; \emptyset$$

and so we can find $x_0 \in C$, such that

$$\xi_1(x_0) \;\in\; E.$$

It follows that

$$\varphi\big(\gamma(x_0)\big) \;=\; \varphi\big(h_1\big(\xi_1(x_0)\big)\big) \;\geqslant\; c+\varepsilon,$$

which contradicts the choice of $\gamma \in \Gamma$. This proves that

$$K_c^{\varphi} \cap E \neq \emptyset.$$

□

With a slight restriction on the sets C_0 and C (which though is satisfied in almost all applications of interest), we can have the above theorem with the $\mathrm{PS_c}$-condition replaced by the $\mathrm{C_c}$-condition.

THEOREM 5.2.5

If C is a nonempty, bounded, closed, convex set in X, C_0 is the relative boundary of C and E is a nonempty, closed subset in X, C_0 and E are linking in X, $\varphi \in C^1(X)$,

$$\sup_{x \in C_0} \varphi(x) \leqslant \inf_{x \in E} \varphi(x),$$

φ satisfies the $\mathrm{C_c}$-condition, where

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \sup_{x \in C} \varphi\big(\gamma(x)\big),$$

with $C \supseteq C_0$ as in Definition 5.2.1 and

$$\Gamma \stackrel{df}{=} \big\{\gamma \in C(C; X) : \gamma|_{C_0} = id_{C_0}\big\},$$

then $c \geqslant \inf_{x \in E} \varphi(x)$ and c is a critical value of φ. Moreover, if

$$c = \inf_{x \in E} \varphi(x),$$

then $K_c^{\varphi} \cap E \neq \emptyset$.

PROOF As in the proof of Theorem 5.2.4, we check that

$$c \geqslant \inf_{x \in E} \varphi(x)$$

and first we prove the theorem when

$$c = \inf_{x \in E} \varphi(x).$$

Without any loss of generality, we may assume that $0 \in C$. Suppose that

$$K_c^{\varphi} \cap E = \emptyset.$$

Since K_c^{φ} is compact, we can find a neighbourhood U of K_c^{φ}, such that

$$U \cap E = \emptyset.$$

Then let $\varepsilon > 0$ and

$$h\colon [0,1] \times X \longrightarrow X$$

be as postulated by Theorem 5.1.34 and choose $\gamma \in \Gamma$, such that

$$\varphi\big(\gamma(x)\big) \leqslant c + \varepsilon \qquad \forall\, x \in C.$$

We introduce $\eta\colon [0,1]\times C_0 \longrightarrow X$, by

$$\eta(t,x) \stackrel{df}{=} \gamma\big((1-t)x\big).$$

We have

$$\eta(0,x) = \gamma(x) = x \quad \text{and} \quad \eta(1,x) = \gamma(0),$$

hence $\eta \in \mathcal{D}_{C_0}$, where $\mathcal{D}_{C_0}$ is the set of contractions (deformations to a point) of C_0. We define

$$\xi\colon [0,1]\times C_0 \longrightarrow X,$$

by

$$\xi(t,x) \stackrel{df}{=} \begin{cases} h(2t,x) & \text{if} \quad t \in \left[0,\frac{1}{2}\right], \\ h\big(1,\eta(2t-1,x)\big) & \text{if} \quad t \in \left[\frac{1}{2},1\right]. \end{cases}$$

Evidently ξ is continuous and

$$\xi(0,x) = h(0,x) = x \quad \text{and} \quad \xi(1,x) = h\big(1,\eta(1,x)\big) = h\big(1,\gamma(0)\big).$$

Therefore $\xi \in \mathcal{D}_{C_0}$. Also note that by Theorem 5.1.34, we have

$$h(2t,x) = x \quad \text{or} \quad \varphi\big(h(2t,x)\big) < \varphi(x) \leqslant c.$$

Therefore in both cases we have that $h(2t,x) \notin E$. Moreover, from the choice of $\gamma \in \Gamma$, we have

$$\gamma(x) \in \varphi^{c+\varepsilon} \qquad \forall\, x \in C.$$

Since C is convex and we have assumed that $0 \in C$, we have

$$2(1-t)x \in C \qquad \forall\, t \in \left[\frac{1}{2},1\right]$$

and so

$$\gamma\big(2(1-t)x\big) \in \varphi^{c+\varepsilon} \qquad \forall\, t \in \left[\frac{1}{2},1\right].$$

Then Theorem 5.1.34**(d)** implies

$$h\big(1,\eta(2t-1,x)\big) \in \varphi^{c-\varepsilon} \cup U \quad \text{and} \quad \big(\varphi^{c-\varepsilon}\cup U\big)\cap E = \emptyset.$$

So it follows that ξ is a contraction of C_0 in $X \setminus E$, a contradiction to the fact that the sets C_0 and E are linking in X (see Remark 5.2.2). This proves that

$$K_c^{\varphi} \cap E \neq \emptyset.$$

When

$$c > \inf_{x\in E} \varphi(x),$$

the proof is similar as that of Theorem 5.2.4 but using this time Theorem 5.1.34. ∎

REMARK 5.2.6 We emphasize that in both Theorems 5.2.4 and 5.2.5, the inequality

$$\sup_{x\in C_0} \varphi(x) \;\leqslant\; \inf_{x\in E} \varphi(x)$$

is not strict. Equality is a possibility and in that case we refer to "relaxed boundary conditions."

With appropriate choices of the sets C_0 and E, from Theorem 5.2.5 we can recover the classical results of critical point theory, strengthened in the sense that the PS_c-condition is replaced by the C_c-condition and we have relaxed boundary conditions.

The first result that we deduce from Theorem 5.2.5 is the mountain pass theorem.

COROLLARY 5.2.7 (Mountain Pass Theorem) *If* $\varphi \in C^1(X)$, $x_0, x_1 \in X$ *with* $\|x_0 - x_1\|_X > r > 0$,

$$\max\big\{\varphi(x_0), \varphi(x_1)\big\} \;\leqslant\; \inf_{\|x\|_X = r} \varphi(x)$$

and φ *satisfies the* C_c*-condition, where*

$$c \;\stackrel{df}{=}\; \inf_{\gamma_0\in\Gamma_0} \max_{t\in[0,1]} \varphi\big(\gamma_0(t)\big)$$

and

$$\Gamma_0 \;\stackrel{df}{=}\; \big\{\gamma_0 \in C\big([0,1];X\big) :\; \gamma(0) = x_0,\; \gamma(1) = x_1\big\},$$

then $c \geqslant \inf_{\|x\|_X = r} \varphi(x)$ *and* c *is a critical value of* φ. *Moreover, if*

$$c \;=\; \inf_{\|x\|_X = r} \varphi(x),$$

then there exists a critical point x *of* φ *with* $\varphi(x) = c$ *and* $\|x\|_X = r$.

PROOF Let

$$C_0 \;=\; \big\{x_0, x_1\big\} \quad\text{and}\quad E \;=\; \partial B_r(x_0) = \big\{x \in X :\; \|x - x_0\|_X = r\big\}.$$

From Example 5.2.3**(a)** we know that C_0 and E link in X and

$$C \;\stackrel{df}{=}\; \big\{x \in X :\; (1-t)x_0 + tx_1,\; t \in [0,1]\big\}.$$

Also note that if

$$\gamma \;\in\; \Gamma \;\stackrel{df}{=}\; \big\{\gamma \in C(C;X) :\; \gamma(x_0) = x_0,\; \gamma(x_1) = x_1\big\},$$

and

$$\gamma_0(t) \;=\; \gamma\big((1-t)x_0 + tx_1\big) \;=\; \gamma(x) \qquad \forall\, x \in C,$$

then $\gamma_0 \in \Gamma_0$ and conversely. So we can apply Theorem 5.2.5 and obtain the corollary. ∎

COROLLARY 5.2.8 (Saddle Point Theorem)
If $\varphi \in C^1(X)$, $X = Y \oplus V$ *with* $\dim Y < +\infty$, *there exists* $r > 0$, *such that*

$$\sup_{\substack{x \in Y \\ \|x\|_X = r}} \varphi(x) \leqslant \inf_{x \in V} \varphi(x)$$

and φ *satisfies the* $\mathrm{C_c}$*-condition, where*

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{\substack{x \in Y \\ \|x\|_X \leqslant r}} \varphi\big(\gamma(x)\big)$$

and

$$\Gamma \stackrel{df}{=} \left\{\gamma \in C\big(\overline{B}_r(0) \cap Y; X\big) : \ \gamma|_{\partial B_r(0) \cap Y} = id_{\partial B_r(0) \cap Y}\right\},$$

then $c \geqslant \inf_{x \in V} \varphi(x)$ *and* c *is a critical value of* φ. *Moreover, if*

$$c = \inf_{x \in V} \varphi(x),$$

then there exists a critical point x *of* φ *with* $\varphi(x) = c$ *and* $x \in V$.

PROOF Let $C_0 = \partial B_r(0) \cap Y$, $C = \overline{B}_r(0) \cap Y$ and $E = V$. Then by virtue of Example 5.2.3**(b)** the sets C_0 and E are linking in X. So we apply Theorem 5.2.5 and have the corollary. □

COROLLARY 5.2.9 (Generalized Mountain Pass Theorem)
If $\varphi \in C^1(X)$, $X = Y \oplus V$ *with* $\dim Y < +\infty$, *there exist* $0 < \varrho < r_1$, $0 < r_2$ *and* $e \in V$ *with* $\|e\|_X = 1$, *such that for*

$$C_0 \stackrel{df}{=} \left\{\lambda e + y : \ y \in Y \text{ and } \lambda \in \{0, r_1\} \text{ or } \|y\|_X = r_2\right\},$$

$$C \stackrel{df}{=} \left\{\lambda e + y : \ y \in Y, \ \lambda \in [0, r_1], \ \|y\|_X \leqslant r_2\right\} \quad \text{and} \quad E \stackrel{df}{=} \partial B_\varrho(0) \cap V,$$

we have

$$\max_{x \in C_0} \varphi(x) \leqslant \inf_{x \in E} \varphi(x)$$

and φ *satisfies the* $\mathrm{C_c}$*-condition, where* $c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{x \in C} \varphi\big(\gamma(x)\big)$ *and*

$$\Gamma \stackrel{df}{=} \left\{\gamma \in C(C; X) : \ \gamma|_{C_0} = id_{C_0}\right\},$$

then $c \geqslant \inf_{x \in E} \varphi(x)$ *and* c *is a critical value of* φ. *Moreover, if*

$$c = \inf_{x \in E} \varphi(x),$$

then there exists a critical point x *of* φ *with* $\varphi(x) = c$ *and* $x \in V$.

PROOF The sets C_0 and E are linking in X (see Example 5.2.3**(c)**). So we can apply Theorem 5.2.5. □

Finally a straightforward application of Theorem 4.6.33 gives the following result.

THEOREM 5.2.10
If $\varphi \in C^1(X)$ is bounded from below and satisfies the C_c-condition with

$$c = \inf_{x \in X} \varphi(x),$$

then there exists $x \in X$, such that

$$\varphi(x) = \inf_{x \in X} \varphi(x)$$

(hence $x \in X$ is a critical point of φ).

REMARK 5.2.11 Recall that if $\varphi \in C^1(X)$ is bounded from below, then the PS-condition and the C-condition are equivalent.

Next we consider functionals $\varphi\colon X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ of the form

$$\varphi = \varphi_1 + \varphi_2 \quad \text{with} \quad \varphi_1 \in C^1(X) \quad \text{and} \quad \varphi_2 \in \Gamma_0(X). \tag{5.19}$$

THEOREM 5.2.12
If φ is as in (5.19), φ satisfies the G-PS-*condition (see Definition 5.1.37), C is a compact submanifold of X with relative boundary $\partial C = C_0$, $E \subseteq X$ is a nonempty, C_0 and E are linking in X through C and*

$$\sup_{x \in C_0} \varphi(x) < \inf_{x \in E} \varphi(x),$$

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{x \in C} \varphi\big(\gamma(x)\big),$$

with

$$\Gamma \stackrel{df}{=} \left\{\gamma \in C(C;X) : \ \gamma|_{C_0} = id_{C_0}\right\},$$

then c is a critical value of φ if $c < +\infty$ and $c \geqslant \inf_{x \in E} \varphi(x)$.

PROOF Clearly $c \geqslant \inf_{x \in E} \varphi(x)$. Equip Γ with the supremum metric

$$d_\infty(\gamma, \xi) \stackrel{df}{=} \max_{x \in C} \big\|\gamma(x) - \xi(x)\big\|_X \qquad \forall\ \gamma, \xi \in \Gamma.$$

Then (Γ, d_∞) is a complete metric space. Let $\vartheta\colon \Gamma \longrightarrow \overline{\mathbb{R}}$ be defined by

$$\vartheta(\gamma) \stackrel{df}{=} \max_{x \in C} \varphi\big(\gamma(x)\big).$$

It is easy to check that ϑ is lower semicontinuous. Also we claim that for every $\gamma \in \operatorname{dom}\vartheta = \{\gamma \in \Gamma : \ \vartheta(\gamma) < +\infty\}$, the function $\varphi \circ \gamma$ is continuous on C. To this end note that φ_2 is bounded on $\gamma(C)$, hence φ_2 is bounded and lower semicontinuous on $\overline{\operatorname{conv}}\,\gamma(C)$. Evidently it suffices to show that $\varphi_2|_{\overline{\operatorname{conv}}\,\gamma(C)}$ is continuous. Let $x_0 \in \overline{\operatorname{conv}}\,\gamma(C)$ and let U be a neighbourhood of x_0, such that

$$\varphi_2(u) \ \leqslant \ \mu < +\infty \qquad \forall\ u \in U \cap \overline{\operatorname{conv}}\,\gamma(C).$$

Without any loss of generality, we may assume that

$$x_0 \ = \ 0 \quad \text{and} \quad \varphi_2(x_0) \ = \ 0.$$

If $\lambda \in [0,1]$ and $u \in \lambda U \cap \overline{\operatorname{conv}}\,\gamma(C)$, we have

$$\varphi_2(u) \ \leqslant \ (1-\lambda)\varphi_2(0) + \lambda\varphi_2\left(\frac{u}{\lambda}\right) \ = \ \lambda\varphi_2\left(\frac{u}{\lambda}\right) \ \leqslant \ \lambda\mu,$$

hence

$$\limsup_{\substack{u\to 0\\ u\in\overline{\operatorname{conv}}\,\gamma(C)}} \varphi_2(u) \ \leqslant \ 0.$$

On the other hand from the lower semicontinuity of φ_2, we have

$$\liminf_{\substack{u\to 0\\ u\in\overline{\operatorname{conv}}\,\gamma(C)}} \varphi_2(u) \ \geqslant \ 0.$$

Therefore, it follows that $\varphi_2|_{\overline{\operatorname{conv}}\,\gamma(C)}$ is continuous and so $\varphi \circ \gamma \in C(C)$.

Suppose that c is not a critical value of φ and let

$$\varepsilon_0 \ \stackrel{df}{=} \ \inf_{x\in E} \varphi(x) - \max_{x\in C_0} \varphi(x) \ > \ 0.$$

Let $\varepsilon \in \big(0, \varepsilon_0\big)$ be as in Theorem 5.1.42 and choose $\varepsilon' < \varepsilon$, such that

$$\max_{x\in C_0} \varphi(x) \ < \ c - \varepsilon'.$$

Applying Theorem 4.6.1, we obtain $\gamma \in \Gamma$, such that

$$\vartheta(\gamma) \ \leqslant \ c + \varepsilon' \quad \text{and} \quad -\varepsilon d_\infty(\gamma,\xi) \ \leqslant \ \vartheta(\xi) - \vartheta(\gamma) \qquad \forall\ \xi \in \Gamma. \tag{5.20}$$

Let

$$D \ \stackrel{df}{=} \ \gamma(C) \quad \text{and} \quad D_0 \ \stackrel{df}{=} \ \big\{\gamma(x) : \ x \in C, \varphi\big(\gamma(x)\big) \in [c-\varepsilon', c+\varepsilon']\big\}.$$

By virtue of the continuity of $\varphi \circ \gamma$, we see that D_0 is compact and from the choice of $\varepsilon' > 0$, we have that

$$C_0 \cap D_0 \ = \ \emptyset.$$

Clearly

$$D_0 \ \subseteq \ \varphi^{c+\varepsilon} \setminus \varphi^{c-\varepsilon}.$$

Let $h_t\colon X \longrightarrow X$, $t \in [0, t_0]$, be the homotopy postulated by Theorem 5.1.42 and let us set $\xi \stackrel{df}{=} h_t \circ \gamma$. Because $h_t|_{C_0} = id_{C_0}$, we have that $\xi \in \Gamma$. From Theorem 5.1.42**(a)**, we have

$$d_\infty(\gamma, \xi) \;\leqslant\; t. \tag{5.21}$$

Note that

$$\vartheta(\xi) \;=\; \max_{x \in C} \varphi\big(\xi(x)\big) \;>\; c - \varepsilon'.$$

Hence

$$\vartheta(\xi) \;=\; \max_{x \in C} \varphi\big(\xi(x)\big) \;=\; \max_{x \in C} \varphi\big((h_t \circ \gamma)(x)\big) \;=\; \max_{y \in D_0} \varphi\big(h_t(y)\big).$$

Then Theorem 5.1.42**(d)** implies that

$$\vartheta(\xi) - \vartheta(\gamma) \;=\; \max_{y \in D_0} \varphi\big(h_t(y)\big) - \max_{x \in D_0} \varphi(x) \;\leqslant\; -2\varepsilon t. \tag{5.22}$$

Comparing (5.20), (5.21) and (5.22), we reach a contradiction. This proves that c is a critical value of φ. □

With suitable choices of the sets C and E, we obtain generalizations of the mountain pass theorem, saddle point theorem and generalized mountain pass theorem.

THEOREM 5.2.13 (Mountain Pass Theorem)
If $\varphi = \varphi_1 + \varphi_2$ is as in (5.19), φ satisfies the G-PS-*condition and there exist $x \in X$ and $r > 0$, such that*

$$\max\big\{\varphi(x), \varphi(0)\big\} \;<\; \inf_{\|x\|_X = r} \varphi(x),$$

$$c \;\stackrel{df}{=}\; \inf_{\gamma_0 \in \Gamma_0} \max_{t \in [0,1]} \varphi\big(\gamma_0(t)\big),$$

where

$$\Gamma_0 \;\stackrel{df}{=}\; \big\{\gamma_0 \in C\big([0,1]; X\big) :\; \gamma(0) = 0,\; \gamma(1) = x\big\},$$

then $c \geqslant \inf_{\|x\|_X = r} \varphi(x)$ and c is a critical value of φ.

COROLLARY 5.2.14
If $\varphi = \varphi_1 + \varphi_2$ is as in (5.19), φ satisfies the G-PS-*condition, 0 is a local minimizer of φ and there exists $x \neq 0$, such that $\varphi(x) \leqslant \varphi(0)$,*
then φ has a critical point different from x and 0.
In particular, if φ has two local minima, then φ has at least three critical points.

THEOREM 5.2.15 (Saddle Point Theorem)
If $\varphi = \varphi_1 + \varphi_2$ *is as in (5.19),* φ *satisfies the* G-PS-*condition,* $X = Y \oplus V$ *with* $\dim Y < +\infty$, *there exists* $r > 0$, *such that*

$$\max_{x \in \partial B_r(0) \cap Y} \varphi(x) \ < \ \inf_{x \in V} \varphi(x),$$

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{x \in C} \varphi\big(\gamma(x)\big),$$

where

$$C \stackrel{df}{=} \overline{B}_r(0) \cap Y$$

and

$$\Gamma \stackrel{df}{=} \left\{\gamma \in C\big(C; X\big) : \ \gamma|_{\partial B_r(0) \cap Y} = id_{\partial B_r(0)} \cap Y\right\},$$

then $c \geqslant \inf_{x \in V} \varphi(x)$ *and* c *is a critical value of* φ.

THEOREM 5.2.16 (Generalized Mountain Pass Theorem)
If $\varphi = \varphi_1 + \varphi_2$ *is as in (5.19),* φ *satisfies the* G-PS-*condition,* $X = Y \oplus V$ *with* $\dim Y < +\infty$, *there exist* $0 < \varrho < r_1$, $0 < r_2$ *and* $e \in V$ *with* $\|e\|_X = 1$, *such that for*

$$C_0 \stackrel{df}{=} \left\{\lambda e + y : \ y \in Y \text{ and } \lambda \in \left\{0, r_1\right\} \text{ or } \|y\|_X = r_2\right\},$$

$$C \stackrel{df}{=} \left\{\lambda e + y : \ y \in Y, \ \lambda \in [0, r_1], \ \|y\|_X \leqslant r_2\right\}$$

and

$$E \stackrel{df}{=} \partial B_\varrho(0) \cap V,$$

we have

$$\max_{x \in C_0} \varphi(x) \ < \ \inf_{x \in E} \varphi(x)$$

and

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{x \in C} \varphi\big(\gamma(x)\big),$$

where

$$\Gamma \stackrel{df}{=} \left\{\gamma \in C(C; X) : \ \gamma|_{C_0} = id_{C_0}\right\},$$

then $c \geqslant \inf_{x \in E} \varphi(x)$ *and* c *is a critical value of* φ.

Finally a straightforward application of Theorem 4.6.1 (with $\varepsilon = \frac{1}{n}$, $\lambda = 1$), produces the following result.

THEOREM 5.2.17
If $\varphi = \varphi_1 + \varphi_2$ *is as in (5.19),* φ *satisfies the* G-PS-*condition and it is bounded from below,*
then $c = \inf_{x \in X} \varphi(x)$ *is attained and so it is a critical value of* φ.

5.3 Structure of the Critical Set

In this section we go beyond the existence of critical points and produce additional information about the fine structure of the functional near them. Different types of critical points of a functional φ sometimes can be distinguished by the topological structure of their neighbourhoods in the sublevel sets of φ. It turns out that such information about the topological type in some cases is already available from the minimax characterization of the corresponding critical value. One should note that φ has to be only C^1 as opposed to Morse theory, which has the same goal but requires more regularity on φ.

So let X be a Banach space and $\varphi \in C^1(X)$. We keep the notation introduced in Section 5.1. So recall that for $c \in \mathbb{R}$, we set

$$\begin{aligned} K^{\varphi} &\stackrel{df}{=} \big\{x \in X : \ \varphi'(x) = 0\big\}, \\ K_c^{\varphi} &\stackrel{df}{=} \big\{x \in K^{\varphi} : \varphi(x) = c\big\}, \\ \varphi^c &\stackrel{df}{=} \big\{x \in X : \ \varphi(x) \leqslant c\big\}, \\ \varphi_c &\stackrel{df}{=} \big\{x \in X : \ \varphi(x) \geqslant c\big\}, \\ \overset{\circ}{\varphi}{}^c &\stackrel{df}{=} \big\{x \in X : \ \varphi(x) < c\big\}. \end{aligned}$$

DEFINITION 5.3.1 *Suppose that $x_0 \in K_c^{\varphi}$.*

(a) *We say that x_0 is a **local minimizer** of φ, if there exists an open neighbourhood U of x_0, such that $\varphi(x_0) \leqslant \varphi(x)$ for all $x \in U$.*

(b) *We say that x_0 is of **mountain pass type**, if for all open neighbourhoods U of x_0, $U \cap \overset{\circ}{\varphi}{}^c \neq \emptyset$ and the set $U \cap \overset{\circ}{\varphi}{}^c$ is not path-connected, where $c = \varphi(x_0)$.*

REMARK 5.3.2 Note that we do not require that the critical point is isolated.

We start with a topological lemma.

LEMMA 5.3.3
If (Y, d_Y) is a metric space, C and V are nonempty subsets of Y, such that C is compact, V is open and $C \subseteq \overline{V}$ and $\big\{U(y)\big\}_{y \in C}$ is an open cover of C, such that $y \in U(y)$ and for every $y \in C$, the set $U(y) \cap V$ is path connected, then there exists a finite, disjoint, open cover $\big\{U_i\big\}_{i=1}^{N}$ of C, such that for each $i \in \big\{1, \dots, N\big\}$, the set $U_i \cap V$ is contained in a path connected component $\Big(\bigcup\limits_{y \in C} U(y)\Big) \cap V$.

PROOF Clearly we may assume that C is a strict subset of Y or otherwise there is nothing to prove. Because C is compact, we can find a finite subcover $\{U(y_i)\}_{i=1}^{M}$ of the open cover $\{U(y)\}_{y\in C}$. We let

$$\delta \stackrel{df}{=} \min_{y\in C} \max_{i\in\{1,\dots,M\}} d_{_Y}\big(y, C\setminus U(y_i)\big).$$

We claim that $\delta > 0$. Indeed, if $\delta = 0$, then for every $m \geqslant 1$, we can find $y_m \in C$, such that

$$d_{_Y}\big(y_m, U(y_i)^c\big) \leqslant \frac{1}{m} \qquad \forall\, i \in \{1,\dots,M\}.$$

Because C is compact, we may assume that

$$y_m \longrightarrow y \quad \text{as } m \to +\infty,$$

for some $y \in C$. Then

$$d_{_Y}\big(y, U(y_i)^c\big) \leqslant 0 \qquad \forall\, i \in \{1,\dots,M\}.$$

Hence $y \in U(y_i)^c$ for all $i \in \{1,\dots,M\}$ and so

$$y \in \left[\bigcup_{i=1}^{M} U(y_i)\right]^c \subseteq C^c,$$

a contradiction. Therefore $\delta > 0$ and from the definition of $\delta > 0$, we see that

$$B_\delta(y) = \{y' \in Y :\ d_{_Y}(y', y) < \delta\} \subseteq U(\overline{y}) \qquad \forall\, y \in C,$$

for some $\overline{y} \in C$. Next on C we define an equivalent relation $\sim$ as follows: $y \sim y'$ if and only if there exist finitely many points $\{y_i\}_{i=0}^{k+1}$, such that

$$d_{_Y}(y_i, y_{i+1}) < \delta, \quad y_0 = y \quad \text{and} \quad y_{k+1} = y'.$$

Because C is compact, there are only finitely many equivalence classes denoted by $C_1, \dots, C_N$. Let

$$W_i \stackrel{df}{=} \left\{y \in Y :\ d_{_Y}(y, C_i) < \frac{\delta}{4}\right\} \qquad \forall\, i \in \{1,\dots,N\}.$$

Evidently $W_i \cap W_j = \emptyset$ for $i \neq j$ and

$$C \subseteq \bigcup_{i=1}^{N} W_i.$$

It remains to show that each set $W_i \cap V$ is contained in a path connected component of $\left(\bigcup_{y\in C} U(y)\right) \cap V$.

We define another equivalence relation $\sim_1$ on C as follows: $y \sim_1 y$ if and only if y and y' belong to the same path connected component of

$$\Big(\bigcup_{y \in C} U(y) \Big) \cap V.$$

Fix $i \in \{1, \ldots, N\}$ and let $y, y' \in W_i \cap V$. We need to show that

$$y \ \sim_1 \ y'.$$

From the definition of W_i, we can find a finite sequence $\{y_r\}_{r=1}^{k+1} \subseteq C_i$, such that

$$d_{_Y}(y, y_0) \ < \ \frac{\delta}{4}$$

and

$$d_{_Y}(y', y_{k+1}) \ < \ \frac{\delta}{4}, \quad \text{and} \quad d_{_Y}(y_r, y_{r+1}) \ < \ \delta, \qquad \forall \ r \in \{0, \ldots, k\}.$$

Let

$$\varepsilon \ \stackrel{df}{=} \ \delta - \max_{r \in \{1, \ldots, k\}} d_{_Y}(y_r, y_{r+1}) > 0.$$

Since by hypothesis $C \subseteq \overline{V}$, we can find a finite sequence $\{x_r\}_{r=0}^{k+1} \subseteq V$, such that

$$d_{_Y}(y_r, x_r) \ < \ \frac{\varepsilon}{2} \ < \ \delta \qquad \forall \ r \in \{0, \ldots, k+1\}$$

and

$$d_{_Y}(x_r, x_{r+1}) \ < \ \delta \qquad \forall \ r \in \{0, \ldots, k\}.$$

From the choice of $\delta > 0$, we have

$$y, x_0 \ \in \ \big(B_\delta(y_0) \cap V\big) \ \subseteq \ \big(U(\overline{y}_0) \cap V\big),$$

for some $\overline{y}_0 \in C$. By virtue of our hypothesis concerning the cover $\big\{U(y)\big\}_{y \in C}$, we have that $y \sim_1 x_0$. In a similar fashion we show that $x_{k+1} \sim_1 y'$.

Finally fix $r \in \{0, \ldots, k\}$. From the triangle inequality, we have

$$d_{_Y}(y_r, x_{r+1}) \ \leqslant \ d_{_Y}(y_r, y_{r+1}) + d_{_Y}(y_{r+1}, x_{r+1}) \ < \ \delta - \varepsilon + \frac{\varepsilon}{2} \ < \ \delta.$$

Therefore we can find $\overline{y}_r \in C$, such that

$$x_r, x_{r+1} \ \in \ \big(B_\delta(y_r) \cap V\big) \ \subseteq \ \big(U(\overline{y}_r) \cap V\big)$$

and so

$$x_r \sim_1 x_{r+1} \qquad \forall \ r \in \{0, \ldots, k\}.$$

Summing up we have established that

$$y \sim_1 x_0 \sim_1 x_1 \sim_1 \ldots \sim_1 x_{k+1} \sim_1 y'.$$

This completes the proof of the lemma. ∎

We will also need the following trivial variant of the standard deformation theorem (see Theorem 5.1.30).

LEMMA 5.3.4
If $\varphi \in C^1(X)$ satisfies the PS*-condition, $\varepsilon_0 > 0$, $c \in \mathbb{R}$ and U and V are two open neighbourhoods of K_c^φ, such that*

$$\overline{U} \subseteq V \quad \text{and} \quad d_Y(\partial V, U) > 0,$$

then there exist $\varepsilon \in (0, \varepsilon_0]$ and a φ-decreasing, locally Lipschitz homotopy of homeomorphisms $h\colon [0,1] \times X \longrightarrow X$, such that

(a) $h_1\big(\varphi^{c+\varepsilon} \setminus U\big) \subseteq \varphi^{c-\varepsilon}$;

(b) $h\big([0,1] \times \overline{U}\big) \subseteq V$;

(c) $h(t,x) = x$ *for all* $(t,x) \in [0,1] \times \big(\varphi^{c-\varepsilon_0} \cup \varphi_{c+\varepsilon_0}\big)$.

Now we can have the first structural result for the critical set K_c^φ.

THEOREM 5.3.5
If $\varphi \in C^1(X)$ satisfies the PS*-condition, x_0, x_1 are two distinct points in X,*

$$\Gamma \stackrel{df}{=} \big\{\gamma \in C\big([0,1]; X\big) : \ \gamma(0) = x_0, \ \gamma(1) = x_1\big\},$$

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi\big(\gamma(t)\big)$$

and

$$c > \eta \stackrel{df}{=} \max\big\{\varphi(x_0), \varphi(x_1)\big\},$$

then $K_c^\varphi \neq \emptyset$ and one of the following conditions holds:

(a) *K_c^φ contains a local minimizer of φ; or*

(b) *K_c^φ contains a critical point of mountain pass type.*

PROOF Since $c > \eta$, we can find $r > 0$, such that

$$\inf_{\|x\|_X = r} \varphi(x) > \eta.$$

So by virtue of the mountain pass theorem (see Corollary 5.2.7), we have that $K_c^\varphi \neq \emptyset$. To prove the alternative part of the theorem, we argue by contradiction. Suppose that K_c^φ contains no local minimizers nor critical points of mountain pass type. Then for any $x \in K_c^\varphi$ we can find a neighbourhood $U(x)$ of x, such that the set $U(x) \cap \overset{\circ}{\varphi}{}^c$ is path connected. Since φ satisfies the PS-condition, we have that K_c^φ is compact. Because we do not have local minimizers, then

$$K_c^\varphi = \overline{\overset{\circ}{\varphi}{}^c}.$$

The collection $\{U(x)\}_{x\in K_c^\varphi}$ is of course an open cover of K_c^φ. So we can apply Lemma 5.3.3 and produce open sets $U_1,\dots,U_N$, such that

$$K_c^\varphi \ \subseteq\ \bigcup_{i=1}^N U_i,$$

$$\text{if}\quad U_i\cap U_j\neq 0\quad\text{then}\quad i=j$$

and $U_i\cap \overset{\circ}{\varphi}{}^c$ is contained in a path connected component of

$$\Big(\bigcup_{x\in K_c^\varphi} U(x)\Big)\cap \overset{\circ}{\varphi}{}^c\ .$$

If $V=\bigcup\limits_{i=1}^N U_i$, then $d_Y\big(\partial V,K_c^\varphi\big)>0$. Also set $\widehat{U}\overset{df}{=}\bigcup\limits_{x\in K_c^\varphi}U(x)$ and define

$$\varepsilon_0\ \overset{df}{=}\ \frac{1}{2}(c-\eta),$$

$$\delta\ \overset{df}{=}\ \frac{1}{8}\min\left\{d_Y\big((\partial\widehat{U})\cup\{x_0,x_1\},K_c^\varphi\big),d_Y\big(\partial V,K_c^\varphi\big)\right\}$$

and

$$U\ \overset{df}{=}\ \big\{x\in X:\ d_Y(x,K_c^\varphi)<\delta\big\}.$$

Then with the above data, we can apply Lemma 5.3.4 and obtain $\varepsilon\in(0,\varepsilon_0]$ and a φ-decreasing, locally Lipschitz homotopy of homeomorphisms h_t as postulated by the lemma. Choose $\gamma\in\Gamma$, such that

$$\varphi\big(\gamma(t)\big)\ \leqslant\ c+\varepsilon\qquad\forall\ t\in[0,1]$$

and define

$$\Sigma\ \overset{df}{=}\ \big\{t\in[0,1]:\ \gamma(t)\notin U\big\},$$
$$D\ \overset{df}{=}\ \big(\widehat{U}\cap K_c^\varphi\big)\cup h\big(1,\gamma(\Sigma)\big).$$

Evidently $x_0,x_1\in D$. Let D_0 be the path connected component of D which contains x_0. We will show that $x_1\in D_0$. If $\Sigma=[0,1]$, then we are done. So suppose that $\Sigma\neq[0,1]$. Note that Σ is closed and let

$$t^*\ \overset{df}{=}\ \sup\big\{t\in\Sigma:\ h\big(1,\gamma(t)\big)\in D_0\big\}.$$

Suppose that $t^*<1$. Note that 0 belongs in the relative interior of Σ in $[0,1]$. Therefore $0<t^*<1$. Let $\big[t_1^*,t_2^*\big]$ be the component of Σ containing t^*.

If $t_1^*<t^*$, then $t^*=t_2^*$. Suppose that $t^*=t_1^*$. Then $\gamma(t_1^*)\in\partial U$ and so $h\big(1,\gamma(t_1^*)\big)\in\operatorname{int}D$. Hence there exists $\overline{\varepsilon}>0$, such that

$$B_{\overline{\varepsilon}}\big(h\big(1,\gamma(t_1^*)\big)\big)\ \subseteq\ D$$

and we must have

$$B_\varepsilon\big(h\big(1,\gamma(t_1^*)\big)\big)\cap D_0\neq\emptyset,$$

which means that

$$h\big(1,\gamma(t_1^*)\big)\ =\ h\big(1,\gamma(t^*)\big)\ \in\ D_0.$$

It follows that

$$h\big(1,\gamma(t_2^*)\big)\ \in\ D_0,$$

which means that

$$t^*\ \geqslant\ t_2^*\ >\ t_1^*,$$

a contradiction. Therefore $t^*=t_2^*$ and

$$h\big(1,\gamma(t^*)\big)\ \in\ D_0.$$

This means that

$$\gamma(t^*)\ \in\ \partial U.$$

Let $i_0\in\big(1,\dots,N\big)$ be such that

$$d_{_Y}\big(\gamma(t^*),U_{i_0}\cap K_c^\varphi\big)\ =\ \delta.$$

Let

$$\widehat{t}\ \stackrel{df}{=}\ \sup\big\{t\in[0,1]:\ \gamma(t)\in\overline{U\cap U_{i_0}}\big\}.$$

Because $t^*=t_2^*$, we must have

$$\widehat{t}\ \in\ (t^*,1).$$

Also

$$\gamma(\widehat{t})\ \in\ \partial\big(U\cap U_{i_0}\big)$$

and so $\gamma(\widehat{t})\in\Sigma$. Then

$$y\ =\ h\big(1,\gamma(\widehat{t})\big)\ \in\ U_{i_0}\cap\overset{\circ}{\varphi}{}^c$$

and

$$u\ =\ h\big(1,\gamma(t^*)\big)\ \in\ U_{i_0}\cap\overset{\circ}{\varphi}{}^c\ .$$

Since $U_{i_0}\cap\overset{\circ}{\varphi}{}^c$ is contained in a path connected component of

$$\Big(\bigcup_{x\in K_c^\varphi}U(x)\Big)\cap\overset{\circ}{\varphi}{}^c,$$

we have that $y\sim_1 u$ (see the proof of Lemma 5.3.3). Moreover, $u\in D_0$ and so $x_0\sim_1 u\sim_1 y$. From this it follows that $t^*\geqslant\widehat{t}>t^*$, a contradiction. Therefore we infer that $t^*=1$, which implies that $e_1\in D_0$. But this contradicts the definition of c since

$$D_0\ \subseteq\ D\ \subseteq\ \overset{\circ}{\varphi}{}^c\ .$$

This concludes the proof of the theorem. □

Another structural result for the set of critical points is the following one.

THEOREM 5.3.6
If $\varphi \in C^1(X)$ *satisfies the* PS*-condition,* 0 *is a local minimizer of* φ *with* $\varphi(0) = 0$ *and* φ *has a second local minimizer* $x_1 \neq 0$,
then one of the following holds:

(a) *there is a critical point* x *of* φ*, which is a saddle point (i.e., every neighbourhood of* x *contains points* y, u*, such that* $\varphi(y) < \varphi(x) < \varphi(u)$*);*

(b) *the origin and* x_1 *can be path connected in any neighbourhood of the set of local minimizers* x *of* φ *with* $\varphi(x) = 0$ *(then necessarily* $\varphi(x_1) = 0$*).*

PROOF Let

$$\Gamma \stackrel{df}{=} \big\{\gamma \in C\big([0,1];X\big) : \ \gamma(0) = 0, \ \gamma(1) = x_1\big\}$$

and let us set

$$c \stackrel{df}{=} \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi\big(\gamma(t)\big).$$

Suppose that K_c^φ consists entirely of local minimizers of φ. So for every $x \in K_c^\varphi$ we can find a neighbourhood $U(x)$ of x, such that

$$\varphi(x) \ = \ c \ \leqslant \ \varphi(y) \qquad \forall\ y \in U(x)$$

(see Definition 5.3.1**(a)**). Let $\widehat{U} \stackrel{df}{=} \bigcup_{x\in K_c^\varphi} U(x)$. For any neighbourhood $\widetilde{U}$ of K_c^φ let $\varepsilon > 0$ and h_t be as in Theorem 5.1.34 with $\varepsilon_0 = 1$ and $U = \widehat{U} \cap \widetilde{U}$. Choose $\gamma \in \Gamma$, such that

$$\varphi\big(\gamma(t)\big) \ \leqslant \ c + \varepsilon \qquad \forall\ t \in [0,1].$$

Let us set $\gamma_0 \stackrel{df}{=} h(1,\cdot) \circ \gamma$. Clearly $\gamma_0 \in \Gamma$ and we have

$$\gamma_0\big([0,1]\big) \ \subseteq \ h\big(1, \varphi^{c+\varepsilon}\big) \ \subseteq \ \varphi^{c-\varepsilon} \cup U \ \subseteq \ \varphi^{c-\varepsilon} \cup \widehat{U}.$$

But the sets $\varphi^{c-\varepsilon}$ and $\widehat{U}$ are disjoint. So we must have

$$\gamma_0\big([0,1]\big) \ \subseteq \ \varphi^{c-\varepsilon} \quad \text{or} \quad \gamma_0\big([0,1]\big) \ \subseteq \ U.$$

Since the first inclusion contradicts the definition of c, we must have

$$\gamma_0\big([0,1]\big) \ \subseteq \ U \ \subseteq \ \widetilde{U},$$

hence 0 and x_1 can be path connected in any neighbourhood $\widetilde{U}$ of K_c^φ. ∎

5.4 Multiple Critical Points

In this section we present some useful results on the existence of multiple critical points for a smooth functional. The most powerful results in this direction can be obtained for functionals which are invariant under a group of symmetries. However, first we will present multiplicity results in which no symmetry conditions are assumed. Instead we employ a splitting condition for the functionals, which is known as "local linking at 0."

Let X be a Banach space and X^* its topological dual.

DEFINITION 5.4.1 *Suppose that $X = Y \oplus V$. We say that $\varphi \in C^1(X)$ has a **local linking at** 0, if there exists $r > 0$, such that*

$$\varphi(y) \leqslant 0 \qquad \forall\, y \in Y,\ \|y\|_X \leqslant r$$

and

$$\varphi(v) \geqslant 0 \qquad \forall\, v \in V,\ \|v\|_X \leqslant r.$$

REMARK 5.4.2 It is clear from the above definition that

$$\langle \varphi'(0), h \rangle_X = 0 \qquad \forall\, h \in Y$$

and

$$\langle \varphi'(0), u \rangle_X = 0 \qquad \forall\, u \in V.$$

Hence

$$\langle \varphi'(0), x \rangle_X = 0 \qquad \forall\, x \in X,$$

i.e., $\varphi'(0) = 0$ in X^*. So 0 is a critical point of φ. The notion of local linking generalizes the notions of local minimum and of local maximum. If $X = H$ is a Hilbert space, $\varphi \in C^2(H)$, $\varphi(0) = 0$ and 0 is a nondegenerate critical point (i.e., the self-adjoint operator $\varphi''(0)$ is invertible), then clearly φ has a local linking at 0.

We start with a geometric lemma which complements Example 5.2.3**(c)**.

LEMMA 5.4.3
If $X = Y \oplus V$ with $\dim Y < +\infty$, $0 < \dim V \leqslant +\infty$, $e \in V$, $\|e\|_X = 1$,

$$C \stackrel{df}{=} \{x = \lambda e + y :\ y \in Y,\ \lambda \geqslant 0,\ \|x\|_X \leqslant 1\},$$

$r > 0$ and $\gamma\colon C \longrightarrow X$ is a continuous map, such that

$$\gamma(y) = y \qquad \forall\, y \in Y,\ \|y\|_X \leqslant 1 \tag{5.23}$$

and

$$\|\gamma(x)\|_X \geqslant r > 0 \qquad \forall\, x \in C,\ \|x\|_X = 1, \tag{5.24}$$

then for any $\varrho \in (0,r)$, there exists $\widehat{x} \in C$, such that

$$\mathrm{proj}_{_Y}\big(\gamma(\widehat{x})\big) \;=\; 0 \quad \textit{and} \quad \big\|\mathrm{proj}_{_Y}(\widehat{x})\big\|_{_X} \;=\; \varrho,$$

where $\mathrm{proj}_{_Y} \in \mathcal{L}(X)$ is the projection onto Y (it exists since by hypothesis Y is finite dimensional).

PROOF Let $Z = Y \oplus \mathbb{R}e$ and consider the map $\xi\colon C \longrightarrow Z$, defined by

$$\xi(x) \;\stackrel{df}{=}\; \mathrm{proj}_{_Y}\big(\gamma(x)\big) + \big\|(id_{_X} - \mathrm{proj}_{_Y})\gamma(x)\big\|_{_X} e.$$

To prove the lemma, it suffices to show that there exists $\widehat{x} \in C$, such that

$$\xi(\widehat{x}) \;=\; \varrho e.$$

From (5.23) and (5.24), we see that

$$\xi(x) \;\neq\; \varrho e \qquad \forall\, x \in \partial C.$$

So the Brouwer degree $D_B\big(\xi, \mathrm{int}\, C, \varrho e\big)$ is defined. Let

$$C_1 \;\stackrel{df}{=}\; \big\{(y,0) \in Z :\; y \in Y,\; \|y\|_{_X} \leqslant 1\big\}$$

and

$$C_2 \;\stackrel{df}{=}\; \big\{x \in C :\; \|x\|_{_X} = 1\big\}.$$

Then

$$\partial C \;=\; C_1 \cup C_2.$$

Recall that Brouwer's degree depends only on the boundary values of the function ξ. So we consider $\xi|_{_{\partial C}}$. From the definition of γ and ξ, we see that

$$\xi(x) \;=\; x \qquad \forall\, x \in C_1$$

and

$$\big\|\xi(x)\big\|_{_X} \;\geqslant\; r \;>\; 0 \qquad \forall\, x \in C_2.$$

Also let $\eta\colon \partial C \longrightarrow X$ be defined by

$$\eta(x) \;\stackrel{df}{=}\; \begin{cases} x & \text{if} \quad x \in C_1, \\ \dfrac{\xi(x)}{\|\xi(x)\|_{_X}} & \text{if} \quad x \in C_2. \end{cases}$$

Evidently $\eta \in C\big(\partial C; X\big)$. Also note that the functions ξ and η are homotopic in $Z \setminus \big\{\varrho e\big\}$ through the homotopy

$$h_1(t,x) \;\stackrel{df}{=}\; t\xi(x) + (1-t)\eta(x) \qquad \forall\, t \in [0,1].$$

Note that

$$\eta(C_2) \subseteq C_2 \quad \text{and} \quad \eta|_{\partial C_2} = id_{\partial C_2}.$$

Since C_2 is homeomorphic to a ball, there is a continuous deformation $h_2(t,x)$ connecting $\eta|_{C_2}$ and id_{C_2}, such that

$$h_2(t,x) = x \qquad \forall\, t \in [0,1] \quad \text{and} \quad x \in \partial C_2.$$

Therefore $\xi|_{\partial C}$ is homotopic to the identity in $Z \setminus \{\varrho e\}$ and so

$$D_B\big(\xi, \operatorname{int} C, \varrho e\big) = D_B\big(id_C, \operatorname{int} C, \varrho e\big) = 1.$$

Hence $\varrho e \in \xi(C)$ and we are done. □

REMARK 5.4.4 The above lemma implies that ∂C and $V \cap \partial B_\varrho(0)$ are linking in X (see Definition 5.2.1).

We will need the following lemma. As in Section 5.1, by V we denote a pseudogradient vector field associated to $\varphi \in C^1(X)$ on the set

$$\big\{x \in X : \ \varphi'(x) \neq 0\big\}$$

(see Definition 5.1.16 and Theorem 5.1.19).

LEMMA 5.4.5
If $\varphi \in C^1(X)$ satisfies the PS*-condition, $x_0 \in X$ is a unique global minimizer of φ (i.e., $\varphi(x_0) < \varphi(x)$ for all $x \neq x_0$), $y \neq x_0$ is such that*

$$\varphi'(y) \neq 0$$

and φ has no critical value in $\big(\varphi(x_0), \varphi(y)\big)$,
then the negative pseudogradient flow, defined by

$$\begin{cases} \dot{u}(t) = -\dfrac{V(u(t))}{\|V(u(t))\|_X^2}, \\ u(0) = y \end{cases} \tag{5.25}$$

exists for a maximal finite time $t \in \big[0, b(y)\big]$ and $u\big(b(y)\big) = x_0$.

PROOF By replacing φ by

$$\varphi_1(x) \overset{df}{=} \varphi(x + x_0) - \varphi(x_0),$$

if necessary, we may assume that

$$x_0 = 0 \quad \text{and} \quad \varphi(x_0) = 0.$$

From the definition of the pseudogradient vector field V (see Definition 5.1.16) and (5.25), we have

$$\begin{aligned} \frac{d}{dt}\varphi\big(u(t)\big) &= \big\langle \varphi'\big(u(t)\big), \dot{u}(t)\big\rangle_X \\ &= \left\langle \varphi'\big(u(t)\big), -\frac{V(u(t))}{\|V(u(t))\|_X^2}\right\rangle_X \\ &\leqslant -\frac{\|\varphi'(u(t))\|_{X^*}^2}{\|V(u(t))\|_X^2} < -\frac{1}{4}. \end{aligned} \tag{5.26}$$

Then from Theorem 5.1.21 and (5.26), we have that the flow $u(t)$ exists on a maximal open interval $(0,b)$ and by integrating (5.26) and since $\varphi(x_0)=0$ is a strict global minimum of φ, we see that $b \leqslant 4\varphi(y)$. Moreover, we have

$$0 < \varphi\big(u(t)\big) < \varphi(y) \qquad \forall\, t \in (0,b).$$

We claim that

$$u(t) \longrightarrow 0 \quad \text{as } t \to b^-.$$

First assume that φ' is bounded away from zero along the flow u.

So there exists $\delta > 0$, such that

$$\big\|\varphi'\big(u(t)\big)\big\|_{X^*} \geqslant \delta \qquad \forall\, t \in (0,b).$$

Then

$$\delta \leqslant \big\|\varphi'\big(u(t)\big)\big\|_{X^*} \leqslant \big\|V\big(u(t)\big)\big\|_X$$

and so

$$\int_0^b \big\|\dot{u}(t)\big\|_X\, dt \leqslant \frac{1}{\delta^2}\int_0^b \big\|V\big(u(t)\big)\big\|_X\, dt < +\infty,$$

a contradiction to Theorem 5.1.21.

So φ' cannot be bounded away from zero along the flow u.

Hence we can find $t_n \longrightarrow b^-$, such that

$$\big\|\varphi'\big(u(t_n)\big)\big\|_{X^*} \longrightarrow 0.$$

Since the sequence $\big\{\varphi\big(u(t_n)\big)\big\}_{n\geqslant 1}$ is bounded and φ satisfies the PS-condition, a subsequence of $\big\{u(t_n)\big\}_{n\geqslant 1}$ converges to a critical point of φ. Since by hypothesis φ has no critical value in $\big(0,\varphi(y)\big]$, this critical point is 0. Therefore

$$\lim_{t\to b^-} \varphi\big(u(t)\big) = \varphi(x_0) = \varphi(0) = 0$$

and from the Ekeland variational principle (see Theorem 4.6.1), we conclude that

$$u(t) \longrightarrow x_0 = 0 \quad \text{as } t \to b^-.$$

□

Now we are ready for the first multiplicity result.

THEOREM 5.4.6
If $X = Y \oplus V$ and $\dim Y < +\infty$, $\varphi \in C^1(X)$, $\varphi(0) = 0$, φ is bounded from below,

$$\inf_{x \in X} \varphi(x) \ < \ 0,$$

φ satisfies the PS*-condition and it has a local linking at* 0,
then φ has at least two nonzero critical points.

PROOF From Theorem 5.2.10, we know that there exists $x_0 \in X$, such that

$$\varphi(x_0) \ = \ \inf_{x \in X} \varphi(x) \ < \ 0 \ = \ \varphi(0).$$

So $x_0 \neq 0$ and it is a critical point of φ. Also because of the local linking condition, we know that 0 is another critical point of φ. Suppose that 0 and x_0 are the only critical points of φ.

Case 1. $0 < \dim Y$, $0 < \dim V$.

Without any loss of generality, we may assume that $r = 1 < \|x_0\|_X$. Then for every $y \in Y$, with $\|y\|_X = 1$, we have

$$\varphi'(y) \ \neq \ 0$$

and so by Lemma 5.4.5, the flow u of (5.23) exists on a maximal open interval $\big(0, b(y)\big)$ with $b(y) \leqslant -4\varphi(x_0)$.

Also we claim that there exists $\delta > 0$, such that

$$\overset{\circ}{\varphi}^{\varphi(x_0)+\delta} \ \subseteq \ B_{\frac{\|x_0\|_X}{2}}(x_0). \tag{5.27}$$

Indeed, if no such $\delta > 0$ exists, we can find a minimizing sequence $\{y_n\}_{n \geqslant 1}$ for φ, such that

$$\|y_n - x_0\|_X \ \geqslant \ \frac{\|x_0\|_X}{2} \qquad \forall \ n \geqslant 1.$$

By virtue of Corollary 4.6.8, we can find another minimizing sequence $\{x_n\}_{n \geqslant 1}$ of φ, such that

$$\varphi'(x_n) \ \longrightarrow \ 0 \quad \text{and} \quad \frac{\|x_0\|_X}{4} \ \leqslant \ \|x_n - x_0\|_X \qquad \forall \ n \geqslant 1.$$

Since φ satisfies the PS-condition, we may assume that

$$x_n \ \longrightarrow \ \widehat{x} \quad \text{in } X.$$

Then $\widehat{x}$ is a critical point of φ different from 0 and x_0, since

$$\varphi(\widehat{x}) \;=\; \inf_{x\in X}\varphi(x) < 0 = \varphi(0) \quad \text{and} \quad \frac{\|x_0\|_X}{4} \;\leqslant\; \|\widehat{x}-x_0\|_X\,,$$

a contradiction to our hypothesis in the beginning of the proof.

Then by choosing $\delta > 0$ in (5.27) small enough, we can find a unique $t = t(y) < b(y)$, such that

$$\varphi\big(x\big(t(y)\big)\big) \;=\; \varphi(x_0)+\delta.$$

The uniqueness property (which is a consequence of (5.24)), implies that the map $y \longmapsto t(y)$ is continuous.

Let $e \in V$ be such that $\|e\|_X = 1$. Consider the set

$$C \;\stackrel{df}{=}\; \big\{x = \lambda e + y :\; y\in Y,\; \lambda \geqslant 0,\; \|x\|_X \leqslant 1\big\}.$$

We define $\gamma_0 \in C\big(\partial C; X\big)$ in the following way:

$$\begin{aligned}\gamma_0(y) &\;\stackrel{df}{=}\; y \qquad \forall\; y\in Y,\; \|y\|_X \leqslant 1,\\ \gamma_0(e) &\;\stackrel{df}{=}\; x_0.\end{aligned}$$

Also, any $x \in \partial C$, $x \neq e$ with $\|x\|_X = 1$ has a unique representation

$$x \;=\; \lambda e + \mu y,$$

with $\lambda \in [0,1]$, $y \in Y$, $\|y\|_X = 1$ and $\mu \in (0,1]$ (i.e., the coefficients λ, μ and the vector y are unique). Then we define

$$\gamma_0(\lambda e+\mu y) \;\stackrel{df}{=}\; u\big(2st(y)\big) \qquad \forall\; s\in\left[0,\frac{1}{2}\right]$$

(recall that u is the pseudogradient flow; see (5.25)). Hence

$$\gamma_0\left(\frac{1}{2}e+\mu y\right) \;=\; u\big(t(y)\big).$$

Recall that

$$\varphi\big(x\big(t(y)\big)\big) \;=\; \varphi(x_0)+\delta.$$

So from (5.27), it follows that

$$\left\|\frac{1}{2}e+\mu y-x_0\right\|_X \;\leqslant\; \frac{1}{2}\,\|x_0\|_X\,.$$

Also define

$$\gamma_0(\lambda e+\mu y) \;\stackrel{df}{=}\; (2\lambda-1)x_0+(2-2\lambda)u\big(t(y)\big) \qquad \forall\;\lambda\in\left[\frac{1}{2},1\right].$$

As λ moves from $\frac{1}{2}$ to 1, $\gamma_0(\lambda e + \mu y)$ traverses the line segment from $u\big(t(y)\big)$ to x_0 and so we have

$$\big\|\gamma_0(\lambda e + \mu y) - x_0\big\|_X \;\leqslant\; \frac{1}{2}\,\|x_0\|_X \qquad \forall\ \lambda \in \left[\frac{1}{2}, 1\right].$$

Clearly γ_0 is continuous and

$$\varphi|_{\gamma_0(\partial C)} \;\leqslant\; 0.$$

Moreover, for some $r \leqslant 1$, we have

$$\big\|\gamma_0(x)\big\|_X \;\geqslant\; r \;>\; 0 \qquad \forall\ \|x\|_X = 1.$$

Let us fix $\varrho \in (0, r)$. Then by Lemma 5.4.3, the sets ∂C and $V \cap \partial B_\varrho(0)$ are linking in X, i.e., for any continuous extension γ of γ_0 to all of C, we have

$$\gamma(C) \cap \big(V \cap \partial B_\varrho(0)\big) \;\neq\; \emptyset.$$

Let

$$\Gamma \;\stackrel{df}{=}\; \big\{\gamma \in C(C; X) :\ \gamma|_{\partial C} = \gamma_0\big\}$$

and

$$c \;\stackrel{df}{=}\; \inf_{\gamma\in\Gamma} \max_{x\in C} \varphi\big(\gamma(x)\big).$$

From Theorem 5.2.4, we know that c is a critical value of φ. If $c > 0$, then we have a second nonzero critical point of φ (since for the other nonzero critical point x_0, we have $\varphi(x_0) < 0$). If $c = 0$, then from Theorem 5.2.4, we know that

$$K_c^\varphi \cap \big(V \cap \partial B_\varrho(0)\big) \;\neq\; \emptyset$$

and so again we have a second nonzero critical point of φ (distinct from x_0).

Case 2. $\dim Y = 0$.

Let $x_0 \neq 0$ be a minimizer of φ and suppose that it is the only nonzero critical point of φ. Then we can find $r > 0$ small enough, such that

$$\varphi(x) \;\geqslant\; \beta_0 \;>\; 0 \qquad \forall\ \|x\|_X = r.$$

Since $\varphi(x_0) < \varphi(0) = 0$ and φ satisfies the PS-condition, we can apply the mountain pass theorem (see Corollary 5.2.7) and obtain a second nonzero critical point of φ.

Case 3. $\dim V = 0$. (In this case we may allow $\dim Y = +\infty$.)

In this case we have

$$\varphi(x) \;\leqslant\; -\beta_0 \;<\; 0 \qquad \forall\ \|x\|_X = r,$$

with $r > 0$ small enough. Applying the mountain pass theorem (see Corollary 5.2.7) on $-\varphi$, we obtain a second nonzero critical point of φ. □

From the proof of Theorem 5.4.6, we deduce a somewhat stronger result.

THEOREM 5.4.7
If $X = Y \oplus V$ *with* $\dim Y < +\infty$, $\dim V > 0$, $\varphi \in C^1(X)$, $\varphi(0) = 0$, $\inf_{x \in X} \varphi(x) < 0$, φ *is bounded from below and satisfies the* PS*-condition, has a local linking at* 0 *and has only finite number of critical points, each a local minimizer of* φ *and on them we have* $\varphi < 0$,
then φ *has another nonzero critical point.*

PROOF We argue indirectly. Suppose that φ has no other nonzero critical point. Let $\{x_k\}_{k=1}^N$ be the critical points of φ, where $\varphi < 0$. Choose $\varrho > 0$ small enough so that the open balls $B_\varrho(x_k)$, $k \in \{1, \dots, N\}$ are mutually disjoint and they do not contain the origin. In the local linking condition, we can take $r > 0$ small enough so that $\overline{B}_r(0)$ is disjoint from all the open balls $B_\varrho(x_k)$. As in the proof of Theorem 5.4.6, we can find $\delta > 0$, small enough, so that the component D_k containing x_k of $\overset{\circ}{\varphi}^{\varphi(x_k)+\delta}$ is contained in $B_\varrho(x_k)$ for $k \in \{1, \dots, N\}$.

Let $y \in Y$ with $\|y\|_X = r$. Then $\varphi'(y) \neq 0$ and considering the negative pseudogradient flow starting at y (see (5.25)), we know that it exists for a finite time

$$b(y) \leqslant -4 \min \varphi.$$

There is a unique time $t = t(y)$, such that

$$x\big(t(y)\big) \text{ first touches } \bigcup_{k=1}^N \partial D_k.$$

By renumbering things if necessary, we may assume that

$$x\big(t(y)\big) \in \partial D_0.$$

Then

$$x(t) \in D_0 \qquad \forall\, t \in \big(t(y), b(y)\big).$$

So the flow which starts at y ends up in D_0. From the continuous dependence of the flow on the initial condition, it follows that for $v \in V$ close to y on $\partial B_r(0)$, the negative pseudogradient flow starting at v will end up in D_0. Since the set $Y \cap \partial B_r(0)$ is connected, starting from any point in that set, the negative pseudogradient flow ends up in D_0. Then we can continue as in the proof of Theorem 5.4.6 (follow the argument after (5.27)) and produce a nonzero critical point of φ where $\varphi \geqslant 0$. □

COROLLARY 5.4.8
If $X = Y \oplus V$, $\dim Y < +\infty$, $\dim V > 0$ *and* $\varphi \in C^1(X)$ *satisfies all the conditions of Theorem 5.4.6 and in addition* φ *is even,*
then φ *has at least two pairs* $\big(\{\pm x\}, \{\pm w\}\big)$ *of nonzero critical points.*

To state another consequence of Theorem 5.4.6, we need to introduce some notions and state a result, which is the starting point of the so-called "Morse Theory."

DEFINITION 5.4.9 *Let H be a Hilbert space, $\varphi \in C^2(H)$ and x is a critical point of φ.*

(a) *We say that x is **nondegenerate**, if $\varphi''(x)$ is invertible.*

(b) *The **Morse index** of x is defined as the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite.*

THEOREM 5.4.10 (Morse Lemma)
If H is a Hilbert space, $\varphi \in C^2(H)$ and 0 is a nondegenerate critical point of φ,
then there exists a Lipschitz continuous homeomorphism h of a neighbourhood W of 0 onto a neighbourhood U of 0, such that $h(0) = 0$ and

$$\varphi\big(h(x)\big) \;=\; \varphi(0) + \frac{1}{2}\big(\varphi''(0)x, x\big)_{H}.$$

REMARK 5.4.11 Let H_+ be the subspace of H where $\varphi''(0)$ is positive definite and H_- the subspace of H where $\varphi''(0)$ is negative definite. Since 0 is a nondegenerate critical point, we have that

$$H \;=\; H_+ \oplus H_-.$$

Let P_{H_+} be the orthogonal projection on H_+. If on H we use the equivalent norm

$$|x|_{H} \;=\; \big|\big\langle \varphi''(0)x, x\big\rangle_{H}\big| \qquad \forall\, x \in H,$$

then we can equivalently rewrite the conclusion of Morse lemma as

$$\varphi\big(h(x)\big) \;=\; \varphi(0) + \frac{1}{2}\big|\mathrm{P}_{H}(x)\big|_{H}^{2} + \frac{1}{2}\big|\big(id_{H} - \mathrm{P}_{H}\big)(x)\big|_{H}^{2}. \tag{5.28}$$

From (5.28), it is clear that if 0 is a nondegenerate critical point of $\varphi \in C^2(H)$ with a finite Morse index, then the splitting in the local linking condition is assured. So we can state the following theorem.

THEOREM 5.4.12
If H is a Hilbert space, $\varphi \in C^2(H)$ is bounded from below, 0 is a nondegenerate critical point of φ with finite Morse index and

$$\inf_{x \in H} \varphi(x) \;<\; \varphi(0) \;=\; 0,$$

then φ has at least two nonzero critical points.

Thus far we have assumed that in the direct sum decomposition

$$X \;=\; Y \oplus V, \quad \text{with } \dim Y < +\infty.$$

Now we drop this requirement. We start with a quantitative deformation lemma, which is an immediate consequence of Corollary 5.1.28.

LEMMA 5.4.13
If $\varphi \in C^1(X)$, $C \subseteq X$, $\varepsilon, \delta > 0$, $c \in \mathbb{R}$ *and*

$$\big\|\varphi'(x)\big\|_{X^*} \;\geqslant\; \frac{4\varepsilon}{\delta} \qquad \forall\, x \in \varphi^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big) \cap C_{2\delta},$$

then there exists a φ*-decreasing and locally Lipschitz homotopy of homeomorphisms* h_t*, such that*

(a) $\varphi\big(h(t,x)\big) < c$ *for all* $(t,x) \in [0,1] \times \big(\varphi^c \cap C\big)$;

(b) $h\big(1, \varphi^{c+\varepsilon} \cap C\big) \subseteq \varphi^{c-\varepsilon}$;

(c) $\big\|h(t,x) - x\big\|_X \leqslant \delta$ *for all* $(t,x) \in [0,1] \times X$.

Now suppose that

$$X \;=\; Y \oplus V.$$

Consider two sequences $\{Y_n\}_{n\geqslant 1}$ and $\{V_n\}_{n\geqslant 1}$ of subsets of Y and V respectively, such that

$$Y_1 \;\subseteq\; Y_2 \;\subseteq\; \ldots \;\subseteq\; Y_n \;\subseteq\; \ldots \;\subseteq\; Y,$$

$$V_1 \;\subseteq\; V_2 \;\subseteq\; \ldots \;\subseteq\; V_n \;\subseteq\; \ldots \;\subseteq\; V$$

and

$$\overline{\bigcup_{n=1}^{\infty} Y_n} \;=\; Y \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} V_n} \;=\; V.$$

For a given multiindex $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we set

$$X_\alpha \;=\; Y_{\alpha_1} \oplus V_{\alpha_2}.$$

Recall that on the space of multiindices $\mathbb{N}^2$ we use the coordinatewise ordering, defined by

$$\alpha \;\leqslant\; \beta \iff \Big[\alpha_1 \;\leqslant\; \beta_1,\; \alpha_2 \;\leqslant\; \beta_2\Big].$$

DEFINITION 5.4.14 *A sequence of multiindices* $\{\alpha_n\}_{n\geqslant 1} \subseteq \mathbb{N}^2$ *is* ***admissible****, if for every* $\alpha \in \mathbb{N}^2$ *there is an integer* $n_0 \geqslant 1$ *such that* $\alpha_n \geqslant \alpha$*, for every* $n \geqslant n_0$*. Also for a given function* $\psi\colon X \longrightarrow \mathbb{R}$ *and multiindex* $\alpha \in \mathbb{N}^2$*, by* ψ_α *we denote the restriction of* ψ *on* X_α.

We will use the following generalization of the Palais-Smale compactness condition.

DEFINITION 5.4.15 *Let $\varphi \in C^1(X)$. We say that φ satisfies the* $\mathrm{PS}^*_{\mathrm{c}}$-***condition**, if for every sequence $\{x_{\alpha_n}\}_{n\geqslant 1} \subseteq X$, such that $\{\alpha_n\}_{n\geqslant 1} \subseteq \mathbb{N}^2$ is admissible and $x_{\alpha_n} \in X_{\alpha_n}$ for $n \geqslant 1$,*

$$\varphi\big(x_{\alpha_n}\big) \longrightarrow c \quad \text{and} \quad \varphi'_{\alpha_n}(x_{\alpha_n}) \longrightarrow 0,$$

we can extract a strongly convergent subsequence. We say that φ satisfies the PS^**-condition if it satisfies* $\mathrm{PS}^*_{\mathrm{c}}$*-condition at every level $c \in \mathbb{R}$.*

REMARK 5.4.16 If

$$Y_n \;=\; X \quad \text{and} \quad V_n \;=\; \{0\} \qquad \forall\, n \geqslant 1,$$

then the above definition reduces to the usual Palais-Smale condition (local and global). In the previous definitions and in what follows we can replace $\mathbb{N}^2$ by any directed set.

DEFINITION 5.4.17 *Let C and D be two closed subsets in X. Then $C \prec_\infty D$, if there exists $\beta \in \mathbb{N}^2$, such that for every $\alpha \geqslant \beta$, we can find $h_\alpha \in C\big([0,1] \times X_\alpha; X_\alpha\big)$, such that*

(a) *$h_\alpha(0,x) = x$ for all $x \in X_\alpha$;*

(b) *$h_\alpha(1,x) \in D$ for all $x \in X_\alpha \cap C$.*

This definition with the help of the corresponding Palais-Smale type condition in Definition 5.4.15 can extend the deformation lemma to the present Galerkin-type setting.

LEMMA 5.4.18
If $\varphi \in C^1(X)$, $c \in \mathbb{R}$, $r > 0$, U is a neighbourhood of K_c^φ and φ satisfies the $\mathrm{PS}^*_{\mathrm{c}}$*-condition,*
then for all $\varepsilon > 0$ small enough, we have

$$\varphi^{c+\varepsilon} \setminus U \;\prec_\infty\; \varphi^{c-\varepsilon}$$

and the corresponding homotopies $h_\alpha \in C\big([0,1] \times X_\alpha; X_\alpha\big)$ satisfy

$$\big\|h_\alpha(t,x) - x\big\|_X \;\leqslant\; r \qquad \forall\, (t,x) \in [0,1] \times X_\alpha$$

and

$$\varphi\big(h_\alpha(t,x)\big) \;<\; c \qquad \forall\, (t,x) \in [0,1] \times \big(\varphi_\alpha^c \setminus U\big).$$

PROOF Since φ satisfies the PS_c^*-condition, we can find $\varrho > 0$ and $\beta \in \mathbb{N}^2$, such that for every $\alpha \geqslant \beta$, we have

$$\big\|\varphi'_\alpha(x)\big\|_{X^*} \;\geqslant\; \varrho \qquad \forall\, x \in \varphi_\alpha^{-1}\big([c-2\varrho, c+2\varrho]\big) \cap \overline{\big(X_\alpha \setminus U\big)}_{2\varrho},$$

where

$$\overline{A}_\delta \;\stackrel{df}{=}\; \big\{x \in X :\; d_{_X}(x, A) \leqslant \delta\big\}.$$

Choose

$$\delta \;\stackrel{df}{=}\; \min\left\{\frac{\varrho}{2}, 4, r\right\} \quad \text{and} \quad \varepsilon \in \left(0, \frac{\delta\varrho}{4}\right]$$

and apply Lemma 5.4.13 with $C = X_\alpha \setminus U$. □

LEMMA 5.4.19
If $\varphi \in C^1(X)$ is bounded below,

$$c \;=\; \inf_{x \in X} \varphi(x)$$

and φ satisfies the PS_c^*-condition,
then c is a critical value of φ.

PROOF Suppose that c is not a critical value of φ. Then by virtue of Lemma 5.4.18 (with $U = \emptyset$), we can find $\varepsilon > 0$, such that

$$\varphi^{c+\varepsilon} \;\prec_\infty\; \varphi^{c-\varepsilon}.$$

From the definition of c, we see that for all $\alpha \in \mathbb{N}^2$, $\varphi_\alpha^{c-\varepsilon}$ is empty, while for all $\alpha \in \mathbb{N}^2$ large enough, $\varphi_\alpha^{c+\varepsilon}$ is nonempty. This contradicts Definition 5.4.17. □

LEMMA 5.4.20
If $\varphi \in C^1(X)$ is bounded from below and satisfies the PS^*-condition,
then φ is weakly coercive.

PROOF We proceed by contradiction. Suppose that φ is not weakly coercive. Then

$$\xi \;\stackrel{df}{=}\; \sup\big\{\lambda \in \mathbb{R} :\; \varphi^\lambda \text{ is bounded}\big\} \;<\; +\infty.$$

Because of the PS^*-condition, K_ξ^φ is bounded. Let U be a bounded neighbourhood of K_ξ^φ. By Lemma 5.4.18 with $r = 1$, we can find $\varepsilon > 0$, such that

$$\varphi^{\xi+\varepsilon} \setminus U \;\prec_\infty\; \varphi^{\xi-\varepsilon}.$$

Note that the set $\varphi^{\xi+\varepsilon} \setminus U$ is unbounded, while

$$\varphi^{\xi-\varepsilon} \;\subseteq\; \overline{B}_R(0),$$

for some $R > 0$. Since

$$\left\|h_\alpha(t,x) - x\right\|_X \ \leqslant\ 1 \qquad \forall\ (t,x) \in [0,1] \times X_\alpha,$$

for all $\alpha \in \mathbb{N}^2$ large enough, we have

$$\varphi_\alpha^{\xi+\varepsilon} \setminus U \ \subseteq\ \overline{B}_{R+1}(0),$$

hence

$$\varphi^{\xi+\varepsilon} \setminus U \ \subseteq\ \overline{B}_{R+1}(0),$$

a contradiction. □

Now we are ready for the version of Theorem 5.4.7, where we drop the condition that the component space Y is finite dimensional. Note that if φ has a local linking at 0 and satisfies the PS*-condition, then we assume that the same direct sum decomposition of X holds for the two properties and that

$$\dim Y_n \ <\ +\infty \quad \text{and} \quad \dim V_n \ <\ +\infty \qquad \forall\ n \geqslant 1.$$

THEOREM 5.4.21
If $\varphi \in C^1(X)$ and satisfies the following assumptions:

(i) φ has a local linking at 0;

(ii) φ satisfies the PS**-condition;*

(iii) φ is bounded (i.e., maps bounded sets into bounded sets);

(iv) φ is bounded from below and

$$c \ \stackrel{df}{=}\ \inf_{x \in X} \varphi(x) \ <\ 0,$$

then φ has at least two distinct nontrivial critical points.

PROOF In the decomposition $X = Y \oplus V$ we assume that $\dim Y > 0$ and $\dim V > 0$, since the analysis of the other cases is similar. From Lemma 5.4.19, we know that there exists $x_0 \in X$, such that

$$\varphi(x_0) \ =\ \inf_{x \in X} \varphi(x).$$

Evidently $x_0 \neq 0$. Suppose that $\{0, x_0\}$ are the only critical points of φ. Let $\varrho < \frac{\|x_0\|_X}{3}$ be such that

$$\overline{B}_\varrho(x_0) \ \subseteq\ \varphi^{\frac{c}{2}}. \tag{5.29}$$

Apply Lemma 5.4.18 to the functions φ and $\psi = -\varphi$. Then we can find $\varepsilon \in \left(0, -\frac{c}{2}\right)$, such that

$$\varphi^\varepsilon \setminus B_{\frac{\varrho}{3}}(0) \ \prec_\infty\ \varphi^{-\varepsilon}, \tag{5.30}$$

so

$$\psi^{\varepsilon} \setminus B_{\frac{\varrho}{3}}(0) \ \prec_{\infty} \ \psi^{-\varepsilon} \tag{5.31}$$

and

$$\varphi^{c+\varepsilon} \setminus B_{\frac{\varrho}{3}}(x_0) \ \prec_{\infty} \ \varphi^{c-\varepsilon} \ = \ \emptyset. \tag{5.32}$$

Also we can assume that the corresponding homotopies

$$h_{\alpha} \in C\big([0,1] \times X_{\alpha}; X_{\alpha}\big)$$

exist for $\alpha \geqslant (m_0, m_0)$ and satisfy

$$\big\|h_{\alpha}(t,x) - x\big\|_X \ \leqslant \ \frac{\varrho}{2} \qquad \forall \ (t,x) \in [0,1] \times X_{\alpha}. \tag{5.33}$$

Since φ satisfies the PS*-condition, we can find $m_1 \geqslant m_0$ and $\delta > 0$, such that

$$\big\|\varphi'_{\alpha}(x)\big\|_{X^*} \ \geqslant \ \delta \qquad \forall \ x \in \varphi_{\alpha}^{-1}\big([c+\varepsilon, -\varepsilon]\big). \tag{5.34}$$

Let us fix $n \geqslant m_1$ and let

$$\alpha \stackrel{df}{=} (n,n).$$

From (5.29) and (5.32), we have that

$$\varphi_{\alpha}^{c+\varepsilon} \ \subseteq \ X_{\alpha} \cap \overline{B}_{\varrho}(x_0) \ \subseteq \ \varphi_{\alpha}^{\frac{\varepsilon}{2}}. \tag{5.35}$$

Let

$$\begin{aligned} S_n^1 &\stackrel{df}{=} \big\{y \in Y_n : \ \|y\|_X = \varrho\big\}, \\ S_n^2 &\stackrel{df}{=} \big\{v \in V_n : \ \|v\|_X = \varrho\big\}. \end{aligned}$$

Using (5.30) and (5.34), we can have a homotopy

$$\xi \in C\big([0,1] \times S_n^2; X_{\alpha}\big),$$

such that

$$\varphi\big(\xi(t,x)\big) \ < \ 0 \qquad \forall \ (t,x) \in (0,1] \times S_n^2 \tag{5.36}$$

and

$$\varphi\big(\xi(1,x)\big) \ = \ c + \varepsilon \qquad \forall \ x \in S_n^2. \tag{5.37}$$

Also let

$$\begin{aligned} D_n^1 &\stackrel{df}{=} \big\{y \in Y_n : \ \|y\|_X \leqslant \varrho\big\}, \\ D_n^2 &\stackrel{df}{=} \big\{v \in V_n : \ \|v\|_X \leqslant \varrho\big\}. \end{aligned}$$

From (5.35), we infer that there exists $\eta \in C\big(D_n^2; X_{\alpha}\big)$, such that

$$\eta(x) \ = \ \xi(1,x) \qquad \forall \ x \in S_n^2 \tag{5.38}$$

and

$$\eta\big(D_n^2\big) \;\subseteq\; X_\alpha \cap \overline{B}_\varrho(x_0) \;\subseteq\; \varphi_\alpha^{\frac{c}{2}}. \tag{5.39}$$

Let

$$E \;\stackrel{df}{=}\; [0,1] \times D_n^2$$

and define

$$\sigma\colon \partial E \longrightarrow \varphi_\alpha^0$$

as follows

$$\sigma(t,x) \;\stackrel{df}{=}\; \begin{cases} x & \text{if} \quad (t,x) \in \{0\} \times D_n^2, \\ \xi(t,x) & \text{if} \quad (t,x) \in (0,1) \times S_n^2, \\ \eta(x) & \text{if} \quad (t,x) \in \{1\} \times D_n^2. \end{cases}$$

From Lemma 5.4.20, we know that we can find $R > 0$, such that

$$\varphi^0 \;\subseteq\; \overline{B}_R(0).$$

By Dugundji's extension theorem (see Theorem 3.1.11), we can find

$$\widehat{\sigma} \;\in\; C(E; X_\alpha),$$

such that

$$\widehat{\sigma}|_{\partial E} \;=\; \sigma \quad \text{and} \quad \sup_{x \in E} \big(\varphi \circ \widehat{\sigma}\big)(x) \;\leqslant\; c_0 \;=\; \sup_{x \in \overline{B}_R(0)} \varphi(x) \;<\; +\infty \tag{5.40}$$

(recall that φ maps bounded sets into bounded sets).

Let ϑ (depending on α) be the homotopy corresponding to (5.31). We claim that if $S = \vartheta(1, S_n^1)$, for any extension $\overline{\sigma} \in C(E; X_\alpha)$ of σ, we have

$$\overline{\sigma}(E) \cap S \;\neq\; \emptyset,$$

i.e., the sets ∂E and S are linking in X through E. Suppose that this is not the case. Then

$$\vartheta(1,y) \;\neq\; \overline{\sigma}(t,v) \qquad \forall\, t \in [0,1],\; y \in S_n^1,\; v \in S_n^2. \tag{5.41}$$

From (5.31), (5.33), (5.36) and (5.37), we have that (5.41) holds for all $t \in [0,1]$, $y \in D_n^1$, $v \in S_n^2$.

Also from (5.38) and (5.39), it follows that (5.41) holds for $t = 1$ and for all $y \in D_n^1$, $v \in D_n^2$. Then, if

$$U_0 \;\stackrel{df}{=}\; \overset{\circ}{D}{}_n^1 \times \overset{\circ}{D}{}_n^2$$

and

$$\gamma(t,x) \;\stackrel{df}{=}\; \vartheta(1,y) - \overline{\sigma}(t,v),$$

from the properties of Brouwer's degree, we have

$$D_B\big(\gamma(0,\cdot), U_0, 0\big) \;=\; D_B\big(\gamma(1,\cdot), U_0, 0\big) \;=\; 0. \tag{5.42}$$

From (5.33), we obtain that

$$\vartheta(t,y) \neq v \qquad \forall\, t \in [0,1],\ y \in D_n^1,\ v \in S_n^2. \tag{5.43}$$

From (5.31), we see that (5.43) holds for all $t \in [0,1]$, $y \in S_n^1$, $v \in D_n^2$. Let us define

$$g \in C\big([0,1] \times U_0; X_\alpha\big),$$

by

$$g(t,x) \stackrel{df}{=} \vartheta(t,y) - v,$$

with

$$x = y + v.$$

Using (5.42) and the homotopy invariance of Brouwer's degree, we have

$$\begin{aligned} 0 &= D_B\big(g(1,\cdot), U_0, 0\big) = D_B\big(g(0,\cdot), U_0, 0\big) \\ &= D_B\big(\mathrm{proj}_{Y_n} - \mathrm{proj}_{V_n}, U_0, 0\big) \neq 0, \end{aligned}$$

a contradiction (here proj_{Y_n} and proj_{V_n} are the projection operators on Y_n and V_n respectively). So the linking property holds.

Let

$$\Gamma \stackrel{df}{=} \big\{\overline{\sigma} \in C\big(E; X_\alpha\big) :\ \overline{\sigma}_{\partial E} = \sigma\big\}$$

and define

$$\widehat{c} \stackrel{df}{=} \inf_{\overline{\sigma} \in \Gamma} \sup_{w \in E} \varphi\big(\overline{\sigma}(w)\big).$$

From Theorem 5.2.4, we know that $\widehat{c}$ is a critical value of φ. Also from (5.40) and the previous argument, we have that

$$\varepsilon \leqslant \widehat{c} \leqslant c_0.$$

Since φ satisfies the PS*-condition, we can find $m_2 \geqslant m_1$ and $\beta > 0$, such that for $\alpha \geqslant (m_2, m_2)$, we have

$$\big\|\varphi'_\alpha(x)\big\|_X \geqslant \beta > 0 \qquad \forall\, x \in \varphi_\alpha^{-1}\big([\varepsilon, c_0]\big). \tag{5.44}$$

Then (5.44) contradicts the fact that $\widehat{c} \in [\varepsilon, c_0]$ is a critical value of φ. Therefore we conclude that φ has at least one more (nontrivial) critical point. □

COROLLARY 5.4.22

If $\varphi \in C^1(X)$ satisfies the PS**-condition, maps bounded sets into bounded sets and has a global minimizer and a local maximizer,*
then φ has a third critical point.

If φ is not bounded from below but has a local linking at 0, we can still find a nontrivial critical point.

THEOREM 5.4.23
If $\varphi \in C^1(X)$ satisfies the following assumptions:

(i) φ has a local linking at 0 and $Y \neq \{0\}$;

(ii) φ satisfies the PS^*-condition;

(iii) φ maps bounded sets into bounded sets;

(iv) for every $m \geqslant 1$, if $x \in Y_m \oplus V$ and $\|x\|_X \longrightarrow +\infty$, we have $\varphi(x) \longrightarrow -\infty$ (i.e., $-\varphi|_{Y_m \oplus V}$ is weakly coercive),

then φ has at least one nontrivial critical point.

PROOF We suppose that $\dim Y = +\infty$ and $\dim V > 0$. The other cases are similar and simpler.

Suppose that 0 is the only critical point of φ (recall that due to the local linking condition the origin is a critical point of φ). As before using Lemma 5.4.18 on the functions φ and $\psi = -\varphi$, we obtain $\varepsilon > 0$, such that for $\varrho > 0$, we have

$$\varphi^{\varepsilon} \setminus B_{\frac{\varrho}{3}}(0) \prec_{\infty} \varphi^{-\varepsilon} \tag{5.45}$$

and

$$\psi^{\varepsilon} \setminus B_{\frac{\varrho}{3}}(0) \prec_{\infty} \psi^{-\varepsilon}. \tag{5.46}$$

We can assume that the corresponding homotopies exist for $\alpha \geqslant (m_0, m_0)$ and satisfy Lemma 5.4.18 with $r = \frac{\varrho}{2}$ (see (5.33)). Because φ satisfies the PS^*-condition, we can find $m_1 \geqslant m_0$ and $\delta > 0$, such that for $\alpha \geqslant (m_1, m_1)$, we have

$$\big\|\varphi'_{\alpha}(x)\big\|_{X^*} \geqslant \delta \qquad \forall\, x \in \varphi_{\alpha}^{-1}\big((-\infty, -\varepsilon]\big). \tag{5.47}$$

Due to the weak coercivity of $-\varphi|_{Y_{m_1+1} \oplus V}$ (see hypothesis (iv)), we can find $R > 0$, such that

$$\varphi(x) \leqslant -\varepsilon \qquad \forall\, x \in Y_{m_1+1} \oplus V,\ \|x\|_X \geqslant \frac{R}{\sqrt{2}}. \tag{5.48}$$

Let

$$\mu \stackrel{df}{=} \inf_{\|x\|_X \leqslant R} \varphi(x). \tag{5.49}$$

Since φ is bounded, we see that μ is finite. Without any loss of generality, we may assume that

$$Y_{m_1} \neq Y_{m_1+1}$$

and the norm on Y_{m_1+1} is Euclidean. Let us set $\alpha = (m_1, n)$ with $n \geqslant m_1 + 1$ fixed. Using (5.45) and (5.47), we obtain a homotopy

$$\xi \in C\big([0,1] \times S_n^2; X_{\alpha}\big),$$

such that

$$\varphi\big(\xi(t,x)\big) \;<\; 0 \qquad \forall\ (t,x) \in (0,1] \times S_n^2 \tag{5.50}$$

and

$$\varphi\big(\xi(1,x)\big) \;=\; \mu - 1 \qquad \forall\ x \in S_n^2. \tag{5.51}$$

The sets S_n^1 and S_n^2 are as in the proof of Theorem 5.4.21.

Also let D_n^1, D_n^2 and E be as in the proof of Theorem 5.4.21 and define $\sigma \in C\big(\partial E; X_\alpha\big)$, by

$$\sigma(t,x) \;\stackrel{df}{=}\; \begin{cases} x & \text{if} \quad (t,x) \in \{0\} \times D_n^2, \\ \xi(t,x) & \text{if} \quad (t,x) \in \big(0, \frac{1}{2}\big] \times S_n^2, \\ 2(1-t)\xi(1,x) + (2t-1)u_0 & \text{if} \quad (t,x) \in \big(\frac{1}{2}, 1\big) \times S_n^2, \\ u_0 & \text{if} \quad (t,x) \in \{1\} \times D_n^2, \end{cases}$$

where $u_0 \in Y_{m_1+1} \setminus Y_{m_1}$ and $\|u_0\|_X \geqslant R$. From (5.48), (5.49), (5.50) and (5.51), we have that $\sigma(\partial E) \subseteq \varphi^0$. Moreover, $\sigma(\partial E) \subseteq Y_{m_1+1} \oplus V_n$. So there exists a continuous extension $\widehat{\sigma}\colon E \longrightarrow Y_{m_1+1} \oplus V_n$ of σ, such that

$$\sup_{x \in E} \widehat{\sigma}(x) \;\leqslant\; c_0 \;=\; \inf_{x \in Y_{m_1+1} \oplus V} \varphi(x). \tag{5.52}$$

From hypotheses (iii) and (iv), we see that c_0 is finite. As before, we can verify that ∂E and $S = \vartheta(1, S_n^1)$ are linking in X via E. Here ϑ (depending on $\beta = (n,n)$) is the homotopy determined from (5.46).

Finally let

$$\Gamma \;\stackrel{df}{=}\; \big\{\overline{\sigma} \in C\big(E; X_\beta\big) :\ \overline{\sigma}|_{\partial E} = \sigma\big\}$$

and define

$$\widehat{c} \;\stackrel{df}{=}\; \inf_{\overline{\sigma} \in \Gamma} \sup_{w \in E} \varphi\big(\overline{\sigma}(w)\big).$$

Again $\varepsilon \leqslant \widehat{c} \leqslant c_0$ and arguing as in the last part of the proof of Theorem 5.4.21, we reach a contradiction.

Therefore φ has at least one nontrivial critical point. □

Next we present a generalization of the saddle point theorem (see Corollary 5.2.8).

THEOREM 5.4.24

If $X = Y \oplus V$ with $\dim Y < +\infty$, $Y = W + \mathbb{R}e$ with $e \neq 0$, $\varphi \in C^1(X)$, there exists $r > 0$, such that

$$-\infty \;\leqslant\; a \;\leqslant\; \inf_{x \in \mathbb{R}_+ e \oplus V} \varphi(x) \;\leqslant\; \sup_{x \in \partial B(0) \cap Y} \varphi(x) \;\leqslant\; \inf_{x \in V} \varphi(x)$$

and φ satisfies the PS*-condition,*
then φ admits at least two distinct critical points x_0 and x_1, such that

$$-\infty \;<\; a \;\leqslant\; \varphi(x_0) \;\leqslant\; \sup_{x \in \partial B(0) \cap Y} \varphi(x) \;\leqslant\; \inf_{x \in V} \varphi(x) \;\leqslant\; \varphi(x_1). \tag{5.53}$$

PROOF Evidently the critical point x_1 is given by the saddle point theorem (see Corollary 5.2.8).

Next let $C_0 = \emptyset$, $C = \partial B_r(0) \cap Y$, $E = \mathbb{R}_+ e \oplus V$, $D = V$. We check that C_0 and E are linking in $X \setminus D$ and so Theorem 5.2.4 provides a second critical point x_0, such that (5.53) hold. If $\varphi(x_0) = \varphi(x_1)$, then

$$\varphi(x_0) \;=\; \sup_{x \in \partial B(0) \cap Y} \varphi(x) \;=\; \inf_{x \in V} \varphi(x) \;=\; \varphi(x_1)$$

and so $x_0 \in \partial B_r(0) \cap Y$, $x_1 \in V$. Hence $x_0 \neq x_1$. □

In the last part of this section, we prove some multiplicity results concerning the critical points of functionals possessing a $\mathbb{Z}_2$-symmetry. To treat such functionals we use the so-called Krasnoselskii's genus, which is a particular case of a topological index. A topological index associated with a compact group G acting on a manifold M is a mapping from the closed G-invariant subsets of M into $\mathbb{N} \cup \{+\infty\}$, which satisfies certain properties (see Proposition 5.4.29 below). The index helps to distinguish different critical points and so produce multiplicity results. Here we focus on the particular index known as "Krasnoselskii's index," which is suitable in treating functionals which are even (so in this case the compact group is $G = \{id_{_M}, -id_{_M}\} \equiv \mathbb{Z}_2$).

DEFINITION 5.4.25 *Let*

$$\mathcal{T} \stackrel{df}{=} \{A \subseteq X : \; 0 \notin A, \; A \textit{ is closed and symmetric}\}.$$

The ***Krasnoselskii genus*** *is the map* $\gamma \colon \mathcal{T} \longrightarrow \mathbb{N} \cup \{+\infty\}$, *defined by*

$$\gamma(A) \stackrel{df}{=} \begin{cases} 0 & \text{if} \quad A = \emptyset, \\ \inf\{k \geqslant 1 : \; \exists h \in C(A; \mathbb{R}^k) \text{ odd}, \\ \qquad h(x) \neq 0 \text{ for all } x \in A\} & \text{if} \quad A \neq \emptyset. \end{cases}$$

If for $A \neq \emptyset$ *no such integer* $k \geqslant 1$ *exists, we set* $\gamma(A) = +\infty$.

REMARK 5.4.26 Note that symmetry with respect to the origin of the set $A \in \mathcal{T}$ and oddness of the map h are nothing but invariance properties with respect to a representation of the group $\mathbb{Z}_2$.

The notion of genus generalizes the notion of dimension of a linear space. To see this we need to recall a result from degree theory, known as the Borsuk-Ulam theorem (see, e.g., Denkowski, Migórski & Papageorgiou (2003*b*, p. 198)).

THEOREM 5.4.27 (Borsuk-Ulam Theorem)
If $U \subseteq \mathbb{R}^N$ *is an open, bounded, symmetric neighbourhood of the origin,* $M < N$ *and* $h \colon \partial U \longrightarrow \mathbb{R}^M$ *is continuous and odd,*
then $0 \in \varphi(\partial U)$.

Using this theorem, we can prove the following result.

PROPOSITION 5.4.28
If U is any bounded, symmetric neighbourhood of the origin in X, then $\gamma(\partial U) = \dim X$.

PROOF If $0 < \dim X < +\infty$, then by taking $h = id_X$ in Definition 5.4.25, we see that $\gamma(\partial U) \leqslant \dim X$. Then by virtue of Theorem 5.4.27, we conclude that $\gamma(\partial U) = \dim X$.

If $\dim X = +\infty$, let X_n be an n-dimensional subspace of X. From the definition of γ, we see that

$$\gamma\big(\partial U \cap X_n\big) \;=\; n \;\leqslant\; \gamma(\partial U) \qquad \forall\, n \geqslant 1.$$

So $\gamma(\partial U) = +\infty = \dim X$.

Finally for $X = \{0\}$, we see that $\partial U = \emptyset$ and so by definition $\gamma(\emptyset) = 0$. □

In the next proposition we show that γ satisfies all the properties of a topological index.

PROPOSITION 5.4.29
If $A, A_1, A_2 \in \mathcal{T}$ and $h \in C(X;X)$ is an odd map, then

(a) *$\gamma(A) \geqslant 0$ and $\gamma(A) = 0$ if and only if $A = \emptyset$;*

(b) *γ is monotone, i.e., if $A_1 \subseteq A_2$, then $\gamma(A_1) \leqslant \gamma(A_2)$;*

(c) *γ is subadditive, i.e., $\gamma(A_1 \cup A_2) \leqslant \gamma(A_1) + \gamma(A_2)$;*

(d) *γ is supervariant, i.e., $\gamma(A) \leqslant \gamma\big(\overline{h(A)}\big)$;*

(e) *γ is continuous, i.e., if $A \in \mathcal{T}$ is compact, then $\gamma(A) < +\infty$ and there exists an open set U, such that $A \subseteq U$, $\overline{U} \in \mathcal{T}$ and $\gamma(A) = \gamma\big(\overline{U}\big)$*

PROOF **(a)** Follows from Definition 5.4.25.

(b) Suppose that $\gamma(A_2) < +\infty$ or otherwise we are done. Let $m \stackrel{df}{=} \gamma(A_2)$. Then we can find $h \in C\big(A_2; \mathbb{R}^m \setminus \{0\}\big)$ which is also odd. Let $h_1 \stackrel{df}{=} h|_{A_1}$. Then $h_1 \in C\big(A_1; \mathbb{R}^m \setminus \{0\}\big)$ and it is odd. Therefore $\gamma(A_1) \leqslant m$.

(c) Again we suppose that $m_1 = \gamma(A_1) < +\infty$ and $m_2 = \gamma(A_2) < +\infty$. By definition we can find odd maps

$$h_1 \;\in\; C\big(A_1; \mathbb{R}^{m_1} \setminus \{0\}\big) \quad \text{and} \quad h_2 \;\in\; C\big(A_2; \mathbb{R}^{m_2} \setminus \{0\}\big).$$

Invoking the Tietze extension theorem (see Theorem A.1.14), we can find two odd maps

$$\widehat{h}_1 \in C\big(X;\mathbb{R}^{m_1}\big) \quad \text{and} \quad \widehat{h}_2 \in C\big(X;\mathbb{R}^{m_2}\big),$$

such that

$$\widehat{h}_1|_{A_1} = h_1 \quad \text{and} \quad \widehat{h}_2|_{A_2} = h_2.$$

Let us set

$$\widehat{h}(x) \stackrel{df}{=} \big(\widehat{h}_1(x), \widehat{h}_2(x)\big) \qquad \forall\, x \in X.$$

Then $\widehat{h} \in C\big(X;\mathbb{R}^{m_1+m_2}\big)$ is odd and $\widehat{h}|_{A_1 \cup A_2} \neq 0$. So

$$\gamma(A_1 \cap A_2) \leqslant m_1 + m_2$$

(see Definition 5.4.25).

(d) If $\widehat{h} \in C\big(\overline{h(A)};\mathbb{R}^m \setminus \{0\}\big)$ is an odd map, then $\widehat{h} \circ h \in C\big(A;\mathbb{R}^m \setminus \{0\}\big)$ is an odd map too. So it follows that

$$\gamma(A) \leqslant \gamma\big(\overline{h(A)}\big).$$

(e) Since $A \in \mathcal{T}$ is compact, we can find $r > 0$, such that $A \cap \overline{B}_r(0) = \emptyset$. Consider the family of open sets $\big\{D_r(x) = B_r(x) \cup B_r(-x)\big\}_{x \in A}$. Evidently this is an open cover of A and so because of the compactness of A, we can find a finite subcover $\big\{D_r(x_k)\big\}_{k=1}^m$. Let $\{\varphi_k\}_{k=1}^m$ be a continuous partition of unity subordinate to the cover $\big\{D_r(x_k)\big\}_{k=1}^m$, i.e., $\varphi_k \in C(A)$ with

$$\operatorname{supp}\varphi_k \subseteq D_r(x_k), \quad 0 \leqslant \varphi_k \leqslant 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \varphi_k(x) = 1 \qquad \forall\, x \in A.$$

We may assume that each φ_k is even (otherwise replace φ_k by $\widehat{\varphi}_k(x) = \frac{1}{2}\big(\varphi_k(x) + \varphi_k(-x)\big)$ for all $x \in A$). Also note that from the choice of $r > 0$, we have

$$B_r(x_k) \cap B_r(-x_k) = \emptyset \qquad \forall\, k \in \big\{1,\dots,m\big\}.$$

Consider the map $h = (h_k)_{k=1}^m \in C(X;\mathbb{R}^m)$, defined by

$$h_k(x) \stackrel{df}{=} \begin{cases} \varphi_k(x) & \text{if} \quad x \in B_r(x_k), \\ -\varphi_k(x) & \text{if} \quad x \in B_r(-x_k). \end{cases}$$

Evidently h is odd and $h|_A \neq 0$. So $\gamma(A) \leqslant m < +\infty$. Suppose $\gamma(A) = m_0 < +\infty$. Let $h \in C\big(A;\mathbb{R}^{m_0} \setminus \{0\}\big)$ be odd. We may assume that $h \in C(X;\mathbb{R}^{m_0})$ (by the Tietze extension theorem; see Theorem A.1.14). Because A is compact, $h(A)$ is compact too. We can find a bounded symmetric neighbourhood $\widehat{U}$ of $h(A)$, such that $\overline{\widehat{U}} \subseteq \mathbb{R}^{m_0} \setminus \{0\}$. Let $U = h^{-1}\big(\widehat{U}\big)$. By construction $h\big(\overline{U}\big) \neq 0$ and $\gamma\big(\overline{U}\big) \leqslant m_0$. Since $S \subseteq U$, we have

$$\gamma\big(\overline{U}\big) = m_0 = \gamma(A)$$

(see **(b)**). ∎

COROLLARY 5.4.30
If $A, A_1, A_2 \in \mathcal{T}$,
then

(a) *if* A_1 *and* A_2 *are homeomorphic with respect to an odd homeomorphism, then* $\gamma(A_1) = \gamma(A_2)$;

(b) *if* $\gamma(A_1), \gamma(A_2) < +\infty$, *then* $\gamma\big(\overline{A_2 \setminus A_1}\big) \geqslant \gamma(A_2) - \gamma(A_1)$;

(c) $\gamma(A) \leqslant \dim X$;

(d) *if* $\gamma(A) > m$, $1 \leqslant m < +\infty$ *and* $\mathrm{proj}_Y \in \mathcal{L}(X;Y)$ *is the projection operator on the subspace* Y *of* X *with* $\dim Y = m$, *then*

$$A \cap (id_X - \mathrm{proj}_Y)(X) \neq \emptyset.$$

REMARK 5.4.31 If $A \in \mathcal{T}$ is finite, then $\gamma(A) = 1$. Indeed choose

$$h \in C\big(A; \mathbb{R} \setminus \{0\}\big),$$

such that

$$h(\pm x_k) = \pm 1,$$

where $A = \{\pm x\}_{k=1}^m$.

On $\mathcal{T}$ we can consider the Hausdorff metric h_0, defined by

$$h_0(A_1, A_2) \stackrel{df}{=} \max\left\{ \sup_{a_1 \in A_1} d_X(a_1, A_2), \sup_{a_2 \in A_2} d_X(a_2, A_1) \right\}.$$

It is well known (see, e.g., Hu & Papageorgiou (1997, p. 6)) that $(\mathcal{T}, h_0)$ is a complete metric space. Let $\mathcal{T}^c$ be the subspace of $\mathcal{T}$ consisting of compact sets and let

$$\mathcal{T}_k^c \stackrel{df}{=} \overline{\big\{A \in \mathcal{T}^c : \gamma(A) \geqslant k\big\}},$$

the closure taken in the metric space $(\mathcal{T}^c, h_0)$. The metric spaces $\big(\mathcal{T}^c, h_0\big)$ and $\big(\mathcal{T}_k^c, h_0\big)$ are complete.

LEMMA 5.4.32
If $A \in \mathcal{T}_k^c$, *then* $\gamma(A) \geqslant k$.

PROOF Let $\{A_n\}_{n \geqslant 1} \subseteq \mathcal{T}_k^c$ be such that

$$A_n \xrightarrow{h_0} A.$$

By Proposition 5.4.29**(e)**, we can find $\delta > 0$, such that

$$\gamma(A) = \gamma\big(\overline{A}_\delta\big),$$

where

$$\overline{A}_\delta \stackrel{df}{=} \{x \in X : d_X(x, A) \leqslant \delta\}.$$

Because

$$A_n \xrightarrow{h_0} A,$$

we can find $n_0 = n_0(\delta) \geqslant 1$, such that

$$A_n \subseteq \overline{A}_\delta \qquad \forall\, n \geqslant n_0.$$

Then from Proposition 5.4.29**(b)**, we have

$$k \leqslant \gamma(A_n) \leqslant \gamma(\overline{A}_\delta) = \gamma(A) \qquad \forall\, n \geqslant n_0.$$

□

We will derive the multiplicity result using the general setting in the last part of Section 5.1. So suppose that

$$\varphi = \varphi_1 + \varphi_2, \quad \text{with} \quad \varphi_1 \in C^1(X) \quad \text{and} \quad \varphi_2 \in \Gamma_0(X) \tag{5.54}$$

(see Definition 4.2.1). Observe that if φ_1 and φ_2 are both even then $\varphi_1'(0) = 0$ and $\varphi_2(0)$ is a global minimizer of φ_2. So in this case $x = 0$ is necessarily a critical point of φ (see Definition 5.1.35).

THEOREM 5.4.33
If $\varphi = \varphi_1 + \varphi_2$ is as in (5.54), φ satisfies the G-PS-*condition (see Definition 5.1.37), $\varphi(0) = 0$, φ_1 and φ_2 are even,*

$$c_k \stackrel{df}{=} \inf_{A \in \mathcal{T}_k^c} \sup_{x \in A} \varphi(x)$$

and

$$-\infty < c_k < 0 \qquad \forall\, k \in \{1, \ldots, m\},$$

then φ has at least m distinct pairs $\{\pm x_m\}_{k=1}^m$ of nontrivial critical points.

PROOF Let $1 \leqslant k \leqslant m$ and suppose that

$$c_k = \ldots = c_{k+i} = c,$$

for some $i \geqslant 0$. Because $c < 0$, we see that

$$0 \notin K_c^\varphi.$$

We claim that

$$\gamma(K_c^\varphi) \geqslant i + 1.$$

We proceed by contradiction. So suppose that

$$\gamma(K_c^{\varphi}) \leqslant i.$$

Let $r > 0$ be such that

$$\gamma\big(\big(\overline{K_c^{\varphi}}\big)_{2r}\big) = \gamma(K_c^{\varphi})$$

(see Proposition 5.4.29**(e)**). Define

$$\xi\colon \mathcal{T}_k^c \longrightarrow \mathbb{R},$$

by

$$\xi(A) = \sup_{x\in A} \varphi(x).$$

We claim that ξ is lower semicontinuous on $(\mathcal{T}_k^c, h_0)$. To this end let

$$A_n \xrightarrow{h_0} A \quad \text{in } (\mathcal{T}_k^c, h_0).$$

Let $\widehat{x} \in A$ be such that

$$\xi(A) = \varphi(\widehat{x}) = \sup_{x\in A} \varphi(x).$$

We can find $\widehat{x}_n \in A_n$, such that

$$\widehat{x}_n \longrightarrow \widehat{x}.$$

Then

$$\varphi(\widehat{x}_n) \leqslant \xi(A_n) \qquad \forall\, n \geqslant 1$$

and so

$$\xi(A) = \varphi(\widehat{x}) \leqslant \liminf_{n\to+\infty} \varphi(\widehat{x}_n) \leqslant \liminf_{n\to+\infty} \xi(A_n),$$

which proves the lower semicontinuity of ξ. Let

$$\varepsilon_0 \stackrel{df}{=} \min\{1, r, -c\} \quad \text{and} \quad U \stackrel{df}{=} \operatorname{int}\big(\overline{K_c^{\varphi}}\big)_r.$$

Apply Theorem 5.1.42 to obtain $\varepsilon \in (0, \varepsilon_0)$ for which the postulates of that theorem hold. Choose $A_1 \in \mathcal{T}_{k+i}^c$, such that

$$\xi(A_1) \leqslant c + \varepsilon^2.$$

Since

$$c + \varepsilon^2 \leqslant c + \varepsilon < 0,$$

we have that $0 \notin A_1$ and from Lemma 5.4.32, it follows that

$$\gamma(A_1) \geqslant k + i.$$

Let

$$A_2 \stackrel{df}{=} \overline{A_1 \setminus (K_c^{\varphi})_{2r}}.$$

Then

$$\xi(A_2) \leqslant c + \varepsilon^2$$

and by virtue of Corollary 5.4.30**(b)**, we have

$$k = k + i - i \leqslant \gamma(A_1) - \gamma\big((K_c^\varphi)_{2r}\big) \leqslant \gamma(A_2).$$

Invoking Theorem 4.6.1 with ε^2 and $\lambda = \frac{1}{\varepsilon}$, we obtain $A \in \mathcal{T}_k^c$, such that

$$\xi(A) \leqslant c + \varepsilon^2, \quad h_0(A, A_2) \leqslant \varepsilon \tag{5.55}$$

and

$$-\varepsilon h_0(A, B) \leqslant \xi(B) - \xi(A) \qquad \forall\ B \in \mathcal{T}_k^c. \tag{5.56}$$

Because $\varepsilon < r$ and $A \in \mathcal{T}_k^c$, we have

$$A \cap \overline{\big(K_c^\varphi\big)}_r = \emptyset \quad \text{and} \quad \xi(A) \geqslant c.$$

Apply Theorem 5.1.42 (see also Remark 5.1.43) to produce an odd homotopy

$$h_t \colon A \longrightarrow X.$$

Since

$$c + \varepsilon^2 < 0,$$

we have that

$$0 \notin A \quad \text{and} \quad \gamma(A) \geqslant k.$$

Let

$$B = h_t(A),$$

for $t \in [0, 1]$ small enough. Then from Proposition 5.4.29**(d)**, we have that

$$\gamma(B) \geqslant \gamma(A) \geqslant k.$$

So $B \in \mathcal{T}_k^c$ and from Theorem 5.1.42, (5.55) and (5.56), we have

$$-\varepsilon t \leqslant -\varepsilon h_0(A, B) \leqslant \xi(B) - \xi(A) \leqslant -2\varepsilon t,$$

which is a contradiction. Therefore

$$\gamma(K_c^\varphi) \geqslant i + 1.$$

In particular $\gamma(K_{c_k}^\varphi) \geqslant 1$ and so each $K_{c_k}^\varphi$ has at least two points x_k and $-x_k$. This gives the theorem, if all c_k are distinct. If not, then $i > 0$ for some $k \geqslant 1$. Hence

$$\gamma\big(K_{c_k}^\varphi\big) \geqslant 2$$

and

$$K_{c_k}^\varphi \text{ is an infinite set}$$

(see Remark 5.4.31). ∎

THEOREM 5.4.34
If $\varphi = \varphi_1 + \varphi_2$ is as in (5.54), φ satisfies the G-PS-*condition (see Definition 5.1.37), $\varphi(0) = 0$, φ_1, φ_2 are even and*

(i) there exists a subspace V of X of finite codimension and $\beta, r > 0$, such that $\varphi|_{\partial B_r(0) \cap V} \geqslant \beta$;

(ii) there exists a finite dimensional subspace Y of X with $\dim Y > \operatorname{codim} V$, such that $\varphi|_Y$ is weakly anticoercive (i.e., $\varphi(x) \longrightarrow -\infty$ as $\|x\|_X \to +\infty$, $x \in Y$),

then φ has at least $\dim Y - \operatorname{codim} V$ distinct pairs of nontrivial critical points.

PROOF Assume that for some $d > 0$, φ^{-d} contains no critical points of φ (otherwise there are infinitely many critical points and so there is nothing to prove). Let $R > r$ be such that

$$\varphi|_{\partial B_R^Y} \leqslant -d$$

(see hypothesis (ii)). Let

$$m \stackrel{df}{=} \operatorname{codim} V, \quad k \stackrel{df}{=} \dim Y, \quad B \stackrel{df}{=} B_R^Y = \{x \in Y : \|x\|_X \leqslant R\}.$$

For $j \in \{1, \dots, k\}$, we introduce the following objects:

$$\mathcal{F} \stackrel{df}{=} \{\eta \in C(B; X) : \eta \text{ is odd and } \eta|_{\partial B} \text{ is homotopic to } id|_{\partial B} \text{ in } \varphi^{-d} \text{ by an odd homotopy}\},$$

$$\Gamma_j \stackrel{df}{=} \{\eta(B \setminus U) : \eta \in \mathcal{F}, U \text{ is open in } B \text{ and symmetric, } U \cap \partial B = \emptyset \text{ and for each } Z \subseteq U \text{ with } Z \in \mathcal{T} \text{ we have } \gamma(Z) \leqslant k - j\},$$

$$\Delta_j \stackrel{df}{=} \{A \subseteq X : A \in \mathcal{T}, A \text{ is compact and for each open } W, \text{ such that } A \subseteq W, \text{ there exists } A_0 \in \Gamma_j \text{ with } A_0 \subseteq W\}.$$

Note that

$$B \in \Delta_j \quad \text{with } A_0 = B, \quad U = \emptyset \quad \text{and} \quad \eta = id|_B$$

so that

$$\Delta_j \neq \emptyset \qquad \forall\, j \in \{1, \dots, k\}.$$

We introduce the following numbers:

$$c_j \stackrel{df}{=} \inf_{A \in \Delta_j} \sup_{x \in A} \varphi(x) \qquad \forall\, j \in \{1, \dots, k\}.$$

Using standard topological arguments (see Szulkin (1986, Lemma 4.5 and Lemma 4.6)), we obtain:

(a) $c_j \geqslant \beta$ for all $j \in \{m+1, \ldots, k\}$.

(b) $\Delta_{j+1} \subseteq \Delta_j$ for all $j \in \{1, \ldots, k-1\}$.

(c) If $A \in \Delta_j$, C is a closed and symmetric set, such that $A \subseteq \operatorname{int} C$ and $\psi\colon C \longrightarrow X$ is an odd map, such that $\psi|_{C \cap \varphi^{-d}}$ is homotopic to $id|_{C \cap \varphi^{-d}}$ by an odd homotopy, then $\psi(A) \in \Delta_j$.

(d) If $D \in \mathcal{T}$ is compact, $\gamma(D) \leqslant l$ and $\varphi|_D > -d$, then there exists a number $\delta > 0$, such that for each $A \in \Delta_{j+l}$ we have $A \setminus \operatorname{int}\left(\overline{D}\right)_\delta \in \Delta_j$.

By virtue of **(a)** and **(b)**, we have $\beta \leqslant c_{m+1} \leqslant \ldots \leqslant c_k$. Suppose that

$$c_{j+1} = \ldots = c_{j+1+l} = c \text{ for some } l \in \{0, \ldots, k-m-1\}.$$

Because φ is even, we have that K_c^φ is symmetric and of course it is compact (since φ satisfies the generalized PS-condition). Note that $c > 0$ (since $c \geqslant \beta > 0$) and so

$$0 \notin K_c^\varphi$$

(recall that $\varphi(0) = 0$). Hence $K_c^\varphi \in \mathcal{T}$.

We shall prove that

$$l+1 \leqslant \gamma(K_c^\varphi).$$

Suppose that this is not the case. So $\gamma(K_c^\varphi) \leqslant l$. By virtue of Proposition 5.4.29**(e)**, we can find $\delta > 0$, such that

$$\gamma(K_c^\varphi) = \gamma\left(\overline{(K_c^\varphi)_\delta}\right).$$

Let $\varepsilon_0 > 0$ and

$$U \stackrel{df}{=} \operatorname{int}\left(\overline{(K_c^\varphi)_\delta}\right).$$

According to the Deformation Theorem (see Theorem 5.1.42; see also Remark 5.1.43), we can find $\varepsilon \in (0, \varepsilon_0)$ and a continuous map $\eta\colon [0,1] \times X \longrightarrow X$ satisfying statements **(a)**–**(h)** of that theorem. In particular, we have that η has the semigroup property

$$\eta(s, \cdot) \circ \eta(t, \cdot) = \eta(s+t, \cdot) \qquad \forall\ s, t \in [0,1],\ s+t \leqslant 1$$

and

$$\eta(t, \cdot) \text{ is an odd homeomorphism} \qquad \forall\ t \in [0,1].$$

From Theorem 5.1.42**(b)** and **(c)**, we have that

$$\eta(1, x) = x \qquad \forall\ x \in \varphi^{c+\varepsilon_0} \setminus \varphi^{c-\varepsilon_0}$$

and

$$\eta(1, \varphi^{c+\varepsilon} \setminus U) \subseteq \varphi^{c-\varepsilon}.$$

We can find $A \in \Delta_{j+l}$, such that

$$\sup_{x \in A} \varphi(x) \leqslant c + \varepsilon.$$

Because K_c^φ is compact, $\gamma(K_c^\varphi) \leqslant l$ and $\varphi|_{K_c^\varphi} > -d$, by **(d)** above and if $\delta > 0$ is small enough, we have that

$$A \setminus U \in \Delta_j$$

(recall the definition of U). Moreover,

$$A \setminus U \subseteq \varphi^{c+\varepsilon} \setminus U$$

and so

$$\eta(1, A \setminus U) \subseteq \varphi^{c-\varepsilon}.$$

Thus if ε_0 is sufficiently small we have

$$\eta(1, x) = x \qquad \forall\, x \in \varphi^{-d}$$

and so by **(c)** above, $\eta(1, A \setminus U) \in \Delta_j$. We have

$$\sup\{\varphi(x) : x \in \eta(1, A \setminus U)\} \leqslant c - \varepsilon,$$

which contradicts the fact that

$$\sup\{\varphi(x) : x \in \eta(1, A \setminus U)\} \geqslant c.$$

Therefore

$$\gamma(K_c^\varphi) \geqslant l + 1.$$

In particular then $\gamma(K_{c_j}^\varphi) \geqslant 1$ and so by Remark 5.1.43, each $K_{c_j}^\varphi$ has at least two antipodal points $\pm x_j$. This produces the claimed number of critical points of φ if all c_j are distinct. If they are not, then $l > 0$ for some j, so $\gamma(K_{c_j}^\varphi) > 1$ and by Remark 5.1.43, φ has infinite number of critical points (see Remark 5.4.31). □

COROLLARY 5.4.35

If the hypotheses of Theorem 5.4.34 hold with (ii) replaced by:

(ii)' for any positive integer k, there is a k-dimensional subspace Y of X, such that $\varphi|_Y$ is weakly anticoercive,

then φ has infinitely many distinct pairs of nontrivial critical points.

5.5 Lusternik-Schnirelman Theory and Abstract Eigenvalue Problems

The Lusternik-Schnirelman theory generalizes to arbitrary smooth functions the theory of eigenvalues of quadratic forms developed by R. Courant.

Let us start by recalling the min-max principles which characterize the eigenvalues of a self-adjoint compact operator. Suppose that H is a separable Hilbert space. Let $A \in \mathcal{L}_c(H)$ be a self-adjoint operator (i.e., A is linear, compact and self-adjoint). From Theorem 3.1.38, we know that $\sigma(A)$ consists of at most a countably infinite number of real eigenvalues $\{\lambda_k\}_{k\geqslant 1}$ with $\lambda = 0$ the only possible limit point. Moreover, the multiplicity of λ_k is equal to

$$\dim N(A - \lambda_k id_{_H}) < +\infty$$

and

$$Ax \;=\; \sum_{k=1}^{\infty} \lambda_k \,(x, x_k)_{_H}\, x_k,$$

where $\{x_k\}_{k\geqslant 1}$ is an orthonormal sequence of eigenvectors and λ_k is repeated $\dim N(A - \lambda_k id_{_H})$-times. The eigenvalues of A can be characterized by min-max principles. In particular, if the positive eigenvalues λ_k^+ are ordered in decreasing order (with multiplicities repeated), then

$$\lambda_k^+ \;=\; \sup_{S_m \in \mathcal{L}_k} \min_{x\in S_m} (Ax, x)_{_H}\,. \tag{5.57}$$

Here S_m denotes the boundary of an arbitrary m-dimensional unit ball in H, i.e.,

$$S_m \;\stackrel{df}{=}\; S \cap H_m,$$

$$S \;\stackrel{df}{=}\; \partial B_1(0) \;=\; \big\{x \in H :\; \|x\|_{_H} = 1\big\}$$

and H_m is an arbitrary m-dimensional subspace of H. Also $\mathcal{L}_k$ is the set of all S_m with $m \geqslant k$.

It is natural to try to extend (5.57) to general smooth functionals φ by finding topological analogs of the sets S_m and $\mathcal{L}_k$. We already did this in the previous section for functions exhibiting symmetry (even functionals), using the notion of genus (see Definition 5.4.25). Here we deal with general smooth functionals, not necessarily even. First we recall a basic topological notion.

DEFINITION 5.5.1 *Let Y be a Hausdorff topological space and $A \subseteq Y$, $A \neq \emptyset$. We say that A is **contractible** in Y, if there exist $h \in C\big([0,1]\times A; Y\big)$ and $y_0 \in Y$, such that*

$$h(0,x) \;=\; x \quad \textit{and} \quad h(1,x) \;=\; y_0 \qquad \forall\, x \in A.$$

REMARK 5.5.2 According to the above definition A is contractible if and only if the identity map on A is null homotopic (i.e., $id_{A} \simeq 0$).

DEFINITION 5.5.3 *Let Y be a Hausdorff topological space. The **Lusternik-Schnirelman category** relative to Y is the map*

$$cat_{Y}(A)\colon 2^{Y} \longrightarrow \mathbb{N}_0 \cup \{+\infty\},$$

defined by

$$cat_{Y}(\emptyset) \stackrel{df}{=} 0,$$

$$cat_{Y}(A) \stackrel{df}{=} \min\Big\{k \in \mathbb{N}:\ A \subseteq \bigcup_{m=1}^{k} A_m \text{ is closed and contractible for } m = 1, \dots, k\Big\},$$

$$cat_{Y}(A) \stackrel{df}{=} +\infty \quad \text{if } A \neq \emptyset \text{ and does not admit such a finite cover.}$$

REMARK 5.5.4 Since closedness and contractibility are preserved by homeomorphisms, we get the same value for homeomorphic Y or homeomorphic, closed $A \subseteq Y$. However, in the definition it is important to specify the space with respect to which the Lusternik-Schnirelman category is defined, since a set may be contractible in larger space Y_1 but not in Y. So if Y is embedded continuously in Y_1, then

$$cat_{Y_1}(A) \leqslant cat_{Y}(A) \qquad \forall\ A \subseteq Y.$$

PROPOSITION 5.5.5
If Y and Z are Hausdorff topological spaces and $A, C \subseteq Y$, then

(a) $cat_{Y}(A) \leqslant cat_{Y}(C) \qquad \forall\ A \subseteq C$;

(b) $cat_{Y}(A \cup C) \leqslant cat_{Y}(A) + cat_{Y}(C)$;

(c) *if ξ is null homotopic to the identity on A, then* $cat_{Y}(A) \leqslant cat_{Y}\big(\xi(A)\big)$;

(d) *if $z \in Z$, then* $cat_{Y \times Z}\big(A \times \{z\}\big) = cat_{Y}(A)$.

PROOF Statements **(a)**, **(b)** and **(d)** follow immediately from Definition 5.5.3. So it remains to show statement **(c)**. Suppose that

$$k \stackrel{df}{=} cat_{Y}\big(\xi(A)\big) < +\infty$$

or otherwise there is nothing to prove. Let $\{C_m\}_{m=1}^{k}$ be closed, contractible sets in Y which cover $\xi(A)$. Then we can find corresponding deformations $\{h_m\}_{m=1}^{k}$. Let $h \in C\big([0,1] \times A; Y\big)$ be such that

$$h(0, \cdot) = id_{A} \quad \text{and} \quad h(1, \cdot) = \xi$$

(it exists since ξ is null homotopic to id_{A}). If

$$h_1(\cdot) \stackrel{df}{=} h(1,\cdot),$$

let us set

$$A_m \stackrel{df}{=} h_1^{-1}(C_m) \qquad \forall\, m \in \{1,\dots,k\}.$$

Evidently the sets A_m are closed and they cover A. Also let

$$\widehat{h}_m\colon [0,1]\times A_m \longrightarrow Y$$

be defined by

$$\widehat{h}_m(t,x) \stackrel{df}{=} \begin{cases} h(2t,x) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ h_m\big(2t-1, h_1(x)\big) & \text{if} \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly $\widehat{h}_m \in C\big([0,1]\times A_m; Y\big)$, $\widehat{h}_m(0,\cdot) = id_{A_m}$ and $\widehat{h}_m(1,\cdot) = \widehat{y}_m$. Then according to Definition 5.5.3, we have

$$cat_{_Y}(A) \;\leqslant\; k \;=\; cat_{_Y}\big(\xi(A)\big).$$

□

Let us recall another important topological notion.

DEFINITION 5.5.6 *A metric space Y is said to be an **absolute neighbourhood retract** (**ANR** for short), if for every metric space V, every closed $D \subseteq V$ and every $\varphi \in C(D;Y)$, there exists a continuous extension of φ to some neighbourhood U of D. If it is always possible to extend to all of V, then we say that Y is an **absolute retract** (**AR** for short).*

REMARK 5.5.7 Evidently an AR is an ANR but the converse need not be true. Finite products of ANRs are an ANR, while for AR this is true for arbitrary products. Also by Dugundji's extension theorem (see Theorem 3.1.11), every convex subset of a normed space is an AR. Moreover, if Y is a Banach space then $\partial B_1(0)$ is an ANR. If in addition $\dim Y = +\infty$, then $\partial B_1(0)$ is an AR. In a Banach space an ANR is an AR if and only if it is contractible (see Palais (1966)).

PROPOSITION 5.5.8
If Y is an ANR and A is a closed subset of Y, then there exists a neighbourhood U of A, such that

$$cat_{_Y}(A) \;=\; cat_{_Y}\big(\overline{U}\big).$$

PROOF If

$$cat_{_Y}(A) \;=\; +\infty,$$

then the result is clear (see Proposition 5.5.5**(a)**). So suppose that

$$k \;\stackrel{df}{=}\; cat_{_Y}(A) < +\infty.$$

Let $\{A_m\}_{m=1}^{k}$ be a covering of A by closed contractible in Y sets. It suffices to show that each A_m has a neighbourhood U_m, such that $\overline{U}_m$ is contractible in Y. Since A_m is contractible in Y, we can find

$$h_m \;\in\; C\big([0,1]\times A_m; Y\big) \quad \text{and} \quad \widehat{y}_m \in Y,$$

such that

$$h_m(0,x) \;=\; x \quad \text{and} \quad h_m(1,x) \;=\; \widehat{y}_m \qquad \forall\, x \in A_m,\; m \in \big(1,\ldots,k\big).$$

Let

$$W \;\stackrel{df}{=}\; [0,1]\times Y \quad \text{and} \quad E_m \;\stackrel{df}{=}\; \big([0,1]\times A_m\big) \cup \big(\{0\}\times Y\big) \cup \big(\{1\}\times Y\big).$$

The set E_m is closed in W. Let $u_m \colon E_m \longrightarrow Y$ be defined by

$$u_m(t,x) \;\stackrel{df}{=}\; \begin{cases} h_m(t,x) & \text{if} \quad t \in [0,1],\; x \in A_m, \\ x & \text{if} \quad t = 0,\; x \in Y, \\ \widehat{y}_m & \text{if} \quad t = 1,\; x \in Y. \end{cases}$$

Clearly $u_m \in C(W;Y)$. Since Y is ANR, u_m admits a continuous extension $\widehat{u}_m$, defined on a neighbourhood V_m of E_m. Since $[0,1]\times Y$ is a metric space (hence normal), we can assume that V_m is closed and we can find a neighbourhood U_m of A_m, such that $[0,1]\times\overline{U}_m \subseteq V_m$. Since

$$\widehat{u}_m(0,x) \;=\; x \quad \text{and} \quad \widehat{u}_m(1,x) \;=\; \widehat{y}_m \qquad \forall\, x \in \overline{U}_m,$$

we infer that $\overline{U}_m$ is contractible in Y. Then

$$\overline{U} \;=\; \bigcup_{m=1}^{k} \overline{U}_m$$

is a closed neighbourhood of A and

$$k \;=\; cat_{_Y}(A) \;\leqslant\; cat_{_Y}\big(\overline{U}\big) \;\leqslant\; k.$$

□

REMARK 5.5.9 It is easy to see that if A is a compact subset of Y, then $cat_{_Y}(A) < +\infty$. Propositions 5.5.5 and 5.5.8 imply that when Y is an ANR, then $cat_{_Y}$ is a topological index.

Let us find the Lusternik-Schnirelman category of some standard sets.

EXAMPLE 5.5.10 **(a)** Let X be a Banach space and $A = \overline{B}_r(0)$. Then

$$cat_{_A}(A) \;=\; 1.$$

To see this note that

$$h(t,x) \;\overset{df}{=}\; (1-t)x$$

deforms A into $\{0\}$.

(b) Let X be a Banach space and $A \overset{df}{=} \partial B_r(0)$. Then if $\dim X = +\infty$, we have $cat_{_A}(A) = 1$ and if $\dim X < +\infty$, we have $cat_{_A}(A) = 2$. Indeed if $\dim X = +\infty$, then A is a retract of $\overline{B}_r(0)$. Let ξ be such a retraction and set

$$h(t,x) \;\overset{df}{=}\; \xi\big((1-t)x\big).$$

Then $h \in C\big([0,1]\times A; X\big)$ deforms A into $\big\{\xi(0)\big\}$ and so we have $cat_{_Y}(A) = 1$. On the other hand, if $\dim X < +\infty$, then we may assume that $X = \mathbb{R}^N$. A neighbourhood $\overline{U}$ of the north pole is contractible to it and $A\backslash U$ is contractible to the south pole. So we have $cat_{_Y}(A) \leqslant 2$. But as a consequence of Borsuk's antipodal theorem, we know that A is not contractible in itself. Therefore $cat_{_Y}(A) = 2$.

(c) Consider the real projective space $P^m \overset{df}{=} S^m/\mathbb{Z}_2$, where

$$S^m \;\overset{df}{=}\; \big\{x \in \mathbb{R}^{m+1} :\; \|x\|_{\mathbb{R}^{m+1}} = 1\big\}.$$

Then we have

$$cat_{_{P^k}}(P^m) \;=\; m+1 \qquad \forall\; k \geqslant m.$$

Also if X is a uniformly convex Banach space and $P^\infty(X)$ is the infinite dimensional projective space obtained by identifying the antipodal points of the unit sphere $S = \big\{x \in X : \|x\|_X = 1\big\}$, i.e.,

$$P^\infty(X) \;\overset{df}{=}\; S/\mathbb{Z}_2,$$

then

$$cat_{_{P^\infty(X)}}\big(P^m(X)\big) \;=\; m+1 \quad \text{and} \quad cat_{_{P^\infty(X)}}\big(P^\infty(X)\big) \;=\; +\infty.$$

For details we refer to Schwartz (1969, Chapter V).

(d) If $A = T^m = \mathbb{R}^m/\mathbb{Z}^m$ is the m-torus, then $cat_{_{T^m}} = m+1$. For details see Schwartz (1969, Chapter V).

REMARK 5.5.11 The determination of the Lusternik-Schnirelman category of a subset A of Y is, in general, a rather complicated matter which is based on nontrivial results of cohomology and homotopy theories. That is why it is often preferable to use other topological indices (such as the genus). Nevertheless, we should point out that the Lusternik-Schnirelman category is maximal among all topological indices which are invariant under homeomorphisms and satisfy the properties of Proposition 5.5.5.

Next we show that the Lusternik-Schnirelman category can be employed to find critical levels of min-max type. So let X be a Banach space and for $k \geqslant 1$, we define

$$\mathcal{A}_k \stackrel{df}{=} \big\{A \subseteq X : A \text{ is compact and } cat_X(A) \geqslant k\big\}.$$

Clearly $\{\mathcal{A}_k\}_{k\geqslant 1}$ is a decreasing sequence, i.e.,

$$\mathcal{A}_k \subseteq \mathcal{A}_{k-1} \qquad \forall\, k \geqslant 2.$$

For a given $\varphi \in C^1(X)$, we set

$$c_k \stackrel{df}{=} \inf_{A\in\mathcal{A}_k} \max_{x\in X} \varphi(x). \tag{5.58}$$

Evidently $-\infty \leqslant c_1 \leqslant c_2 \leqslant \ldots < +\infty$.

THEOREM 5.5.12
If $\varphi \in C^1(X)$ *satisfies the* PS*-condition and for some* $k \geqslant 1$*, we have* $c_k > -\infty$*,*
then c_k *is a critical value of* φ*.*
Moreover if $c = c_k = c_{k+1} = \ldots = c_{k+m}$*, then*

$$cat_X(K_c^\varphi) \geqslant m+1. \tag{5.59}$$

PROOF It is enough to show (5.59) and then by Definition 5.5.3,

$$K_c^\varphi \neq \emptyset.$$

Note that because of the PS-condition, K_c^φ is compact. According to Proposition 5.5.8, we can find a neighbourhood U of K_c^φ, such that

$$cat_X\big(\overline{U}\big) = cat_X\big(K_c^\varphi\big).$$

Applying Theorem 5.1.34, we obtain $\varepsilon \in (0,1)$ and let $A \in \mathcal{A}_{k+m}$ be such that

$$\max_{x\in X} \varphi(x) \leqslant c+\varepsilon.$$

Let

$$C \stackrel{df}{=} A \setminus U.$$

Then using Proposition 5.5.5, we have

$$\begin{aligned} k+m \;&\leqslant\; cat_X(A) \;\leqslant\; cat_X\big(C \cup \overline{U}\big) \\ &\leqslant\; cat_X(C) + cat_X(\overline{U}) \;=\; cat_X(C) + cat_X(K_c^{\varphi}). \end{aligned} \tag{5.60}$$

If h_t is the φ-decreasing, locally Lipschitz homotopy of homeomorphisms postulated by Theorem 5.1.34 and

$$D \;\stackrel{df}{=}\; h(1, C),$$

then

$$D \;\subseteq\; \varphi^{c-\varepsilon}.$$

So

$$\max_{x \in D} \varphi(x) \;\leqslant\; c - \varepsilon$$

and by the definition of c, we have that

$$cat_X(D) \;\leqslant\; k-1.$$

Then Proposition 5.5.5**(c)** implies that

$$cat_X(C) \;\leqslant\; cat_X(D) \;\leqslant\; k-1. \tag{5.61}$$

Using (5.61) in (5.60), we obtain

$$m+1 \;\leqslant\; cat_X\big(K_c^{\varphi}\big).$$

□

Another result which can be deduced in the same way from the properties of the Lusternik-Schnirelman category and Theorem 5.1.34 is the following one. We leave the details to the reader.

PROPOSITION 5.5.13
If $\varphi \in C^1(X)$, $a < b$, φ *satisfies the* PS_c*-condition for all* $c \in [a, b]$, $K_b^{\varphi} = \emptyset$ *and* $cat_X\big(\varphi^a\big) < +\infty$,
then $cat_X\big(\varphi^b\big) < +\infty$.

Before passing to the material related to nonlinear eigenvalue problems, let us state the relation between the Lusternik-Schnirelman category and the genus.

Let X be a Banach space and let $P^{\infty}(X)$ be the infinite dimensional projective space over X furnished with the quotient topology with respect to the canonical projection $\mathrm{proj}_{P^{\infty}(X)} : X \longrightarrow P^{\infty}(X)$, defined by

$$\mathrm{proj}_{P^{\infty}(X)}(x) \;\stackrel{df}{=}\; \big\{x, -x\big\} \qquad \forall\, x \in X \setminus \{0\}.$$

The next theorem was proved by Rabinowitz (1973).

THEOREM 5.5.14
If X is a Banach space and $A \in \mathcal{T}$ (see Definition 5.4.25), then

$$cat_{P^\infty(X)}\big(\mathrm{proj}_{P^\infty(X)}(A)\big) \;=\; \gamma(A).$$

COROLLARY 5.5.15
If X is a Banach space, then

$$cat_{\mathrm{proj}_{P^\infty(X)}(\partial B_1(0))}\big(\mathrm{proj}_{P^\infty(X)}(\partial B_1(0))\big) \;=\; \gamma\big(\partial B_1(0)\big) \;=\; \dim X.$$

In the rest of this section we deal with the method of Lagrange multipliers in constrained minimization which leads to nonlinear eigenvalue problems. So the problem under consideration is the following:

$$\inf_{x\in C} f(x), \quad \text{where} \quad C \;=\; \big\{x \in X :\; g(x) = 0\big\}. \tag{5.62}$$

In the next definition we abstract the idea that a tangent vector to a surface is the velocity vector of a trajectory in the surface.

DEFINITION 5.5.16 *Let X be a Banach space, $C \subseteq X$ and $x_0 \in C$.*

(a) *A **curve** in C through x_0 is a continuous map $c\colon (-\varepsilon,\varepsilon) \longrightarrow X$, such that*

$$c(t) \;\in\; C \qquad \forall\, t \in (-\varepsilon,\varepsilon),$$

$c(0) = x_0$ and $c'(0)$ exists.

(b) *A vector $v \in X$ is a **tangent vector** to C at x_0 if and only if $v = c'(0)$ for some curve c as in* **(a)**.

(c) *If the set of all tangent vectors to C at x_0 form a vector space, it is denoted by $T_{x_0}(C)$ and called the **tangent space** to C at x_0. Its translation $x_0 + T_{x_0}(C)$ is called the **tangent plane** to C at x_0.*

Using this intuitive definition, we can proceed to introduce the notion of a manifold. First we isolate those points of C for which a local parametrization (local coordinates) exists.

DEFINITION 5.5.17 *Let X be a Banach space and $C \subseteq X$. A point $x_0 \in C$ is said to be **regular**, if $T_{x_0}(C)$ exists and it is closed and there exists a neighbourhood V of the origin in $T_{x_0}(C)$ and a homeomorphism $\vartheta\colon V \longrightarrow C$ on a neighbourhood of U of x_0.*

REMARK 5.5.18 According to this definition $U = \vartheta(V)$ is a neighbourhood of x_0. Then each $x \in U$ can be described as $x = \vartheta(v)$ for some $v \in V$. We call $v \in V$ the ***local coordinate*** of $x \in U$ and the pair (V, ϑ) is a ***local parametrization*** (local coordinate system) to C at x_0. Suppose that x_1 and x_2 are two regular points in C with corresponding local coordinate systems (V_1, ϑ_1) and (V_2, ϑ_2) respectively. Let

$$U_1 \stackrel{df}{=} \vartheta_1(V_1) \quad \text{and} \quad U_2 \stackrel{df}{=} \vartheta_2(V_2).$$

Then every $x \in U_1 \cap U_2$ has local coordinate $v_1 = \vartheta_1^{-1}(x)$ for the local system (V_1, ϑ_1) and $v_2 = \vartheta_2^{-1}(x)$ for the local system (V_2, ϑ_2). Then the changes of coordinates between the two local systems is described by $v_1 = \vartheta_1^{-1}\big(\vartheta_2(v_2)\big)$ (i.e., by the map $\vartheta_1^{-1} \circ \vartheta_2$).

DEFINITION 5.5.19 *Let X be a Banach space and $C \subseteq X$. We say that C is a (**topological**) **manifold**, if every point of C is regular. We say that C is a C^k-**manifold**, if C is a manifold and the changes of local coordinates $\vartheta_1^{-1} \circ \vartheta_2$ are C^k-maps.*

REMARK 5.5.20 For a manifold C, the changes of local coordinates $\vartheta_1^{-1} \circ \vartheta_2$ are homeomorphisms. Similarly for C^k-manifold all the maps $\vartheta_1^{-1} \circ \vartheta_2$ are C^k-diffeomorphisms.

EXAMPLE 5.5.21 It is routine to check that if X is a Banach space and $\varphi \in C^1(X)$, then $C = \varphi^{-1}(r)$ is a C^1-manifold provided that r is not a critical value of φ. The next theorem gives the generalization of this fact (see also Theorem 4.2.7).

THEOREM 5.5.22 (Lusternik Theorem)
If X and Y are two Banach spaces, $\varphi \in C^1(X; Y)$,

$$C \stackrel{df}{=} \big\{x \in X : \ \varphi(x) = 0\big\}$$

and for every $x \in C$, $\varphi'(x)$ is surjective and has a complemented kernel $N\big(\varphi'(x)\big)$ (i.e., $X = N\big(\varphi'(x)\big) \oplus V$ for some closed subspace V of X),
then C is a C^1-manifold and for every $x \in C$, we have

$$T_x(C) \ = \ N\big(\varphi'(x)\big).$$

PROOF Let $x \in C$. By hypothesis, we have

$$X \ = \ N\big(\varphi'(x)\big) \oplus V.$$

It follows that there exists a continuous projection $\mathrm{proj}_{N(\varphi'(x))}$ onto $N\big(\varphi'(x)\big)$. Let

$$\psi \colon N\big(\varphi'(x)\big) \oplus V \longrightarrow Y$$

be defined by

$$\psi(u,v) \stackrel{df}{=} \varphi(x+u+v) \qquad \forall\, u \in N\big(\varphi'(x)\big),\ v \in V.$$

Note that

$$\psi(0,0) = \varphi(x)$$

and ψ is continuously differentiable with

$$D_1\psi(0,0) = \varphi'(x)|_{N(\varphi'(x))} = 0 \in \mathcal{L}\big(N\big(\varphi'(x)\big);Y\big)$$

and

$$D_2\psi(0,0) = \varphi'(x)|_V \in \mathcal{L}(V;Y).$$

By the inverse function theorem (see Theorem 4.1.32), $D_2\psi(0,0)$ is an isomorphism between V and Y.

Consider the equation

$$\psi(u,v) = \varphi(x) = 0.$$

The properties of ψ permit the use of the implicit function theorem (see Theorem 4.1.27). So we can find $\delta > 0$ and a uniquely determined C^1-map

$$\xi\colon N\big(\varphi'(x)\big) \cap B_\delta(0) \longrightarrow V,$$

such that

$$0 = \varphi(x) = \psi\big(u,\xi(u)\big) \qquad \forall\, u \in N\big(\varphi'(x)\big) \cap B_\delta(0),$$

$$\xi(0) = 0 \quad \text{and} \quad \xi'(0) = -D_2\psi(0,0)^{-1}D_1\psi(0,0).$$

Hence $\xi'(0) = 0$ and so

$$\lim_{u\to 0} \frac{\|\xi(u)\|_X}{\|u\|_X} = 0.$$

Now let

$$\eta\colon N\big(\varphi'(x)\big) \cap B_\delta(0) \longrightarrow X$$

be defined by

$$\eta(u) \stackrel{df}{=} x+u+\xi(u).$$

Evidently η is continuous. By construction

$$0 = \varphi(x) = \psi\big(u,\xi(u)\big) = \varphi\big(x+u+\xi(u)\big)$$

and so we see that

$$x+u+\xi(u) \in C,$$

i.e., η maps into C. Because u and $\xi(u)$ are in complementary subspaces of X, we have that η is injective. Hence η is invertible on

$$W \stackrel{df}{=} \big\{x+u+\xi(u) :\ u \in N\big(\varphi'(x)\big) \cap B_\delta(0)\big\} \subseteq C$$

and so

$$\eta^{-1}\big(x+u+\xi(u)\big) \;=\; x+u.$$

Recall that

$$R\big(\mathrm{proj}_{N(\varphi'(x))}\big) \;=\; N\big(\varphi'(x)\big) \quad \text{and} \quad N\big(\mathrm{proj}_{N(\varphi'(x))}\big) \;=\; V.$$

So we can write that

$$\eta^{-1}\big(x+u+\xi(u)\big) \;=\; x+\mathrm{proj}_{N(\varphi'(x))}\big(u+\xi(u)\big), \tag{5.63}$$

which shows that η^{-1} is continuous. Therefore η is a homeomorphism of $N\big(\varphi'(x)\big)\cap B_\delta(0)$ onto W. In fact from the properties of ξ and (5.63), we conclude that η is a diffeomorphism of $N\big(\varphi'(x)\big)\cap B_\delta(0)$ onto W. This implies that C is C^1-manifold.

Next let $w\in T_x(C)$. Let c be the corresponding curve, such that

$$\varphi\big(c(t)\big) \;=\; 0 \qquad \forall\, t\in(-\varepsilon,\varepsilon),$$

and

$$c(0) \;=\; x, \quad c'(0) \;=\; w$$

(see Definition 5.5.17). Then by the chain rule, we have

$$\varphi'(x)w \;=\; 0,$$

i.e., $w\in N\big(\varphi'(x)\big)$. On the other hand, if $w\in N\big(\varphi'(x)\big)$, then

$$c(t) \;=\; \varphi\big(tw\big), \quad t\in(-\varepsilon,\varepsilon),$$

is a curve satisfying the requirements of Definition 5.5.17. Hence $w\in T_x(C)$ and we conclude that

$$T_x(C) \;=\; N\big(\varphi'(x)\big).$$

□

REMARK 5.5.23 Recall that every finite dimensional subspace F of X is complemented. Similarly if a closed subspace Z of X has finite codimension, then Z is complemented. Of course in a Hilbert space every closed subspace is complemented (by the orthogonal complement). In general, however, every Banach space that is not isomorphic to a Hilbert space has closed subspaces that are not complemented. If X and Y are Banach spaces and $A\in\mathcal{L}(X;Y)$ is surjective, then it is easy to check that A has a right inverse if and only if $N(A)$ is complemented.

The next result establishes the method of Lagrange multipliers for constrained optimization problems in Banach spaces.

THEOREM 5.5.24
If X and Y are two Banach spaces, $f\colon X \longrightarrow \mathbb{R}$ and $g\colon X \longrightarrow Y$ are Fréchet differentiable maps, $x_0 \in X$ is a finite local minimizer of the constrained problem (5.62) and $g'(x_0) \in \mathcal{L}(X;Y)$ is surjective with $N\big(g'(x_0)\big)$ complemented in X,
then there exists $y^* \in Y^*$, such that $x_0 \in X$ is a critical point of

$$h(x) \stackrel{df}{=} f(x) - \langle y^*, g(x)\rangle_Y ,$$

i.e., $f'(x_0) = y^* \circ g'(x_0)$.

PROOF By virtue of Theorem 5.5.22, if

$$C \stackrel{df}{=} \big\{x \in X :\ g(x) = 0\big\},$$

we have

$$T_{x_0}(C) \ = \ N\big(g'(x_0)\big).$$

Let $w \in N\big(g'(x_0)\big)$. We can find a curve c, such that

$$g\big(c(t)\big) \ = \ 0 \qquad \forall\ t \in (-\varepsilon, \varepsilon),$$

$$c(0) \ = \ x_0 \quad \text{and} \quad c'(0) \ = \ w.$$

Then by the chain rule, we have $g'(x_0)w = 0$. Let

$$h(t) \stackrel{df}{=} f\big(c(t)\big) \qquad \forall\ t \in (-\varepsilon, \varepsilon).$$

By hypothesis h has a local minimum at $t = 0$. Therefore

$$f'(0) \ = \ 0$$

and so

$$\big\langle f'(x_0), w\big\rangle_X \ = \ 0 \qquad \forall\ w \in N\big(g'(x_0)\big).$$

So we have that

$$N\big(g'(x_0)\big) \ \subseteq \ N\big(f'(x_0)\big). \tag{5.64}$$

By hypotheses

$$X \ = \ N\big(g'(x_0)\big) \oplus V,$$

with a closed subspace V of X. Note that

$$L \ = \ g'(x_0)|_V$$

is an isomorphism. Also let proj_V be the projection operator onto V. Because of (5.64), we have that

$$f'(x_0) \circ \mathrm{proj}_V \ = \ f'(x_0)$$

and

$$f'(x_0) \circ \mathrm{proj}_{_V} : V \longrightarrow \mathbb{R}$$

is a continuous, linear functional (i.e., $f'(x_0) \circ \mathrm{proj}_{_V} \in V^*$). Then

$$y^* \;=\; f'(x_0) \circ \mathrm{proj}_{_V} \circ L^{-1} \in Y^*.$$

We have

$$\big(y^* \circ g'(x_0)\big)(x) \;=\; \big(y^* \circ g'(x_0)\big)\big(\mathrm{proj}_{_V}(x)\big) \;=\; \big(y^* \circ L\big)\big(\mathrm{proj}_{_V}(x)\big) \;=\; f'(x_0)(x),$$

so

$$f'(x_0) \;=\; y^* \circ g'(x_0).$$

□

We can generalize the above theorem by weakening the assumptions on the constraint function g. To do this we need the following partial generalization of Theorem 5.5.22.

THEOREM 5.5.25
If X and Y are two Banach spaces, $\varphi\colon X \longrightarrow Y$ is Fréchet differentiable, $\varphi'(x_0) \in \mathcal{L}(X;Y)$ is surjective (i.e., φ is regular at x_0) and

$$C \;\stackrel{df}{=}\; \big(x \in X :\; \varphi(x) = \varphi(x_0)\big),$$

then

$$T_{x_0}(C) \;=\; N\big(\varphi'(x_0)\big)$$

and there exist a neighbourhood U of x_0, a number $\beta > 0$ and mapping $U \ni u \longmapsto \xi(u) \in X$, such that

$$\varphi(x_0) \;=\; \varphi\big(u + \xi(u)\big) \quad \text{and} \quad \big\|\xi(u)\big\|_X \;\leqslant\; \beta\big\|\varphi(u) - \varphi(x_0)\big\|_Y \qquad \forall\, u \in U.$$

Using this theorem, we can have the following generalization of Theorem 5.5.24.

THEOREM 5.5.26
If X and Y are two Banach spaces, $f\colon X \longrightarrow \mathbb{R}$ is Fréchet differentiable at x_0, $g\colon X \longrightarrow Y$ is continuously Fréchet differentiable at x_0 with $g'(x_0) \in \mathcal{L}(X;Y)$ having closed range and $x_0 \in C \stackrel{df}{=} \big\{x \in X :\; g(x) = 0\big\}$ is a finite local minimizer of the constrained problem (5.62),
then there exist $\lambda_0 \in \mathbb{R}$ and $y^* \in Y^*$ not both equal to zero, such that

$$\lambda_0 f'(x_0) \;=\; y^* \circ g'(x_0) \quad \text{in } X^*. \tag{5.65}$$

Moreover, if $g'(x_0)$ is surjective (i.e., g is regular at x_0), then we have that $\lambda_0 \neq 0$ and so we can take $\lambda_0 = 1$.

PROOF In the degenerate case when $g'(x_0) \in \mathcal{L}(X;Y)$ is not surjective, by a well known corollary of the Hahn-Banach theorem (see, e.g., Denkowski, Migórski & Papageorgiou (2003*a*, p. 263)), we can satisfy (5.65) with $\lambda_0 = 0$. Otherwise, using Theorem 5.5.25, we can show that (5.64) holds. Also since now we have that

$$R\big(g'(x_0)\big) \;=\; Y,$$

by virtue of Lemma 3.1.42, we have

$$\begin{aligned} R\big(g'(x_0)^*\big) \;&=\; N\big(g'(x_0)\big)^{\perp} \\ =\; \big\{x^* \in X^* :\; \langle x^*, x\rangle_X = 0 \text{ for all } x \in N\big(g'(x_0)\big)\big\}&. \end{aligned}$$

Then from (5.64), we see that

$$f'(x_0) \;\in\; N\big(g'(x_0)\big)^{\perp} \;=\; R\big(g'(x_0)^*\big)$$

and so for some $y^* \in Y^*$, we have

$$f'(x_0) \;=\; g'(x_0)^* y^*.$$

Therefore

$$\big\langle f'(x_0), x\big\rangle_X \;=\; \big\langle g'(x_0)^* y^*, x\big\rangle_X \;=\; \big\langle y^*, g'(x_0)x\big\rangle_Y \qquad \forall\, x \in X,$$

so

$$f'(x_0) \;=\; y^* \circ g'(x_0) \quad \text{in } X^*,$$

i.e., (5.65) holds with $\lambda_0 = 1$. □

REMARK 5.5.27 In Theorems 5.5.27 and 5.5.26 the quantities $\lambda_0 \in \mathbb{R}$ and $y^* \in Y^*$ are called ***Lagrange multipliers***.

If Y is finite dimensional, say $Y = \mathbb{R}^N$, then every $y^* \in Y^*$ is characterized uniquely by an N-tuple of numbers $(\lambda_1, \ldots, \lambda_N)$ and so Theorem 5.5.24 (see also Theorem 5.5.26) reduces to the well known theorem from multivariable calculus on the existence of Lagrange multipliers.

THEOREM 5.5.28
If $f\colon \mathbb{R}^k \longrightarrow \mathbb{R}$ is a differentiable function, $g\colon \mathbb{R}^k \longrightarrow \mathbb{R}^N$ is continuously differentiable, f attains at $x_0 \in \mathbb{R}^k$ a local minimum on the set

$$C \;\stackrel{df}{=}\; \big\{x \in \mathbb{R}^k :\; g(x) = 0\big\}$$

and x_0 is a regular point of g (i.e., $g'(x_0) \in \mathcal{L}\big(\mathbb{R}^k; \mathbb{R}^N\big)$ is surjective), then we can find $\widehat{\lambda} = \big\{\lambda_i\big\}_{i=1}^N \in \mathbb{R}^N$, such that

$$\frac{\partial f}{\partial x_j}(x_0) \;=\; \sum_{i=1}^{N} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \qquad \forall\, j \in \big\{1, \ldots, k\big\}.$$

REMARK 5.5.29 According to the above theorem, the local extrema of f subject to the constraint $x \in C \stackrel{df}{=} \{x \in \mathbb{R}^k : \ g(x) = 0\}$ are included in the solutions $\big(\widehat{\lambda} = (\lambda_i)_{i=1}^N, x_0\big) \in \mathbb{R}^N \times \mathbb{R}^k$ of the algebraic system

$$\begin{cases} \dfrac{\partial f}{\partial x_j}(x_0) = \lambda_1 \dfrac{\partial g_1}{\partial x_j}(x_0) + \ldots + \lambda_N \dfrac{\partial g_N}{\partial x_j}(x_0) & \text{for } j \in \{1, \ldots, k\}, \\ g_i(x_0) \ = \ 0 & \text{for } i \in \{1, \ldots, N\}. \end{cases}$$

This is a system of $k + N$ equations with $k + N$ unknowns.

Let us see how we can use the method of Lagrange multipliers to establish the existence of eigenvalues for certain linear, bounded, self-adjoint operators in Hilbert spaces.

EXAMPLE 5.5.30 Let H be a Hilbert space and suppose that $A \in \mathcal{L}(H)$ is self-adjoint. Let

$$f(x) \ \stackrel{df}{=} \ \big(Ax, x\big)_H, \quad g(x) \ \stackrel{df}{=} \ \|x\|_H - 1$$

and

$$C \ \stackrel{df}{=} \ \{x \in H : \ g(x) = 0\}.$$

Suppose that $x_0 \in H$, $\|x_0\|_H = 1$ is a solution of the minimization problem

$$\min_{x \in C} f(x).$$

Then by virtue of Theorem 5.5.24 with $Y = \mathbb{R}$, we can find $\lambda \in \mathbb{R}$, such that

$$f'(x_0) \ = \ \lambda g'(x_0),$$

hence $Ax_0 = \lambda x_0$. So $x_0 \in H$ is an eigenvector of A with corresponding real eigenvalue $\lambda \in \mathbb{R}$. Here we have used the fact that every $x \in C$ is a regular point of g.

Next we extend the notion of critical point to constrained functionals.

DEFINITION 5.5.31 *Let X be a Banach space, $C \subseteq X$ and*

$$\varphi \colon X \longrightarrow \mathbb{R}.$$

We say that φ has a ***critical point with respect to*** *C at x_0, if $x_0 \in C$ and for every curve $c \colon (-\varepsilon, \varepsilon) \longrightarrow X$, such that $c(t) \in C$ for all $t \in (-\varepsilon, \varepsilon)$, $c(0) = x_0$ and $c'(0)$ exists, we have*

$$\frac{d}{dt} \varphi\big(c(t)\big)\big|_{t=0} \ = \ 0. \tag{5.66}$$

REMARK 5.5.32 If $\operatorname{int} C \neq \emptyset$ and $x_0 \in \operatorname{int} C$, then x_0 is a usual critical point of φ, often called a ***free critical point*** of φ.

From (5.66) and the chain rule, we have the following criterion.

PROPOSITION 5.5.33
If X is a Banach space, $C \subseteq X$, $\varphi\colon X \longrightarrow \mathbb{R}$ is a function which is Fréchet differentiable at $x_0 \in C$ and the set C has a tangent space $T_{x_0}(C)$ at x_0, then x_0 is a critical point of φ with respect to C if and only if $\varphi'(x_0)|_{T_{x_0}(C)} = 0$.

We can have another criterion based on Lagrange multipliers.

PROPOSITION 5.5.34
If X and Y are two Banach spaces, $f\colon X \longrightarrow \mathbb{R}$ is Fréchet differentiable, $g\colon X \longrightarrow Y$ is a continuously Fréchet differentiable function, $x_0 \in X$,

$$C \stackrel{df}{=} \big\{x \in X :\ g(x) = 0\big\}$$

and $g'(x_0) \in \mathcal{L}(X;Y)$ is surjective with $N\big(g'(x_0)\big)$ complemented in X, then f has a critical point with respect to C at x_0 if and only if there exists $y^* \in Y^*$, such that

$$f'(x_0) \ = \ y^* \circ g'(x_0) \quad \text{in } X^*.$$

PROOF "$\Longrightarrow$": From Theorem 5.5.22 (see also Theorem 5.5.25), we know that

$$T_{x_0}(C) \ = \ N\big(g'(x_0)\big).$$

So from Proposition 5.5.33, we know that

$$f'(x_0)v \ = \ 0 \qquad \forall\ v \in N\big(g'(x_0)\big),$$

hence

$$N\big(g'(x_0)\big) \ \subseteq \ N\big(f'(x_0)\big).$$

From this inclusion, arguing as in the proof of Theorem 5.5.26, we obtain $y^* \in Y^*$, such that

$$f'(x_0) \ = \ y^* \circ g'(x_0) \quad \text{in } X^*. \tag{5.67}$$

"$\Longleftarrow$": If (5.67) holds and $c\colon (-\varepsilon, \varepsilon) \longrightarrow X$ is a curve, such that $c(t) \in C$ for all $t \in (-\varepsilon, \varepsilon)$, $c(0) = x_0$ and $c'(0)$ exists. Then $g\big(c(t)\big) = 0$ for all $t \in (-\varepsilon, \varepsilon)$ and $g'(x_0)c'(0) = 0$. Because of (5.67), we have that

$$\big\langle f'(x_0), c'(0)\big\rangle_X \ = \ 0$$

and so

$$\frac{d}{dt} f\big(c(t)\big)|_{t=0} \ = \ 0,$$

which means that x_0 is a critical point of f with respect to C. □

5.6 Remarks

5.1: Deformation techniques along the gradient flow already appeared in 1960's in the works of Browder (1965*a*) and Schwartz (1964). In the deformation approach, the Palais-Smale condition (PS-condition; see Definition 5.1.5**(a)**) and the pseudogradient vector field (see Definition 5.1.16) play a central role and were introduced by Palais & Smale (1964) and Palais (1966) respectively. The slight extension of the PS-condition, the Cerami condition (C-condition; see Definition 5.1.5**(b)**), was introduced by Cerami (1978). Both conditions replaced the compactness of the domain of the energy functionals. The connection between the PS-condition and coercivity (see Theorem 5.1.13) is due to Costa & Silva (1991). Analogous results can be found in the works of Čaklović, Li & Willem (1990), Goeleven (1993) and Gasiński & Papageorgiou (2005) (for nonsmooth functionals). For the Theorem 5.1.21 we refer to Cartan (1967, p. 122). The basic deformation result is due to Clark (1972). Proposition 5.1.25 and its consequences (see Corollaries 5.1.26, 5.1.27, 5.1.28) are quantitative versions of the result of Clark due to Willem (1996). Theorem 5.1.32 is due to Du (1991). The second deformation theorem (see Theorem 5.1.33) is due to Rothe (1973), Marino & Prodi (1975), Chang (1981) and Wang (1987) (see also the book of Chang (1993, p. 23)). The deformation theorem based on the Cerami condition (see Theorem 5.1.34) is due to Bartolo, Benci & Fortunato (1983). A nonsmooth generalization of it can be found in Gasiński & Papageorgiou (2005). The extension of the deformation approach to functionals of the form $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in C^1(X)$ and $\varphi_2 \in \Gamma_0(X)$ (see Theorem 5.1.42) is due to Szulkin (1986). In Gasiński & Papageorgiou (2005) we find a further generalization of Szulkin's theory to functionals $\varphi = \varphi_1 + \varphi_2$ with φ_1 being locally Lipschitz.

5.2: The notion of linking sets (see Definition 5.2.1) is essentially due to Benci & Rabinowitz (1979). The main minimax results which generate the others by appropriate choice of the sets (see Theorems 5.2.4 and 5.2.5) are generalized versions of a theorem which can be found in Struwe (1990, p. 118). The mountain pass theorem (see Corollary 5.2.7) is due to Ambrosetti & Rabinowitz (1973). The saddle point theorem (see Corollary 5.2.8) is due to Rabinowitz (1978*b*), while the generalized mountain pass theorem (see Corollary 5.2.9) is due to Rabinowitz (1978*a*) (see also Rabinowitz (1986)). The extensions to functionals of the form $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in C^1(X)$ and $\varphi_2 \in \Gamma_0(X)$ presented in the last part of Section 5.2 are due to Szulkin (1986). Additional results and developments of the deformation and minimax techniques, with applications to boundary value problems, can be found in Ambrosetti (1992), Du (1991), Fang & Ghoussoub (1992), Ghoussoub (1993*a*, 1993*b*), Mawhin & Willem (1989), Nirenberg (1981, 1989), Papageorgiou & Papageorgiou (2004), Ramos & Rebelo (1994) and Silva (1991).

5.3: Theorem 5.3.5 is due to Hofer (1985) (for a generalization see also Hofer (1988)). Theorem 5.3.6 can be found in Struwe (1990, p. 129) and is also related to results of Chang (1993) and Pucci & Serrin (1985, 1987). Similar results can be found in Hofer (1984, 1986) and Manes & Micheletti (1973).

5.4: The notion of local linking (see Definition 5.4.1) was introduced by Liu & Li (1984) under the stronger assumption that $\dim Y < +\infty$ and $\varphi(v) \geqslant r > 0$ for all $v \in V$ with $\|v\|_V = r$. Theorems 5.4.6 and 5.4.7 are due to Brézis & Nirenberg (1991). For the proof of Theorem 5.4.10 (Morse lemma) we refer to Schwartz (1969, p. 136). Theorems 5.4.21 and 5.4.23 are due to Li & Willem (1995). Theorem 5.4.24 is due to Gasiński & Papageorgiou (2005). The notion of genus (see Definition 5.4.25) was introduced by Krasnoselskii (1964*b*) (see also Krasnoselskii & Zabreiko (1984, p. 385)). Here we use the definition of genus given by Coffman (1969). The equivalence of the definition of Coffman and of the original definition of Krasnoselskii can be found in Rabinowitz (1973). A presentation of the various properties of the genus can be found in Mawhin & Willem (1989), Rabinowitz (1986) and Struwe (1990). The proof of Theorem 5.5.25 can be found in Ioffe & Tihomirov (1979, p. 30). Theorems 5.4.33 and 5.4.34 are due to Szulkin (1986) and generalize earlier results of Rabinowitz (1986) with $\varphi_2 = 0$.

5.5: The Lusternik-Schnirelman category (see Definition 5.5.3) is the first example of a topological index and was introduced by Lusternik & Schnirelman (1934), who also proved the basic multiplicity result, Theorem 5.5.12. Here we give a restricted definition of a manifold C by requiring that C lies in a fixed Banach space X. A more general definition can be found in Lang (1972). Theorems 5.5.22 and 5.5.24 are due to Lusternik (1934). In this section, crucial is the assumption that $R\big(g'(x_0)\big) = Y$ (regularity of the map $g\colon X \longrightarrow Y$ at x_0). So let us give a criterion for such surjectivity to hold (see Yosida (1978, p. 208)).

THEOREM 5.6.1
If X and Y are two Banach spaces and $A\colon X \supseteq D(A) \longrightarrow Y$ is a closed and densely defined, linear operator,
then A is surjective (i.e., $R(A) = Y$) if and only if A^* has a continuous inverse (i.e., there exists $c > 0$ such that $\|A^*y^*\|_{X^*} \geqslant c\,\|y^*\|_{Y^*}$ for all $y^* \in D(A^*)$).

EXAMPLE 5.6.2 If $X = H$ is a Hilbert space and $A\colon H \supseteq D(A) \longrightarrow H$ is a closed, densely defined, linear operator, such that there exists $c > 0$ for which we have

$$\big(Ax, x\big)_H \;\geqslant\; c\,\|x\|_H^2 \qquad \forall\, x \in D(A)$$

(i.e., A is strongly monotone), then $R(A) = H$.

Additional results on constrained infinite dimensional optimization can be found in Blanchard & Brüning (1992), Ioffe & Tihomirov (1979), Vaĭnberg (1973) and Zeidler (1985*b*).

Chapter 6

Eigenvalue Problems and Maximum Principles

In this chapter we continue our investigation of the methods and techniques used in the study of linear and nonlinear elliptic partial differential equations. First we analyze the spectrum of linear elliptic differential operators and then we pass to certain nonlinear ones, namely the p-Laplacian. Our analysis involves energy methods which are based on the critical point theory developed in the previous chapter. In the last two sections we develop the other distinct technique in the study of stationary partial differential equations which is based on maximum principles. This method leads to pointwise deductions and for this reason requires higher smoothness in the functions and it is different from the integral-based energy methods which are developed in the framework of Sobolev spaces.

In Section 6.1 we study linear eigenvalue problems with weight for general linear elliptic differential operators in divergence form. As a particular case we obtain a complete description of the spectrum of the negative Laplacian with Dirichlet or Neumann boundary conditions. We also obtain variational characterizations of the eigenvalues via minimax expressions (Courant's theory). At the end of the section we present the basic maximum principles for linear elliptic partial differential equations (weak and strong (Hopf) maximum principles).

In Section 6.2 we pass to nonlinear elliptic partial differential operators and examine the spectrum of the p-Laplacian. The relevant eigenvalue problem is nonlinear and we use energy methods (critical point theory; see Chapter 5) to study it. We determine the beginning of the spectrum of the negative p-Laplacian with Dirichlet or Neumann boundary conditions and establish the existence of a principal eigenvalue. Our investigations involve also some basic nonlinear regularity results.

In Section 6.3 we examine the ordinary (one dimensional) p-Laplacian differential operator. We consider both the scalar and vector ordinary p-Laplacian differential operators under Dirichlet, Neumann or periodic boundary conditions.

In Section 6.4 we develop versions of the maximum principle when the differential operator is nonlinear and involves the p-Laplacian. We also state the nonlinear Picone's identity and illustrate how it can be used in the study of nonlinear eigenvalue problems.

Finally in Section 6.5 we prove comparison results involving the p-Laplacian. In contrast to the linear case, where the comparison principles are an easy consequence of the maximum principle, the nonlinear case is more complicated due to the degeneracy of the operator.

6.1 Linear Elliptic Operators

In this section we develop the theory of linear eigenvalue problems with weights. This theory is an important tool in the study of semilinear boundary value problems. The theory is based on the spectral properties of compact self-adjoint operators on a Hilbert space. Some first results in this direction were obtained in Section 3.1. Here we continue this study and obtain additional results which eventually will be used in the case of linear elliptic differential operators.

The mathematical framework of our analysis is a Hilbert space H and a linear compact self-adjoint operator $A\colon H \longrightarrow H$ (i.e., $A \in \mathcal{L}_c(H)$ and it is self-adjoint).

PROPOSITION 6.1.1
If $\lambda_1 \stackrel{df}{=} \sup_{\|x\|_H=1} \big(Ax,x\big)_H > 0$,
then there exists $u_1 \in H$ *with* $\|u_1\|_H = 1$, *such that*

$$Au_1 = \lambda_1 u_1 \quad \textit{and} \quad \big(Au_1,u_1\big)_H = \lambda_1.$$

PROOF Clearly $\lambda_1 < +\infty$. Let

$$\{x_n\}_{n\geqslant 1} \subseteq \partial B_1(0) = \big\{x \in H :\ \|x\|_H = 1\big\}$$

be a sequence, such that

$$\big(Ax_n,x_n\big)_H \nearrow \lambda_1.$$

By passing to a subsequence if necessary, we may assume that

$$x_n \stackrel{w}{\longrightarrow} u_1 \quad \text{in } H.$$

Since A is compact, passing to a next subsequence if necessary, we may also assume that

$$Ax_n \longrightarrow Au_1 \quad \text{in } H.$$

Hence $\big(Au_1,u_1\big)_H = \lambda_1$. Let

$$S \stackrel{df}{=} \lambda_1 id_H - A.$$

Then

$$\big(Su_1, u_1\big)_H \;=\; 0 \quad \text{and} \quad \big(Sx, x\big)_H \;\geqslant\; 0 \qquad \forall\, x \in H.$$

Let $h \in H$ and let $x = u_1 + th$ with $t \in \mathbb{R}$. We have

$$\begin{aligned} 0 \;\leqslant\; \big(Sx, x\big)_H \;&=\; \big(Su_1, u_1\big)_H + 2t\big(Su_1, h\big)_H + t^2\big(Sh, h\big)_H \\ &=\; t^2\big(Sh, h\big)_H + 2t\big(Su_1, h\big)_H. \end{aligned}$$

Let $t = r\big(Su_1, h\big)_H$ with $r \in \mathbb{R}$. We obtain

$$0 \;\leqslant\; r\big(Su_1, h\big)_H^2\big[r\big(Sh, h\big)_H + 2\big].$$

Taking $|r| > 0$ small enough, we see that we must have

$$\big(Su_1, h\big)_H \;=\; 0 \qquad \forall\, h \in H.$$

So $Au_1 = \lambda_1 u_1$. ∎

In a similar fashion we show the next proposition.

PROPOSITION 6.1.2
If $\lambda_{-1} = \inf\limits_{\|x\|_H = 1} \big(Ax, x\big)_H < 0$,
then there exists $u_{-1} \in H$ *with* $\|u_{-1}\|_H = 1$, *such that*

$$Au_{-1} = \lambda_{-1} u_{-1} \quad \textit{and} \quad \big(Au_{-1}, u_{-1}\big)_H \;=\; \lambda_{-1}.$$

REMARK 6.1.3 From Proposition 3.1.54 we know that λ_1 is the largest eigenvalue of A and λ_{-1} is the smallest one.

By passing to the orthogonal complements of the eigenspaces $\mathbb{R}u_1$, $\mathbb{R}u_{-1}$, we can produce the other eigenvalues of A in a similar fashion. The whole process is based on the following elementary fact from the theory of bounded linear operators on a Hilbert space.

LEMMA 6.1.4
If V *is an* A*-invariant subspace of* H *(i.e.,* $A(V) \subseteq V$*),*
then $A \in \mathcal{L}_c(V^\perp)$ *and it is self-adjoint.*

PROPOSITION 6.1.5
If $\lambda_n \stackrel{df}{=} \sup\Big\{\big(Ax, x\big)_H :\; \|x\|_H = 1,\; x \perp u_k \text{ for } k \in \{1, \ldots, n-1\}\Big\} > 0$,
then there exists $u_n \in H$ *with* $\|u_n\|_H = 1$, *such that*

$$Au_n \;=\; \lambda_n u_n \quad \textit{and} \quad \big(Au_n, u_n\big)_H \;=\; \lambda_n.$$

PROPOSITION 6.1.6
If $\lambda_{-n} \stackrel{df}{=} \inf\left\{ \left(Ax,x\right)_H : \ \|x\|_H = 1, \ x \perp u_{-k} \text{ for } k \in \{1,\ldots,n-1\} \right\} < 0,$
then there exists $u_{-n} \in H$ *with* $\|u_{-n}\|_H = 1$, *such that*

$$Au_{-n} \ = \ \lambda_{-n} u_{-n} \quad \text{and} \quad \left(Au_{-n}, u_{-n}\right)_H \ = \ \lambda_{-n}.$$

REMARK 6.1.7 The expressions for λ_n and λ_{-n} in the above propositions have the drawback that they require knowledge of all previous positive (respectively negative) eigenvalues. This is remedied in the next two propositions.

PROPOSITION 6.1.8
For every integer $n \geqslant 1$, *we have*

$$\lambda_n \ = \ \inf_{Y \in \mathcal{L}_{n-1}} \ \sup_{\substack{x \,\in\, Y^{\perp} \\ \|x\|_H \,=\, 1}} \ \left(Ax,x\right)_H, \tag{6.1}$$

where $\mathcal{L}_{n-1}$ *is the set of all* $(n-1)$*-dimensional subspaces* Y *of* H.

PROOF Let m_n be the right hand side of (6.1). If

$$Y \ \stackrel{df}{=} \ \text{span}\,\{u_k\}_{k=1}^{n-1},$$

then from Proposition 6.1.5, we infer that $m_n \leqslant \lambda_n$. On the other hand, let $\{v_k\}_{k=1}^{n-1}$ be mutually orthogonal vectors and let us set

$$Y \ \stackrel{df}{=} \ \text{span}\,\{v_k\}_{k=1}^{n-1}.$$

Let

$$u \ \stackrel{df}{=} \ \sum_{i=1}^{n} \beta_i u_i$$

be such that

$$\left(u, v_k\right)_H \ = \ 0 \qquad \forall\, k \in \{1,\ldots,n-1\} \quad \text{and} \quad \sum_{i=1}^{n} \beta_i^2 \ = \ 1$$

(i.e., $\|u\|_H = 1$, the vector is normalized). We have

$$\left(Au,u\right)_H \ = \ \sum_{i=1}^{n} \beta_i^2 \lambda_i \ \geqslant \ \lambda_n,$$

so

$$\sup_{\substack{x \,\in\, Y^{\perp} \\ \|x\|_H \,=\, 1}} \ \left(Ax,x\right)_H \ \geqslant \ \lambda_n \qquad \forall\, Y \in \mathcal{L}_{n-1}$$

and thus $m_n \geqslant \lambda_n$. Therefore we conclude that $\lambda_n = m_n$. ∎

In a similar fashion we obtain analogous expressions for the negative eigenvalues.

PROPOSITION 6.1.9
For every integer $n \geqslant 1$, we have

$$\lambda_{-n} = \sup_{Y \in \mathcal{L}_{n-1}} \inf_{\substack{x \in Y^{\perp} \\ \|x\|_H = 1}} \big(Ax, x\big)_H. \tag{6.2}$$

Still we detect something unsatisfactory in (6.1) and (6.2). Namely in (6.1) (respectively (6.2)), first we perform a maximization (respectively minimization) over an infinite dimensional subspace of H. In the next two propositions we fix this.

PROPOSITION 6.1.10
For every integer $n \geqslant 1$, we have

$$\lambda_n = \sup_{Y \in \mathcal{L}_n} \inf_{\substack{x \in Y \\ \|x\|_H = 1}} \big(Ax, x\big)_H. \tag{6.3}$$

PROOF Let ξ_n be the right hand side of (6.3). Let $Y = \operatorname{span}\{u_k\}_{k=1}^n$. Let $x \in Y$ be such that $\|x\|_H = 1$. We have $x = \sum_{k=1}^n \beta_k u_k \in Y$, with $\|x\|_H^2 = \sum_{k=1}^n \beta_k^2 = 1$. So

$$\big(Ax, x\big)_H = \sum_{k=1}^n \beta_k^2 \lambda_k \geqslant \lambda_n,$$

and hence $\xi_n \geqslant \lambda_n$.

On the other hand, if $Y \in \mathcal{L}_n$, we choose $x \in Y$, $\|x\|_H = 1$, such that

$$x \perp u_k \qquad \forall\, k \in \big\{1, \ldots, n-1\big\}.$$

From Proposition 6.1.5, we know that $\big(Ax, x\big)_H \leqslant \lambda_n$, so $\xi_n \leqslant \lambda_n$. Therefore we conclude that $\lambda_n = \xi_n$. ∎

Similarly for the negative eigenvalues.

PROPOSITION 6.1.11
For every integer $n \geqslant 1$, we have

$$\lambda_{-n} = \inf_{Y \in \mathcal{L}_n} \sup_{\substack{x \in Y \\ \|x\|_H = 1}} \big(Ax, x\big)_H. \tag{6.4}$$

REMARK 6.1.12 In the above characterizations the inf and sup operations are attained. So they can be replaced by min and max.

Next we will use the above abstract results to describe the solutions of a linear eigenvalue problem. So let $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain (i.e., a bounded connected open set). We introduce the linear differential operator in divergence form, defined by

$$Lx \stackrel{df}{=} -\sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}\left(a_{ij}(z)\frac{\partial x}{\partial z_i}\right) + a_0(z)x,$$

where $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, $a_0 \in L^{\frac{N}{2}}(\Omega)$ and $a_0(z) \geqslant 0$ for almost all $z \in \Omega$. We also assume that the operator L is ***uniformly elliptic*** in Ω, namely there exists $c_0 > 0$, such that

$$\sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \geqslant c_0 \|\xi\|^2_{\mathbb{R}^N} \quad \text{for a.a. } z \in \Omega \text{ and all } \xi = (\xi_k)_{k=1}^N \in \mathbb{R}^N.$$

Uniform ellipticity of the operator L means that for almost all $z \in \Omega$, the $N \times N$-matrix $B(z) = \big(a_{ij}(z)\big)_{i,j=1}^N$ is symmetric, positive definite with the smallest eigenvalue greater or equal to $c_0 > 0$. If

$$a_{ij} = \delta_{ij} \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad \forall\, i, j \in \{1, \ldots, N\},$$

then $L = -\Delta$.

We consider the following weighted linear eigenvalue problem:

$$\begin{cases} Lx = \lambda m x & \text{in } \Omega, \\ x|_{\partial\Omega} = 0. \end{cases} \tag{6.5}$$

Here $m\colon \Omega \longrightarrow \mathbb{R}$ is the weight function and it is assumed to belong in $L^\infty(\Omega)$. The important feature of (6.5), which we want to emphasize, is that the weight function m can change sign in Ω.

Consider the bilinear form $b\colon H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$, defined by

$$b(x, y) \stackrel{df}{=} \int_\Omega \left[\sum_{i,j=1}^{N} a_{ij}(z)D_i x D_j y + a_0(z)xy\right] dz.$$

Since by hypothesis $a_0 \in L^{\frac{N}{2}}(\Omega)$, $x, y \in L^{2^*}(\Omega)$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if} \quad N \geqslant 3 \\ +\infty & \text{if} \quad N = 2 \end{cases}$$

and because $\frac{2}{N} + \frac{2}{2^*} = 1$, from the generalized Hölder's inequality (see Theorem A.2.27), we have that $a_0 xy \in L^1(\Omega)$ and so

$$\int_\Omega a_0(z)xy\, dz$$

is well defined.

Note that the bilinear form $b(\cdot,\cdot)$ is:

(a) symmetric, i.e.,

$$b(x,y) = b(y,x) \qquad \forall\ x,y \in H_0^1(\Omega);$$

(b) continuous, since

$$\big|b(x,y)\big| \ \leqslant\ c_1 \|x\|_{H_0^1(\Omega)} \|y\|_{H_0^1(\Omega)},$$

for some $c_1 > 0$;

(c) coercive, since

$$b(x,x) \ \geqslant\ c_2 \|x\|^2_{H_0^1(\Omega)},$$

for some $c_2 > 0$ (by virtue of the strong ellipticity condition and since $a_0 \geqslant 0$).

Hence the bilinear form b defines an equivalent inner product on $H_0^1(\Omega)$. Let $h \in L^2(\Omega)$. Since by hypothesis $m \in L^\infty(\Omega)$, the function

$$y \longmapsto \int_\Omega mhy\,dz$$

is linear continuous on $H_0^1(\Omega)$. So by the Riesz representation theorem, we can find $Ah \in H_0^1(\Omega)$, such that

$$b\big(Ah,y\big) \ =\ \int_\Omega mhy\,dz \qquad \forall\ y \in H_0^1(\Omega).$$

The map $A\colon L^2(\Omega) \longrightarrow H_0^1(\Omega) \subseteq L^2(\Omega)$ is clearly linear, bounded and self-adjoint. Moreover, exploiting the compactness of the embedding $H_0^1(\Omega) \subseteq L^2(\Omega)$, we can easily check that A is compact. Using the operator A, problem (6.5) can be equivalently rewritten as

$$b(x,y) \ =\ \lambda b\big(Ax,y\big)_H \qquad \forall\ y \in H_0^1(\Omega),$$

hence

$$Ax \ =\ \frac{1}{\lambda}x.$$

Since A is compact, self-adjoint on the Hilbert space $H = L^2(\Omega)$, we can apply the earlier abstract results to determine the eigenvalues and eigenfunctions of problem (6.5).

First let us give the definition of eigenvalue and eigenfunction for the differential operator L.

DEFINITION 6.1.13 **(a)** *We say that $x \in H_0^1(\Omega)$ is a* ***weak solution*** *of the boundary value problem (6.5), if*

$$b(x,y) \;=\; \lambda \int_\Omega mxy\,dz \qquad \forall\, y \in H_0^1(\Omega).$$

(b) *We say that $\lambda \in \mathbb{R}$ is an* ***eigenvalue*** *of L, if problem (6.5) has a nontrivial weak solution $x \in H_0^1(\Omega)$. The nonzero solutions are the eigenfunctions corresponding to the eigenvalue λ.*

REMARK 6.1.14 Since

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}\left(a_{ij}(\cdot)\frac{\partial x}{\partial z_i}\right) \;\in\; H^{-1}(\Omega) \qquad \forall\, x \in H_0^1(\Omega)$$

(see Theorem 2.4.57) and $a_0 x \in L^{\frac{2^*}{2^*-1}}(\Omega) = L^{(2^*)'}(\Omega) \subseteq H^{-1}(\Omega)$ ($\frac{1}{2^*} + \frac{1}{(2^*)'} = 1$; generalized Hölder's inequality; see Theorem A.2.27), from Definition 6.1.13**(a)**, it follows that

$$Lx(z) \;=\; \lambda m(z)x(z) \quad \text{for a.a. } z \in Z$$

(strong solution).

From Propositions 6.1.10 and 6.1.11, we have the following one.

PROPOSITION 6.1.15
The linear eigenvalue problem (6.5) has a double sequence of eigenvalues

$$\ldots \leqslant \lambda_{-n} \leqslant \ldots \leqslant \lambda_{-1} < 0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots,$$

which have the following variational characterizations

$$\frac{1}{\lambda_n} \;=\; \sup_{Y \in \mathcal{L}_n} \inf_{\substack{x \in Y \\ \|x\|_2 = 1}} \int_\Omega mx^2\,dz, \tag{6.6}$$

$$\frac{1}{\lambda_{-n}} \;=\; \inf_{Y \in \mathcal{L}_n} \sup_{\substack{x \in Y \\ \|x\|_2 = 1}} \int_\Omega mx^2\,dz. \tag{6.7}$$

The corresponding eigenfunctions u_n are such that

$$b(u_n,y) \;=\; \lambda_n \int_\Omega mu_n y\,dz \qquad \forall\, y \in H_0^1(\Omega),$$

$$\|u_n\|_2 \;=\; 1 \quad \textit{and} \quad \frac{1}{\lambda_n} \;=\; \int_\Omega mu_n^2\,dz$$

and similarly for the eigenfunctions of λ_{-n}.

REMARK 6.1.16 These eigenvalues do not have accumulation points, except $+\infty$ and $-\infty$.

In what follows

$$\Omega_+ \stackrel{df}{=} \big\{z \in \Omega :\ m(z) > 0\big\}, \quad \Omega_- \stackrel{df}{=} \big\{z \in \Omega :\ m(z) < 0\big\}$$

and by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

PROPOSITION 6.1.17
(a) *If* $|\Omega_+|_N = 0$, *then there is no* λ_n *for* $n \geqslant 1$;

(b) *If* $|\Omega_-|_N = 0$, *then there is no* λ_{-n} *for* $n \geqslant 1$;

(c) *If* $|\Omega_+|_N > 0$, *then* $\lambda_n \longrightarrow +\infty$;

(d) *If* $|\Omega_-|_N > 0$, *then* $\lambda_{-n} \longrightarrow -\infty$.

PROOF **(a)** and **(b)** follow at once from Proposition 6.1.15.

(c) Let $B_1, \ldots, B_n$ be a family of pairwise disjoint open balls in Ω, such that

$$\big|B_i \cap \Omega_+\big|_N \ > \ 0 \qquad \forall\, i \in \big\{1, \ldots, n\big\}.$$

Let $\{u_i\}_{i=1}^n \subseteq C_c^\infty(\Omega)$, with

$$\operatorname{supp} u_i \ \subseteq \ B_i \quad \text{and} \quad \int\limits_\Omega m u_i^2 \, dz \ = \ 1.$$

Then let us set

$$Y \ \stackrel{df}{=} \ \operatorname{span}\{u_i\}_{i=1}^n \in \mathcal{L}_n.$$

For $x = \sum\limits_{i=1}^n \beta_i u_i \in Y$, we have

$$\int\limits_\Omega m x^2 \, dz \ = \ \sum_{i=1}^n \beta_i^2 \int\limits_\Omega m u_i^2 \, dz \ = \ \sum_{i=1}^n \beta_i^2$$

and

$$b(x,x) \ = \ \sum_{i=1}^n \beta_i^2 b(u_i, u_i) \ \leqslant \ c \sum_{i=1}^n \beta_i^2,$$

with $c = \max\limits_{i \in \{1,\ldots,n\}} b(u_i, u_i)$. Therefore

$$\int\limits_\Omega m x^2 \, dz \ \geqslant \ \frac{1}{c} b(x,x) \qquad \forall\ x \in Y \in \mathcal{L}_n.$$

Then the result follows from this inequality and **(b)**.

(d) The proof is similar to that of **(c)** using this time (6.7). ∎

The eigenvalues $\{\lambda_{\pm n}\}_{n\geqslant 1}$ depend on the weight function $m \in L^\infty(\Omega)$. Proposition 6.1.15 establishes the basic features of this dependence.

PROPOSITION 6.1.18
(a) *If* $m_1, m_2 \in L^\infty(\Omega)$, $m_1 \leqslant m_2$ *and* $\lambda_{\pm n}(m_1)$, $\lambda_{\pm n}(m_2)$ *exist, then* $\lambda_{\pm n}(m_2) \leqslant \lambda_{\pm n}(m_1)$.

(b) *If the inequality* $m_1 \leqslant m_2$ *is strict on a set of positive measure, then* $\lambda_{\pm n}(m_2) < \lambda_{\pm n}(m_1)$.

Next let us examine the eigenfunctions corresponding to these eigenvalues. First, from the spectral theorem, for compact, self-adjoint operators (see Theorem 3.1.57), we have the following result.

PROPOSITION 6.1.19
The eigenfunctions $\{u_n\}_{n\geqslant 1}$ *and* $\{u_{-n}\}_{n\geqslant 1}$ *form orthonormal bases for* $L^2(\Omega)$ *and* $\left\{\frac{u_n}{\sqrt{\lambda_n}}\right\}_{n\geqslant 1}$ *and* $\left\{\frac{u_{-n}}{\sqrt{\lambda_{-n}}}\right\}_{n\geqslant 1}$ *are orthonormal bases of* $H_0^1(\Omega)$ *equipped with the inner product* $b(\cdot,\cdot)$.

The next result is a theorem of the Krein-Rutman-type for the eigenvalue problem (6.5). First let us recall the Krein-Rutman theorem.

THEOREM 6.1.20 (Krein-Rutman Theorem)
If X *is an ordered Banach space with positive cone* K, $X = \overline{K - K}$, $A \in \mathcal{L}_c(X)$, $A(K) \subseteq K$ *(i.e.,* A *is a positive operator) and*

$$r(A) \stackrel{df}{=} \max_{\lambda\in\sigma_X(A)} |\lambda| > 0,$$

then $r(A)$ *is an eigenvalue with a positive eigenvector.*

In the context of the eigenvalue problem (6.5) this theorem takes the following form.

THEOREM 6.1.21
(a) *If* $|\Omega_+|_N > 0$, *then* $\lambda_1 > 0$ *is simple and* $u_1(z) > 0$ *for almost all* $z \in \Omega$.

(b) *If* $|\Omega_-|_N > 0$, *then* $\lambda_{-1} < 0$ *is simple and* $u_1(z) > 0$ *for almost all* $z \in \Omega$.

PROOF **(a)** Let $u \in H_0^1(\Omega)$ be an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. At this stage we do not know whether or not some eigenvalues λ_i are equal to λ_1 and so in general u is a linear combination of u_i's. We have

$$b(u,v) = \lambda_1 \int_\Omega muv\,dz \qquad \forall\, v \in H_0^1(\Omega).$$

Suppose that u does not have a constant sign. Then $u^+, u^- \neq 0$ and from Proposition 2.4.27, we know that $u^+, u^- \in H_0^1(\Omega)$ and

$$b(u,u) \;=\; b(u^+,u^+) + b(u^-,u^-).$$

We have that

$$b(u^+,u^+) \;>\; 0 \quad \text{and} \quad b(u^-,u^-) \;>\; 0.$$

Note that

$$\frac{1}{\lambda_1} \;=\; \frac{\int_\Omega mu^2\,dz}{b(u,u)} \;=\; \frac{\int_\Omega m(u^+)^2\,dz + \int_\Omega m(u^-)^2\,dz}{b(u^+,u^+) + b(u^-,u^-)}.$$

By virtue of Proposition 6.1.1, it follows that

$$\frac{1}{\lambda_1} \;=\; \frac{\int_\Omega m(u^+)^2\,dz}{b(u^+,u^+)} \quad \text{and} \quad \frac{1}{\lambda_1} \;=\; \frac{\int_\Omega m(u^-)^2\,dz}{b(u^-,u^-)},$$

so u^+ and u^- are both eigenfunctions corresponding to $\lambda_1 > 0$. Then from Stampacchia (1966, p. 238) (if $m \geqslant 0$, $m \neq 0$, from the classical maximum principle), we obtain that

$$u^+(z) \;>\; 0 \quad \text{and} \quad u^-(z) \;>\; 0 \quad \text{for a.a. } z \in \Omega,$$

a contradiction. This proves that u has a constant sign and so we may assume that

$$u(z) \;\geqslant\; 0 \quad \text{for a.a. } z \in \Omega.$$

As above from Stampacchia (1966) (or the maximum principle if $m \geqslant 0$, $m \neq 0$), we infer that

$$u(z) \;>\; 0 \quad \text{for a.a. } z \in \Omega.$$

Now let us show that $\lambda_1 > 0$ is simple, i.e., it has multiplicity 1. To this end let $\widehat{u}_1$ and $\widehat{u}_2$ be two eigenfunctions corresponding to the eigenvalue $\lambda_1 > 0$. For every $\mu \in \mathbb{R}$, the function $\widehat{u}_1 + \mu\widehat{u}_2$ has constant sign. Let

$$C_+ \;\stackrel{df}{=}\; \big\{\mu \in \mathbb{R} :\; \widehat{u}_1 + \mu\widehat{u}_2 \geqslant 0\big\},$$
$$C_- \;\stackrel{df}{=}\; \big\{\mu \in \mathbb{R} :\; \widehat{u}_1 + \mu\widehat{u}_2 \leqslant 0\big\}.$$

Evidently the sets C_+ and C_- are nonempty, closed and

$$C_+ \cup C_- \;=\; \mathbb{R}.$$

So we can find $\mu_0 \in C_+ \cap C_-$. Then

$$\widehat{u}_1 + \mu_0\widehat{u}_2 \;=\; 0,$$

which shows that $\widehat{u}_1$ and $\widehat{u}_2$ are linearly dependent, i.e., the eigenspace corresponding to $\lambda_1 > 0$ is one dimensional.

(b) The proof is similar to that of **(a)**. □

REMARK 6.1.22 The simplicity of $\lambda_1 > 0$ and $\lambda_{-1} > 0$ implies that

$$0 \;<\; \lambda_1 \;<\; \lambda_2 \;\leqslant\; \lambda_3 \ldots$$

and

$$\ldots \;\leqslant\; \lambda_{-3} \;\leqslant\; \lambda_{-2} \;<\; \lambda_{-1} \;<\; 0.$$

This analysis produces as a special case a complete description of the eigenelements of the negative Laplacian with Dirichlet boundary conditions, i.e., of $\big(-\Delta, H_0^1(\Omega)\big)$. So we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta x(z) = \lambda x(z) \quad \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \tag{6.8}$$

with $\lambda \in \mathbb{R}$.

THEOREM 6.1.23
Problem (6.8) has countably many eigenvalues

$$0 \;<\; \lambda_1 \;<\; \lambda_2 \;\leqslant\; \lambda_3 \;\leqslant\; \ldots,$$

with $\lambda_n \longrightarrow +\infty$ *and an orthonormal basis* $\{u_n\}_{n\geqslant 1}$ *of* $L^2(\Omega)$ *of eigenfunctions, with* $\left\{\frac{u_n}{\sqrt{\lambda_n}}\right\}_{n\geqslant 1}$ *being an orthonormal basis of* $H_0^1(\Omega)$. *The eigenvalues* $\{\lambda_n\}_{n\geqslant 1}$ *have the following variational characterizations:*

$$\lambda_1 \;=\; \min_{\substack{x \,\in\, H_0^1(\Omega) \\ x \,\neq\, 0}} \frac{\|Dx\|_2^2}{\|x\|_2^2}, \tag{6.9}$$

$$\lambda_n \;=\; \min_{\substack{x \,\in\, H_0^1(\Omega) \\ x \,\perp\, \{u_1, \ldots, u_{n-1}\}}} \frac{\|Dx\|_2^2}{\|x\|_2^2}, \tag{6.10}$$

$$\lambda_n \;=\; \min_{Y \in \mathcal{L}_n} \max_{\substack{x \,\in\, Y \\ x \,\neq\, 0}} \frac{\|Dx\|_2^2}{\|x\|_2^2}. \tag{6.11}$$

The first eigenvalue is simple and the corresponding eigenfunction does not change sign and so we can choose u_1, *such that* $u_1 \geqslant 0$.

PROPOSITION 6.1.24
If $u \in H_0^1(\Omega) \setminus \{0\}$ *satisfies the equation*

$$\|Du\|_2^2 \;=\; \lambda_1 \|u\|_2^2,$$

then u *is an eigenfunction corresponding to* λ_1, *i.e.,*

$$u \;=\; \vartheta u_1,$$

for some $\vartheta \in \mathbb{R} \setminus \{0\}$.

PROOF Let $y \in H_0^1(\Omega)$ and $t > 0$ be such that

$$u + ty \;\neq\; 0.$$

Then $u + ty \in H_0^1(\Omega)$ and by (6.9), we have

$$\frac{\|Du\|_2^2}{\|u\|_2^2} \;=\; \lambda_1 \;\leqslant\; \frac{\|D(u+ty)\|_2^2}{\|u+ty\|_2^2}.$$

Without any loss of generality, we may assume that $\|u\|_2 = 1$. So

$$\lambda_1\left(2t\int_\Omega uy\,dz + t^2\,\|y\|_2^2\right) \;\leqslant\; 2t\int_\Omega \big(Du, Dy\big)_{\mathbb{R}^N}\,dz + t^2\,\|Dy\|_2^2\,.$$

Dividing with $2t$ and then letting $t \to 0^+$, we obtain

$$\lambda_1\int_\Omega uy\,dz \;\leqslant\; \int_\Omega \big(Du, Dy\big)_{\mathbb{R}^N}\,dz \qquad \forall\; y \in H_0^1(\Omega),$$

so

$$\begin{cases} -\Delta u(z) = \lambda_1 u(z) & \text{for a.a. } z \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Therefore $u = \vartheta u_1$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$. □

We can say more about the eigenfunctions of $\big(-\Delta, H_0^1(\Omega)\big)$, but to do this we need to recall the basic regularity result from elliptic equations. Detailed proofs can be found in Evans (1998, Section 6.3) and Gilbarg & Trudinger (2001, Sections 8.3 and 8.4).

For a bounded domain $\Omega \subseteq \mathbb{R}^N$, we consider the following boundary value problem:

$$L_0 x \;=\; h \quad \text{in } \Omega, \tag{6.12}$$

where L_0 is the linear differential operator in divergence form, given by

$$L_0 x \;\stackrel{df}{=}\; -\sum_{i,j=1}^N \frac{\partial}{\partial z_j}\left(a_{ij}(z)\frac{\partial x}{\partial z_i}\right) + \sum_{i=1}^N b_i(z)\frac{\partial x}{\partial z_i} + a_0(z)x.$$

The first result is about interior (local) regularity of the weak solutions of problem (6.12).

THEOREM 6.1.25
If $a_{ij} \in C(\Omega)$, $b_i, a_0 \in L^\infty(\Omega)$ for $i, j \in \{1, \ldots, N\}$, $h \in L^2(\Omega)$ and $x \in H^1(\Omega)$ is a weak solution of (6.12),
then $x \in H^2_{\mathrm{loc}}(\Omega)$ (i.e., $x \in H^2(\Omega')$ for each open set $\Omega' \subset\subset \Omega$) and for each open set $\Omega' \subset\subset \Omega$, we have the estimate

$$\|x\|_{H^2(\Omega')} \;\leqslant\; c\left(\|h\|_{L^2(\Omega)} + \|x\|_{L^2(\Omega)}\right)$$

and the constant $c > 0$ depends only on $\Omega' \subset\subset \Omega$ and the coefficients of L_0.

REMARK 6.1.26 Note that we do not require $x \in H_0^1(\Omega)$, i.e., (6.12) need not be a Dirichlet problem. Since $x \in H^2_{\mathrm{loc}}(\Omega)$, we have

$$L_0 x(z) \;=\; h(z) \quad \text{for a.a. } z \in \Omega.$$

By strengthening the regularity of the coefficients and of the right hand side, we can improve the local regularity of the weak solution of (6.12). Namely, if

$$a_{ij}, b_i, a_0 \in C^{m+1}(\Omega) \qquad \forall\, i,j \in \{1,\dots,N\} \quad \text{and} \quad h \in H^m(\Omega),$$

then $x \in H^{m+2}_{\mathrm{loc}}(\Omega)$ and for each $\Omega' \subset\subset \Omega$, we have

$$\|x\|_{H^{m+2}(\Omega')} \;\leqslant\; c\left(\|h\|_{H^m(\Omega)} + \|x\|_{H^m(\Omega)}\right),$$

with $c > 0$ depending only on m, Ω', Ω and the coefficients of L_0.

Finally, if

$$a_{ij}, b_i, a_0 \in C^{\infty}(\Omega) \qquad \forall\, i,j \in \{1,\dots,N\} \quad \text{and} \quad h \in C^{\infty}(\Omega),$$

then the weak solutions $x \in H^1(\Omega)$ of (6.12) are in $C^{\infty}(\Omega)$. Again we emphasize that no assumptions are made about $\partial\Omega$. So any possible singularities of u on the boundary $\partial\Omega$ do not propagate in the interior.

Now we examine regularity of the weak solutions of (6.12) up to the boundary.

THEOREM 6.1.27
If $a_{i,j} \in C^1(\overline{\Omega})$, $b_i, a_0 \in L^{\infty}(\Omega)$ *for* $i,j \in \{1,\dots,N\}$, $h \in L^2(\Omega)$, $\partial\Omega$ *is a* C^2*-manifold and* $x \in H_0^1(\Omega)$ *is a weak solution of (6.12) with Dirichlet boundary conditions,*
then $x \in H^2(\Omega)$ *and*

$$\|x\|_{H^2(\Omega)} \;\leqslant\; c\left(\|h\|_{L^2(\Omega)} + \|x\|_{L^2(\Omega)}\right),$$

with $c > 0$ *depending only on* Ω *and the coefficients of* L_0.

REMARK 6.1.28 If the weak solution $x \in H_0^1(\Omega)$ is unique, then the estimate simplifies and becomes

$$\|x\|_{H^2(\Omega)} \;\leqslant\; c\,\|h\|_{L^2(\Omega)}.$$

Again by strengthening the regularity of the coefficients and of the boundary $\partial\Omega$, we can improve the conclusion of the theorem. So if $a_{i,j}, b_i, a_0 \in C^{m+1}(\overline{\Omega})$ for $i,j \in \{1,\dots,N\}$, $h \in H^m(\Omega)$ and $\partial\Omega$ is a C^{m+2}-manifold, then $x \in H^{m+2}(\Omega)$ and

$$\|x\|_{H^{m+2}(\Omega)} \;\leqslant\; c\left(\|h\|_{H^m(\Omega)} + \|x\|_{L^2(\Omega)}\right),$$

with $c > 0$ depending only on m, Ω and the coefficients of L_0. Moreover, if the weak solution is unique, then

$$\|x\|_{H^{m+2}(\Omega)} \leqslant c\|h\|_{H^m(\Omega)}.$$

Finally, if $a_{i,j}, b_i, a_0 \in C^\infty(\overline{\Omega})$ for $i, j \in \{1, \dots, N\}$, $h \in L^\infty(\Omega)$ and $\partial\Omega$ is a C^∞-manifold, then $x \in C^\infty(\overline{\Omega})$.

Having these regularity results, we can complete Theorem 6.1.23.

THEOREM 6.1.29
The eigenfunctions $\{u_n\}_{n\geqslant 1}$ of problem (6.8) obtained in Theorem 6.1.23 belong in $H_0^1(\Omega) \cap C^\infty(\Omega)$. Moreover, if $\partial\Omega$ is a C^∞-manifold, then

$$u_n \in C^\infty(\overline{\Omega}) \qquad \forall\, n \geqslant 1.$$

REMARK 6.1.30 Any $x \in L^2(\Omega)$ can be expanded in terms of these eigenfunctions,

$$x = \sum_{k=1}^{\infty} (x, u_k)_2 \, u_k,$$

where by $(\cdot,\cdot)_2$ we denote the inner product in $L^2(\Omega)$, i.e.,

$$(g, h)_2 \stackrel{df}{=} \int_\Omega g(z)h(z)\, dz \qquad \forall\, g, h \in L^2(\Omega).$$

Moreover, if $x \in H_0^1(\Omega)$, then

$$\begin{aligned} (Dx, Dx)_2 &= \int_\Omega \big(Dx(z), Dx(z)\big)_{\mathbb{R}^N} dz \\ &= \sum_{k=1}^{\infty} \lambda_k \, (x, u_k)_2^2 . \end{aligned}$$

So those $x \in L^2(\Omega)$ which do not belong in $H_0^1(\Omega)$ can be characterized by the fact that the series

$$\sum_{k=1}^{\infty} \lambda_k \, (x, u_k)_2^2$$

diverges.

In fact in a similar fashion, we can have the spectrum of the negative Laplacian with Neumann boundary conditions (i.e., of $\big(-\Delta, H^1(\Omega)\big)$). So we consider the following linear eigenvalue problem on the bounded domain Ω with C^1-boundary (so that the divergence theorem holds; see Theorem A.4.1):

$$\begin{cases} -\Delta x(z) = \lambda x(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial x}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.13}$$

THEOREM 6.1.31
If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^1-boundary $\partial\Omega$, then the eigenvalue problem (6.13) has countably many eigenvalues

$$0 \;=\; \lambda_0 \;<\; \lambda_1 \;\leqslant\; \ldots \;\leqslant\; \lambda_k \;\leqslant\; \ldots,$$

with $\lambda_k \longrightarrow +\infty$ and corresponding eigenfunctions $\{u_k\}_{k\geqslant 0}$ which form an orthonormal basis of $L^2(\Omega)$, while $\left\{u_0, \frac{u_k}{\sqrt{\lambda_k}}\right\}_{k\geqslant 1}$ form an orthonormal basis of $H^1(\Omega)$. The elements $\left\{(\lambda_k, u_k)\right\}_{k\geqslant 0}$ have the following variational characterizations:

$$\begin{aligned}
\lambda_{k-1} &= \min_{Y\in\mathcal{L}_k} \max_{\substack{x\,\in\,Y\\ x\,\neq\,0}} \frac{\|Dx\|_2^2}{\|x\|_2^2} \\
&= \max_{Y\in\mathcal{L}_{k-1}} \min_{\substack{x\,\in\,Y^\perp\\ x\,\neq\,0}} \frac{\|Dx\|_2^2}{\|x\|_2^2} \\
&= \frac{\|Du_{k-1}\|_2^2}{\|u_{k-1}\|_2^2}.
\end{aligned}$$

REMARK 6.1.32 Any $x \in L^2(\Omega)$ can be expanded in terms of these eigenfunctions, namely

$$x \;=\; \sum_{k=0}^{\infty} (x, u_k)_2 \, u_k.$$

Moreover, if $x \in H^1(\Omega)$, then

$$\begin{aligned}
\big(Dx, Dx\big)_2 &= \int_\Omega \big(Dx(z), Dx(z)\big)_{\mathbb{R}^N}\, dz \\
&= \sum_{k=0}^{\infty} \lambda_k \, (x, u_k)_2^2 \,.
\end{aligned}$$

So those $x \in L^2(\Omega)$ which do not belong to $H^1(\Omega)$ can be characterized by the fact that the series

$$\sum_{k=0}^{\infty} \lambda_k \, (x, u_k)_2^2 \quad \text{diverges.}$$

For both eigenvalue problems (6.8) and (6.13), the eigenvalues depend monotonically on the domain Ω, i.e., if $\Omega_1 \subseteq \Omega_2$, then

$$\lambda_k(\Omega_2) \leqslant \lambda_k(\Omega_1) \qquad \forall\; k \geqslant 1.$$

Next we prove the maximum principles for second order elliptic partial differential operators. The starting point of the maximum principles is the following fact from calculus. If Ω is an open set, $\varphi \in C^2(\Omega)$ and φ attains its maximum on Ω at $z_0 \in \Omega$, then

$$\nabla\varphi(z_0) \;=\; 0 \quad \text{and} \quad D^2\varphi(z_0) \;=\; \left(\frac{\partial\varphi(z_0)}{\partial z_i \partial z_j}\right)_{i,j=1}^{N} \;\leqslant\; 0. \tag{6.14}$$

The relations (6.14) are pointwise and so are the deductions based on them. For this reason the methods differ from the integral-based energy methods in the study of elliptic partial differential equations. Hence, we will consider an elliptic differential operator L in nondivergence form and we will assume that the solutions are C^2. So the linear differential operator L has the following form:

$$Lx(z) \;\stackrel{df}{=}\; -\sum_{i,j=1}^{N} a_{ij}(z)\frac{\partial^2 x}{\partial z_i \partial z_j}(z) + \sum_{i=1}^{N} b_i(z)\frac{\partial x}{\partial z_i}(z) + a_0(z)x(z). \tag{6.15}$$

We assume the following conditions concerning the coefficients in the operator L:

$\underline{H(L)}$

(*i*) $a_{ij} \in C(\Omega) \cap L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for $i, j \in \{1, \ldots, N\}$;

(*ii*) there exists constant $c_0 > 0$, such that for almost all $z \in \Omega$ and all $\xi = (\xi_i)_{i=1}^N \in \mathbb{R}^N$, we have

$$\sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \;\geqslant\; c_0 \,\|\xi\|_{\mathbb{R}^N}^2$$

(uniform ellipticity condition);

(*iii*) $b_i, a_0 \in C(\Omega) \cap L^\infty(\Omega)$ for $i \in \{1, \ldots, N\}$.

REMARK 6.1.33 If

$$a_{ij} \;\in\; C^1(\Omega) \qquad \forall\, i, j \in \{1, \ldots, N\},$$

then any linear differential operator in divergence form can be put in the nondivergence form (6.15).

The next two theorems are called "Weak Minimum Principles." In the first we require that $a_0 \equiv 0$ and in the second that $a_0 \geqslant 0$.

THEOREM 6.1.34 (Weak Maximum Principle I)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypothesis $H(L)$ with $a_0 \equiv 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$Lx(z) \leqslant 0 \qquad \forall\, z \in \Omega,$$

then x attains its maximum on $\partial\Omega$, i.e.,

$$\max_{z\in\overline{\Omega}} x(z) = \max_{z\in\partial\Omega} x(z).$$

PROOF First we consider the case that

$$Lx(z) < 0 \quad \text{for a.a. } z \in \Omega.$$

We argue indirectly. Suppose that at some $z_0 \in \Omega$, the function x attains its maximum. We have

$$\nabla x(z_0) = 0 \quad \text{and} \quad D^2x(z_0) \leqslant 0.$$

By hypotheses $H(L)(i)$ and (ii), the matrix $A = \big(a_{ij}(z_0)\big)_{i,j=1}^N$ is symmetric and positive definite, hence it is diagonalizable, i.e., there exists an orthogonal $N \times N$-matrix $U = (u_{ij})_{i,j=1}^N$, such that

$$UAU^T = \operatorname{diag}(d_k)_{k=1}^N, \tag{6.16}$$

with $d_k > 0$ for all $k \in \{1, \dots, N\}$.

Let

$$v \stackrel{df}{=} z_0 + U(z - z_0) \in \mathbb{R}^N.$$

Then

$$z - z_0 = U^{-1}(v - z_0)$$

and so

$$\frac{\partial x}{\partial z_i} = \sum_{k=1}^N \frac{\partial x}{\partial v_k} u_{ki} \qquad \forall\, i \in \{1, \dots, N\}$$

and

$$\frac{\partial^2 x}{\partial z_i \partial z_j} = \sum_{k,m=1}^N \frac{\partial^2 x}{\partial v_k \partial v_m} u_{ki} u_{mj} \qquad \forall\, i, j \in \{1, \dots, N\}.$$

So, using also (6.16), at $z_0 \in \Omega$, we have

$$\begin{aligned} \sum_{i,j=1}^N a_{ij}(z_0) \frac{\partial^2 x}{\partial z_i \partial z_j}(z_0) &= \sum_{i,j=1}^N \sum_{k,m=1}^N a_{ij}(z_0) \frac{\partial^2 x}{\partial v_k \partial v_m}(z_0) u_{ki} u_{mj} \\ &= \sum_{k=1}^N d_k \frac{\partial^2 x}{\partial v_k^2}(z_0) \leqslant 0, \end{aligned} \tag{6.17}$$

since $d_k > 0$ and $\frac{\partial^2 x}{\partial v_k^2}(z_0) \leqslant 0$.

Using (6.17) and recalling that

$$\nabla x(z_0) \;=\; \left(\frac{\partial x}{\partial z_i}(z_0)\right)_{i=1}^{N} \;=\; 0,$$

we have

$$Lx(z_0) \;=\; -\sum_{i,j=1}^{N} a_{ij}(z_0)\frac{\partial^2 x}{\partial z_i \partial z_j}(z_0) + \sum_{i=1}^{N} b_i(z_0)\frac{\partial x}{\partial z_i}(z_0) \;\geqslant\; 0. \tag{6.18}$$

But by hypothesis $Lx(z_0) < 0$, a contradiction to (6.18).

Now for the general case assume that

$$Lx(z) \;\leqslant\; 0 \qquad \forall\, z \in \Omega.$$

Let $\varepsilon > 0$ and $\lambda > 0$ and define

$$x_\varepsilon(z) \;\stackrel{df}{=}\; x(z) + \varepsilon e^{\lambda z_1} \qquad \forall\, z \in \Omega.$$

From the uniform ellipticity condition (see hypothesis $H(L)(ii)$), we have

$$\sum_{i,j=1}^{N} a_{ij}(z_0)\xi_i\xi_j \;\geqslant\; c_0 \left\|\xi\right\|_{\mathbb{R}^N}^2 \qquad \forall\, \xi = (\xi_i)_{i=1}^N \in \mathbb{R}^N.$$

Taking

$$\xi \;= e_i \;=\; (0, \ldots, 0, 1, 0, \ldots, 0),$$

we obtain that

$$a_{ii}(z_0) \;\geqslant\; c_0 \qquad \forall\, i \in \{1, \ldots, N\}.$$

Then

$$\begin{aligned} Lx_\varepsilon(z) \;&=\; Lx(z) + \varepsilon L e^{\lambda z_1} \;\leqslant\; \varepsilon e^{\lambda z_1}\big[-\lambda^2 a_{11}(z_0) + \lambda b_1(z_0)\big] \\ &\leqslant\; \varepsilon e^{\lambda z_1}\big[-\lambda^2 c_0 + \lambda\big\|\widehat{b}\big\|_\infty\big] \;<\; 0 \qquad \forall\, z \in \Omega, \end{aligned}$$

where $\widehat{b} = (b_i)_{i=1}^N$, provided we choose $\lambda > 0$ large enough. Then from the first part of the proof, we have that

$$\max_{z \in \overline{\Omega}} x_\varepsilon(z) \;=\; \max_{z \in \partial\Omega} x_\varepsilon(z).$$

Letting $\varepsilon \searrow 0$, we conclude that

$$\max_{z \in \overline{\Omega}} x(z) \;=\; \max_{z \in \partial\Omega} x(z).$$

□

COROLLARY 6.1.35
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ with $a_0 \equiv 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$Lx(z) \geqslant 0 \qquad \forall\, z \in \Omega,$$

then x attains its minimum on $\partial\Omega$, i.e.,

$$\min_{z \in \overline{\Omega}} x(z) = \min_{z \in \partial\Omega} x(z).$$

COROLLARY 6.1.36
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ with $a_0 \equiv 0$, $x, y \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$Lx(z) \leqslant Ly(z) \qquad \forall\, z \in \Omega,$$

$$x(z) \leqslant y(z) \qquad \forall\, z \in \partial\Omega,$$

then

$$x(z) \leqslant y(z) \qquad \forall\, z \in \overline{\Omega}.$$

Moreover, if $Lx(z) = Ly(z)$ for all $z \in \Omega$ and $x(z) = y(z)$ for all $z \in \partial\Omega$, then

$$x(z) = y(z) \qquad \forall\, z \in \overline{\Omega}.$$

COROLLARY 6.1.37
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ and we consider the Dirichlet problem

$$\begin{cases} Lx(z) = h(z) & \forall\, z \in \Omega, \\ x|_{\partial\Omega} = g, \end{cases}$$

with $h \in C(\Omega)$ and $g \in C(\partial\Omega)$,
then the Dirichlet problem has a unique solution $x \in C^2(\Omega) \cap C(\overline{\Omega})$ (classical solution).

Motivated from Theorem 6.1.34 and Corollary 6.1.35, we make the following definition.

DEFINITION 6.1.38 *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let L be as in (6.15). A function $y \in C^2(\Omega)$ is said to be a **lower solution** or **subsolution** of the equation*

$$Lx(z) = 0 \qquad \forall\, z \in \Omega,$$

*if $Ly(z) \leqslant 0$ for all $z \in \Omega$ and is said to be an **upper solution** or **supersolution**, if $Ly(z) \geqslant 0$ for all $z \in \Omega$. In the particular case where $L = -\Delta$ (i.e., $a_{ij} = \delta_{ij}$, $b_i = 0$, $a_0 = 0$ for all $i, j \in \{1, \ldots, N\}$), then a lower solution is called **subharmonic** and an upper solution is called **superharmonic**.*

REMARK 6.1.39 The above nomenclature when $L = -\Delta$ is motivated by the following simple observation. Let $x \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$ be a harmonic function, i.e.,

$$-\Delta x(z) \;=\; 0 \qquad \forall\, z \in \Omega.$$

If $y \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$ is a lower solution (i.e., $-\Delta y(z) \leqslant 0$ for all $z \in \Omega$), such that $y = x$ on $\partial\Omega$, then by Corollary 6.1.36, we have

$$y(z) \;\leqslant\; x(z) \qquad \forall\, z \in \overline{\Omega}.$$

Therefore a subharmonic function always lies below a harmonic function with the same boundary values. Similarly for superharmonic functions.

In the next Theorem, we remove the restriction that $a_0 \equiv 0$.

THEOREM 6.1.40 (Weak Maximum Principle II)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ with $a_0(z) \geqslant 0$ for all $z \in \Omega$, $x \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$ and

$$Lx(z) \;\leqslant\; 0 \qquad \forall\, z \in \Omega$$

(i.e., x is a lower solution for $Lx = 0$),
then

$$\max_{z \in \overline{\Omega}} x(z) \;\leqslant\; \max_{z \in \partial\Omega} x^+(z),$$

where $x^+ \stackrel{df}{=} \max\big\{x, 0\big\}$.

PROOF Consider the open set

$$V \;\stackrel{df}{=}\; \big\{z \in \Omega :\; x(z) > 0\big\}.$$

If $L_1 x = Lx - a_0 x$, we have

$$L_1 x(z) \;\leqslant\; -a_0(z)x(z) \;\leqslant\; 0 \qquad \forall\, z \in V.$$

Note that the operator L_1 has no zero order term. So if $V \neq \emptyset$, we can apply Theorem 6.1.34 and obtain

$$\max_{z \in \overline{V}} x(z) \;=\; \max_{z \in \partial V} x(z) \;=\; \max_{z \in \partial\Omega} x^+(z),$$

hence

$$\max_{z \in \overline{\Omega}} x(z) \;=\; \max_{z \in \partial\Omega} x^+(z).$$

If $V = \emptyset$, then $x \leqslant 0$ on Ω and so

$$\max_{z \in \overline{\Omega}} x(z) \;\leqslant\; \max_{z \in \partial\Omega} x^+(z).$$

$\square$

COROLLARY 6.1.41
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ with $a_0(z) \geqslant 0$ for all $z \in \Omega$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$ and $Lx(z) \geqslant 0$ for all $z \in \Omega$ (i.e., x is an upper solution for $Lx = 0$),
then

$$\min_{z \in \overline{\Omega}} x(z) \ \geqslant \ -\max_{z \in \partial\Omega} x^-(z),$$

where $x^- \stackrel{df}{=} \max\{-x, 0\}$.

REMARK 6.1.42 In particular if $Lx(z) = 0$ for all $z \in \Omega$, then

$$\max_{z \in \overline{\Omega}} |x(z)| \ = \ \max_{z \in \partial\Omega} |x(z)|.$$

Next we prove the so-called "Strong Maximum Principle." First we establish an interesting auxiliary result known as Hopf's lemma.

LEMMA 6.1.43 (Hopf Lemma)
(a) *If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, L satisfies hypotheses $H(L)$ with $a_0 \equiv 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$Lx(z) \ \leqslant \ 0 \qquad \forall \ z \in \Omega,$$

$z_0 \in \partial\Omega$,

$$x(z_0) \ > \ x(z) \qquad \forall \ z \in \Omega$$

and there exists a ball $B \subseteq \Omega$, such that $z_0 \in \partial B$ (interior ball condition at $z_0 \in \partial\Omega$),
then $\frac{\partial u}{\partial n}(z_0) > 0$ with n being the outer unit normal to B at z_0.

(b) *If the hypotheses of* **(a)** *hold but now $a_0(z) \geqslant 0$ for all $z \in \Omega$,*
then the same condition is valid provided $x(z_0) \geqslant 0$.

PROOF Suppose that

$$a_0(z) \ \geqslant \ 0 \qquad \forall \ z \in \Omega.$$

Also let $B = B_r(\overline{z})$ for some $\overline{z} \in \Omega$ and $r > 0$. We introduce the auxiliary function

$$u(z) \ = \ e^{-\mu \|z - \overline{z}\|^2_{\mathbb{R}^N}} - e^{-\mu r^2},$$

where $\mu > 0$ is a constant to be determined. We have

$$\begin{aligned} Lu(z) \ = \ & e^{-\mu \|z-\overline{z}\|^2_{\mathbb{R}^N}} \Bigg[-4\mu^2 \sum_{i,j=1}^{N} a_{ij}(z)(z_i - \overline{z}_i)(z_j - \overline{z}_j) \\ & +2\mu \sum_{i=1}^{N} \big(a_{ii}(z) - b_i(z)(z_i - \overline{z}_i)\big) \Bigg] + a_0(z) \Big(e^{-\mu \|z-\overline{z}\|^2_{\mathbb{R}^N}} - e^{-\mu r^2} \Big). \end{aligned} \tag{6.19}$$

From the uniform ellipticity condition (see hypothesis $H(L)(ii)$), we have

$$\sum_{i,j=1}^{N} a_{ij}(z)(z_i - \overline{z}_i)(z_j - \overline{z}_j) \;\geqslant\; c_0 \|z - \overline{z}\|^2_{\mathbb{R}^N}.$$

Using this in (6.19), we obtain

$$Lu(z) \;\leqslant\; e^{-\mu\|z-\overline{z}\|^2_{\mathbb{R}^N}} \Big[-4\mu^2 c_0 \|z - \overline{z}\|^2_{\mathbb{R}^N} + 2\mu \operatorname{tr} A(z) + 2\mu \big\|\widehat{b}\big\|_\infty \|z - \overline{z}\|_{\mathbb{R}^N} + \|a_0\|_\infty \Big], \tag{6.20}$$

where

$$A(z) \;=\; \big(a_{ij}(z)\big)^N_{i,j=1} \quad \text{and} \quad \widehat{b}(z) \;=\; \big(b_i(z)\big)^N_{i=1}.$$

Note that $\operatorname{tr} A(\cdot) \in L^\infty(\Omega)$. We consider the open annulus

$$D \;\stackrel{df}{=}\; B_r(\overline{z}) \setminus \overline{B}_{\frac{r}{2}}(\overline{z}).$$

Then from (6.20), for all $z \in D$, we have

$$Lu(z) \;\leqslant\; e^{-\mu\|z-\overline{z}\|^2_{\mathbb{R}^N}} \Big(-\mu^2 c_0 \|z - \overline{z}\|^2_{\mathbb{R}^N} + 2\mu \big\|\operatorname{tr} A\big\|_\infty + \mu \big\|\widehat{b}\big\|^r_\infty + \|a_0\|_\infty \Big).$$

Choosing $\mu > 0$ large enough, we see that

$$Lu(z) \;\leqslant\; 0 \qquad \forall\, z \in D.$$

Since by hypothesis

$$x(z_0) \;>\; x(z) \qquad \forall\, z \in \Omega,$$

we can find $\varepsilon > 0$ small enough so that

$$x(z) + \varepsilon u(z) \;\leqslant\; x(z_0) \qquad \forall\, z \in \partial B_{\frac{r}{2}}(\overline{z}). \tag{6.21}$$

Also note that because $u \equiv 0$ on $\partial B_r(\overline{z})$, we have

$$x(z) + \varepsilon u(z) \;\leqslant\; x(z_0) \qquad \forall\, z \in \partial B_r(\overline{z}). \tag{6.22}$$

By hypothesis $Lx(z) \leqslant 0$ for all $z \in \Omega$ and recall that $Lu(z) \leqslant 0$ for all $z \in D$. So

$$L\big(x + \varepsilon u - x(z_0)\big)(z) \;=\; -a_0(z)x(z_0) \;\leqslant\; 0 \qquad \forall\, z \in D.$$

In addition from (6.21) and (6.22), we have

$$x(z) + \varepsilon u(z) - x(z_0) \;\leqslant\; 0 \qquad \forall\, z \in \partial D.$$

Applying Theorem 6.1.40, we obtain

$$x(z) + \varepsilon u(z) - x(z_0) \;\leqslant\; 0 \qquad \forall\, z \in \overline{D}.$$

But

$$x(z_0) + \varepsilon u(z_0) - x(z_0) \;=\; 0.$$

Hence

$$\frac{\partial}{\partial n}\big(x + \varepsilon u - x(z_0)\big)(z_0) \;=\; \frac{\partial x}{\partial n}(z_0) + \varepsilon \frac{\partial u}{\partial n}(z_0) \;\geqslant\; 0$$

and so

$$\frac{\partial x}{\partial n}(z_0) \;\geqslant\; -\varepsilon \frac{\partial u}{\partial n}(z_0) \;=\; -\frac{\varepsilon}{r}\big(Du(z_0), z_0 - \overline{z}\big)_{\mathbb{R}^N} \;=\; 2\mu\varepsilon r e^{-\mu r^2} \;>\; 0$$

(recall that $n(z_0) = \frac{z_0 - \overline{z}}{r}$). □

Using the above lemma, we can prove the strong maximum principles. In the first $a_0 \equiv 0$ and in the second $a_0 \geqslant 0$. Recall that a ***domain*** in $\mathbb{R}^N$ is an open and connected set in $\mathbb{R}^N$.

THEOREM 6.1.44 (Strong Maximum Principle I)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, L satisfies hypotheses $H(L)$ with $a_0 \equiv 0$, $x \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$,

$$Lx(z) \;\leqslant\; 0 \qquad \forall\, z \in \Omega$$

and x attains its maximum over $\overline{\Omega}$ at an interior point (i.e., in Ω), then x is constant on $\overline{\Omega}$.

PROOF Suppose that x is not constant. Let us set

$$m \;\stackrel{df}{=}\; \max_{z \in \overline{\Omega}} x(z)$$

(note that $m \geqslant 0$ if $a_0 \neq 0$). Then

$$V \;\stackrel{df}{=}\; \big\{z \in \Omega :\; x(z) < m\big\} \;\neq\; \emptyset$$

and

$$\partial V \cap \Omega \;\neq\; \emptyset.$$

Let $\overline{z} \in V$ be such that

$$d_x\big(\overline{z}, \partial V\big) \;<\; d_x\big(\overline{z}, \partial\Omega\big)$$

and let $B_r(\overline{z})$ $(r > 0)$ be the largest ball, such that $B_r(\overline{z}) \subseteq V$. Then we can find $z_0 \in \partial B_r(\overline{z})$, such that

$$x(z_0) \;=\; m.$$

Applying Lemma 6.1.43**(a)**, we obtain $\frac{\partial x}{\partial n}(z_0) > 0$. On the other hand since x attains its maximum on $z_0 \in \Omega$, we have

$$\nabla x(z_0) \;=\; 0,$$

hence $\frac{\partial x}{\partial n}(z_0) = 0$, a contradiction. □

COROLLARY 6.1.45
If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, L satisfies hypotheses $H(L)$ with $a_0 \equiv 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lx(z) \geqslant 0 \qquad \forall\, z \in \Omega$$

and x attains its minimum over $\overline{\Omega}$ at an interior point (i.e., in Ω), then x is constant on $\overline{\Omega}$.

Similarly using Lemma 6.1.43**(b)**, we can have the strong maximum principle when $a_0 \geqslant 0$.

THEOREM 6.1.46 (Strong Maximum Principle II)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, L satisfies hypotheses $H(L)$ with $a_0 \geqslant 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lx(z) \leqslant 0 \qquad \forall\, z \in \Omega$$

and x attains a nonnegative maximum over $\overline{\Omega}$ at an interior point (i.e., in Ω), then x is constant on $\overline{\Omega}$.

COROLLARY 6.1.47
If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, L satisfies hypotheses $H(L)$ with $a_0 \geqslant 0$, $x \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lx(z) \geqslant 0 \qquad \forall\, z \in \Omega$$

and x attains a nonpositive minimum over $\overline{\Omega}$ at an interior point (i.e., in Ω), then x is constant on $\overline{\Omega}$.

REMARK 6.1.48 In Theorem 6.1.46 and Corollary 6.1.47, the assumption that $a_0 \geqslant 0$ is essential. If it is not satisfied, then the result fails. To see this consider the equation

$$x''(t) + x(t) = 0 \qquad \forall\, t \in (0, \pi).$$

Then the solution $x(t) = \sin t$ for $t \in (0, \pi)$ attains a positive maximum at $t = \frac{\pi}{2} \in (0, \pi)$.

We conclude this section with Harnack's inequality, which states that the values of the nonnegative solutions of $Lx = 0$ are comparable. For a proof of this theorem we refer to the book of Gilbarg & Trudinger (2001, Section 8.8).

THEOREM 6.1.49 (Harnack Inequality)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded, open set, L satisfies hypotheses $H(L)$, $x \in C^2(\Omega)$, $x \geqslant 0$, $Lx(z) = 0$ for all $z \in \Omega$ and $U \subset\subset \Omega$ is a connected set, then there exists $c > 0$, such that

$$\sup_{z \in U} x(z) \leqslant c \inf_{z \in U} x(z),$$

for some constant $c > 0$ dependent only on U and the coefficients of L.

6.2 The Partial p-Laplacian

In this section we study the spectral properties of the partial p-Laplacian differential operator. So let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $m \in L^\infty(\Omega)$ be such that

$$\big|\{z \in \Omega :\ m(z) > 0\}\big|_N \ > \ 0,$$

where by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. We emphasize that in general m may change sign. We consider the following weighted nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}\left(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) = \lambda m(z)|x(z)|^{p-2}x(z) & \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0. \end{cases} \tag{6.23}$$

DEFINITION 6.2.1 *We say that a number $\lambda \in \mathbb{R}$ is an **eigenvalue** of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$, if (6.23) admits a nontrivial solution $x \in W_0^{1,p}(\Omega)$, which is known as an **eigenfunction** corresponding to λ.*

PROPOSITION 6.2.2
There exists the least (first or principal) eigenvalue $\lambda_1 > 0$ with a corresponding eigenfunction $u_1 \in W_0^{1,p}(\Omega)$, $u_1 \geqslant 0$.

PROOF Let

$$\lambda_1 \ \stackrel{df}{=} \ \inf\left\{\|Dx\|_p^p :\ x \in W_0^{1,p}(\Omega),\ \int_\Omega m|x|^p\,dz = 1\right\}.$$

Evidently $\lambda_1 \geqslant 0$. Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a minimizing sequence, i.e.,

$$\|Dx_n\|_p^p \ \searrow \lambda_1$$

and

$$\int_\Omega m|x_n|^p\,dz \ = \ 1 \qquad \forall\ n \geqslant 1.$$

By virtue of Poincaré's inequality (see Theorem 2.5.4), the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So we may assume that

$$\begin{aligned} x_n &\stackrel{w}{\longrightarrow} u_1 && \text{in } W_0^{1,p}(\Omega), \\ x_n &\longrightarrow u_1 && \text{in } L^p(\Omega), \\ x_n(z) &\longrightarrow u_1(z) && \text{for a.a. } z \in \Omega, \\ |x_n(z)| &\leqslant k(z) && \text{for a.a. } z \in \Omega \text{ and all } n \geqslant 1, \end{aligned}$$

with $k \in L^p(\Omega)$. So

$$\int_\Omega m|x_n|^p\,dz \;\longrightarrow\; \int_\Omega m|u_1|^p\,dz,$$

hence

$$\int_\Omega m|u_1|^p\,dz \;=\; 1. \tag{6.24}$$

Also from the weak lower semicontinuity of the norm in $L^p\big(\Omega;\mathbb{R}^N\big)$, we have

$$\big\|Du_1\big\|_p^p \;\leqslant\; \liminf_{n\to+\infty} \big\|Dx_n\big\|_p^p,$$

i.e., $\big\|Du_1\big\|_p^p = \lambda_1$. From (6.24), we have $u_1 \neq 0$ and so $\lambda_1 > 0$. Invoking the Lagrange multiplier rule (see Theorem 5.5.24), we can find $\lambda \in \mathbb{R}$, such that

$$A(u_1) \;=\; \lambda m|u_1|^{p-2}u_1 \quad \text{in } W^{-1,p'}(\Omega), \tag{6.25}$$

where $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = \big(W_0^{1,p}(\Omega)\big)^*$, (with $\frac{1}{p} + \frac{1}{p'} = 1$) is the nonlinear operator defined by

$$\big\langle A(x), y\big\rangle_{W_0^{1,p}(\Omega)} \;\stackrel{df}{=}\; \int_\Omega \big\|Dx(z)\big\|_{\mathbb{R}^N}^{p-2}\big(Dx(z), Dy(z)\big)_{\mathbb{R}^N}\,dz.$$

Note that

$$\operatorname{div}\big(\big\|Du_1(\cdot)\big\|_{\mathbb{R}^N}^{p-2}Du_1(\cdot)\big) \;\in\; W^{-1,p'}(\Omega)$$

(see Theorem 2.4.57). So from (6.25), we have that

$$\begin{aligned} &\big\langle -\operatorname{div}\big(\big\|Du_1\big\|_{\mathbb{R}^N}^{p-2}\big)Du_1, v\big\rangle_{W_0^{1,p}(\Omega)} \\ &= \int_\Omega \lambda m|u_1|^{p-2}u_1 v\,dz \qquad \forall\; v \in W_0^{1,p}(\Omega), \end{aligned} \tag{6.26}$$

hence

$$\begin{cases} -\operatorname{div}\big(\,\|Du_1(z)\|_{\mathbb{R}^N}^{p-2}\,Du_1(z)\big) = \lambda m(z)\big|u_1(z)\big|^{p-2}u_1(z) & \text{for a.a. } z \in \Omega, \\ u_1|_{\partial\Omega} = 0. \end{cases}$$

Moreover, by using in (6.26) as a test function $u_1 \in W_0^{1,p}(\Omega)$, we obtain

$$\lambda_1 \;=\; \big\|Du_1\big\|_p^p \;=\; \lambda\int_\Omega m|u_1|^p\,dz \;=\; \lambda.$$

Therefore u_1 is an eigenfunction corresponding to $\lambda_1 > 0$. Finally note that if we replace u_1 by $|u_1|$, it is still a solution of the minimization problem and so it is an eigenfunction corresponding to $\lambda_1 > 0$. Thus we may assume that $u_1(z) \geqslant 0$ for almost all $z \in \Omega$. ∎

Next we will determine some regularity properties of the eigenfunction u_1. For this reason we need to prove some nonlinear regularity results. We start with a result on Sobolev functions and distributions due to Brézis & Browder (1982). For the proof we refer to that paper.

LEMMA 6.2.3
If $x \in W_0^{1,p}(\Omega)$ *(with* $p \in (1,+\infty)$*),* $T \in W^{-1,p'}(\Omega) \cap L^1_{\rm loc}(\Omega)$ *(where* $\frac{1}{p} + \frac{1}{p'} = 1$*) and for some* $g \in L^1(\Omega)$*, we have*

$$g(z) \leqslant T(z)x(z) \quad \text{for a.a. } z \in \Omega,$$

then $T(\cdot)x(\cdot) \in L^1(\Omega)$ *and*

$$\langle T, x\rangle_{W_0^{1,p}(\Omega)} = \int_\Omega T(z)x(z)\,dz.$$

LEMMA 6.2.4
If $x \in W_0^{1,p}(\Omega)$ *is such that*

$$\Delta_p x = \operatorname{div}\left(\|Dx\|_{\mathbb{R}^N}^{p-2} Dx\right) \in L^1_{\rm loc}(\Omega),$$

there exist $\sigma \in [1, p^*)$*, where*

$$p^* \stackrel{df}{=} \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geqslant N, \end{cases}$$

$r \in \left[1, \frac{p^*}{p}\right)$*,* $c > 0$ *and* $a \in L^{r'}(\Omega)_+$ *(*$\frac{1}{r} + \frac{1}{r'} = 1$*), such that*

$$-x(z)\Delta_p x(z) \leqslant c\big|x(z)\big|^\sigma + a(z)\big|x(z)\big| \quad \text{for a.a. } z \in \Omega, \tag{6.27}$$

then $x \in L^{p_n}(\Omega)$ *for all* $n \geqslant 0$*, where*

$$p_0 \stackrel{df}{=} \begin{cases} p^* & \text{if} \quad p < N, \\ 2\max\{pr, \sigma\} & \text{if} \quad p \geqslant N \end{cases}$$

and

$$p_{n+1} \stackrel{df}{=} p_0 + \frac{p_0}{p}\min\left\{p_n - \sigma, \frac{p_n}{r} - 1\right\}.$$

PROOF Clearly $x \in L^{p_0}(\Omega)$. Suppose that $x \in L^{p_n}(\Omega)$ for some $n \geqslant 0$. We will show that $x \in L^{p_{n+1}}(\Omega)$. We consider the following sequence of truncations of x:

$$x_k(z) \stackrel{df}{=} \begin{cases} -k & \text{if} \quad x(z) \leqslant -k, \\ x(z) & \text{if} \quad -k \leqslant x(z) \leqslant k, \\ k & \text{if} \quad k \leqslant x(z), \end{cases} \qquad \forall\, k \geqslant 1.$$

Also let us set

$$\beta \stackrel{df}{=} p\frac{p_{n+1}-p_0}{p_0}.$$

Since $p_0 \leqslant p_n$ for all $n \geqslant 0$, we have $\beta \geqslant 0$. From (6.27), we have

$$-\big|x_k(z)\big|^{\beta}x_k(z)\Delta_p x(z) \;\leqslant\; c\big|x(z)\big|^{\sigma+\beta}+a(z)\big|x(z)\big|^{\beta+1} \quad \text{for a.a. } z\in\Omega. \tag{6.28}$$

Note that by Hölder's inequality (see Theorem A.2.27), we have

$$\int_\Omega \big(c|x|^{\sigma+\beta}+a|x|^{\beta+1}\big)\,dz \;\leqslant\; c\,\|x\|_{\sigma+\beta}^{\sigma+\beta}+\|a\|_{r'}\,\|x\|_{r(\beta+1)}^{\beta+1}. \tag{6.29}$$

Since $p_n=\max\big(\sigma+\beta, r(\beta+1)\big)$, from (6.29), it follows that

$$\int_\Omega \big(c|x|^{\sigma+\beta}+a|x|^{\beta+1}\big)\,dz \;\leqslant\; \zeta\big(\|x\|_{p_n}^{p_n}+1\big), \tag{6.30}$$

with

$$\zeta \stackrel{df}{=} \big(c+\|a\|_{r'}\big)\big(|\Omega|_N+1\big) \;>\; 0.$$

Moreover, since

$$|x_k|^{\beta}x_k \;\in\; W_0^{1,p}(\Omega) \quad\text{and}\quad \Delta_p x \;\in\; W^{-1,p'}(\Omega)\cap L^1_{\rm loc}(\Omega)$$

(by hypothesis) and from (6.28), we see that $|x_k|^{\beta}x_k\Delta_p x$ is minorized by an $L^1(\Omega)$-function, we can use Lemma 6.2.3 and infer that

$$|x_k|^{\beta}x_k\Delta_p x \;\in\; L^1(\Omega)$$

and

$$-\int_\Omega |x_k|^{\beta}x_k\Delta_p x\,dz \;=\; \big\langle -\Delta_p x, |x_k|^{\beta}x_k\big\rangle_{W_0^{1,p}(\Omega)}.$$

We have

$$\begin{aligned}
-\int_\Omega |x_k|^{\beta}x_k\Delta_p x\,dz \;&=\; \int_\Omega \big(D\big(|x_k|^{\beta}x_k\big),Dx\big)_{\mathbb{R}^N}\big\|Dx\big\|_{\mathbb{R}^N}^{p-2}\,dz\\
&\geqslant\; (\beta+1)\int_\Omega \big\|Dx_k\big\|_{\mathbb{R}^N}^{p}|x_k|^{\beta}\,dz\\
&=\; (\beta+1)\Big(\frac{p}{\beta+p}\Big)^p\int_\Omega \big\|D\big(|x_k|^{\frac{\beta}{p}}x_k\big)\big\|_{\mathbb{R}^N}^{p}\,dz\\
&\geqslant\; \frac{1}{c_0^p}\Big(\frac{p}{\beta+p}\Big)^p\|x_k\|_{p_0\frac{\beta+p}{p}}^{\beta+p} \;=\; \frac{1}{c_0^p}\Big(\frac{p_0}{p_{n+1}}\Big)^p\|x\|_{p_{n+1}}^{p\frac{p_{n+1}}{p_0}},
\end{aligned} \tag{6.31}$$

where $c_0>0$ is the constant for the embedding $W_0^{1,p}(\Omega)\subseteq L^{p_0}(\Omega)$, i.e.,

$$\|x\|_{L^{p_0}(\Omega)} \;\leqslant\; c_0\,\|x\|_{W_0^{1,p}(\Omega)} \qquad \forall\; x\in W_0^{1,p}(\Omega).$$

Then from (6.28), (6.30) and (6.31), we obtain

$$\|x_k\|_{p_{n+1}}^{p\frac{p_{n+1}}{p_0}} \leqslant \widehat{c}_0 p_{n+1}^p \big(\|x\|_{p_n}^{p_n} + 1\big),$$

with $\widehat{c}_0 \stackrel{df}{=} \zeta\big(\frac{c_0}{p_0}\big)^p$. Because

$$x_k(z) \longrightarrow x(z) \quad \text{for a.a. } z \in \Omega \quad \text{as } k \to +\infty$$

and by the induction hypothesis $x \in L^{p_n}(\Omega)$, we have that

$$x_k \stackrel{w}{\longrightarrow} x \quad \text{in } L^{p_{n+1}}(\Omega)$$

and so $x \in L^{p_{n+1}}(\Omega)$. □

Now we consider the sequence $\{r_n\}_{n\geqslant 1}$, defined by

$$\begin{aligned} r_0 &\stackrel{df}{=} p_0, \\ r_{n+1} &\stackrel{df}{=} \big(r_n + r(p-1)\big)\delta \qquad \forall\, n \geqslant 0, \end{aligned}$$

with $\delta \stackrel{df}{=} \frac{p_0}{rp}$.

LEMMA 6.2.5

If the hypotheses of Lemma 6.2.4 hold, then $x \in L^{r_n}(\Omega)$ for all integers $n \geqslant 0$ and

$$\|x\|_{r_{n+1}}^{\frac{r_{n+1}}{\delta r}} \leqslant c r_{n+1}^p \|x\|_{r_n}^{\frac{r_n}{r}},$$

where $c > 0$ is a constant depending on N, p, σ, r, $\|a\|_{r'}$, c and $\|x\|_{p_0}$.

PROOF For every $n \geqslant 1$, we have

$$p_n \geqslant p_0 + s_0 \sum_{k=1}^{n} \delta^k = p_0 + s_0 \delta \frac{\delta^n - 1}{\delta - 1},$$

where $s_0 = \min\big\{r(p_0 - \sigma), p_0 - r\big\}$. Since $s_0 > 0$ and $\delta > 1$ we see that

$$p_n \longrightarrow +\infty.$$

So Lemma 6.2.4 implies that $x \in L^{\vartheta}(\Omega)$ for all $\vartheta \in [1, +\infty)$. Hence $|x|^{\sigma-1} \in L^{r'}(\Omega)$ and $\big\||x|^{\sigma-1}\big\|_{r'} \leqslant \eta$ with $\eta > 0$ depending only on N, p, σ, r, $\|a\|_{r'}$, c and $\|x\|_{p_0}$ (see the proof of Lemma 6.2.4). We have

$$\int_\Omega \Big(c|x|^{\sigma-1} + a|x|^{\frac{r_n}{r}}\Big)\, dz \leqslant \big(c\eta + \|a\|_{r'}\big) \|x\|_{r_n}^{\frac{r_n}{r}}.$$

If in (6.28) we replace β by $\frac{r_n}{r} - 1$ and arguing as in the proof of Lemma 6.2.4, we obtain the desired inequality. □

Now we are ready for the first regularity result.

THEOREM 6.2.6
If all the hypotheses of Lemma 6.2.4 hold, then $x \in L^\infty(\Omega)$ *and*

$$\|x\|_\infty \leqslant \eta_1$$

with the constant $\eta_1 > 0$ *depending on* N, p, σ, r, $\|a\|_{r'}$, c *and* $\|x\|_{p_0}$.

PROOF Let

$$h_n \stackrel{df}{=} r_n \ln\left(\|x\|_{r_n}\right) \quad \text{and} \quad \mu_n \stackrel{df}{=} r \ln\left(c r_{n+1}^p\right) \qquad \forall\, n \geqslant 0.$$

Then from Lemma 6.2.5, we have

$$h_{n+1} \leqslant \delta(h_n + \mu_n),$$

so

$$h_n \leqslant \delta^n h_0 + \sum_{k=1}^{n} \delta^k \mu_{n-k} \qquad \forall\, n \geqslant 1.$$

Since

$$r_n = \delta^n p_0 + \delta r(p-1)\frac{\delta^n - 1}{\delta - 1},$$

we have

$$\delta^n p_0 \leqslant r_n \leqslant \delta^n \overline{c},$$

with $\overline{c} \stackrel{df}{=} p_0 + \delta r \frac{p-1}{\delta - 1}$. Hence

$$\mu_n \leqslant \tau_1 + (n+1)\tau_2,$$

with $\tau_1 \stackrel{df}{=} r \ln(c\overline{c}^p)$ and $\tau_2 \stackrel{df}{=} pr \ln \delta$.

It follows that

$$\sum_{k=1}^{n} \delta^k \mu_{n-k} \leqslant \zeta \delta^n,$$

with $\zeta \stackrel{df}{=} \left(\tau_1 + \tau_2 \frac{\delta}{\delta - 1}\right)\frac{\delta}{\delta - 1}$.

Then we have

$$\|x\|_{r_n} \leqslant \exp\left(\frac{h_n}{\delta^n p_0}\right) \leqslant \exp\left(\frac{h_0 + \zeta}{p_0}\right),$$

so

$$\|x\|_\infty = \lim_{n \to +\infty} \|x\|_{r_n} \leqslant \exp\left(\frac{h_0 + \zeta}{p_0}\right) < +\infty.$$

□

Using the above theorem, we can have a stronger regularity result. This result is a particular case of a theorem of Lieberman (1988) (cf. also Di Benedetto (1983) and Tolksdorf (1984)).

THEOREM 6.2.7
If $x \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ *and* $\Delta_p x \in L^r(\Omega)$ *with* $r \in [1, +\infty]$, *then*

(a) *if* $r = +\infty$, *we have* $x \in C_0^{1,\beta}(\overline{\Omega})$ *with* $\beta \in (0,1)$ *and*

$$\|x\|_{C_0^{1,\beta}(\overline{\Omega})} \leqslant c$$

with $c > 0$; *both* β *and* c *depend only on* N, p, $\|x\|_{\infty}$ *and* $\|\Delta_p x\|_{\infty}$;

(b) *if* $r > Np'$ *(with* $\frac{1}{p} + \frac{1}{p'} = 1$*), then for every* $\Omega' \subset\subset \Omega$, *we have* $x \in C_0^{1,\beta}(\overline{\Omega'})$ *and*

$$\|x\|_{C_0^{1,\beta}(\overline{\Omega'})} \leqslant \widehat{c},$$

for some $\beta \in (0,1)$ *and* $\widehat{c} > 0$; *both* β *and* $\widehat{c}$ *depend only on* N, p, $\|x\|_{\infty}$, $\|\Delta_p x\|_r$ *and* Ω'.

To fully describe the first eigenfunction u_1 associated to $\lambda_1 > 0$ (see Proposition 6.2.2), we will need a nonlinear version of the Strong (Hopf) Maximum Principle (see Theorem 6.1.44). The result is due to Vázquez (1984, Theorem 5), where the interested reader can find its proof.

THEOREM 6.2.8
If $x \in C^1(\Omega)$, $x(z) \geqslant 0$ *for all* $z \in \Omega$, $x \neq 0$, $\Delta_p x \in L^2_{\mathrm{loc}}(\Omega)$ *and*

$$\Delta_p x(z) \leqslant \xi\big(x(z)\big) \quad \text{for a.a. } z \in \Omega,$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *is a continuous, increasing function, such that*

$$\xi(0) = 0$$

and

$$\text{either } \xi(r_0) = 0 \text{ for some } r_0 > 0 \text{ or } \int_0^1 \frac{1}{(r\xi(r))^{\frac{1}{p}}}\, dr = +\infty,$$

then

$$x(z) > 0 \qquad \forall\, z \in \Omega.$$

Moreover, if $x \in C^1\big(\Omega \cup \{z_0\}\big)$, $z_0 \in \partial\Omega$ *and* $x(z_0) = 0$, *then*

$$\frac{\partial u}{\partial n}(z_0) < 0$$

(with n *being the exterior unit normal on* $\partial\Omega$ *at* z_0*).*

Having these general results we can establish the properties of u_1.

THEOREM 6.2.9
If $u \in W_0^{1,p}(\Omega)$ is an eigenfunction corresponding to an eigenvalue $\lambda \geqslant \lambda_1$ in (6.25),
then $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. In particular, if u_1 is an eigenfunction corresponding to the principal eigenvalue $\lambda_1 > 0$, then

$$u_1(z) \ > \ 0 \qquad \forall \ z \in \Omega$$

and

$$\frac{\partial u_1}{\partial n}(z) \ < \ 0 \qquad \forall \ z \in \partial\Omega.$$

PROOF The regularity of u follows from Theorem 6.2.7**(a)**. Also from Proposition 6.2.2, we know that $u_1 \geqslant 0$, $u_1 \neq 0$. We have

$$\Delta_p u_1(z) \ \leqslant \ \lambda_1 \|m\|_\infty u_1(z)^{p-1} \quad \text{for a.a. } z \in \Omega.$$

So we can apply Theorem 6.2.8 (with $\xi(r) = \lambda_1 \|m\|_\infty r^{p-1}$) and conclude that the thesis of the theorem holds. □

REMARK 6.2.10 If on $C_0^1(\overline{\Omega})$ we consider the pointwise partial ordering (i.e., $x \leqslant y$ if and only if $x(z) \leqslant y(z)$ for all $z \in \overline{\Omega}$), which is induced by the positive cone

$$C_0^1(\overline{\Omega})_+ \ \stackrel{df}{=} \ \big\{x \in C_0^1(\overline{\Omega}) : \ x(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\big\},$$

then

$$\operatorname{int} C_0^1(\overline{\Omega})_+ \ = \ \Big\{x \in C_0^1(\Omega) : \ x(z) > 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\Big\}.$$

So Theorem 6.2.9 implies that $u_1 \in \operatorname{int} C_0^1(\overline{\Omega})_+$.

The function $v \longmapsto \|v\|_{\mathbb{R}^N}^p$ (with $p > 1$) is strictly convex and so for every $v_1, v_2 \in \mathbb{R}^N$, $v_1 \neq v_2$, we have

$$p\,\|v_1\|_{\mathbb{R}^N}^{p-2}\,\big(v_1, v_2 - v_1\big)_{\mathbb{R}^N} \ < \ \|v_2\|_{\mathbb{R}^N}^2 - \|v_1\|_{\mathbb{R}^N}^p . \tag{6.32}$$

Sometimes it is useful to express the strictness in inequality (6.32) in a more precise way. This is done in the next lemma, the proof of which can be found in Lindqvist (1990, 1992).

LEMMA 6.2.11
(a) *If* $p \in (1,2)$ *and* $v_1, v_2 \in \mathbb{R}^N$,
then

$$p\,\|v_1\|_{\mathbb{R}^N}^{p-2}\,(v_1, v_2 - v_1)_{\mathbb{R}^N} + \frac{3p(p-1)}{16}\,\frac{\|v_2 - v_1\|_{\mathbb{R}^N}^2}{(\|v_1\|_{\mathbb{R}^N} + \|v_2\|_{\mathbb{R}^N})^{2-p}} \leqslant \|v_2\|_{\mathbb{R}^N}^p - \|v_1\|_{\mathbb{R}^N}^p .$$

(b) *If* $p \geqslant 2$ *and* $v_1, v_2 \in \mathbb{R}^N$,
then

$$p\,\|v_1\|_{\mathbb{R}^N}^{p-2}\,(v_1, v_2 - v_1)_{\mathbb{R}^N} + \frac{\|v_2 - v_1\|_{\mathbb{R}^N}^2}{2^{p-1} - 1} \leqslant \|v_2\|_{\mathbb{R}^N}^p - \|v_1\|_{\mathbb{R}^N}^p .$$

REMARK 6.2.12 In the proof of Lemma 6.2.11**(b)** the central role plays the so-called ***Clarkson's inequality***, which says that for $p \geqslant 2$ and for all $v_1, v_2 \in \mathbb{R}^N$, we have

$$\frac{1}{2^{p-1}}\left[\|v_1 + v_2\|_{\mathbb{R}^N}^p + \|v_1 - v_2\|_{\mathbb{R}^N}^p\right] \leqslant \|v_1\|_{\mathbb{R}^N}^p + \|v_2\|_{\mathbb{R}^N}^p .$$

Another related useful inequality is the following one. For the proof see Cheng (1998).

LEMMA 6.2.13
(a) *If* $p \in (1,2)$ *and* $v_1, v_2 \in \mathbb{R}^N$,
then

$$\frac{p-1}{2}\,\|v_1 - v_2\|_{\mathbb{R}^N}^2\,(\|v_1\|_{\mathbb{R}^N} + \|v_2\|_{\mathbb{R}^N})^{p-2} \leqslant \left(\|v_1\|_{\mathbb{R}^N}^{p-2} v_1 - \|v_2\|_{\mathbb{R}^N}^{p-2} v_2, v_1 - v_2\right)_{\mathbb{R}^N} \leqslant 2^{2-p}\,\|v_1 - v_2\|_{\mathbb{R}^N}^p .$$

(b) *If* $p \geqslant 2$ *and* $v_1, v_2 \in \mathbb{R}^N$,
then

$$c_1\,\|v_1 - v_2\|_p^p \leqslant \left(\|v_1\|_{\mathbb{R}^N}^{p-2} v_1 - \|v_2\|_{\mathbb{R}^N}^{p-2} v_2, v_1 - v_2\right)_{\mathbb{R}^N} \leqslant c_2\,\|v_1 - v_2\|_{\mathbb{R}^N}^p\,(\|v_1\|_{\mathbb{R}^N} + \|v_2\|_{\mathbb{R}^N})^{p-2},$$

for some constants $c_1, c_2 > 0$ *depending only on* $p \geqslant 2$.

Using these inequalities, we can establish the simplicity of the principal eigenvalue $\lambda_1 > 0$.

PROPOSITION 6.2.14
The first (principal) eigenvalue $\lambda_1 > 0$ of $\big(-\Delta, W_0^{1,p}(\Omega)\big)$ is simple, i.e., if $u, v \geqslant 0$ are eigenfunctions corresponding to the eigenvalue $\lambda_1 > 0$, then u and v are proportional.

PROOF Let $u, v \in W_0^{1,p}(\Omega)$ be two eigenfunctions corresponding to the eigenvalue $\lambda_1 > 0$. Then from Theorem 6.2.9, we know that $u, v \in C_0^1(\overline{\Omega})$ and

$$u(z) > 0 \quad \text{and} \quad v(z) > 0 \qquad \forall\, z \in \Omega.$$

Also we have

$$\int_\Omega \big\|Du\big\|_{\mathbb{R}^N}^{p-2} \big(Du, D\vartheta\big)_{\mathbb{R}^N} dz = \lambda_1 \int_\Omega m|u|^{p-2}u\vartheta\, dz \quad \forall\, \vartheta \in W_0^{1,p}(\Omega) \tag{6.33}$$

and

$$\int_\Omega \big\|Dv\big\|_{\mathbb{R}^N}^{p-2} \big(Dv, D\eta\big)_{\mathbb{R}^N} dz = \lambda_1 \int_\Omega m|v|^{p-2}v\eta\, dz \quad \forall\, \eta \in W_0^{1,p}(\Omega). \tag{6.34}$$

Let

$$\vartheta \stackrel{df}{=} \frac{u^p - v^p}{u^{p-1}} \quad \text{and} \quad \eta \stackrel{df}{=} \frac{v^p - u^p}{v^{p-1}}.$$

Clearly $\vartheta, \eta \in L^\infty(\Omega)$ and

$$D\vartheta = \left[1 + (p-1)\left(\frac{v}{u}\right)^p\right] Du - p\left(\frac{v}{u}\right)^{p-1} Dv$$

and

$$D\eta = \left[1 + (p-1)\left(\frac{u}{v}\right)^p\right] Dv - p\left(\frac{u}{v}\right)^{p-1} Du$$

and so $\vartheta, \eta \in W_0^{1,p}(\Omega)$. Therefore we can use them as test functions in (6.33) and (6.34). We use ϑ in (6.33) and η in (6.34) and then add the resulting equalities. We obtain

$$\begin{aligned}
&\int_\Omega \left(\left[1 + (p-1)\left(\frac{v}{u}\right)^p\right] \|Du\|_{\mathbb{R}^N}^p + \left[1 + (p-1)\left(\frac{u}{v}\right)^p\right] \|Dv\|_{\mathbb{R}^N}^p\right) dz \\
&= \int_\Omega \left(p\left(\frac{v}{u}\right)^{p-1} \|Du\|_{\mathbb{R}^N}^{p-2} \big(Du, Dv\big)_{\mathbb{R}^N} \right. \\
&\qquad \left. + p\left(\frac{u}{v}\right)^{p-1} \|Dv\|_{\mathbb{R}^N}^{p-2} \big(Dv, Du\big)_{\mathbb{R}^N}\right) dz.
\end{aligned}$$

Note that

$$D\ln u \;=\; \frac{1}{u}Du.$$

So the last equality becomes

$$\begin{aligned}
&\int_\Omega (u^p - v^p)\left(\left\|D\ln u\right\|_{\mathbb{R}^N}^p - \left\|D\ln v\right\|_{\mathbb{R}^N}^p\right) dz \\
&= \int_\Omega pv^p \left\|D\ln u\right\|_{\mathbb{R}^N}^{p-2} \left(D\ln u, D\ln v - D\ln u\right)_{\mathbb{R}^N} dz \\
&\quad + \int_\Omega pu^p \left\|D\ln v\right\|_{\mathbb{R}^N}^{p-2} \left(D\ln v, D\ln u - D\ln v\right)_{\mathbb{R}^N} dz.
\end{aligned}$$

If $p \in (1,2)$, we apply Lemma 6.2.11**(a)** and obtain

$$-\frac{3p(p-1)}{16}\int_\Omega \left(v^p + u^p\right)\frac{\|D\ln u - D\ln v\|_{\mathbb{R}^N}^2}{(\|D\ln u\|_{\mathbb{R}^N} + \|D\ln v\|_{\mathbb{R}^N})^{2-p}}\, dz \;=\; 0,$$

so

$$-\frac{3p(p-1)}{16}\int_\Omega \left(\frac{1}{u^p} + \frac{1}{v^p}\right)\frac{\|vDu - uDv\|_{\mathbb{R}^N}^2}{(v\,\|Du\|_{\mathbb{R}^N} + u\,\|Dv\|_{\mathbb{R}^N})^{2-p}}\, dz \;=\; 0$$

and thus

$$v(z)Du(z) \;=\; u(z)Dv(z) \quad \text{for a.a. } z \in \Omega.$$

If $p \geqslant 2$, we apply Lemma 6.2.11**(a)** and obtain

$$-\frac{1}{2^{p-1}}\int_\Omega \left(u^p + v^p\right)\left\|D\ln v - D\ln u\right\|_{\mathbb{R}^N}^p dz \;=\; 0,$$

so

$$-\frac{1}{2^{p-1}}\int_\Omega \left(\frac{1}{v^p} + \frac{1}{u^p}\right)\left\|vDu - uDv\right\|_{\mathbb{R}^N}^p dz \;=\; 0$$

and thus

$$v(z)Du(z) \;=\; u(z)Dv(z) \quad \text{for a.a. } z \in \Omega.$$

Therefore for any $p \in (1,+\infty)$, we have

$$v(z)Du(z) \;=\; u(z)Dv(z) \quad \text{for a.a. } z \in \Omega,$$

which implies that

$$u \;=\; \beta v,$$

for some $\beta > 0$. ∎

Next we show that any eigenfunction $u \in C_0^1(\overline{\Omega})$ (see Theorem 6.2.9) associated to an eigenvalue $\lambda \neq \lambda_1$ must change sign.

PROPOSITION 6.2.15
If $u \in C_0^1(\overline{\Omega})$ is an eigenfunction corresponding to an eigenvalue $\lambda > \lambda_1$, then u changes sign (i.e., $u^+ \neq 0$, $u^- \neq 0$) and if

$$\Omega_+ \stackrel{df}{=} \{z \in \Omega : \ u(z) > 0\},$$
$$\Omega_- \stackrel{df}{=} \{z \in \Omega : \ u(z) < 0\},$$

then we have

$$\min\{|\Omega_+|_N, |\Omega_-|_N\} \geqslant \left(\lambda c^p \|m\|_\infty\right)^\sigma,$$

with a constant $c > 0$ independent of u and λ and

$$\sigma \stackrel{df}{=} \begin{cases} -\frac{N}{p} & \text{if} \quad p < N, \\ -2 & \text{if} \quad p \geqslant N. \end{cases}$$

PROOF Let u_1 be the eigenfunction corresponding to $\lambda_1 > 0$ with

$$\|Du_1\|_p = 1 \quad \text{and} \quad u_1(z) > 0 \qquad \forall\, z \in \Omega.$$

We proceed indirectly. Suppose that u has constant sign. Without any loss of generality, we may assume that

$$u \geqslant 0$$

(the analysis is similar if we assume that $u \leqslant 0$) and that $\|Du\|_p = 1$. From Theorem 6.2.9, we have that $u \in C_0^1(\overline{\Omega})$. So invoking Theorem 6.2.8, we infer that

$$u(z) > 0 \qquad \forall\, z \in \Omega.$$

Then

$$\frac{u_1^p - u^p}{u_1^{p-1}} \in W_0^{1,p}(\Omega) \quad \text{and} \quad \frac{u^p - u_1^p}{u^{p-1}} \in W_0^{1,p}(\Omega)$$

and using them as test functions, we obtain

$$\begin{aligned} &\left\langle -\Delta_p u_1, \frac{u_1^p - u^p}{u_1^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} + \left\langle -\Delta_p u, \frac{u^p - u_1^p}{u^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \\ &= \left\langle \lambda_1 m u_1^{p-1}, \frac{u_1^p - u^p}{u_1^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} + \left\langle \lambda_1 m u^{p-1}, \frac{u^p - u_1^p}{u^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \\ &= (\lambda_1 - \lambda) \int_\Omega m\left(u_1^p - u^p\right) dz \geqslant 0 \end{aligned} \tag{6.35}$$

(see the proof of Proposition 6.2.14).

On the other hand, since

$$1 \;=\; \big\|Du_1\big\|_p^p \;=\; \lambda_1 \int\limits_\Omega m u_1^p \, dz$$

and

$$1 \;=\; \big\|Du\big\|_p^p \;=\; \lambda \int\limits_\Omega m u^p \, dz,$$

we have

$$(\lambda_1 - \lambda)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda}\right) \;<\; 0. \tag{6.36}$$

Comparing (6.35) and (6.36), we reach a contradiction. So u changes sign.

Using as a test function $u^+ \in W_0^{1,p}(\Omega)$, from (6.25), we have

$$\big\|Dv^+\big\|_p^p \;=\; \lambda \int\limits_\Omega m|v^+|^p \, dz \;\leqslant\; \lambda \, \|m\|_\infty \, \big\|v^+\big\|_p^p. \tag{6.37}$$

If

$$p_0^* \;\stackrel{df}{=}\; \begin{cases} p^* = \frac{Np}{N-p} & \text{if} \quad p < N, \\ 2p & \text{if} \quad p \geqslant N, \end{cases}$$

then, since $1 < p < p_0^*$, we have

$$\big\|v^+\big\|_p \;\leqslant\; \big\|v^+\big\|_{p_0^*} \, |\Omega_+|^{\frac{1}{p} - \frac{1}{p_0^*}},$$

so

$$\big\|v^+\big\|_p^p \;\leqslant\; \big\|v^+\big\|_{p_0^*}^p \, |\Omega_+|^{1 - \frac{p}{p_0^*}}. \tag{6.38}$$

Using (6.38) in (6.37), we obtain

$$\big\|Dv^+\big\|_p^p \;\leqslant\; \lambda \, \|m\|_\infty \, \big\|v^+\big\|_{p_0^*}^p \, |\Omega_+|^{1 - \frac{p}{p_0^*}}. \tag{6.39}$$

From Theorem 2.5.3, we know that

$$\big\|v^+\big\|_{p_0^*}^p \;\leqslant\; c^p \big\|Dv^+\big\|_p^p,$$

for some constant $c(p, N) > 0$. Then from (6.39), we have

$$|\Omega_+|_N \;\geqslant\; \big(\lambda c^p \, \|m\|_\infty\big)^\sigma,$$

where

$$\sigma \;\stackrel{df}{=}\; \begin{cases} -\frac{N}{p} & \text{if} \quad p < N, \\ -2 & \text{if} \quad p \geqslant N. \end{cases}$$

Similarly for $|\Omega_-|_N$. This completes the proof of the proposition. □

REMARK 6.2.16 In fact an estimate analogous to the one obtained in Proposition 6.2.15 holds at every connected component of $\{z \in \Omega : u(z) \neq 0\}$. So if Ω' is such a connected component, then

$$|\Omega'|_N \geqslant \left(\lambda \|m\|_\infty c_0^p\right)^\beta,$$

for some constant $c_0(p, N) > 0$ and with

$$\beta \stackrel{df}{=} \begin{cases} -\frac{N}{p} & \text{if} \quad p < N, \\ -1 & \text{if} \quad p = N, \\ 1 - \frac{p}{N} & \text{if} \quad p > N. \end{cases}$$

If $\lambda = \lambda_1$, then there is only one connected component, namely the domain Ω itself and so we have an estimation (from below) of λ_1 in terms of $|\Omega|_N$ and $\|m\|_\infty$, namely

$$\lambda_1 \geqslant \frac{|\Omega|_N^{\frac{1}{\beta}}}{\|m\|_\infty c_0^p}.$$

We can use Proposition 6.2.15, to show that $\lambda_1 > 0$ is isolated.

PROPOSITION 6.2.17
The first eigenvalue $\lambda_1 > 0$ of $\left(-\Delta, W_0^{1,p}(\Omega), m\right)$ is isolated, i.e., there exists $\varepsilon > 0$, such that there is no eigenvalue in the open interval $(\lambda_1, \lambda_1 + \varepsilon)$.

PROOF We argue by contradiction. Suppose that the proposition is not true. Then we can find eigenvalues $\widehat{\lambda}_n$ for $n \geqslant 1$, such that

$$\widehat{\lambda}_n \searrow \lambda_1.$$

Let $\{\widehat{u}_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be associated normalized (i.e., with $\|\widehat{u}_n\|_{W^{1,p}(\Omega)} = 1$ for $n \geqslant 1$) eigenfunctions. Let $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ be the nonlinear operator, defined by

$$\left\langle A(x), y \right\rangle_{W_0^{1,p}(\Omega)} = \int_\Omega \left\|Dx\right\|_{\mathbb{R}^N}^{p-2} \left(Dx, Dy\right)_{\mathbb{R}^N} dz \qquad \forall\, x, y \in W_0^{1,p}(\Omega).$$

It is easy to see that A is monotone (see Lemma 6.2.13) and demicontinuous. Hence A is maximal monotone (see Proposition 3.2.19). For all $n \geqslant 1$, we have

$$A(\widehat{u}_n) = \widehat{\lambda}_n m |\widehat{u}_n|^{p-2} \widehat{u}_n \quad \text{in } W^{-1,p'}(\Omega). \tag{6.40}$$

By passing to a subsequence, if necessary, we may assume that

$$\begin{aligned} \widehat{u}_n &\stackrel{w}{\longrightarrow} u && \text{in } W_0^{1,p}(\Omega), \\ \widehat{u}_n &\longrightarrow u && \text{in } L^p(\Omega), \\ \widehat{u}_n(z) &\longrightarrow u(z) && \text{for a.a. } z \in \Omega, \\ \left|\widehat{u}_n(z)\right| &\leqslant k(z) && \text{for a.a. } z \in \Omega \text{ and all } n \geqslant 1, \end{aligned}$$

with $k \in L^p(\Omega)_+$. Then the sequence $\{|\widehat{u}_n|^{p-2}\widehat{u}_n\}_{n\geqslant 1} \subseteq L^{p'}(\Omega)$ is bounded and

$$|\widehat{u}_n(z)|^{p-2}\widehat{u}_n(z) \longrightarrow |\widehat{u}(z)|^{p-2}\widehat{u}(z) \quad \text{for a.a. } z \in \Omega.$$

So we infer that

$$|\widehat{u}_n(\cdot)|^{p-2}\widehat{u}_n(\cdot) \longrightarrow |\widehat{u}(\cdot)|^{p-2}\widehat{u}(\cdot) \quad \text{in } L^{p'}(\Omega).$$

We have

$$\langle A(\widehat{u}_n), \widehat{u}_n - u\rangle_{W_0^{1,p}(\Omega)} = \lambda_1 \int_\Omega m|\widehat{u}_n|^{p-2}\widehat{u}_n(\widehat{u}_n - u)\,dz.$$

Since

$$\lambda_1 \int_\Omega m|\widehat{u}_n|^{p-2}\widehat{u}_n(\widehat{u}_n - u)\,dz \longrightarrow 0,$$

we obtain

$$\lim_{n\to+\infty} \langle A(\widehat{u}_n), \widehat{u}_n - u\rangle_{W_0^{1,p}(\Omega)} = 0.$$

But A being maximal monotone, it is generalized pseudomonotone (see Proposition 3.2.47) and so

$$\langle A(\widehat{u}_n), \widehat{u}_n\rangle_{W_0^{1,p}(\Omega)} \longrightarrow \langle A(u), u\rangle_{W_0^{1,p}(\Omega)},$$

hence

$$\|D\widehat{u}_n\|_p \longrightarrow \|Du\|_p.$$

Since

$$D\widehat{u}_n \xrightarrow{w} Du \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big)$$

and the Lebesgue space $L^p\big(\Omega;\mathbb{R}^N\big)$ is uniformly convex, from the Kadec-Klee property (see Remark A.3.22), we have

$$D\widehat{u}_n \longrightarrow Du \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big)$$

and so

$$\widehat{u}_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega).$$

It follows that

$$A(\widehat{u}_n) \xrightarrow{w} A(u) \quad \text{in } W^{-1,p'}(\Omega).$$

Passing to the limit as $n \to +\infty$ in (6.40), we obtain

$$A(u) = \lambda_1|u|^{p-2}u \quad \text{in } W^{-1,p'}(\Omega) \quad \text{and} \quad \|u\|_{W_0^{1,p}(\Omega)} = 1,$$

which implies that u is an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. By Theorem 6.2.9, we have that $u \in C_0^1(\overline{\Omega})$ and that

$$|u|(z) > 0 \qquad \forall\, z \in \Omega.$$

Without any loss of generality we may assume that

$$u(z) \ > \ 0 \qquad \forall\ z \in \Omega.$$

Then for a given $\varepsilon > 0$, we can find $\xi_\varepsilon > 0$ and a compact set $C_\varepsilon \subseteq \Omega$, such that

$$u(z) \ \geqslant \ 2\xi_\varepsilon \ > \ 0 \qquad \forall\ z \in C_\varepsilon.$$

Also, since

$$\widehat{u}_n(z) \ \longrightarrow \ u(z) \quad \text{for a.a. } z \in \Omega,$$

by Egorov's theorem (see Theorem A.2.10), we can find a compact set $D_\varepsilon \subseteq \Omega$ with $\big|\Omega \setminus D_\varepsilon\big| \leqslant \frac{\varepsilon}{2}$, such that

$$u_n(z) \ \longrightarrow \ u(z) \quad \text{uniformly on } D_\varepsilon.$$

So we can find $n_0 = n_0(\varepsilon) \geqslant 1$, such that

$$\big|\widehat{u}_n(z) - u(z)\big| \ \leqslant \ \xi_\varepsilon \qquad \forall\ n \geqslant n_0,\ z \in D_\varepsilon.$$

It follows that

$$\widehat{u}_n(z) \ \geqslant \ \xi_\varepsilon \qquad \forall\ n \geqslant n_0,\ z \in C_\varepsilon \cap D_\varepsilon.$$

Note that, if $\varepsilon < |\Omega|_N$, then

$$\big|C_\varepsilon \cap D_\varepsilon\big|_N \ \geqslant \ |\Omega|_N - \varepsilon \ > \ 0. \tag{6.41}$$

Fix $n \geqslant n_0$ and let

$$\Omega_-^n \ \stackrel{df}{=} \ \big\{z \in \Omega :\ \widehat{u}_n(z) < 0\big\}.$$

Then from Proposition 6.2.15, we have

$$\big|\Omega_-^n\big|_N \ \geqslant \ \big(\lambda_1 c^p \, \|m\|_\infty\big)^\sigma \ > \ 0. \tag{6.42}$$

So, if we choose $\varepsilon \stackrel{df}{=} \frac{1}{2}\big(\lambda_1 c^p \, \|m\|_\infty\big)^\sigma$, then from (6.41) and (6.42), we have

$$|\Omega|_N \ \geqslant \ \big|\Omega_+^n \cup \Omega_-^n\big|_N \ = \ \big|\Omega_+^n\big|_N + \big|\Omega_-^n\big|_N \ > \ |\Omega|_N,$$

a contradiction. This proves that $\lambda_1 > 0$ is isolated. □

Therefore summarizing the situation about the first element in the spectrum of $\big(-\Delta_p, W_0^{1,p}(\Omega), m\big)$, we can state the following theorem.

THEOREM 6.2.18

The nonlinear eigenvalue problem (6.23) has a least (first or principal) eigenvalue $\lambda_1 > 0$, which is isolated and simple (i.e., the associated eigenspace is one dimensional). Moreover, a corresponding eigenfunction u belongs in $C_0^1(\overline{\Omega})$, does not change sign (i.e., $|u|(z) > 0$ for all $z \in \Omega$) and so we can always assume that $u(z) > 0$ for all $z \in \Omega$.

The next question is what can be said about the higher part of the spectrum of $\big(-\Delta_p, W_0^{1,p}(\Omega), m\big)$? In this direction we use the Lusternik-Schnirelman theory discussed in Section 5.5. In fact we apply this theory using as topological index the Krasnoselskii genus γ (see Definition 5.4.25). So we consider the functionals

$$\varphi_1, \varphi_2 \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R},$$

defined by

$$\varphi_1(x) \stackrel{df}{=} \frac{1}{p}\big\|Dx\big\|_p^p \quad \text{and} \quad \varphi_2(x) \stackrel{df}{=} \frac{1}{p}\int_\Omega m|x|^p\,dz \qquad \forall\, x \in W_0^{1,p}(\Omega).$$

Evidently $\varphi_1, \varphi_2 \in C^1\big(W_0^{1,p}(\Omega)\big)$. Let us set

$$\varphi(x) \stackrel{df}{=} \varphi_1(x)^2 - \varphi_2(x) \qquad \forall\, x \in W_0^{1,p}(\Omega).$$

Then $\varphi \in C^1\big(W_0^{1,p}(\Omega)\big)$. If x is a nontrivial critical point of φ, i.e., $\varphi'(x) = 0$, then

$$\frac{2}{p}\big\|Dx\big\|_p^p A(x) \;=\; m|x|^{p-2}x \quad \text{in } W^{-1,p'}(\Omega) \tag{6.43}$$

and so using as a test function $\frac{1}{p}x$, we obtain

$$\frac{2}{p^2}\big\|Dx\big\|_p^{2p} \;=\; \frac{1}{p}\int_\Omega m|x|^p\,dz.$$

So the critical value is given by

$$c \;=\; -\frac{1}{p^2}\big\|Dx\big\|_p^{2p} \;<\; 0.$$

Then from (6.43), we see that x is an eigenfunction of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$ associated to the eigenvalue

$$\lambda \;=\; \frac{1}{2\sqrt{-c}}.$$

Conversely, if $x \in W_0^{1,p}(\Omega)$ is an eigenfunction of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$ corresponding to an eigenvalue $\lambda > 0$, then if we set

$$u \stackrel{df}{=} \frac{x}{(2\lambda\varphi_1(x))^{\frac{1}{p}}},$$

we can check that u is an eigenfunction corresponding to the eigenvalue

$$\lambda' \;=\; \frac{1}{2\varphi_1(u)}$$

and u is also a critical point of φ with critical value $c = -\frac{1}{4\lambda^2}$.

Let

$$\mathcal{T}_n^c \overset{df}{=} \Big\{ C \subseteq W_0^{1,p}(\Omega) : \ 0 \notin C, \ C \text{ is compact, symmetric and } \gamma(C) \geqslant n \Big\},$$

with γ being the Krasnoselskii genus (see Definition 5.4.25). We consider the sequence

$$c_n \overset{df}{=} \inf_{C \in \mathcal{T}_n^c} \max_{x \in C} \varphi(x). \tag{6.44}$$

LEMMA 6.2.19
For every $n \geqslant 1$, c_n defined by (6.44) is a critical value of φ and we have

$$-\infty < \inf_{x \in W_0^{1,p}(\Omega)} \varphi(z) = c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n \leqslant \ldots < 0 = \varphi(0).$$

PROOF According to the Lusternik-Schnirelman theory (see Theorem 5.5.12 with topological index the Krasnoselskii genus), we need to show that φ is bounded from below, satisfies the PS-condition and for every $n \geqslant 1$, there exists a compact, symmetric set $C \subseteq W_0^{1,p}(\Omega)$ with $\gamma(C) = n$, such that

$$\sup_{x \in C} \varphi(x) < 0.$$

Note that

$$\begin{aligned} \varphi(x) &= \frac{1}{p^2} \|Dx\|_p^{2p} - \frac{1}{p} \int_\Omega m|x|^p \, dz \\ &\geqslant \frac{1}{p^2} \|Dx\|_p^{2p} - \frac{1}{p} \|m\|_\infty \|x\|_p^p \\ &\geqslant \frac{1}{p^2} \|Dx\|_p^{2p} - \frac{1}{p\lambda_1} \|m\|_\infty \|Dx\|_p^p \qquad \forall \ x \in W_0^{1,p}(\Omega). \end{aligned} \tag{6.45}$$

Since $\|Dx\|_p$ is an equivalent norm on $W_0^{1,p}(\Omega)$, from (6.45) we infer that φ is coercive. Hence it is bounded from below and satisfies the PS-condition.

For a given $n \geqslant 1$, let $v_1, \ldots, v_n$ be functions in $C_c^1(\Omega)$, such that

$$\operatorname{supp} v_i \cap \operatorname{supp} v_j = \emptyset \qquad \forall \ i \neq j$$

and

$$\int_\Omega m|v_i|^p \, dz > 0.$$

We can generate such a family by taking disjoint, open balls $\{B_i\}_{i=1}^n$ in Ω, such that

$$\left|B_i \cap \left\{z \in \Omega : \ m(z) > 0\right\}\right|_N \ > \ 0 \qquad \forall \ i \in \{1, \dots, n\}$$

and then approximating the $L^p(\Omega)$ function $\chi_{B_i \cap \{m>0\}}$. Let

$$Y_n \stackrel{df}{=} \operatorname{span}\{v_i\}_{i=1}^n.$$

For every $y \in Y_n$, $y = \sum_{i=1}^N \beta_i v_i$, $\beta_i \in \mathbb{R}$, we have

$$\varphi_2(y) \ = \ \sum_{i=1}^n |\beta_i|^p.$$

Hence the map

$$y \longmapsto \varphi_2(y)^{\frac{1}{p}}$$

is a norm in Y_n and because Y_n is finite dimensional all norms are equivalent and so we can find $c > 0$, such that

$$c\varphi_1(y) \ \leqslant \ \varphi_2(y) \ \leqslant \ \frac{1}{c}\varphi_1(y) \qquad \forall \ y \in Y_n.$$

Consider the compact set $C \subseteq W_0^{1,p}(\Omega)$, defined by

$$C \stackrel{df}{=} \left\{y \in Y_n : \frac{c^2}{4} \leqslant \varphi_2(y) \leqslant \frac{c^2}{3}\right\}.$$

We have

$$\sup_{y \in C} \varphi(y) \ \leqslant \ -\frac{c^2}{8} \ < \ 0.$$

Moreover, since Y_n is isomorphic to $\mathbb{R}^n$, we can identify C with an annulus C' in $\mathbb{R}^n$, such that

$$\partial B_1 \ \subseteq \ C' \ \subseteq \ \mathbb{R}^n \setminus \{0\}.$$

Therefore $\gamma(C) = n$ and we have proved the lemma. □

REMARK 6.2.20 The functional φ has an infinity of critical points. Indeed, if $i < j$ and $c_i = c_j = c$, then $\gamma(K_c^\varphi) \geqslant j - i + 2$, where

$$K_c^\varphi \stackrel{df}{=} \left\{x \in W_0^{1,p}(\Omega) : \ \varphi(x) = c, \ \varphi'(x) = 0\right\}$$

and so K_c^φ is infinite (see Remark 5.4.31). However, this does not mean that the sequence $\{c_n\}_{n \geqslant 1}$ is infinite.

LEMMA 6.2.21
If the sequence $\{c_n\}_{n\geqslant 1}$ *is defined by (6.44),*
then $c_n \longrightarrow 0$.

PROOF Let

$$M \stackrel{df}{=} \big\{x \in W_0^{1,p}(\Omega) :\ \varphi(x) \leqslant 0\big\}.$$

We need to show that for every $\varepsilon > 0$, we can find an integer $n_\varepsilon \geqslant 0$, such that

$$-\varepsilon \ \leqslant\ \sup_{x\in C} \varphi(x) \qquad \forall\ C \in \mathcal{T}^c_{n_\varepsilon},$$

with $C \subseteq M$. From the proof of Lemma 6.2.19, we know that

$$\varphi(x) \ \longrightarrow\ +\infty \quad \text{as } \|x\|_{W^{1,p}(\Omega)} \to +\infty.$$

So M is bounded in $W_0^{1,p}(\Omega)$. Since the canonical injection

$$i\colon W_0^{1,p}(\Omega) \longrightarrow L^p(\Omega)$$

is compact, for a given $\vartheta > 0$, we can find a finite dimensional subspace Y_ϑ of $L^p(\Omega)$ and a map $h_\vartheta\colon M \longrightarrow Y_\vartheta$, such that

$$\sup_{y\in M} \big\|y - h_\vartheta(y)\big\|_p \ \leqslant\ \vartheta.$$

Let us set

$$\widehat{h}_\vartheta(y) \ =\ \frac{1}{2}\big(h_\vartheta(y) - h_\vartheta(-y)\big).$$

Because M is symmetric, we see that

$$\widehat{h}_\vartheta\colon M \longrightarrow Y_\vartheta$$

and it is continuous, odd and satisfies

$$\sup_{y\in M} \big\|y - \widehat{h}_\vartheta(y)\big\|_p \ \leqslant\ \vartheta.$$

The set M being bounded in $W_0^{1,p}(\Omega)$, it is relatively compact in $L^p(\Omega)$ and so for a given $\varepsilon > 0$, we can find $\vartheta_\varepsilon > 0$, such that

$$\Big|\varphi_2(y) - \varphi_2\big(\widehat{h}_{\vartheta_\varepsilon}(y)\big)\Big| \ \leqslant\ \frac{\varepsilon}{2} \qquad \forall\ y \in M.$$

Let $\delta_\varepsilon > 0$ be such that

$$\varphi_2(y) \ \leqslant\ \frac{\varepsilon}{2} \qquad \forall\ \|y\|_p \leqslant \delta_\varepsilon.$$

So if $y \in M$, with

$$\big\|\widehat{h}_{\vartheta_\varepsilon}(y)\big\|_p \ \leqslant\ \delta_\varepsilon,$$

we have

$$\varphi_2(y) \leqslant \left|\varphi_2(y) - \varphi_2\big(\widehat{h}_{\vartheta_\varepsilon}(y)\big)\right| + \varphi_2\big(\widehat{h}_{\vartheta_\varepsilon}(y)\big) < \varepsilon.$$

Hence, if the set

$$C \subseteq M \cap \big\{y \in W_0^{1,p}(\Omega) \colon \varphi_2(y) \geqslant \varepsilon\big\}$$

is compact and symmetric, we have

$$\widehat{h}_{\vartheta_\varepsilon}(C) \subseteq \big\{y \in Y_{\vartheta_\varepsilon} : \|y\|_p \geqslant \delta_\varepsilon\big\}.$$

Since the set $\widehat{h}_{\vartheta_\varepsilon}(C) \subseteq L^p(\Omega)$ is compact and symmetric, we have

$$\gamma_p\big(\widehat{h}_{\vartheta_\varepsilon}(C)\big) = \dim Y_{\vartheta_\varepsilon},$$

with γ_p being the Krasnoselskii's genus in $L^p(\Omega)$ (see Proposition 5.4.28). Note that

$$\gamma(C) \leqslant \gamma_p\big(\widehat{h}_{\vartheta_\varepsilon}(C)\big)$$

(see Proposition 5.4.29**(d)**). So

$$\gamma(C) \leqslant \dim Y_{\vartheta_\varepsilon}.$$

Hence for all compact and symmetric sets $C \subseteq M$, such that

$$\gamma(C) \geqslant \dim Y_{\vartheta_\varepsilon} + 1,$$

we can find $y_0 \in C$, such that

$$\inf_{y \in C} \varphi_2(y) \leqslant \varphi_2(y_0) < \varepsilon$$

and since $\varphi(y) \geqslant -\varphi_2(y)$, we have

$$\sup_{y \in C} \varphi(y) \geqslant -\inf_{y \in C} \varphi_2(y) > -\varepsilon,$$

which proves the lemma. □

Using Lemma 6.2.19 and Lemma 6.2.21 and recalling the discussion before Lemma 6.2.19, we can state the following theorem concerning the spectrum of $\big(-\Delta_p, W_0^{1,p}(\Omega), m\big)$.

THEOREM 6.2.22

The nonlinear eigenvalue problem (6.23) has a sequence $\{\lambda_n\}_{n \geqslant 1}$ *of positive eigenvalues, such that* $\lambda_n \longrightarrow +\infty$.

REMARK 6.2.23 If $\big|\{z \in \Omega : \ m(z) < 0\}\big|_N > 0$, then if in the above analysis we replace m by $-m$, we see that (6.23) has a sequence $\{\lambda_{-n}\}_{n\geqslant 1}$ of negative eigenvalues, such that $\lambda_{-n} \longrightarrow -\infty$. From (6.44) and since $\lambda_n = \frac{1}{2\sqrt{-c_n}}$, we see that

$$\lambda_n \ = \ \inf_{C\in\mathcal{T}_n^c} \ \sup_{\substack{y \ \in \ C \\ \varphi_2(y) \ = \ 1}} \ \varphi_1(y).$$

Therefore

$$\lambda_1 \ = \ \inf_{\substack{y \ \in \ W_0^{1,p}(\Omega) \\ \varphi_2(y) \ = \ 1}} \ \varphi_1(y),$$

which is what we obtained in Proposition 6.2.2. It is easy to check that the eigenvalues of (6.23) form a closed set in $\mathbb{R}$ and the set of the corresponding normalized eigenfunctions form a compact set in $W_0^{1,p}(\Omega)$.

However, if $p \neq 2$ (nonlinear case), we do not know if $\{\lambda_n\}_{n\geqslant 1}$ are the only eigenvalues of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$. The eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ are known as the ***variational*** or ***Lusternik-Schnirelman eigenvalues*** of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$. If (λ, u) is an eigenelement of (6.23) and

$$\Omega_0(u) \ \stackrel{df}{=} \ \big\{z \in \Omega : u(z) = 0\big\},$$

then by $N(u)$ we denote the number of connected components of the open set $\Omega \setminus \Omega_0(u)$. Let

$$N(\lambda) \ \stackrel{df}{=} \ \max\Big\{N(u) : \ (\lambda, u) \text{ is an eigenelement of (6.23)}\Big\}.$$

Using Remark 6.2.16, we can easily verify that

$$\lambda_{N(\lambda)} \ \leqslant \ \lambda.$$

So if $\lambda_n < \lambda_{n+1}$, then $N(u_{\lambda_n}) \leqslant n$, which is the nonlinear analog of Courant's nodal theorem (see also Theorem 6.6.1).

Because $\lambda_1 > 0$ is isolated, we have

$$\widehat{\lambda}_2 \ \stackrel{df}{=} \ \inf\Big\{\lambda : \ \lambda \text{ is an eigenvalue of } \big(-\Delta_p, W_0^{1,p}(\Omega), m\big), \ \lambda > \lambda_1\Big\} \ > \ \lambda_1.$$

THEOREM 6.2.24
$\lambda_2 = \widehat{\lambda}_2$.

PROOF Since the set of eigenvalues of $\big(-\Delta_p, W_0^{1,p}(\Omega), m\big)$ is closed, we have that $\widehat{\lambda}_2 > 0$ is an eigenvalue of (6.23). If u is an eigenfunction corresponding to $\widehat{\lambda}_2$, then u must change sign (see Proposition 6.2.15). Hence $2 \leqslant N(u)$ and so $\lambda_2 \leqslant \lambda_{N(\widehat{\lambda}_2)}$. But from Remark 6.2.23, we know that $\lambda_{N(\widehat{\lambda}_2)} \leqslant \lambda_2$ and so we conclude that $\lambda_2 = \widehat{\lambda}_2$. ∎

REMARK 6.2.25 If we have two weighted functions $m_1, m_2 \in L^\infty(\Omega)$, $m_1 \leqslant m_2$, $m_1 \neq m_2$ and $\big|\{m_1 > 0\}\big|_N > 0$, then if by $\lambda_1(m_1)$ (respectively $\lambda_1(m_2)$) we denote the first positive eigenvalue of (6.23) with weight m_1 (respectively m_2), we have

$$\lambda_1(m_1) \ > \ \lambda_1(m_2)$$

(see Proposition 6.2.2). Using the variational characterization of $\lambda_2 > 0$ established in Theorem 6.2.24, we can show that if $m_1(z) < m_2(z)$ for almost all $z \in \{m_1 > 0\}$, then $\lambda_2(m_2) < \lambda_2(m_1)$ (see Anane & Tsouli (1996)).

In the last part of this section we establish a few basic things about the spectrum of $-\Delta_p$ with Neumann boundary conditions (i.e., of $\big(-\Delta_p, W^{1,p}(\Omega)\big)$). So the nonlinear eigenvalue problem under consideration is the following:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\big) = \lambda |x(z)|^{p-2} x(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial x}{\partial n_p} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.46}$$

Here $\frac{\partial x}{\partial n_p} = \|Dx\|_{\mathbb{R}^N}^{p-2}(Dx, n)_{\mathbb{R}^N}$ with n being the outward unit normal on $\partial\Omega$.

Clearly $\lambda = 0$ is an eigenvalue of (6.46) with the constant functions as eigenfunctions. More precisely, we have the following result.

PROPOSITION 6.2.26
$\lambda = 0$ is the first eigenvalue of $\big(-\Delta_p, W^{1,p}(\Omega)\big)$ and it is isolated and simple.

PROOF First note that (6.46) cannot have negative eigenvalues. Indeed, if $\lambda < 0$ is an eigenvalue of (6.46), then by multiplying with $x(z)$, integrating over Ω and using the nonlinear Green's identity (see Theorem 2.4.53), we obtain

$$\|Dx\|_p^p \ = \ \lambda \|x\|_p^p,$$

a contradiction since $\lambda < 0$.

The simplicity of $\lambda = 0$ is a direct consequence of the fact that

$$0 \ = \ \inf_{\substack{x \in W^{1,p}(\Omega) \\ x \neq 0}} \frac{\|Dx\|_p^p}{\|x\|_p^p}.$$

Finally suppose that $\lambda = 0$ is not isolated. We can find a sequence $\{\lambda_n\}_{n \geqslant 1}$ of positive eigenvalues of (6.46), such that $\lambda_n \searrow 0$. Consider a sequence $\{u_n\}_{n \geqslant 1}$ of associated eigenfunctions. From Theorems 6.2.6 and 6.2.7**(a)**, we have that $u_n \in C^1(\overline{\Omega})$. Assume without any loss of generality that $\|u_n\|_p = 1$ for $n \geqslant 1$. We have

$$\lambda_n \ = \ \frac{\|Du_n\|_p^p}{\|u_n\|_p^p} \ = \ \|Du_n\|_p^p \ \searrow \ 0.$$

So the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. Therefore by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega), \\ u_n &\longrightarrow u \quad \text{in } L^p(\Omega) \end{aligned}$$

(recall that the embedding $W^{1,p}(\Omega) \subseteq L^p(\Omega)$ is compact; see Theorem 2.5.17). We have

$$\|u\|_p \;=\; 1 \quad \text{and} \quad \|Du\|_p \;=\; 0$$

and so

$$u \;=\; \pm \frac{1}{|\Omega|_N^{\frac{1}{p}}}.$$

Using as a test function $y \equiv 1 \in W^{1,p}(\Omega)$ in (6.46), we obtain

$$\int_\Omega \big|u_n(z)\big|^{p-2} u_n(z)\, dz \;=\; 0 \qquad \forall\, n \geqslant 1.$$

Passing to the limit as $n \to +\infty$, we have

$$\int_\Omega \big|u(z)\big|^{p-2} u(z)\, dz \;=\; 0,$$

a contradiction. □

We will characterize the first nonzero element of the spectrum of

$$\big(-\Delta_p, W^{1,p}(\Omega)\big).$$

Suppose that $\lambda > 0$ is an eigenvalue of (6.46) with a corresponding eigenfunction $u \in C^1(\overline{\Omega})$. Integrating (6.46) and using the nonlinear Green's identity (see Theorem 2.4.53), we obtain

$$\int_\Omega \big|u(z)\big|^{p-2} u(z)\, dz \;=\; 0.$$

So we are led to the consideration of the following nonempty, closed, symmetric and pointed cone:

$$C(p) \stackrel{df}{=} \left\{ x \in W^{1,p}(\Omega) : \int_\Omega \big|x(z)\big|^{p-2} x(z)\, dz = 0 \right\}.$$

Note that if $p = 2$, then $C(p)$ is the orthogonal complement of the kernel of the operator $-\Delta$ with Neumann boundary conditions (which is $\mathbb{R}$). We intersect $C(p)$ with the unit sphere of $L^p(\Omega)$ and obtain

$$C_1(p) \stackrel{df}{=} \left\{ x \in W^{1,p}(\Omega) : \|x\|_p = 1,\ \int_\Omega \big|x(z)\big|^{p-2} x(z)\, dz = 0 \right\}.$$

Let $\vartheta_p \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the strictly convex, C^1-map, defined by

$$\vartheta_p(x) \;=\; \|Dx\|_p^p \qquad \forall\, x \in W^{1,p}(\Omega).$$

We consider the following minimization problem:

$$\lambda_1(p) \;=\; \inf_{x \in C_1(p)} \vartheta_p(x). \tag{6.47}$$

PROPOSITION 6.2.27
$\lambda_1(p) > 0$ and we can find $x \in C_1(p)$, such that $\vartheta_p(x) = \lambda_1(p)$.

PROOF Let $\{x_n\}_{n \geqslant 1} \subseteq C_1(p)$ be a minimizing sequence for problem (6.47). Evidently the sequence $\{x_n\}_{n \geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$\begin{aligned} x_n &\xrightarrow{w} x && \text{in } W_0^{1,p}(\Omega), \\ x_n &\longrightarrow x && \text{in } L^p(\Omega), \\ x_n(z) &\longrightarrow x(z) && \text{for a.a. } z \in \Omega, \\ \big|x_n(z)\big| &\leqslant k(z) && \text{for a.a. } z \in \Omega, \end{aligned}$$

with $k \in L^p(\Omega)_+$.

Via the Lebesgue dominated convergence theorem (see Theorem A.2.2), we have that

$$\int_\Omega \big|x(z)\big|^{p-2} x(z)\, dz \;=\; 0 \quad \text{and} \quad \|x\|_p \;=\; 1,$$

i.e., $x \in C_1(p)$. Moreover, exploiting the weak lower semicontinuity of the norm functional, we have

$$\|Dx\|_p^p \;\leqslant\; \lambda_1(p),$$

hence

$$\|Dx\|_p^p \;=\; \lambda_1(p).$$

Note that since $x \in C_1(p)$, x is nonconstant and so

$$\lambda_1(p) \;=\; \|Dx\|_p^p \;>\; 0.$$

□

The above proposition leads to the following Poincaré-Wirtinger inequality.

COROLLARY 6.2.28 (Poincaré-Wirtinger Inequality)
If $x \in C(p)$ and $\lambda_1 = \lambda_1(p) > 0$,
then $\lambda_1 \|x\|_p^p \leqslant \|Dx\|_p^p$.

Next we show that for $p \geqslant 2$, $\lambda_1 = \lambda_1(p) > 0$ is the first nonzero eigenvalue of $\big(-\Delta_p, W^{1,p}(\Omega)\big)$.

THEOREM 6.2.29
If $p \geqslant 2$,
then $\lambda_1 > 0$ *is the first nonzero eigenvalue of* $\big(-\Delta_p, W^{1,p}(\Omega)\big)$.

PROOF Let $x \in C_1(p)$ be a solution of problem (6.47). By the Lagrange multiplier rule, we know that we can find $a, b, c \in \mathbb{R}$, not all of them equal to zero, such that

$$ap \int_\Omega \big\|Dx(z)\big\|_{\mathbb{R}^N}^{p-2} \big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} dz + bp \int_\Omega \big|x(z)\big|^{p-2} x(z)y(z)\, dz$$
$$+ c(p-1) \int_\Omega \big|x(z)\big|^{p-2} y(z)\, dz \;=\; 0 \qquad \forall\, y \in W^{1,p}(Z). \tag{6.48}$$

Let $y = c$ and recall that

$$\int_\Omega \big|x(z)\big|^{p-2} x(z)\, dz \;=\; 0$$

(since $x \in C_1(p)$). So we obtain

$$c^2(p-1) \int_\Omega \big|x(z)\big|^{p-2}\, dz \;=\; 0$$

and this implies that $c = 0$. Using this in (6.48), we have

$$ap \int_\Omega \big\|Dx(z)\big\|_{\mathbb{R}^N}^{p-2} \big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} dz + bp \int_\Omega \big|x(z)\big|^{p-2} x(z)y(z)\, dz \;=\; 0$$
$$\forall\, y \in W^{1,p}(\Omega).$$

If $a = 0$, then

$$bp \int_\Omega \big|x(z)\big|^{p-2} x(z)y(z)\, dz \;=\; 0 \qquad \forall\, y \in W^{1,p}(\Omega).$$

Let $y = x$. We obtain

$$bp\, \|x\|_p^p \;=\; 0,$$

i.e., $bp = 0$ and so $b = 0$, a contradiction, since a, b, c are not all equal to zero. So $a \neq 0$ and without any loss of generality, we can assume that $a = 1$. Then

$$\int_\Omega \big\|Dx(z)\big\|_{\mathbb{R}^N}^{p-2} \big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} dz + b \int_\Omega \big|x(z)\big|^{p-2} x(z)y(z)\, dz \;=\; 0$$
$$\forall\, y \in W^{1,p}(\Omega).$$

As before, we let $y = x$ and obtain

$$\|Dx\|_p^p + b\,\|x\|_p^p \;=\; 0,$$

hence

$$b \;=\; -\lambda_1 \;=\; -\lambda_1(p).$$

Using the nonlinear Green's identity (see Theorem 2.4.53), we see that $x \in C_1(p)$ solves (6.46) with $\lambda = \lambda_1$.

Therefore $\lambda_1 > 0$ is an eigenvalue of $\big(-\Delta_p, W^{1,p}(\Omega)\big)$ and clearly there is no eigenvalue in the open interval $(0, \lambda_1)$. □

Finally we consider the weighted eigenvalue problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|_{\mathbb{R}^N}^{p-2}\, Dx(z)\big) = \lambda m(z)\big|x(z)\big|^{p-2}x(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial x}{\partial n_p} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.49}$$

with $m \in L^\infty(\Omega)$. We define

$$\widehat{\lambda}(m) \;\stackrel{df}{=}\; \inf\left\{\big\|Dx\big\|_p^p :\; x \in W^{1,p}(\Omega),\; \int_\Omega m(z)\big|x(z)\big|^p\,dz = 1\right\}.$$

The following theorem is due to Huang (1990) and Godoy, Gossez & Paczka (2002).

THEOREM 6.2.30
(a) *If* $\int_\Omega m(z)\,dz < 0$,
then $\widehat{\lambda}(m) > 0$ *and* $\widehat{\lambda}(m)$ *is the unique nonzero eigenvalue, which is simple and has a positive eigenfunction (i.e.,* $\widehat{\lambda}(m)$ *is principal) and there is no eigenvalue in the interval* $\big(0, \widehat{\lambda}(m)\big)$.

(b) *If* $\int_\Omega m(z)\,dz \geqslant 0$,
then $\widehat{\lambda}(m) = 0$ *and when* $\int_\Omega m(z)\,dz = 0$, *we have that* 0 *is the unique eigenvalue with a positive eigenfunction.*

REMARK 6.2.31 If $\int_\Omega m(z)\,dz > 0$, then there is no eigenvalue $\lambda > 0$ with positive eigenfunctions.

6.3 The Ordinary p-Laplacian

In this section we examine the spectrum of the ordinary p-Laplacian (both the scalar and the vector ordinary p-Laplacian) under Dirichlet, Neumann and periodic boundary conditions. We start with the scalar operator and Dirichlet boundary conditions. So for $p > 1$ we consider the following scalar eigenvalue problem:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \lambda |x(t)|^{p-2}x(t) & \text{for a.a. } t \in T, \\ x(0) = x(b) = 0, \end{cases} \tag{6.50}$$

with $T = [0, b]$, $b > 0$, $\lambda \in \mathbb{R}$ and x' stands for $\frac{dx}{dt}$.

A simple integration argument shows that a necessary condition for (6.50) to have a nontrivial solution is that $\lambda \geqslant 0$. So throughout the analysis of problem (6.50), we will assume that $\lambda \geqslant 0$.

DEFINITION 6.3.1 *We say that $\lambda \geqslant 0$ is an **eigenvalue** of the negative scalar ordinary p-Laplacian with Dirichlet boundary conditions (still denoted by $\big(-\Delta_p, W_0^{1,p}\big((0,b)\big)\big)$), if problem (6.50) admits a nontrivial solution $x \in W_0^{1,p}\big((0,b)\big)$ which is known as an **eigenfunction** corresponding to the eigenvalue $\lambda \geqslant 0$.*

In what follows $\mathbb{N}_0 \stackrel{df}{=} \mathbb{N} \cup \{0\}$ (the set of nonnegative integers) and for $r \in \mathbb{R}$

$$\langle r \rangle \stackrel{df}{=} \begin{cases} +\infty & r \in 2\mathbb{N}_0, \\ \min\big\{n \in \mathbb{N}_0 : \ n \geqslant r\big\} & \text{otherwise.} \end{cases} \tag{6.51}$$

We start by establishing the complete regularity of the eigenfunctions in problem (6.50).

PROPOSITION 6.3.2
If $x \in W_0^{1,p}\big((0,b)\big)$ is an eigenfunction for problem (6.50), then $x \in C^{\alpha}(T) \cap C^{\langle p \rangle}\big(T \setminus Z(x)\big)$, where

$$Z(x) \stackrel{df}{=} \big\{t \in T : \ x'(t) = 0\big\} \quad \text{and} \quad \alpha \stackrel{df}{=} \min\left\{\left\langle \frac{2-p}{p-1} \right\rangle + 1, \langle p \rangle\right\}.$$

Moreover, if $p > 2$, then x is not everywhere twice differentiable in $(0, b)$.

PROOF Let $y \in C^1(T)$ be defined by

$$y(t) \stackrel{df}{=} \int_0^t |x(s)|^{p-2}x(s)\, ds.$$

Then for all $\vartheta \in C_c^\infty\big((0,b)\big)$, we have

$$\begin{aligned}\big\langle -\big(|x'|^{p-2}x'\big)', \vartheta\big\rangle_{W_0^{1,p}((0,b))} &= \lambda\big\langle |x|^{p-2}x, \vartheta\big\rangle_{W_0^{1,p}((0,b))} \\ &= \lambda \langle y', \vartheta\rangle_{W_0^{1,p}((0,b))},\end{aligned}$$

so by integration by parts, we have

$$\big\langle |x'|^{p-2}x', \vartheta'\big\rangle_{W_0^{1,p}((0,b))} = -\lambda \langle y, \vartheta'\rangle_{W_0^{1,p}((0,b))}. \tag{6.52}$$

Since $\vartheta \in C_c^\infty\big((0,b)\big)$ is arbitrary, from (6.52), it follows that

$$\big|x'(t)\big|^{p-2}x'(t) = -\lambda y(t) + c \quad \text{for a.a. } t \in T,$$

with $c \in \mathbb{R}$, so

$$|x'|^{p-2}x' \in C^1(T). \tag{6.53}$$

The map $\psi_p\colon \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\psi_p(r) \stackrel{df}{=} \begin{cases} |r|^{p-2}r & \text{if} \quad r \neq 0 \\ 0 & \text{if} \quad r = 0 \end{cases}$$

is a homeomorphism for $p \in (1, +\infty)$ and so from (6.53), it follows that $x' \in C(T)$, hence $x \in C^1(T)$.

Suppose that $x \in C^r(T)$ (respectively $x \in C^r\big(T \setminus Z(x)\big)$) for $r \geqslant 1$. Then

$$\big|x(\cdot)\big|^{p-2}x(\cdot) \in C^{r_0}(T)$$

(respectively $\big|x(\cdot)\big|^{p-2}x(\cdot) \in C^{r_0}\big(T \setminus Z(x)\big)$), where

$$r_0 \stackrel{df}{=} \min\big\{r, \langle p\rangle - 2\big\}$$

and so $y \in C^{r_0+1}(T)$ (respectively $y \in C^{r_0+1}\big(T \setminus Z(x)\big)$).

If

$$u(t) \stackrel{df}{=} \big|x'(t)\big|^{p-2}x'(t),$$

since $\psi_p^{-1}(r) = |r|^{p_0}r$, with $p_0 = \frac{2-p}{p-1} = p' - 2$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), we have that

$$x'(t) = \big|u(t)\big|^{p_0}u(t) \qquad \forall\, t \in T$$

and so

$$x'(t) = 0 \iff u(t) = 0.$$

Hence $x \in C^m(T)$ (respectively $x \in C^{r_0+2}\big(T \setminus Z(x)\big)$), where

$$m = \min\big\{\langle p_0\rangle + 1, r_0 + 2\big\}.$$

Continuing this procedure as far as $r \leqslant m$ (respectively $r \leqslant r_0 + 2$), we have the first result of the proposition.

To prove the second part of the theorem assume that $p > 2$ and $x \in C^2\big((0,b)\big)$. Then we have

$$\big(|x'|^{p-2}x'\big)' \;=\; (p-2)|x'|^{p-2}x'' + |x'|^{p-2}x'' \;=\; (p-1)|x'|^{p-2}x''.$$

So we obtain

$$-(p-1)\big|x'(t)\big|^{p-2}x''(t) \;=\; \big|x(t)\big|^{p-2}x(t) \quad \text{for a.a. } t \in T. \tag{6.54}$$

But there exists a point $t_0 \in (0,b)$, where x attains its positive maximum (or negative minimum), i.e.,

$$x'(t_0) \;=\; 0 \quad \text{and} \quad x(t_0) \;>\; 0 \quad (\text{or } x(t_0) < 0).$$

These facts compared with (6.54) lead to a contradiction. □

REMARK 6.3.3 If $p > 2$, then $\alpha = 1$ and if $p \in (1,2]$, then $\alpha \geqslant 2$. In particular we have $x \in C^1(T)$ for $p > 2$ and $x \in C^2(T)$ for $p \in (1,2]$.

For $n \geqslant 1$, let

$$\begin{aligned} S_n(T) &\stackrel{df}{=} \Big\{x \in C_0^1(T) :\ x \text{ has exactly } n-1 \text{ simple zeros in } (0,b)\Big\},\\ S_n^+(T) &\stackrel{df}{=} \big\{x \in S_n(T) :\ x'(0) > 0\big\},\\ S_n^-(T) &\stackrel{df}{=} -S_n^+(T). \end{aligned}$$

From (6.50), we see that the first eigenvalue $\lambda_1 > 0$ is given by

$$\lambda_1 \;=\; \inf_{\substack{x \,\in\, W_0^{1,p}((0,b))\\ \|x\|_p \,=\, 1}} \|x'\|_p^p \;=\; \inf_{\substack{x \,\in\, W_0^{1,p}((0,b))\\ x \,\neq\, 0}} \frac{\|x'\|_p^p}{\|x\|_p^p}. \tag{6.55}$$

THEOREM 6.3.4
There is a unique sequence of functions $u_k \in S_k^+$, $k \geqslant 1$, with maximum value 1 on $(0,b)$, such that $\big\{(\lambda_n, \mu u_n)\big\}_{n \geqslant 1}$ are eigenelements of (6.50), where $\mu \in \mathbb{R} \setminus \{0\}$ and

$$\lambda_n \;=\; \Big(\frac{n}{b}\Big)^p \lambda_1 \;=\; \Big(\frac{n}{b}\Big)^p (p-1)\bigg[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\bigg]^p. \tag{6.56}$$

Moreover, for $m \in \big\{0, 1, \ldots, n-1\big\}$, we have

$$u_n(t) \;=\; (-1)^m u_1\big(nt - mb\big) \qquad \forall\ t \in \bigg[\frac{bm}{n}, \frac{b(m+1)}{n}\bigg]. \tag{6.57}$$

PROOF From Remark 6.3.3, we know that every eigenfunction $u \in C(T)$ satisfies $u \in C^1(T)$ if $p > 2$ and $u \in C^2(T)$ if $p \in (1, 2]$.

Suppose that (λ, u) is an eigenelement of (6.50). Then multiplying (6.50) with $u(t)$ and then integrating over T, we obtain

$$\|u'\|_p^p \;=\; \lambda \|u\|_p^p,$$

i.e., $\lambda > 0$.

Also multiplying (6.50) with $u'(t)$ and integrating over $[0, t]$, $t \in (0, 1)$, after integration by parts, we have

$$(p-1)\big|u'(t)\big|^p + \lambda\big|u(t)\big|^p \;=\; (p-1)\big|u'(0)\big|^p. \tag{6.58}$$

Since $t \in (0, b)$ was arbitrary and $u \neq 0$, we must have

$$u'(0) \;\neq\; 0.$$

Now we will construct u explicitly. Since $u'(0) \neq 0$, we may assume that

$$u'(0) \;=\; \vartheta \;>\; 0.$$

Then we can find $t_1 \in (0, b)$, such that

$$u'(t) > 0 \qquad \forall\; t \in (0, t_1)$$

and from (6.58) solving for $u'(t)$, we have

$$u'(t) \;=\; \left(\vartheta^p - \frac{\lambda}{p-1}u(t)^p\right)^{\frac{1}{p}} \qquad \forall\; t \in [0, t_1]. \tag{6.59}$$

Then from the inverse function theorem, it follows that

$$t \;=\; \int_0^{u(t)} \frac{ds}{(\vartheta^p - \frac{\lambda s^p}{p-1})^{\frac{1}{p}}}.$$

This formula remains valid as long as $u(t)$ remains smaller than the first positive zero of the function

$$s \longmapsto \varphi(\vartheta, s) \stackrel{df}{=} \vartheta^p - \frac{\lambda s^p}{p-1}.$$

This zero is given by

$$z(\vartheta) \;=\; \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p}} \vartheta.$$

Because $z(\vartheta)$ is simple, we can define $\xi(\vartheta)$, by

$$\xi(\vartheta) \;\stackrel{df}{=}\; \int_0^{z(\vartheta)} \frac{ds}{(\vartheta^p - \frac{\lambda s^p}{p-1})^{\frac{1}{p}}}.$$

We have

$$u\big(\xi(\vartheta)\big) \;=\; z(\vartheta) \quad \text{and} \quad u'\big(\xi(\vartheta)\big) \;=\; 0.$$

Because

$$\vartheta^p \;=\; \frac{\lambda z(\vartheta)^p}{p-1},$$

we can write

$$\xi(\vartheta) \;=\; \xi_\lambda \;=\; c\bigg(\frac{p-1}{\lambda}\bigg)^{\frac{1}{p}} \quad \text{with} \quad c \overset{df}{=} \int\limits_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}}. \tag{6.60}$$

From (6.59), we see that u is decreasing on some interval $[\xi_\lambda, \eta)$ and so

$$t - \xi_\lambda \;=\; -\int\limits_{u(t)}^{z(\vartheta)} \frac{ds}{(\vartheta^p - \frac{\lambda s^p}{p-1})^{\frac{1}{p}}}$$

and this formula remains valid as long as u is decreasing, in particular as long as u is positive. If

$$t_2 \in (0, \xi_\lambda) \quad \text{and} \quad t_3 = 2\xi_\lambda - t_2,$$

then

$$t_2 \;=\; \int\limits_0^{u(t_2)} \frac{ds}{\varphi(\vartheta, s)^{\frac{1}{p}}}, \quad \xi_\lambda - t_2 \;=\; -\int\limits_{u(t_3)}^{z(\vartheta)} \frac{ds}{\varphi(\vartheta, s)^{\frac{1}{p}}} \quad \text{and} \quad u(t_2) = u(t_3).$$

So $t = \xi_\lambda$ is an axis of symmetry for $u|_{[0,2\xi_\lambda]}$ and $t = 2\xi_\lambda$ is a center of symmetry for $u|_{[0,4\xi_\lambda]}$. It follows that

$$u \text{ is } 4\xi_\lambda\text{-periodic on } \mathbb{R}_+.$$

The necessary and sufficient condition for $u|_{_T}$ to be a solution of (6.50) is that $\frac{b}{2\xi_\lambda} = n \in \mathbb{N}$, so $2n\xi_\lambda = b$ and thus

$$\lambda \;=\; \bigg(\frac{2n}{b}\bigg)^p (p-1)\bigg[\int_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}}\bigg]^p$$

(see (6.60)). So we have produced the sequence given by (6.56).

The number of zeros of u is given by $b\big(\frac{1}{2\xi_\lambda} - 1\big)$ and since (6.50) is homogeneous, it follows that (6.57) holds. The uniqueness of the sequence $\big\{(\lambda_n, u_n)\big\}_{n\geqslant 1}$ is a consequence of the above construction. □

REMARK 6.3.5 In the variational expression for $\lambda_1 > 0$ (see (6.55)), the infimum is attained at

$$x \;=\; u_1.$$

Some authors write

$$\pi_p \;\stackrel{df}{=}\; 2\int_0^{(p-1)^{\frac{1}{p}}} \frac{ds}{(1-\frac{s^p}{p-1})^{\frac{1}{p}}} \;=\; \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})}$$

and then the sequence of eigenvalues λ_n produced in Theorem 6.3.4 is given by

$$\lambda_n \;=\; \left(\frac{n\pi_p}{b}\right)^p \qquad \forall\; n \geqslant 1.$$

Note that for $p = 2$, $\pi_p = \pi$ and so we recover the eigenvalues of the negative scalar ordinary Laplacian with Dirichlet boundary conditions, which are

$$\lambda_n \;=\; \left(\frac{n\pi}{b}\right)^2 \qquad \forall\; n \geqslant 1.$$

To derive the eigenvalues of the Neumann and periodic problems, we need to study the following initial value problem:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \lambda|x(t)|^{p-2}x(t) \quad \text{for a.a. } t \in \mathbb{R}, \\ x(t_0) = a,\;\; x'(t_0) = c, \end{cases} \tag{6.61}$$

for some $t_0 \in \mathbb{R}$. Here $a, c \in \mathbb{R}$. We have the following existence result for problem (6.61).

PROPOSITION 6.3.6
For any $\lambda \geqslant 0$, problem (6.61) has a unique solution defined on $\mathbb{R}$.

PROOF For $\lambda = 0$, the result follows from direct integration. So we may assume that $\lambda > 0$. Also since the equation is autonomous, without any loss of generality, we may assume that $t_0 = 0$. The existence of a local solution on an interval $(-\varepsilon, \varepsilon)$ (for $\varepsilon > 0$) follows from a simple application of Schauder's fixed point theorem on the compact operator

$$K\colon W^{1,p}\big((0,b)\big) \longrightarrow \mathbb{R},$$

defined by

$$K(x)(t) \;\stackrel{df}{=}\; ct - \int_0^t \psi_p^{-1}\left(\lambda\int_0^s \psi_p\big(x(\tau)\big)\,d\tau\right)ds \qquad \forall\; t \in T,$$

where as before $\psi_p\colon \mathbb{R} \longrightarrow \mathbb{R}$ is the homeomorphism, defined by

$$\psi_p(r) \stackrel{df}{=} \begin{cases} |r|^{p-2}r & \text{if} \quad r \neq 0, \\ 0 & \text{if} \quad r = 0; \end{cases}$$

recall that

$$\psi_p^{-1} \;=\; \psi_{p'} \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1.$$

We now show that this local solution is unique. We work only on $\mathbb{R}_+$.

Case 1. $a = c = 0$.

The solution x satisfies in its domain of definition

$$\frac{|x'(t)|^p}{p'} + \lambda \frac{\big|x(t)\big|^p}{p} \;=\; \frac{|c|^p}{p} + \lambda \frac{|a|^p}{p} \;=\; 0,$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. From this we deduce that

$$x \;\equiv\; 0.$$

Case 2. $a = 0$, $c \neq 0$.

Let x, y be two local solutions of (6.61) with the above initial conditions. Then

$$\psi_p\big(x'(t)\big) - \psi_p\big(y'(t)\big) \;=\; \lambda \int_0^t \big[\psi_p\big(y(s)\big) - \psi_p\big(x(s)\big)\big]\, ds$$

$$= \;\lambda \int_0^t s^{p-1} \left[\psi_p\left(\frac{y(s)}{s}\right) - \psi_p\left(\frac{x(s)}{s}\right)\right] ds. \tag{6.62}$$

Note that

$$\frac{x(s)}{s},\; \frac{y(s)}{s} \;\longrightarrow\; c \quad \text{as } s \to 0.$$

Since

$$\psi_p \text{ is a diffeomorphism on } \mathbb{R}^N \setminus \{0\},$$

we can find $\xi_1, \xi_2 > 0$, such that

$$\xi_1 \big|x'(t) - y'(t)\big| \;\leqslant\; \lambda\xi_2 \int_0^t s^{p-1} \left|\frac{y(s)}{s} - \frac{x(s)}{s}\right| ds,$$

so

$$\xi_1 \left\|v'\right\|_{C([0,\varepsilon])} \;\leqslant\; \lambda\xi_2 \varepsilon^{p-1} \left\|v\right\|_{C([0,\varepsilon])}. \tag{6.63}$$

Since

$$v(0) \;=\; 0, \quad v'(0) \;=\; 0 \quad \text{and} \quad v(t) \;=\; \int_0^t v'(s)\,ds,$$

from (6.63) and choosing $\varepsilon > 0$ small enough, we have

$$\left(\xi_1 - \lambda\xi_2\varepsilon^p\right)\|v'\|_{C([0,\varepsilon])} \;\leqslant\; 0,$$

from which we conclude that $v = 0$, hence $x = y$.

Case 3. $a \neq 0$, $c = 0$.

We reduce this case to Case 2. Namely, we rewrite (6.61) as the following equivalent system

$$\begin{cases} x' = \psi_{p'}(y), \\ y' = -\lambda\psi_p(x). \end{cases} \tag{6.64}$$

From the second equation in (6.64), we have

$$-x \;=\; \psi_{p'}\left(\frac{y'}{\lambda}\right).$$

Using this in the first equation of (6.64), we see that y satisfies

$$\begin{cases} -\left(\psi_{p'}(y')\right)' + \lambda^{p'-1}\psi_{p'}(y) = 0 \quad \text{on } T, \\ y(0) = 0,\ y'(0) = -\lambda\psi_p(a) \neq 0. \end{cases}$$

So we have reduced the problem to Case 2.

Case 4. $a, c \neq 0$.

Consider system (6.64). Since

$$x(0) \;\neq\; 0 \quad \text{and} \quad y(0) \;\neq\; 0,$$

we have

$$x(t), y(t) \neq 0,$$

for $t > 0$ small enough and so ψ_p and $\psi_{p'}$ are C^1-functions on that subinterval. Then the uniqueness follows from the standard ordinary differential equations theory.

So the solution of (6.61) is locally unique. Since for any solution x, we have

$$\frac{|x'(t)|^p}{p'} + \frac{|x(t)|^p}{p} \;=\; \frac{|c|^p}{p'} + \frac{|a|^p}{p},$$

we see that the solution x extends to all of $\mathbb{R}$ and so local uniqueness becomes global uniqueness. ∎

Using the above proposition, we can determine the spectrum of the negative scalar p-Laplacian with Neumann and periodic boundary conditions. We start with the Neumann problem. So we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \lambda|x(t)|^{p-2}x(t) & \text{for a.a. } t \in T, \\ x'(0) = x'(b) = 0. \end{cases} \tag{6.65}$$

Observe that $\lambda = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (the constant functions). Moreover, any nonconstant eigenfunction $u \in C^1(T)$ satisfies

$$\int_0^b |u(t)|^{p-2}u(t)\,dt \;=\; 0.$$

Therefore u must change sign. Since (6.65) is autonomous, Proposition 6.3.6 implies that the positive eigenvalues of (6.65) are the same with those of the Dirichlet problem and the corresponding eigenfunctions are translations of the Dirichlet eigenfunctions.

THEOREM 6.3.7
The eigenvalues of problem (6.65) are given by

$$\begin{aligned} \mu_0 &= 0, \\ \mu_n &= \left(\frac{n}{b}\right)^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p \qquad \forall\ n \geqslant 1 \end{aligned}$$

and the corresponding eigenfunctions are

$$\begin{aligned} v_0(t) &= c, \quad c \in \mathbb{R} \setminus \{0\}, \\ v_n(t) &= u_n\left(t - \frac{b}{2n}\right) \qquad \forall\ n \geqslant 1, \end{aligned}$$

where u_n are Dirichlet eigenfunctions.

Finally let us consider the periodic problem:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \lambda|x(t)|^{p-2}x(t) & \text{for a.a. } t \in T, \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{6.66}$$

A similar reasoning as for the Neumann problem yields the following result.

THEOREM 6.3.8
The eigenvalues of problem (6.66) are given by

$$\begin{aligned} \nu_0 &= 0, \\ \nu_n &= \left(\frac{2n}{b}\right)^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p \qquad \forall\ n \geqslant 1 \end{aligned}$$

and the corresponding eigenfunctions are

$$\begin{aligned} w_0(t) &= c, \quad c \in \mathbb{R} \setminus \{0\}, \\ w_n(t) &= u_n(t - t_n) \qquad \forall\ n \geqslant 1, \end{aligned}$$

with $t_n \in \mathbb{R}$ arbitrary and u_n being Dirichlet eigenfunctions.

REMARK 6.3.9 If $p = 2$ (linear case), then the eigenvalues for the Neumann problem are given by

$$\mu_n = \left(\frac{n\pi}{b}\right)^2 \qquad \forall\ n \geqslant 0$$

and the eigenvalues for the periodic problem are given by

$$\nu_n = \left(\frac{2\pi n}{b}\right)^2 \qquad \forall\ n \geqslant 0.$$

Since the eigenvalue problems (6.50), (6.65) and (6.66) are variational, we can produce a sequence of eigenvalues and eigenfunctions using the Lusternik-Schnirelman theory (see Section 5.5). It can be shown that the Lusternik-Schnirelman eigenelements coincide with those obtained in Theorems 6.3.4, 6.3.7 and 6.3.8. For details we refer to Drábek & Manásevich (1999).

In the last part of this section we consider the vector ordinary p-Laplacian. So now the function x is $\mathbb{R}^N$-valued. We consider the following nonlinear vector eigenvalue problem:

$$\begin{cases} -\big(\|x'(t)\|_{\mathbb{R}^N}^{p-2} x'(t)\big)' = \lambda \|x(t)\|_{\mathbb{R}^N}^{p-2} x(t) & \text{for a.a. } t \in T, \\ x(0) = x(b) = 0. \end{cases} \tag{6.67}$$

Exploiting the variational structure of (6.67) and using the Lusternik-Schnirelman theory (see Section 5.5), we can produce the eigenvalues of (6.67).

THEOREM 6.3.10
The eigenvalues of problem (6.67) are

$$\lambda_n = \left(\frac{n}{b}\right)^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p \qquad \forall\ n \geqslant 1$$

and the corresponding eigenfunctions are

$$\widehat{u}_n(t) = a u_n(t) \qquad \forall\ n \geqslant 1,$$

with $a \in \mathbb{R}^N$ and with u_n being scalar Dirichlet eigenfunctions.

For the periodic problem, the situation is more involved. So consider the following vector eigenvalue problem:

$$\begin{cases} -\big(\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t)\big)' = \lambda\|x(t)\|_{\mathbb{R}^N}^{p-2}x(t) & \text{for a.a. } t \in T, \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{6.68}$$

As before we see that $\lambda = 0$ is an eigenvalue of (6.68) with corresponding eigenspace $\mathbb{R}^N$ (the constant functions). In addition to (6.68), we also consider the following problem:

$$\begin{cases} -\big(\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t)\big)' - \mu\|x(t)\|_{\mathbb{R}^N}^{p-2}x(t) = h(t) & \text{for a.a. } t \in T, \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{6.69}$$

with $h \in L^1\big(T;\mathbb{R}^N\big)$. We want to determine those $\mu \in \mathbb{R}$ for which problem (6.69) has at least one solution for each $h \in L^1\big(T;\mathbb{R}^N\big)$. In analogy to the linear theory, we call this set the ***resolvent set*** of the negative vector ordinary p-Laplacian with periodic boundary conditions and denote it by $\varrho(p,N)$. Also denote by $\sigma(p,N)$ the set of eigenvalues λ of problem (6.68). We have the following result relating the two sets $\varrho(p,N)$ and $\sigma(p,N)$.

PROPOSITION 6.3.11
$\varrho(p,N)^c \subseteq \sigma(p,N)$.

PROOF First we show that (6.69) has a solution for every $h \in L^{p'}\big(T;\mathbb{R}^N\big)$. To this end note that

$$\sigma(p,N) \text{ is closed.}$$

So if

$$\lambda \notin \sigma(p,N),$$

then we can find $\varepsilon > 0$ small enough, such that

$$\big[\lambda-\varepsilon, \lambda+\varepsilon\big] \cap \sigma(p,N) \ = \ \emptyset.$$

For a given $g \in L^{p'}\big(T;\mathbb{R}^N\big)$, we consider the following auxiliary periodic problem:

$$\begin{cases} -\big(\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t)\big)' + \varepsilon\|x(t)\|_{\mathbb{R}^N}^{p-2}x(t) = g(t) & \text{for a.a. } t \in T, \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{6.70}$$

If

$$K\colon W^{1,p}_{\text{per}}\big((0,b);\mathbb{R}^N\big) \longrightarrow L^{p'}\big(T;\mathbb{R}^N\big)$$

is defined by

$$K(x)(\cdot) \ \stackrel{df}{=} \ \|x(\cdot)\|_{\mathbb{R}^N}^{p-2}x(\cdot) \qquad \forall\ x \in W^{1,p}_{\text{per}}\big((0,b);\mathbb{R}^N\big),$$

then the operator

$$W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big) \ni x \longmapsto A(x)+\varepsilon K(x) \in \big(W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big)\big)^*$$

is maximal monotone, strictly monotone, coercive; hence it is surjective. So problem (6.70) has a unique solution. Consider the operator

$$V_\varepsilon\colon L^{p'}\big(T;\mathbb{R}^N\big) \longrightarrow W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big)$$

which to each $g \in L^{p'}\big(T;\mathbb{R}^N\big)$ assigns the unique solution of (6.70). We will show that

$$V_\varepsilon \text{ is completely continuous}$$

(hence compact too since $L^{p'}\big(T;\mathbb{R}^N\big)$ is reflexive). To this end suppose that $\{g_n\}_{n\geqslant 1} \subseteq L^{p'}\big(T;\mathbb{R}^N\big)$ is a sequence, such that

$$g_n \xrightarrow{w} g \quad \text{in } L^{p'}\big(T;\mathbb{R}^N\big)$$

for some $g \in L^{p'}\big(T;\mathbb{R}^N\big)$ and let

$$x_n \stackrel{df}{=} V_\varepsilon(g_n) \qquad \forall\, n \geqslant 1.$$

Clearly the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big)$ is bounded and so we may assume that

$$\begin{aligned} &x_n \xrightarrow{w} x \quad \text{in } W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big),\\ &x_n \longrightarrow x \quad \text{in } C\big(T;\mathbb{R}^N\big). \end{aligned}$$

We have

$$\begin{aligned} &\big\langle A(x_n), x_n - x\big\rangle_{W^{1,p}_{\rm per}((0,b);\mathbb{R}^N)} + \varepsilon \int_0^b \big\|x_n(t)\big\|^{p-2}_{\mathbb{R}^N}\big(x_n(t),(x_n-x)(t)\big)_{\mathbb{R}^N}\,dt\\ &= \int_0^b \big(g_n(t),(x_n-x)(t)\big)_{\mathbb{R}^N}\,dt. \end{aligned}$$

Since

$$\int_0^b \big\|x_n(t)\big\|^{p-2}_{\mathbb{R}^N}\big(x_n(t),(x_n-x)(t)\big)_{\mathbb{R}^N}\,dt \longrightarrow 0$$

and

$$\int_0^b \big(g_n(t),(x_n-x)(t)\big)_{\mathbb{R}^N}\,dt \longrightarrow 0,$$

we obtain

$$\lim_{n\to+\infty} \big\langle A(x_n), x_n - x\big\rangle_{W^{1,p}_{\rm per}((0,b);\mathbb{R}^N)} \;=\; 0.$$

From this as in the proof of Proposition 6.2.17, we infer that

$$x_n \;\longrightarrow x \quad \text{in } W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big).$$

Then in the limit, we have

$$A(x) + \varepsilon K(x) \;=\; g,$$

i.e., $x = V_\varepsilon(g)$. Hence

$$V_\varepsilon(g_n) \;\longrightarrow\; V_\varepsilon(g) \quad \text{in } W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big)$$

and we have proved the complete continuity of V_ε.

Now we consider the compact homotopy

$$H\colon [0,1] \times W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big) \;\longrightarrow\; W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big),$$

defined by

$$H(\beta,x) \;\stackrel{df}{=}\; V_\varepsilon\big(\lambda K(x) + \beta\big(h + \varepsilon K(x)\big)\big).$$

We will show that there exists $r > 0$, such that

$$0 \;\notin\; x - H(\beta,x) \qquad \forall\, \beta \in (0,1),\ \|x\|_{W^{1,p}((0,b);\mathbb{R}^N)} = r.$$

Suppose that this is not true. Then we can find a sequence

$$\big\{(\beta_n,x_n)\big\}_{n\geqslant 1} \;\subseteq\; (0,1) \times W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big),$$

such that

$$x_n \;=\; H(\beta_n,x_n) \qquad \forall\, n \geqslant 1$$

and

$$\|x_n\|_{W^{1,p}((0,b);\mathbb{R}^N)} \;\longrightarrow\; +\infty.$$

Let

$$y_n \;\stackrel{df}{=}\; \frac{x_n}{\|x_n\|_{W^{1,p}((0,b);\mathbb{R}^N)}} \qquad \forall\, n \geqslant 1.$$

We may assume that

$$\begin{aligned} &y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big),\\ &y_n \longrightarrow y \quad \text{in } C\big(T;\mathbb{R}^N\big),\end{aligned}$$

for some $y \in W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big)$. For every $n \geqslant 1$, we have

$$\begin{cases} -\big(\big\|y_n'(t)\big\|^{p-2}_{\mathbb{R}^N} y_n'(t)\big)' + \varepsilon\big\|y_n(t)\big\|^{p-2}_{\mathbb{R}^N} y_n(t) \\ \qquad = \lambda\big\|y_n(t)\big\|^{p-2}_{\mathbb{R}^N} y_n(t) + \beta_n\big(h(t) + \varepsilon\big\|y_n(t)\big\|^{p-2}_{\mathbb{R}^N} y_n(t)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in T, \\ y_n(0) = y_n(b),\ y_n'(0) = y_n'(b). \end{cases} \tag{6.71}$$

As we did earlier (cf., e.g., the proof of Proposition 6.2.17), we can show that

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{W^{1,p}_{\mathrm{per}}((0,b);\mathbb{R}^N)} = 0,$$

from which we infer that

$$y_n \longrightarrow y \quad \text{in } W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big).$$

So if we pass to the limit as $n \to +\infty$ in (6.71), we obtain

$$\begin{cases} -\big(\|y'(t)\|^{p-2}_{\mathbb{R}^N} y'(t)\big)' + \varepsilon\|y(t)\|^{p-2}_{\mathbb{R}^N} y(t) \\ \qquad = \lambda\|y(t)\|^{p-2}_{\mathbb{R}^N} y(t) + \beta\varepsilon\|y(t)\|^{p-2}_{\mathbb{R}^N} y(t) \quad \text{for a.a. } t \in T, \\ y(0) = y(b),\ y'(0) = y'(b). \end{cases} \tag{6.72}$$

Since $\|y\|_{W^{1,p}((0,b);\mathbb{R}^N)} = 1$, we have $y \neq 0$ and so from (6.72), it follows that $y \in C^1_{\mathrm{per}}\big(T;\mathbb{R}^N\big)$ is an eigenfunction of the negative vector ordinary p-Laplacian with periodic boundary conditions with corresponding eigenvalue

$$\lambda - (1-\beta)\varepsilon \in \sigma(p,N),$$

which contradicts the choice of $\varepsilon > 0$. Therefore, we can find $r > 0$ large enough, so that

$$x - H(t,x) \neq 0 \qquad \forall\, x \in \partial B_r(0),$$

where

$$\partial B_r(0) = \big\{x \in W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big) : \|x\|_{W^{1,p}((0,b);\mathbb{R}^N)} = r\big\}.$$

From the homotopy invariance of the Leray-Schauder degree, we have

$$\pm 1 = d_{LS}\big(id - H(0,x), B_r(0), 0\big) = d_{LS}\big(id - H(1,x), B_r(0), 0\big),$$

since $\lambda \notin \sigma(p,N)$. So we can find $x \in W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big)$, such that

$$x = H(1,x) \quad \text{and} \quad \|x\|_{W^{1,p}((0,b);\mathbb{R}^N)} < r.$$

Evidently this is a solution of (6.69) when $h \in L^{p'}\big(T;\mathbb{R}^N\big)$.

Finally let $h \in L^1\big(T;\mathbb{R}^N\big)$ and choose a sequence $\{h_n\}_{n\geqslant 1} \subseteq L^{p'}\big(T;\mathbb{R}^N\big)$, such that

$$h_n \longrightarrow h \quad \text{in } L^1\big(T;\mathbb{R}^N\big).$$

Let $x_n \in C^1_{\mathrm{per}}\big(T;\mathbb{R}^N\big)$ for $n \geqslant 1$ be solutions of (6.69), when h is replaced by h_n. Then as before we can show that

$$x_n \longrightarrow x \quad \text{in } W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big).$$

Therefore in the limit, we have that

$$\begin{cases} -\big(\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t)\big)' = \lambda\|x(t)\|_{\mathbb{R}^N}^{p-2}x(t) + h(t) & \text{for a.a. } t \in T, \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases}$$

i.e., problem (6.69) has a solution $x \in C^1_{\rm per}\big(T;\mathbb{R}^N\big)$. This proves that

$$\lambda \ \in \ \varrho(p,N)$$

and so we have proved the proposition. □

Let $u \in C^1_{\rm per}\big(T;\mathbb{R}^N\big)$ be an eigenfunction for problem (6.68). Taking inner product with $u(t)$, integrating over T and using integration by parts, we obtain

$$\lambda \ = \ \frac{\|u'\|_p^p}{\|u\|_p^p} \ \geqslant \ 0.$$

So all the eigenvalues of (6.68) are nonnegative and since $0 \in \sigma(p,N)$, we infer that 0 is the smallest eigenvalue. Moreover, it is clear from (6.68) that all the eigenvalues of the scalar problem ($N = 1$) are also eigenvalues of the vector problem ($N > 1$). So we obtain the eigenvalues:

$$\begin{aligned} \nu_0 \ &= \ 0, \\ \nu_n \ &= \ \left(\frac{2n}{b}\right)^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p \qquad \forall\ n \geqslant 1. \end{aligned}$$

But in contrast to the Dirichlet problem, the periodic problem has more eigenvalues when $N > 1$ (vector case). For example consider

$$\xi_n \ \stackrel{df}{=} \ \left(\frac{2\pi n}{b}\right)^p \qquad \forall\ n \geqslant 1$$

with corresponding eigenfunctions

$$v_n(t) \ \stackrel{df}{=} \ \left(0,\dots,0,\cos\frac{2\pi n}{b}t, 0\dots,0,\sin\frac{2\pi n}{b}t, 0,\dots,0\right).$$

More generally, if $n \geqslant 1$, every nontrivial solution v of

$$v''(t) + \left(\frac{2\pi n}{b}\right)^p v(t) \ = \ 0 \quad \text{for a.a. } t \in T,$$

such that

$$\big(v'(t), v(t)\big)_{\mathbb{R}^N} \ = \ 0 \qquad \forall\ t \in T,$$

is an eigenfunction corresponding to the eigenvalue ξ_n.

Integrating (6.68) over T, we see that, if $\lambda \in \sigma(p,N) \setminus \{0\}$ and u is an associated eigenfunction, we have

$$\int_0^b \big\|u(t)\big\|_{\mathbb{R}^N}^{p-2} u(t)\,dt \;=\; 0.$$

So following the analysis of the Neumann problem for the partial p-Laplacian (see Section 6.2), we define

$$C(p,N) \stackrel{df}{=} \left\{x \in W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big) : \int_0^b \big\|x(t)\big\|_{\mathbb{R}^N}^{p-2} x(t)\,dt = 0\right\},$$

$$C_1(p,N) \stackrel{df}{=} \left\{x \in C(p,N) : \; \|x\|_p = 1\right\},$$

$$\vartheta_{p,N}(x) \stackrel{df}{=} \|x'\|_p^p \qquad \forall\, x \in W^{1,p}_{\rm per}\big((0,b);\mathbb{R}^N\big).$$

Evidently $\vartheta_{p,N}$ is strictly convex and C^1. We consider the minimization problem

$$\lambda_1(p,N) \;=\; \inf_{x \in C_1(p,N)} \vartheta_{p,N}(x).$$

As in Proposition 6.2.27, we have the following result.

PROPOSITION 6.3.12
$\lambda_1(p,N) > 0$ and we can find $x \in C_1(p,N)$, such that

$$\vartheta_{p,N}(x) \;=\; \lambda_1(p,N).$$

This proposition leads to the following Poincaré-Wirtinger-type inequality.

COROLLARY 6.3.13
If $x \in C(p,N)$ and $\lambda_1 = \lambda_1(p,N) > 0$,
then

$$\lambda_1 \|x\|_p^p \;\leqslant\; \|x'\|_p^p.$$

If $p \geqslant 2$, then we can have the following characterization of

$$\lambda_1 \;=\; \lambda_1(p,N) \;>\; 0.$$

The proof is similar to that of Theorem 6.2.29.

THEOREM 6.3.14
If $p \geqslant 2$,
then $\lambda_1 > 0$ is the first nonzero eigenvalue of the negative vector ordinary p-Laplacian with periodic boundary conditions.

REMARK 6.3.15 The complete structure of $\sigma(p,N)$ when $N > 1$ (vector case) is far from being understood and more research is needed in this direction.

6.4 Maximum Principles

In the last part of Section 6.1, we proved maximum principles for second order, linear, uniformly elliptic differential operators. In this section we derive analogous results for nonlinear differential operators involving the partial p-Laplacian.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$ and let $a \in L^\infty(\Omega)$ be a weight function. Using the notation introduced in the previous section, by $\psi_p \colon \mathbb{R} \longrightarrow \mathbb{R}$ we denote the homeomorphism, defined by

$$\psi_p(r) \stackrel{df}{=} \begin{cases} |r|^{p-2}r & \text{if} \quad r \neq 0, \\ 0 & \text{if} \quad r = 0. \end{cases}$$

In this section we study the following nonlinear $(p-1)$-homogeneous differential operator:

$$V_p(x) \stackrel{df}{=} -\Delta_p x + a(z)\psi_p(x).$$

Let $h \in W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) and consider the equation:

$$V_p(x) \;=\; h. \tag{6.73}$$

We start by introducing the notions of weak solutions, weak upper solutions and weak lower solutions for problem (6.73).

DEFINITION 6.4.1 **(a)** *A function $x \in W^{1,p}(\Omega)$ is said to be a* ***weak solution*** *of (6.73), if*

$$\int_\Omega \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz + \int_\Omega a\psi_p(x)y \, dz \;=\; \langle h, y\rangle_{W_0^{1,p}(\Omega)} \;\; \forall y \in W_0^{1,p}(\Omega).$$

(b) *A function $x \in W^{1,p}(\Omega)$ is said to be a* ***weak upper solution*** *of (6.73), if*

$$\int_\Omega \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz + \int_\Omega a\psi_p(x)y \, dz \;\geqslant\; \langle h, y\rangle_{W_0^{1,p}(\Omega)}$$
$$\forall \; y \in W_0^{1,p}(\Omega), \; y \geqslant 0.$$

(c) *A function $x \in W^{1,p}(\Omega)$ is said to be a* ***weak lower solution*** *of (6.73), if*

$$\int_\Omega \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz + \int_\Omega a\psi_p(x)y \, dz \;\leqslant\; \langle h, y\rangle_{W_0^{1,p}(\Omega)}$$
$$\forall \; y \in W_0^{1,p}(\Omega), \; y \geqslant 0.$$

DEFINITION 6.4.2 *We say that the differential operator V_p satisfies the* ***maximum principle****, if every weak solution $x \in W^{1,p}(Z)$ of*

$$\begin{cases} V_p(x) = f, \\ x|_{\partial\Omega} \geqslant 0, \end{cases}$$

with $f \in L^\infty(\Omega)$, $f \geqslant 0$, satisfies

$$x(z) \geqslant 0 \quad \text{for a.a. } z \in \Omega.$$

We say that V_p satisfies the ***strong maximum principle****, if in addition*

$$x(z) > 0 \quad \text{for a.a. } z \in \Omega,$$

when $f \neq 0$.

We examine the first eigenvalue of the differential operator V_p with Dirichlet boundary conditions (i.e., of $\big(V_p, W_0^{1,p}(\Omega)\big)$). So we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\big) + a(z)|x(z)|^{p-2}x(z) \\ \qquad = \lambda|x(z)|^{p-2}x(z) \quad \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \tag{6.74}$$

with $\lambda \in \mathbb{R}$.

To establish the existence of a principal eigenvalue for (6.74), we will need the following lemma. Let

$$D(I) \stackrel{df}{=} \left\{(x,y) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : \ x \geqslant 0, \ y \geqslant 0, \ \frac{x}{y}, \frac{y}{x} \in L^\infty(\Omega)\right\}$$

and consider the function

$$I(x,y) \stackrel{df}{=} \left\langle A(x), \frac{x^p - y^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} - \left\langle A(y), \frac{x^p - y^p}{y^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \quad \forall x, y \in D(I).$$

Here $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is the nonlinear maximal monotone, strictly monotone operator, defined by

$$\big\langle A(x), y\big\rangle_{W_0^{1,p}(\Omega)} \stackrel{df}{=} \int_\Omega \|Dx(z)\|_{\mathbb{R}^N}^{p-2} \big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} dz \quad \forall x, y \in W_0^{1,p}(\Omega)$$

(see the proof of Proposition 6.2.2). Evidently $A(x) = -\Delta_p x$ and note that because of Theorem 2.4.57, we have

$$-\Delta_p x \ = \ -\operatorname{div}\big(\|Dx\|_{\mathbb{R}^N}^{p-2} Dx\big) \ \in \ W^{-1,p'}(\Omega).$$

Concerning the functional $I(x,y)$ defined on $D(I)$, we have the following result (see also the proof of Proposition 6.2.15).

LEMMA 6.4.3
For all $(x,y) \in D(I)$, $I(x,y) \geqslant 0$ and $I(x,y) = 0$ if and only if $x = \mu y$ for some $\mu \geqslant 0$.

PROOF First we prove the nonnegativity of the functional $I(x,y)$ on the domain $D(I)$. So let $(x,y) \in D(I)$. We have

$$D\left(\frac{x^p}{y^{p-1}}\right) = p\left(\frac{x}{y}\right)^{p-1} Dx - (p-1)\left(\frac{x}{y}\right)^p Dy$$

and

$$D\left(\frac{y^p}{x^{p-1}}\right) = p\left(\frac{y}{x}\right)^{p-1} Dy - (p-1)\left(\frac{y}{x}\right)^p Dx.$$

So we obtain

$$\begin{aligned}
&\left\langle A(x), \frac{x^p - y^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \\
&= \int_\Omega \Bigg[-p\left(\frac{y}{x}\right)^{p-1} \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N} \\
&\qquad + \left(1 + (p-1)\left(\frac{y}{x}\right)^p\right) \|Dx\|_{\mathbb{R}^N}^p \Bigg] dz \\
&= \int_\Omega \Bigg[p\left(\frac{y}{x}\right)^{p-1} \|Dx\|_{\mathbb{R}^N}^{p-2} \left(\|Dx\|_{\mathbb{R}^N} \|Dy\|_{\mathbb{R}^N} - (Dx, Dy)_{\mathbb{R}^N}\right) \\
&\qquad + \left(1 + (p-1)\left(\frac{y}{x}\right)^p\right) \|Dx\|_{\mathbb{R}^N}^p - p\left(\frac{y}{x}\right)^{p-1} \|Dx\|_{\mathbb{R}^N}^{p-1} \|Dy\|_{\mathbb{R}^N} \Bigg] dz.
\end{aligned}$$

An analogous expression is obtained for $\left\langle A(x), \frac{x^p - y^p}{y^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)}$ by interchanging the roles of $x, y \in D(I)$. So finally exploiting the symmetry of the expression, we can write that

$$I(x,y) = \int_\Omega h_1\left(\frac{y}{x}, Dx, Dy\right) dz + \int_\Omega h_2\left(\frac{y}{x}, \|Dx\|_{\mathbb{R}^N}, \|Dy\|_{\mathbb{R}^N}\right) dz,$$

where

$$\begin{aligned}
h_1(r,\xi,\eta) &\stackrel{df}{=} p\left(r^{p-1} \|\xi\|_{\mathbb{R}^N}^{p-2} + r^{1-p} \|\eta\|_{\mathbb{R}^N}^{p-2}\right) \left(\|\xi\|_{\mathbb{R}^N} \|\eta\|_{\mathbb{R}^N} - (\xi,\eta)_{\mathbb{R}^N}\right) \\
&\qquad \forall\ r \geqslant 0,\ \xi, \eta \in \mathbb{R}^N, \\
h_2(r,\tau,s) &\stackrel{df}{=} \left(1 + (p-1)r^p\right)\tau^p + \left(1 + (p-1)r^{-p}\right)s^p \\
&\qquad - pr^{p-1}\tau^{p-1}s - pr^{1-p}s^{p-1}\tau \qquad \forall\ r, \tau, s \geqslant 0.
\end{aligned}$$

By virtue of the Cauchy-Schwarz inequality $h_1 \geqslant 0$. We will show that $h_2 \geqslant 0$ too. Note that

$$h_2(r,0,s) \;\geqslant\; 0 \quad \text{and} \quad \text{if } h_2(r,0,s)=0, \text{ then } s=0.$$

If $\tau \neq 0$, setting $z \stackrel{df}{=} \big(\frac{s}{r\tau}\big)^p$, we obtain

$$h_2(r,\tau,s) \;=\; \tau^p\big(r^p\gamma_1(z)+\gamma_2(z)\big),$$

with

$$\gamma_1(z) \;\stackrel{df}{=}\; z^p - pz + p - 1 \quad \text{and} \quad \gamma_2(z) \;\stackrel{df}{=}\; (p-1)z^p - pz^{p-1}.$$

Evidently $\gamma_1, \gamma_2 \geqslant 0$ and so

$$I(x,y) \;\geqslant\; 0 \qquad \forall\, (x,y) \in D(I).$$

The functions γ_1 and γ_2 vanish at $z=1$. So

$$h_2(t,\tau,s) \;=\; 0 \text{ if and only if } s \;=\; r\tau.$$

Then if $I(x,y)=0$, we have

$$h_1\big(\frac{y}{x}, Dx, Dy\big) \;=\; 0 \quad \text{and} \quad h_2\big(\frac{y}{x}, \|Dx\|_{\mathbb{R}^N}, \|Dy\|_{\mathbb{R}^N}\big) \;=\; 0 \quad \text{for a.a. } z \in \Omega$$

and this is equivalent to saying that

$$x(z)\big\|Dy(z)\big\|_{\mathbb{R}^N} \;=\; y(z)\big\|Dx(z)\big\|_{\mathbb{R}^N} \quad \text{for a.a. } z \in \Omega$$

and

$$\big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} \;=\; \big\|Dx(z)\big\|_{\mathbb{R}^N}\big\|Dy(z)\big\|_{\mathbb{R}^N} \quad \text{for a.a. } z \in \Omega.$$

It follows that

$$\big\|xDy - yDx\big\|^2_{\mathbb{R}^2} \;=\; 0 \quad \text{for a.a. } z \in \Omega,$$

from which it follows that $x = cy$ with $c \geqslant 0$. □

REMARK 6.4.4 The nonnegativity of $I(x,y)$ on $D(I)$ can also be deduced by considering the functional $\psi\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}}_+$, defined by

$$\psi(x) \;\stackrel{df}{=}\; \begin{cases} \frac{1}{p}\big\|Dx^{\frac{1}{p}}\big\|_p^p & \text{if } x^{\frac{1}{p}} \in W_0^{1,p}(\Omega),\ x \geqslant 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then it is easy to see that $\psi \in \Gamma_0\big(L^1(\Omega)\big)$. Moreover, if by ψ'_G we denote the Gâteaux derivative of ψ, we have

$$0 \;\leqslant\; \big\langle \psi'_G(x^p) - \psi'_G(y^p), x^p - y^p\big\rangle_{W_0^{1,p}(\Omega)} \;=\; I(x,y) \qquad \forall\, x,y \in D(I).$$

Using Lemma 6.4.3, we can have an analog of Theorem 6.2.18 for the eigenvalue problem (6.74).

THEOREM 6.4.5
The nonlinear eigenvalue problem (6.74) has a unique eigenvalue $\lambda = \lambda_{1,p}(a)$, *defined by*

$$\lambda_{1,p}(a) \stackrel{df}{=} \inf_{\substack{x \in W_0^{1,p}(\Omega) \\ \|x\|_p = 1}} \left[\|Dx\|_p^p + \int_\Omega a(z)|x(z)|^p \, dz \right], \tag{6.75}$$

with the property of exhibiting a positive eigenfunction $\widehat{u}_1 \in W_0^{1,p}(\Omega)$. *Moreover,* $\lambda_{1,p}(a)$ *is simple, isolated and* $\widehat{u}_1 \in C_0^1(\overline{\Omega})$ *with* $\frac{\partial \widehat{u}_1}{\partial n} < 0$ *on* $\partial\Omega$ *(where* n *is the outer unit normal on* $\partial\Omega$*).*

PROOF Since $a \in L^\infty(\Omega)$, we may assume without any loss of generality that $a \geqslant 0$. Indeed, if this is not the case we replace a and λ in (6.74) by

$$A_M(z) \stackrel{df}{=} a(z) + M \quad \text{and} \quad \lambda_M \stackrel{df}{=} \lambda + M$$

respectively, with $M \stackrel{df}{=} \|a\|_\infty$. Since $a \geqslant 0$, the function

$$\eta \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}_+,$$

defined by

$$\eta(x) \stackrel{df}{=} \|Dx\|_p^p + \int_\Omega a|x|^p dz,$$

is convex, lower semicontinuous (hence weakly lower semicontinuous too) and

$$\eta(x) \geqslant \|Dx\|_p^p,$$

which by Poincaré's inequality implies that η is coercive. So we can find $\widehat{u}_1 \in W_0^{1,p}(\Omega)$ with $\|\widehat{u}_1\|_p = 1$, such that

$$\lambda_{1,p}(a) = \eta(\widehat{u}_1)$$

(see (6.75)). Then by the Lagrange multiplier rule (see Theorem 5.5.24), we have that $\widehat{u}_1 \in W_0^{1,p}(\Omega)$ is a solution of (6.74). Using Theorems 6.2.6 and 6.2.7**(a)** (nonlinear regularity theory), we have that $\widehat{u}_1 \in C_0^1(\overline{\Omega})$. Moreover, it is clear from (6.75) that $\widehat{u}_1$ does not change sign and so we may assume that

$$\widehat{u}_1(z) \geqslant 0 \qquad \forall\, z \in \Omega$$

and $\widehat{u}_1 \neq 0$. Invoking Theorem 6.2.8, we have that

$$\widehat{u}_1(z) > 0 \qquad \forall\, z \in \Omega \quad \text{and} \quad \frac{\partial \widehat{u}_1}{\partial n}(z) < 0 \qquad \forall\, z \in \partial\Omega.$$

For the simplicity of $\lambda_{1,p}(a)$, note that, if $u, v \in W_0^{1,p}(\Omega)$ is any pair of positive eigenfunctions corresponding to $\lambda_{1,p}(a)$, then

$$I(u,v) \;=\; 0$$

and so by Lemma 6.4.3, we have $u = cv$ which establishes that the corresponding eigenspace is one dimensional.

For the uniqueness of $\lambda_{1,p}(a)$, let $\lambda \neq \lambda_{1,p}(a)$ be an eigenvalue of (6.74) (hence $\lambda > \lambda_{1,p}(a)$), which has a positive eigenfunction $u \in W_0^{1,p}(\Omega)$, i.e.,

$$V_p(u) \;=\; \lambda\psi_p(u) \quad \text{and} \quad u(z) \;>\; 0 \quad \text{for a.a. } z \in \Omega.$$

As for $\widehat{u}_1$, we have $u \in C_0^1(\overline{\Omega})$ and

$$u(z) \;>\; 0 \qquad \forall\, z \in \Omega.$$

Assume that $\|u\|_p = 1$. Then

$$0 \;\leqslant\; I(\widehat{u}_1, u) \;=\; \int_\Omega \big(\lambda_{1,p}(a) - \lambda\big)\big(\widehat{u}_1^p - u^p\big)\,dz \;=\; \lambda_{1,p}(a) - \lambda \;<\; 0,$$

a contradiction. This proves the uniqueness of $\lambda_{1,p}(a)$.

Finally we show that $\lambda_{1,p}(a)$ is isolated. If this is not the case, then we can find eigenelements (λ_n, u_n) of (6.74), such that

$$\lambda_n \;\searrow\; \lambda_{1,p}(a).$$

We may assume that $\|u_n\|_{W_0^{1,p}(\Omega)} = 1$ and so at least for a subsequence, we have

$$\begin{aligned} u_n &\xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega), \\ u_n &\longrightarrow u \quad \text{in } L^p(\Omega). \end{aligned}$$

Then since

$$A(u_n) + a\psi_p(u_n) \;=\; \lambda_n\psi_p(u_n) \qquad \forall\, n \geqslant 1,$$

we have

$$\int_\Omega a\psi_p(u_n)(u_n - u)\,dz \;\longrightarrow\; 0$$

and

$$\int_\Omega \psi_p(u_n)(u_n - u)\,dz \;\longrightarrow\; 0$$

and so

$$\big\langle A(u_n), u_n - u\big\rangle_{W_0^{1,p}(\Omega)} \;\longrightarrow\; 0,$$

which by virtue of the maximal monotonicity of A implies that

$$\big\langle A(u_n), u_n\big\rangle_{W_0^{1,p}(\Omega)} \longrightarrow \big\langle A(u), u\big\rangle_{W_0^{1,p}(\Omega)},$$

hence

$$\|Du_n\|_p \longrightarrow \|Du\|_p .$$

Since

$$Du_n \stackrel{w}{\longrightarrow} Du \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big),$$

we infer that

$$Du_n \longrightarrow Du \quad \text{in } L^p\big(\Omega;\mathbb{R}^N\big)$$

and so

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega).$$

In the limit as $n \to +\infty$, we have

$$A(u) + a\psi_p(u) \;=\; \lambda_{1,p}(a)\psi_p(u)$$

and so

$$u \;=\; \widehat{u}_1.$$

Arguing as in the last part of the proof of Proposition 6.2.17, we can show that

$$\big|\{u_n < 0\}\big|_N \;\geqslant\; \vartheta \;>\; 0,$$

for some $\vartheta > 0$ and all $n \geqslant n_0$. This contradicts the fact that $\widehat{u}_1 > 0$. □

We can use the above theorem to obtain characterizations of the maximum principle and strong maximum principle for the operator V_p.

THEOREM 6.4.6
Let $V_p(x) \stackrel{df}{=} -\Delta_p x + a\psi_p(x)$. The following statements are equivalent:

(a) *the operator V_p satisfies the maximum principle;*

(b) *the operator V_p satisfies the strong maximum principle;*

(c) *$\lambda_{1,p}(a) > 0$ (see (6.75));*

(d) *there exists $x \in W_0^{1,p}(\Omega)$, $x(z) > 0$ for almost all $z \in \Omega$, such that $V_p(x) = f \in L^\infty(\Omega)$, $f \geqslant 0$, $f \neq 0$ (i.e., x is a positive strict upper solution of the equation $V_p(x) = 0$);*

(e) *for every $f \in L^\infty(\Omega)_+$, there exists a unique weak solution of the equation $V_p(x) = f$, which is nonnegative.*

PROOF "**(a)**$\Longrightarrow$**(b)**": Let $x \in W^{1,p}(\Omega)$ be a weak solution of

$$-\Delta_p x + a\psi_p(x) \;=\; f \quad \text{in } \Omega,$$

with $f \in L^\infty(\Omega)$, $f \geqslant 0$, $f \neq 0$, such that $x \geqslant 0$ (see Definition 6.4.2). From Theorems 6.2.6 and 6.2.7**(a)**, we have that $x \in C^1(\overline{\Omega})$ and because

$$x(z) \;\geqslant\; 0 \qquad \forall\, z \in \overline{\Omega},$$

we have

$$-\Delta_p x(z) + \|a\|_\infty \psi_p\big(x(z)\big) \;\geqslant\; 0 \quad \text{for a.a. } z \in \Omega.$$

Invoking Theorem 6.2.8, we obtain

$$x(z) \;>\; 0 \qquad \forall\, z \in \overline{\Omega}.$$

"**(b)**$\Longrightarrow$**(c)**": By Theorem 6.4.5, we have a positive eigenfunction $\widehat{u}_1 \in C^1(\overline{\Omega})$ corresponding to the eigenvalue $\lambda_{1,p}(a)$ of the operator V_p. If $\lambda_{1,p}(a) \leqslant 0$, we have

$$-\Delta_p\big(-\widehat{u}_1\big)(z) + a(z)\psi_p\big(-\widehat{u}_1(z)\big) \;=\; -\lambda_{1,p}(a)\psi_p\big(\widehat{u}_1(z)\big) \;\geqslant\; 0 \quad \text{for a.a. } z \in \Omega.$$

So by hypothesis, we have

$$-\widehat{u}_1(z) \;\geqslant\; 0 \qquad \forall\, z \in \overline{\Omega},$$

a contradiction. Therefore $\lambda_{1,p}(a) > 0$.

"**(c)**$\Longrightarrow$**(d)**": Just take $x = \widehat{u}_1$, the eigenfunction corresponding to the eigenvalue $\lambda_{1,p}(a)$.

"**(d)**$\Longrightarrow$**(c)**": Let $\widehat{u}_1 \in C^1(\overline{\Omega})$ be the positive eigenfunction corresponding to $\lambda_{1,p}(a)$ (see Theorem 6.4.5) and let $x \in W_0^{1,p}(\Omega)$ be the positive strict upper solution existing by hypothesis. We have $\big(\widehat{u}_1, x\big) \in D(I)$ and in particular $\frac{x}{\widehat{u}_1} \in L^\infty(\Omega)$. Let $v = c\widehat{u}_1$ with $c \geqslant \left\| \frac{x}{\widehat{u}_1} \right\|_\infty$. Suppose that $\lambda_{1,p}(a) \leqslant 0$. From the choice of $c > 0$ and since $f \geqslant 0$, we have

$$\begin{aligned} \left\langle -\Delta_p x, \frac{v^p - x^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega a\big(v^p - x^p\big)\, dz \\ = \int_\Omega f\left(\frac{v^p - x^p}{x^{p-1}}\right) dz \;\geqslant\; 0. \end{aligned} \tag{6.76}$$

Also

$$\left\langle -\Delta_p v, \frac{v^p - x^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \;=\; \int_\Omega -a\big(v^p - x^p\big)\, dz$$

$$+ \int_\Omega \lambda_{1,p}(a)\big(v^p - x^p\big)\,dz. \tag{6.77}$$

Therefore

$$\begin{aligned} I(v,x) &= \left\langle -\Delta_p v, \frac{v^p - x^p}{v^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} - \left\langle -\Delta_p x, \frac{v^p - x^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \\ &\leqslant \int_\Omega \lambda_{1,p}(a)\big(v^p - x^p\big)\,dz \leqslant 0 \end{aligned}$$

(see (6.76) and (6.77)) and recall that we have assumed $\lambda_{1,p}(a) \leqslant 0$. From Lemma 6.4.3, it follows that

$$I(v,x) = 0 \quad \text{and} \quad v = c_0 x,$$

for some $c_0 > 0$. Then

$$\lambda_{1,p}(a)\psi_p\big(v(z)\big) = c_0^{p-1} f(z) \quad \text{for a.a. } z \in \Omega,$$

a contradiction since $\lambda_{1,p}(a) \leqslant 0$, $c_0 > 0$ and $f \geqslant 0$, $f \neq 0$.

"**(c)$\Longrightarrow$(a)**": Suppose that $x \in W_0^{1,p}(\Omega)$ is a weak solution of

$$-\Delta_p x + a\psi_p(x) = f,$$

with $f \in L^\infty(\Omega)$, $f \geqslant 0$ and $x|_{\partial\Omega} \geqslant 0$. Using as a test function $-x^- \in W^{1,p}(\Omega)$ (see Proposition 2.4.27), we obtain

$$\big\|Dx^-\big\|_p^p + \int_\Omega a|x^-|^p\,dz = \int_\Omega f(-x^-)\,dz \leqslant 0,$$

so $\lambda_{1,p}(a) \leqslant 0$, a contradiction to the hypothesis (see Theorem 6.4.5).

"**(c)$\Longrightarrow$(e)**": Consider the C^1-functional $\vartheta\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\vartheta(x) \stackrel{df}{=} \|Dx\|_p^p + \int_\Omega a|x|^p\,dz - p\int_\Omega fx\,dz \qquad \forall\, x \in W_0^{1,p}(\Omega),$$

with $f \in L^\infty(\Omega)$, $f \geqslant 0$. Since $\lambda_{1,p}(a) > 0$, we have that there exists $\xi > 0$, such that

$$\|Dx\|_p^p + \int_\Omega a|x|^p\,dz \geqslant \xi\,\|x\|_{W^{1,p}(\Omega)}^p.$$

So we have

$$\vartheta(x) \geqslant \xi\,\|x\|_{W^{1,p}(\Omega)}^p - p\int_\Omega fx\,dz,$$

so ϑ is coercive.

From the compactness of the embedding $W_0^{1,p}(\Omega) \subseteq L^p(\Omega)$, we infer that ϑ is weakly lower semicontinuous. So by the Weierstrass theorem, we can find $x \in W_0^{1,p}(\Omega)$, such that

$$\vartheta(x) \;=\; \min_{y \in W_0^{1,p}(\Omega)} \vartheta(y),$$

so

$$\vartheta'(x) \;=\; 0. \tag{6.78}$$

From (6.78), it follows that $x \in W_0^{1,p}(\Omega)$ is a solution of

$$-\Delta_p x(z) + a(z)\big|x(z)\big|^{p-2}x(z) \;=\; f(z) \quad \text{for a.a. } z \in \Omega.$$

Using as a test function $-x^- \in W_0^{1,p}(\Omega)$, we obtain

$$\lambda_{1,p}(a)\,\big\|x^-\big\|_p^p \;\leqslant\; \big\|Dx^-\big\|_p^p + \int_\Omega a|x^-|^p\,dz \;=\; \int_\Omega f(-x^-)\,dz \;\leqslant\; 0,$$

so $x^- = 0$, i.e., $x \geqslant 0$.

Now we will show that this nonnegative solution is unique. Let $x, y \in W_0^{1,p}(\Omega)$ be two such solutions. As before $x, y \in C^1(\overline{\Omega})$ and from Theorem 6.2.8, we have

$$x(z) \;>\; 0 \quad \text{and} \quad y(z) \;>\; 0 \quad \text{and} \quad z \in \Omega$$

and

$$\frac{\partial x}{\partial n}(z) \;<\; 0 \quad \text{and} \quad \frac{\partial y}{\partial n}(z) \;<\; 0 \qquad \forall\, z \in \partial\Omega.$$

Then $(x, y) \in D(I)$ and

$$\begin{aligned} 0 \;\leqslant\; I(x,y) \;&=\; \left\langle -\Delta_p x, \frac{x^p - y^p}{x^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} - \left\langle -\Delta_p y, \frac{x^p - y^p}{y^{p-1}} \right\rangle_{W_0^{1,p}(\Omega)} \\ &=\; \int_\Omega f \frac{y^{p-1} - x^{p-1}}{x^{p-1}y^{p-1}}\big(x^p - y^p\big)\,dz \;\leqslant\; 0, \end{aligned}$$

so

$$I(x,y) \;=\; 0,$$

hence $x = cy$ for some $c > 0$ (see Lemma 6.4.3).

From this we obtain

$$c^{p-1} f \;=\; f.$$

If $f \neq 0$, then $c = 1$ and so $x = y$.
If $f = 0$, then because $\lambda_{1,p}(a) > 0$, $x = 0$ is the only solution.

"**(e)**$\Longrightarrow$**(d)**": Obvious. ∎

For the next theorem, we will need the so-called Picone's identity. For $p = 2$ this identity says the following.

THEOREM 6.4.7 (Picone Identity for $p = 2$)
If $x > 0$, $y \geqslant 0$ *are differentiable functions,*
then

$$\begin{aligned} &\|Dy\|_{\mathbb{R}^N}^2 + \frac{y^2}{x^2}\|Dx\|_{\mathbb{R}^N}^2 - 2\frac{y}{x}(Dx, Dy)_{\mathbb{R}^N} \\ &= \|Dy\|_{\mathbb{R}^N}^2 - \left(D\left(\frac{y^2}{x}\right), Dx\right)_{\mathbb{R}^N} \geqslant 0 \quad \text{a.e. on } \Omega. \end{aligned} \tag{6.79}$$

This identity was extended recently for $p > 1$ not necessarily equal to 2 by Allegretto & Huang (1998).

THEOREM 6.4.8 (Picone Identity for $p > 1$)
If $x > 0$, $y \geqslant 0$ *are differentiable functions,*

$$\begin{aligned} L_1(x, y) &\stackrel{df}{=} \|Dy\|_{\mathbb{R}^N}^p + (p-1)\frac{y^p}{x^p}\|Dx\|_{\mathbb{R}^N}^p - p\frac{y^{p-1}}{x^{p-1}}\left(Dy, \|Dx\|_{\mathbb{R}^N}^{p-2} Dx\right)_{\mathbb{R}^N}, \\ R_1(x, y) &\stackrel{df}{=} \|Dy\|_{\mathbb{R}^N}^p + \left(D\left(\frac{y^p}{x^{p-1}}\right), \|Dy\|_{\mathbb{R}^N}^{p-2} Dy\right)_{\mathbb{R}^N}, \end{aligned}$$

then

$$L_1(x, y) = R_1(x, y) \geqslant 0 \quad \text{a.e. on } \Omega$$

and

$$L_1(x, y) = 0 \quad \text{for a.a. } z \in \Omega \quad \text{if and only if} \quad D\left(\frac{y}{x}\right) = 0 \quad \text{for a.a. } z \in \Omega,$$

i.e., $y = cx$ for some $c \geqslant 0$.

The next theorem is an interesting consequence of this identity.

THEOREM 6.4.9
If $u \in C^1(\Omega)$ *is a positive supersolution for the operator V_p and $a_0 \in L^\infty(\Omega)$ satisfies*

$$a_0(z) \geqslant a(z) \quad \text{for a.a. } z \in \Omega,$$

then the equation

$$-\operatorname{div}\left(\left\|Dx(z)\right\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) + a_0(z)\left|x(z)\right|^{p-2} x(z) = 0 \quad \text{for a.a. } z \in \Omega$$

has a positive solution too.

PROOF Let $\vartheta \in C_c^\infty(\Omega)$. From Picone's identity for $p > 1$ (see Theorem 6.4.8), we have

$$\begin{aligned} 0 \leqslant \int_\Omega L_1(\vartheta^\pm, u)\, dz &= \int_\Omega \|D\vartheta^\pm\|_{\mathbb{R}^N}^p \, dz - \int_\Omega \frac{(\vartheta^\pm)^p}{u^{p-1}}\big(-\Delta_p u\big)\, dz \\ &\leqslant \int_\Omega \big(\|D\vartheta^\pm\|^p + a(\vartheta^\pm)^p\big)\, dz. \end{aligned} \tag{6.80}$$

In (6.80) equalities hold only if

$$\vartheta^\pm = 0 \quad \text{or} \quad \vartheta^\pm = cu, \quad \text{with } c > 0.$$

Evidently this last option cannot happen. Therefore we have

$$0 \leqslant \int_\Omega \big(\|D\vartheta\|_{\mathbb{R}^N}^p + a|\vartheta|^p\big)\, dz \leqslant \int_\Omega \big(\|D\vartheta\|_{\mathbb{R}^N}^p + a_0|\vartheta|^p\big)\, dz \qquad \forall\, \vartheta \in C_c^\infty(\Omega).$$

We claim that there exists $\xi > 0$, such that

$$\xi \|\vartheta\|^p \leqslant \int_\Omega \big(\|D\vartheta\|_{\mathbb{R}^N}^p + a_0|\vartheta|^p\big)\, dz \qquad \forall\, \vartheta \in C_c^\infty(\Omega). \tag{6.81}$$

Suppose that (6.81) is not true. Then for every $n \geqslant 1$, we can find $\vartheta_n \in C_c^\infty(\Omega)$, such that $\|\vartheta_n\| = 1$ and

$$\int_\Omega \big(\|D\vartheta_n\|_{\mathbb{R}^N}^p + a_0|\vartheta_n|^p\big)\, dz \leqslant \frac{1}{n},$$

so because of (6.80) and since $a_0 \geqslant a$, we have

$$0 \leqslant \int_\Omega \big(\|D\vartheta_n\|_{\mathbb{R}^N}^p + a|\vartheta_n|^p\big)\, dz \leqslant \frac{1}{n}. \tag{6.82}$$

By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} &\vartheta_n \xrightarrow{w} \vartheta \quad \text{in } W^{1,p}(\Omega), \\ &\vartheta_n \longrightarrow \vartheta \quad \text{in } L^p(\Omega). \end{aligned}$$

Passing to the limit as $n \to +\infty$ in (6.82), we obtain

$$\int_\Omega \big(\|D\vartheta\|_{\mathbb{R}^N}^p + a|\vartheta|^p\big) = 0,$$

so $\vartheta = 0$, a contradiction to the fact that $\|\vartheta_n\| = 1$ for $n \geqslant 1$.

This proves that (6.81) must hold. Hence

$$\lambda_{1,p}(a) \ > \ 0.$$

Consider functions

$$f_n \in C_c^1(\Omega), \quad f_n \geqslant 0, \quad f_n \neq 0, \quad \|f_n\|_\infty \longrightarrow 0$$

and the Dirichlet problem:

$$\begin{cases} -\mathrm{div}\left(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) + a_0(z)|x(z)|^{p-2}x(z) = f_n(z) \quad \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0. \end{cases}$$

By Theorem 6.4.6, for every $n \geqslant 1$, this problem has a solution $x_n \in C\big(\overline{\Omega}\big)$, such that

$$x_n(z) \ > \ 0 \qquad \forall\, z \in \Omega.$$

Fix $z_0 \in \Omega$. Then by Harnack's inequality (see Theorem 6.1.49 and note that it is also valid for the p-Laplacian; see Trudinger (1967)), we can find $\xi > 0$, such that

$$x_n(z_0) \ \geqslant \ \xi \ > \ 0.$$

From Theorem 6.4.8, we see that the sequence $\{x_n\}_{n\geqslant 1} \subseteq C_0^{1,\beta}\big(\overline{\Omega}\big)$ is bounded ($\beta \in (0,1)$). Since

$$\text{the embedding } C_0^{1,\beta}\big(\overline{\Omega}\big) \subseteq C_0^1(\overline{\Omega}) \text{ is compact,}$$

we can assume that

$$x_n \ \longrightarrow \ x \quad \text{in } C_0^1(\overline{\Omega}).$$

Note that

$$x(z_0) \ \geqslant \ \xi$$

and so $x \neq 0$. Also

$$\begin{cases} -\mathrm{div}\left(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) + a_0(z)|x(z)|^{p-2}x(z) = 0 \quad \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0, \ x \geqslant 0, \ x \neq 0. \end{cases}$$

Invoking Theorem 6.4.9, we conclude that

$$x(z) \ > \ 0 \qquad \forall\, z \in \Omega.$$

□

6.5 Comparison Principles

If $x, y \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and

$$-\Delta x \leqslant -\Delta y \quad \text{with } x|_{\partial\Omega} \leqslant y|_{\partial\Omega},$$

then using the strong maximum principle (see Theorem 6.1.44 and Corollary 6.1.45), we can show that $x \leqslant y$ in Ω. The situation is more involved with the p-Laplacian due to the degeneracy of the operator. In this section we derive such comparison principles for the p-Laplacian.

The mathematical setting remains unchanged. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary and let $a \in L^\infty(\Omega)$ be a weight function. We deal with the differential operator

$$V_p(x) \stackrel{df}{=} -\Delta_p(x) + a\psi_p(x),$$

where

$$\psi_p \colon \mathbb{R} \longrightarrow \mathbb{R}$$

is the homeomorphism, defined by

$$\psi_p(r) \stackrel{df}{=} \begin{cases} |r|^{p-2}r & \text{if} \quad r \neq 0, \\ 0 & \text{if} \quad r = 0, \end{cases}$$

with $p \in (1, +\infty)$. If $a \equiv 0$, then we have the p-Laplacian.

DEFINITION 6.5.1 *We say that the operator V_p satisfies the **weak comparison principle**, if*

$$V_p(x) \leqslant V_p(y) \quad \text{in } W^{-1,p'}(\Omega)$$

and

$$x|_{\partial\Omega} \leqslant y|_{\partial\Omega} \quad \text{for some } x, y \in W^{1,p}(\Omega)$$

implies

$$x \leqslant y \quad \text{in } \Omega.$$

A first easy situation where the weak comparison principle holds is when $a \geqslant 0$. This is an easy consequence of the following proposition.

PROPOSITION 6.5.2
If $f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that:

(i) for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z, \zeta)$ is measurable;

(ii) for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta)$ is continuous, nondecreasing;

(iii) for almost all $z \in \Omega$ *and all* $\zeta \in \mathbb{R}$, *we have*

$$\big|f(z,\zeta)\big| \leqslant a(z) + c|\zeta|^{p-1},$$

with $a \in L^{p'}(\Omega)_+$ *(with* $\frac{1}{p} + \frac{1}{p'} = 1$*),* $c > 0$ *and* $x, y \in W^{1,p}(\Omega)$ *satisfy*

$$-\Delta_p x + f\big(\cdot, x(\cdot)\big) \leqslant -\Delta_p y + f\big(\cdot, y(\cdot)\big) \quad \text{in } W^{-1,p'}(\Omega),$$

with $x|_{\partial\Omega} \leqslant y|_{\partial\Omega}$,

then $x \leqslant y$ *in* Ω.

PROOF Note that

$$(x-y)^+ \in W_0^{1,p}(\Omega).$$

Using this as a test function, we obtain

$$\begin{aligned}&\int_{\{x>y\}} \Big(\|Dx\|_{\mathbb{R}^N}^{p-2}\, Dx - \|Dy\|_{\mathbb{R}^N}^{p-2}\, Dy, Dx - Dy\Big)_{\mathbb{R}^N}\, dz \\ &= \int_{\{x>y\}} \big(f\big(z, y(z)\big) - f\big(z, x(z)\big)\big)(x-y)(z)\, dz.\end{aligned}$$

By hypothesis

$$\int_{\{x>y\}} \big(f\big(z, y(z)\big) - f\big(z, x(z)\big)\big)(x-y)(z)\, dz \leqslant 0.$$

So using Lemma 6.2.13, we obtain $(x-y)^+ = 0$ and so $x \leqslant y$. □

COROLLARY 6.5.3
If $a \geqslant 0$,
then V_p *satisfies the weak comparison principle.*

REMARK 6.5.4 In fact the condition $a \geqslant 0$ is both necessary and sufficient for the weak comparison principle to hold. Moreover, in this case the operator

$$V_p^{-1} \colon W^{-1,p'}(\Omega) \longrightarrow L^p(\Omega)$$

is well defined, compact and order preserving (for details see García-Melián & Sabina de Lis (1998)).

Another situation where the weak comparison principle holds is given in the next theorem.

THEOREM 6.5.5
If $\lambda_{1,p}(a) > 0$, $x, y \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$,

$$V_p(x), V_p(y) \in L^{\infty}(\Omega), \quad x|_{\partial\Omega}, y|_{\partial\Omega} \in C^2(\partial\Omega),$$

$$V_p(x) \leqslant V_p(y) \text{ almost everywhere on } \Omega,$$

$$x|_{\partial\Omega} \leqslant y|_{\partial\Omega}$$

and

$$V_p(y) \geqslant 0 \quad \text{almost everywhere on } \Omega,$$

with $y|_{\partial\Omega} \geqslant 0$,
then $x \leqslant y$.
Moreover, if

$$x|_{\partial\Omega} = y|_{\partial\Omega} = 0,$$

then the same conclusion is valid under the less restrictive hypotheses that $x, y \in W^{1,p}(Z)$,

$$V_p(x), V_p(y) \in L^{\infty}(\Omega),$$

$$V_p(x) \leqslant V_p(y) \quad \text{almost everywhere on } \Omega$$

and

$$V_p(y) \geqslant 0 \quad \text{almost everywhere on } \Omega.$$

PROOF If $y = 0$, then using as test function $x^+ \in W_0^{1,p}(\Omega)$, we obtain $x^+ = 0$ and so

$$x \leqslant 0.$$

Therefore, we may assume that $y \neq 0$. In fact since by hypothesis

$$V_p(y) \geqslant 0 \quad \text{with } y|_{\partial\Omega} \geqslant 0,$$

using as a test function $-y^- \in W_0^{1,p}(\Omega)$, we obtain

$$\left\|Dy^-\right\|_p^p + \int_{\Omega} a|y^-|^p \, dz \leqslant 0,$$

hence $y^- = 0$ and so

$$y \geqslant 0, \quad y \neq 0.$$

From Theorem 6.2.7**(a)**, we know that $x, y \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Moreover, Theorem 6.2.8 implies that

$$y(z) > 0 \qquad \forall\, z \in \Omega$$

and

$$\frac{\partial y}{\partial n}(z) < 0 \qquad \forall\, z \in \partial\Omega,\ y(z) = 0.$$

This means that $y \in \operatorname{int} C^1(\overline{\Omega})_+$, where

$$C^1(\overline{\Omega})_+ \stackrel{df}{=} \{y \in C^1(\overline{\Omega}) : \ y(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$$

and so y is an order unit in the ordered Banach space $C^1(\overline{\Omega})$. Hence we can find $c > 1$, such that $x \leqslant cy$. Let

$$h \ = \ V_p(y) \in L^\infty(\Omega) \quad \text{and} \quad g \ = \ y|_{\partial\Omega} \in C^1(\partial\Omega)$$

and consider the following boundary value problem:

$$\begin{cases} V_p(w) = -\Delta_p w + a\varphi_0(w) = h & \text{in } \Omega, \\ w|_{\partial\Omega} = g & \text{in } \partial\Omega, \end{cases} \tag{6.83}$$

It is easy to prove that $v = x$ is a lower solution of (6.83) and $u = cy$ is an upper solution of (6.83). Then using standard truncation and penalization techniques (see, e.g., Gasiński & Papageorgiou (2005, Section 4.5)), we obtain a solution $w \in W^{1,p}(\Omega) \cap L^p(\Omega)$ of (6.83), such that

$$x(z) \ \leqslant \ w(z) \ \leqslant \ cy(z) \quad \text{for a.a. } z \in \Omega.$$

Moreover, since by hypothesis $\lambda_{1,p}(a) > 0$ and $h \geqslant 0$, from Theorem 6.4.6, we have that $w \geqslant 0$.

Claim. Problem (6.83) has a unique nonnegative solution in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Suppose that $w_1, w_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are two nonnegative solutions of (6.83). Then from Theorems 6.2.7 and 6.2.8, we have that

$$\frac{w_1}{w_2}, \frac{w_2}{w_1} \in L^\infty(\Omega).$$

A careful reading of the proof of Lemma 6.4.3 reveals that its conclusions remain valid even if the two functions do not vanish on $\partial\Omega$ and only have the same value. Then

$$0 \ \leqslant \ I(w_1, w_2) \ = \ \int_\Omega h\left(\frac{w_2^{p-1} - w_1^{p-1}}{w_1^{p-1} w_2^{p-1}}\right)\left(w_1^p - w_2^p\right) dz \ \leqslant \ 0$$

and so

$$I(w_1, w_2) \ = \ 0,$$

which, by Lemma 6.4.3, implies that $w_2 = \gamma w_1$ for some $\gamma > 0$. Hence

$$\gamma^{p-1} h \ = \ h,$$

which means that $\gamma = 1$ and so $w_1 = w_2$.

Using the Claim, we infer that $w = y$ and so we conclude that $x \leqslant y$.

The last statement of the theorem is a consequence of Theorem 6.2.6. □

Next we derive strong comparison principles (i.e., we show that $x < y$) for the p-Laplacian differential operator (i.e., we assume that $a \equiv 0$).

THEOREM 6.5.6 (Strong Comparison Principle I)
If $x, y \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $h, g \in L^\infty(\Omega)$ *are such that*

$$-\operatorname{div}\left(\left\|Dx(z)\right\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) = h(z) \quad \text{for a.a. } z \in \Omega, \tag{6.84}$$

$$-\operatorname{div}\left(\left\|Dy(z)\right\|_{\mathbb{R}^N}^{p-2} Dy(z)\right) = g(z) \quad \text{for a.a. } z \in \Omega, \tag{6.85}$$

$x \leqslant y$ *and* $h \leqslant g$ *almost everywhere on* Ω *and the set*

$$K \stackrel{df}{=} \left\{z \in \Omega : \ x(z) = y(z)\right\}$$

is compact,
then $K = \emptyset$.

PROOF From nonlinear regularity theory (see Theorem 6.2.7), we know that $x, y \in C^{1,\beta}\left(\overline{\Omega}\right)$ for some $\beta \in (0,1)$. We argue indirectly. Suppose that K is nonempty and compact. We can find an open set U with C^2-boundary ∂U, such that

$$K \subseteq U \subseteq \overline{U} \subseteq \Omega$$

and

$$x(z) < y(z) \qquad \forall\, z \in U \setminus K.$$

For a given $\varepsilon > 0$, let $x_\varepsilon, y_\varepsilon \in W^{1,p}(U)$ be the unique solutions of

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon + \left\|Dx_\varepsilon(z)\right\|_{\mathbb{R}^N}^2\right)^{\frac{p-2}{2}} Dx_\varepsilon(z)\right) = h(z) & \text{for a.a. } z \in \Omega, \\ x_\varepsilon|_{\partial U} = x|_{\partial U} \end{cases}$$

and

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon + \left\|Dy_\varepsilon(z)\right\|_{\mathbb{R}^N}^2\right)^{\frac{p-2}{2}} Dy_\varepsilon(z)\right) = g(z) & \text{for a.a. } z \in \Omega, \\ y_\varepsilon|_{\partial U} = y|_{\partial U} \end{cases}$$

respectively. The result of Lieberman (1988) that gave us Theorem 6.2.7 also implies that $x_\varepsilon, y_\varepsilon \in C^{1,\gamma}(U)$ for some $\gamma \in (0,1)$ independent of $\varepsilon \in (0,1]$ and are bounded. Similarly $\{x_\varepsilon\}_{\varepsilon \in (0,1]}, \{y_\varepsilon\}_{\varepsilon \in (0,1]} \in L^p\left(U; \mathbb{R}^N\right)$ are bounded. Therefore we may assume that

$$\lim_{\varepsilon \searrow 0} x_\varepsilon = x \quad \text{and} \quad \lim_{\varepsilon \searrow 0} y_\varepsilon = y$$

weakly in $W^{1,p}(U)$ and strongly in $C^{1,\beta}(U)$ for $\beta \in (0,\gamma)$ (recall that the embedding $C^{1,\gamma}(U) \subseteq C^{1,\beta}(U)$ is compact for $\beta \in (0,\gamma)$). Let V be an open subset of Ω, such that

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

and let

$$m_V \stackrel{df}{=} \min_{z\in\partial V}\big(y-x\big)(z) \;>\; 0 \quad \text{and} \quad w_\varepsilon \stackrel{df}{=} y_\varepsilon - x_\varepsilon.$$

Choose $\varepsilon > 0$ small enough, such that

$$\|y-y_\varepsilon\|_{L^\infty(V)} \;<\; \frac{m_V}{4} \quad \text{and} \quad \|x-x_\varepsilon\|_{L^\infty(V)} \;<\; \frac{m_V}{4}.$$

It follows that

$$y_\varepsilon - x_\varepsilon \;>\; \frac{m_V}{2} \quad \text{on } \partial V \quad \text{and} \quad y_\varepsilon - x_\varepsilon \;<\; \frac{m_V}{2} \quad \text{on } \partial K.$$

Because of the mean value theorem, we have

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial z_i}\left(a_{ij}^{\varepsilon}\frac{\partial w_\varepsilon}{\partial z_j}\right) \;=\; h-g \;\geqslant\; 0 \quad \text{in } V,$$

where

$$\begin{aligned} a_{ij}^{\varepsilon} \;=\; & \Big(\varepsilon + \big\|\lambda_i Dy_\varepsilon + (1-\lambda_i)Dx_\varepsilon\big\|_{\mathbb{R}^N}^2\Big)^{\frac{p-4}{2}} \times \\ & \times\Bigg[\delta_{ij}\Big(\varepsilon + \big\|\lambda_i Dy_\varepsilon + (1-\lambda_i)Dx_\varepsilon\big\|_{\mathbb{R}^N}^2\Big) \\ & \quad + (p-2)\left(\lambda_i\frac{\partial y_\varepsilon}{\partial z_i} + (1-\lambda_i)\frac{\partial x_\varepsilon}{\partial z_i}\right)\left(\lambda_i\frac{\partial y_\varepsilon}{\partial z_j} + (1-\lambda_j)\frac{\partial x_\varepsilon}{\partial z_j}\right)\Bigg] \end{aligned} \tag{6.86}$$

and $\lambda_i \in (0,1)$,

$$\delta_{ij} \stackrel{df}{=} \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

We set

$$\begin{aligned} r \;&\stackrel{df}{=}\; \min_{z\in V} w_\varepsilon(z) \;=\; \min_{z\in V}\big(y_\varepsilon - x_\varepsilon\big)(z), \\ K_r \;&\stackrel{df}{=}\; \big\{z\in V:\; w_\varepsilon(z) = \big(y_\varepsilon - x_\varepsilon\big)(z) = r\big\}. \end{aligned}$$

Evidently

$$K_r \subseteq V \text{ is nonempty, compact}$$

and

$$Dy_\varepsilon \;=\; Dx_\varepsilon \quad \text{on } K_r.$$

So from (6.86), we have

$$a_{ij}^{\varepsilon} \;=\; \Big(\varepsilon + \|Dy_\varepsilon\|_{\mathbb{R}^N}^2\Big)^{\frac{p-4}{2}}\left[\delta_{ij}\Big(\varepsilon + \|Dy_\varepsilon\|_{\mathbb{R}^N}^2\Big) + (p-\varepsilon)\frac{\partial y_\varepsilon}{\partial z_i}\frac{\partial x_\varepsilon}{\partial z_j}\right].$$

Also since a_{ij}^{ε} are the components of the Hessian of the convex function

$$\widehat{\xi} = (\xi_i)_{i=1}^N \longmapsto \frac{1}{p}\left(\varepsilon + \|\widehat{\xi}\|_{\mathbb{R}^N}^2\right)^{\frac{p}{2}}$$

at the point $\widehat{\xi} = Dy_\varepsilon$, we have

$$\sum_{i,j=1}^N a_{ij}^{\varepsilon}\xi_i\xi_j \geqslant \mu\|\widehat{\xi}\|_{\mathbb{R}^N}^2 \qquad \forall\, \widehat{\xi} = (\xi_i)_{i=1}^N \in \mathbb{R}^N, \tag{6.87}$$

for some $\mu > 0$. We choose an open neighbourhood W_r of the compact set K_r, such that

$$\overline{W_r} \subseteq V,$$

$$y_\varepsilon - x_\varepsilon > r \quad \text{on } \partial W_r$$

and (6.87) is still valid with $\mu > 0$ replaced by $\frac{\mu}{2}$ (recall that Dy_ε and Dx_ε are continuous). Then by the strong maximum principle (see Theorem 6.1.44), we conclude that

$$w_\varepsilon = y_\varepsilon - x_\varepsilon \text{ is constant on } W_r,$$

a contradiction. This proves that $K = \emptyset$. □

The next strong comparison theorem gives additional information about the normal derivatives on the boundary, reminiscent of the Hopf boundary lemma for linear differential operators (see Lemma 6.1.43).

THEOREM 6.5.7 (Strong Comparison Principle II)
If $x, y \in C_0^1(\overline{\Omega})$, $h, g \in L^\infty(\Omega)$ *are such that*

$$-\text{div}\left(\|Dx(z)\|_{\mathbb{R}^N}^{p-2}Dx(z)\right) = h(z) \quad \textit{for a.a. } z \in \Omega, \tag{6.88}$$

$$-\text{div}\left(\|Dy(z)\|_{\mathbb{R}^N}^{p-2}Dy(z)\right) = g(z) \quad \textit{for a.a. } z \in \Omega, \tag{6.89}$$

$$0 \leqslant h(z) \leqslant g(z) \quad \textit{for a.a. } z \in \Omega$$

and the set

$$C \stackrel{df}{=} \{z \in \Omega : h(z) = g(z) \textit{ for a.a. } z \in \Omega\}$$

has empty interior,
then

$$0 \leqslant x(z) < y(z) \qquad \forall\, z \in \Omega$$

and

$$\frac{\partial x}{\partial n}(z) < \frac{\partial y}{\partial n}(z) \qquad \forall\, z \in \partial\Omega.$$

PROOF Note that the set

$$\big\{z \in \Omega : \ g(z) > 0 \text{ for a.a. } z \in \Omega\big\}$$

has positive measure and so

$$y \ \neq \ 0.$$

Therefore we can apply Theorem 6.2.8 and obtain

$$y(z) \ > \ 0 \quad \text{and} \quad \frac{\partial y}{\partial n}(z) \ < \ 0 \qquad \forall \ z \in \partial\Omega.$$

Moreover, from Theorem 6.5.5, we have that

$$x \ \leqslant \ y \quad \text{in } \Omega$$

and we can easily check that

$$x \ \geqslant \ 0.$$

Consider the coincidence set

$$K \ \stackrel{df}{=} \ \big(z \in \Omega : x(z) = y(z)\big)$$

and suppose that it is not empty. Then we can find a sequence $\{z_n\}_{n \geqslant 1} \subseteq K$ and $z_0 \in \partial\Omega$, such that

$$z_n \ \longrightarrow \ z$$

(recall that $x|_{\partial\Omega} = y|_{\partial\Omega} = 0$). Since $x, y \in C^1(\overline{\Omega})$, we have that

$$\frac{\partial x}{\partial n}(z_0) \ = \ \frac{\partial y}{\partial n}(z_0) \ = \ -\mu^2 \ < \ 0. \tag{6.90}$$

The function $w = y - x$ satisfies

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(a_{ij}\frac{\partial w}{\partial z_j}\right) \ = \ g - h \ \geqslant \ 0,$$

with $a_{ij} = a_{ij}^0$ (see (6.86) with $\varepsilon = 0$). Then

$$a_{ij}(z_0) \ = \ \left|\frac{\partial y}{\partial n}(z_0)\right|^{p-4}\left[\delta_{ij}\left|\frac{\partial y}{\partial n}(z_0)\right|^2 + (p-2)\frac{\partial y}{\partial z_i}(z_0)\frac{\partial y}{\partial z_j}(z_0)\right].$$

Exploiting the continuity of Dx and Dy, we can find a ball $B \subseteq \Omega$, such that $z_0 \in \partial B$ and the elliptic operator, defined by the a_{ij}, is uniformly elliptic in B. Hence once again by the strong maximum principle (see Theorem 6.1.44), we infer that either $w = 0$, which cannot happen since by hypothesis $\operatorname{int} C = \emptyset$ or

$$w \ > \ 0 \quad \text{in } B \quad \text{and} \quad \frac{\partial w}{\partial n}(z_0) \ < \ 0,$$

which contradicts (6.90). Therefore

$$x(z) \;<\; y(z) \qquad \forall\, z \in \Omega.$$

Finally since

$$\frac{\partial x}{\partial n} \;\leqslant\; \frac{\partial y}{\partial n} \;\leqslant\; 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \frac{\partial y}{\partial n}(z_0) \;<\; 0,$$

as above we conclude that

$$\frac{\partial x}{\partial n}(z) \;<\; \frac{\partial y}{\partial n}(z) \qquad \forall\, z \in \partial\Omega.$$

□

A related problem deals with the oscillation/nonoscillation of the solutions (Sturmian theory). Picone's identity (see Theorem 6.4.8) can be useful in this direction.

THEOREM 6.5.8
If $a_1, a_2 \in L^\infty(\Omega)$, $a_2 \leqslant a_1 \leqslant 0$ *almost everywhere on* Ω, $a_1 \neq a_2$ *and* $x \in C_0^1(\overline{\Omega})$ *is a positive solution of*

$$\begin{cases} -\operatorname{div}\left(\left\|Dx(z)\right\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) + a_1(z)\left|x(z)\right|^{p-2} x(z) = 0 & \text{for a.a. } z \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases}$$

then any solution $y \in C_0^1(\overline{\Omega})$ *of*

$$\begin{cases} -\operatorname{div}\left(\left\|Dy(z)\right\|_{\mathbb{R}^N}^{p-2} Dy(z)\right) + a_2(z)\left|y(z)\right|^{p-2} y(z) = 0 & \text{for a.a. } z \in \Omega, \\ y|_{\partial\Omega} = 0 \end{cases}$$

must change sign.

PROOF We argue indirectly. So assume without any loss of generality that

$$y(z) \;>\; 0 \qquad \forall\, z \in \Omega$$

(see Theorem 6.4.9). We have

$$0 \;\leqslant\; \int_\Omega L(x,y)\, dz \;=\; \int_\Omega \left(a_2 - a_1\right) x^p\, dz \;\leqslant\; 0,$$

so

$$x \;=\; cy,$$

for some $c > 0$. But this contradicts the hypothesis that $a_1 \neq a_2$. So we conclude that y must change sign. □

6.6 Remarks

6.1: The eigenvalue theory of symmetric elliptic differential operators is standard and can be found in many books on partial differential equations. Indicatively we mention the books of Courant & Hilbert (1953, 1989) (a classical), Di Benedetto (1995), Evans (1998), Jost (2002), Protter & Weinberger (1967) and Renardy & Rogers (1993). The minimax expressions for the eigenvalues (see Theorem 6.1.23) can be traced back to Courant (see Courant & Hilbert (1953, 1989)). Our analysis of the weighted eigenvalue problem with an indefinite weight follows the work of de Figueiredo (1982). The results on the spectrum of self-adjoint elliptic operators can be viewed as an infinite dimensional generalization of the well known result from linear algebra which says that a real, symmetric matrix has real eigenvalues and an orthonormal basis of eigenvectors. For nonsymmetric, elliptic operators in nondivergence form

$$Lx \stackrel{df}{=} -\sum_{i,j=1}^{N} a_{ij} \frac{\partial x}{\partial z_i \partial z_j} + \sum_{i=1}^{N} b_i \frac{\partial x}{\partial z_i} + a_0 x,$$

with $a_{ij} = a_{ji}, a_{ij}, b_i, a_0 \in C^\infty(\Omega)$, the principal eigenvalue of L is real, simple and the corresponding eigenfunction does not change sign in Ω. Another important result of the linear eigenvalue theory is the so-called Courant's nodal domain theorem (see Courant & Hilbert (1953)).

THEOREM 6.6.1 (Courant Nodal Domain Theorem)
If $0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ *are the eigenvalues of* $\big(-\Delta, H_0^1(\Omega)\big)$, $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ *are the corresponding eigenfunctions and*

$$S_n \stackrel{df}{=} \big\{z \in \Omega : \ u_n(z) = 0\big\}$$

(the ***nodal set*** *of* u_n*),*
then $\Omega \setminus S_n$ *has at most* n *components.*

We should point out that the eigenfunctions of $-\Delta$ (Dirichlet or Neumann) have the so-called ***unique continuation property***, namely if u is an eigenfunction, then the set

$$\big\{z \in \Omega : \ u(z) = 0\big\}$$

has an empty interior.

The regularity theory of elliptic differential equations is a very broad subject. Here we limited ourselves to the case where the coefficients a_{ij} are regular. The general case of bounded measurable coefficients a_{ij} was resolved independently by De Giorgi (1957) and Nash (1958), who employed different methods. Later Moser (1960) produced a more compact proof based on a version of the Harnack inequality. The result of De Giorgi-Nash for elliptic equations with discontinuous coefficients says the following.

THEOREM 6.6.2
If $x \in H^1(\Omega)$ *is a weak solution of*

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial x}{\partial z_i}(z)\right) = 0 \quad \text{in } \Omega,$$

where the coefficients a_{ij} *are bounded, measurable and satisfy*

$$c\,\|\xi\|^2_{\mathbb{R}^N} \leqslant \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \qquad \forall\ z \in \Omega,\ \xi = (\xi_i)_{i=1}^N \in \mathbb{R}^N,$$

$$\left|a_{ij}(z)\right| \leqslant M \qquad \forall\ z \in \Omega,$$

with $0 < c < M < +\infty$,
then x *is Hölder continuous.*

Additional results in this direction can be found in Stampacchia (1965).

The maximum principle for harmonic and subharmonic functions goes back to Gauss. Many books contain expositions of the maximum principle. We mention those by Courant & Hilbert (1953, 1989), Evans (1998), Gilbarg & Trudinger (2001), Nečas (1967), Protter & Weinberger (1967) and Renardy & Rogers (1993). The strong maximum principle (see Lemma 6.1.43 and Theorem 6.1.44) is essentially due to Hopf (1927). See also Hopf (1952), Oleinik (1952) and Pucci (1952). The maximum principle can be used to prove a result on the removability of isolated singularities of the Laplace operator.

PROPOSITION 6.6.3
If $z_0 \in \mathbb{R}^N$ *(*$N \geqslant 2$*),* $r > 0$ *and* $u\colon B_r(z_0) \setminus \{z_0\} \longrightarrow \mathbb{R}$ *is bounded and harmonic,*
then there exists a harmonic function $\widehat{u}\colon B_r(z_0) \longrightarrow \mathbb{R}$*, such that*

$$\widehat{u}|_{B_r(z_0)\setminus\{z_0\}} = u.$$

REMARK 6.6.4 Consider the problem

$$\begin{cases} \Delta u(z) = 0 \quad \text{in } B_1(0) \setminus \{0\}, \\ u|_{\partial B_1(0)} = 0,\ u(0) = 1. \end{cases}$$

Then by virtue of Proposition 6.6.3, this problem has no solution. Indeed if u is such a solution, then from the above proposition u admits a harmonic extension $\widehat{u}\colon B_1(0) \longrightarrow \mathbb{R}$, such that

$$\widehat{u}|_{\partial B_1(0)} = 0.$$

Then $\widehat{u} \equiv 0$ and so

$$\widehat{u}(0) = 0 \neq 1.$$

6.2: The systematic study of the p-Laplacian differential operator started in the mid-eighties. First de Thélin (1986) proved that there exists only one first eigenfunction among the radial functions in a ball. Then the study was extended to more general domains and variational results were obtained by Anane (1987, 1987–1988), Bhattacharya (1988), Guedda & Véron (1988), Lindqvist (1990, 1992). Theorem 6.2.6 is a particular case of a more general result due to Ladyzhenskaya & Uraltseva (1968, p. 286). The variational characterization of λ_2 given in Theorem 6.2.24 is due to Anane & Tsouli (1996). More on the nonlinear weighted eigenvalue problem can be found in Binding, Drábek & Huang (1997*a*), Godoy, Gossez & Paczka (2002) and Huang (1990). The eigenfunctions of $-\Delta_p$ in general do not have the unique continuation property. See Martio (1988) for a counterexample. However, if we assume that the unique continuation property holds, then we can have a generalization of Courant's nodal domain theorem (see Theorem 6.6.1) to the case of the p-Laplacian. Namely we have the following result due to Drábek & Robinson (2002).

THEOREM 6.6.5
If $-\Delta_p$ satisfies the unique continuous property and u_{λ_n} is an eigenfunction associated with the eigenvalue $\lambda_n > 0$,
then u_{λ_n} has at most n nodal domains (recall that a nodal domain is defined as a maximal connected, open subset (connected component) of $\{z \in \Omega : u_{\lambda_n}(z) \neq 0\}$).

As we already mentioned, the Fredholm alternative may not hold for $-\Delta_p$, with $p \neq 2$ (see Binding, Drábek & Huang (1997*b*), Drábek & Holubová (2001) for the partial p-Laplacian and del Pino, Drábek & Manásevich (1999) for the ordinary p-Laplacian).

6.3: The regularity properties of the eigenfunctions of the nonnegative scalar ordinary p-Laplacian with Dirichlet boundary conditions were established by Ôtani (1984). The complete sequence $\{\lambda_n\}_{n\geqslant 1}$ of eigenvalues for problem (6.50) was obtained by Guedda & Véron (1988) and del Pino, Elgueta & Manásevich (1989). The eigenvalues for the Neumann problem (see (6.65)) and for periodic problem (see (6.66)) can be found in Drábek & Manásevich (1999), where a more general nonlinear eigenvalue problem is studied, namely

$$-\big(|x'(t)|^{p-2}x'(t)\big)' \;=\; \lambda|x(t)|^{r-2}x(t) \quad \text{for a.a. } t \in T = [0,b],$$

with $p, r \in (1, +\infty)$. The weighted eigenvalue problems with Dirichlet and periodic bounded conditions were studied by Zhang (2001). He examined the following nonlinear eigenvalue problem:

$$-\big(|x'(t)|^{p-2}x'(t)\big)' \;=\; \big(\lambda + a(t)\big)|x(t)|^{p-2}x(t) \quad \text{for a.a. } t \in T = [0,b],$$

with $a \in L^1(T)$, $p \in (1, +\infty)$ and with Dirichlet or periodic boundary conditions. Zhang proved that the Dirichlet eigenvalue problem has a sequence of eigenvalues

$$-\infty \ < \ \lambda_1(a) \ < \ \lambda_2(a) \ < \ \ldots \ < \ \lambda_n(a) \ < \ \ldots,$$

with

$$\lambda_n(a) \ \longrightarrow \ +\infty.$$

All these eigenvalues are simple and to the first (principal) eigenvalue $\lambda_1(a)$ corresponds an eigenfunction $\widehat{u}_1$ which satisfies $\widehat{u}_1 > 0$ for all $t \in (0, b)$.

For the periodic eigenvalue problem, there is a sequence of eigenvalues

$$-\infty \ < \ \overline{\lambda}_0(a) \ < \ \underline{\lambda}_2(a) \ \leqslant \ \overline{\lambda}_2(a) \ < \ \ldots \ < \ \underline{\lambda}_{2n}(a) \ \leqslant \ \overline{\lambda}_{2n}(a) \ < \ \ldots,$$

with $\underline{\lambda}_{2n} \longrightarrow +\infty$ (hence $\overline{\lambda}_{2n} \longrightarrow +\infty$).

If $p = 2$, then $\left\{\overline{\lambda}_0(a), \underline{\lambda}_{2n}(a), \overline{\lambda}_{2n}(a)\right\}_{n \geqslant 1}$ cover all the eigenvalues of the weighted periodic eigenvalue problem.

If $p \neq 2$, we do not know if this is the case. The eigenvalues $\overline{\lambda}_0(a)$, $\underline{\lambda}_{2n}(a)$, $\overline{\lambda}_{2n}(a)$ have the following monotonicity property with respect to the weight function $a \in L^1(T)$. Namely, if $a, c \in L^1(T)$, $a(t) \leqslant c(t)$ for almost all $t \in T$ and the inequality is strict on a set of positive measure, then

$$\overline{\lambda}_0(a) \ > \ \overline{\lambda}_0(c), \quad \underline{\lambda}_{2n}(a) \ > \ \underline{\lambda}_{2n}(c) \quad \text{and} \quad \overline{\lambda}_{2n}(a) \ > \ \overline{\lambda}_{2n}(a) \qquad \forall \ n \geqslant 1.$$

Moreover, the functions $a \longmapsto \overline{\lambda}_0(a)$, $a \longmapsto \underline{\lambda}_{2n}(a)$ and $a \longmapsto \overline{\lambda}_{2n}(a)$ are continuous on $L^1(T)$ with the norm topology.

The spectrum of the vector ordinary p-Laplacian was studied by Manásevich & Mawhin (1998, 2000), Mawhin (2001) and Zhang (2000). Proposition 6.3.11 was established by Manásevich & Mawhin (1998), as a corollary of a more general existence theorem. Here we give a direct proof.

6.4: Maximum principle-type results for the p-Laplacian were obtained by Allegretto & Huang (1999) (see Theorem 6.4.9) and García-Melián & Sabina de Lis (1998) (see Theorems 6.4.5 and 6.4.6). Lemma 6.4.3 is due to Anane (1987). Maximum principles for more general quasilinear operators can be found in Damascelli (1998). Nonlinear analogs of Harnack's inequality in terms of the p-Laplacian can be found in Damascelli (1998), Tolksdorf (1983) and Trudinger (1967).

There is also the so-called antimaximum principle. Namely, consider the following problem:

$$\begin{cases} -\text{div}\left(\left\|Dx(z)\right\|_{\mathbb{R}^N}^{p-2} Dx(z)\right) = \lambda \left|x(t)\right|^{p-2} x(t) + h(z) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in \Omega, \\ Bx = 0. \end{cases} \tag{6.91}$$

Here Bx represents either Dirichlet or Neumann boundary conditions, $p \in (1, +\infty)$ and $h \in L^\infty(\Omega)$.

THEOREM 6.6.6 (Antimaximum Principle)
If $h \in L^\infty(\Omega)$, $h(z) \geqslant 0$ *for almost all* $z \in \Omega$, *then there exists* $\delta = \delta(h) > 0$, *such that for all* $\lambda \in (\lambda_1, \lambda_1 + \delta)$ *any solution* $x \in C^1(\overline{\Omega})$ *of (6.91) satisfies*

$$x(z) \ < \ 0 \qquad \forall \ z \in \Omega.$$

For $p = 2$, the antimaximum principle was proved by Clément & Peletier (1979), Hess (1981) and Godoy, Gossez & Paczka (1999) (the last two works with a weight function). For $p \neq 2$, the antimaximum principle was proved by Fleckinger, Gossez, Takač & de Thélin (1995) and Godoy, Gossez & Paczka (2002) (problems with a weight).

6.5: Comparison results involving the p-Laplacian were proved by García-Melián & Sabina de Lis (1998) (see Theorem 6.5.5), Guedda & Véron (1989), (see Theorems 6.5.6 and 6.5.7) and Allegretto & Huang (1999) (see Theorem 6.5.8). Additional results can be found in Allegretto & Huang (1998) and for more general quasilinear degenerate operators in Damascelli (1998).

Chapter 7

Fixed Point Theory

Fixed point theorems are the basic mathematical tools in showing the existence of solutions in various kinds of equations and inclusions. Fixed point theory is in the heart of nonlinear analysis since it provides the necessary tools to have existence theorems in many different nonlinear problems. It employs its tools from both analysis and topology and for this reason we have the informal (and imprecise) classification to "metric fixed point theory" and "topological fixed point theory."

Section 7.1 deals with the metric fixed point theory. Roughly speaking this includes all results for which the metric structure of the underlying space and/or the metric properties of the map involved play a crucial role. The most characteristic example of such a result is the celebrated Banach's contraction principle. In fact many of the results in this group are outgrowths of this theorem. Essentially Banach's fixed point theorem is a clever abstraction of the well known from the theory of differential equations method of successive approximations. We present several generalizations of Banach's theorem, we examine nonexpansive maps and we also present Caristi's theorem, which is a remarkable deep generalization of the contraction principle.

In Section 7.2, we examine the other group of fixed point theorems, which constitute the so-called "topological fixed point theory." Now crucial are the topological properties of the space and/or of the map involved. In particular the notion of compactness is basic in our considerations. The archetypical topological fixed point theorems are Brouwer's fixed point theorem and its infinite dimensional generalization, the Schauder-Tychonoff fixed point theorem. Our presentation is centered around these two fundamental results.

In many applications, order induced by a particular binary relation plays a central role. So in Section 7.3 we examine how order structures enter in fixed point theory. Special emphasis is given to ordered Banach spaces and the different types of order cones which characterize them. Also we introduce the notion of fixed point index, which is the main tool in obtaining fixed point theorems of the cone expansion and compression type.

Finally in Section 7.4 we show that most of the fixed point theorems for single valued maps have their counterparts for set valued maps (multifunctions). Such results are important in many applications, such as optimization and mathematical economics.

7.1 Metric Fixed Point Theory

In this section we present some fixed point theorems in which geometric conditions on the underlying space and/or mappings play a crucial role. So the results presented in this section are within the framework of at least a metric space, usually of a Banach space, and the analysis involves both the topological and the geometrical structure of the space combined with metric constraints on the map.

We start with the celebrated "Banach Contraction Principle," which, since its appearance in Banach's thesis in 1922, found many applications in different parts of mathematical analysis. First a definition.

DEFINITION 7.1.1 *Let (X, d_X) be a metric space and let $\varphi\colon X \longrightarrow X$. We say that φ is:*

(a) *a **contraction**, if there is $\beta \in (0,1)$, such that*

$$d_X\big(\varphi(x), \varphi(y)\big) \leqslant \beta d_X(x,y) \qquad \forall\, x, y \in X;$$

(b) *a **nonexpansive map**, if*

$$d_X\big(\varphi(x), \varphi(y)\big) \leqslant d_X(x,y) \qquad \forall\, x, y \in X.$$

THEOREM 7.1.2 (Banach Fixed Point Theorem) *If (X, d_X) is a complete metric space and $\varphi\colon X \longrightarrow X$ is a contraction, then φ has a unique fixed point $y \in X$ and*

$$\varphi^{(n)}(x) \longrightarrow y \qquad \forall\, x \in X,$$

where the sequence $\big\{\varphi^{(n)}(x)\big\}_{n\geqslant 1}$ is defined inductively by

$$\varphi^{(0)}(x) = x \quad \text{and} \quad \varphi^{(n+1)}(x) = \varphi\big(\varphi^{(n)}(x)\big).$$

PROOF Let $x \in X$ and consider the sequence of iterates $\big\{\varphi^{(n)}(x)\big\}_{n\geqslant 1}$. Then for any $n, m \geqslant 1$, we have

$$d_X\big(\varphi^{(n)}(x), \varphi^{(n+m)}(x)\big) \leqslant \sum_{k=n}^{n+m-1} d_X\big(\varphi^k(x), \varphi^{(k+1)}(x)\big)$$

$$\leqslant \sum_{k=n}^{n+m-1} \beta^k d_X\big(x, \varphi(x)\big) \leqslant \frac{\beta^n}{1-\beta} d_X\big(x, \varphi(x)\big)$$

(since $\beta \in (0,1)$). Because

$$\frac{\beta^n}{1-\beta} \longrightarrow 0,$$

it follows that

$$\left\{\varphi^{(n)}(x)\right\}_{n\geqslant 1} \text{ is a Cauchy sequence in } X$$

and so exploiting the completeness of (X, d_X), we have that

$$\varphi^{(n)}(x) \longrightarrow y \quad \text{in } X.$$

Then

$$\varphi^{(n+1)}(x) = \varphi\big(\varphi^{(n)}(x)\big) \longrightarrow \varphi(y)$$

and so

$$\varphi(y) = y,$$

i.e., $y \in X$ is a fixed point of φ. If y and y' are both fixed points of φ, then

$$d_X(y, y') = d_X\big(\varphi(y), \varphi(y')\big) \leqslant \beta d_X(y, y'),$$

a contradiction unless $y = y'$. So the fixed point of φ is unique. □

REMARK 7.1.3 From the above proof we have that

$$d_X\left(\varphi^{(n)}(x), \varphi^{(n+m)}(x)\right) \leqslant \frac{\beta^n}{1-\beta} d_X\left(x, \varphi(x)\right) \qquad \forall\ m \geqslant 1.$$

Hence

$$\begin{aligned} d_X\left(\varphi^{(n)}(x), y\right) &= \lim_{m\to+\infty} d_X\left(\varphi^{(n)}(x), \varphi^{(n+m)}(x)\right) \\ &\leqslant \frac{\beta^n}{1-\beta} d_X\left(x, \varphi(x)\right). \end{aligned}$$

This inequality gives an estimate for the error in the n-th iteration starting from any $x \in X$. Also for any $x \in X$, we have

$$d_X(x, y) \leqslant \frac{1}{1-\beta} d_X\left(x, \varphi(x)\right).$$

EXAMPLE 7.1.4 **(a)** In Theorem 7.1.2 we cannot assume that the map is nonexpansive instead of contraction. Let $X = c_0$, the space of all $\mathbb{R}$-valued sequences $\widehat{x} = \{x_n\}_{n\geqslant 1}$, such that

$$x_n \longrightarrow 0.$$

Furnished with the supremum norm

$$\|\widehat{x}\|_\infty = \sup_{n\geqslant 1} |x_n|,$$

$X = c_0$ becomes a Banach space. The map

$$\varphi\colon X \longrightarrow X$$

defined by

$$\varphi(\widehat{x}) \stackrel{df}{=} \big(1, x_1, x_2, \dots\big) \qquad \forall\, x \in X$$

is an isometry but clearly it is fixed point free.

(b) Let $T = [0, b]$, Y be a separable Banach space and

$$f\colon T \times Y \longrightarrow Y$$

be a function, such that for all $y \in Y$, the function $t \longmapsto f(t, y)$ is measurable and

$$\big\|f(t, y) - f(t, z)\big\|_Y \leqslant k(t)\, \|y - z\|_Y \quad \text{for a.a. } t \in T \text{ and all } y, z \in Y,$$

where $k \in L^1(T)_+$. Consider the Cauchy problem

$$\begin{cases} x'(t) = f\big(t, x(t)\big) & \text{for a.a. } t \in T, \\ x(0) = y_0 \in Y. \end{cases}$$

We claim that this problem has a unique solution. Indeed note that this existence problem is equivalent to the fixed point problem

$$x \;=\; \varphi(x), \quad x \in X = C(T; Y),$$

where

$$\varphi\colon X \longrightarrow X$$

is defined by

$$\varphi(x) \stackrel{df}{=} y_0 + \int_0^t f\big(s, x(s)\big)\, ds \qquad \forall\, t \in T, x \in X.$$

On X we consider the equivalent norm

$$|x| \stackrel{df}{=} \max_{t \in T} \Big[\big\|x(t)\big\|_Y e^{-\lambda t}\Big],$$

with $\lambda > 0$. Then we obtain

$$\big|\varphi(x)(t) - \varphi(z)(t)\big| \leqslant \frac{\|k\|_1}{\lambda} e^{\lambda t} |x - z| \qquad \forall\, t \in T,\ x, z \in X,$$

so

$$\big|\varphi(x) - \varphi(z)\big| \leqslant \frac{\|k\|_1}{\lambda} |x - z| \qquad \forall\, x, z \in X.$$

If we choose $\lambda > \|k\|_1$, we see that φ is a contraction for the $|\cdot|$-norm and so we can apply Theorem 7.1.2. This remetrization trick is often used in applying Theorem 7.1.2.

We can use Theorem 7.1.2 to establish the following invariance of domain result.

PROPOSITION 7.1.5
If X is a Banach space, $U \subseteq X$ is an open set and $\varphi\colon U \longrightarrow X$ is a contraction,
then

(a) *$\psi = id_X - \varphi\colon U \longrightarrow X$ is an open map;*

(b) *$\psi\colon U \longrightarrow \psi(U)$ is a homeomorphism.*

PROOF **(a)** Let $x \in U$ and $r > 0$ be such that

$$B_r(x) \;=\; \Big\{ y \in X :\; \|y - x\|_X < r \Big\} \;\subseteq\; U.$$

Choose any $y \in B_{(1-\beta)r}\big(\psi(x)\big)$, where $\beta \in (0,1)$ is the contraction constant of the map φ. We will show that $y \in \psi\big(B_r(x)\big)$. To this end let

$$h(z) \;\stackrel{df}{=}\; \varphi(z) + y \qquad \forall\; z \in \overline{B}_r(x).$$

We have

$$\begin{aligned} \big\|h(z) - x\big\|_X \;&=\; \big\|\varphi(z) + y - x\big\|_X \\ &\leqslant\; \big\|\varphi(z) - \varphi(x)\big\|_X + \big\|\varphi(x) + y - x\big\|_X \\ &\leqslant\; \beta\big\|z - x\big\|_X + \big\|\psi(x) - y\big\|_X \\ &<\; \beta r + (1-\beta)r \;=\; r. \end{aligned}$$

Therefore

$$h\colon \overline{B}_r(x) \;\longrightarrow\; B_r(x)$$

and since h is a contraction (with constant β), we can apply Theorem 7.1.2 and obtain $u \in B_r(x)$, such that $u = h(u)$, hence

$$\psi(u) \;=\; y,$$

which proves that ψ is open.

(b) Since $\psi\colon U \longrightarrow \psi(U)$ is a continuous, open bijection, it is a homeomorphism. ∎

REMARK 7.1.6 If $\varphi\colon X \longrightarrow X$ is a contraction, then

$$\psi \;=\; id_X - \varphi\colon X \longrightarrow X$$

is a homeomorphism of X onto itself.

There are many generalizations of Theorem 7.1.2, which are obtained by weakening the contraction property of the map φ.

THEOREM 7.1.7
If (X, d_X) is a complete metric space and $\varphi\colon X \longrightarrow X$, then φ has a unique fixed point $y \in X$ and

$$\varphi^{(n)}(x) \longrightarrow y \qquad \forall\, x \in X$$

provided that one of the following conditions is satisfied:

(a) $d_X\big(\varphi^{(m)}(x), \varphi^{(m)}(z)\big) \leqslant \beta d_X(x, z)$ *for some $m \geqslant 1$, some $\beta \in (0, 1)$ and all $x, z \in X$;*

(b) $d_X\big(\varphi^{(m)}(x), \varphi^{(m)}(z)\big) \leqslant \beta_m d_X(x, z)$ *for all $x, z \in X$ and with $\sum_{m=1}^{\infty} \beta_m < +\infty$;*

(c) *X is compact and $d_X\big(\varphi(x), \varphi(z)\big) < d_X(x, z)$ for all $x, z \in X$;*

(d) $d_X\big(\varphi(x), \varphi(z)\big) \leqslant \beta\big[d_X\big(x, \varphi(x)\big) + d_X\big(z, \varphi(z)\big)\big]$ *for some $\beta \in \big(0, \frac{1}{2}\big)$ and all $x, z \in X$.*

PROOF **(a)** Let

$$\vartheta(x) \stackrel{df}{=} d_X\big(x, \varphi^{(m)}(x)\big).$$

Then $\big\{\vartheta\big(\varphi^{(n)}(x)\big)\big\}_{n \geqslant 1}$ is a Cauchy sequence in X and so

$$\lim_{n \to +\infty} \vartheta\big(\varphi^{(n)}(x)\big) = u,$$

for some $u \in X$. We have

$$\vartheta\big(\varphi^{(n)}(x)\big) \leqslant \beta^{r_n} \max_{k \leqslant n-1} \vartheta\big(\varphi^{(k)}(x)\big) \qquad \forall\, n \geqslant 1,$$

with r_n being the largest nonnegative integer bigger or equal to $\frac{n}{m}$. Therefore $u = 0$. Also $\big\{\varphi^{(n)}(x)\big\}_{n \geqslant 1}$ is a Cauchy sequence, so

$$\varphi^{(n)}(x) \longrightarrow y.$$

We have

$$d_X\big(y, \varphi^{(m)}(y)\big) = 0,$$

hence

$$\varphi(y) = \varphi^{(m)}\big(\varphi(y)\big)$$

and since φ^m has only one fixed point, we conclude that $y = \varphi(y)$.

(b) Now let $\vartheta(x) = d_{_X}\big(x, \varphi(x)\big)$. We have

$$\vartheta\big(\varphi^{(n)}(x)\big) \;\leqslant\; \beta_n \vartheta(x)$$

and so

$$\vartheta\big(\varphi^{(n)}(x)\big) \;\longrightarrow\; 0.$$

Also

$$\operatorname{diam}\Big(\big\{\varphi^{(n)}(x)\big\}_{n\geqslant m}\Big) \;\leqslant\; c\sum_{n=m}^{\infty}\beta_n$$

for some $c > 0$, so

$$\operatorname{diam}\Big(\big\{\varphi^{(n)}(x)\big\}_{n\geqslant m}\Big) \;\longrightarrow\; 0$$

and thus we infer that

$$\big\{\varphi^{(n)}(x)\big\}_{n\geqslant 1} \text{ is a Cauchy sequence.}$$

Therefore

$$\varphi^{(n)}(x) \;\longrightarrow\; y.$$

We obtain

$$\vartheta(y) \;=\; d_{_X}\big(y, \varphi(y)\big) \;=\; 0$$

and so $y = \varphi(y)$.

(c) Again let

$$\vartheta(x) \;\stackrel{df}{=}\; d_{_X}\big(x, \varphi(x)\big).$$

Evidently the sequence $\big\{\vartheta\big(\varphi^{(n)}(x)\big)\big\}_{n\geqslant 1}$ is decreasing and so we have

$$\vartheta\big(\varphi^{(n)}(x)\big) \;\longrightarrow\; u.$$

Also due to the compactness of X, we may assume that

$$\varphi^{(n)}(x) \;\longrightarrow\; y \quad \text{in } X.$$

Therefore

$$\vartheta(y) \;=\; d_{_X}\big(y, \varphi(y)\big) \;=\; 0$$

and so $y = \varphi(y)$.

(d) As before let

$$\vartheta(x) \;\stackrel{df}{=}\; d_{_X}\big(x, \varphi(x)\big).$$

We have that $\big\{\vartheta\big(\varphi^{(n)}(x)\big)\big\}_{n\geqslant 1}$ is a Cauchy sequence and so

$$\vartheta\big(\varphi^{(n)}(x)\big) \;\longrightarrow\; u.$$

Since

$$\vartheta\big(\varphi^{(n+1)}(x)\big) \leqslant \beta\vartheta\big(\varphi^{(n)}(x)\big) + \beta\vartheta\big(\varphi^{(n+1)}(x)\big),$$

we have

$$\vartheta\big(\varphi^{(n+1)}(x)\big) \leqslant \gamma\vartheta\big(\varphi^{(n)}(x)\big),$$

with $\gamma = \frac{\beta}{1-\beta} < 1$ and thus

$$\vartheta\big(\varphi^{(n+1)}(x)\big) \leqslant \gamma^{n+1}\vartheta(x),$$

so $u = 0$. Moreover, $\big\{\varphi^{(n)}(x)\big\}_{n\geqslant 1}$ is a Cauchy sequence and so

$$\varphi^{(n)}(x) \longrightarrow y.$$

Note that in this case φ need not be continuous, but we have

$$\vartheta(y) = \lim_{n\to+\infty} d_x\big(\varphi^{(n)}(x), \varphi(y)\big) \leqslant \beta \lim_{n\to+\infty} \vartheta\big(\varphi^{(n-1)}(x)\big) + \beta\vartheta(y),$$

so $\vartheta(y) = 0$ (since $\beta < \frac{1}{2}$). Hence $y = \varphi(y)$.

Clearly in all four cases the fixed point $y \in X$ is unique. □

EXAMPLE 7.1.8 **(a)** A function φ satisfying the condition in Theorem 7.1.7**(a)** (or 7.1.7**(b)**) need not be continuous. Indeed suppose that $X = \mathbb{R}$ and let

$$\varphi(x) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Evidently φ is not continuous and we have $\varphi^{(2)}(x) = 1$ for all $x \in \mathbb{R}$.

(b) We give an example of a continuous function which is not a contraction but $\varphi^{(n)}$ is a contraction for some sufficiently large $n \geqslant 1$. Let $T = [0, b]$, $X = C(T)$ and consider the map $\varphi\colon X \longrightarrow X$, defined by

$$\varphi(x)(t) \stackrel{df}{=} \int_0^t x(s)\,ds.$$

Then φ is not a contraction if $b > 1$ and

$$\varphi^{(n)}(x)(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x(s)\,ds \qquad \forall\ n \geqslant 1.$$

Hence for $n \geqslant 1$ sufficiently large $\varphi^{(n)}$ is a contraction.

(c) In Theorem 7.1.7**(c)** the compactness of X is important. Consider

$$X \;=\; [1,+\infty) \quad \text{and} \quad \varphi(x) \;=\; x+\frac{1}{x}.$$

Then

$$\big|\varphi(x)-\varphi(y)\big| \;<\; |x-y| \qquad \forall\, x,y\in X,$$

φ is not a contraction and φ does not have a fixed point.

Another generalization of Theorem 7.1.2 is given below.

THEOREM 7.1.9
If (X,d_X) is a complete metric space, $\varphi\colon X\longrightarrow X$, $\gamma\colon \mathbb{R}_+\longrightarrow\mathbb{R}_+$ satisfies

$$\gamma(t) \;<\; t \qquad \forall\, t>0$$

and

$$\limsup_{t\to t_0^+}\gamma(t) \;\leqslant\; \gamma(t_0)$$

and

$$d_X\big(\varphi(x),\varphi(y)\big) \;\leqslant\; \gamma\big(d_X(x,y)\big) \qquad \forall\, x,y\in X,$$

then φ has a unique fixed point $y\in X$ and for every $x\in X$, we have

$$\varphi^{(n)}(x) \;\longrightarrow\; y.$$

PROOF Let

$$x_n \;\stackrel{df}{=}\; \varphi^{(n)}(x),$$

and

$$\vartheta_n \;\stackrel{df}{=}\; d_X(x_n,x_{n+1}) \;=\; d_X\big(\varphi^{(n)}(x),\varphi^{(n+1)}(x)\big).$$

We may assume that $\vartheta_n>0$ for all $n\geqslant 1$. For every $n>1$, we have

$$\vartheta_n \;=\; d_X\big(\varphi(x_{n-1}),\varphi(x_n)\big) \;\leqslant\; \gamma(\vartheta_{n-1}) \;<\; \vartheta_{n-1}. \tag{7.1}$$

So the sequence $\{\vartheta_n\}_{n\geqslant 1}\subseteq\mathbb{R}_+$ is decreasing, hence the limit $\vartheta=\lim\limits_{n\to+\infty}\vartheta_n$ exists. From (7.1) and the properties of γ, we have that $\vartheta\leqslant\gamma(\vartheta)$ and so $\vartheta=0$. Now we prove that $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence. Suppose that this is not the case. We can find $\varepsilon>0$, such that for each $k\geqslant 1$, there exist n_k and m_k, such that

$$k \;\leqslant\; m_k \;\leqslant\; n_k \quad \text{and} \quad d_X\big(x_{m_k},x_{n_k}\big) \;\geqslant\; \varepsilon. \tag{7.2}$$

We may assume that n_k is the smallest integer satisfying (7.2). We set

$$\xi_k \;=\; d_X\left(x_{m_k}, x_{n_k}\right) \qquad \forall\, k \geqslant 1.$$

Then

$$\varepsilon \;\leqslant\; \xi_k \;\leqslant\; d_X\left(x_{m_k}, x_{n_k-1}\right) + d_X\left(x_{n_k-1}, x_{n_k}\right) \;\leqslant\; \varepsilon + \vartheta_{n_k-1}$$

and so

$$\xi_k \;\longrightarrow\; \varepsilon \quad \text{as } k \to +\infty.$$

Also, we have

$$\begin{aligned}\varepsilon \;&\leqslant\; \xi_k \;\leqslant\; d_X\left(x_{m_k}, x_{m_k+1}\right) + d_X\left(x_{m_k+1}, x_{n_k+1}\right) + d_X\left(x_{n_k+1}, x_{n_k}\right)\\ &\leqslant\; \vartheta_{m_k} + \gamma(\xi_k) + \vartheta_{n_k},\end{aligned}$$

so

$$\varepsilon \;\leqslant\; \limsup_{n\to+\infty} \gamma(\xi_k) \;\leqslant\; \gamma(\varepsilon) \;<\; \varepsilon,$$

a contradiction. Therefore

$$x_n \;=\; \varphi^{(n)}(x)$$

is a Cauchy sequence and

$$x_n \;\longrightarrow\; y,$$

for some $y \in X$. We have

$$d_X\left(x_n, \varphi(y)\right) \;\leqslant\; \gamma\big(d_X(x_{n-1}, y)\big),$$

so

$$d_X\left(y, \varphi(y)\right) \;=\; 0,$$

i.e., $y = \varphi(y)$. By virtue of the properties of γ, this fixed point is unique. □

Now we turn our attention to nonexpansive maps (see Definition 7.1.1**(b)**). A nonexpansive map on a complete metric space need not have a fixed point and if it has a fixed point, it need not be unique.

EXAMPLE 7.1.10 **(a)** Let X be a Banach space, $x_0 \in X \setminus \{0\}$ and $\varphi\colon X \longrightarrow X$ is defined by

$$\varphi(x) \;=\; x + x_0.$$

Then φ is clearly nonexpansive but it cannot have a fixed point (see also Example 7.1.4**(a)**).

(b) Let X be a Banach space and

$$\varphi \;=\; id_X.$$

Then φ is nonexpansive and of course every $x \in X$ is a fixed point of φ.

PROPOSITION 7.1.11
If X is a Banach space, $C \subseteq X$ is a nonempty, bounded, closed and convex set and $\varphi\colon C \longrightarrow C$ is nonexpansive,
then

$$\inf_{x \in C} \|x - \varphi(x)\|_X \;=\; 0.$$

PROOF Fix $u \in C$ and $\varepsilon \in (0,1)$ and consider the map

$$\varphi_\varepsilon\colon C \;\longrightarrow\; C$$

defined by

$$\varphi_\varepsilon(x) \;=\; \varepsilon u + (1-\varepsilon)\varphi(x).$$

Then

$$\|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)\|_X \;\leqslant\; (1-\varepsilon)\|\varphi(x) - \varphi(y)\|_X \;\leqslant\; (1-\varepsilon)\,\|x-y\|_X\,,$$

i.e., φ_ε is a contraction.

Applying Theorem 7.1.2, we obtain $x_\varepsilon \in C$, such that

$$x_\varepsilon \;=\; \varphi_\varepsilon(x_\varepsilon).$$

We have

$$\begin{aligned} \|x_\varepsilon - \varphi(x_\varepsilon)\|_X \;&=\; \|\varepsilon u + (1-\varepsilon)\varphi(x_\varepsilon) - \varphi(x_\varepsilon)\|_X \\ =\; \varepsilon\|u - \varphi(x_\varepsilon)\|_X \;&\leqslant\; \varepsilon \operatorname{diam} C. \end{aligned}$$

Let $\varepsilon \searrow 0$ to finish the proof. □

COROLLARY 7.1.12
If X is a Banach space, $C \subseteq X$ is a nonempty, compact, convex set and $\varphi\colon C \longrightarrow C$ is nonexpansive,
then φ has a fixed point.

Next we introduce some special sets which are important in the search for fixed points of nonexpansive maps.

DEFINITION 7.1.13 *Let X be a Banach space, $C \subseteq X$ a nonempty, closed, convex set and $\varphi\colon C \longrightarrow C$.*

(a) *We say that $K \subseteq C$ is **invariant** under φ (or **φ-invariant**), if*

$$\varphi(K) \;\subseteq\; K.$$

(b) *A nonempty, closed, convex set $K \subseteq C$ is said to be **minimal invariant** under φ, if K is invariant under φ and K has no nonempty, closed and convex proper subsets which are φ-invariant.*

REMARK 7.1.14 We can modify Definition 7.1.13**(b)** in obvious ways so that it applies to smaller families of sets (for example minimal φ-invariant weakly (or weakly*) compact sets). A decreasing sequence of nonempty, closed, convex, φ-invariant sets may be obtained by setting

$$K_0 = C \quad \text{and} \quad K_{n+1} \;=\; \overline{\text{conv}}\,\varphi(K_n) \qquad \forall\, n \geqslant 1.$$

We set

$$\widehat{K} \;\stackrel{df}{=}\; \bigcap_{n=1}^{\infty} K_n.$$

The set $\widehat{K}$ is closed, convex and φ-invariant. But it may be empty. Of course this situation cannot occur if C is weakly compact.

PROPOSITION 7.1.15
If X is a Banach space, $C \subseteq X$ is a nonempty, weakly compact, convex set and $\varphi\colon C \longrightarrow C$,
then there exists a nonempty, closed, convex set $\widehat{K} \subseteq C$ which is minimal φ-invariant.

PROOF Let $\mathcal{T}$ be the family of all nonempty, closed, convex subsets of C which are φ-invariant. We order $\mathcal{T}$ by reverse inclusion, namely if $K_1, K_2 \in \mathcal{T}$, then

$$K_1 \;\leqslant\; K_2 \quad\Longleftrightarrow\quad K_2 \subseteq K_1.$$

By the finite intersection property for the weak topology, every chain in $\mathcal{T}$ has an upper bound (namely the intersection of the elements in the chain). So by the Kuratowski-Zorn lemma, $\mathcal{T}$ has a maximal element $\widehat{K} \in \mathcal{T}$. Evidently $\widehat{K}$ is φ-invariant. □

REMARK 7.1.16 An analogous result also holds for nonempty, w^*-closed, convex subsets of a w^*-compact set C in a dual Banach space. Note that if $\widehat{K} \subseteq C$ is a nonempty, closed, convex and minimal φ-invariant set, then

$$\widehat{K} \;=\; \overline{\text{conv}}\,\varphi(\widehat{K}).$$

If $\widehat{K} \in \mathcal{T}$ in Proposition 7.1.15 is a singleton, i.e., $\widehat{K} = \{y\}$, then

$$\varphi(y) \;=\; y,$$

i.e., it is a fixed point of φ. For many years it remained an open question whether a nonexpansive map $\varphi\colon C \longrightarrow C$ on a nonempty, w-compact, convex set C has a singleton minimal φ-invariant set $\widehat{K} \subseteq C$. Alspach (1981) settled this question negatively in 1981, by furnishing the following example.

EXAMPLE 7.1.17 Let $X = L^1[0,1]$,

$$C \stackrel{df}{=} \left\{ h \in X : \ 0 \leqslant h(t) \leqslant 2 \text{ for a.a. } t \in [0,1], \ \int_0^1 h(s)\, ds = 1 \right\}$$

and let $\varphi \colon C \longrightarrow C$ be defined by

$$\begin{cases} \min\left\{2h(2t), 2\right\} & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ \max\left\{0, 2h(2t-1) - 2\right\} & \text{if} \quad \frac{1}{2} < t \leqslant 1. \end{cases}$$

It can be shown that C is nonempty, weakly compact, convex in $L^1[0,1]$ and φ is an isometry (in particular it is nonexpansive). However φ is fixed point free.

So more structure is needed to guarantee fixed points for nonexpansive maps. For this reason we introduce the following definition.

DEFINITION 7.1.18 *Let X be a Banach space and let $C \subseteq X$ be a nonempty, bounded, closed, convex set. A point $x \in C$ is said to be* ***diametral****, if*

$$\operatorname{diam} C \ = \ \sup_{c \in C} \|c - x\|_X \, .$$

We say that C has ***normal structure****, if for any given bounded, closed, convex set $K \subseteq C$ containing more than one point, there exists a nondiametral $x \in C$.*

THEOREM 7.1.19
Every nonempty, compact, convex set C in a Banach space X has normal structure.

PROOF If C (with $\operatorname{diam} C > 0$) does not have normal structure, then for any $x_1 \in C$ we can find $x_2 \in C$, such that

$$\|x_1 - x_2\|_X \ = \ \operatorname{diam} C.$$

Since C is convex, we have

$$\frac{x_1 + x_2}{2} \ \in \ C.$$

So there exists $x_3 \in C$, such that

$$\left\| \frac{x_1 + x_2}{2} - x_3 \right\|_X \ = \ \operatorname{diam} C.$$

This way we produce a sequence $\{x_n\}_{n\geqslant 1} \subseteq C$, such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x_{n+1} \right\|_X \;=\; \operatorname{diam} C.$$

Therefore

$$\begin{aligned} \operatorname{diam} C \;&=\; \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x_{n+1} \right\|_X \\ &=\; \left\| \frac{x_1 - x_{n+1}}{n} + \frac{x_2 - x_{n+1}}{n} + \ldots + \frac{x_n - x_{n+1}}{n} \right\|_X \\ &\leqslant\; \frac{1}{n} \sum_{k=1}^{n} \|x_k - x_{n+1}\|_X \;\leqslant\; \operatorname{diam} C, \end{aligned}$$

so

$$\|x_k - x_{n+1}\|_X \;=\; \operatorname{diam} C \;>\; 0.$$

Therefore $\{x_n\}_{n\geqslant 1} \subseteq C$ is a sequence with no convergent subsequence, a contradiction to the fact that C is compact. □

THEOREM 7.1.20
Every nonempty, bounded, closed, convex set C of a uniformly convex Banach space X has normal structure.

PROOF Without any loss of generality we may assume that

$$C \;\subseteq\; \overline{B}_1,$$

where

$$\overline{B}_1 \;=\; \big\{x \in X :\; \|x\|_X \leqslant 1\big\}.$$

Let C_1 be a nonempty, closed, convex subset of C with $\operatorname{diam} C_1 > 0$ and let $x_1 \in C$ and $\varepsilon = \frac{1}{2}$. We can find $x_2 \in C_1$, such that

$$\frac{1}{2} \operatorname{diam} C_1 \;\leqslant\; \|x_2 - x_1\|_X .$$

Then for any $x \in X$, we have

$$\begin{aligned} \left\| x - \frac{x_1 + x_2}{2} \right\|_X \;&=\; \left\| \frac{x - x_1}{2} + \frac{x - x_2}{2} \right\|_X \\ \leqslant \operatorname{diam} C_1 &\left(1 - \delta\left(\frac{1}{2}\right)\right), \end{aligned}$$

where $\delta(\varepsilon)$ is the modulus of uniform convexity of X. Since $\delta\left(\frac{1}{2}\right) > 0$, we conclude that C has normal structure. □

To prove the main fixed point theorem for nonexpansive maps, we need to introduce some geometric quantities associated with a set $C \subseteq X$.

DEFINITION 7.1.21 *Let X be a Banach space and D, E nonempty subsets of X.*

(a) *For a given $x \in X$, the **radius** of D relative to x is given by*

$$r_x(D) \overset{df}{=} \sup_{y \in D} \|x - y\|_X .$$

(b) *The **Chebyshev radius** of D relative to E is given by*

$$r_E(D) \overset{df}{=} \inf_{x \in E} r_x(D).$$

(c) *The **Chebyshev center** of D relative to E is given by*

$$C_E(D) \overset{df}{=} \{x \in E : \ r_x(D) = r_E(D)\}.$$

REMARK 7.1.22 If $D = E$, then for **(b)** and **(c)** in the above definition we use the notion $r(D)$ and $C(D)$ and the phrase "relative to E" is dropped. If $y \in E$ belongs in $C_E(D)$, then

$$B_{r_y(D)}(y) \ \supseteq \ D$$

and no ball centered at any point of E with smaller radius has this property. Clearly for any $x \in D$, we have

$$r(D) \ \leqslant \ r_x(D) \ \leqslant \ \operatorname{diam} D.$$

Moreover, $r_x \colon D \longrightarrow X$ is continuous and $C(D)$ is closed, convex.

The main fixed point theorem for nonexpansive maps is the following.

THEOREM 7.1.23
If X is a Banach space, $C \subseteq X$ is a nonempty, w-compact, convex set with normal structure and $\varphi \colon C \longrightarrow C$ is nonexpansive,
then φ has a fixed point, i.e., there exists $x \in C$, such that $\varphi(x) = x$.

PROOF By virtue of Proposition 7.1.15, we may assume that C is minimal φ-invariant. Then

$$C \ = \ \overline{\operatorname{conv}}\, \varphi(C)$$

(see Remark 7.1.16). Let $x \in C(C)$, i.e.,

$$r_x(C) \ = \ r(C).$$

Because

$$\big\|\varphi(z)-\varphi(x)\big\|_X \;\leqslant\; \|z-x\|_X \;\leqslant\; r(C) \qquad \forall\, z\in C,$$

we have

$$\varphi(C) \;\subseteq\; B_{r(C)}\big(\varphi(x)\big),$$

so

$$C \;=\; \overline{\operatorname{conv}}\,\varphi(C) \;\subseteq\; B_{r(C)}\big(\varphi(x)\big)$$

and thus

$$r_{\varphi(x)}(C) \;=\; r(C).$$

This means that $\varphi(x)\in C(C)$ and so $C(C)$ is φ-invariant, which in view of the normal structure contradicts the minimality of C (see Remark 7.1.22), unless $\operatorname{diam} C=0$. So $C=\{x_0\}$ is a fixed point of φ. □

In the above theorem the hypothesis that C has normal structure cannot be dispensed with as the following example illustrates.

EXAMPLE 7.1.24 Let $X=C\big([0,1]\big)$ and

$$C \;\stackrel{df}{=}\; \Big\{x\in C\big([0,1]\big):\; x(0)=0,\; x(1)=1,\; 0\leqslant x(t)\leqslant 1 \text{ for all } t\in[0,1]\Big\}.$$

Then

$$r(C) \;=\; \operatorname{diam} C \;=\; 1 \quad\text{and}\quad C(C)=C.$$

So C does not have the normal structure. Let $\varphi\colon C\longrightarrow C$ be defined by

$$\varphi(x)(t) \;=\; tx(t) \qquad \forall\, t\in[0,1].$$

Then φ is nonexpansive but does not have a fixed point.

Recall that, if

$$C \;\stackrel{df}{=}\; \overline{B}_c \;=\; \big\{x\in X:\; \|x\|_X\leqslant c\big\},$$

the c-radial retraction is the map $r_c\colon X\longrightarrow C$, defined by

$$r_c(x) \;\stackrel{df}{=}\; \begin{cases} x & \text{if} \quad x\in C,\\ \dfrac{cx}{\|x\|_X} & \text{if} \quad x\notin C. \end{cases}$$

From a well known result of de Figueiredo & Karlovitz (1967), if $\dim X\geqslant 3$, then X is a Hilbert space if and only if r_c is nonexpansive. Of course when $\dim X=1$ or $\dim X=2$, then r_c is always nonexpansive. Using this fact we can prove the following alternative theorem for nonexpansive maps on a Hilbert space.

THEOREM 7.1.25
If H is a Hilbert space,

$$C \stackrel{df}{=} \overline{B}_c = \{x \in H : \|x\|_H \leqslant c\}$$

and $\varphi: C \longrightarrow H$ is nonexpansive,
then at least one of the following properties holds:

(a) *φ has a fixed point;*

(b) *there is an $x \in \partial C$ and $\lambda \in (0,1)$, such that $x = \lambda\varphi(x)$.*

PROOF The map

$$r_c \circ \varphi: C \longrightarrow C$$

is nonexpansive. Since a Hilbert space is uniformly convex, because of Theorem 7.1.20, C has normal structure. So we can apply Theorem 7.1.23 and obtain $x \in C$, such that

$$x = (r_c \circ \varphi)(x).$$

If $\varphi(x) \in C$, then $x = \varphi(x)$. If $\varphi(x)$ does not belong to C, then

$$x = \frac{c\varphi(x)}{\|\varphi(x)\|_H} \in \partial C$$

and so for

$$\lambda = \frac{c}{\|\varphi(x)\|_H} < 1,$$

we have $x = \lambda\varphi(x)$. ∎

By virtue of Theorem 7.1.25, we see that by imposing conditions on φ which prevent the second possibility to occur, we can have fixed points for nonexpansive maps on Hilbert spaces.

COROLLARY 7.1.26
If H is a Hilbert space,

$$C \stackrel{df}{=} \overline{B}_c = \{x \in H : \|x\|_H \leqslant c\},$$

$\varphi: C \longrightarrow H$ is nonexpansive and for all $x \in \partial C$ one of the following conditions holds:

(i) $\|\varphi(x)\|_H \leqslant \|x\|_H$;

(ii) $\|\varphi(x)\|_H \leqslant \|x - \varphi(x)\|_H$;

(iii) $\|\varphi(x)\|_H^2 \leqslant \|x\|_H^2 + \|x - \varphi(x)\|_H^2$;

(iv) $\big(x, \varphi(x)\big)_H \leqslant \|x\|_H^2$,

then φ has a fixed point.

PROOF Let us prove the corollary when (i) is in effect. The other cases are proved similarly. Suppose that for some $x \in \partial C$ and some $\lambda \in (0,1)$, we have

$$x \;=\; \lambda\varphi(x).$$

Then

$$\|x\|_H \;\leqslant\; \lambda\,\|x\|_H \;<\; \|x\|_H\,,$$

a contradiction. Hence **(b)** in Theorem 7.1.25 cannot occur and so φ must have a fixed point. □

REMARK 7.1.27 In Corollary 7.1.26, case (i) is known as ***Rothe's fixed point theorem***, while case (iii) is known as ***Altman's fixed point theorem***.

Recall that in Theorem 4.6.9, we proved a deep generalization of Theorem 7.1.2 (the Banach contraction principle), known as Caristi's fixed point theorem. Let us recall the statement of that result.

THEOREM 7.1.28 (Caristi Fixed Point Theorem)
If (X, d_X) *is a complete metric space,*

$$\varphi\colon X \;\longrightarrow\; \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$$

is a proper, lower semicontinuous, bounded below function,

$$F\colon X \longrightarrow 2^X \setminus \{\emptyset\}$$

is a multifunction and for all $x \in X$*, there exists*

$$y \;\in\; F(x),$$

such that

$$\varphi(y) \;\leqslant\; \varphi(x) - d_X(y,x),$$

then F *has a fixed point, i.e., there exists* $x \in X$*, such that*

$$x \;\in\; F(x).$$

REMARK 7.1.29 Theorem 7.1.28 is equivalent to the Ekeland variational principle in the form of Corollary 4.6.3 (see Proposition 4.6.11).

7.2 Topological Fixed Point Theory

In this section we prove fixed point theorems in which the topological structure of the problem plays a central role.

DEFINITION 7.2.1 *A Hausdorff topological space X is said to have the **fixed point property** (**FPP** for short), if every continuous map $\varphi\colon X \longrightarrow X$ has a fixed point.*

The following result is an easy consequence of the above definition.

PROPOSITION 7.2.2
If X and Y are homeomorphic Hausdorff topological spaces and X has the FPP,
then Y has the FPP too.

DEFINITION 7.2.3 *Let X be a Hausdorff topological space and $C \subseteq X$. We say that C is a **retract** of X, if there is a continuous map $r\colon X \longrightarrow C$, such that*

$$r|_C \;=\; id_C.$$

*The map r is called a **retraction**.*

REMARK 7.2.4 Every nonempty, closed and convex subset of a normed space X is a retract. This is an immediate consequence of Dugundji's extension theorem (see Theorem 3.1.11). Each retract is closed.

PROPOSITION 7.2.5
If X is a Hausdorff topological space with the FPP and C is a retract of X, then C has the FPP.

PROOF Let $r\colon X \longrightarrow C$ be the retraction map and let $\varphi\colon C \longrightarrow C$ be a continuous map. Then $\varphi \circ r\colon X \longrightarrow X$ is continuous and since X has the FPP we can find $x \in X$, such that

$$x \;=\; \varphi\big(r(x)\big).$$

Because

$$\big(\varphi \circ r\big)(X) \;\subseteq\; C,$$

we have $x \in C$ and so

$$r(x) \;=\; x \quad \text{and} \quad x \;=\; \varphi(x).$$

□

REMARK 7.2.6 Recall that if X is a normed space and

$$\overline{B}_r \stackrel{df}{=} \{x \in X : \|x\|_X \leqslant r\}, \quad r > 0,$$

then ∂B_r is a retract of $\overline{B}_r$ if and only if $\dim X = +\infty$.

Now we are ready for the first major topological fixed point theorem.

THEOREM 7.2.7 (Brouwer Fixed Point Theorem)
If X is a finite dimensional normed space,
then

(a) $\overline{B}_1 = \{x \in X : \|x\|_X \leqslant 1\}$ *has the FPP;*

(b) *every compact, convex subset C of X has the FPP.*

PROOF **(a)** We proceed by contradiction. Suppose that there exists a continuous map

$$\varphi : \overline{B}_1 \longrightarrow \overline{B}_1,$$

which is fixed point free. Let

$$r : \overline{B}_1 \longrightarrow \partial\overline{B}_1$$

be a map defined as follows: for each $x \in \overline{B}_1$ extend the line segment from $\varphi(x)$ through x to the point of intersection with ∂B_1. Call this point $r(x)$. Clearly r is continuous and

$$r|_{\partial B_1} = id_{\partial B_1},$$

i.e., r is a retraction. But this is a contradiction (see Remark 7.2.6).

(b) Let $r > 0$ be large enough so that

$$C \subseteq \overline{B}_r.$$

We know that C is a retract of $\overline{B}_r$ and $\overline{B}_r$ is homeomorphic to $\overline{B}_1$. Then from **(a)** and Proposition 7.2.2 and Proposition 7.2.5, we conclude that C has the FPP. ∎

Can we extend this result to infinite dimensional normed spaces? The answer is negative. To show this we need the following result due to Klee (1956).

PROPOSITION 7.2.8
If X is an infinite dimensional normed space and $C \subseteq X$ is compact,
then $X \setminus C$ and X are homeomorphic.

THEOREM 7.2.9
If X is a normed space and $\overline{B}_1 = \{x \in X : \|x\|_X \leqslant 1\}$, then $\overline{B}_1$ has the FPP if and only if $\dim X < +\infty$.

PROOF "$\Longrightarrow$": We argue indirectly. Suppose that X is infinite dimensional. Then by virtue of Proposition 7.2.8 there exists a homeomorphism

$$\vartheta\colon X \longrightarrow X \setminus \{0\}.$$

For each $u_0 \in \partial\overline{B}_1$ map the line segment $[0, u_0]$ linearly onto the line segment $\big[0, \vartheta^{-1}(u_0)\big]$ and this way we can have a continuous map

$$g\colon \overline{B}_1 \longrightarrow X,$$

such that

$$g(0) = 0.$$

Now let

$$r\colon \overline{B}_1 \longrightarrow \partial\overline{B}_1$$

be defined by

$$r(x) \stackrel{df}{=} \frac{(\vartheta \circ g)(x)}{\|(\vartheta \circ g)(x)\|_X} \qquad \forall\, x \in \overline{B}_1.$$

Note that this map is well defined since

$$\vartheta(X) = X \setminus \{0\}$$

and clearly is continuous. If $u_0 \in \partial B_1$, then

$$\big(\vartheta \circ g\big)(u_0) = \vartheta\big(\vartheta^{-1}(u_0)\big) = u_0$$

and so

$$r|_{\partial B_1} = id_{\partial B_1},$$

i.e., r is a retraction. Hence

$$x \longmapsto -r(x)$$

is a continuous, fixed point free map from $\overline{B}_1$ into itself, a contradiction. So we conclude that $\dim X < +\infty$.

"$\Longleftarrow$": This is Theorem 7.2.7**(a)**. ∎

In the next example we produce a fixed point free homeomorphism of the closed unit ball of an infinite dimensional Hilbert space onto itself.

EXAMPLE 7.2.10 Let H be a separable Hilbert space and

$$\overline{B}_1 \;=\; \big\{x \in H :\; \|x\|_H \leqslant 1\big\}.$$

Since H is separable, we can find a countable orthonormal basis $\{e_n\}_{n\geqslant 1}$. Consider the transformation $R\colon H \longrightarrow H$, defined by

$$R(e_n) \;=\; e_{n+1} \qquad \forall\; n \geqslant 1$$

and extending linearly to all of H, i.e.,

$$R(x) \;\stackrel{df}{=}\; \sum_{n=1}^{\infty} \lambda_n R(e_n) \;=\; \sum_{n=1}^{\infty} \lambda_n e_{n+1} \qquad \forall\; x = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Evidently R is an isomorphism and a homeomorphism of ∂B_1 onto itself. We consider the map $\varphi\colon H \longrightarrow H$, defined by

$$\varphi(x) \;\stackrel{df}{=}\; \frac{1}{2}\,(1 - \|x\|_H)\, e_1 + R(x) \qquad \forall\; x \in H.$$

It is easy to check that φ is a homeomorphism which maps $\overline{B}_1$ into itself. We claim that φ has no fixed points in $\overline{B}_1$. Suppose that there exists $x_0 \in \overline{B}_1$, such that

$$\varphi(x_0) \;=\; x_0.$$

We have

$$x_0 - R(x_0) \;=\; \frac{1}{2}\,(1 - \|x_0\|_H)\, e_1.$$

If $x_0 = 0$, then

$$\frac{1}{2}\,(1 - \|x_0\|_H)\, e_1 \;=\; 0$$

and so

$$\|x_0\|_H \;=\; 1,$$

a contradiction.

If $\|x_0\|_H = 1$, then

$$x_0 \;=\; R(x_0),$$

a contradiction since from its definition R cannot have a fixed point on ∂B_1.

Finally suppose that $0 < \|x_0\|_H < 1$. We have

$$x_0 \;=\; \sum_{n=1}^{\infty} \lambda_n e_n, \quad \text{with} \quad \sum_{n=1}^{\infty} |\lambda_n|^2 \;=\; \|x_0\|_H^2 \;<\; 1.$$

Then

$$\lambda_1 \;=\; \lambda_2 \;=\; \ldots \;=\; \lambda_n \;=\; \ldots$$

and so

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \;=\; +\infty,$$

a contradiction.

Theorem 7.2.9 and Example 7.2.10 illustrate that in order to extend Theorem 7.2.7 to infinite dimensional normed spaces, we need to restrict the maps $\varphi\colon C \longrightarrow C$ that we consider. The next theorem is the infinite dimensional version of Brouwer's theorem (see Theorem 7.2.7).

THEOREM 7.2.11 (Schauder Fixed Point Theorem) *If X is a normed space, $C \subseteq X$ is a nonempty, convex (not necessarily closed) set and $\varphi\colon C \longrightarrow C$ is a continuous map, such that*

$$\overline{\varphi(C)} \subseteq X \quad \text{is compact,}$$

then φ has at least one fixed point.

PROOF By virtue of Theorem 3.1.10, for a given $\varepsilon > 0$, we can find a continuous map

$$\varphi_\varepsilon\colon C \longrightarrow C,$$

such that

$$\|\varphi_\varepsilon(x) - \varphi(x)\|_X < \varepsilon \qquad \forall\, x \in C$$

and

$$\varphi_\varepsilon(C) \subseteq D \subseteq C,$$

where D is a finite dimensional, closed ball. Since

$$\varphi_\varepsilon(D) \subseteq D,$$

we can apply Theorem 7.2.9**(b)**, to obtain $x_\varepsilon \in D$, such that

$$\varphi_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Now let $\varepsilon_n \searrow 0$ and set

$$x_n \stackrel{df}{=} x_{\varepsilon_n} \qquad \forall\, n \geqslant 1.$$

Then

$$\|x_n - \varphi(x_n)\|_X \leqslant \frac{1}{n}$$

and because of our hypothesis on φ we may assume that

$$\varphi(x_n) \longrightarrow y \quad \text{in } X,$$

hence

$$x_n \longrightarrow y \quad \text{in } X.$$

So in the limit as $n \to +\infty$, we obtain

$$\|y - \varphi(y)\|_X = 0,$$

hence $\varphi(y) = y$. ∎

Using the same approximation technique, we can prove the so-called ***Borsuk's fixed point theorem***. First let us recall the finite dimensional version of the result (see Dugundji & Granas (1982, p. 46)).

THEOREM 7.2.12
If U is a bounded, convex, symmetric neighbourhood of the origin in $\mathbb{R}^N$ and $\varphi\colon \overline{U} \longrightarrow \mathbb{R}^N$ is a continuous map, such that $\varphi|_{\partial U}$ is odd,
then φ has at least one fixed point.

THEOREM 7.2.13 (Borsuk Fixed Point Theorem)
If X is a normed space, $U \subseteq X$ is a bounded, convex, symmetric neighbourhood of the origin and $\varphi\colon \overline{U} \longrightarrow X$ is a compact and odd map
then φ has at least one fixed point.

PROOF Because of Theorem 3.1.10, we can find a finite, symmetric set $F \subseteq X$, such that $\varphi\big(\overline{U}\big) \subseteq F_{(\varepsilon)}$, where

$$F \;=\; \{u_i\}_{i=1}^m \quad \text{and} \quad F_{(\varepsilon)} \stackrel{df}{=} \bigcup_{i=1}^m B_\varepsilon(u_i).$$

Let $\eta_i\colon F_{(\varepsilon)} \longrightarrow \mathbb{R}$ be the map

$$\eta_i(x) \stackrel{df}{=} \max\big\{0, \varepsilon - \|x - u_i\|_X\big\} \qquad \forall\, i \in \{1, \ldots, m\}$$

and let us set

$$p_\varepsilon(x) \stackrel{df}{=} \frac{1}{\sum_{i=1}^m \eta_i(x)} \sum_{i=1}^m \eta_i(x) u_i \qquad \forall\, x \in F_{(\varepsilon)}.$$

As in the proof of Theorem 3.1.10**(a)**, we can check that

$$\text{the map } p_\varepsilon\colon F_{(\varepsilon)} \longrightarrow \operatorname{conv} F \subseteq U \text{ is compact and odd}$$

and

$$\big\|x - p_\varepsilon(x)\big\|_X \;<\; \varepsilon \qquad \forall\, x \in F_{(\varepsilon)}.$$

Then $p \circ \varphi_\varepsilon|_{\partial U}$ is odd. Let X_ε be a finite dimensional subspace of X, such that

$$\big(p_\varepsilon \circ \varphi\big)\big(\overline{U}\big) \;\subseteq\; X_\varepsilon.$$

Let

$$\widehat{\varphi}_\varepsilon \stackrel{df}{=} p_\varepsilon \circ \varphi|_{\overline{U} \cap X_\varepsilon}.$$

Applying Theorem 7.2.12, we obtain x_ε, such that $\widehat{\varphi}_\varepsilon(x_\varepsilon) = x_\varepsilon$. Then

$$\big\|x_\varepsilon - \varphi_\varepsilon(x_\varepsilon)\big\|_X \;<\; \varepsilon$$

and as in the proof of Theorem 7.2.11, by considering $\varepsilon_n \searrow 0$ and using the compactness of φ, we can produce a fixed point of φ. ∎

There is the following extension of Theorem 7.2.11 to locally convex spaces.

THEOREM 7.2.14 (Schauder-Tychonoff Fixed Point Theorem)
Any nonempty, compact, convex set in a locally convex space has the FPP.

REMARK 7.2.15 Another equivalent formulation of this theorem is the following: "Let X be a locally convex space, $C \subseteq X$ a nonempty, closed, convex set and $\varphi\colon C \longrightarrow C$ a continuous map, such that $\overline{\varphi(C)}$ is compact. Then φ has at least one fixed point."

In analogy to the metric nonlinear alternative theorem (see Theorem 7.1.25), we can have a topological alternative theorem, known as the ***Leray-Schauder alternative principle*** or ***Schaeffer fixed point theorem***.

THEOREM 7.2.16 (Leray-Schauder Alternative Principle)
If X is a normed space, $C \subseteq X$ a nonempty, convex set and $\varphi\colon C \longrightarrow C$ a compact map,
then at least one of the following properties holds:

(a) *φ has a fixed point;*

(b) *the set $S \overset{df}{=} \{x \in C :\ x = \lambda\varphi(x)$ for some $\lambda \in (0,1)\}$ is unbounded.*

PROOF Suppose that S is bounded. Let $M > 0$ be such that

$$\|x\|_X < M \quad \text{if } x = \lambda\varphi(x) \text{ for some } \lambda \in (0,1). \tag{7.3}$$

Let us set

$$\widehat{\varphi} = r_M \circ \varphi,$$

where r_M is the M-radial retraction. Observe that $\widehat{\varphi}\colon \overline{B}_M \longrightarrow \overline{B}_M$, where

$$\overline{B}_M = \{x \in X :\ \|x\|_X \leqslant M\}.$$

Let

$$K = \overline{\operatorname{conv}}\,\widehat{\varphi}\big(\overline{B}_M\big).$$

Since φ is compact and r_M is continuous, $\widehat{\varphi}$ is compact too and so $K \subseteq X$ is compact and convex. Moreover, we have $\widehat{\varphi}\colon K \longrightarrow K$. Applying Theorem 7.2.11, we obtain $x \in K$, such that

$$\widehat{\varphi}(x) = x.$$

If $x \in K$ is not a fixed point of φ, then $\big\|\widehat{\varphi}(x)\big\|_X > M$ and so

$$x = \lambda\varphi(x) \quad \text{with} \quad \lambda = \frac{M}{\|\varphi(x)\|_X} < 1. \tag{7.4}$$

But $\|x\|_X = \big\|\widehat{\varphi}(x)\big\|_X = M$. So comparing (7.3) with (7.4), we reach a contradiction. Thus $\varphi(x) = x$. ∎

REMARK 7.2.17 The above theorem is the basis of the informal principle, which says that, "if we can produce *a priori* bounds for the solutions of nonlinear partial differential equations, under the assumption that such solutions exist, then indeed these solutions exist (briefly *a priori* estimates imply existence)." This is the so-called "method of *a priori* estimates." Compared to Theorem 7.2.11, Theorem 7.2.16 has the advantage that we do not have to assume that C is compact (for example C can be the whole space X).

As in the metric case (see Corollary 7.1.26), using Theorem 7.2.16 we can derive many known fixed point theorems by imposing conditions on the map φ which prevent the occurrence of property **(b)** in Theorem 7.2.16.

COROLLARY 7.2.18
If X is a normed space, $C \subseteq X$ is a nonempty, convex set, $U \subseteq C$ is open, $\varphi\colon \overline{U} \longrightarrow C$ is compact and for all $x \in \partial U$ one of the following conditions holds:

(i) $\|\varphi(x)\|_X \leqslant \|x\|_X$;

(ii) $\|\varphi(x)\|_X^2 \leqslant \|\varphi(x) - x\|_X^2 + \|x\|_X^2$,

then φ has a fixed point.

REMARK 7.2.19 In the above corollary, case (i) is known as ***Rothe's fixed point theorem*** and case (ii) as ***Altman's fixed point theorem***.

Using the measure of noncompactness, we can have a generalization of Theorem 7.2.11 in which the compactness condition on the map φ is relaxed. First let us recall the definitions of the two measures of noncompactness which we will use.

DEFINITION 7.2.20 *Let X be a Banach space and $\mathcal{B}$ the collection of all bounded subsets of X.*

(a) *The **Kuratowski measure of noncompactness** $\alpha\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is defined by*

$$\alpha(C) = \inf\Big\{d > 0 :\ C \text{ admits a finite cover by sets of diameter not bigger than } d\Big\}.$$

(b) *The **ball** (or **Hausdorff**) **measure of noncompactness** $\beta\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is defined by*

$$\beta(C) = \inf\Big\{r > 0 :\ C \text{ can be covered by finitely many balls of radius } r\Big\}.$$

In the next proposition we have gathered some basic properties of the above two measures of noncompactness. For details we refer to Denkowski, Migórski & Papageorgiou (2003*b*, pp. 14–15).

PROPOSITION 7.2.21
If X is an infinite dimensional Banach space, $\mathcal{B}$ is the collection of all bounded subsets of X, $\gamma\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is either α or β,
then

(a) $\gamma(C) = 0$ *if and only if* $\overline{C}$ *is compact;*

(b) $\gamma(\lambda C) = |\lambda|\gamma(C)$ *for all* $\lambda \in \mathbb{R}$
and $\gamma(C_1 + C_2) \leqslant \gamma(C_1) + \gamma(C_2)$ *(γ is a seminorm);*

(c) *if* $C_1 \subseteq C_2$, *then* $\gamma(C_1) \leqslant \gamma(C_2)$ *(monotonicity);*

(d) $\gamma(C_1 \cup C_2) \leqslant \max\{\gamma(C_1), \gamma(C_2)\}$ *(semi-additivity);*

(e) $\big|\gamma(C_1) - \gamma(C_2)\big| \leqslant kh(C_1, C_2)$, *with*

$$k = \begin{cases} 2 & \text{if } \gamma = \alpha, \\ 1 & \text{if } \gamma = \beta \end{cases}$$

and h being the Hausdorff semimetric on $\mathcal{B}$;
in particular

$$\gamma(C) = \gamma(\overline{C});$$

(f) $\gamma(C) = \gamma(\operatorname{conv} C)$.

REMARK 7.2.22 It is clear from Definition 7.2.20 that we have

$$\beta(C) \leqslant \alpha(C) \leqslant 2\beta(C) \qquad \forall\, C \in \mathcal{B}.$$

Also

$$\alpha(B_r) = \alpha\big(\partial B_r\big) = 2r \qquad \forall\, r > 0,$$

and

$$\beta(B_r) = \beta\big(\partial B_r\big) = r \qquad \forall\, r > 0,$$

provided that X is infinite dimensional. Here

$$B_r = \big\{x \in X : \|x\|_X < r\big\}.$$

Also by virtue of Proposition 7.2.21**(a)** and **(b)**, the same is true for

$$B_r(x) = x + B_r, \quad x \in X.$$

Using the measure of noncompactness, we can extend Cantor's theorem (see Theorem A.1.11) as follows.

PROPOSITION 7.2.23
If X is a Banach space, $\mathcal{B}$ is the collection of all bounded subsets of X, $\gamma\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is either α or β, $\{C_n\}_{n\geqslant 1} \subseteq \mathcal{B}$ is a decreasing sequence of nonempty closed sets, such that

$$\gamma(C_n) \searrow 0,$$

then

$$\widehat{C} \stackrel{df}{=} \bigcap_{n=1}^{\infty} C_n \text{ is a nonempty and compact set.}$$

PROOF Choose $x_n \in C_n$ for $n \geqslant 1$. Then from Proposition 7.2.21**(a)** and **(b)** we have

$$\gamma\big(\{x_n\}_{n\geqslant 1}\big) = \gamma\big(\{x_n\}_{n\geqslant k}\big) \qquad \forall\, k \geqslant 1$$

and

$$\gamma\big(\{x_n\}_{n\geqslant k}\big) \subseteq \gamma(C_k).$$

So

$$\gamma\big(\{x_n\}_{n\geqslant 1}\big) = 0$$

and this implies that the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is relatively compact (see Proposition 7.2.21**(a)**). We may assume that

$$x_n \longrightarrow x \quad \text{in } X.$$

Evidently $x \in \widehat{C}$ and so $\widehat{C} \neq \emptyset$. Also

$$\gamma(\widehat{C}) \leqslant \gamma(C_n) \qquad \forall\, n \geqslant 1,$$

hence $\gamma(\widehat{C}) = 0$. Since $\widehat{C}$ is closed, we conclude that $\widehat{C}$ is compact. ∎

DEFINITION 7.2.24 *Let X be a Banach space, $\mathcal{B}$ the collection of all bounded subsets of X, $\gamma\colon \mathcal{B} \longrightarrow \mathbb{R}_+$ is either α or β, $C \subseteq X$ and $\varphi\colon C \longrightarrow X$ a continuous map.*

(a) *φ is γ-**Lipschitz**, if*

$$\gamma\big(\varphi(C)\big) \leqslant k\gamma(C) \qquad \forall\, C \in \mathcal{B},$$

for some $k > 0$;

(b) *φ is γ-**contraction**, if it is γ-Lipschitz with constant $k < 1$;*

(c) *φ is γ-**condensing**, if*

$$\gamma\big(\varphi(C)\big) < \gamma(C) \qquad \forall\, C \in \mathcal{B},\ \text{with } \gamma(C) > 0.$$

REMARK 7.2.25 Recall that in a Hilbert space H, the c-radial retraction

$$r_c\colon H \longrightarrow \overline{B}_c$$

is nonexpansive. In particular then it is a α-Lipschitz with constant $k = 1$. The same is true for the metric projection map on a closed, convex set $C \subseteq H$.

The next result extends Proposition 3.1.17.

PROPOSITION 7.2.26
If X is a Banach space, $C \subseteq X$ is a nonempty, closed, bounded set and $\varphi\colon C \longrightarrow X$ is γ-condensing with γ being either α or β,
then $id_X - \varphi$ is proper (see Definition 3.1.13) and maps closed subsets of C onto closed sets in X.

PROOF Let $K \subseteq X$ be a nonempty compact set and let

$$D \stackrel{df}{=} (id_X - \varphi)^{-1}(K).$$

Since $id_X - \varphi$ is continuous, D is closed. Moreover, we have

$$\gamma(D) \leqslant \gamma(K) + \gamma\big(\varphi(D)\big) = \gamma\big(\varphi(D)\big)$$

(see Proposition 7.2.21**(a)**). Since φ is γ-condensing, it follows that

$$\gamma(D) = 0$$

and so D is compact.

Next let $B \subseteq C$ be a closed set and consider

$$x_n \in (id_X - \varphi)(B),$$

such that

$$x_n \longrightarrow x \quad \text{in } X.$$

We have

$$x_n = (id_X - \varphi)(u_n) \qquad \forall\, n \geqslant 1,$$

with $u_n \in B$. The set $K = \overline{\{x_n\}}_{n\geqslant 1}$ is compact and so

$$(id_X - \varphi)^{-1}(K) \text{ is compact.}$$

Therefore, we may assume that

$$u_n \longrightarrow u \in B,$$

for some $u \in B$. Due to the continuity of $id_X - \varphi$, we have

$$x = (id_X - \varphi)(u)$$

and so

$$x \in \big(id_X - \varphi\big)(B),$$

i.e., $\big(id_X - \varphi\big)(B)$ is closed. ∎

Now we are ready to have another generalization of Theorem 7.2.11.

THEOREM 7.2.27 (Sadovskii Fixed Point Theorem)
If X is a Banach space, $C \subseteq X$ is a nonempty, bounded, closed, convex set and $\varphi\colon C \longrightarrow C$ is a γ-condensing map,
then φ has at least one fixed point.

PROOF First suppose that φ is a γ-contraction with constant $k \in (0,1)$. Define a decreasing sequence $\{C_n\}_{n\geqslant 1}$ of closed subsets of C as follows:

$$C_1 \stackrel{df}{=} C \quad \text{and} \quad C_{n+1} \stackrel{df}{=} \overline{\text{conv}}\,\varphi(C_n) \qquad \forall\, n \geqslant 1.$$

Using Proposition 7.2.21**(f)** and the fact that φ is γ-Lipschitz with constant $k < 1$, we have

$$\begin{aligned}\gamma(C_{n+1}) &= \gamma\big(\overline{\text{conv}}\,\varphi(C_n)\big) = \gamma\big(\varphi(C_n)\big)\\ &\leqslant k\gamma(C_n) \leqslant \ldots \leqslant k^{n+1}\gamma(C_1) \longrightarrow 0.\end{aligned}$$

Invoking Proposition 7.2.23, we obtain that

$$\widehat{C} = \bigcap_{n=1}^{\infty} C_n$$

is a nonempty and compact set. Moreover, $\widehat{C}$ is also convex and $\varphi\colon \widehat{C} \longrightarrow \widehat{C}$. Applying Theorem 7.2.11, we obtain $x \in \widehat{C}$, such that $\varphi(x) = x$.

Now we remove the additional hypothesis that φ is a γ-contraction. So suppose that φ is only γ-condensing. Without any loss of generality, we may assume that $0 \in C$. Let $\{k_n\}_{n\geqslant 1} \subseteq (0,1)$ and assume that $k_n \longrightarrow 1$. From the first part of the proof, we can find $x_n \in C$, such that

$$k_n\varphi(x_n) = x_n \qquad \forall\, n \geqslant 1.$$

Hence

$$x_n - \varphi(x_n) = (k_n - 1)\varphi(x_n) \longrightarrow 0.$$

But from Proposition 7.2.26, we know that $(id_X - \varphi)(C)$ is closed and so

$$0 \in (id_X - \varphi)(C).$$

Therefore there exists $x \in C$, such that

$$\varphi(x) = x.$$

∎

7.3 Partial Order and Fixed Points

The purpose of this section is to establish the importance of order structures in fixed point theory.

DEFINITION 7.3.1 *Let X be a nonempty set. A binary relation $\leqslant$ in X is said to be* ***partial order*** *on X, if it satisfies the following properties:*

(a) *$x \leqslant x$ for all $x \in X$ (i.e., $\leqslant$ is reflexive);*

(b) *if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ for all $x, y, z \in X$ (i.e., $\leqslant$ is transitive);*

(c) *if $x \leqslant y$ and $y \leqslant x$ then $x = y$ for all $x, y \in X$ (i.e., $\leqslant$ is antisymmetric).*

The set X equipped with the partial order $\leqslant$ is said to be a ***partially ordered set*** *and is denoted by $(X, \leqslant)$ or simply by X when the partial order $\leqslant$ is clearly understood.*

REMARK 7.3.2 Some authors drop from the definition of partial order the property of reflexivity. We will call such an order ***strict partial order*** and denote it by $<$. We often write $x \geqslant y$ instead of $y \leqslant x$. Also $y < x$ or $x > y$ means that $y \leqslant x$ and $x \neq y$ (it is a strict partial order).

DEFINITION 7.3.3 *Let $(X, \leqslant)$ be a partially ordered set and let $C \subseteq X$ be a nonempty set. An* ***upper bound*** *for C is an element $x \in X$, such that $c \leqslant x$ for all $c \in C$. A* ***maximal element*** *in X is an element $x \in X$, such that if $x' \in X$ and $x \leqslant x'$, then $x = x'$. A* ***lower bound*** *for C and a* ***minimal element*** *of X are defined by reversing the above inequalities. A set $C \subseteq X$ is said to be a* ***bounded above set*** *(respectively a* ***bounded below set****), if there exists an upper (respectively lower) bound for C. If it is both bounded below and above, then we say that C is an* ***order bounded set****. An upper bound $x \in X$ for C is said to be a* ***least upper bound*** *for C (or* ***supremum*** *of C), if every upper bound y for C has the property $x \leqslant y$. The* ***greatest lower bound*** *for C (or* ***infimum*** *of C) is defined similarly. We denote them by $\sup C$ and $\inf C$ respectively. When they exist, then they are necessarily unique and are denoted by $\sup C$ and $\inf C$ respectively. We say that $(X, \leqslant)$ is a* ***lattice****, if for all $x, y \in X$, $\inf\{x, y\}$ and $\sup\{x, y\}$ exist and we say that $(X, \leqslant)$ is a* ***complete lattice****, if $\sup C$ and $\inf C$ exist for every nonempty $C \subseteq X$. Finally a* ***total*** *(or* ***linear****)* ***order*** *$\leqslant$ on X is a partial order with the property that if $x, y \in X$, $x \neq y$, then either $x \leqslant y$ or $y \leqslant x$. A* ***chain*** *in the partially ordered set $(X, \leqslant)$ is a nonempty set $C \subseteq X$ on which $\leqslant$ is a total order.*

REMARK 7.3.4 The maximal elements need not be unique.

The basic existence theorem for maximal elements is the ***Kuratowski-Zorn lemma***.

THEOREM 7.3.5 (Kuratowski-Zorn Lemma)
If $(X, \leqslant)$ is a partially ordered set and every chain of X has an upper bound, then X has at least one maximal element.

DEFINITION 7.3.6 *Let $(X, \leqslant)$ be a partially ordered set. For every $x \in X$ we set*

$$S_+(x) \stackrel{df}{=} \{y \in X : \ x \leqslant y\} \quad \text{and} \quad S_-(x) \stackrel{df}{=} \{y \in X : \ y \leqslant x\}.$$

*The sets $S_+(x)$ and $S_-(x)$ are called respectively the **right** and the **left section** of x. The set*

$$[x, y] \stackrel{df}{=} S_+(x) \cap S_-(y) \ = \ \{z \in X : \ x \leqslant z \leqslant y\}$$

*is the **order interval** determined by x and y. In particular*

$$[x, y] \ \neq \ \emptyset \quad \Longleftrightarrow \quad x \leqslant y.$$

*Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two partially ordered sets and let $\varphi\colon X \longrightarrow Y$. We say that φ is **increasing** (respectively **decreasing**), if $x_1 \leqslant_X x_2$ implies that $\varphi(x_1) \leqslant_Y \varphi(x_2)$ (respectively $\varphi(x_2) \leqslant_Y \varphi(x_1)$). We say that φ is **strictly increasing** (respectively **strictly decreasing**), if $x_1 <_X x_2$ implies that $\varphi(x_1) <_Y \varphi(x_2)$ (respectively $\varphi(x_2) <_Y \varphi(x_1)$).*

REMARK 7.3.7 If $\varphi\colon X \longrightarrow Y$ is increasing (respectively decreasing) and $\widehat{x} \leqslant \varphi(\widehat{x})$ (respectively $\varphi(\widehat{x}) \leqslant \widehat{x}$), then $\varphi\big(S_+(\widehat{x})\big) \subseteq S_+(\widehat{x})$ (respectively $\varphi\big(S_-(\widehat{x})\big) \subseteq S_-(\widehat{x})$). Therefore, if $a \leqslant \varphi(a)$ and $\varphi(b) \leqslant b$, then $\varphi\big([a, b]\big) \subseteq [a, b]$.

The first fixed point theorem using the order structure on X is the following.

THEOREM 7.3.8 (Bourbaki-Kneser Fixed Point Theorem)
If $(X, \leqslant)$ is a partially ordered set, such that every chain has an upper bound (respectively a lower bound) and $\varphi\colon X \longrightarrow X$ is a map, such that $x \leqslant \varphi(x)$ (respectively $\varphi(x) \leqslant x$) for all $x \in X$,
then φ has at least one fixed point.

PROOF Kuratowski-Zorn lemma (see Theorem 7.3.5) implies the existence of a maximal (respectively minimal) element x_0 of X. Then $x_0 \leqslant \varphi(x_0)$ (respectively $\varphi(x_0) \leqslant x_0$) and so we must have $\varphi(x_0) = x_0$ (see Definition 7.3.3). ∎

REMARK 7.3.9 If we strengthen the condition on X and instead assume that every chain has a supremum (respectively an infimum), then we can prove Theorem 7.3.8 without using Theorem 7.3.5. In fact in this case Theorem 7.3.5 is a consequence of Theorem 7.3.8. For details see Zeidler (1985*a*, pp. 506–511).

THEOREM 7.3.10 (Amann Fixed Point Theorem) *If $(X, \leqslant)$ is a partially ordered set, such that every chain has a supremum (respectively infimum), $\varphi\colon X \longrightarrow X$ is increasing and there exists $x_0 \in X$, such that*

$$x_0 \leqslant \varphi(x_0) \quad \textit{(respectively } \varphi(x_0) \leqslant x_0\textit{)},$$

then φ has a least (respectively greatest) fixed point in $S_+(x_0)$ (respectively in $S_-(x_0)$).

PROOF We consider the first case. The second can be treated analogously.

First we show the existence of a fixed point in $S_+(x_0)$. To this end let

$$X_+ = \{x \in X : x \leqslant \varphi(x)\} \cap S_+(x_0).$$

Note that $x_0 \in X_+$. If C is a chain in X_+, it has a supremum b in X and because φ is increasing,

$$c \leqslant \varphi(c) \leqslant \varphi(b) \qquad \forall\, c \in C.$$

Because $b = \sup C$, we must have $b \leqslant \varphi(b)$, which shows that every chain in X_+ has an upper bound in X_+. Applying Theorem 7.3.8, using the partially ordered set $(X_+, \leqslant)$, we obtain

$$x \in S_+(x_0),$$

such that

$$\varphi(x) = x.$$

Next we show the existence of a greatest fixed point in $S_+(x_0)$. Let

$$D_+ = \{x \in X : \varphi(x) = x\} \cap S_+(x_0).$$

Let us set

$$E_+ \stackrel{df}{=} \{z \in X_+ : z \leqslant d \text{ for all } d \in D_+\}.$$

From the first part of the proof we know that

$$E_+ \neq \emptyset \quad \text{and} \quad \varphi(E_+) \subseteq E_+.$$

Moreover, we can easily check that every chain in E_+ has a supremum in E_+. So by virtue of Kuratowski-Zorn lemma (see Theorem 7.3.5), we can find $z_0 \in E_+$, such that

$$\varphi(z_0) = z_0.$$

Evidently z_0 is the least fixed point of φ in $S_+(x_0)$. ∎

An immediate consequence of Theorem 7.3.10 is the following result.

THEOREM 7.3.11 (Tarski Fixed Point Theorem)
If $(X, \leqslant)$ is a partially ordered set which is chain complete (i.e., every chain in X has a supremum and an infimum), $\varphi\colon X \longrightarrow X$ is increasing and there exist $a, b \in X$, such that

$$a \leqslant b, \quad a \leqslant \varphi(a) \quad \text{and} \quad \varphi(b) \leqslant b,$$

then φ has a least and a greatest fixed point in $[a, b]$.

PROOF Just apply Theorem 7.3.10 to the chain complete partially ordered set $\big([a, b], \leqslant\big)$. □

REMARK 7.3.12 In fact the original theorem of Tarski says:

> "If the partially ordered set $(X, \leqslant)$ is a complete lattice and $\varphi\colon X \longrightarrow X$ is increasing, then φ possesses a least and a greatest fixed point."

Clearly this is a particular case of the more general Theorem 7.3.11.

Next we will consider Banach spaces which are partially ordered by a cone.

DEFINITION 7.3.13 *Let X be a Banach space and let $K \subseteq X$ be a nonempty set. We say that K is an **order cone**, if it is closed, convex,*

$$\lambda K \ \subseteq \ K \qquad \forall\ \lambda \geqslant 0$$

and

$$K \cap (-K) \ = \ \{0\}.$$

*We say that the order cone K is **solid**, if $\operatorname{int} K \neq \emptyset$.*

REMARK 7.3.14 An order cone K introduces a partial order on X, namely

$$x \ \leqslant \ y \quad \Longleftrightarrow \quad y - x \in K.$$

As before $x < y$ if and only if $x \leqslant y$ and $x \neq y$. Also, if K is solid, then

$$x \ \ll \ y \quad \Longleftrightarrow \quad y - x \in \operatorname{int} K.$$

By an ***ordered Banach space*** X we mean a Banach space X together with an order cone. The elements of the ordered cone are called ***positive***. Finally we emphasize the difference between an order cone and a cone. Recall that $C \subseteq X$ is a ***cone*** if and only if it is nonempty and

$$\lambda x \in C \qquad \forall\ x \in C,\ \lambda > 0.$$

Evidently every order cone is a cone but the converse is not true in general.

We introduce particular types of order cones which are useful in fixed point theory.

DEFINITION 7.3.15 *Let X be a Banach space and let $K \subseteq X$ be an order cone.*

(a) *We say that K is **reproducing** (or **generating**) if $X = K - K$ and K is **total** if $X = \overline{K - K}$.*

(b) *We say that K is **normal**, if there exists $\delta > 0$, such that*

$$\|x + y\|_X \geqslant \delta \qquad \forall\, x, y \in K,\ \|x\|_X = \|y\|_X = 1,$$

i.e.,

$$\inf_{x,y \in K \cap \partial B_1} \|x + y\|_X > 0.$$

(c) *We say that K is **regular** if every increasing sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ which is order bounded converges, i.e., if*

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant b \qquad \forall\, n \geqslant 1,$$

with $b \in X$, then $x_n \longrightarrow x$.

(d) *We say that K is **fully regular**, if every increasing sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ which is norm bounded converges, i.e., if*

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$$

and

$$\|x_n\|_X \leqslant M \qquad \forall\, n \geqslant 1,$$

for some $M > 0$, then $x_n \longrightarrow x$.

(e) *We say that K is **minihedral**, if for any $x, y \in X$, $\sup\{x, y\}$ exists.*

(f) *We say that K is **strongly minihedral**, if every $C \subseteq X$, which is bounded from above, has a supremum.*

REMARK 7.3.16 If K is reproducing, every element $u \in X$ can be written as $u = x - y$ with $x, y \in K$. Geometrically normality means that the angle between two positive unit vectors is bounded away from π, i.e., a normal cone cannot be too large. Clearly an order cone K is regular (respectively fully regular) if and only if every decreasing and order bounded (respectively norm bounded) sequence is convergent. Similarly an order cone K is minihedral if and only if for all $x, y \in X$, $\inf\{x, y\}$ exists and K is strongly minihedral if and only if every $C \subseteq X$ which is bounded below has an infimum. Finally note that K is minihedral if and only if for every $C \subseteq X$ finite, we have $\sup C$ and $\inf C$ exist.

Next we prove some results which elaborate further on the notions introduced in Definition 7.3.15.

PROPOSITION 7.3.17
If X is a Banach space and $K \subseteq X$ is an order cone which is solid, then K is reproducing.

PROOF Let $x_0 \in \operatorname{int} K$ and choose $r > 0$, such that

$$B_r(x_0) \;=\; \left\{x \in X :\; \|x - x_0\|_X < r\right\} \;\subseteq\; K.$$

So if $x \in X \setminus \{0\}$, then

$$y \;=\; x_0 + r\frac{x}{\|x\|_X} \in K$$

and

$$x \;=\; \frac{\|x\|_X}{r}(y - x_0) \in K - K.$$

Therefore K is reproducing. □

The next proposition provides useful criteria for normality of an order cone.

PROPOSITION 7.3.18
If X is a Banach space and $K \subseteq X$ is an order cone, then the following properties are equivalent:

(a) *K is normal;*

(b) *there exists $\beta > 0$, such that*

$$\|x + y\|_X \;\geqslant\; \beta \max\left\{\|x\|_X, \|y\|_X\right\} \qquad \forall\; x, y \in K;$$

(c) *the norm of X is semimonotone, i.e., there exists $\xi > 0$, such that if $0 \leqslant x \leqslant y$, then $\|x\|_X \leqslant \xi \|y\|_X$;*

(d) *there exists an equivalent norm $\|\cdot\|_1$ on X which is monotone, i.e., if $0 \leqslant x \leqslant y$, then $\|x\|_1 \leqslant \|y\|_1$;*

(e) *if $x_n \leqslant u_n \leqslant y_n$ for all $n \geqslant 1$ and $x_n, y_n \longrightarrow x$, then $u_n \longrightarrow x$;*

(f) *the set $\left(\overline{B}_1 + K\right) \cap \left(\overline{B}_1 - K\right)$ is norm bounded;*

(g) *the order intervals*

$$[x, y] \;=\; \left\{u \in X :\; x \leqslant u \leqslant y\right\}$$

are bounded for all $x, y \in X$.

PROOF "**(a)⟹(b)**": We may assume that $\|x\|_X = 1$ and $\|y\|_X \leqslant 1$. First

$$1 \;=\; \|x\|_X \;\leqslant\; \|x+y\|_X + \|y\|_X\,.$$

So using the fact that K is normal and the above estimate, we have

$$\begin{aligned}\|x+y\|_X \;&=\; \left\|x + \frac{y}{\|y\|_X} - \frac{1-\|y\|_X}{\|y\|_X}y\right\|_X\\ &\geqslant\; \left\|x+\frac{y}{\|y\|_X}\right\|_X - 1 + \|y\|_X \;\geqslant\; \delta - 1 + \|y\|_X\\ &\geqslant\; \delta - \|x+y\|_X\,.\end{aligned}$$

Therefore it follows that $\|x+y\|_X \geqslant \frac{\delta}{2}$. So **(b)** holds with $\beta = \frac{\delta}{2}$.

"**(b)⟹(c)**": Suppose that the implication is not true. Then we can find two sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subseteq K$, such that

$$0 \;\leqslant\; x_n \;\leqslant\; y_n \qquad \forall\, n \geqslant 1$$

and

$$0 \;<\; n\,\|y_n\|_X \;<\; \|x_n\|_X \qquad \forall\, n \geqslant 1.$$

Let us set

$$u_n \;\stackrel{df}{=}\; \frac{x_n}{\|x_n\|_X} + \frac{y_n}{n\,\|y_n\|_X} \quad\text{and}\quad v_n \;\stackrel{df}{=}\; -\frac{x_n}{\|x_n\|_X} + \frac{y_n}{n\,\|y_n\|_X} \qquad \forall\, n \geqslant 1.$$

Evidently

$$u_n, v_n \;\in\; K \qquad \forall\, n \geqslant 1$$

and

$$\|u_n\|_X \;\geqslant\; 1 - \frac{1}{n}, \quad \|v_n\|_X \;\geqslant\; 1 - \frac{1}{n}.$$

Because of **(b)**, we have

$$\frac{2}{n} \;=\; \|u_n + v_n\|_X \;\geqslant\; \beta\left(1 - \frac{1}{n}\right).$$

Let $n \to +\infty$ to obtain that $0 \geqslant \beta > 0$, a contradiction.

"**(c)⟹(d)**": For every $x \in X$, let

$$\|x\|_1 \;=\; \inf_{y\leqslant x} \|y\|_X + \inf_{z\geqslant x} \|z\|_X\,.$$

We will show that $\|\cdot\|_1$ is a norm on X. First note that $\|0\|_1 = 0$. Conversely, if $\|x\|_1 = 0$, then for a given $\varepsilon > 0$, we can find $y, z \in X$, such that

$$y \;\leqslant\; x \;\leqslant\; z \quad\text{and}\quad \|y\|_X, \|z\|_X \;\leqslant\; \varepsilon.$$

Because of **(c)** and the fact that $x - y \leqslant z - y$, we have

$$\|x\|_X \leqslant \|x-y\|_X + \|y\|_X \leqslant \xi \|z-y\|_X + \|y\|_X \leqslant (2\xi+1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we infer that $\|x\|_X = 0$, hence $x = 0$.

Clearly for all $\lambda \in \mathbb{R}$ and all $x \in X$, we have

$$\|\lambda x\|_1 = |\lambda| \, \|x\|_1 .$$

Next let $x, u \in X$. Then for a given $\varepsilon > 0$, we can find $y_1, z_1, y_2, z_2 \in X$, such that

$$y_1 \leqslant x \leqslant z_1 \quad \text{and} \quad y_2 \leqslant u \leqslant z_2$$

and

$$\|y_1\|_X + \|z_1\|_X \leqslant \|x\|_1 + \varepsilon \quad \text{and} \quad \|y_2\|_X + \|z_2\|_X \leqslant \|u\|_1 + \varepsilon.$$

Since

$$y_1 + y_2 \leqslant x + u \leqslant z_1 + z_2,$$

we have

$$\|x+u\|_1 \leqslant \|y_1 + y_2\|_X + \|z_1 + z_2\|_X$$

and so

$$\|x+u\|_1 \leqslant \|y_1\|_X + \|z_1\|_X + \|y_2\|_X + \|z_2\|_X \leqslant \|x\|_1 + \|u\|_1 + 2\varepsilon.$$

Let $\varepsilon \searrow 0$ to conclude that

$$\|x+u\|_X \leqslant \|x\|_1 + \|u\|_1$$

and we have proved that $\|\cdot\|_1$ is a norm on X.

In fact $\|\cdot\|_1$ is monotone. Indeed, if $0 \leqslant x \leqslant u$, then

$$\inf_{y \leqslant x} \|y\|_X = \inf_{y \leqslant u} \|y\|_X = 0$$

and so

$$\|x\|_1 = \inf_{z \geqslant x} \|z\|_X \leqslant \inf_{z \geqslant u} \|z\|_X = \|u\|_1 .$$

It remains to show that $\|\cdot\|_X$ and $\|\cdot\|_1$ are equivalent norms. Clearly

$$\|x\|_1 \leqslant 2\|x\|_X \qquad \forall\, x \in X.$$

On the other hand for every $y \leqslant x \leqslant z$, we have

$$\|x\|_X \leqslant \|x-y\|_X + \|y\|_X \leqslant \xi \|z-y\|_X + \|y\|_X \leqslant (\xi+1)\big(\|y\|_X + \|z\|_X\big),$$

so

$$\|x\|_X \leqslant (\xi+1)\|x\|_1 .$$

This proves that $\|\cdot\|_X$ and $\|\cdot\|_1$ are equivalent norms on X.

"**(d)**$\Longrightarrow$**(e)**": Since

$$0 \leqslant u_n - x_n \leqslant y_n - x_n \qquad \forall\, n \geqslant 1,$$

we have

$$\|u_n - x_n\|_X \leqslant \|y_n - x_n\|_X \qquad \forall\, n \geqslant 1.$$

So

$$u_n \longrightarrow x \quad \text{in } X.$$

"**(e)**$\Longrightarrow$**(f)**": Suppose that the set $\big(\overline{B}_1+K\big)\cap\big(\overline{B}_1-K\big)$ is unbounded. We can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq \big(\overline{B}_1 + K\big) \cap \big(\overline{B}_1 - K\big)$, such that $\|u_n\|_X \longrightarrow +\infty$. Evidently

$$x_n \leqslant u_n \leqslant y_n,$$

with some $x_n, y_n \in \overline{B}_1$. Let

$$v_n \stackrel{df}{=} \frac{x_n}{\|u_n\|_X}, \quad z_n \stackrel{df}{=} \frac{u_n}{\|u_n\|_X}, \quad w_n \stackrel{df}{=} \frac{y_n}{\|u_n\|_X} \qquad \forall\, n \geqslant 1.$$

We have

$$v_n \leqslant z_n \leqslant w_n$$

and

$$v_n, w_n \longrightarrow 0 \quad \text{in } X.$$

So by **(e)**, we have that $z_n \longrightarrow 0$, a contradiction to the fact that $\|z_n\|_X = 1$ for all $n \geqslant 1$.

"**(f)**$\Longrightarrow$**(g)**": Since by hypothesis the set $\big(\overline{B}_1 + K\big) \cap \big(\overline{B}_1 - K\big)$ is bounded, we can find $r > 0$, such that

$$\big(\overline{B}_1 + K\big) \cap \big(\overline{B}_1 - K\big) \subseteq r\overline{B}_1.$$

Let $x, y \in X$, $x \leqslant y$ and $u \in [x, y]$. Let us set

$$\eta \stackrel{df}{=} \max\big\{\, \|x\|_X, \|y\|_X \,\big\}.$$

Then

$$\frac{1}{\eta} u \in \big(\overline{B}_1 + K\big) \cap \big(\overline{B}_1 - K\big) \subseteq r\overline{B}_1$$

and so $\|u\|_X \leqslant r\eta$. This proves the norm boundedness of the order interval $[x, y]$.

"**(g)**$\Longrightarrow$**(a)**": Suppose that the implication is not true. We can find two sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subseteq K \cap \partial B_1$, such that

$$\|x_n + y_n\|_X < \frac{1}{4^n} \qquad \forall\, n \geqslant 1.$$

Let us set

$$u_n \stackrel{df}{=} \frac{x_n}{\|x_n + y_n\|_X^{\frac{1}{2}}} \quad \text{and} \quad v_n \stackrel{df}{=} \frac{x_n + y_n}{\|x_n + y_n\|_X^{\frac{1}{2}}} \qquad \forall\, n \geqslant 1.$$

Then

$$0 \leqslant u_n \leqslant v_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|v_n\|_X \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} < +\infty.$$

It follows that

$$\sum_{n=1}^{\infty} v_n = v \in X,$$

$$0 \leqslant u_n \leqslant v_n \leqslant v$$

and

$$\|u_n\|_X > 2^n.$$

So the order interval $[0, v]$ is unbounded, which contradicts **(g)**. □

PROPOSITION 7.3.19
If X is a Banach space and $K \subseteq X$ is an order cone, then

$$\Big[K \text{ is fully regular}\Big] \implies \Big[K \text{ is regular}\Big] \implies \Big[K \text{ is normal}\Big].$$

PROOF First suppose that K is regular and we will show that K is normal. Suppose that the implication is not true. Then we can find two sequences

$$\{x_n\}_{n \geqslant 1}, \{y_n\}_{n \geqslant 1} \subseteq K \cap \partial B_1,$$

such that

$$\|x_n + y_n\|_X \leqslant \frac{1}{2^n}.$$

Let us set

$$u_m \stackrel{df}{=} \sum_{n=1}^{m} x_n \leqslant \sum_{n=1}^{m} (x_n + y_n)$$

and since

$$\sum_{n=1}^{\infty} \|x_n + y_n\|_X \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} < +\infty,$$

we have that

$$\sum_{n=1}^{m} (x_n + y_n) \nearrow \sum_{n=1}^{\infty} (x_n + y_n) \in K.$$

Then the sequence $\{u_m\}_{m\geqslant 1}$ is increasing, order bounded and so by virtue of the regularity of K we must have that

$$u_m \longrightarrow u \quad \text{in } X \quad \text{as } m \to +\infty,$$

for some $u \in X$. But

$$\|u_{m+1} - u_m\|_X = \|x_{m+1}\|_X = 1 \qquad \forall\, m \geqslant 1,$$

a contradiction. This proves the normality of K.

Next assume that K is fully regular and we will show that K is normal. Suppose that the implication is not true. As before, we can find two sequences

$$\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subseteq K \cap \partial B_1,$$

such that

$$\|x_n + y_n\|_X \leqslant \frac{1}{2^n}.$$

Let us set

$$u_{2m} = \sum_{n=1}^{2m} (x_n + y_n) \qquad \forall\, m \geqslant 1$$

and

$$u_{2m+1} = u_{2m} + x_{2m+1} \qquad \forall\, m \geqslant 1.$$

Clearly the sequence $\{u_m\}_{n\geqslant 1} \subseteq K$ is increasing and norm bounded and so because K is fully regular, we have that

$$u_m \longrightarrow x \quad \text{in } X \quad \text{as } m \to +\infty,$$

for some $x \in X$. But note that

$$\|u_{2m+1} - u_{2m}\|_X = \|x_{2m+1}\|_X = 1,$$

a contradiction. This proves the normality of K.

Next assume that K is fully regular and we will show that K is regular. Consider an increasing sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ which is order bounded, i.e., there exists $y \in X$, such that

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant y \qquad \forall\, n \geqslant 1.$$

Then because of the previous implication and Proposition 7.3.18**(c)**, we have

$$\|y - x_n\|_X \leqslant \beta \|y - x_1\|_X \qquad \forall\, n \geqslant 1.$$

So the sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ is increasing and norm bounded and because of the full regularity of K we have

$$x_n \longrightarrow x \quad \text{in } X,$$

for some $x \in X$. This proves the regularity of K. ∎

In ordered Banach spaces the positive linear functionals (i.e., $x^* \in X^*$, such that $\langle x^*, x\rangle_X \geqslant 0$ for all $x \in K$) play a role similar in importance to that of the linear functionals on subspaces. For this reason, we introduce the following definition.

DEFINITION 7.3.20 *Let X be a Banach space and $K \subseteq X$ an order cone. The set*

$$K^* \stackrel{df}{=} \left\{x^* \in X^* : \langle x^*, x\rangle_X \geqslant 0 \text{ for all } x \in K\right\}$$

*is called the **dual cone** of K.*

REMARK 7.3.21 Clearly K^* is a cone but it may not be an order cone, because the condition

$$K^* \cap (-K^*) = \{0\}$$

may be violated (see Definition 7.3.13). It is easy to check that

$$K^* \text{ is an order cone} \iff K \text{ is total.}$$

We will show that the notions of normality and reproducing are dual for the cones K and K^*. First an auxiliary result.

LEMMA 7.3.22
If X is a Banach space, K is an order cone and

$$C \stackrel{df}{=} K \cap \overline{B}_1,$$

then

$$\left[K \text{ is reproducing}\right] \iff \left[r\overline{B}_1 \subseteq \overline{C - C} \text{ for some } r > 0\right]$$
$$\iff \left[\eta\overline{B}_1 \subseteq C - C \text{ for some } \eta > 0\right].$$

PROOF First suppose that K is reproducing and we will show that

$$r\overline{B}_1 \subseteq \overline{C - C},$$

for some $r > 0$. Since $X = K - K$, we have

$$X = \sum_{n=1}^{\infty} n(C - C);$$

hence by Baire category theorem (see Theorem A.1.10), we have that

$$\operatorname{int}\left(\overline{C - C}\right) \neq \emptyset.$$

So we can find $r > 0$, such that

$$\overline{B}_r \;=\; r\overline{B}_1 \subseteq \overline{C - C}.$$

Next suppose that $r\overline{B}_1 \subseteq \overline{C - C}$ for some $r > 0$ and we will show that

$$\eta\overline{B}_1 \;\subseteq\; C - C,$$

for some $\eta > 0$. Let

$$\eta \;=\; \frac{r}{2}$$

and choose

$$x_n \;\in\; \frac{1}{2^n}(C - C),$$

such that

$$\left\| x - \sum_{k=1}^{n} x_n \right\|_X \;<\; \frac{1}{2^{n-1}}.$$

Then we see that

$$\eta\overline{B}_1 \;\subseteq\; C - C.$$

Finally, if $\eta\overline{B}_1 \subseteq C - C$ for some $\eta > 0$, then obviously K is reproducing. □

Using the above lemma we can establish the duality between the notions of normality and reproducing.

THEOREM 7.3.23 (Krein Theorem)
If X is a Banach space, $K \subseteq X$ is an order cone and $K^* \subseteq X^*$ the dual cone,
then

(a) *K is reproducing if and only if K^* is normal;*

(b) *K is normal if and only if K^* is reproducing.*

PROOF **(a)** First suppose that K is reproducing.
By virtue of Lemma 7.3.22, there exists $\eta > 0$, such that

$$\eta\overline{B}_1 \;\subseteq\; C - C,$$

where

$$C \;\stackrel{df}{=}\; K \cap \overline{B}_1.$$

Let $0 \leqslant x^* \leqslant y^*$ (see Remark 7.3.21) and $x \in X$. Then

$$\frac{\eta x}{\|x\|_X} \;\in\; \eta\overline{B}_1,$$

so

$$\frac{\eta x}{\|x\|_X} \;=\; u - v,$$

for some $u, v \in C$ and thus

$$\langle x^*, x\rangle_X \;=\; \frac{1}{\eta}\,\|x\|_X\,\langle x^*, u - v\rangle_X\,.$$

Hence

$$\langle x^*, x\rangle_X \;\leqslant\; \frac{1}{\eta}\,\|x\|_X\,\langle x^*, u\rangle_X \;\leqslant\; \frac{1}{\eta}\,\|x\|_X\,\langle y^*, u\rangle_X \;\leqslant\; \frac{1}{\eta}\,\|x\|_X\,\|y^*\|_{X^*}\,.$$

Since $x \in X$ was arbitrary, we obtain that

$$\|x^*\|_{X^*} \;\leqslant\; \frac{1}{\eta}\,\|y^*\|_{X^*}$$

and this by virtue of Proposition 7.3.18 implies that K^* is normal.

Next suppose that K^* is normal. We argue indirectly. Suppose that K is not reproducing. Then $\overline{C - C}$ is not a neighbourhood to the origin (see Lemma 7.3.22). So for every $n \geqslant 1$, we can find $x \in X$, such that

$$\|x\|_X \;<\; \frac{1}{n} \quad \text{and} \quad x \;\notin\; \overline{C - C}$$

and by the strong separation theorem (see Theorem A.3.2), we can find $x^* \in X^*$, such that

$$\langle x^*, x\rangle_X \;>\; 1 \quad \text{and} \quad \langle x^*, u\rangle_X \;<\; 1 \qquad \forall\, u \in C - C.$$

In particular then $\|x^*\|_{X^*} > n$. We claim that

$$x^* \;\in\; \big(\overline{B}_1^* + K^*\big) \cap \big(\overline{B}_1^* - K^*\big), \tag{7.5}$$

where

$$\overline{B}_1^* \;=\; \big\{x^* \in X^* :\; \|x^*\|_{X^*} \leqslant 1\big\}.$$

Suppose that

$$x^* \;\notin\; \overline{B}_1^* + K^*.$$

Since $\overline{B}_1^*$ is w^*-compact and K^* is w^*-closed, the set $\overline{B}_1^* + K^*$ is w^*-closed. So by the separation theorem for convex sets (see Theorem A.3.1), we can find $u \in X$, such that

$$\langle x^*, u\rangle_X \;\geqslant\; 1 \;\geqslant\; \langle y^*, u\rangle_X \qquad \forall\, y^* \in \overline{B}_1^* + K^*.$$

Note that

$$0 \;\in\; K^* \cap \overline{B}_1^*.$$

So

$$\langle y^*, u\rangle_X \leqslant 1 \qquad \forall\, y^* \in \overline{B}_1^*$$

and

$$\langle y^*, u\rangle_X \leqslant 1 \qquad \forall\, y^* \in K^*.$$

Hence $\|u\|_X \leqslant 1$ and $u \in -K$, i.e.,

$$u \in \overline{B}_1 \cap (-K) = -C.$$

But then

$$\langle x^*, u\rangle_X < 1,$$

a contradiction. So

$$x^* \in \overline{B}_1^* + K^*.$$

If

$$x^* \notin \overline{B}_1^* - K^*,$$

then $-x^* \notin \overline{B}_1^* + K^*$ and so arguing as above, we conclude that $x^* \in \overline{B}_1^* - K^*$. This proves (7.5). But then because of Proposition 7.3.18, we reach a contradiction, since

$$\|x^*\|_{X^*} > n.$$

This proves that K is reproducing.

(b) First suppose that K is normal. We argue indirectly. So suppose that K^* is not reproducing. Let

$$C^* \stackrel{df}{=} K^* \cap \overline{B}_1^*.$$

Since $C^* - C^*$ is w^*-closed, we have that $C^* - C^*$ is not a neighbourhood of the origin and so for any given $n \geqslant 1$, we can find $u \in X$, such that

$$\|u\|_X > n \quad \text{and} \quad \langle y^*, u\rangle_X < 1 \qquad \forall\, y^* \in C^* - C^*.$$

Note that

$$\operatorname{int}\left(\overline{B}_1 + K\right) \neq \emptyset \quad \text{and} \quad \operatorname{int}\left(\overline{B}_1 - K\right) \neq \emptyset.$$

So as above we can use the separation theorem (see Theorem A.3.1) and obtain

$$x \in \left(\overline{B}_1 + K\right) \cap \left(\overline{B}_1 - K\right),$$

a contradiction.

Next suppose that K^* is reproducing. Then from part **(a)**, we have that

$$K^{**} = \left\{x^{**} \in X^{**} : \langle x^{**}, x^*\rangle_{X^*} \geqslant 0 \text{ for all } x^* \in K^*\right\}$$

is normal. Because $K \subseteq K^{**}$, we conclude that K is normal too. ∎

Using the above theorem we can improve the first implication of Proposition 7.3.19, provided we strengthen the structure of the space X.

PROPOSITION 7.3.24
If X is a weakly complete Banach space and $K \subseteq X$ is an order cone, then K is fully regular if and only if K is regular.

PROOF "$\Longrightarrow$": This follows from Proposition 7.3.19.

"$\Longleftarrow$": Let $\{x_n\}_{n\geqslant 1} \subseteq X$ be an increasing sequence which is norm bounded. Then for every $x^* \in K^*$, the real sequence $\left\{ \langle x^*, x_n\rangle_X \right\}_{n\geqslant 1}$ converges. Since K is normal (see Proposition 7.3.19), we have that K^* is reproducing (see Theorem 7.3.23**(b)**). So it follows that for every $x^* \in X^*$, the real sequence $\left\{ \langle x^*, x_n\rangle_X \right\}_{n\geqslant 1}$ converges. Then by virtue of the weak completeness of X, we have that

$$x_n \xrightarrow{w} x \quad \text{in } X,$$

for some $x \in X$. Since for every $x^* \in K^*$, the sequence $\left\{ \langle x^*, x_n\rangle_X \right\}_{n\geqslant 1} \subseteq \mathbb{R}_+$ is increasing, we infer that

$$\langle x^*, x_n\rangle_X \leqslant \langle x^*, x\rangle_X \qquad \forall\, n \geqslant 1,\ x^* \in K^*,$$

so

$$x_n \leqslant x \qquad \forall\, n \geqslant 1.$$

Therefore the sequence $\{x_n\}_{n\geqslant 1}$ is order bounded and because of the regularity of K, we have that

$$x_n \longrightarrow x \quad \text{in } X.$$

□

REMARK 7.3.25 Recall that every reflexive Banach space is weakly complete. In fact for reflexive Banach space, we can show that:

$$\Big[K \text{ is fully regular}\Big] \iff \Big[K \text{ is regular}\Big] \iff \Big[K \text{ is normal}\Big].$$

For a proof we refer to Guo & Lakshmikantham (1988, pp. 10–12).

PROPOSITION 7.3.26
If X is a separable Banach space and $K \subseteq X$ is a regular, minihedral order cone,
then K is strongly minihedral.

PROOF Let $C \subseteq X$ be a nonempty set which is bounded above. Let $\{x_n\}_{n\geqslant 1} \subseteq X$ be a sequence which is dense in C and let us set

$$u_n \stackrel{df}{=} \sup\{x_k\}_{k=1}^n \qquad \forall\, n \geqslant 1.$$

Since K is minihedral, u_n exists. As C is bounded above, we have

$$u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n \leqslant \ldots \leqslant y \qquad \forall\, n \geqslant 1,$$

for some $y \in X$. Because K is regular, we have that

$$u_n \longrightarrow u \quad \text{in } X,$$

for some $u \in X$. Note that

$$x_n \leqslant u \qquad \forall\, n \geqslant 1.$$

If $x \in C$, we can find a subsequence of $\{x_n\}_{n\geqslant 1}$, which converges strongly to x. Therefore $x \leqslant u$ and so u is an upper bound for the set C. If z is any other upper bound of C, then

$$u_n \leqslant z \qquad \forall\, n \geqslant 1$$

and so $u \leqslant z$. Therefore $u = \sup C$. □

Combining Proposition 7.3.26 and Remark 7.3.25, we obtain the following result.

COROLLARY 7.3.27
If X is a separable reflexive Banach space and $K \subseteq X$ is a normal, minihedral order cone,
then K is strongly minihedral.

Let us give some examples of cones illustrating the notions introduced in Definition 7.3.15.

EXAMPLE 7.3.28 **(a)** Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with finite Lebesgue measure and let $X = L^p(\Omega)$ with $p \in [1, +\infty]$. We consider

$$K \stackrel{df}{=} \big\{u \in L^p(\Omega) :\ u(z) \geqslant 0 \text{ for a.a. } z \in \Omega\big\}.$$

Then K is an order cone. This order cone is reproducing (recall that $u = u^+ - u^-$) and normal (since the norm $\|\cdot\|_p$ is monotone; see Proposition 7.3.18). If $p = +\infty$, then K is solid, but for $p \in [1, +\infty)$, we have $\operatorname{int} K = \emptyset$. Using the Lebesgue dominated convergence theorem (see Theorem A.2.2), we could see that K is fully regular for every $p \in [1, +\infty]$ (this is trivial if $p \in (1, +\infty)$, since $L^p(\Omega)$ is reflexive and K is normal; see Remark 7.3.25). Clearly K is also minihedral for all $p \in [1, +\infty]$ and moreover, if $p \in [1, +\infty)$, then because $L^p(\Omega)$ is separable, K is also strongly minihedral (see Proposition 7.3.26).

(b) Let $D \subseteq \mathbb{R}^N$ be compact and let $X = C(D)$. We consider

$$K \stackrel{df}{=} \big\{u \in C(D) :\ u(z) \geqslant 0 \text{ for all } z \in D\big\}.$$

Then K is an order cone which is solid, generating and normal. K is not regular. To see this let $D = [0,1]$ and let

$$x_n(t) \stackrel{df}{=} 1 - t^n \qquad \forall\, n \geqslant 1,\ t \in [0,1].$$

Then

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant y \qquad \forall\, n \geqslant 1,$$

with

$$y(t) = 1 \qquad \forall\, t \in [0,1].$$

But the sequence $\{x_n\}_{n \geqslant 1}$ does not converge in $C\big([0,1]\big)$, since

$$x(t) \longrightarrow x(t) \qquad \forall\, t \in [0,1],$$

where

$$x(t) \stackrel{df}{=} \begin{cases} 1 & \text{if} \quad t \neq 1, \\ 0 & \text{if} \quad t = 0. \end{cases}$$

Also the order cone K is minihedral but not strongly minihedral. To see this let $D = [0,2]$ and let

$$C \stackrel{df}{=} \Big\{ u \in C\big([0,2]\big) :\ u(t) < 1 \text{ for } t \in (0,1) \text{ and } u(t) < 2 \text{ for } t \in (1,2) \Big\}.$$

Then C is order bounded but $\sup D$ does not exist.

(c) Let $X = C^1\big([0,2\pi]\big)$ equipped with the norm

$$\|x\|_{C^1} \stackrel{df}{=} \|x\|_\infty + \|x'\|_\infty$$

and let

$$K \stackrel{df}{=} \Big\{ u \in C^1\big([0,2\pi]\big) :\ u(t) \geqslant 0,\ \text{for all } t \in [0,2\pi] \Big\}.$$

It is easy to see that K is an order cone, which is solid and generating (indeed $u = y - x$, where $y(t) = M > \|x\|_\infty$ for all $t \in [0,2\pi]$ and $x(t) = M - u(t)$ for all $t \in [0,2\pi]$).

But K is not normal. To see this note that if K was normal, then by Proposition 7.3.18, we can find $\beta > 0$, such that, if $0 \leqslant u \leqslant v$, then $\|u\|_{C^1} \leqslant \beta \|v\|_{C^1}$. Let

$$u_n(t) \stackrel{df}{=} 1 - \cos(nt) \quad \text{and} \quad v_n(t) \stackrel{df}{=} 2 \qquad \forall\, t \in [0,2\pi],\ n \geqslant 1.$$

Then

$$0 \leqslant u_n \leqslant v_n, \quad \|u_n\|_{C^1} = 2 + n \quad \text{and} \quad \|v_n\|_{C^1} = 2$$

and so we cannot have

$$\|u_n\|_{C^1} \leqslant \beta \|v_n\|_{C^1} \qquad \forall\, n \geqslant 1.$$

Before passing to some fixed point theorems, let us state a proposition, which contains some useful properties of the elements of K and of K^*.

PROPOSITION 7.3.29
If X is a Banach space, $K \subseteq X$ is an order cone and $K^* \subseteq X^*$ is the dual cone,
then

(a) *if K^* is nontrivial (i.e., $K^* \neq \emptyset$), then $x \in K$ if and only if*

$$\langle x^*, x \rangle_X \geqslant 0 \qquad \forall\, x^* \in K^*;$$

if $x \in K \setminus \{0\}$, then there exists $\widehat{x}^* \in K^*$, such that

$$\langle \widehat{x}^*, x \rangle_X > 0;$$

(b) *if K is solid, then $x \in \operatorname{int} K$ if and only if*

$$\langle x^*, x \rangle_X > 0 \qquad \forall\, x^* \in K^* \setminus \{0\};$$

(c) *if X is separable, then there exists $\widehat{x}^* \in K^*$, such that*

$$\langle \widehat{x}^*, x \rangle_X > 0 \qquad \forall\, x \in K \setminus \{0\}.$$

PROOF **(a)** Let $z \notin K$. Then by the strong separation theorem for convex sets (see Theorem A.3.2), we can find $x^* \in X^*$, such that

$$\langle x^*, z \rangle_X < \eta \leqslant \langle x^*, x \rangle_X \qquad \forall\, x \in K.$$

Since $0 \in K$, we see that $\eta \leqslant 0$ and so

$$\langle x^*, z \rangle_X < 0.$$

Moreover, for all $x \in K$ and all $n \geqslant 1$, we have $\eta \leqslant \langle x^*, nx \rangle_X$, hence

$$\frac{\eta}{n} \leqslant \langle x^*, x \rangle_X .$$

Let $n \to +\infty$ to obtain that

$$\langle x^*, x \rangle_X \geqslant 0 \qquad \forall\, x \in K.$$

Therefore $x^* \in K^* \setminus \{0\}$. This proves **(a)**.

(b) First suppose that $x \in \operatorname{int} K$. We will show that

$$\langle x^*, x \rangle_X > 0 \qquad \forall\, x^* \in K^* \setminus \{0\}.$$

Since $x \in \operatorname{int} K$, we can find $r > 0$, such that

$$B_r(x) \;=\; \big\{y \in X :\; \|y - x\|_X < r\big\} \;\subseteq\; K.$$

So

$$x \pm rz \geqslant 0 \qquad \forall\, z \in \overline{B}_1 = \big\{z \in X :\; \|z\|_X \leqslant 1\big\}.$$

Thus from part **(a)**, we have that

$$\big\langle x^*, x \pm rz\big\rangle_X \;\geqslant\; 0 \qquad \forall\, x^* \in K^* \setminus \{0\},$$

so

$$\langle x^*, x\rangle_X \;\geqslant\; r\,\|x^*\|_X \;>\; 0.$$

Next suppose that

$$\langle x^*, x\rangle_X \;>\; 0 \qquad \forall\, x^* \in K^* \setminus \{0\}$$

and we will show that $x \in \operatorname{int} K$. To proceed indirectly, suppose that $x \notin \operatorname{int} K$. Let

$$K_1 \;\stackrel{df}{=}\; \big\{y \in X :\; y = \lambda x,\; \lambda \geqslant 0\big\}.$$

Evidently K_1 is also an order cone in X and

$$K_1 \cap \operatorname{int} K \;=\; \emptyset.$$

So by virtue of the weak separation theorem for convex sets (see Theorem A.3.1), we can find $x^* \in X^* \setminus \{0\}$, such that

$$\langle x^*, y\rangle_X \;\leqslant\; \eta \;\leqslant\; \langle x^*, z\rangle_X \qquad \forall\, y \in K_1,\; z \in K.$$

Since $0 \in K_1 \cap K$, we see that $\eta = 0$ and so $\langle x^*, x\rangle_X \leqslant 0$, a contradiction.

(c) Since X is separable, the dual unit ball $\overline{B}_1^* = \big\{x^* \in X^* :\; \|x^*\|_{X^*} \leqslant 1\big\}$ furnished with the w^*-topology is compact metrizable. So we can find a sequence $\{x_n^*\}_{n\geqslant 1}$ dense in $K^* \cap \overline{B}_1^*$. Let

$$u^* \;\stackrel{df}{=}\; \sum_{n=1}^{\infty} \frac{1}{n^2} x_n^*.$$

Evidently $u^* \in K^*$. We show that

$$\langle u^*, x\rangle_X \;>\; 0 \qquad \forall\, x \in K \setminus \{0\}.$$

If $\langle u^*, x\rangle_X = 0$ for some $x \in K \setminus \{0\}$, then

$$\langle x^*, x\rangle_X \;=\; 0 \qquad \forall\, x^* \in K^*,$$

which contradicts part **(a)**. ∎

Now we state some fixed points for maps defined on ordered Banach spaces. The results that we prove are in the spirit of Theorem 7.3.10 and reveal that there is a balance between the conditions on the order cone $K \subseteq X$ and on increasing maps φ. The stronger the hypotheses on K, the more general φ can be and vice versa.

THEOREM 7.3.30
If X is a Banach space, $K \subseteq X$ is an order cone, $x_0 < y_0$, $\varphi\colon [x_0, y_0] \longrightarrow X$ is an increasing map, such that

$$x_0 \leqslant \varphi(x_0), \qquad \varphi(y_0) \leqslant y_0$$

and one of the following hypotheses holds:

(i) K is normal and φ is condensing; or

(ii) K is regular and φ is demicontinuous, i.e.,

$$\text{if } x_n \longrightarrow x, \text{ then } \varphi(x_n) \xrightarrow{w} \varphi(x),$$

then φ has a minimal fixed point $\underline{x}$ and a maximal fixed point $\overline{x}$ in $[x_0, y_0]$ and

$$\underline{x} = \lim_{n\to+\infty} x_n, \quad \text{and} \quad \overline{x} = \lim_{n\to+\infty} y_n$$

where

$$x_n \stackrel{df}{=} \varphi(x_{n-1}), \quad y_n \stackrel{df}{=} \varphi(y_{n-1}) \qquad \forall\, n \geqslant 1$$

and

$$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant x \leqslant \overline{x} \leqslant \ldots \leqslant y_n \leqslant \ldots \leqslant y_1 \leqslant y_0.$$

PROOF If $x_n = \varphi(x_{n-1})$, $y_n = \varphi(y_{n-1})$ for $n \geqslant 1$, from the properties of φ, we see that

$$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant y_n \leqslant \ldots \leqslant y_1 \leqslant y_0. \tag{7.6}$$

First suppose that hypothesis (i) is in effect. The sequence $C = \{x_n\}_{n\geqslant 1}$ is bounded (see Proposition 7.3.18) and

$$C = \varphi(C) \cup \{x_0\},$$

hence

$$\gamma(C) = \gamma\big(\varphi(C)\big),$$

which by virtue of the fact that φ is condensing implies that $\gamma(C) = 0$ (see Definition 7.2.24**(c)**), i.e., C is relatively compact in X (see Proposition 7.2.21). So we can find a subsequence $\{x_{n_k}\}_{k\geqslant 1}$ of $\{x_n\}_{n\geqslant 1}$, such that

$$x_{n_k} \longrightarrow \underline{x}.$$

From (7.6), it is clear that

$$x_n \leqslant \underline{x} \leqslant y_n \qquad \forall\, n \geqslant 1.$$

For $m > n_k$, we have

$$0 \leqslant \underline{x} - x_m \leqslant \underline{x} - x_{n_k}$$

and because K is by hypothesis normal, there is $\beta > 0$, such that

$$\|\underline{x} - x_m\|_X \leqslant \beta \|\underline{x} - x_{n_k}\|_X$$

(see Proposition 7.3.18), so

$$x_m \longrightarrow \underline{x} \quad \text{as } m \to +\infty.$$

From the equality $\varphi(x_{n-1}) = x_n$, if we pass to the limit as $n \to +\infty$, we obtain $\varphi(\underline{x}) = \underline{x}$.

Next suppose that hypothesis (ii) is in effect. Because K is regular, we have

$$x_n \longrightarrow \underline{x} \quad \text{in } X.$$

Then from the equality

$$\varphi(x_{n-1}) = x_n \qquad \forall\, n \geqslant 1$$

and the demicontinuity of φ, we obtain

$$\varphi(\underline{x}) = \underline{x}.$$

In a similar fashion, in both cases, we show that

$$y_n \longrightarrow \overline{x} \quad \text{in } X \quad \text{and} \quad \varphi(\overline{x}) = \overline{x}.$$

Finally we show that $\overline{x}$ and $\underline{x}$ are the maximal and minimal fixed points of φ on $[x_0, y_0]$. Suppose that $u \in [x_0, y_0]$ is such that

$$\varphi(u) = u.$$

Then since φ is increasing, we have

$$x_1 \leqslant \varphi(u) = u \leqslant y_1$$

and continuing this way, we obtain

$$x_n \leqslant u \leqslant y_n \qquad \forall\, n \geqslant 1.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\underline{x} \leqslant u \leqslant \overline{x},$$

which proves the theorem. □

THEOREM 7.3.31
If X is a Banach space, $K \subseteq X$ is an order cone, which is strongly minihedral, $x_0 < y_0$ and $\varphi\colon [x_0, y_0] \longrightarrow X$ is an increasing map, such that

$$x_0 \leqslant \varphi(x_0) \quad \text{and} \quad \varphi(y_0) \leqslant y_0,$$

then φ has a minimal fixed point $\underline{x}$ and a maximal fixed point $\overline{x}$ in $[x_0, y_0]$.

PROOF Let

$$C \stackrel{df}{=} \Big\{ x \in [x_0, y_0] : \ x \leqslant \varphi(x) \Big\}.$$

Clearly $x_0 \in C$ and y_0 is an upper bound of C. Because K is strongly minihedral, we have that $\sup C = \overline{x}$ exists. We show that $\overline{x}$ is the maximal fixed point of φ on $[x_0, y_0]$. We have

$$x_0 \leqslant x \leqslant \overline{x} \leqslant y_0 \qquad \forall\, x \in C$$

and so since φ is increasing, we have

$$x_0 \leqslant \varphi(x_0) \leqslant \varphi(x) \leqslant \varphi(\overline{x}) \leqslant \varphi(y_0) \leqslant y_0,$$

hence

$$x \leqslant \varphi(\overline{x}) \qquad \forall\, x \in C$$

and thus

$$\sup C = \overline{x} \leqslant \varphi(\overline{x}). \tag{7.7}$$

On the other hand, since $\overline{x} \leqslant \varphi(\overline{x})$ and φ is increasing, we have

$$\varphi(\overline{x}) \leqslant \varphi\big(\varphi(\overline{x})\big),$$

i.e., $\varphi(\overline{x}) \in C$. So we have

$$\varphi(\overline{x}) \leqslant \sup C = \overline{x}$$

and it follows that $\varphi(\overline{x}) = \overline{x}$ (see (7.7)). If $u \in [x_0, y_0]$ is a fixed point of φ, then $u \in C$ and so $u \leqslant \overline{x}$. This proves the maximality of $\overline{x}$.

A similar argument using the set

$$C_1 \stackrel{df}{=} \Big\{ x \in [x_0, y_0] : \ \varphi(x) \leqslant x \Big\}$$

gives us $\underline{x} = \inf C_1$ (see Remark 7.3.16), which is the minimal fixed point of φ on $[x_0, y_0]$. □

REMARK 7.3.32 We emphasize that in Theorem 7.3.31, φ need not be continuous. For this reason we do not claim that $\overline{x}$ and $\underline{x}$ are obtained via a monotone iteration process. If φ is continuous, then the iteration processes converge.

THEOREM 7.3.33
If X is a Banach space, $K \subseteq X$ is an order cone which is minihedral, $x_0 < y_0$ and $\varphi\colon [x_0, y_0] \longrightarrow X$ is an increasing map, such that

$$x_0 \leqslant \varphi(x_0), \quad \varphi(y_0) \leqslant y_0$$

and $\varphi\big([x_0, y_0]\big)$ is relatively compact in X,
then φ has a minimal fixed point $\underline{x}$ and a maximal fixed point $\overline{x}$ in $[x_0, y_0]$.

PROOF Let

$$C \stackrel{df}{=} \Big\{x \in \varphi\big([x_0, y_0]\big) : \ x \leqslant \varphi(x)\Big\}.$$

Since

$$\varphi(x_0) \leqslant \varphi\big(\varphi(x_0)\big),$$

we see that $\varphi(x_0) \in C$ and so $C \neq \emptyset$. Let D be a chain in the partially ordered set C. Because

$$D \subseteq \varphi\big([x_0, y_0]\big),$$

we have that D is relatively compact and so separable. Let $\{x_n\}_{n\geqslant 1}$ be a sequence dense in D and let us set

$$y_n \stackrel{df}{=} \sup\{x_k\}_{k=1}^n \in D \qquad \forall\, n \geqslant 1$$

(in fact y_n equals one of the x_k since D is a chain). By virtue of the relative compactness of D, we can find a subsequence $\{y_{n_k}\}_{k\geqslant 1}$ of $\{y_n\}_{n\geqslant 1}$, such that

$$y_{n_k} \longrightarrow \overline{y} \quad \text{in } X.$$

Because

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n \leqslant \ldots,$$

we have that

$$x_n \leqslant y_n \leqslant \overline{y} \qquad \forall\, n \geqslant 1 \tag{7.8}$$

and

$$\overline{y} \in \overline{D} \subseteq \overline{C} \subseteq \overline{\varphi\big([x_0, y_0]\big)} \subseteq [x_0, y_0].$$

From (7.8) and the density of $\{x_n\}_{n\geqslant 1}$ in D, we have that

$$x \leqslant \overline{y} \qquad \forall\, x \in D$$

and so

$$x \leqslant \varphi(x) \leqslant \varphi(\overline{y}) \qquad \forall\, x \in D.$$

So $\varphi(\overline{y})$ is an upper bound of the set D. On the other hand, since

$$y_n \leqslant \varphi(\overline{y}) \qquad \forall\, n \geqslant 1,$$

we have

$$\overline{y} \leqslant \varphi(\overline{y})$$

and so

$$\varphi(\overline{y}) \leqslant \varphi\big(\varphi(\overline{y})\big),$$

which proves that $\varphi(\overline{y})$ is an upper bound of D in C. Invoking Kuratowski-Zorn lemma (see Theorem 7.3.5), we can find a maximal element $\overline{x} \in C$ of C. Then

$$\overline{x} \leqslant \varphi(\overline{x})$$

and so

$$\varphi(\overline{x}) \leqslant \varphi\big(\varphi(\overline{x})\big),$$

i.e., $\varphi(\overline{x}) \in C$, which by virtue of the maximality of $\overline{x}$ implies that

$$\varphi(\overline{x}) = \overline{x}.$$

We claim that $\overline{x}$ is the maximal fixed point of φ in $[x_0, y_0]$. To this end let u be a fixed point of φ in $[x_0, y_0]$. Since K is by hypothesis minihedral, $v = \sup\{\overline{x}, u\}$ exists. From the facts that

$$\overline{x} \leqslant v \quad \text{and} \quad u \leqslant v,$$

we have

$$\overline{x} = \varphi(\overline{x}) \leqslant \varphi(v) \quad \text{and} \quad u = \varphi(u) \leqslant \varphi(v).$$

Therefore $v \leqslant \varphi(v)$ and so

$$\varphi(v) \leqslant \varphi\big(\varphi(v)\big),$$

which implies that $\varphi(v) \in C$. By the maximality of $\overline{x}$ in C we must have

$$\varphi(v) = \overline{x}$$

and so

$$v \leqslant \overline{x},$$

i.e.,

$$\overline{x} \text{ is the maximal fixed point of } \varphi \text{ in } [x_0, y_0].$$

In a similar fashion using the set

$$C_1 \stackrel{df}{=} \Big\{x \in \varphi\big([x_0, y_0]\big) : \varphi(x) \leqslant x\Big\},$$

we obtain a minimal fixed point of φ in $[x_0, y_0]$. □

REMARK 7.3.34 Again we emphasize that the map φ need not be continuous.

COROLLARY 7.3.35
If X is a Banach space, $K \subseteq X$ is an order cone, which is minihedral and normal, $x_0 < y_0$ and $\varphi\colon [x_0, y_0] \longrightarrow X$ is an increasing map, which maps bounded sets to relatively compact sets and

$$x_0 \leqslant \varphi(x_0) \quad \text{and} \quad \varphi(y_0) \leqslant y_0,$$

then φ has a minimal fixed point $\underline{x}$ and a maximal fixed point $\overline{x}$ in $[x_0, y_0]$.

To produce additional fixed point theorems for maps on ordered Banach spaces as well as theorems on the existence of multiple fixed points, we need to introduce some tools from degree theory.

Recall that if X is a Hausdorff topological space and $C \subseteq X$ is a nonempty set, then C is a ***retract*** of X, if there is a continuous map $r\colon X \longrightarrow C$, such that

$$r|_C = id_C$$

(see Definition 7.2.3). Every nonempty, closed, convex subset of a Banach space is a retract and every retract is a closed set but not necessarily convex. The set $\partial B_1 = \{x \in X : \|x\|_X = 1\}$ is a retract of $\overline{B}_1 = \{x \in X : \|x\|_X = 1\}$ if $\dim X = +\infty$. Also if C is a retract of X and D is a retract of C, then D is a retract of X.

DEFINITION 7.3.36 *Let X be a Banach space and consider the family*

$$M \stackrel{df}{=} \Big\{(\varphi, U, K) : K \subseteq X \text{ is a retract, } U \subseteq K \text{ is a bounded, relatively open set, } \varphi\colon \overline{U} \longrightarrow K \text{ is compact and } \mathrm{Fix}(\varphi) \cap \partial U = \emptyset\Big\},$$

where

$$\mathrm{Fix}(\varphi) \stackrel{df}{=} \big\{u \in \overline{U} : \varphi(x) = x\big\}.$$

*If $r\colon X \longrightarrow K$ is a retraction, then for every $(\varphi, U, K) \in M$, we can define the $\mathbb{Z}$-valued **fixed point index of φ over U with respect to** K, by*

$$i(\varphi, U, K) \stackrel{df}{=} d_{LS}\big(id_X - \varphi \circ r, r^{-1}(U), 0\big),$$

where d_{LS} denotes the Leray-Schauder degree.

REMARK 7.3.37 Using the properties of the Leray-Schauder degree (namely the homotopy invariance and excision properties), we can check that the above definition is independent of the retraction used. It extends the Leray-Schauder degree (just consider the case where $K = X$).

The fixed point index inherits the properties of the Leray-Schauder degree. So we can state the following theorem. We omit its straightforward proof.

THEOREM 7.3.38

On the set M of admissible triples, there is a unique $\mathbb{Z}$-valued function $i(\varphi, U, K)$ called the fixed point index of φ on U with respect to K, which satisfies:

(a) *Normalization: if*

$$\varphi(x) \;=\; u_0 \in U \qquad \forall\, x \in \overline{U},$$

then

$$i(\varphi, U, K) \;=\; 1.$$

(b) *Additivity: if U_1 and U_2 are disjoint open subsets of U and*

$$\mathrm{Fix}\,(\varphi) \cap \big(\overline{U} \setminus (U_1 \cup U_2)\big) \;=\; \emptyset,$$

then

$$i(\varphi, U, K) \;=\; i(\varphi, U_1, K) + i(\varphi, U_2, K)$$

(c) *Homotopy invariance: if $h\colon [0,1] \times \overline{U} \longrightarrow K$ is compact and*

$$h(t, x) \;\neq\; x \qquad \forall\, (t, x) \in [0,1] \times \partial U,$$

then

$$i\big(h(t, \cdot), U, K\big) \text{ is independent of } t \in [0,1].$$

(d) *Reduction property: if K_1 is a retract of K and $\varphi(\overline{U}) \subseteq K_1$, then*

$$i(\varphi, U, K) \;=\; i(\varphi, U \cap K_1, K_1).$$

(e) *Dependence on boundary values: if $\varphi = \psi$ on ∂U, then*

$$i(\varphi, U, K) \;=\; i(\psi, U, K).$$

(f) *Excision property: if U_0 is an open subset of U and*

$$\mathrm{Fix}\,(\varphi) \cap \big(\overline{U} \setminus U_0\big) \;=\; \emptyset,$$

then

$$i(\varphi, U, K) \;=\; i(\varphi, U_0, K).$$

(g) *Solution property: if*

$$i(\varphi, U, K) \neq \emptyset,$$

then there exists $x \in U$, such that

$$\varphi(x) \;=\; x.$$

REMARK 7.3.39 As in the case with the Leray-Schauder degree, the fixed point index can be extended to all γ-condensing maps (see Definition 7.2.24**(c)**). For details we refer to Nussbaum (1971).

In what follows all topological notions are understood with respect to the relative norm topology on K.

PROPOSITION 7.3.40
If $(\varphi, U, K) \in M$ *(see Definition 7.3.36) and there exists* $u \in U$*, such that*

$$\varphi(x) - x \neq t(x-u) \qquad \forall\, t > 0,\ x \in \partial U,$$

then

$$i(\varphi, U, K) = 1.$$

PROOF Let

$$h(t,x) \stackrel{df}{=} (1-t)\varphi(x) + tu \qquad \forall\, t \in [0,1],\ x \in \overline{U}.$$

We claim that

$$h(t,x) \neq x \qquad \forall\, (t,x) \in [0,1] \times \partial U.$$

Indeed, if there exist $t_1 \in [0,1]$ and $x_1 \in \partial U$, such that

$$(1-t_1)\varphi(x_1) + t_1 u = x_1,$$

then $t_1 \neq 1$ and so

$$\varphi(x_1) - x_1 = \frac{t_1}{1-t_1}(x_1 - u) \quad \text{with } \frac{t_1}{1-t_1} > 0,$$

which contradicts our hypothesis. So

$$(1-t)\varphi(x) + tu \neq x \qquad \forall\, t \in [0,1],\ x \in \overline{U}.$$

Hence by the homotopy invariance property (see Theorem 7.3.38**(c)**), we have

$$i(\varphi, U, K) = i(u, U, K) = 1$$

(see Theorem 7.3.38**(a)**). □

PROPOSITION 7.3.41
If $(\varphi, U, K) \in M$ *(see Definition 7.3.36) and there exists* $u \in K \setminus \overline{U}$*, such that*

$$\varphi(x) - x \neq t(x-u) \qquad \forall\, t > 0,\ x \in \partial U,$$

then

$$i(\varphi, U, K) = 0.$$

PROOF As before let

$$h(t,x) \stackrel{df}{=} (1-t)\varphi(x) + tu \qquad \forall\ t \in [0,1],\ x \in \overline{U}$$

and we have

$$h(t,x) \neq x \qquad \forall\ (t,x) \in [0,1] \times \partial U.$$

So from the homotopy invariance of the fixed point index, we have

$$i(\varphi, U, K) = i(u, U, K) = 0.$$

□

PROPOSITION 7.3.42
If X is a Banach space, $K \subseteq X$ is an order cone, $U \subseteq X$ is a bounded open set, $\varphi\colon K \cap \overline{U} \longrightarrow K$ and $\psi\colon K \cap \partial U \longrightarrow K$ are compact maps and

(i) $\inf\left\{\left\|\psi(x)\right\|_X :\ x \in \partial(K \cap U) = K \cap \partial U\right\} > 0$;

(ii) $x - \varphi(x) \neq t\psi(x)$ *for all* $t \geqslant 0$ *and all* $x \in \partial(K \cap U) = K \cap \partial U$,

then

$$i(\varphi, K \cap U, K) = 0.$$

PROOF By virtue of Theorem 3.1.12, we can find a compact map

$$\widehat{\psi}\colon K \cap \overline{U} \longrightarrow K,$$

such that

$$\widehat{\psi}(K \cap \overline{U}) \subseteq \overline{\operatorname{conv}}\,\psi(K \cap \partial U). \tag{7.9}$$

Let us set

$$C \stackrel{df}{=} \psi(K \cap \partial U).$$

Then

$$\overline{\operatorname{conv}}\,C = \overline{D},$$

where

$$D \stackrel{df}{=} \left\{y = \sum_{k=1}^{n} \lambda_k y_k :\ y_k \in C,\ \lambda_k \geqslant 0,\ \sum_{k=1}^{n} \lambda_k = 1,\ n \geqslant 1\right\}.$$

We claim that

$$\inf_{y \in D} \|y\|_X > 0. \tag{7.10}$$

Let

$$X_0 \stackrel{df}{=} \overline{\operatorname{span}}\,C.$$

Since C is relatively compact, X_0 is a separable Banach subspace of X. Also $K_0 = K \cap X_0$ is an order cone in X_0 and

$$\overline{\operatorname{conv} C} \subseteq K_0.$$

From Proposition 7.3.29**(a)**, we know that there exists $u_0^* \in K_0^*$, such that

$$\langle u_0^*, y \rangle_X > 0 \qquad \forall\, y \in K_0 \setminus \{0\}.$$

We will show that

$$\inf_{y \in C} \langle u_0^*, y \rangle_X = m > 0. \tag{7.11}$$

If $m = 0$, then we can find a sequence $\{y_k\}_{k \geqslant 1} \subseteq C$, such that

$$\langle u_0^*, y_k \rangle_X \searrow m = 0 \quad \text{as } k \to +\infty.$$

Since C is relatively compact (recall that ψ is compact), we may assume that

$$y_k \longrightarrow y \in K_0 \quad \text{as } k \to +\infty$$

and so

$$\langle u_0^*, y_k \rangle_X \longrightarrow \langle u_0^*, y \rangle_X,$$

which implies that $y = 0$ and so

$$\|y_k\|_X \longrightarrow 0,$$

a contradiction to hypothesis (i). This proves that (7.11) is true.

For any $y \in D$, we have

$$\langle u_0^*, y \rangle_X = \sum_{k=1}^{n} \lambda_k \langle u_0^*, y_k \rangle_X \geqslant m \sum_{k=1}^{n} \lambda_k = m$$

(see (7.11)), so

$$\langle u_0^*, y \rangle_X \geqslant m \qquad \forall\, y \in \overline{D}. \tag{7.12}$$

Because $\overline{D} = \overline{\operatorname{conv} C}$, the set $\overline{D}$ is compact in X and so we can find $y_0 \in \overline{D}$, such that

$$\inf_{y \in \overline{D}} \|y\|_X = \|y_0\|_X \geqslant \frac{1}{\|u_0^*\|_X} \langle u_0^*, y_0 \rangle_X \geqslant m > 0$$

(see (7.12)), so $y_0 \neq 0$. This proves that (7.10) is true. From (7.9) and (7.10), we obtain

$$\inf_{x \in K \cap \overline{U}} \big\|\widehat{\psi}(x)\big\|_X = \eta > 0. \tag{7.13}$$

Suppose that

$$i(\varphi, K \cap U, K) \neq 0.$$

Then because of hypothesis (ii) and the homotopy invariance property, we have

$$i\big(\varphi + t\widehat{\psi}, K \cap U, K\big) \;=\; i\big(\varphi, K \cap U, K\big) \;\neq\; 0 \qquad \forall\, t > 0.$$

In particular then for $t_0 > \frac{\vartheta+\mu}{\eta}$, where

$$\vartheta \;=\; \sup_{x \in K \cap \overline{U}} \|x\|_X \quad \text{and} \quad \mu \;=\; \sup_{x \in K \cap \overline{U}} \big\|\varphi(x)\big\|_X,$$

we have

$$i\big(\varphi + t_0\widehat{\psi}, K \cap U, K\big) \;\neq\; 0$$

and so by the solution property (see Theorem 7.3.38**(g)**), we have that there exists $x_0 \in K \cap U$, such that

$$\varphi(x_0) + t_0\widehat{\psi}(x_0) \;=\; x_0.$$

Then

$$t_0 \;=\; \frac{\|x - \varphi(x_0)\|_X}{\|\widehat{\psi}(x_0)\|_X} \;\leqslant\; \frac{\vartheta + \mu}{\eta},$$

a contradiction. □

COROLLARY 7.3.43

If X is a Banach space, $K \subseteq X$ is an order cone, $U \subseteq X$ is a bounded open set, $\varphi\colon K \cap \overline{U} \longrightarrow K$ is compact and there exists $x_0 > 0$, such that

$$x - \varphi(x) \;\neq\; tx_0 \qquad \forall\, t \geqslant 0,\; x \in K \cap \partial U,$$

then

$$i\big(\varphi, K \cap U, K\big) \;=\; 0.$$

PROOF Let

$$\psi(x) \;\stackrel{df}{=}\; x_0 \qquad \forall\, x \in K \cap \partial U$$

and apply Proposition 7.3.42. □

PROPOSITION 7.3.44

If X is a Banach space, $K \subseteq X$ is an order cone, $U \subseteq X$ is a bounded open set, $\varphi\colon K \cap \overline{U} \longrightarrow K$ is compact and

(i) $\inf\big\{\big\|\varphi(x)\big\|_X :\; x \in K \cap \partial U\big\} > 0$;

(ii) $\varphi(x) = \lambda x$ *for all* $\lambda \in [0,1]$ *and* $x \in K \cap \partial U$,

then

$$i(\varphi, U, K) \;=\; 0.$$

PROOF Suppose that there exist $t_0 > 0$ and $x_0 \in K \cap \partial U$, such that

$$x_0 - \varphi(x_0) \;=\; t_0 \varphi(x_0).$$

Then

$$\varphi(x_0) \;=\; \lambda_0 x_0,$$

with $\lambda_0 = \frac{1}{1+t_0} \in [0,1]$, a contradiction to hypothesis (ii). Therefore

$$x - \varphi(x) \;\neq\; t\varphi(x) \qquad \forall\, t > 0,\; x \in K \cap \partial U.$$

So we can apply Proposition 7.3.42 with $\psi = \varphi|_{K \cap \partial U}$ and finish the proof. □

DEFINITION 7.3.45 *Let X be a Banach space, $C \subseteq X$ a nonempty, closed and convex set, $\xi\colon C \longrightarrow \mathbb{R}$ a continuous, convex function and $\vartheta\colon C \longrightarrow \mathbb{R}$ a continuous, concave function. For given $\lambda, \mu \in \mathbb{R}$, we define*

$$\begin{aligned} C(\xi, \lambda) &\stackrel{df}{=} \big\{x \in C :\; \xi(x) \leqslant \lambda\big\}, \\ C(\vartheta, \mu) &\stackrel{df}{=} \big\{x \in C :\; \vartheta(x) \geqslant \mu\big\}, \\ C(\xi, \vartheta, \lambda, \mu) &\stackrel{df}{=} C(\xi, \lambda) \cap C(\vartheta, \mu). \end{aligned}$$

PROPOSITION 7.3.46
If X is a Banach space, $C \subseteq X$ is a nonempty, closed, convex set, $\xi\colon C \longrightarrow \mathbb{R}$ is a continuous, convex function, $\vartheta\colon C \longrightarrow \mathbb{R}$ is a continuous, concave function, $\lambda, \mu \in \mathbb{R}$, the set $\big\{x \in C :\; \vartheta(x) < \mu\big\}$ is nonempty and bounded,

$$\{x \in C(\xi, \vartheta, \lambda, \mu) :\; \vartheta(x) > \mu\} \;\neq\; \emptyset,$$

the function

$$\varphi\colon \overline{\big\{x \in C :\; \vartheta(x) < \mu\big\}} \;\longrightarrow\; C$$

is compact,

$$\vartheta\big(\varphi(x)\big) \;>\; \mu \qquad \forall\, x \in C(\xi, \vartheta, \lambda, \mu)$$

and

$$\vartheta\big(\varphi(x)\big) \;>\; \mu \qquad \forall\, x \in C(\vartheta, \mu),\ \text{with } \xi\big(\varphi(x)\big) > \lambda,$$

then

$$i(\varphi, U, C) \;=\; 0,$$

where

$$U \;\stackrel{df}{=}\; \big\{x \in C :\; \vartheta(x) < \mu\big\}.$$

PROOF By hypothesis, the set U is nonempty, bounded and open in C. Let

$$u \;\in\; \big\{u \in C(\xi, \vartheta, \lambda, \mu) :\; \vartheta(u) > \mu\big\}.$$

We have

$$\xi(u) \leqslant \lambda \quad \text{and} \quad \vartheta(u) > \mu.$$

So $u \in C \setminus \overline{U}$. We claim that

$$\varphi(x) - x \neq t(x-u) \qquad \forall\, t \geqslant 0,\ x \in \partial U.$$

Indeed, if there exist $t_1 \geqslant 0$ and $x_1 \in \partial U$, such that

$$\varphi(x_1) - x_1 = t_1(x_1 - u),$$

we have

$$\vartheta(x_1) = \mu \quad \text{and} \quad x_1 = \frac{1}{1+t_1}\varphi(x_1) + \frac{t_1}{1+t_1}u.$$

If

$$\xi\big(\varphi(x_1)\big) \leqslant \lambda,$$

then exploiting the convexity of ξ, we have

$$\xi(x_1) \leqslant \frac{1}{1+t_1}\xi\big(\varphi(x_1)\big) + \frac{t_1}{1+t_1}\xi(u) \leqslant \lambda.$$

Hence by virtue of the hypotheses, we have

$$\vartheta\big(\varphi(x_1)\big) > \mu.$$

Using the concavity of ϑ, we obtain

$$\mu = \vartheta(x_1) \geqslant \frac{1}{1+t_1}\vartheta\big(\varphi(x_1)\big) + \frac{t_1}{1+t_1}\vartheta(u) > \mu,$$

a contradiction. If

$$\xi\big(\varphi(x_1)\big) > \lambda,$$

then from the hypotheses, we have

$$\vartheta\big(\varphi(x_1)\big) > \mu.$$

So

$$\mu = \vartheta(x_1) \geqslant \frac{1}{1+t_1}\vartheta\big(\varphi(x_1)\big) + \frac{t_1}{1+t_1}\vartheta(u) > \mu,$$

again a contradiction. So the hypotheses of Proposition 7.3.41 are satisfied and we have

$$i(\varphi, U, C) = 0.$$

□

PROPOSITION 7.3.47
If X is a Banach space, $C \subseteq X$ is a nonempty, closed, convex set, $\xi\colon C \longrightarrow \mathbb{R}$ is a continuous, convex map, $\vartheta\colon C \longrightarrow \mathbb{R}$ is a continuous, concave function, $\lambda, \mu \in \mathbb{R}$, the set $\big\{x \in C :\ \xi(x) < \lambda\big\}$ is nonempty and bounded,

$$\big\{x \in C(\xi, \vartheta, \lambda, \mu) :\ \xi(x) < \lambda\big\} \neq \emptyset,$$

the map $\varphi\colon \overline{\big\{x \in C :\ \xi(x) < \lambda\big\}} \longrightarrow C$ is compact,

$$\xi\big(\varphi(x)\big) \ < \ \lambda \qquad \forall\ x \in C(\xi, \vartheta, \lambda, \mu)$$

and

$$\xi\big(\varphi(x)\big) \ < \ \lambda \qquad \forall\ x \in C(\xi, \lambda),\ \text{with } \vartheta\big(\varphi(x)\big) < \mu,$$

then

$$i(\varphi, U, C) \ = \ 1,$$

where

$$U \ \stackrel{df}{=} \ \big\{x \in X :\ \xi(x) < \lambda\big\}.$$

PROOF By hypothesis, the set U is nonempty, bounded and open in C. Let

$$u \ \in \ \big\{u \in C(\xi, \vartheta, \lambda, \mu) :\ \xi(u) < \lambda\big\}.$$

We have

$$\xi(u) \ < \ \lambda \quad \text{and} \quad \vartheta(u) \ \geqslant \ \mu.$$

So $u \in U$. We claim that

$$\varphi(x) - x \ \neq \ t(x - u) \qquad \forall\ t \geqslant 0,\ x \in \partial U.$$

Indeed, if there exist $t_1 \geqslant 0$ and $x_1 \in \partial U$, such that

$$\varphi(x_1) - x_1 \ = \ t_1(x_1 - u),$$

we have

$$\xi(x_1) = \lambda \quad \text{and} \quad x_1 = \frac{1}{1 + t_1}\varphi(x_1) + \frac{t_1}{1 + t_1}u.$$

If $\vartheta\big(\varphi(x_1)\big) \geqslant \mu$, then exploiting the concavity of ϑ, we have

$$\vartheta(x_1) \ \geqslant \ \frac{1}{1 + t_1}\vartheta\big(\varphi(x_1)\big) + \frac{t_1}{1 + t_1}\vartheta(u) \ \geqslant \ \mu.$$

Hence by virtue of the hypotheses, we have

$$\xi\big(\varphi(x_1)\big) \ < \ \lambda.$$

Using the convexity of ξ, we obtain

$$\lambda \ = \ \xi(x_1) \ \leqslant \ \frac{1}{1 + t_1}\xi\big(\varphi(x_1)\big) + \frac{t_1}{1 + t_1}\xi(u) \ < \ \lambda,$$

a contradiction.

If $\vartheta\big(\varphi(x_1)\big) < \mu$, then from the hypotheses, we have

$$\xi\big(\varphi(x_1)\big) \;<\; \lambda.$$

So

$$\lambda \;=\; \xi(x_1) \;\leqslant\; \frac{1}{1+t_1}\xi\big(\varphi(x_1)\big) + \frac{t_1}{1+t_1}\xi(u) \;<\; \lambda,$$

again a contradiction. So the hypotheses of Proposition 7.3.40 are satisfied and we have

$$i(\varphi, U, C) = 1.$$

□

Using these degree theoretical tools, we can prove some more existence and multiplicity results for fixed points, based on the order structure of X. We start with the existence results which employ conditions of cone expansion and compression.

THEOREM 7.3.48

If X is a Banach space, $K \subseteq X$ is an order cone, $U_1, U_2 \subseteq X$ are two nonempty, bounded open sets, such that

$$0 \in U_1 \quad \textit{and} \quad \overline{U}_1 \subseteq U_2,$$

$\varphi\colon K \cap \big(\overline{U}_2 \setminus U_1\big) \longrightarrow K$ is a compact map and one of the following two conditions holds:

(i) $\varphi(x) \not\geqslant x$ for all $x \in K \cap \partial U_1$ and $\varphi(x) \not\leqslant x$ for all $x \in K \cap \partial U_2$; or

(ii) $\varphi(x) \not\leqslant x$ for all $x \in K \cap \partial U_1$ and $\varphi(x) \not\geqslant x$ for all $x \in K \cap \partial U_2$,

then φ has at least one fixed point in $K \cap \big(U_2 \setminus \overline{U}_1\big)$.

PROOF Invoking Theorem 3.1.12, we can find a compact map

$$\widehat{\varphi}\colon K \cap \overline{U}_2 \;\longrightarrow\; K,$$

such that

$$\widehat{\varphi}\big|_{K\cap(\overline{U}_2\setminus U_1)} \;=\; \varphi.$$

First suppose that hypothesis (i) is in effect (i.e., we have cone expansion). Then

$$\varphi(x) \;\neq\; \lambda x \qquad \forall\, \lambda \geqslant 1,\ x \in K \cap \partial U_1. \tag{7.14}$$

Indeed, if there exist $\lambda_0 \geqslant 1$ and $x_0 \in K \cap \partial U_1$, such that

$$\varphi(x_0) \;=\; \lambda_0 x_0,$$

then

$$\varphi(x_0) \ \geqslant \ x_0,$$

which contradicts hypothesis (i). So (7.14) is true. Considering the homotopy

$$h(t,x) \ \stackrel{df}{=} \ t\widehat{\varphi}(x) \qquad \forall\ (t,x) \in [0,1] \times \big(K \cap \overline{U}_1\big),$$

we see that

$$h(t,x) \ \neq \ x \qquad \forall\ (t,x) \in [0,1] \times \partial U_1$$

and so by the homotopy invariance property, we obtain

$$i\big(\widehat{\varphi}, K \cap U_1, K\big) \ = \ 1. \tag{7.15}$$

On the other hand let $u_0 > 0$ be arbitrary. We have

$$x - \varphi(x) \ \neq \ tu_0 \qquad \forall\ t \geqslant 0,\ x \in K \cap \partial U_2. \tag{7.16}$$

Indeed, if there exist $t_1 \geqslant 0$ and $x_1 \in K \cap \partial U_2$, such that

$$x_1 - \varphi(x_1) \ = \ t_1 u_0,$$

then

$$\varphi(x_1) \ \leqslant \ x_1,$$

which contradicts hypothesis (i). So (7.16) is true and we can use Corollary 7.3.43 and infer that

$$i\big(\widehat{\varphi}, K \cap U_2, K\big) \ = \ 0. \tag{7.17}$$

From (7.15), (7.17) and the additivity property of the fixed point index, we have

$$i\big(\widehat{\varphi}, K \cap \big(U_2 \setminus \overline{U}_1\big), K\big) \ = \ i\big(\widehat{\varphi}, K \cap U_2, K\big) - i\big(\widehat{\varphi}, K \cap U_1, K\big) \ = \ -1. \tag{7.18}$$

Because of (7.18) and the solution property of the fixed point index, we can find $x \in U_2 \setminus \overline{U}_1$, such that $\varphi(x) = x$.

If hypothesis (ii) is in effect, a similar reasoning gives

$$i\big(\widehat{\varphi}, K \cap U_1, K\big) \ = \ 0 \quad \text{and} \quad i\big(\widehat{\varphi}, K \cap U_2, K\big) \ = \ 1$$

and so from the additivity property of the fixed point index, it follows that

$$i\big(\widehat{\varphi}, K \cap \big(U_2 \setminus \overline{U}_1\big), K\big) \ = \ 1. \tag{7.19}$$

Therefore because of the solution property of the fixed point index, from (7.19) we can assert the existence of $x \in U_2 \setminus \overline{U}_1$, such that $\varphi(x) = x$. □

THEOREM 7.3.49
If X is a Banach space, $K \subseteq X$ is an order cone, $U_1, U_2 \subseteq X$ are two nonempty, bounded, open sets, such that

$$0 \in U_1 \quad \text{and} \quad \overline{U}_1 \subseteq U_2,$$

$\varphi\colon K \cap \left(\overline{U}_2 \setminus U_1\right) \longrightarrow K$ is a compact map and one of the following two conditions holds:

(i) $\left\|\varphi(x)\right\|_X \leqslant \|x\|_X$ *for all* $x \in K \cap \partial U_1$ *and* $\left\|\varphi(x)\right\|_X \geqslant \|x\|_X$ *for all* $x \in K \cap \partial U_2$*; or*

(ii) $\left\|\varphi(x)\right\|_X \geqslant \|x\|_X$ *for all* $x \in K \cap \partial U_1$ *and* $\left\|\varphi(x)\right\|_X \leqslant \|x\|_X$ *for all* $x \in K \cap \partial U_2$,

then φ has at least one fixed point in $K \cap \left(\overline{U}_2 \setminus U_1\right)$.

PROOF We will do the proof when condition (i) is in effect. The argument is similar if condition (ii) holds. Invoking Theorem 3.1.12, we can find a compact map $\widehat{\varphi}\colon K \cap \overline{U}_2 \longrightarrow K$, such that

$$\widehat{\varphi}\big|_{K \cap (\overline{U}_2 \setminus U_1)} \;=\; \varphi.$$

We may assume that φ has no fixed points on $K \cap \partial U_2$ and $K \cap \partial U_1$. We claim that

$$\varphi(x) \;\neq\; \lambda x \qquad \forall\, \lambda \geqslant 1,\; x \in K \cap \partial U_1.$$

Indeed, suppose that there exist $\lambda_0 \geqslant 1$ and $x_0 \in K \cap \partial U_1$, such that

$$\varphi(x_0) \;=\; \lambda_0 x_0.$$

Then

$$\left\|\varphi(x_0)\right\|_X \;=\; \lambda_0 \left\|x_0\right\|_X \;>\; \left\|x_0\right\|_X,$$

which contradicts (i). So from the homotopy invariance of the fixed point index, we have

$$i\big(\widehat{\varphi}, K \cap U_1, K\big) \;=\; 1. \tag{7.20}$$

Also we claim that

$$\varphi(x) \;=\; \lambda x \qquad \forall\, x \in K \cap \partial U_2,\; \lambda \in (0, 1].$$

Indeed, suppose that there exist $\lambda_1 \in (0, 1)$ and $x_1 \in K \cap \partial U_2$, such that

$$\varphi(x_1) \;=\; \lambda_1 x_1.$$

Then

$$\left\|\varphi(x_1)\right\|_X \;=\; \lambda_1 \left\|x_1\right\|_X \;<\; \left\|x_1\right\|_X,$$

contradicting condition (i). Moreover, because of (i), we have

$$\inf_{x\in K\cap U_2} \big\|\varphi(x)\big\|_X \;\geqslant\; \inf_{x\in K\cap \partial U_2} \|x\|_X \;>\; 0.$$

Invoking Proposition 7.3.44, we obtain

$$i\big(\widehat{\varphi}, K\cap U_2, K\big) \;=\; 0. \tag{7.21}$$

Then from (7.19) and (7.21) and the additivity property of the fixed point index, we have

$$i\big(\widehat{\varphi}, K\cap \big(\overline{U}_2\setminus U_1\big), K\big) \;=\; i\big(\widehat{\varphi}, K\cap U_2, K\big) - i\big(\widehat{\varphi}, K\cap U_1, K\big) \;=\; -1 \;\neq\; 0.$$

The solution property of the fixed point index implies the existence of at least one fixed point on $\overline{U}_2\setminus U_1$. □

THEOREM 7.3.50

If X is a Banach space, $C\subseteq X$ is a nonempty, bounded, closed, convex set, $U_1, U_2\subseteq C$ are both relatively open, $\overline{U}_1\cap\overline{U}_2=\emptyset$, $\varphi\colon C\longrightarrow C$ is a compact map, such that

(i) there exists $u_1\in U_1$, such that

$$\varphi(x)-x \;\neq\; \lambda(x-u_1) \qquad \forall\ \lambda\geqslant 0,\ x\in\partial U_1;$$

(ii) there exists $u_2\in U_2$, such that

$$\varphi(x)-x \;\neq\; \lambda(x-u_2) \qquad \forall\ \lambda\geqslant 0,\ x\in\partial U_2,$$

then φ has at least three fixed points $x_1, x_2, x_3\in C$, such that $x_1\in U_1$, $x_2\in U_2$ and $x_3\in C\setminus\big(\overline{U_1\cup U_2}\big)$.

PROOF We have $i(\varphi, C, C)=1$. Also from Proposition 7.3.40, it follows that

$$i(\varphi, U_1, C) \;=\; i(\varphi, U_2, C) \;=\; 1.$$

So exploiting the additivity of the fixed point index, we have

$$\begin{aligned} i\big(\varphi, C\setminus\big(\overline{U_1\cup U_2}\big), C\big) \;&=\; i(\varphi, C, C) - i(\varphi, U_1, C) - i(\varphi, U_2, C) \\ &=\; 1-1-1 \;=\; -1 \;\neq\; 0. \end{aligned}$$

Therefore there exist $x_1\in U_1$, $x_2\in U_2$ and $x_3\in C\setminus\big(\overline{U_1\cup U_2}\big)$, such that

$$\varphi(x_k) \;=\; x_k \qquad \forall\ k\in\big\{1,2,3\big\}.$$

□

COROLLARY 7.3.51
If X is a Banach space, $C \subseteq X$ is a nonempty, bounded, closed, convex set, $\xi\colon C \longrightarrow X$ is a continuous, convex function, $\vartheta\colon C \longrightarrow X$ is a continuous, concave function, $\lambda, \mu \in \mathbb{R}$,

$$U_1 \stackrel{df}{=} \big\{x \in C :\ \xi(x) < \lambda\big\} \neq \emptyset,$$
$$U_2 \stackrel{df}{=} \big\{x \in C :\ \vartheta(x) > \mu\big\} \neq \emptyset,$$

$\overline{U}_1 \cap \overline{U}_2 = \emptyset$ and $\varphi\colon C \longrightarrow C$ is compact and

(i) $\xi\big(\varphi(x)\big) < \lambda$ for all $x \in C$ with $\xi(x) = \lambda$;

(ii) $\vartheta\big(\varphi(x)\big) > \mu$ for all $x \in C$ with $\vartheta(x) = \mu$,

then φ has at least three distinct fixed points $x_1, x_2, x_3 \in C$, such that

$$\xi(x_1) \ < \ \lambda, \quad \mu \ < \ \vartheta(x_2), \quad \lambda \ < \ \xi(x_3) \quad \text{and} \quad \vartheta(x_3) < \mu.$$

PROOF Choose $u_1 \in U_1$. We have $\xi(u_1) < \lambda$. Suppose that there exist $t \geqslant 0$ and $x \in \partial U_1$, such that $\varphi(x) - x = t(x - u_1)$. Then

$$\xi(x) \ = \ \lambda \quad \text{and} \quad x \ = \ \frac{1}{1+t}\varphi(x) + \frac{1}{1+t}u_1.$$

Exploiting the convexity of ξ and hypothesis (i), we obtain that

$$\lambda \ = \ \xi(x) \ \leqslant \ \frac{1}{1+t}\xi\big(\varphi(x)\big) + \frac{t}{1+t}\xi(u_1) \ < \ \lambda,$$

a contradiction. So condition (i) in Theorem 7.3.50 holds. Similarly, choosing $u_2 \in U_2$, we can show that condition (ii) in Theorem 7.3.50 holds. Applying Theorem 7.3.50, we obtain at least three distinct fixed points for φ. □

THEOREM 7.3.52
If X is a Banach space, $K \subseteq X$ is an order cone, $\varphi\colon K \longrightarrow K$ is a compact map, $\xi\colon K \longrightarrow \mathbb{R}$ is a continuous convex map, $\vartheta\colon K \longrightarrow \mathbb{R}$ is a continuous, concave map, $\lambda, \mu \in \mathbb{R}$ and

(i) $0 \in \big\{x \in K :\ \vartheta(x) < \mu\big\}$ and the set $\big\{x \in K :\ \vartheta(x) < \mu\big\}$ is bounded;

(ii) $\big\{x \in K(\xi, \vartheta, \lambda, \mu) :\ \vartheta(x) > \mu\big\} \neq 0$ and $\vartheta\big(\varphi(x)\big) > \mu$ for all $x \in K(\xi, \vartheta, \lambda, \mu)$;

(iii) $\vartheta\big(\varphi(x)\big) > \mu$ for all $x \in K(\vartheta, \mu)$ with $\xi\big(\varphi(x)\big) > \lambda$;

(iv) $i(\varphi, K_r, K) = 1$ for $r > 0$ small enough and $i(\varphi, K_R, K) = 1$ for $R > 0$ large enough, where for any $\eta > 0$, $K_\eta \stackrel{df}{=} K \cap B_\eta$,

then φ has at least three fixed points in K.

PROOF Let

$$U \stackrel{df}{=} \big\{x \in K : \ \vartheta(x) < \mu\big\}.$$

Then from Proposition 7.3.46, we have that

$$i(\varphi, U, K) \ = \ 0.$$

Because

$$0 \ \in \ \big\{x \in K : \ \vartheta(x) < \mu\big\}$$

and the set $\big\{x \in K : \ \vartheta(x) < \mu\big\}$ is bounded (see hypothesis (i)), we can find $r > 0$ small enough and $R > 0$ large enough, such that

$$\overline{K}_r \ \subseteq \ U \ \subseteq \ \overline{U} \ \subseteq \ K_R.$$

Using the additivity property of the fixed point index, we have

$$i\big(\varphi, U \setminus \overline{K}_r, K\big) \ = \ i(\varphi, U, K) - i(\varphi, K_r, K) \ = \ 0 - 1 \ = \ -1 \ \neq \ 0$$

and

$$i\big(\varphi, K_R \setminus \overline{U}, K\big) \ = \ i(\varphi, K_R, K) - i(\varphi, U, K) \ = \ 1 - 0 \ = \ 1 \ \neq \ 0.$$

So φ has at least three fixed points $x_1, x_2, x_3 \in K$, such that

$$x_1 \in K_r, \quad x_2 \in U \setminus \overline{K}_r \quad \text{and} \quad x_3 \in K_R \setminus \overline{U}.$$

□

Similarly using Proposition 7.3.47, we obtain the following multiplicity result.

THEOREM 7.3.53
If X is a Banach space, $K \subseteq X$ is an order cone, $\varphi\colon K \longrightarrow K$ is a compact map, $\xi\colon K \longrightarrow \mathbb{R}$ is a continuous, convex function, $\vartheta\colon K \longrightarrow \mathbb{R}$ is a continuous, concave function, $\lambda, \mu \in \mathbb{R}$ and

(i) $0 \in \big\{x \in K : \ \xi(x) < \lambda\big\}$ *and the set* $\big\{x \in K : \ \xi(x) < \lambda\big\}$ *is bounded;*

(ii) $\big\{x \in K(\xi, \vartheta, \lambda, \mu) : \ \xi(x) < \lambda\big\} \neq \emptyset$ *and* $\xi\big(\varphi(x)\big) < \lambda$ *for all* $x \in K(\xi, \vartheta, \lambda, \mu)$;

(iii) $\xi\big(\varphi(x)\big) < \lambda$ *for all* $x \in K(\xi, \lambda)$ *with* $\vartheta\big(\varphi(x)\big) < \mu$;

(iv) $i(\varphi, K_r, K) = 0$ *for* $r > 0$ *small enough and* $i(\varphi, K_R, K) = 0$ *for* $R > 0$ *large enough, where for any* $\eta > 0$, $K_\eta \stackrel{df}{=} K \cap B_\eta$,

then φ has at least three fixed points in K.

THEOREM 7.3.54
If X is a Banach space, $C \subseteq X$ is a nonempty, bounded, closed, convex set, $\gamma, \xi\colon C \longrightarrow \mathbb{R}$ are continuous, convex functions, $\vartheta, \eta\colon C \longrightarrow \mathbb{R}$ are continuous, concave functions,

$$\vartheta(x) \;\leqslant\; \xi(x) \qquad \forall\, x \in C,$$

$\varphi\colon C \longrightarrow C$ is a compact map, $\lambda, \mu, \nu, \varrho \in \mathbb{R}$ with $\nu < \lambda$ and the following conditions hold:

(i) $\big\{x \in C(\gamma, \vartheta, \lambda, \mu) :\; \vartheta(x) > \mu\big\} \neq \emptyset$ *and* $\vartheta\big(\varphi(x)\big) > \mu$ *for all* $x \in C(\gamma, \vartheta, \lambda, \mu)$;

(ii) $\big\{x \in C(\xi, \eta, \lambda, \mu) :\; \xi(x) < \lambda\big\} \neq \emptyset$ *and* $\xi\big(\varphi(x)\big) < \lambda$ *for all* $x \in C(\xi, \eta, \nu, \varrho)$;

(iii) $\vartheta\big(\varphi(x)\big) > \mu$ *for all* $x \in C(\vartheta, \mu)$ *with* $\gamma\big(\varphi(x)\big) > \lambda$;

(iv) $\xi\big(\varphi(x)\big) < \lambda$ *for all* $x \in C(\xi, \nu)$ *with* $\eta\big(\varphi(x)\big) < \varrho$,

then φ has at least three fixed points $x_1, x_2, x_3 \in C$ with

$$\xi(x_1) \;<\; \nu, \quad \mu \;<\; \vartheta(x_2), \quad \nu \;<\; \xi(x_3) \quad \text{and} \quad \vartheta(x_3) \;<\; \mu.$$

PROOF Let

$$U_1 \;\stackrel{df}{=}\; \big\{x \in C :\; \xi(x) < \nu\big\}.$$

Then from hypotheses (ii) and (iv) and Proposition 7.3.47, we have that

$$i(\varphi, U_1, C) \;=\; 1.$$

Also let

$$U_2 \;\stackrel{df}{=}\; \big\{x \in C :\; \vartheta(x) > \mu\big\}.$$

Then from hypotheses (i) and (iii) and Proposition 7.3.47, we have that

$$i(\varphi, U_2, C) \;=\; 1.$$

Since $\xi \leqslant \vartheta$ and $\nu < \lambda$, we have

$$\overline{U}_1 \cap \overline{U}_2 \;=\; \emptyset.$$

Therefore

$$\begin{aligned} i\big(\varphi, C \setminus \big(\overline{U_1 \cup U_2}\big)\big) \;&=\; i\big(\varphi, C, C\big) - i(\varphi, U_1, C) - i(\varphi, U_2, C) \\ &=\; 1 - 1 - 1 \;=\; -1 \;\neq\; 0. \end{aligned}$$

So we can find $x_1 \in U_1$, $x_2 \in U_2$ and $x_3 \in C \setminus \big(\overline{U_1 \cup U_2}\big)$, such that

$$\varphi(x_k) \;=\; x_k \qquad \forall\, k \in \big\{1, 2, 3\big\}.$$

□

THEOREM 7.3.55 (Amann Three Fixed Points Theorem) *If X is a Banach space, $K \subseteq X$ is an order cone which is normal and solid, $y_1, u_1, y_2, u_2 \in X$ are such that*

$$y_1 \ < \ u_1 \ < \ y_2 \ < \ u_2$$

and $\varphi\colon [y_1, y_2] \longrightarrow X$ is compact and strongly increasing (i.e., if $x < y$, then $\varphi(x) \ll \varphi(y)$; see Remark 7.3.14) and

$$y_1 \ \leqslant \ \varphi(y_1), \quad \varphi(u_1) \ < \ u_1, \quad y_2 \ < \ \varphi(y_2), \quad \varphi(u_2) \ \leqslant \ u_2,$$

then φ has at least three fixed points $x_1, x_2, x_3 \in [y_1, y_2]$, such that

$$y_1 \ \leqslant \ x_1 \ \ll \ u_1, \quad y_2 \ \ll \ x_2 \ \leqslant \ u_2, \quad y_2 \ \not\leqslant \ x_3 \ \not\leqslant \ u_1.$$

PROOF Let

$$C \stackrel{df}{=} [y_1, y_2].$$

Then C is nonempty, closed, convex and bounded (due to the normality of K; see Proposition 7.3.18). Let

$$u_1' \stackrel{df}{=} \varphi(u_1), \quad y_2' \stackrel{df}{=} \varphi(y_2).$$

Because φ is strongly increasing, we have

$$y_1 \ \ll \ u_1' \ \ll \ y_2' \ \ll \ u_2.$$

Let $\xi, \vartheta\colon C \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \xi(x) &\stackrel{df}{=} \inf\big\{t \geqslant 0 : \ x - y_1 \leqslant t(u_1' - y_1)\big\}, \\ \vartheta(x) &\stackrel{df}{=} \sup\big(t \geqslant 0 : \ x - y_1 \leqslant t(y_2' - y_1)\big). \end{aligned}$$

Clearly

$$\vartheta(x) \ < \ +\infty \qquad \forall\, x \in C$$

and ξ is convex, while ϑ is concave.

Claim 1. ξ and ϑ are continuous functions.

We show that ξ is continuous. The proof of the continuity of ϑ is similar. Suppose that $\{x_n\}_{n \geqslant 1} \subseteq X$ is a sequence, such that

$$x_n \ \longrightarrow \ x,$$

for some $x \in X$. Let

$$e \stackrel{df}{=} u_1' - y_1.$$

Then

$$x - y_1 \ \leqslant \ \xi(x)e.$$

For a given $\varepsilon > 0$, since $\varepsilon e \gg 0$, we can find $\delta > 0$, such that

$$B_\delta(\varepsilon e) \ = \ \big\{x \in X : \ \|x - \varepsilon e\|_X < \delta\big\} \ \subseteq \ K.$$

We can find $N \geqslant 1$, such that

$$\varepsilon e + (x - x_n) \ \in \ B_\delta(\varepsilon e) \ \subseteq \ K \qquad \forall \ n \geqslant N.$$

So it follows that

$$x_n - y_1 \ = \ x_n - x + x - y_1 - \varepsilon e + \varepsilon e \ \leqslant \ \big(\xi(x) + \varepsilon\big)e \qquad \forall \ n \geqslant N,$$

hence

$$\xi(x_n) \ \leqslant \ \xi(x) + \varepsilon \qquad \forall \ n \geqslant N$$

and thus

$$\limsup_{n \to +\infty} \xi(x_n) \ \leqslant \ \xi(x). \tag{7.22}$$

On the other hand assume that

$$\liminf_{n \to +\infty} \xi(x_n) \ = \ t \ \geqslant \ -\infty.$$

We can find a subsequence $\{n_k\}_{k \geqslant 1} \subseteq \{n\}$, such that $\xi(x_{n_k}) \longrightarrow t$. Since

$$0 \ \leqslant \ x_{n_k} - y_1 \ \leqslant \ \xi(x_{n_k})e$$

($e = u_1' - y_1 \gg 0$), we obtain that $t > -\infty$. Also $x_{n_k} - y_1 \leqslant \xi(x_{n_k})e$ implies that

$$x - y_1 \ \leqslant \ te,$$

hence

$$\xi(x) \ \leqslant \ t,$$

i.e.,

$$\xi(x) \ \leqslant \ \liminf_{n \to +\infty} \xi(x_n). \tag{7.23}$$

From (7.22) and (7.23), we infer the continuity of ξ. So the Claim is proved.

Next let

$$U_1 \ \stackrel{df}{=} \ \big\{x \in C : \ \xi(x) < 1\big\} \quad \text{and} \quad U_2 \ \stackrel{df}{=} \ \big\{x \in C : \ \vartheta(x) > 1\big\}.$$

Both sets are open and nonempty, since $y_1 \in U_1$ and $u_2 \in U_2$. We will show that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Indeed, for any $x \in \overline{U}_1$, we have $\xi(x) \leqslant 1$, hence

$$x - y_1 \ \leqslant \ \xi(x)(u_1' - y_1) \ \leqslant \ u_1' - y_1$$

and so $x \leqslant u_1'$, i.e., $x \in [y_1, u_1']$. Then $\overline{U}_1 \subseteq [y_1, u_1']$. On the other hand, for any $x \in [y_1, u_1']$, we have that $x - y_1 \leqslant u_1' - y_1$ and so $\xi(x) \leqslant 1$. Also

$$\xi\left(\frac{1}{n}y_1 + \left(1 - \frac{1}{n}\right)x\right) \;\leqslant\; \frac{1}{n}\xi(y_1) + \left(1 - \frac{1}{n}\right)\xi(x) \;<\; 1,$$

so $\xi(x) \leqslant 1$, i.e., $x \in \overline{U}_1$ and finally $[y_1, u_1'] \subseteq \overline{U}_1$, i.e., $\overline{U}_1 = [y_1, u_1']$.

Similarly, we have that $\overline{U}_2 = [y_2', u_2]$. Note that $[y_1, u_1'] \cap [y_2', u_2] = \emptyset$.

Finally we will show that hypotheses (i) and (ii) of Corollary 7.3.51 hold. For $x \in C$ with $\xi(x) = 1$, we have $x \leqslant u_1'$. Then

$$\varphi(x) \;\leqslant\; \varphi(u_1') \;\ll\; u_1',$$

hence

$$\varphi(x) - y_1 \;\ll\; u_1' - y_1$$

and so there exists $\delta_1 > 0$, such that

$$\varphi(x) - y_1 \;\leqslant\; (1 - \delta_1)(u_1' - y_1).$$

This implies that

$$\xi\big(\varphi(x)\big) \;\leqslant\; (1 - \delta_1) \;<\; 1.$$

Similarly for $x \in C$, with $\vartheta(x) = 1$, we have $y_2' \leqslant x$. Then

$$y_2' \;\ll\; \varphi(y_2') \;\leqslant\; \varphi(x),$$

hence

$$y_2' - y_1 \;\leqslant\; \varphi(x) - y_1$$

and so there exists $\delta_2 > 0$, such that

$$(1 + \delta_2)(y_2' - y_1) \;\leqslant\; \varphi(x) - y_1.$$

This implies that

$$1 \;<\; (1 + \delta_2) \;\leqslant\; \vartheta\big(\varphi(x)\big).$$

Therefore all the hypotheses of Corollary 7.3.51 are satisfied. Accordingly we obtain at least three fixed points x_1, x_2, x_3 of φ, such that

$$y_1 \;\leqslant\; x_1 \;\ll\; u_1, \quad y_2 \;\ll\; x_2 \;\leqslant\; u_2 \quad \text{and} \quad y_2 \;\not\leqslant\; x_3 \;\not\leqslant\; u_1.$$

□

REMARK 7.3.56 In the above proof we have expressed the boundary of the order interval $[y_1, y_2]$ in terms of a convex and of a concave functions, both continuous.

7.4 Fixed Points of Multifunctions

Most of the known metric and topological fixed point results for single valued maps can be extended to the case of set valued maps (multifunctions). In this section we present a few such characteristic generalizations. We will make an effort to have this section self contained. For further details on the theory of multifunctions we refer to Hu & Papageorgiou (1997, 2000).

First let us fix our notation concerning hyperspaces (sets of sets). If X is a Hausdorff topological space, we set

$$\begin{aligned} P_f(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, closed}\}, \\ \widehat{P}_f(X) &\stackrel{df}{=} P_f(X) \cup \{\emptyset\}, \\ P_k(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, compact}\}. \end{aligned}$$

If X is a normed space, we also have

$$\begin{aligned} P_{fc}(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, closed, convex}\}, \\ P_{bf}(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, bounded, closed}\}, \\ P_{bf(c)}(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, bounded, closed (and convex)}\}, \\ P_{(w)kc}(X) &\stackrel{df}{=} \{A \subseteq X : \ A \text{ is nonempty, (weakly-)compact, convex}\}. \end{aligned}$$

In what follows, for a Hausdorff topological space X, if $x \in X$, then by $\mathcal{N}(x)$ we denote the filter of neighbourhoods of x. Also if (X, d_X) is a metric space, $r > 0$ and $x \in X$, then

$$B_r(x) \stackrel{df}{=} \{y \in X : \ d_X(x, y) < r\}.$$

Moreover, if X is a normed space as before

$$B_r \stackrel{df}{=} \{y \in X : \ \|y\|_X < r\}$$

and

$$B_r(x) \stackrel{df}{=} \{y \in X : \ \|y - x\|_X < r\}.$$

Finally, we will use the convention that in a metric space (X, d_X),

$$d_X(x, \emptyset) = +\infty \qquad \forall\, x \in X.$$

DEFINITION 7.4.1 *Let (X, d_X) be a metric space and $A, C \in 2^X \setminus \{\emptyset\}$. We set*

$$h(A,C) \stackrel{df}{=} \max\left\{\sup_{a\in A} d_X(a,C),\ \sup_{c\in C} d_X(c,A)\right\},$$

*where $h(A,C) = +\infty$ is allowed. The extended real number $h(A,C)$ is called the **Hausdorff distance** between A and C relative to the metric d_X.*

REMARK 7.4.2 It is easy to see that h satisfies the triangle inequality and that

$$h(A,C) = 0 \iff \overline{A} = \overline{C}.$$

So h is a metric on $P_f(X)$, known as the ***Hausdorff metric***. Moreover, $(\widehat{P}_f(X), h)$ is also a metric space and $\emptyset$ is an isolated point of it. If d_X is bounded, then so is h on $P_f(X)$. It can be shown that if (X, d_X) is a complete metric, then so is $(P_f(X), h)$ and $P_k(X)$ is a closed subspace of it. Moreover, if X is a Banach space, then $P_{fc}(X)$, $P_{bf(c)}(X)$ and $P_{kc}(X)$ are also closed subspaces of $(P_f(X), h)$ (hence they are complete for the Hausdorff metric). If (X, d_X) is separable, then so is $(P_k(X), h)$. Therefore, if (X, d_X) is a Polish space (see Definition A.2.29**(a)**), then so is $(P_k(X), h)$. From Definition 7.4.1, it is clear that

$$h(A,C) = \inf\{\varepsilon > 0 :\ A \subseteq C_\varepsilon \text{ and } C \subseteq A_\varepsilon\},$$

where for any $D \in 2^X \setminus \{\emptyset\}$ and any $\varepsilon > 0$,

$$D_\varepsilon \stackrel{df}{=} \{y \in X :\ d_X(y,D) < \varepsilon\} = \bigcup_{y\in D} B_\varepsilon(y).$$

Therefore

$$h(A,C) = \sup\{|d_X(y,A) - d_X(y,C)| :\ y \in X\} \qquad \forall\ A, C \in 2^X \setminus \{\emptyset\}.$$

Finally, if X is a normed space and $A, C \in P_{bfc}(X)$, then the following ***Hörmander's formula*** holds

$$h(A,C) = \sup\{|\sigma_X(y^*, A) - \sigma_X(y^*, C)| :\ \|y^*\|_{X^*} \leqslant 1\},$$

where σ_X is the so-called ***support function***, defined by

$$\sigma_X(y^*, D) \stackrel{df}{=} \sup\{\langle y^*, y\rangle_X :\ y \in D\} \qquad \forall\ D \in 2^X \setminus \{\emptyset\}.$$

We start with a set valued generalization of Banach's contraction principle (see Theorem 7.1.2).

THEOREM 7.4.3 (Nadler Fixed Point Theorem) *If (X, d_X) is a complete metric space and $F\colon X \longrightarrow P_f(X)$ is an h-contraction (i.e., $h(F(x), F(y)) \leqslant k d_X(x,y)$ for all $x, y \in X$ with $k \in [0,1)$), then F has a fixed point, i.e., there exists $x \in X$, such that $x \in F(x)$.*

PROOF Choose $k_1 \in (k, 1)$ and $x_0 \in X$. Then we pick $x_1 \in F(x_0)$ with

$$d_X(x_0, x_1) \ > \ 0.$$

If no such x_1 exists, then x_0 is a fixed point of F and we are done. We have

$$d_X\big(x_1, F(x_1)\big) \ \leqslant \ h\big(F(x_0), F(x_1)\big) \ < \ k_1 d_X(x_0, x_1)$$

and so we can find $x_2 \in F(x_1)$, such that

$$d_X(x_1, x_2) \ < \ k_1 d_X(x_0, x_1).$$

Inductively, we produce a sequence $\{x_n\}_{n \geqslant 1}$, such that

$$x_{n+1} \ \in \ F(x_n) \qquad \forall \ n \geqslant 1$$

and

$$d_X(x_n, x_{n+1}) \ < \ k_1^n d_X(x_0, x_1) \qquad \forall \ n \geqslant 1. \tag{7.24}$$

From the inequality in (7.24), it follows that $\{x_n\}_{n \geqslant 1} \subseteq X$ is a Cauchy sequence and so

$$x_n \ \longrightarrow \ x \quad \text{in } X,$$

for some $x \in X$. Then

$$d_X\big(x_n, F(x)\big) \ \leqslant \ h\big(F(x_n), F(x)\big) \ < \ k d_X(x_n, x) \ \longrightarrow \ 0.$$

So

$$d_X\big(x, F(x)\big) \ = \ 0$$

and because $F(x)$ is closed, we have $x \in F(x)$. □

REMARK 7.4.4 In contrast to Theorem 7.1.2, the fixed point $x \in X$ in Theorem 7.4.3 is not unique. Indeed, if

$$F(x) \ = \ X \qquad \forall \ x \in X,$$

then every $x \in X$ is a fixed point of F. Evidently the set of fixed points of F (denoted by $\operatorname{Fix}(F)$) is closed.

We have a stability result for multivalued contractions with bounded values.

PROPOSITION 7.4.5
If (X, d_X) is a complete metric space, $F_1, F_2 \colon X \longrightarrow P_{bf}\big(X\big)$ are h-contractions with the same constant $k \in [0, 1)$ and $\operatorname{Fix}(F_i)$ denotes the set of fixed points of F_i (for $i = 1, 2$),
then

$$h\big(\operatorname{Fix}(F_1), \operatorname{Fix}(F_2)\big) \ \leqslant \ \frac{1}{1-k} \sup_{x \in X} h\big(F_1(x), F_2(x)\big).$$

PROOF Let $\varepsilon > 0$ and choose $\xi > 0$, such that

$$\xi \sum_{n=1}^{\infty} nk^n \ < \ 1.$$

Let us set

$$\varepsilon_1 \ = \ \xi\varepsilon \frac{1}{1-k}.$$

Pick $x_0 \in \mathrm{Fix}\,(F_1)$ and then choose $x_1 \in F_2(x_0)$, such that

$$d_{_X}(x_0, x_1) \ \leqslant \ h\big(F_1(x_0), F_2(x_0)\big) + \varepsilon. \tag{7.25}$$

Because

$$h\big(F_2(x_1), F_2(x_0)\big) \ \leqslant \ kd_{_X}(x_1, x_0),$$

we can find $x_2 \in F_2(x_1)$, such that

$$d_{_X}(x_2, x_1) \ \leqslant \ kd_{_X}(x_1, x_0) + k\varepsilon_1.$$

Inductively, we construct a sequence $\{x_n\}_{n\geqslant 1}$, such that

$$x_{n+1} \ \in \ F_2(x_n) \qquad \forall\ n \geqslant 1 \tag{7.26}$$

and

$$d_{_X}(x_{n+1}, x_n) \ \leqslant \ kd_{_X}(x_n, x_{n+1}) + k^n\varepsilon_1 \qquad \forall\ n \geqslant 1. \tag{7.27}$$

From the inequality (7.27), we obtain

$$d_{_X}(x_{n+1}, x_n) \ \leqslant \ k^n d_{_X}(x_1, x_0) + nk^n\varepsilon_1,$$

so

$$\sum_{n=m}^{\infty} d_{_X}(x_{n+1}, x_n) \ \leqslant \ \frac{k^m}{1-k} d_{_X}(x_1, x_0) + \varepsilon_1 \sum_{n=m}^{\infty} nk^n \tag{7.28}$$

and thus $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence. So we have

$$x_n \ \longrightarrow \ x \quad \text{in } X,$$

for some $x \in X$ and from (7.26), it follows that $x \in \mathrm{Fix}\,(F_2)$. Moreover, from (7.28) and (7.25), we have

$$\begin{aligned} d_{_X}(x_0, x) \ &\leqslant \ \sum_{n=0}^{\infty} d_{_X}(x_n, x_{n+1}) \ \leqslant \ \frac{1}{1-k} d_{_X}(x_0, x_1) + \varepsilon_1 \sum_{n=1}^{\infty} nk^n \\ &\leqslant \ \frac{1}{1-k}\big(h\big(F_1(x_0), F_1(x_1)\big) + 2\varepsilon\big). \end{aligned}$$

Revising the roles of F_1 and F_2 in the above argument we conclude that

$$h\big(\mathrm{Fix}\,(F_1), \mathrm{Fix}\,(F_2)\big) \ \leqslant \ \frac{1}{1-k} \sup_{x\in X} h\big(F_1(x), F_2(x)\big).$$

□

COROLLARY 7.4.6
If (X, d_X) *is a complete metric space,* $F_n, F\colon X \longrightarrow P_{bf}\big(X\big)$ *for* $n \geqslant 1$ *are* h*-contractions with the same constant* $k \in [0,1)$ *and*

$$\sup_{x\in X} h\big(F_n(x), F(x)\big) \;\longrightarrow\; 0,$$

then

$$h\big(\mathrm{Fix}\,(F_n), \mathrm{Fix}\,(F)\big) \;\longrightarrow\; 0.$$

Next we prove some topological fixed point theorems. We start with a definition.

DEFINITION 7.4.7 *Let* X *be a vector space.*

(a) *A subset* $C \subseteq X$ *is said to be* ***finitely closed****, if its intersection with every finite dimensional flat* $Y \subseteq X$ *(i.e.,* $Y = x + L$ *for some* $x \in X$ *and some finite dimensional subspace* L *of* X*) is closed in the Euclidian topology of* Y*.*

(b) *A family* $\{C_i\}_{i\geqslant 1}$ *of sets of* X *is said to have the* ***finite intersection property****, if the intersection of each finite subfamily in not empty.*

(c) *Let* $C \subseteq X$ *be a nonempty set and let* $F\colon C \longrightarrow 2^X$ *be a multifunction. We say that* F *is a* ***Knaster-Kuratowski-Mazurkiewicz multifunction*** *(a* ***KKM-multifunction*** *for short), if for every finite set* $\{x_k\}_{k=1}^m \subseteq C$*, we have*

$$\mathrm{conv}\,\{x_k\}_{k=1}^m \;\subseteq\; \bigcup_{k=1}^m F(x_k).$$

The basic property of KKM-multifunctions is given in the next theorem.

THEOREM 7.4.8
If X *is a vector space,* $C \subseteq X$ *is nonempty and* $F\colon C \longrightarrow 2^X$ *is a KKM-multifunction, such that for each* $x \in C$*,* $F(x)$ *is finitely closed,*
then the family $\big\{F(x) :\; x \in C\big\}$ *has the finite intersection property.*

PROOF We proceed indirectly. Suppose that

$$\bigcap_{k=1}^n F(x_k) \;=\; \emptyset,$$

for some $n \geqslant 1$ and some $x_1, \ldots, x_n \in C$. Let

$$Y \;=\; \mathrm{span}\,\{x_k\}_{k=1}^n,$$

let d_Y be the Euclidian metric in Y and

$$D \;=\; \operatorname{conv}\{x_k\}_{k=1}^n \subseteq Y.$$

By hypothesis, each set $Y \cap F(x_k)$ is closed. Because

$$\bigcap_{k=1}^n \big(Y \cap F(x_k)\big) \;=\; \emptyset,$$

we see that the function

$$\xi(x) \;\stackrel{df}{=}\; \sum_{k=1}^n d_Y\big(x, Y \cap F(x_k)\big)$$

is not zero for any $x \in D$. So we can define the continuous function

$$\vartheta\colon D \;\longrightarrow\; D$$

by

$$\vartheta(x) \;\stackrel{df}{=}\; \frac{1}{\xi(x)} \sum_{k=1}^n d_Y\big(x, Y \cap F(x_k)\big)x_k.$$

By Brouwer's fixed point theorem (see Theorem 7.2.7), we can find $x_0 \in D$, such that

$$\vartheta(x_0) \;=\; x_0.$$

Let

$$J \;\stackrel{df}{=}\; \big\{k :\; d_Y\big(x_0, Y \cap F(x_k)\big) \neq 0\big\}.$$

Then

$$x_0 \;\notin\; \bigcup_{k\in J} F(x_k).$$

But

$$x_0 \;=\; \vartheta(x_0) \;\in\; \operatorname{conv}\{x_k\}_{k\in J} \;\subseteq\; \bigcup_{k\in J} F(x_k)$$

(since F is a KKM-multifunction), a contradiction. □

An immediate consequence of the above theorem is the following result.

THEOREM 7.4.9

If X is a Hausdorff topological space, $C \subseteq X$ is a nonempty set, $F\colon C \longrightarrow 2^X$ is a KKM-multifunction, such that for each $x \in C$, $F(x)$ is closed and for at least one $x_0 \in C$, $F(x_0)$ is compact,
then

$$\bigcap_{x\in C} F(x) \neq \emptyset.$$

We can have the same conclusion in a different setting, which avoids imposing a compactness condition on the sets $F(x)$ and instead uses an auxiliary multifunction.

THEOREM 7.4.10
If X is a vector space, $C \subseteq X$ is a nonempty set, $F\colon C \longrightarrow 2^X$ is a KKM-multifunction, $G\colon C \longrightarrow 2^X$ is another multifunction, such that

$$G(x) \subseteq F(x) \qquad \forall\, x \in C$$

and

$$\bigcap_{x\in C} G(x) = \bigcap_{x\in C} F(x)$$

and there is another topology on X, such that $G(x)$ is compact for every $x \in C$, then

$$\bigcap_{x\in C} F(x) \neq \emptyset.$$

REMARK 7.4.11 KKM-multifunctions can be used to produce an alternative proof of the Schauder-Tychonoff fixed point theorem (see Theorem 7.2.14). For details we refer to Dugundji & Granas (1982, pp. 74–75).

LEMMA 7.4.12
If X is a vector space, $C \subseteq X$ is a nonempty set and $F\colon C \longrightarrow 2^X$ is a multifunction, such that

$$x \notin \operatorname{conv} F(x) \qquad \forall\, x \in C,$$

then

$$x \longmapsto G(x) = X \setminus F^{-1}(x) \text{ is a KKM-multifunction.}$$

PROOF Let $\{x_k\}_{k=1}^n \subseteq X$. We have

$$X \setminus \bigcup_{k=1}^{n} G(x_k) = \bigcap_{k=1}^{n} \big(X \setminus G(x_k)\big) = \bigcap_{k=1}^{n} F^{-1}(x_k).$$

It follows that $y \in X$ satisfies

$$y \notin \bigcup_{k=1}^{n} G(x_k) \iff y \in \bigcap_{k=1}^{n} F^{-1}(x_k)$$

and this is equivalent to

$$y \in C \quad \text{and} \quad x_k \in F(y) \qquad \forall\, k \in \{1, \ldots, n\}.$$

Now, let

$$y \in \operatorname{conv}\{x_k\}_{k=1}^n.$$

We claim that

$$y \in \bigcup_{k=1}^n G(x_k).$$

If this is not the case, then from the previous argument, we know that

$$y \in C \quad \text{and} \quad x_k \in F(y) \qquad \forall\, k \in \{1, \ldots, n\}.$$

Therefore

$$\operatorname{conv}\{x_k\}_{k=1}^n \subseteq \operatorname{conv} F(y),$$

so, from the choice of y,

$$y \in \operatorname{conv} F(y),$$

a contradiction.

Therefore, we have

$$\operatorname{conv}\{x_k\}_{k=1}^n \subseteq \bigcup_{k=1}^n G(x_k)$$

and so we conclude that G is a KKM-multifunction. □

This lemma can be used to establish the existence of maximal elements for binary relations, which are irreflexive but not necessarily transitive. The result is useful in mathematical economics.

PROPOSITION 7.4.13

If X is a Hausdorff topological vector space, $C \subseteq X$ is a nonempty, compact, convex set, $\prec$ is an irreflexive binary relation on C and

(i) for every $x \in C$, $x \notin \operatorname{conv}\{y \in C : x \prec y\}$;

(ii) for every $x \in C$, the lower section $\{y \in C : y \prec x\}$ is open in C,

then the set of $\prec$-maximal elements in C is nonempty and compact.

PROOF Let

$$F(x) \stackrel{df}{=} \{y \in C : x \prec y\} \qquad \forall\, x \in C.$$

Because of hypothesis (i) and Lemma 7.4.12,

$$x \longmapsto G(x) = X \setminus F^{-1}(x) \text{ is a KKM-multifunction.}$$

Hence $S\colon C \longrightarrow 2^C$, defined by

$$S(x) \stackrel{df}{=} C \cap G(x) = C \setminus F^{-1}(x),$$

is a KKM-multifunction too. Moreover, because of hypothesis (ii),

$$\text{the set } F^{-1}(x) \text{ is open in } C$$

and so S has compact values. Applying Theorem 7.4.9, we infer that

$$\bigcap_{x\in C} \big(C \setminus F^{-1}(x)\big) \;=\; \bigcap_{x\in C} S(x) \;\neq\; \emptyset$$

and it is compact.

But the set of $\prec$-maximal elements in C is just $\bigcap\limits_{x\in C} \big(C \setminus F^{-1}(x)\big)$. □

The above theorem leads to the following existence result important in the theory of variational inequalities.

THEOREM 7.4.14

If X is a locally convex space, $C \subseteq X$ is a nonempty, compact and convex set and

$$u\colon C \;\longrightarrow\; X^*$$

is a map, such that the function

$$(x,y) \;\longmapsto\; \big\langle u(x), y\big\rangle_X$$

is jointly continuous on $C \times C$,
then there exists $x \in C$, such that

$$\big\langle u(x), y-x\big\rangle_X \;\geqslant\; 0 \qquad \forall\, y \in C.$$

PROOF On C we define the irreflexive relation $\prec$ by

$$x \prec y \quad\Longleftrightarrow\quad \big\langle u(x), x-y\big\rangle_X > 0.$$

Note that for every $x \in C$, we have

$$\begin{aligned} x \notin \operatorname{conv}\big\{y \in C :\; x \prec y\big\} \;&=\; \big\{y \in C :\; x \prec y\big\} \\ &=\; \big\{y \in C :\; \big\langle u(x), x-y\big\rangle_X > 0\big\}. \end{aligned}$$

Moreover, the function

$$(x,y) \;\longmapsto\; \big\langle u(y), x-y\big\rangle_X$$

is jointly continuous. So

$$\big\{y \in C :\; y \prec x\big\} \;=\; \big\{y \in C :\; \big\langle u(y), x-y\big\rangle_X < 0\big\}$$

is open. Therefore we can apply Proposition 7.4.13 and obtain a $\prec$-maximal element $x \in C$. We have

$$\big\langle u(x), y-x\big\rangle_X \;\geqslant\; 0 \qquad \forall\, y \in C.$$

□

We can use the above theorem to obtain topological fixed point theorems for multifunctions. First a definition.

DEFINITION 7.4.15 *Let X be a vector space and $C \subseteq X$ be a nonempty set. We say that the multifunction $F\colon C \longrightarrow 2^X \setminus \{\emptyset\}$ is* ***weakly inward****, if*

$$F(x) \cap \overline{I_C(x)} \neq \emptyset \qquad \forall\, x \in C,$$

where $I_C(x)$ is the ***inward set*** *of $x \in C$ with respect to C, defined by*

$$I_C(x) \stackrel{df}{=} \{x + \lambda(y - x) :\ \lambda \geqslant 0 \text{ and } y \in C\}.$$

REMARK 7.4.16 If F maps C into itself, then F is inward, since for any $y \in F(x)$ and for $\lambda = 1$, we have

$$x + \lambda(y - x) = x + y - x = y \in C.$$

Using the above notion we can prove a topological fixed point theorem for multifunctions.

THEOREM 7.4.17
If X is a locally convex space, $C \subseteq X$ is nonempty, compact, convex and $F\colon C \longrightarrow P_{fc}(X)$ is a weakly inward multifunction which is upper semicontinuous (see Definition 3.2.12),
then we can find $x \in C$, such that

$$x \in F(x).$$

PROOF We proceed by contradiction. Suppose that F has no fixed point. For a given $x \in C$,

$$0 \notin x - F(x)$$

and so by the strong separation theorem for convex sets (see Theorem A.3.2), we can find $x^* \in X^* \setminus \{0\}$, such that

$$\langle x^*, x - y \rangle_X < 0 \qquad \forall\, y \in F(x),$$

i.e.,

$$\sigma_X\big(x^*; x - F(x)\big) < 0.$$

Let us set

$$U(x^*) \stackrel{df}{=} \{x \in C :\ \sigma_X\big(x^*; x - F(x)\big) < 0\} \qquad \forall\, x^* \in X^* \setminus \{0\}.$$

Since F is upper semicontinuous, $U(x^*)$ is open and

$$\{U(x^*)\}_{x^* \in X^* \setminus \{0\}} \text{ is a cover of } C.$$

So we can find a finite subcover $\{U(x_k^*)\}_{k=1}^n$ and a corresponding continuous partition of unit $\{\xi_k\}_{k=1}^n$. Let us set

$$u(x) = \sum_{k=1}^{n} \xi_k(x) x_k^*.$$

Evidently

$$\text{the function } (x,y) \longmapsto \langle u(x), y\rangle_X \text{ is jointly continuous on } C \times C.$$

So we can apply Theorem 7.4.14 and obtain $x_0 \in C$, such that

$$\langle u(x_0), y - x_0\rangle_X \geqslant 0 \qquad \forall\, y \in C.$$

By hypothesis there is

$$y \in F(x_0) \cap \overline{I_C(x_0)}.$$

So

$$y = x_0 + \lim_{n\to+\infty} \lambda_n (y_n - x_0),$$

with $\lambda_n \geqslant 0$ and $y_n \in C$. Then

$$y - x_0 = \lim_{n\to+\infty} \lambda_n (y_n - x_0)$$

and so

$$\langle u(x_0), y - x_0\rangle_X = \lim_{n\to+\infty} \lambda_n \langle u(x_0), y_n - x_0\rangle \geqslant 0,$$

a contradiction to the fact that

$$\sigma_X\big(u(x); x - F(x)\big) < 0 \qquad \forall\, x \in C.$$

□

REMARK 7.4.18 It is easy to see that in the above theorem the set $\operatorname{Fix}(F)$ is compact.

Combining Theorem 7.4.17 with Remarks 7.4.16 and 7.4.18 and recalling that for locally compact multifunctions, upper semicontinuity and closedness of the graph are equivalent (see Remark 3.2.13), we obtain the following topological fixed point theorem, which is a multivalued generalization of Theorem 7.2.14.

THEOREM 7.4.19 (Kakutani-Ky Fan Fixed Point Theorem) *If X is a locally convex space, $C \subseteq X$ is a nonempty, compact and convex set and $F\colon C \longrightarrow P_{fc}(C)$ is a multifunction with closed graph, then the set of fixed points of F is nonempty and compact.*

Using ideas from the proof of Theorem 7.4.17, we can prove a third topological fixed point theorem for multifunctions.

THEOREM 7.4.20
If X is a locally convex space, $C \subseteq X$ is a nonempty, compact, convex set and $F\colon C \longrightarrow 2^C \setminus \{\emptyset\}$ is a multifunction with convex values, such that for each $y \in C$, the set

$$F^+(\{y\}) \;=\; \big\{x \in C :\; y \in F(x)\big\}$$

is open,
then there exists $x \in C$, such that

$$x \;\in\; F(x).$$

PROOF The family $\big\{F^+(\{y\})\big\}_{y\in C}$ is an open cover of C. So we can find a finite subcover $\big\{F^+(\{y_k\})\big\}_{k=1}^n$ and a corresponding continuous partition of unity $\{\xi_k\}_{k=1}^n$. Then, if we set

$$u(x) \;\stackrel{df}{=}\; \sum_{k=1}^n \xi_k(x) y_k \qquad \forall\, x \in C,$$

then $u\colon C \longrightarrow C$ is a continuous selector of F. Applying Theorem 7.2.14, we obtain $x \in C$, such that

$$x \;=\; u(x) \;\in\; F(x).$$

□

Next let X, Y be two Banach spaces and let $C \subseteq X$, $D \subseteq Y$ be nonempty, closed and convex sets. In what follows by (D, w) we denote the set D furnished with the relative weak topology of Y. We consider multifunctions $G\colon C \longrightarrow 2^C \setminus \{\emptyset\}$, which admit the following decomposition

$$G \;=\; K \circ N, \tag{7.29}$$

where $N\colon C \longrightarrow 2^D \setminus \{\emptyset\}$ is an upper semicontinuous multifunction into (D, w) and has weakly compact and convex values and $K\colon (D, w) \longrightarrow C$ is sequentially continuous, i.e.,

$$\text{if} \quad y_n \xrightarrow{w} y \quad \text{in } D, \quad \text{then} \quad K(y_n) \longrightarrow K(y) \quad \text{in } C.$$

We also assume that G is compact, i.e., maps bounded sets onto relatively compact sets. If by $\mathcal{D}$ we denote the family of triples (G, U, C), where G and C are as above and $U \subseteq C$ is a nonempty, bounded, relatively open set, such that $\mathrm{Fix}\,(G) \cap \partial U = \emptyset$, then on $\mathcal{D}$ we can define a fixed point index similar to the one introduced in Definition 7.3.36. For details we refer to Bader (2001). In particular we have the following result.

THEOREM 7.4.21
There exists a map $i\colon \mathcal{D} \longrightarrow \mathbb{Z}$, *such that*

(a) *Normalization: If* K *in (7.29) is a constant map, i.e.,* $K(y) = v_0 \notin \partial U$ *for all* $y \in Y$, *then*

$$i(G,U,C) \;=\; \begin{cases} 1 & \text{if} \quad v_0 \in U, \\ 0 & \text{if} \quad v_0 \notin U. \end{cases}$$

(b) *Additivity: If* $\operatorname{Fix}(G) \cap U \subseteq U_1 \cup U_2$, *where* U_1 *and* U_2 *are disjoint open subsets of* U, *then*

$$i(G,U,C) \;=\; i(G,U_1,C) + i(G,U_2,C).$$

(c) *Homotopy invariance: Let* $F\colon C \longrightarrow 2^C \setminus \{\emptyset\}$ *be a multifunction with a decomposition* $F = S \circ L$ *as in (7.29) and assume that* G *and* F *are homotopic in the following sense: "There exists an upper semicontinuous multifunction* $H\colon [0,1] \times C \longrightarrow 2^D \setminus \{\emptyset\}$ *(*D *with the relative weak topology) with weakly compact, convex values, such that* $H(0,\cdot) = N$ *and* $H(1,\cdot) = L$ *and a sequentially continuous map* $u\colon [0,1] \times (D,w) \longrightarrow C$, *such that* $u(0,\cdot) = K$ *and* $u(1,\cdot) = S$." *Let us set*

$$\Psi(t,x) \;\stackrel{df}{=}\; u\big(t, H(t,x)\big)$$

and assume that Ψ *is compact and*

$$x \notin \Psi(t,x) \qquad \forall\ (t,x) \in [0,1] \times \partial U.$$

Then

$$i(G,U,C) \;=\; i(F,U,C).$$

(d) *Decomposition property: For any other decomposition* $G' = K' \circ N'$ *with intermediate set* D' *being a nonempty, closed, convex set of a Banach space* Y', *such that there exists a sequentially continuous map* $p\colon (D,w) \longrightarrow (D',w)$ *with* $N' = p \circ N$ *and* $K' \circ p = K$, *we have*

$$i(G,U,C) \;=\; i(G',U,C).$$

(e) *Solution property: if* $i(G,U,C) \neq 0$, *then* $\operatorname{Fix}(G) \cap U \neq \emptyset$.

REMARK 7.4.22 We emphasize that in the above theorem as well as in the ones that follow, the multifunction G need not have convex values.

Using Theorem 7.4.21, we can derive fixed point principles for multifunctions that need not have convex values.

THEOREM 7.4.23
If X is a Banach space, $G\colon \overline{B}_R \longrightarrow 2^X \setminus \{\emptyset\}$ is a compact multifunction with a decomposition as in (7.29),
then at least one of the following statements holds:

(a) *there exists $x_0 \in \partial B_R$ and $\lambda \in (0,1)$, such that $x_0 \in \lambda G(x_0)$; or*

(b) $\operatorname{Fix}(G) \neq \emptyset$.

PROOF Let $r\colon X \longrightarrow \overline{B}_R$ be a retraction and obtain a decomposition

$$G \circ r \;=\; K \circ (N \circ r).$$

Suppose that

$$\operatorname{Fix}(G) \cap \partial \overline{B}_R \;=\; \emptyset$$

(or otherwise we are done).

Then $i(G \circ r, B_R, X)$ is defined. Let $u\colon [0,1] \times (D, w) \longrightarrow X$ be defined by

$$u(t,y) \;\stackrel{df}{=}\; tK(y).$$

Then using u we see that $G \circ r$ and $D \circ (N \circ r) = 0$ are homotopic in the sense of Theorem 7.4.21 provided **(a)** is not valid. Then by the normalization property, we have

$$i(G \circ r, B_R, X) \;=\; i(0, B_R, X) \;=\; 1.$$

Then by virtue of the solution property, we can find $x \in B_R$, such that $x \in (G \circ r)(x) = G(x)$. □

Similarly we can show the following nonlinear alternative theorem.

THEOREM 7.4.24
If X is a Banach space, $G\colon C \longrightarrow 2^C \setminus \{\emptyset\}$ is a compact multifunction with a decomposition as in (7.29) and $0 \in C$,
then at least one of the following statements holds:

(a) *G has a fixed point; or*

(b) *the set $S \stackrel{df}{=} \{x \in C :\; x \in \lambda G(x)$ for some $0 < \lambda < 1\}$ is unbounded.*

REMARK 7.4.25 Theorem 7.4.24 is a multivalued generalization of Leray-Schauder alternative principle (see Theorem 7.2.16). The remarkable feature of Theorem 7.4.24 is that the multifunction G need not have convex values.

7.5 Remarks

7.1: Theorem 7.1.2 is of course due to Banach (1922) and it is an abstraction of the classical method of successive approximation due to Liouville and Picard. The renorming technique used in Example 7.1.4**(b)** was introduced by Bielecki (1956). Proposition 7.1.5 is essentially due to Schauder (1933) (he had some additional unnecessary hypotheses). There have been many variants and extensions of Banach's contraction principle. Most of them are of limited interest. In Theorem 7.1.7 we have collected the most useful descendants of Banach's fixed point theorem. Theorem 7.1.7**(a)** is an observation resulting from the proof of Theorem 7.1.2. Theorem 7.1.7**(b)** is due to Weissinger (1952), Theorem 7.1.7**(c)** is due to Edelstein (1962) and Theorem 7.1.7**(d)** is due to Kannan (1969). Theorem 7.1.9 is due to Boyd & Wong (1969). The first fixed point theorem for nonexpansive maps in a noncompact setting was proved by Browder (1965*b*) and Göhde (1965) independently. They proved that a nonexpansive self-map of a nonempty, bounded, closed, convex set of a uniformly convex Banach space has a fixed point. Soon thereafter Kirk (1965) proved the same result under the slightly weaker assumption that the Banach space X is reflexive and the set C, on which the nonexpansive map is defined, is nonempty, bounded, closed, convex and has normal structure (see Definition 7.1.18). This geometric property was introduced by Brodskii & Milman (1948) to study fixed points of isometries and it is a property shared by all uniformly convex Banach spaces. Theorem 7.1.23 is essentially due to Kirk (1965). Theorem 7.1.28 is due to Caristi (1976) and it is equivalent to Ekeland's variational principle (see Corollary 4.6.3) and to the following theorem of Takahashi (1991).

THEOREM 7.5.1

If (X, d_X) is a complete metric space, $\varphi\colon X \longrightarrow \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous, bounded below function and for every $y \in X$, such that

$$\inf_{x \in X} \varphi(x) \;<\; \varphi(y),$$

we can find $u \in X$, such that

$$u \neq y \quad \text{and} \quad \varphi(u) + d_X(y, u) \leqslant \varphi(y),$$

then there exists $\widehat{x} \in X$, such that

$$\varphi(\widehat{x}) \;=\; \inf_{x \in X} \varphi(x).$$

Detailed treatment of the metric fixed point theory can be found in the books of Dugundji & Granas (1982), Goebel & Kirk (1990), Istratescu (1981), Smart (1980) and Zeidler (1985*a*).

7.2: Proposition 7.2.5 was first observed by Borsuk (1931). The product of two compact fixed point spaces (i.e., spaces with FFP) need not be a fixed point space (see Bredon (1971)). In contrast, an infinite product of nonempty, compact fixed point spaces will be a fixed point space, provided every finite product of these spaces is a fixed point space. So by virtue of Brouwer's theorem (see Theorem 7.2.7**(b)**), the Hilbert cube $[0,1]^{\mathbb{N}}$ is a fixed point space. Theorem 7.2.7 is one of the oldest and best known results in topology. For $n = 3$ it was proved by Brouwer (1909) and for arbitrary $n \geqslant 1$ by Hadamard (1910). We should also mention similar results obtained earlier by Poincaré (1886) and Bohl (1904). Unlike Banach's contraction principle (see Theorem 7.1.2), Brouwer's theorem does not give any computational scheme for obtaining a fixed point. However, Scarf (1967), under some additional conditions, produced some sort of algorithm for computing a fixed point of a continuous map. In Knaster, Kuratowski & Mazurkiewicz (1929) we can find a simple and short proof of Brouwer's theorem based on combinatorial arguments. They also note that for a continuous map $\varphi\colon \overline{B}_1 \longrightarrow \mathbb{R}^N$ ($\overline{B}_1 = \{x \in \mathbb{R}^N : \|x\|_{\mathbb{R}^N} \leqslant 1\}$), the condition $\varphi(\partial B_1) \subseteq \overline{B}_1$ suffices for the existence of a fixed point. Example 7.2.10 is due to Kakutani (1943). Theorem 7.2.11 is due to Schauder (1930) and Theorem 7.2.13 is due to Borsuk (1931). Theorem 7.2.14 extends Theorem 7.2.11 to locally convex spaces and is due to Tychonoff (1935). Its proof can be found in Dunford & Schwartz (1958, p. 456). There is a weak sequential version of it due to Arino, Gautier & Penot (1984).

THEOREM 7.5.2
If X is a metrizable, locally convex space, $C \subseteq X$ is a nonempty, weakly compact, convex set and $\varphi\colon C \longrightarrow C$ is weakly sequentially continuous, then there exists $x \in C$, such that $\varphi(x) = x$.

Theorem 7.2.16 was first proved for Banach spaces by Leray & Schauder (1934) using degree theory. A direct elementary proof for the general case was provided by Schaefer (1955). Corollary 7.2.18**(a)** is due to Rothe (1938), and Corollary 7.2.18**(b)** is due to Altman (1955). Theorem 7.2.27 is due to Sadovskii (1972) and historical forerunners of it are the fixed theorems of Darbo (1955) and Krasnoselskii (1964*b*, 1964*a*). Finally we mention an interesting consequence of the Schauder-Tychonoff fixed point theorem (see Theorem 7.2.14), which is important in functional analysis.

THEOREM 7.5.3
If X is a locally convex space, $C \subseteq X$ is a nonempty, compact, convex set and $\mathcal{F}$ is a commuting family of continuous affine maps $h\colon C \longrightarrow C$, then $\mathcal{F}$ has a common fixed point, i.e., there exists $x \in C$, such that $h(x) = x$ for all $h \in \mathcal{F}$.

The above theorem was first proved by Markov (1936) using Theorem 7.2.14. Kakutani (1938) produced a direct elementary proof and demonstrated its importance by furnishing several applications of it.

Topological fixed point theory is treated in the books of Brown (1993), Dugundji & Granas (1982), Istratescu (1981), Smart (1980), Zeidler (1985*a*).

7.3: Theorem 7.3.8 was proved by Bourbaki (1940–1949) and Kneser (1950) independently, while Theorem 7.3.10 is due to Amann (1977). Its forerunner is Theorem 7.3.11, proved by Tarski (1955). The theory of ordered Banach spaces and the properties of the various kind of order cones (see Definition 7.3.15) can be found in the classical book of Krasnoselskii (1964*a*). The same topic is also treated in detail in Deimling (1985) and Guo & Lakshmikantham (1988). Theorems 7.3.30 and 7.3.31 can be found in Krasnoselskii (1964*a*), Krasnoselskii & Zabreiko (1984) and Amann (1977), while Theorem 7.3.33 is due to Guo & Sun (1988).

A discussion of the notion of fixed point index (see Definition 7.3.36) can be found in Deimling (1985), Guo & Lakshmikantham (1988), Lloyd (1978), Nussbaum (1971) and Zeidler (1985*a*), while fixed point theorems which employ conditions of cone expansions and compression were first obtained by Krasnoselskii (1964*a*, 1964*b*). Additional results in this direction were obtained by Amann (1976), Guo (1986, 1987), Guo & Sun (1988), Heikkilä & Lakshmikantham (1994), Leggett & Williams (1979), Sun (1991) and Sun & Sun (1986). The theory has important applications in the existence and multiplicity of positive solutions for boundary value problems.

7.4: Theorem 7.4.3 was proved by Nadler (1969) for multivalued contractions with nonempty, bounded, closed values. Soon thereafter Covitz & Nadler (1970) extended the result to multivalued contractions whose values need not be compact. The stability results established in Proposition 7.4.5 and Corollary 7.4.6 are due to Lim (1985). Concerning the structure of the set of fixed points of a multivalued contraction, Ricceri (1987) proved the following result.

THEOREM 7.5.4
If X is a Banach space, $C \subseteq X$ is a nonempty, closed, convex set and $F\colon C \longrightarrow P_{fc}(C)$ is an h-contraction,
then $\operatorname{Fix}(F)$ *is a nonempty set and it is a retract of X. In particular* $\operatorname{Fix}(F)$ *is path-connected.*

Analogous results can be found also in Bressan, Cellina & Fryszkowski (1991) and Górniewicz, Marano & Slosarski (1996). Concerning h-nonexpansive multifunction, we have the following result due to Lim (1974).

THEOREM 7.5.5
If X is a uniformly convex Banach space, $C \subseteq X$ is a nonempty, bounded, closed, convex set and $F\colon C \longrightarrow P_k(C)$ is an h-nonexpansive multifunction (i.e., $h\big(F(x), F(y)\big) \leqslant \|x - y\|_X$ for all $x, y \in C$),
then $\operatorname{Fix}(F) \neq \emptyset$.

Theorem 7.4.8 was first proved by Knaster, Kuratowski & Mazurkiewicz (1929) in the special case where C is the set of vertices of a simplex in $\mathbb{R}^N$. Their approach is combinatorial based on Sprener's lemma. The infinite dimensional version presented in Theorem 7.4.8 is due to Fan (1952). Theorem 7.4.14 is due to Browder (1976, p. 72), but the proof there is different. The finite dimensional forerunner of the result was proved by Hartman & Stampacchia (1966). Weakly inward maps (see Definition 7.4.15) were introduced by Halpern & Bergman (1968) and Theorem 7.4.17 is due to Halpern (1970). Earlier, Browder (1967) had proved a similar result under the stronger condition that $F(x) \subseteq \overline{I_C(x)}$ for all $x \in C$ (strongly inward multifunction). Theorem 7.4.19 is due to Kakutani (1941) (when $X = \mathbb{R}^N$) and Fan (1952, 1960–1961) (when X is a general locally convex space). Theorem 7.4.20 is due to Bader (2001). The fixed point index for multifunctions introduced in Theorem 7.4.21 and the multivalued fixed point principles in Theorems 7.4.23 and 7.4.24 are due to Bader (2001). Additional multivalued formulations of the Leray-Schauder alternative principle can be found in Frigon & Granas (1994) and O'Regan & Precup (2001). Further fixed point results for multifunctions can be found in the books of Border (1985), Górniewicz (1999) and Hu & Papageorgiou (1997, 2000).

Appendix

A.1 Topology

DEFINITION A.1.1 *Let X be a Hausdorff topological space.*
(a) *We say that X is a* ***locally compact space****, if every point $x \in X$ has a relatively compact neighbourhood.*

(b) *We say that X is a* ***σ-compact space****, if*

$$X \;=\; \bigcup_{n=1}^{\infty} K_n,$$

with K_n compact.

REMARK A.1.2 Every finite dimensional space is locally compact and σ-compact. In a locally compact space every point has a local base consisting of relatively compact sets. Also if X is locally compact and σ-compact, then

$$X \;=\; \bigcup_{n=1}^{\infty} K_n,$$

with K_n compact and

$$K_n \subseteq \operatorname{int} K_{n+1} \qquad \forall\; n \geqslant 1.$$

Hence

$$X \;=\; \bigcup_{n=1}^{\infty} \operatorname{int} K_n.$$

THEOREM A.1.3
<u>If</u> (X, τ) is a noncompact, locally compact space (τ is the topology of X) and

$$\begin{aligned} X_\infty &\stackrel{df}{=} X \cup \{\infty\}, \quad \textit{with } \infty \notin X, \\ \tau_\infty &\stackrel{df}{=} \tau \cup \big\{X_\infty \setminus K :\; K \subseteq X \textit{ is compact}\big\}, \end{aligned}$$

<u>then</u> τ_∞ is a Hausdorff topology on X_∞, (X_∞, τ_∞) is a compact space and X is an open dense set in X_∞.

REMARK A.1.4 The space (X_∞, τ_∞) is called the ***Alexandrov one-point compactification*** of X.

DEFINITION A.1.5 *Let (X, τ) be a Hausdorff topological space and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X. We say that $\{U_i\}_{i\in I}$ is a **refinement** of $\{V_j\}_{j\in J}$, if for every $i \in I$, we can find $j \in J$, such that*

$$U_i \subseteq V_j.$$

In this case we write

$$\{U_i\}_{i\in I} \prec \{V_j\}_{j\in J}.$$

*A refinement $\{U_i\}_{i\in I}$ of $\{V_j\}_{j\in J}$ is said to be **precise**, if $I = J$ and*

$$U_i \subseteq V_i \qquad \forall\ i \in I = J.$$

*A cover $\{U_i\}_{i\in I}$ of X is said to be **locally finite**, if every $x \in X$ has a neighbourhood which intersects only finitely many sets U_i.*

PROPOSITION A.1.6
If X is a Hausdorff topological space and the cover $\{V_j\}_{j\in J}$ has a locally finite refinement $\{U_i\}_{i\in I}$,
then it has also a precise locally finite refinement $\{W_k\}_{k\in K}$.
Moreover, if each U_i is open, then each W_k can be chosen to be open too.

DEFINITION A.1.7 **(a)** *A Hausdorff topological space X is said to be **paracompact**, if every open cover of X has a locally finite refinement.*

(b) *A **partition of unity** on a Hausdorff topological space X is a family $\{\psi_i\}_{i\in I}$ of continuous functions $\psi_i \colon X \longrightarrow [0, 1]$, such that*

$$\{\operatorname{supp}\psi_i\}_{i\in I} \text{ is a locally finite closed cover of } X$$

and

$$\sum_{i\in I} \psi_i(x) = 1 \qquad \forall\ x \in X.$$

*If $\{U_i\}_{i\in I}$ is a given open cover of X, then we say that the partition of unity $\{\psi_i\}_{i\in I}$ is **subordinated** to $\{U_i\}_{i\in I}$, if*

$$\operatorname{supp}\psi_i \subseteq U_i \qquad \forall\ i \in I.$$

REMARK A.1.8 Compact and metrizable spaces are paracompact. Moreover, paracompact spaces are normal.

THEOREM A.1.9
If X is a Hausdorff topological space,
then X is paracompact if and only if every open cover of X has a locally finite partition of unity subordinate to it.

THEOREM A.1.10 (Baire Category Theorem)
If X is a complete metric space and

$$X = \bigcup_{n=1}^{\infty} C_n,$$

with C_n being closed for every $n \geqslant 1$,
then there exists $n_0 \geqslant 1$, such that

$$\operatorname{int} C_{n_0} \neq \emptyset.$$

THEOREM A.1.11 (Cantor Intersection Theorem)
A metric space X is complete if and only if for every decreasing sequences $\{C_n\}_{n\geqslant 1}$ of closed sets with

$$\operatorname{diam} C_n \longrightarrow 0,$$

we have that

$$\bigcap_{n=1}^{\infty} C_n \text{ is a singleton.}$$

DEFINITION A.1.12 *Let X be a Hausdorff topological space.*
(a) *We say that X is **regular**, if for every nonempty, closed set $C \subseteq X$ and $x \notin C$, we can find open sets $U, V \subseteq X$, such that*

$$C \subseteq U, \quad x \in V \quad \text{and} \quad U \cap V = \emptyset.$$

(b) *We say that X is **normal**, if for every nonempty, closed sets $C_1, C_2 \subseteq X$, such that*

$$C_1 \cap C_2 = \emptyset,$$

we can find open sets U_1 and U_2, such that

$$C_1 \subseteq U_1, \quad C_2 \subseteq U_2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$

THEOREM A.1.13 (Urysohn Lemma)
A Hausdorff topological space X is normal if and only if for every pair of nonempty disjoint closed sets C_1 and C_2, we can find a continuous function

$$\varphi\colon X \longrightarrow [0,1],$$

such that

$$\varphi|_{C_1} \equiv 0 \quad \text{and} \quad \varphi|_{C_2} \equiv 1.$$

THEOREM A.1.14 (Tietze Extension Theorem)
A Hausdorff topological space is normal if and only if for every nonempty closed $A \subseteq X$ and every continuous function

$$\varphi\colon A \longrightarrow [a,b],$$

we can find a continuous function

$$\widehat{\varphi}\colon X \longrightarrow [a,b],$$

such that

$$\widehat{\varphi}\big|_{A} \;=\; \varphi.$$

DEFINITION A.1.15 *Let (X,d_{X}) and (Y,d_{Y}) be two metric spaces and let $\mathcal{F}$ be a family of continuous maps $f\colon X \longrightarrow Y$. We say that $\mathcal{F}$ is* ***equicontinuous****, if for any given $x \in X$ and $\varepsilon > 0$, we can find*

$$\delta \;=\; \delta(\varepsilon,x) \;>\; 0,$$

such that for all $z \in X$ satisfying

$$d_{X}(x,z) \;<\; \delta,$$

we have that

$$d_{Y}\big(f(x),f(z)\big) \;<\; \varepsilon \qquad \forall\; f \in \mathcal{F}.$$

If $\delta > 0$ can be chosen to be independent of $x \in X$, then we say that $\mathcal{F}$ is ***uniformly equicontinuous****,*

REMARK A.1.16 If X is compact, then every equicontinuous family $\mathcal{F}$ is uniformly equicontinuous.

THEOREM A.1.17 (Arzela-Ascoli Theorem)
If (X,d_{X}) is a compact metric space and $K \subseteq C(X)$,
then

$$K \text{ is relatively compact for the } d_{\infty}\text{-metric on } C(X),$$

if and only if

$$K \text{ is equicontinuous and uniformly bounded,}$$

where

$$d_{\infty}(f,g) \stackrel{df}{=} \sup_{x\in X}\big|f(x)-g(x)\big| \qquad \forall\; f,g \in C(X).$$

A.2 Measure Theory

THEOREM A.2.1 (Fatou Lemma)
Let (Ω, Σ, μ) be a measure space and let $\{f_n : \Omega \longrightarrow \mathbb{R}\}_{n \geqslant 1}$ be a sequence of Σ-measurable functions,
then

(a) *if*

$$f_n(\omega) \geqslant f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and

$$-\infty < \int_\Omega f(\omega)\, d\mu,$$

we have

$$\int_\Omega \liminf_{n \to +\infty} f_n(\omega)\, d\mu \leqslant \liminf_{n \to +\infty} \int_\Omega f_n(\omega)\, d\mu;$$

(b) *if*

$$f_n(\omega) \leqslant f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and

$$\int_\Omega f(\omega)\, d\mu < +\infty,$$

we have

$$\limsup_{n \to +\infty} \int_\Omega f_n(\omega)\, d\mu \leqslant \int_\Omega \limsup_{n \to +\infty} f_n(\omega)\, d\mu.$$

THEOREM A.2.2 (Lebesgue Dominated Convergence Theorem)
If (Ω, Σ, μ) is a measure space,

$$\{f_n : \Omega \longrightarrow \mathbb{R}\}_{n \geqslant 1}$$

is a sequence of Σ-measurable functions and $h : \Omega \longrightarrow \mathbb{R}$ is a μ-integrable function,

$$f_n(\omega) \longrightarrow f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega$$

and

$$|f_n(\omega)| \leqslant h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } n \geqslant 1,$$

then f is μ-integrable and

$$\int_\Omega f_n(\omega)\, d\mu \longrightarrow \int_\Omega f(\omega)\, d\mu.$$

The above two theorems can be extended using the notion of "uniform integrability."

DEFINITION A.2.3 *Let (Ω, Σ, μ) be a measure space and $K \subseteq L^1(\Omega)$. We say that K is* ***uniformly integrable****, if*

$$\lim_{\lambda \to +\infty} \left[\sup_{f \in K} \int\limits_{\{|f| \geqslant \lambda\}} \big|f(\omega)\big| \, d\mu \right] \; = \; 0.$$

REMARK A.2.4 If we have

$$\big|f(\omega)\big| \; \leqslant \; h(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } f \in K,$$

for some $h \in L^1(\Omega)_+$, then $K \subseteq L^1(\Omega)$ is uniformly integrable.

THEOREM A.2.5
If (Ω, Σ, μ) is a finite measure space and $K \subseteq L^1(\Omega)$, then K is uniformly integrable if and only if

(a) *$K \subseteq L^1(\Omega)$ is bounded; and*

(b) *for a given $\varepsilon > 0$, we can find $\delta > 0$, such that*

$$\sup_{f \in K} \int\limits_{A} \big|f(\omega)\big| \, d\mu \; \leqslant \; \varepsilon \qquad \forall \; A \in \Sigma, \; \mu(A) \leqslant \delta.$$

REMARK A.2.6 If μ is nonatomic, then **(a)** follows from **(b)**.

Recall that a set $A \in \Sigma$ is a μ-***atom***, if $\mu(A) > 0$ and for every $B \in \Sigma$ with $B \subseteq A$,

$$\mu(B) \; = \; 0 \quad \text{or} \quad \mu(B) \; = \; \mu(A).$$

A μ-atom is ***finite***, if

$$\mu(A) \; < \; +\infty.$$

A μ-atom is ***infinite***, if

$$\mu(A) \; = \; +\infty.$$

If μ possesses no atoms, then μ is said to be ***nonatomic***.

PROPOSITION A.2.7
If (Ω, Σ, μ) is a finite measure space, $K \subseteq L^1(\Omega)$ is uniformly integrable and $\overline{K}_0$ is the closure of K in the topology of almost everywhere convergence, then K_0 is uniformly integrable.

THEOREM A.2.8 (Extended Fatou Lemma)
If (Ω, Σ, μ) *is a finite measure space and* $\{f_n\}_{n \geqslant 1} \subseteq L^1(\Omega)$ *is a uniformly integrable sequence,*
then

$$\int_\Omega \liminf_{n \to +\infty} f_n(\omega)\, d\mu \leqslant \liminf_{n \to +\infty} \int_\Omega f_n(\omega)\, d\mu$$
$$\leqslant \limsup_{n \to +\infty} \int_\Omega f_n(\omega)\, d\mu \leqslant \int_\Omega \limsup_{n \to +\infty} f_n(\omega)\, d\mu.$$

THEOREM A.2.9 (Vitali Theorem; Extended Dominated Convergence Theorem)
If (Ω, Σ, μ) *is a finite measure space,* $\{f_n\}_{n \geqslant 1} \subseteq L^1(\Omega)$ *is uniformly integrable and*

$$f_n \xrightarrow{\mu} f,$$

then $f \in L^1(\Omega)$ *and*

$$\int_\Omega f_n(\omega)\, d\mu \longrightarrow \int_\Omega f(\omega)\, d\mu.$$

THEOREM A.2.10 (Monotone Convergence Theorem)
If (Ω, Σ, μ) *is a measure space,* $\{f_n \colon \Omega \longrightarrow \mathbb{R}\}_{n \geqslant 1}$ *is a sequence of* Σ*-measurable functions, such that*

$$h(\omega) \leqslant f_n(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega \text{ and all } n \geqslant 1,$$

with

$$-\infty < \int_\Omega h(\omega)\, d\mu$$

and

$$f_n(\omega) \nearrow f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then

$$\int_\Omega f_n(\omega)\, d\mu \nearrow \int_\Omega f(\omega)\, d\mu.$$

THEOREM A.2.11 (Lusin Theorem)
If X *is a complete, separable metric space,* Y *is a separable metric space,* $f \colon X \longrightarrow Y$ *is a Borel measurable function and* μ *is a finite Borel measure on* X*,*
then for a given $\varepsilon > 0$*, we can find a compact set* $K_\varepsilon \subseteq X$*, such that*

$$\mu(K_\varepsilon^c) \leqslant \varepsilon \quad \text{and} \quad f|_{K_\varepsilon} \text{ is continuous.}$$

THEOREM A.2.12 (Egorov Theorem)
If (Ω, Σ, μ) *is a finite measure space and* $\{f_n \colon \Omega \longrightarrow \mathbb{R}\}_{n \geqslant 1}$ *is a sequence of* Σ*-measurable functions,* $f \colon \Omega \longrightarrow \mathbb{R}$ *is a* Σ*-measurable function and*

$$f_n(\omega) \;\longrightarrow\; f(\omega) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

then for a given $\varepsilon > 0$*, we can find* $A \in \Sigma$ *with* $\mu(A^c) < \varepsilon$*, such that*

$$f_n \;\longrightarrow\; f \quad \text{uniformly on } A.$$

THEOREM A.2.13 (Saks Theorem)
If (Ω, Σ, μ) *is a finite measure space and* $\varepsilon > 0$*,*
then Ω *is the union of a finite sequence of pairwise disjoint sets* $\{A_k\}_{k=1}^m \subseteq \Sigma$*, such that* A_k *is an atom or* $\mu(A_k) \leqslant \varepsilon$*.*

REMARK A.2.14 According to Theorem A.2.13, every finite measure space can have at most a countable family of pairwise disjoint atoms.

DEFINITION A.2.15 *Let* $f \colon [a, b] \longrightarrow \mathbb{R}$ *be a function.*

(a) *The* ***total variation*** *of* f *over* $[a, b]$ *is defined by*

$$V_a^b(f) \;\stackrel{df}{=}\; \sup\left\{ \sum_{k=1}^{\infty} \big|f(x_k) - f(x_{k-1})\big| : \; a = x_0 < x_1 < \ldots < x_n = b \right\}.$$

If

$$V_a^b(f) \;<\; +\infty,$$

then we say that f *is of* ***bounded variation*** *over* $[a, b]$*. By* $BV\big([a, b]\big)$*, we denote the space of functions of bounded variation on* $[a, b]$*. If* $T \subseteq \mathbb{R}$ *is any interval, then* $f \colon T \longrightarrow \mathbb{R}$ *is* ***locally of bounded variation****, denoted by* $f \in BV_{\text{loc}}(T)$*, if for all* $a, b \in T$*, we have that* $f \in BV\big([a, b]\big)$*.*

(b) *We say that* f *is* ***absolutely continuous****, if for every* $\varepsilon > 0$*, we can find* $\delta > 0$*, such that if* $\{[c_n, d_n]\}_n$ *is any finite or countable collection of pairwise disjoint, closed intervals in* $[a, b]$ *with*

$$\sum_n (d_n - c_n) < \delta,$$

then

$$\sum_n \big|f(d_n) - f(c_n)\big| \;<\; \varepsilon.$$

By $AC\big([a, b]\big)$ *we denote the space of functions which are absolutely continuous on* $[a, b]$*. If* $T \subseteq \mathbb{R}$ *is any interval, then* $f \colon T \longrightarrow \mathbb{R}$ *is* ***locally absolutely continuous****, denoted by* $f \in AC_{\text{loc}}(T)$*, if for all* $a, b \in T$*, we have that* $f \in AC\big([a, b]\big)$*.*

REMARK A.2.16 Evidently, if $f \in AC\big([a,b]\big)$, then $f \in BV\big([a,b]\big)$ and it is uniformly continuous.

PROPOSITION A.2.17
$f \in BV\big([a,b]\big)$ if and only if f is the difference of two increasing functions. Moreover, if f is also continuous, then both increasing functions can be chosen to be continuous.

PROPOSITION A.2.18
If $f \in BV_{\rm loc}(\mathbb{R})$,
then f' exists almost everywhere, $f' \in L^1_{\rm loc}(\mathbb{R})$ and

$$\int_a^b f'(t)\,dt \;\leqslant\; f(b) - f(a) \qquad \forall\ a,b \in \mathbb{R}.$$

REMARK A.2.19 If $h \in L^1_{\rm loc}(\mathbb{R})$, $c \in \mathbb{R}$ and

$$f(t) \;=\; c + \int_0^t h(s)\,ds \qquad \forall\ s \in \mathbb{R},$$

then $f \in BV_{\rm loc}(\mathbb{R})$ and $f'(t) = h(t)$ for almost all $t \in \mathbb{R}$. Moreover,

$$V_a^b(f) \;=\; \int_a^b \big|h(s)\big|\,ds \;=\; \int_a^b \big|f'(s)\big|\,ds.$$

THEOREM A.2.20 (Fundamental Theorem of Lebesgue Calculus)
If $f \in AC_{\rm loc}(\mathbb{R})$,
then f' exists almost everywhere, belongs in $L^1_{\rm loc}(\mathbb{R})$ and

$$f(t) \;=\; f(a) + \int_a^t f'(s)\,ds \qquad \forall\ t \in \mathbb{R}.$$

THEOREM A.2.21
$f \in AC\big([a,b]\big)$ if and only if $f \in BV\big([a,b]\big) \cap C\big([a,b]\big)$ and for every Lebesgue-null set $C \subseteq [a,b]$, we have

$$\lambda^1\big(f(C)\big) \;=\; 0$$

(λ^1 being the Lebesgue measure on $\mathbb{R}$).

DEFINITION A.2.22 *Let (Ω, Σ, μ) be a measure space and let ν be a signed measure on Σ. We say that ν is **absolutely continuous**, with respect to μ (denoted by $\nu \prec\prec \mu$), if*

$$\mu(A) \;=\; 0 \text{ implies that } \nu(A) \;=\; 0.$$

REMARK A.2.23 If ν is a finite signed measure on Σ, then $\nu \prec\prec \mu$ if and only if for every $\varepsilon > 0$, we can find $\delta > 0$, such that

$$\big|\nu(A)\big| \;<\; \varepsilon \qquad \forall\; A \in \Sigma,\; \mu(A) < \delta.$$

THEOREM A.2.24 (Radon-Nikodym Theorem)
If (Ω, Σ, μ) is a σ-finite measure space, ν is a σ-finite signed measure on Σ and $\nu \prec\prec \mu$,
then there exists a Σ-measurable function $f\colon \Omega \longrightarrow \mathbb{R}$, such that

$$\nu(A) \;=\; \int\limits_A f(\omega)\, d\mu \qquad \forall\; A \in \Sigma.$$

REMARK A.2.25 The function f is called the ***Radon-Nikodym derivative*** of ν with respect to μ and usually is denoted by $\frac{d\nu}{d\mu}$. The Radon-Nikodym derivative is unique modulo a μ-null set. Moreover, if μ and ν are finite, then $f \in L^1(\Omega; \mu)$.

THEOREM A.2.26 (Jensen Inequality)
If (Ω, Σ, μ) is a finite measure space, $J \subseteq \mathbb{R}$ is an interval, $f \in L^1(\Omega)$, $f(\Omega) \subseteq J$, $\varphi\colon J \longrightarrow \mathbb{R}$ is a convex function and $\varphi \circ f \in L^1(\Omega)$,
then

$$\varphi\Big(\frac{1}{\mu(\Omega)} \int\limits_\Omega f(\omega)\, d\mu\Big) \;\leqslant\; \frac{1}{\mu(\Omega)} \int\limits_\Omega \big(\varphi \circ f\big)(\omega)\, d\mu.$$

THEOREM A.2.27 (Hölder Inequality)
If (Ω, Σ, μ) is a measure space, $p \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$, $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$,
then $fg \in L^1(\Omega)$ and

$$\|fg\|_1 \;\leqslant\; \|f\|_p \, \|g\|_{p'}\,.$$

More generally, if $f_k \in L^{p_k}(\Omega)$, $k \in \{1, \ldots, m\}$ with $\frac{1}{p} = \sum\limits_{k=1}^{m} \frac{1}{p_k} \leqslant 1$, then $f_1 \cdot \ldots \cdot f_m \in L^p(\Omega)$ and

$$\|f_1 \cdot \ldots \cdot f_m\|_p \;\leqslant\; \|f_1\|_{p_1} \ldots \|f_m\|_{p_m}\,.$$

THEOREM A.2.28 (Interpolation Inequality)
If (Ω, Σ, μ) *is a measure space,* $f \in L^p(\Omega) \cap L^q(\Omega)$ *with* $p, q \in [1, +\infty]$, $p \leqslant q$,
then $f \in L^r(\Omega)$ *for all* $r \in [p, q]$ *and*

$$\|f\|_r \ \leqslant \ \|f\|_p^t \, \|f\|_q^{1-t},$$

with $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$ $(t \in [0, 1])$.

DEFINITION A.2.29 **(a)** *A topological space* (X, τ) *is said to be a* ***Polish space****, if it is separable and there exists a metric on* X *for which the topology* τ *is complete.*

(b) *A Hausdorff topological space* X *is said to be a* ***Souslin space****, if there exists a Polish space* Y *and a continuous surjection from* Y *to* X.

REMARK A.2.30 Closed and open subsets of Polish spaces are Polish and so are countable products and countable intersections. Every locally compact, σ-compact metrizable space is Polish and a subspace of a Polish space is itself Polish if and only if it is a G_δ-set of its completion (Alexandrov-Mazurkiewicz theorem). Closed and open subsets of Souslin spaces are Souslin and so are countable products, countable intersections and countable unions of Souslin spaces. Let X be a separable Banach space furnished with the weak topology (denoted by X_w). If $\dim X_w = +\infty$, then X_w is a nonmetrizable, Souslin space. Similarly, let $X^*_{w^*}$ be the dual of a separable Banach space X, equipped with the w^*-topology. Then $X^*_{w^*}$ is a Souslin space. The Souslin subspaces of a Polish space are called ***analytic sets***.

THEOREM A.2.31
If Z, X *are two Polish spaces,* $S \in \mathcal{B}(Z \times X)$ *and for all* $z \in Z$,

$$\text{the set } S(z) \stackrel{df}{=} \big\{x \in X : \ (z, x) \in S\big\} \text{ is } \sigma\text{-compact,}$$

then

$$\text{proj}_{_Z} S \ \in \ \mathcal{B}(X).$$

THEOREM A.2.32 (Yankov-von Neumann-Aumann Projection Theorem)
If (Ω, Σ) *is a measurable space,* X *is a Souslin space and* $S \in \Sigma \times \mathcal{B}(X)$,
then

$$\text{proj}_{_\Omega} S \ \in \ \widehat{\Sigma},$$

with $\widehat{\Sigma}$ *being the universal* σ*-field corresponding to* Σ. *If* Σ *is complete, then* $\text{proj}_{_\Omega} S \in \Sigma$.

THEOREM A.2.33 (Yankov-von Neumann-Aumann Selection Theorem)
If (Ω, Σ) *is a complete measurable space,* X *is a Souslin space and* $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ *is a multifunction, such that*

$$\operatorname{Gr} F \;=\; \big\{(\omega, x) \in \Omega \times X :\; x \in F(\omega)\big\} \;\in\; \Sigma \times \mathcal{B}(X),$$

then there exists a Σ*-measurable function* $f\colon \Omega \longrightarrow X$*, such that*

$$f(\omega) \;\in\; F(\omega) \qquad \forall\, \omega \in \Omega.$$

PROPOSITION A.2.34
(a) *If* τ_1 *and* τ_2 *are two comparable Souslin topologies on a set* X*, then*

$$\mathcal{B}\big(X_{\tau_1}\big) \;=\; \mathcal{B}\big(X_{\tau_2}\big).$$

(b) *If* X *and* Y *are two Souslin spaces, then*

$$\mathcal{B}\big(X \times Y\big) = \mathcal{B}(X) \times \mathcal{B}(Y).$$

PROPOSITION A.2.35 (Double Limit Lemma)
Let X *be a metric space and let* $\{x_{m,n}\}_{m,n \geqslant 1} \subseteq X$*. Suppose that*

$$\text{the limit } \lim_{m \to +\infty} \lim_{n \to +\infty} x_{n,m} \text{ exists.}$$

Then there exist subsequences $\{m(n)\}_{n \geqslant 1}$ *and* $\{n(m)\}_{m \geqslant 1}$*, such that*

$$\lim_{m \to +\infty} \lim_{n \to +\infty} x_{m,n} \;=\; \lim_{m \to +\infty} x_{m,n(m)} \;=\; \lim_{n \to +\infty} x_{m(n),n}.$$

THEOREM A.2.36 (Portmanteau Theorem)
If X *is a metric space and* $\{\mu_\alpha\}_{\alpha \in J}$ *is a net in* $M_+^1(X)$ *(see (3.130)), then the following statements are equivalent:*

(a) $\mu_\alpha \xrightarrow{w} \mu$ *in* $M_+^1(X)$*;*

(b) $\displaystyle\int_X f(\omega)\, d\mu_\alpha \longrightarrow \int_X f(\omega)\, d\mu$ *for all bounded, uniformly continuous functions* $f\colon X \longrightarrow \mathbb{R}$*;*

(c) $\limsup_\alpha \mu_\alpha(C) \leqslant \mu(C)$ *for all closed sets* $C \subseteq X$*;*

(d) $\liminf_\alpha \mu_\alpha(U) \geqslant \mu(U)$ *for all open sets* $U \subseteq X$*;*

(e) $\lim_\alpha \mu_\alpha(A) = \mu(A)$ *for all* $A \in \mathcal{B}(X)$ *with* $\mu(\partial A) = 0$ *(such sets are called* ***μ-continuity sets****).*

DEFINITION A.2.37 *Let $\Omega \neq \emptyset$, $\mathcal{S} \subseteq 2^\Omega$ with $\emptyset \in \mathcal{S}$ and $\mathcal{S}$ being stable under finite unions and finite intersections. A function*

$$m\colon 2^\Omega \longrightarrow \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$$

*is said to be a (**Choquet**) $\mathcal{S}$-**capacity**, if*

(i) m is monotone (i.e., if $A \subseteq B \subseteq \Omega$, then $m(A) \leqslant m(B)$);

(ii) if $A_n \nearrow A$ (i.e., $A_n \subseteq A_{n+1}$ and $A = \bigcup\limits_{n=1}^{\infty} A_n$), then

$$m(A_n) \nearrow m(A);$$

(iii) if $A_n \searrow A$ (i.e., $A_n \supseteq A_{n+1}$ and $A = \bigcap\limits_{n=1}^{\infty} A_n$), then

$$m(A_n) \searrow m(A).$$

*We say that a set $A \subseteq \Omega$ is $\mathcal{S}$-m-**capacitable**,, if*

$$m(A) = \sup_{\substack{C \in \mathcal{S}_\delta \\ C \subseteq A}} m(C),$$

where $\mathcal{S}_\delta = \left\{ \bigcap\limits_{k=1}^{\infty} C_k : \ C_k \in \mathcal{S} \right\}$.

DEFINITION A.2.38 *An $\mathcal{S}$-**Souslin** set is a set which belongs in*

$$\mathcal{A}(\mathcal{S}) \stackrel{df}{=} \left\{ A(F) : \ F \in \mathcal{S}^{\mathbb{N}^*} \right\},$$

where $\mathbb{N}^* = \bigcup\limits_{n=1}^{\infty} \mathbb{N}^n$ *and*

$$A(F) = \bigcup_{\{n_k\}_{k \geqslant 1} \in \mathbb{N}^{\mathbb{N}}} \bigcup_{m=1}^{\infty} F(n_1, \ldots, n_m)$$

(this is the Souslin operation corresponding to F).

THEOREM A.2.39 (Choquet Capacitability Theorem)
If $\Omega \neq \emptyset$, $\mathcal{S} \subseteq 2^\Omega$ with $\emptyset \in \mathcal{S}$, $\mathcal{S}$ is stable under finite unions and finite intersections and $m\colon 2^\Omega \longrightarrow \mathbb{R}^*$ is an $\mathcal{S}$-capacity,
then every $\mathcal{S}$-Souslin set is $\mathcal{S}$-m-capacitable.

A.3 Functional Analysis

THEOREM A.3.1 (Weak Separation Theorem)
If X is a locally convex vector space, $A, C \subseteq X$ are two nonempty, convex sets, such that

$$\operatorname{int} A \neq \emptyset \quad \textit{and} \quad \operatorname{int} A \cap C = \emptyset,$$

then there exists $x^* \in X^* \setminus \{0\}$, such that

$$\langle x^*, a \rangle_X \leqslant \langle x^*, c \rangle_X \qquad \forall\ a \in A,\ c \in C.$$

THEOREM A.3.2 (Strong Separation Theorem)
If X is a locally convex space, $A, C \subseteq X$ are two nonempty, disjoint, closed, convex sets and A is compact,
then there exist $x^* \in X^* \setminus \{0\}$ and $\varepsilon > 0$, such that

$$\sup_{a \in A} \langle x^*, a \rangle_X \leqslant \inf_{c \in C} \langle x^*, c \rangle_X - \varepsilon.$$

REMARK A.3.3 Usually in applications of Theorem A.3.2, X is a Banach space endowed with the weak topology

THEOREM A.3.4 (Uniform Boundedness Principle)
If X is a Banach space, Y is a normed space, $\mathcal{F} \subseteq \mathcal{L}(X;Y)$ and

$$\sup_{A \in \mathcal{F}} \|Ax\|_Y < +\infty \qquad \forall\ x \in X,$$

then

$$\sup_{A \in \mathcal{F}} \|A\|_{\mathcal{L}} < +\infty.$$

THEOREM A.3.5 (Open Mapping Theorem)
If X, Y are two Banach spaces and $A \in \mathcal{L}(X;Y)$ is surjective,
then A is an open map.

Recall that a bijective map from one topology space onto another is a homeomorphism if and only if it is both continuous and open. Then Theorem A.3.5 yields the following result.

THEOREM A.3.6 (Banach Theorem)
If X, Y are two Banach spaces and $A \in \mathcal{L}(X;Y)$ is bijective,
then A is an isomorphism (i.e., $A^{-1} \in \mathcal{L}(Y;X)$).

THEOREM A.3.7 (Closed Graph Theorem)
If X, Y are two Banach spaces and $A\colon X \longrightarrow Y$ is a linear operator,
then $A \in \mathcal{L}(X;Y)$ if and only if $\operatorname{Gr} A$ is closed.

THEOREM A.3.8 (Eberlein-Smulian Theorem)
If X is a normed space and $C \subseteq X$,
then

(a) *C is weakly compact if and only if C is weakly sequentially compact;*

(b) *C is relatively weakly compact if and only if C is relatively weakly sequentially compact.*

THEOREM A.3.9 (Alaoglu Theorem)
If X is a normed space and $\overline{B}_1^{X^*} = \{x^* \in X^* : \|x^*\|_{X^*} \leqslant 1\}$,
then $\overline{B}_1^{X^*}$ is a w^*-compact set.

REMARK A.3.10 So every norm-bounded set in X^* is relatively w^*-compact.

A normed linear space is finite dimensional if and only if the convex hull of every compact set is compact. For infinite dimensional spaces the situation is as follows.

THEOREM A.3.11 (Krein-Smulian Theorem)
If X is a Banach space and $C \subseteq X$ is weakly compact,
then $\overline{\text{conv}}\, C$ is weakly compact too.

THEOREM A.3.12 (Mazur Theorem)
If X is a Banach space and $C \subseteq X$ is a norm-compact,
then $\overline{\text{conv}}\, C$ is norm-compact too.

In an infinite dimensional Banach space the weak topology on X is never metrizable. However, on certain particular subsets the relative weak topology is metrizable. The next theorem summarizes those cases.

THEOREM A.3.13
Let X be a Banach space.

(a) *If X is a separable space and $A \subseteq X^*$ is bounded,*
then the relative w^*-topology on A is metrizable.

(b) *If X^* is separable and $A \subseteq X$ is bounded,*
then the relative weak topology on A is metrizable.

(c) *If X is separable and $A \subseteq X$ is weakly compact,*
then the relative weak topology on A is metrizable.

REMARK A.3.14 Recall that, if X^* is separable, then so is X. However the converse is not true (consider $L^1[0,1]$ and $L^\infty[0,1] = L^1[0,1]^*$). But if X reflexive, then X is separable if and only if X^* is separable.

PROPOSITION A.3.15 (Riesz Lemma)
If X is a normed space, $Y \subseteq X$ is a closed subspace and $\lambda \in (0,1)$, then we can find $x_\lambda \in X$, such that

$$\|x_\lambda\|_X = 1 \quad \text{and} \quad d_x(x_\lambda, Y) > \lambda.$$

If X is a normed space and Z is a subspace of X, it is useful to know the duals of Z and of $X/_Z$. The next two propositions identify these two duals.

PROPOSITION A.3.16
If X is a normed space and $Z \subseteq X$ is a subspace endowed with the induced norm, then

(a) *the dual Z^* of Z can be isometrically identified with $X^*/_{Z^\perp}$;*

(b) $w\big(Z, X^*/_{Z^\perp}\big) = w\big(X, X^*\big)|_Z$.

PROPOSITION A.3.17
If X is a normed space and $Z \subseteq X$ is a subspace endowed with the induced norm, then

(a) $\big(X/_Z\big)^*$ *can be isometrically identified with $Z^\perp$;*

(b) $\sigma\big(X/_Z, Z^\perp\big) = w(X, X^*)|_Z$.

PROPOSITION A.3.18
If X is a normed space,

$$x_\alpha \xrightarrow{w} x \quad \text{in } X \quad \text{and} \quad x^*_\alpha \xrightarrow{w^*} x^* \quad \text{in } X^*,$$

then

$$\|x\|_X \leqslant \liminf_\alpha \|x_\alpha\|_X \quad \text{and} \quad \|x^*\|_{X^*} \leqslant \liminf_\alpha \|x^*_\alpha\|_{X^*}$$

(i.e., the norm functional on X is weakly lower semicontinuous and the norm functional on X^* is weakly* lower semicontinuous).

PROPOSITION A.3.19
If X is a normed space and $A \subseteq X$ is convex, then $\overline{A} = \overline{A}^w$.
Hence a convex set is closed if and only if it is weakly closed.

PROPOSITION A.3.20
If X is a reflexive Banach space, then a convex set in X^* is weakly closed if and only if it is weakly sequentially closed.

DEFINITION A.3.21 *A Banach space X is said to be* ***locally uniformly convex****, if for every $\varepsilon > 0$ and $x \in X$ with $\|x\|_X = 1$, we can find $\delta = \delta(\varepsilon, x) > 0$, such that*

$$\text{if } \|x-y\|_X \geqslant \varepsilon, \text{ then } \left\|\frac{x+y}{2}\right\|_X \leqslant 1-\delta \qquad \forall\, y \in X,\ \|y\|_X = 1.$$

If $\delta > 0$ can be chosen independent of x, then we say that X is ***uniformly convex.***

REMARK A.3.22 A uniformly convex Banach space is reflexive (Milman-Pettis theorem). Locally uniformly convex spaces have the Kadec-Klee (or Radon-Riesz) property, namely if

$$x_n \overset{w}{\longrightarrow} x \quad \text{in } X \quad \text{and} \quad \|x_n\|_X \longrightarrow \|x\|_X\,,$$

then

$$x_n \longrightarrow x \quad \text{in } X.$$

Recall that Hilbert spaces are uniformly convex (a consequence of the parallelogram law).

THEOREM A.3.23 (Troyanski Renorming Theorem)
Every reflexive Banach space can be given equivalent norm so that X and X^* are both locally uniformly convex and have Fréchet differentiable norms.

In the next two theorems we identify the dual of certain well known spaces.

THEOREM A.3.24 (Riesz Representation Theorem)
If (Ω, Σ, μ) is a σ-finite measure space, $p \in [1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (if $p = 1$, then $p' = +\infty$),
then $L^{p'}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)^*$ via the isometry

$$u(g)(f) \;=\; \int_\Omega f(\omega) g(\omega)\, d\mu \qquad \forall\ g \in L^{p'}(\Omega),\ f \in L^p(\Omega).$$

So we can write that

$$L^p(\Omega)^* \;=\; L^{p'}(\Omega).$$

THEOREM A.3.25 (Riesz-Markov Theorem)
If X is a locally compact, σ-compact metrizable space,
then $C_0(X)^* = M(X)$.

PROPOSITION A.3.26
If X, Y are two separable, metrizable spaces and $\vartheta\colon X \longrightarrow Y$ is a homeomorphism,
then $\widehat{\vartheta}\colon M_+^1(X) \longrightarrow M_+^1(Y)$, defined by $\widehat{\vartheta}(\mu) = \mu\vartheta^{-1}$, is a homeomorphism too if on $M_+^1(X)$ and $M_+^1(Y)$ we consider the respective weak topologies.

A.4 Calculus and Nonlinear Analysis

THEOREM A.4.1 (Gauss-Green Theorem; Divergence Theorem) *If $\Omega \in \mathbb{R}^N$ is a nonempty, bounded open set with C^1-boundary $\partial\Omega$, then*

$$\int_{\Omega} \operatorname{div} u \, dz \;=\; \int_{\partial\Omega} (u, n)_{\mathbb{R}^N} \, d\sigma \qquad \forall\ u \in C^1\big(\overline{\Omega}; \mathbb{R}^N\big).$$

Hence

$$\int_{\Omega} \frac{\partial u}{\partial z_i} \, dz \;=\; \int_{\partial\Omega} u n_i \, d\sigma \qquad \forall\ u \in C^1\big(\overline{\Omega}; \mathbb{R}^N\big),\ i \in \big\{1, \ldots, N\big\}.$$

REMARK A.4.2 Recall that

$$\operatorname{div} u(z) \;=\; \sum_{k=1}^{N} \frac{\partial u_k(z)}{\partial z_k}$$

and

$$n(z) \;=\; \big(n_k(z)\big)_{k=1}^{N}$$

is the outward unit normal on $\partial\Omega$. If $v \colon \Omega \longrightarrow \mathbb{R}$ is twice partially differentiable, then

$$\nabla v \;=\; \big(\frac{\partial v}{\partial z_1}, \ldots, \frac{\partial v}{\partial z_N}\big)$$

is a partially differentiable vector field and

$$\operatorname{div}(\nabla v) \;=\; \Delta v.$$

Also, if $f \colon \Omega \longrightarrow \mathbb{R}$ is partially differentiable and $u \in C^1\big(\overline{\Omega}; \mathbb{R}^N\big)$, then

$$\operatorname{div}(fu) \;=\; f \operatorname{div} u + (\nabla f, u)_{\mathbb{R}^N}.$$

Theorem A.4.1 leads to the following "integration by parts formula."

COROLLARY A.4.3 (Integration by Parts Formula) *If $\Omega \in \mathbb{R}^N$ is a bounded, open set with a C^1-boundary $\partial\Omega$ and $u, v \in C^1\big(\overline{\Omega}\big)$, then*

$$\int_{\Omega} \frac{\partial u}{\partial z_k} v \, dz + \int_{\Omega} u \frac{\partial v}{\partial z_k} \, dz \;=\; \int_{\partial\Omega} u v n_k \, d\sigma \qquad \forall\ k \in \big\{1, \ldots, N\big\}.$$

Using Corollary A.4.3, we derive the so-called "Green's formulas."

COROLLARY A.4.4 (Green Formula)
If $\Omega \in \mathbb{R}^N$ *is a bounded, open set with* C^1*-boundary* $\partial\Omega$ *and* $u, v \in C^2(\overline{\Omega})$, *then*

(a) $\displaystyle\int_\Omega \Delta u \, dz = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma;$

(b) $\displaystyle\int_\Omega \big(\nabla u, \nabla v\big)_{\mathbb{R}^N} dz + \int_\Omega u\Delta v \, dz = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, d\sigma;$

(c) $\displaystyle\int_\Omega \big(u\Delta v - v\Delta u\big) \, dz = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma.$

PROPOSITION A.4.5 (Young Inequality)
If $p \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $a, b > 0$ *and* $\varepsilon > 0$, *then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'} \quad \text{and} \quad ab \leqslant \frac{\varepsilon}{p}a^p + \frac{1}{\varepsilon p'}b^{p'}.$$

REMARK A.4.6 If $p = p' = 2$, then the first inequality in Proposition A.4.5 is the well known ***Cauchy-Bunyakowski-Schwarz inequality***.

THEOREM A.4.7 (Gronwall Inequality)
If $T = [a, b]$, $u \in C(T)$, $k \in L^1(T)_+$, $h \in L^\infty(T)$, *and*

$$u(t) \leqslant h(t) + \int_0^t k(s)u(s) \, ds \qquad \forall\ t \in T,$$

then

$$u(t) \leqslant h(t) + \int_0^t exp\Big(\int_s^t k(r) \, dr\Big)k(s)h(s) \, ds \qquad \forall\ s \in T.$$

REMARK A.4.8 If $h(t) \equiv h_0$ for all $t \in T$, then

$$u(t) \leqslant h_0 \int_0^t k(s) \, ds \qquad \forall\ t \in T$$

(***Bellman's inequality***).

More details on all the subjects covered in this Appendix and additional references can be found in Denkowski, Migórski & Papageorgiou (2003*a*, 2003*b*).

List of Symbols

Symbol	Page	Meaning
λ^N	p. 1	Lebesgue measure in $\mathbb{R}^N$
$\mu\lfloor A$	p. 3	the restriction of μ on A, defined by $(\mu\lfloor A)(B) \stackrel{df}{=} \mu(A \cap B) \qquad \forall\ B \in 2^X$
$\mu\big\vert_{A}$	p. 3	the restriction of μ on 2^A, defined by $\big(\mu\big\vert_{A}\big)(B) \stackrel{df}{=} \mu(B) \qquad \forall\ B \in 2^A \subseteq 2^X$
Σ_μ	p. 4	the collection of all μ-measurable sets
$\mathcal{B}(X)$	p. 4	the Borel σ-field of X
$\delta(A),\ \mathrm{diam}\,(A)$	p. 7	the diameter of A
λ^N	p. 9	the N-dimensional Lebesgue outer measure
$\mathbb{R}^*$	p. 12	$\mathbb{R}^* \stackrel{df}{=} \mathbb{R} \cup \{\pm\infty\}$
λ^*	p. 12	the Lebesgue outer measure on $\mathbb{R}$
$d_{_X}(A,B)$	p. 22	$d_{_X}(A,B) \stackrel{df}{=} \inf\limits_{\substack{a\,\in\,A\\ b\,\in\,B}} d_{_X}(a,b)\ >\ 0$
A^c	p. 22	$A^c \stackrel{df}{=} X \setminus A$
$\mathcal{T}_\delta(C)$	p. 25	the family of all δ-covers of the set C
$\mu_\delta^{(s)}(C)$	p. 25	$\mu_\delta^{(s)}(C) \stackrel{df}{=} \inf\limits_{\{A_n\}_{n\geqslant 1}\in\mathcal{T}_\delta(C)} \sum\limits_{n=1}^{\infty} \delta(A_n)^s$
$\mu^{(s)}$	p. 25	the Hausdorff s-dimensional outer measure
S_ϑ	p. 30	the Cantor-like set
λ^N	p. 31	Lebesgue N-dimensional outer measure
$L(a,b)$	p. 32	the line in $\mathbb{R}^N$ passing from b in the direction of a
$P(a)$	p. 32	the plane in $\mathbb{R}^N$ passing from the origin, perpendicular to a

$S(a, A)$	p. 33	Steiner symmetrization
Γ	p. 39	the gamma Euler function
$\mathrm{Lip}(f)$	p. 52	Lipschitz constant of a Lipschitz continuous function $f\colon \mathbb{R}^N \supset C \longrightarrow \mathbb{R}^M$
$\mathrm{Gr}\,(f\big\vert_A)$	p. 52	the graph of $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ over A
$\mathcal{L}(X;Y)$	p. 55	the vector space of all bounded linear operators from X into Y
$C_c^\infty(Z)$	p. 57	the space of all $C^\infty(Z)$-functions with compact supports
$\mathrm{jac}\, L$	p. 64	the Jacobian of a linear operator L
Jf	p. 64	the Jacobian of a Lipschitz function f
$W^{1,p}(U)$	p. 81	the Sobolev space $W^{1,p}(U) \stackrel{df}{=} \big\{f \in L^p(U) :\ Df \in L^p\big(U;\mathbb{R}^N\big)\big\}$
$W^{1,p}_{\mathrm{loc}}(U)$	p. 81	the Sobolev space $W^{1,p}_{\mathrm{loc}}(U) \stackrel{df}{=} \big\{f\colon U \longrightarrow \mathbb{R} :\ f\big\vert_V \in W^{1,p}(V)$ for all $V \subset\subset U\big\}$
$V \subset\subset U$	p. 81	V is a bounded open subset of U such that $\overline{V} \subseteq U$
p^*	p. 82	Sobolev critical exponent
K^p	p. 82	$K^p \stackrel{df}{=} \big\{f \in L^{p^*}(\mathbb{R}^N) :\ f \geqslant 0,\ Df \in L^p\big(\mathbb{R}^N;\mathbb{R}^N\big)\big\}$
$\mathrm{cap}_p(C)$	p. 82	p-capacity of C
$\|f\|_{1,p}$	p. 82	the norm of $W^{1,p}(U)$
$\|f\|_{1,\infty}$	p. 82	the norm of $W^{1,\infty}(U)$
X^*	p. 108	the topological dual of the Banach space X
$\langle\cdot,\cdot\rangle_X$	p. 108	the duality brackets for the pair of spaces (X^*, X)
$\|f\|_p$	p. 127	the norm of a strongly measurable function
$W^{1,p}(T;X)$	p. 138	$W^{1,p}(T;X) \stackrel{df}{=} \big\{f \in L^p(T;X) :\ Df \in L^p(T;X)\big\}$
$AC^{1,p}\big(T,X\big)$	p. 138	the space of absolutely continuous function, differentiable almost everywhere with the derivative in $L^p(T;X)$
$W_{\mathrm{pr}}(T)$	p. 144	$W_{\mathrm{pr}}(T) \stackrel{df}{=} \big\{u \in L^p(T;X) :\ u' = Du \in L^r(T;Z)\big\}$

$u_n \xrightarrow{\mathrm{b}} u$	p. 163	biting convergence in $L^1(\Omega; X)$
$\underset{n\to+\infty}{w\text{-}\limsup} A_n$	p. 165	the space of $x \in X$, such that $x = \underset{k\to+\infty}{w\text{-}\lim} x_{n_k}$, with $x_{n_k} \in A_{n_k}$ and $n_1 < n_2 < \dots$
$C_c(Z)$	p. 171	the space of continuous functions on Z with compact support
$C_0(Z)$	p. 171	the space of continuous functions on Z which vanish at infinity
$C_b(Z)$	p. 171	the space of bounded, continuous functions on Z
$M(Z)$	p. 172	the space of all signed measures $m\colon \mathcal{B}(Z) \longrightarrow \mathbb{R}$ which have bounded variation
$\mu_n \xrightarrow{w} \mu$	p. 173	weak convergence of measures
$\alpha = (\alpha_k)_{k=1}^N$	p. 179	multi-index
$\lvert\alpha\rvert$	p. 179	the length of the multi-index α
$C_c^\infty(Z)$	p. 179	the space of $C^\infty(Z)$-functions with compact supports
$\mathcal{D}(Z)$	p. 179	the space of test functions
$\mathcal{D}(Z)^*$	p. 179	the space of distributions
$\lVert u\rVert_{W^{m,p}(Z)}$	p. 181	the norm of $W^{m,p}(Z)$
$W_0^{m,p}(Z)$	p. 181	$W_0^{m,p}(Z) \stackrel{df}{=} \overline{\mathcal{D}(A)}^{\lVert\cdot\rVert_{W^{m,p}(Z)}}$
$H^m(Z)$	p. 182	$H^m(Z) \stackrel{df}{=} W^{m,2}(Z)$
$H_0^m(Z)$	p. 182	$H_0^m(Z) \stackrel{df}{=} W_0^{m,2}(Z)$
φ_ε	p. 183	mollifier
u_ε	p. 183	mollification or regularization of u
$V^q\big(Z, \operatorname{div}\big)$	p. 210	$V^q\big(Z, \operatorname{div}\big) \stackrel{df}{=} \big\{v \in L^q\big(Z; \mathbb{R}^N\big) :\ \operatorname{div} v \in L^q(Z)\big\}$
$\lVert v\rVert_{V^q(Z,\operatorname{div})}$	p. 210	$\lVert v\rVert_{V^q(Z,\operatorname{div})} \stackrel{df}{=} \Big[\lVert v\rVert^q_{L^q(Z;\mathbb{R}^N)} + \big\lVert\operatorname{div} v\big\rVert^q_{L^q(Z)}\Big]^{\frac{1}{q}}$
$\Delta_p u$	p. 211	$\Delta_p u \stackrel{df}{=} \operatorname{div}\big(\lVert Du\rVert^p_{\mathbb{R}^N} Du\big)$ the p-Laplacian
$C_b^k(Z)$	p. 222	the space of all functions $u \in C^k(Z)$, such that $D^\alpha u$ is bounded on Z for all multiindices $\alpha \in \mathbb{N}^N$ with $\lvert\alpha\rvert \leqslant k$
$C_0\big(\overline{Z}\big)$	p. 227	$C_0\big(\overline{Z}\big) \stackrel{df}{=} \big\{u \in C\big(\overline{Z}\big) :\ u\big\rvert_{\partial Z} = 0\big\}$
$D^{1,p}(\mathbb{R}^N)$	p. 231	$D^{1,p}(\mathbb{R}^N) \stackrel{df}{=} \big\{u \in L^{p^*}(\mathbb{R}^N) :\ Du \in L^p\big(\mathbb{R}^N; \mathbb{R}^N\big)\big\}$

$W^{1,p}_{\mathrm{per}}(T;\mathbb{R}^N)$	p. 237	$W^{1,p}_{\mathrm{per}}(T;\mathbb{R}^N) \stackrel{df}{=} \{u \in W^{1,p}(T;\mathbb{R}^N) :\ u(0) = u(b)\}$
$K(D;Y)$	p. 266	the set of compact maps $f\colon D \longrightarrow Y$
$\mathcal{L}_c(X;Y)$	p. 266	$\mathcal{L}_c(X;Y) \stackrel{df}{=} K(X;Y) \cap \mathcal{L}(X;Y)$
id_x	p. 273	identity operator on X
$\mathcal{L}_f(X;Y)$	p. 276	the space of all finite dimensional operators from X into Y equipped with the norm inherited from $\mathcal{L}(X;Y)$
$\operatorname{rank} L$	p. 276	$\operatorname{rank} L \stackrel{df}{=} \dim L(X)$
$\varrho(L)$	p. 278	the resolvent set of $L \in \mathcal{L}(X)$
R	p. 278	the resolvent operator of $L \in \mathcal{L}(X)$
$\sigma(L)$	p. 278	the spectrum of $L \in \mathcal{L}(X)$
$\sigma_p(L)$	p. 278	the point spectrum of $L \in \mathcal{L}(X)$
$A^{\perp}$	p. 285	$A^{\perp} \stackrel{df}{=} \{x^* \in X^* :\ \langle x^*, a\rangle_X = 0 \text{ for all } a \in A\}$
${}^{\perp}C$	p. 285	${}^{\perp}C \stackrel{df}{=} \{x \in X :\ \langle c^*, x\rangle_X = 0 \text{ for all } c^* \in C\}$
$E(\lambda)$	p. 295	the eigenspace corresponding to the eigenvalue λ
$\Phi(X;Y)$	p. 298	the class of Fredholm operators $L \in \mathcal{L}(X;Y)$
$\alpha(L)$	p. 298	the kernel index of an operator L
$\beta(L)$	p. 298	the deficiency index of an operator L
$\operatorname{ind} L$	p. 298	the index of an operator L
$\Phi_+(X;Y)$	p. 298	the class of all semi-Fredholm operators $L \in \mathcal{L}(X;Y)$
$\Phi(X)$	p. 298	the class of Fredholm operators $L \in \mathcal{L}(X)$
$\Phi_+(X)$	p. 298	the class of all semi-Fredholm operators $L \in \mathcal{L}(X)$
$\gamma(L)$	p. 299	the minimum modulus of an operator L
$D(A)$	p. 303	the domain of an operator A
$\operatorname{Gr} A$	p. 303	the graph of an operator A
$F^-(C)$	p. 307	$F^-(C) \stackrel{df}{=} \{y \in Y :\ F(y) \cap C \neq \emptyset\}$, where $F\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ is a multifunction
$F^+(U)$	p. 307	$F^+(U) \stackrel{df}{=} \{y \in Y :\ F(y) \subseteq U\}$, where $F\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ is a multifunction
Z_w	p. 308	the space Z furnished with the weak topology

$\overline{\mathbb{R}}$	p. 311	$\overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$
$\partial\varphi$	p. 311	the subdifferential of φ
$\mathcal{F}$	p. 311	the duality map of X, defined by $$\mathcal{F}(x) \stackrel{df}{=} \left\{x^* \in X^* : \langle x^*, x\rangle_X = \|x\|_X^2 = \|x^*\|_{X^*}^2\right\}$$
$d_B(f, U, 0)$	p. 321	the Brouwer degree of f on U with respect to 0
J_λ	p. 322	the resolvent operator of A
A_λ	p. 322	the Yosida approximation of A
$\langle\cdot,\cdot\rangle_{pp'}$	p. 341	the duality brackets for $\left(L^{p'}\left(T; X^*\right), L^p(T; X)\right)$
$C_b(\mathbb{R})$	p. 369	the space of bounded continuous functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ equipped with the supremum norm
$NBV(\mathbb{R})$	p. 375	the space of all normalized functions of bounded variation with the total variation norm $$\|\vartheta\|_{TV(\mathbb{R})} = (\operatorname{Var}\vartheta)(\mathbb{R}) = \int_{\mathbb{R}} \left|\, d\vartheta(t)\right|.$$
$\varrho(A)$	p. 378	the resolvent set of an operator $A\colon X \supseteq D(A) \longrightarrow X$
R_λ	p. 378	the resolvent operator
S_G^p	p. 423	the set of L^p-selections of the multifunction $G\colon \Omega \longrightarrow 2^Y \setminus \{\emptyset\}$, i.e., $$S_G^p \stackrel{df}{=} \left\{g \in L^p(\Omega; Y) : g(z) \in G(z) \text{ for } \mu\text{-a.a. } z \in \Omega\right\}$$
$M_+^1(E)$	p. 427	the probability measures on E
$M_+^1(E)$	p. 427	the subprobability measures on E
$\widehat{R}(\Omega, E)$	p. 427	the space of transition probabilities on Ω
$\widehat{SR}(\Omega, E)$	p. 427	the space of transition subprobabilities on Ω
$M_+^1(E)_n$	p. 427	the space $M_+^1(E)$ equipped with the topology of narrow convergence
$\mathcal{Y}(\Omega, E, \mu)$	p. 429	the space of Young measures with respect to μ
$S\mathcal{Y}(\Omega, E, \mu)$	p. 429	the space of Young submeasures with respect to μ
δ_u	p. 436	Dirac transition probability associated to u
$\mathcal{N}(\Omega, \Sigma, E)$	p. 438	the set of all normal integrands

$\mathcal{N}_+(\Omega,\Sigma,E)$	p. 438	the set of positive normal integrands
$\widehat{\mathcal{K}}_b(\Omega,\Sigma,E)$	p. 438	the set of all C_b-Carathéodory integrands
$\widehat{\mathcal{K}}_0(\Omega,\Sigma,E)$	p. 438	the set of all C_0-Carathéodory integrands
$Bar(u)$	p. 454	the barycenter of a function u
$f'_G(x)$	p. 468	the Gâteaux derivative of f at x
$f'_F(x)$	p. 470	the Fréchet derivative of f at x
$f'_+(t)$	p. 476	right derivative of f at t
$f'_-(t)$	p. 476	left derivative of f at t
$\operatorname{dom}\varphi$	p. 488	the effective domain of φ
$\operatorname{epi}\varphi$	p. 488	the epigraph of φ
L^{φ}_{λ}	p. 488	the sublevel set of φ
$\Gamma_0(X)$	p. 488	the cone of proper, convex and lower semicontinuous functions
φ^*	p. 512	the Legendre-Fenchel transform (or the conjugate) of φ
$\varphi^{**}(x)$	p. 512	the second conjugate (or biconjugate) of φ
$\sigma_{_C}$	p. 514	the support function of the set C
$\varphi\oplus\psi$	p. 514	the infimal convolution of functions φ,ψ
$\operatorname{sepi}\varphi$	p. 514	the strict epigraph of φ
K^*	p. 517	the dual cone to the cone K
$\varphi'(x_0;h)$	p. 522	the directional derivative of φ at x_0 in the direction $h\in X$
$\partial\varphi(x)$	p. 523	the subdifferential of φ at x
$\partial_\varepsilon\varphi(x)$	p. 540	the ε-subdifferential of φ at x
$\varphi'_\varepsilon(x;h)$	p. 541	$\varphi'_\varepsilon(x;h)\stackrel{df}{=}\inf\limits_{\lambda>0}\dfrac{\varphi(x+\lambda h)-\varphi(x)+\varepsilon}{\lambda}$
$\varphi^0(x;h)$	p. 544	the generalized directional derivative of φ at x in the direction h
$\partial\varphi(x)$	p. 546	the generalized (or Clarke) subdifferential of φ at x
$\mathfrak{B}$	p. 554	bornology
$\mathfrak{B}_G$	p. 554	Gâteaux bornology
$\mathfrak{B}_F$	p. 554	Fréchet bornology

$\mathfrak{B}_H$	p. 554	Hadamard bornology
$\varphi'_{\mathfrak{B}}(x)$	p. 554	$\mathfrak{B}$-derivative of φ (derivative with respect to bornology $\mathfrak{B}$)
$\partial_{\mathfrak{B}}\varphi(x)$	p. 555	$\mathfrak{B}$-subdifferential of φ (subdifferential with respect to bornology $\mathfrak{B}$)
$\partial^{\mathfrak{B}}\varphi(x)$	p. 555	the set of $\mathfrak{B}$-superderivatives of φ at x (with respect to bornology $\mathfrak{B}$)
$\partial_p\varphi(x)$	p. 557	the proximal subdifferential of φ at x
$\partial^-_{\mathfrak{B}}\varphi(x)$	p. 557	the canonical $\mathfrak{B}$-subdifferential of φ at x
$w(V,W)$	p. 568	the weak topology; the smallest topology on V compatible with the dual pair (V,W)
$m(V,W)$	p. 568	Mackey topology; the largest topology on V compatible with the dual pair (V,W)
$D(x,C)$	p. 584	the drop associated with the pair (x,C)
$T_C(x_0)$	p. 599	the tangent cone to C at x_0
$\oint\limits_{\Omega} \varphi_\omega \, d\mu$	p. 603	inf-convolution integral of φ with respect to μ
φ^a	p. 608	the sublevel set of φ at the level a
K^φ	p. 625	the set of critical points of φ
K^φ_c	p. 625	the set of critical points of φ with energy level c
B^Y_r	p. 686	$B^Y_r \stackrel{df}{=} \{x \in Y : \|x\|_X \leqslant R\}$, where Y is a linear subspace of X
$cat_Y(A)$	p. 690	the Lusternik-Schnirelman category
$\mathcal{A}_k$	p. 694	$\mathcal{A}_k \stackrel{df}{=} \{A \subseteq X : A \text{ is compact and } cat_X(A) \geqslant k\}$
$T_{x_0}(C)$	p. 696	the tangent space to C at x_0
$\mathcal{L}_{n-1}$	p. 710	the set of all $(n-1)$-dimensional subspaces Y of H
$(\cdot,\cdot)_2$	p. 721	the inner product in $L^2(\Omega)$
$\frac{\partial x}{\partial n_p}$	p. 754	$\frac{\partial x}{\partial n_p} \stackrel{df}{=} \|Dx\|^{p-2}_{\mathbb{R}^N} \left(Dx, n\right)_{\mathbb{R}^N}$ with n being the outward unit normal on $\partial\Omega$
$\varrho(p,N)$	p. 769	the resolvent set of the negative vector ordinary p-Laplacian with periodic boundary conditions
$\sigma(p,N)$	p. 769	the set of eigenvalues of the negative vector ordinary p-Laplacian with periodic boundary conditions

$C(p,N)$	p. 774	the set of functions $x \in W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big)$ such that $\int\limits_\Omega \big\|x(t)\big\|^{p-2}_{\mathbb{R}^N} x(t)\,dt = 0$
$C_1(p,N)$	p. 774	$C_1(p,N) \stackrel{df}{=} \big\{x \in C(p,N) :\ \|x\|_p = 1\big\}$
$\vartheta_{p,N}$	p. 774	$\vartheta_{p,N}(x) \stackrel{df}{=} \|x'\|^p_p \qquad \forall\ x \in W^{1,p}_{\mathrm{per}}\big((0,b);\mathbb{R}^N\big)$
$r_x(D)$	p. 817	the radius of D relative to x
$r_E(D)$	p. 817	the Chebyshev radius of D relative to E
$C_E(D)$	p. 817	the Chebyshev center of D relative to E
$\alpha(C)$	p. 828	the Kuratowski measure of noncompactness of the set C
$\beta(C)$	p. 828	the ball (Hausdorff) measure of noncompactness of the set C
$(X,\leqslant)$	p. 833	partially ordered set
$S_+(x)$	p. 834	the right section of x in a partially ordered set $(X,\leqslant)$
$S_-(x)$	p. 834	the left section of x in a partially ordered set $(X,\leqslant)$
$[x,y]$	p. 834	the order interval in a partially ordered set $(X,\leqslant)$
$i(\varphi,U,K)$	p. 858	fixed point index of φ over U with respect to K
$P_{fc}\big(X\big)$	p. 877	the class of nonempty, closed and convex subsets of X
$P_{bf}\big(X\big)$	p. 877	the class of nonempty, bounded and closed subsets of X
$P_{bf(c)}X$	p. 877	the class of nonempty, bounded, closed (and convex) subsets of X
$P_{(w)kc}X$	p. 877	the class of nonempty, (weakly-)compact and convex subsets of X
$\mathcal{N}(x)$	p. 877	the filter of neighbourhoods of x
$\mathrm{Fix}\,(F)$	p. 879	the set of fixed points of $F\colon X \longrightarrow 2^X$
$I_C(x)$	p. 886	the inward set of $x \in C$ with respect to C
(X_∞,τ_∞)	p. 896	Alexandrov one-point compactification of X
$\{U_i\}_i \prec \{V_j\}_j$	p. 896	$\{U_i\}_{i\in I}$ is a refinement of $\{V_j\}_{j\in J}$
$V^b_a(f)$	p. 902	the total variation of $f\colon [a,b] \longrightarrow \mathbb{R}$
$BV\big([a,b]\big)$	p. 902	the space of functions of bounded variation on $[a,b]$

$BV_{\mathrm{loc}}(T)$	p. 902	the space of functions $f\colon T \longrightarrow \mathbb{R}$ which are locally of bounded variation
$AC\big([a,b]\big)$	p. 902	the space of absolutely continuous functions from $[a,b]$ into $\mathbb{R}$
$AC_{\mathrm{loc}}(T)$	p. 902	the space of locally absolutely continuous function $f\colon T \longrightarrow \mathbb{R}$
$\nu \prec\prec \mu$	p. 904	measure ν is absolutely continuous with respect to measure μ
$\dfrac{d\nu}{d\mu}$	p. 904	the Radon-Nikodym derivative of measure ν with respect to measure μ

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