Instructor's Solutions Manual to accompany

# Discrete Mathematics, 3e

By

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This manual contains complete solutions to all exercises in *Discrete Mathematics with Graph Theory*, Third Edition, by Edgar G. Goodaire and Michael M. Parmenter. It is intended solely for the use of instructors whom, we trust, will not make it available to students.

The solutions here have been read many times. We think that the number of errors is small, but are always grateful for help in improving accuracy. Our mailing address is below and our email addresses are `edgar@math.mun.ca` and `michael1@math.mun.ca`. Please feel free to contact either of us at any time about any aspect of our book or this solutions manual, both of which have been improved substantially by comments received since the first edition of this book appeared in 1998.

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# Solutions to Exercises

## Exercises 0.1

1. (a) [BB] True (b) [BB] Not valid (c) [BB] False ($0^2$ is not positive.) (d) Not valid
   (e) True (f) Not valid (g) False

2. (a) [BB] True, because both $4 = 2 + 2$ and $7 < \sqrt{50}$ are true statements.
   (b) False, because one of the two statements is false.
   (c) [BB] False, because 5 is not even.
   (d) True, because $16^{-1/4} = \frac{1}{2}$.
   (e) [BB] True, since $9 = 3^2$ is true (or because $3.14 < \pi$).
   (f) True, because $(-4)^2 = 16$ is true.
   (g) [BB] True, because both hypothesis and conclusion are true.
   (h) False, because the hypothesis is true but the conclusion is false.
   (i) [BB] Not a valid mathematical statement.
   (j) True, because both statements are true.
   (k) True, because this is an implication with false hypothesis.
   (l) False, because one of the statements is false while the other is true.
   (m) False, because the hypothesis is true but the conclusion is false.
   (n) [BB] False, because the area of a circle of radius $r$ is not $2\pi r$ and its circumference is not $\pi r^2$.
   (o) False, because the hypothesis of this implication is true, but the conclusion is false.
   (p) [BB] This is true: The hypothesis is true only when $a \geq b$ and $b \geq a$, that is, when $a = b$, and then the conclusion is also true.
   (q) This implication is true because the hypothesis is always false.

3. (a) [BB] If $x > 0$, then $\dfrac{1}{x} > 0$.
   (b) If $a$ and $b$ are rational numbers, then $ab$ is a rational number.
   (c) If $f$ is a differentiable function, then $f$ is continuous.
   (d) [BB] If $\mathcal{G}$ is a graph, then the sum of the degrees of the vertices of $\mathcal{G}$ is even.
   (e) [BB] If $A$ is a matrix and $A \neq 0$, then $A$ is invertible.
   (f) If $P$ is a parallelogram, then the diagonals of $P$ bisect each other.
   (g) If $n$ is an even integer, then $n < 0$.
   (h) If two vectors are orthogonal, then their dot product is 0.
   (i) If $n$ is an integer, then $\frac{n}{n+1}$ is not an integer.
   (j) If $n$ is a natural number, then $n + 3 > 2$.

4. (a) [BB] True (the hypothesis is false).
   (b) True (hypothesis and conclusion are both true).

(c) [BB] True (the hypothesis is false).

(d) False (hypothesis is true, conclusion is false).

(e) [BB] False (hypothesis is true, conclusion is false: $\sqrt{4} = 2$).

(f) True (g) [BB] True (h) True (i) [BB] True (the hypothesis is false: $\sqrt{x^2} = |x|$)
(j) True (k) [BB] False (l) True

5. (a) [BB] $a^2 \leq 0$ and $a$ is a real number (more simply, $a = 0$).

(b) $x$ is not real or $x^2 + 1 \neq 0$ (more simply, $x$ is any number, complex or real).

(c) [BB] $x \neq 1$ and $x \neq -1$.

(d) There exists an integer which is not divisible by a prime.

(e) [BB] There exists a real number $x$ such that $n \leq x$ for every integer $n$.

(f) $(ab)c = a(bc)$ for all $a, b, c$.

(g) [BB] Every planar graph can be colored with at most four colors.

(h) Some Canadian is a fan of neither the Toronto Maple Leafs nor the Montreal Canadiens.

(i) There exists $x > 0$ and some $y$ such that $x^2 + y^2 \leq 0$.

(j) $x \geq 2$ or $x \leq -2$.

(k) [BB] There exist integers $a$ and $b$ such that for all integers $q$ and $r$, $b \neq qa + r$.

(l) [BB] For any infinite set, some proper subset is not finite.

(m) For every real number $x$, there exists an integer $n$ such that $x \leq n < x + 1$.

(n) There exists an integer $n$ such that $\frac{n}{n+1}$ is an integer.

(o) $a > x$ or $a > y$ or $a > z$.

(p) There exists a vector in the plane and there exists a normal to the plane such that the vector is not orthogonal to the normal.

6. (a) [BB] Converse: If $\frac{a}{c}$ is an integer, then $\frac{a}{b}$ and $\frac{b}{c}$ are also integers.
Contrapositive: If $\frac{a}{c}$ is not an integer, then $\frac{a}{b}$ is not an integer or $\frac{b}{c}$ is not an integer.

(b) Converse: $x = \pm 1 \rightarrow x^2 = 1$.
Contrapositive: $x \neq 1$ and $x \neq -1 \rightarrow x^2 \neq 1$.

(c) Converse: If $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$, then $x^2 = x + 1$.
Contrapositive: If $x \neq 1 + \sqrt{5}$ and $x \neq 1 - \sqrt{5}$, then $x^2 \neq x + 1$.

(d) Converse: If $n^2 + n - 2$ is an even integer, then $n$ is an odd integer.
Contrapositive: If $n^2 + 2 - 2$ is an odd integer, then $n$ is an even integer.

(e) [BB] Converse: A connected graph is Eulerian.
Contrapositive: If a graph is not connected, then it is not Eulerian.

(f) Converse: $a = 0$ or $b = 0 \rightarrow ab = 0$.
Contrapositive: $a \neq 0$ and $b \neq 0 \rightarrow ab \neq 0$.

(g) Converse: A four-sided figure is a square.
Contrapositive: If a figure does not have four sides, then it is not a square.

(h) [BB] Converse: If $a^2 = b^2 + c^2$, then $\triangle BAC$ is a right triangle.
Contrapositive: If $a^2 \neq b^2 + c^2$, then $\triangle BAC$ is not a right triangle.

(i) Converse: If $p(x)$ is a polynomial with at least one real root, then $p(x)$ has odd degree.
Contrapositive: If $p(x)$ is a polynomial with no real roots, then $p(x)$ has even degree.

(j) Converse: A set of at most $n$ vectors is linearly independent.
Contrapositive: A set of more than $n$ vectors is not linearly independent.

(k) Converse: If $f$ is not one-to-one, then, for all real numbers $x$ and $y$, $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If $f$ is one-to-one, then there exist real numbers $x$ and $y$ such that $x = y$ or $x^2 + xy + y^2 + x + y \neq 0$.

(l) [BB] Converse: If $f$ is not one-to-one, then there exist real numbers $x$ and $y$ with $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If $f$ is one-to-one, then for all real numbers $x$ and $y$ either $x = y$ or $x^2 + xy + y^2 + x + y = 0$.

7. (a) [BB] There exists a continuous function which is not differentiable.
(b) $2^x \geq 0$ for all real numbers $x$.
(c) [BB] For every real number $x$, there exists a real number $y$ such that $y > x$.
(d) For every set of primes $p_1, p_2, \ldots, p_n$, there exists a prime not in this set.
(e) [BB] For every positive integer $n$, there exist primes $p_1, p_2, \ldots, p_t$ such that $n = p_1 p_2 \cdots p_t$.
(f) For every real number $x > 0$, there exists a real number $a$ such that $a^2 = x$.
(g) [BB] For every integer $n$, there exists an integer $m$ such that $m < n$.
(h) For every real number $x > 0$, there exists a real number $y > 0$ such that $y < x$.
(i) [BB] There exists a polynomial $p$ such that for every real number $x$, $p(x) \neq 0$.
(j) For every pair of real numbers $x$ and $y$ with $x < y$, there exists a rational number $a$ such that $x < a < y$.
(k) For every polynomial $p(x)$ of degree 3, there exists a real number $x$ such that $p(x) = 0$.
(l) There exists a matrix $A \neq 0$ such that $A$ is not invertible.
(m) There exists a real number $x$ such that $x \geq 0$.
(n) For any integer $n$, $n$ is not both even and odd.
(o) For all integers $a$, $b$, $c$, $a^3 + b^3 \neq c^3$.

8. If a given implication "$\mathcal{A} \rightarrow \mathcal{B}$" is false, then $\mathcal{A}$ is true and $\mathcal{B}$ is false. The converse, "$\mathcal{B} \rightarrow \mathcal{A}$" is then true because its hypothesis, $\mathcal{B}$, is false. It is **not** possible for both an implication and its converse to be false.

9. First we remember that $x$ **and** $y$ is true if $x$ and $y$ are both true and false otherwise. Now $p \leftrightarrow q$ means $p \rightarrow q$ **and** $q \rightarrow p$. Also

A. $p \rightarrow q$ is true if $p$ is false or if $p$ is true and $q$ is true.
B. $q \rightarrow p$ is true if $q$ is false or if $q$ is true and $p$ is true.

If $p$ and $q$ are both false, both statements A and B are true, so $p \leftrightarrow q$ is true. Similarly, if both $p$ and $q$ are true, then statements A and B are again true, so $p \leftrightarrow q$ is true. Thus $p \leftrightarrow q$ is true if $p$ and $q$ have the same truth values. Suppose $p$ and $q$ have different truth values. To be specific, say $p$ is true and $q$ is false. If $p$ is true and $q$ is false, we see that statement A is false, so $A$ **and** $B$ is false. Similarly, if $p$ is false and $q$ is true, then statement B is false, so $A$ **and** $B$ is false. This verifies statement $(*)$.

## Exercises 0.2

1. (a) [BB] Hypothesis: $a$ and $b$ are positive numbers.
   Conclusion: $a + b$ is positive.

   (b) Hypothesis: $T$ is a right angled triangle with hypotenuse of length $c$ and the other sides of lengths $a$ and $b$.
   Conclusion: $a^2 + b^2 = c^2$.

   (c) [BB] Hypothesis: $p$ is a prime.
   Conclusion: $p$ is even.

   (d) Hypothesis: $n > 1$ is an integer.
   Conclusion: $n$ is the product of prime numbers.

   (e) Hypothesis: A graph is planar.
   Conclusion: The chromatic number is 3.

2. (a) [BB] $a$ and $b$ are positive is sufficient for $a + b$ to be positive; $a + b$ is positive is necessary for $a$ and $b$ to be positive.

   (b) A right angled triangle has sides of lengths $a, b, c$, $c$ the hypotenuse, is sufficient for $a^2 + b^2 = c^2$; $a^2 + b^2 = c^2$ is necessary for a right angled triangle to have sides of lengths $a, b, c$, $c$ the hypotenuse.

   (c) [BB] $p$ is a prime is sufficient for $p$ to be even; $p$ is even is necessary for $p$ to be prime.

   (d) $n > 1$ an integer is sufficient for $n$ to be the product of primes; $n$ a product of primes is necessary for $n$ to be an integer bigger than 1.

   (e) A graph being planar is sufficient for its chromatic number to be 3. Chromatic number 3 is necessary for a graph to be planar.

3. (a) [BB] $x = -2$ (b) $a = b = -1$ (c) [BB] $x = 4$
   (d) $8, 9, 11, 12$ (e) $\sqrt{2}$ and $\dfrac{1}{\sqrt{2}}$ (f) $x = 5, y = 2$

4. $\mathcal{A}$ can easily be proven false with the counterexample 0. No single counterexample can disprove a statement claiming "there exists" so we prove $\mathcal{B}$ directly. $\mathcal{B}$ is false because the square of a real number is nonnegative.

5. [BB] This statement is true. Suppose the hypothesis, $x$ is an even integer, is true. Then $x = 2k$ for some other integer $k$. Then $x + 2 = 2k + 2 = 2(k + 1)$ is also twice an integer. So $x + 2$ is even. The conclusion is also true.

6. The converse is "$x + 2$ is an even integer $\rightarrow$ $x$ is an even integer." This is true, for suppose that the hypothesis, $x + 2$ is an even integer, is true. Then $x + 2 = 2k$ for some integer $k$, so $x = 2k - 2 = 2(k - 1)$ is also twice an integer. The conclusion is also true.

7. This is true. Let $\mathcal{A}$ be the statement "$x$ is an even integer" and let $\mathcal{B}$ be the statement $x + 2$ is an even integer". In Exercise 5, we showed that $\mathcal{A} \rightarrow \mathcal{B}$ is true and, in Exercise 6, that the converse $\mathcal{B} \rightarrow \mathcal{A}$ is also true. Thus $\mathcal{A} \leftrightarrow \mathcal{B}$ is also true.

8. (a) $\mathcal{A}$ is false: $n = 0$ is a counterexample.

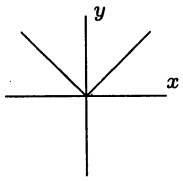
(b) Converse: If $\frac{n}{n+1}$ is not an integer, then $n$ is an integer. This is false: $n = \frac{1}{2}$ is a counterexample ($\frac{n}{n+1} = \frac{1}{3}$).
Contrapositive: If $\frac{n}{n+1}$ is an integer, then $n$ is not an integer. This is false: $n = 0$ is a counterexample.
Negation: There exists an integer $n$ such that $\frac{n}{n+1}$ is an integer. This is true: Take $n = 0$.

9. (a) $n$ prime $\rightarrow 2^n - 1$ prime.
   (b) $n$ prime is sufficient for $2^n - 1$ to be prime.
   (c) $\mathcal{A}$ is false. For example, $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23(89)$ is not. The integer $n = 11$ is a counterexample to $\mathcal{A}$.
   (d) $2^n - 1$ prime $\rightarrow n$ prime.
   (e) The converse of $\mathcal{A}$ is true. To show this, we establish the contrapositive. Thus, we assume $n$ is not prime. Then there exists a pair of integers $a$ and $b$ such that $a > 1$, $b > 1$, and $n = ab$. Using the hint, we can factor $2^n - 1$ as

$$2^n - 1 = (2^a)^b - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1].$$

   Since $a > 1$ and $b > 1$, we have $2^a - 1 > 1$ and $(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1 > 1$, so $2^n - 1$ is the product of two integers both of which exceed one. Hence, $2^n - 1$ is not prime.

10. [BB] The converse is the statement, "A continuous function is differentiable." This is false. The absolute value function whose graph is shown to the right is continuous, but not differentiable at $x = 0$.




11. (a) $2(n-1)$ (b) $n$
    (c) $\mathcal{A}_n \rightarrow \mathcal{A}_{n-1}, \mathcal{A}_{n-1} \rightarrow \mathcal{A}_{n-2}, \ldots, \mathcal{A}_2 \rightarrow \mathcal{A}_1, \mathcal{A}_1 \rightarrow \mathcal{A}_n$.

12. [BB] $\mathcal{A}$ is true. It expresses the fact that every real number lies between two consecutive integers. Statement $\mathcal{B}$ is most definitely false. It asserts that there is a remarkable integer $n$ with the property that every real number lies in the unit interval between $n$ and $n + 1$.

13. $\mathcal{A}$ is false; $\mathcal{B}$ is true. There can be no $y$ with the property described since $y$ is not bigger than $y + 1$; $x = y + 1$ provides a counterexample. To prove $\mathcal{B}$, we note that for every real number $x$, we have $x + 1 > x$ and so $x + 1$ is a suitable $y$.

14. (a) This is false. Suppose such an $n$ exists. Then $q = \dfrac{1}{n+1}$ is rational but $nq$ is not an integer.
    (b) This is true. Given a rational number $q$, there exist integers $m$ and $n$, $n \neq 0$, such that $q = \dfrac{m}{n}$. Then $nq = m$ is an integer.

15. (a) Since $n$ is even, $n = 2k$ for some integer $k$. Thus $n^2 + 3n = 4k^2 + 6k = 2(2k^2 + 3k)$ is even too.
    (b) The converse is the statement $n^2 + 3n$ even $\rightarrow n$ even. This is false and $n = 1$ is a counterexample.

16. (a) [BB] **Case 1:** $a$ is even. In this case, we have one of the desired conclusions.
    **Case 2:** $a$ is odd. In this case, $a = 2m + 1$ for some integer $m$, so $a + 1 = 2m + 2 = 2(m + 1)$ is even, another desired result.
    (b) [BB] $n^2 + n = n(n + 1)$ is the product of consecutive integers one of which must be even; so $n^2 + n$ is even.

17. $n^2 - n + 5 = n(n-1) + 5$. Now either $n-1$ or $n$ is even, since these integers are consecutive. So $n(n-1)$ is even. Since the sum of an even integer and the odd integer 5 is odd, the result follows.

18. [BB] $2x^2 - 4x + 3 = 2(x^2 - 2x) + 3 = 2[(x-1)^2 - 1] + 3 = 2(x-1)^2 + 1$ is the sum of 1 and a nonnegative number. So it is at least 1 and hence positive.

19. For $a^2 - b^2$ to be odd, it is necessary and sufficient for one of $a$ or $b$ to be even while the other is odd. Here's why.

    **Case i:** $a, b$ even.

    In this case, $a = 2n$ and $b = 2m$ for some integers $m$ and $n$, so $a^2 - b^2 = 4n^2 - 4m^2 = 4(n^2 - m^2)$ is even.

    **Case ii:** $a, b$ odd.

    In this case, $a = 2n + 1$ and $b = 2m + 1$ for some integers $m$ and $n$, so $a^2 - b^2 = (4n^2 + 4n + 1) - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m)$ is even.

    **Case iii:** $a$ even, $b$ odd.

    In this case, $a = 2n$ and $b = 2m+1$ for some integers $n$ and $m$, so $a^2 - b^2 = 4n^2 - (4m^2 + 4m + 1) = 4(n^2 - m^2 - m) - 1$ is odd.

    **Case iv:** $a$ odd, $b$ even.

    This is similar to Case iii, and the result follows.

20. [BB] ($\longrightarrow$) To prove this direction, we establish the contrapositive, that is, we prove that $n$ odd implies $n^2$ odd. For this, if $n$ is odd, then $n = 2m + 1$ for some integer $m$. Thus $n^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ is odd.

    ($\longleftarrow$) Here we assume that $n$ is even. Therefore, $n = 2m$ for some integer $m$. So $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$ which is even, as required.

21. We assert that $x + \dfrac{1}{x} \geq 2$ if and only if $x > 0$.

    **Proof.** ($\longrightarrow$) We offer a proof by contradiction. Suppose $x + \dfrac{1}{x} \geq 2$ but $x > 0$ is not true; thus $x \leq 0$. If $x = 0$, $\dfrac{1}{x}$ is not defined, so $x < 0$. In this case, however, $x + \dfrac{1}{x} < 0$, a contradiction.

    ($\longleftarrow$) Conversely, assume that $x > 0$. Note that $(x-1)^2 \geq 0$ implies $x^2 - 2x + 1 \geq 0$, which in turn implies $x^2 + 1 \geq 2x$. Division by the positive number $x$ gives $x + \dfrac{1}{x} \geq 2$ as required. ■

22. [BB] Since $n$ is odd, $n = 2k + 1$ for some integer $k$.

    **Case 1:** $k$ is even.

    In this case $k = 2m$ for some integer $m$, so $n = 2(2m) + 1 = 4m + 1$.

    **Case 2:** $k$ is odd.

    In this case, $k = 2m + 1$ for some integer $m$, so $n = 2(2m + 1) + 1 = 4m + 3$.

    Since each case leads to one of the desired conclusions, the result follows.

23. By Exercise 22, there exists an integer $k$ such that $n = 4k + 1$ or $n = 4k + 3$.

    **Case 1:** $n = 4k + 1$.

    If $k$ is even, there exists an integer $m$ such that $k = 2m$, so $n = 4(2m) + 1 = 8m + 1$, and the desired conclusion is true. If $k$ is odd, there exists an integer $m$ such that $k = 2m + 1$, so $n = 4(2m + 1) + 1 = 8m + 5$, and the desired conclusion is true.

    **Case 2:** $n = 4k + 3$.

    If $k$ is even, there exists an integer $m$ such that $k = 2m$, so $n = 4(2m) + 3 = 8m + 3$, and the desired conclusion is true. If $k$ is odd, there exists an integer $m$ such that $k = 2m + 1$, so $n = 4(2m + 1) + 3 = 8m + 7$, and the desired conclusion is true. In all cases, the desired conclusion is true.

24. [BB] If the statement is false, then there does exist a smallest positive real number $r$. Since $\frac{1}{2}r$ is positive and smaller than $r$, we have reached an absurdity. So the statement must be true.

25. We give a proof by contradiction. If the result is false, then both $a > \sqrt{n}$ and $b > \sqrt{n}$. (Note that the negation of an "or" statement is an "and" statement.) But then $n = ab > \sqrt{n}\sqrt{n} = n$, which isn't true.

26. [BB] Since 0 is an eigenvalue of $A$, there is a nonzero vector $\mathsf{x}$ such that $A\mathsf{x} = 0$. Now suppose that $A$ is invertible. Then $A^{-1}(A\mathsf{x}) = A^{-1}0 = 0$, so $\mathsf{x} = 0$, a contradiction.

27. (a) The given equation is equivalent to $(b - 5)\sqrt{2} = 3 - a$. If $b \neq 5$, then $\sqrt{2} = \frac{3-a}{b-5}$ is a rational number. This is false. Thus $b = 5$, so $a = 3$.

    (b) Note that $(a + b\sqrt{2})^2 = (a^2 + 2b^2) + 2ab\sqrt{2}$. Thus, if $(a + b\sqrt{2})^2 = 3 + 5\sqrt{2}$, then $a^2 + 2b^2 = 3$ and $2ab = 5$, by part (a). The second equation says $a \neq 0$ and $b \neq 0$. Since $a$ and $b$ are integers, it follows that $a^2 \geq 1$ and $b^2 \geq 1$ and $a^2 + 2b^2 \geq 3$ with equality if and only $a = \pm 1 = b$. But then $2ab \neq 5$.

28. [BB] Observe that $(1+a)(1+b) = 1+a+b+ab = 1$. Thus $1+a$ and $1+b$ are integers whose product is 1. There are two possibilities: $1 + a = 1 + b = 1$, in which case $a = b = 0$, or $1 + a = 1 + b = -1$, in which case $a = b = -2$.

29. We offer a proof by contradiction. Suppose $\dfrac{1}{a}$ is not irrational. Then it is rational, so there exist integers $m$ and $n$, $n \neq 0$, such that $\dfrac{1}{a} = \dfrac{m}{n}$. Since $\dfrac{1}{a} \neq 0$, we know also that $m \neq 0$. Now $\dfrac{1}{a} = \dfrac{m}{n}$ implies $a = \dfrac{n}{m}$ is a rational number, a contradiction.

30. We give a proof by contradiction. Assume that $a$ is rational, $b$ is irrational and $a + b$ is rational. Then $a + b = \frac{m}{n}$ for integers $m$ and $n$, $n \neq 0$. Since $a$ is rational, $a = \frac{k}{\ell}$ for integers $k$ and $\ell$, $\ell \neq 0$. Thus

$$b = \frac{m}{n} - a = \frac{m}{n} - \frac{k}{\ell} = \frac{m\ell - kn}{n\ell}$$

    is the quotient of integers with nonzero denominator. This contradicts the fact that $b$ is not rational.

31. [BB] We begin by assuming the negation of the desired conclusion; in other words, we assume that there exist real numbers $x, y, z$ which simultaneously satisfy each of these three equations. Subtracting the second equation from the first we see that $x + 5y - 4z = -2$. Since the third equation we were given says $x + 5y - 4z = 0$, we have $x + 5y - 4z$ equal to both 0 and to $-2$. Thus, the original assumption has led us to a contradiction.

32. (a) [BB] False: $x = y = 0$ is a counterexample.

    (b) False: $a = 6$ is a counterexample.

    (c) [BB] False: $x = 0$ is a counterexample.

    (d) False: $a = \sqrt{2}, b = -\sqrt{2}$ is a counterexample.

    (e) [BB] False: $a = b = \sqrt{2}$ is a counterexample.

    (f) The roots of the polynomial $ax^2 + bx + c$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ is real and not 0, so the formula produces two distinct real numbers $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

    (g) False: $x = \frac{1}{2}$ is a counterexample.

    (h) True: If $n$ is a positive integer, then $n \geq 1$, so $n^2 = n(n) \geq n$.

33. The result is false. A square and a rectangle (which is not a square) have equal angles but not pairwise proportional sides.

34. (a) [BB] Since $n^2+1$ is even, $n^2$ is odd, so $n$ must also be odd. Writing $n = 2k+1$, then $n^2+1 = 2m$ says $4k^2 + 4k + 2 = 2m$, so $m = 2k^2 + 2k + 1 = (k+1)^2 + k^2$ is the sum of two squares as required.

    (b) [BB] We are given that $n^2+1 = 2m$ for $n = 4373$ and $m = 9561565$. Since $n = 2(2186)+1$, our solution to (a) shows that $m = k^2 + (k+1)^2$ where $k = 2186$. Thus, $9561565 = 2186^2 + 2187^2$.

35. (a) $2^{4n+2} + 1 = 4(2^{4n}) + 1 = 4(2^n)^4 + 1$. Applying the given identity with $x = 2^n$, we get

$$\begin{aligned} 2^{4n+2} + 1 &= (2 \cdot 2^{2n} + 2^{n+1} + 1)(2 \cdot 2^{2n} - 2^{n+1} + 1) \\ &= (2^{2n+1} + 2^{n+1} + 1)(2^{2n+1} - 2^{n+1} + 1). \end{aligned}$$

    With $n = 4$, we get $2^{18} + 1 = (2^9 + 2^5 + 1)(2^9 - 2^5 + 1) = 545(481)$.

    (b) $2^{36} - 1 = (2^{18} - 1)(2^{18} + 1) = (2^9 - 1)(2^9 + 1)(545)(481)$ (using the result of part (a))
    $= 511(513)(545)(481)$.

36. If the result is false, then $f(n) = a_0 + a_1 n + \cdots + a_t n^t$ for some $t \geq 1$. Since $f(0) = a_0 = p$ is prime, $f(n) = p + ng(n)$ for $g(n) = a_1 + a_2 n + \cdots + a_t n^{t-1}$. Replacing $n$ by $pn$, we have $f(pn) = p + npg(pn)$. The right hand side is divisible by the prime $p$, hence $f(pn)$ is divisible by $p$. But $f(pn)$ is prime, by hypothesis, so $f(pn) = p$. This means $g(pn) = 0$, contradicting the fact that a polynomial has only finitely many roots.

37. We offer a proof by contradiction. Suppose all the digits occur just a finite number of times. Then there is a number $n_1$ which has the property that after $n_1$ digits in the decimal expansion of $\pi$, the digit 1 no longer occurs. Similarly, there is a number $n_2$ such that after $n_2$ digits, the digit 2 no longer occurs, and so on. In general, for each $k = 1, 2, \ldots, 9$, there is a number $n_k$ such that after $n_k$ digits, the digit $k$ no longer occurs. Let $N$ be the largest of the numbers $n_1, n_2, \ldots, n_k$. Then after $N$ digits in the decimal expansion of $\pi$, the only digit which can appear is 0. This contradicts the fact that the decimal expansion of $\pi$ does not terminate.

38. We have proven in the text that $\sqrt{2}$ is irrational. Thus, if $\sqrt{2}^{\sqrt{2}}$ is rational, we are done (with $a = b = \sqrt{2}$). On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ in which case $a^b = \sqrt{2}^2 = 2$ is rational.

# Chapter 0 Review

1. (a) This implication is true because the hypothesis is always false: $a - b > 0$ and $b - a > 0$ give $a > b$ and $b > a$, which never holds.
   (b) This implication is false: When $a = b$, the hypotheses are true while the conclusion is false.
2. (a) $x$ is a real number and $x \leq 5$.
   (b) For every real number $x$, there exists an integer $n$ such that $n \leq x$.
   (c) There exist positive integers $x$, $y$, $z$ such that $x^3 + y^3 = z^3$.
   (d) There exists a graph with $n$ vertices and $n + 1$ edges whose chromatic number is more than 3.
   (e) There exists an integer $n$ such that for any rational number $a$, $a \neq n$.
   (f) $a \neq 0$ or $b \neq 0$.
3. (a) Converse: If $ab$ is an integer, then $a$ and $b$ are integers.
   Contrapositive: If $ab$ is not an integer, then either $a$ or $b$ is not an integer.
   Negation: There exist integers $a$ and $b$ such that $ab$ is not an integer.
   (b) Converse: If $x^2$ is an even integer, then $x$ is an even integer.
   Contrapositive: If $x^2$ is an odd integer, then $x$ is an odd integer.
   Negation: There exists an even integer $x$ such that $x^2$ is odd.
   (c) Converse: Every graph which can be colored with at most four colors is planar.
   Contrapositive: Every graph which cannot be colored with at most four colors is not planar.
   Negation: There exists a planar graph which cannot be colored with at most four colors.
   (d) Converse: A matrix which equals its transpose is symmetric.
   Contrapositive: If a matrix does not equal its transpose, then it is not symmetric.
   Negation: There exists a symmetric matrix which is not equal to its transpose.
   (e) Converse: A set of at least $n$ vectors is a spanning set.
   Contrapositive: A set of less than $n$ vectors is not a spanning set.
   Negation: There exists a spanning set containing less than $n$ vectors.
   (f) Converse: If $x > -2$ and $x < 1$, then $x^2 + x - 2 < 0$.
   Contrapositive: If $x \leq -2$ or $x \geq 1$, then $x^2 + x - 2 \geq 0$.
   Negation: There exists an $x \leq -2$ or $x \geq 1$ such that $x^2 + x - 2 < 0$.
4. (a) A is false: $a = u = b = 1, v = -1$ provides a counterexample.
   (b) Converse: Given four integers $a, b, u, v$ with $u \neq 0$, $v \neq 0$, if $a = b = 0$, then $au + bv = 0$.
   Negation: There exist integers $a, b, u, v$, $u \neq 0$, $v \neq 0$, with $au + bv = 0$ and $a \neq 0$ or $b \neq 0$.
   Contrapositive: Given four integers $a, b, u, v$ with $u \neq 0$ and $v \neq 0$, if $a \neq 0$ or $b \neq 0$, then $au + bv \neq 0$.

(c) The converse is certainly true since $0u + 0v = 0$.

(d) The negation is true: Take $a = u = b = 1$ and $v = -1$.
The contrapositive of A is false since A is false.

5. (a) There exists a countable set which is infinite.

(b) For all positive integers $n$, $1 \leq n$.

6. (a) This is true. If $x$ is positive, $x + 2$ is positive. In addition, if $x$ is odd, $x + 2$ is odd.

(b) This is false. When $x = -1$, $x + 2 = +1$ is a positive odd integer, while $x$ is not.

7. (a) This statement expresses a well-known property of the real numbers. It is true.

(b) This is false. The conclusion would have us believe that every two real numbers are equal.

8. The desired formula is $ab = \dfrac{(a+b)^2 - (a-b)^2}{4}$ which holds because $(a+b)^2 - (a-b)^2 = (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab$.

9. ($\longrightarrow$) Assume $n^3$ is odd and suppose, to the contrary, that $n$ is even. Thus $n = 2x$ for some integer $x$. But then $n^3 = 8x^3 = 2(4x^3)$ is even, a contradiction. This means that $n$ must be odd.

($\longleftarrow$) Assume $n$ is odd. This means that $n = 2x + 1$ for some integer $x$. Then $n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$ is odd.

10. (a) $a^2 - 5a + 6 = (a-2)(a-3)$ is the product of two consecutive integers, one of which must be even.

(b) The sum $(a^2 - 5b) + (b^2 - 5a)$ is $(a^2 - 5a) + (b^2 - 5b) = (a^2 - 5a + 6) + (b^2 - 5b + 6) - 12$ is the sum of three even integers [using the result of part (a)] and hence even. Thus $b^2 - 5a$ is the difference of even integers and hence even as well.

11. The sum of the angles of a triangle is 180°, so $\angle C = 45°$ and $\angle D = 75°$. Since triangles $ABC$ and $DEF$ are similar, $\dfrac{AC}{AB} = \dfrac{DE}{DF}$, so the length of $AC$ is $|AB| \times \dfrac{DE}{DF} = 12 \times \dfrac{8}{6} = 16$.

12. The rectangle that remains has dimensions 1 by $\tau - 1$. These are in ratio

$$\frac{1}{\tau - 1} = \frac{\tau + 1}{(\tau - 1)(\tau + 1)} = \frac{\tau + 1}{\tau^2 - 1} = \frac{\tau + 1}{\tau} = \tau$$

using twice the fact that $\tau^2 = \tau + 1$.

13. If the result is false, then $x \geq -1$ and $x \leq 2$, so $x + 1 \geq 0$ and $x - 2 \leq 0$. But then $x^2 - x - 2 = (x+1)(x-2) \leq 0$, a contradiction.

14. Suppose, to the contrary, that $-\dfrac{x}{y}$ is the largest negative rational number, where $x$ and $y$ are positive integers. Then $\dfrac{x}{2y}$ is rational and $\dfrac{x}{2y} < \dfrac{x}{y}$, so $-\dfrac{x}{2y} > -\dfrac{x}{y}$. Since $-\dfrac{x}{2y}$ is a negative rational number, we have a contradiction.

15. We use a proof by contradiction which mimics that proof of the irrationality of $\sqrt{2}$ given in Problem 8. Thus, we suppose that $\sqrt{3} = \frac{a}{b}$ is rational and hence the quotient of integers $a$ and $b$ which have no factors in common. Squaring gives $a^2 = 3b^2$ and so $a = 3k$ is a multiple of 3. But then $9k^2 = 3b^2$, so $3k^2 = b^2$. This says that $b$ is also a multiple of 3, contradicting our assumption that $a$ and $b$ have no factors in common.

16. Let the rational numbers be $\frac{a}{b}$ and $\frac{c}{d}$. We may assume that $a, b, c, d$ are positive integers and that $\frac{a}{b} < \frac{c}{d}$. Thus $ad < bc$. The hint suggests that $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$, and this is the case: $\frac{a}{b} < \frac{a+c}{b+d}$ is equivalent to $a(b+d) < b(a+c)$ and $\frac{a+c}{b+d} < \frac{c}{d}$ is equivalent to $(a+c)d < (b+d)c$, both of which are true because $ad < bc$.

17. On a standard checker board, there are 32 squares of one color and 32 of another. Since squares in opposite corners have the same color, the hint shows that our defective board has 32 squares of one color and 30 of the other. Since each domino covers one square of each color, the result follows.

18. (a) We leave the primality checking of $f(1), \ldots, f(39)$ to the reader, but note that $f(40) = 41^2$.

    (b) $f(k^2+40) = 40^2 + 80k^2 + k^4 + 40 + k^2 + 41 = k^4 + 81k^2 + 41^2 = (k^2+41)^2 - k^2 = (k^2+41+k)(k^2+41-k)$.

19. The answer is no, since $333333331 = 19607843 \times 17$.

## Exercises 1.1

1. (a) [BB]

| $p$ | $q$ | $\neg q$ | $(\neg q) \vee p$ | $p \wedge ((\neg q) \vee p)$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |

(b)

| $p$ | $q$ | $\neg p$ | $(\neg p) \rightarrow q$ | $p \wedge q$ | $(p \wedge q) \vee ((\neg p) \rightarrow q)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |

(c)

| $p$ | $q$ | $q \vee p$ | $p \wedge (q \vee p)$ | $\neg\,(p \wedge (q \vee p))$ | $\neg\,(p \wedge (q \vee p)) \leftrightarrow p$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ |

(d) [BB]

| $p$ | $q$ | $r$ | $\neg q$ | $p \vee (\neg q)$ | $\neg\,(p \vee (\neg q))$ | $\neg p$ | $(\neg p) \vee r$ | $(\neg\,(p \vee (\neg q))) \wedge ((\neg p) \vee r)$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |

(e)

| $p$ | $q$ | $r$ | $q \to r$ | $p \to (q \to r)$ | $p \wedge q$ | $(p \wedge q) \vee r$ | $(p \to (q \to r)) \to ((p \wedge q) \vee r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |

2. (a) If $p \to q$ is false, then necessarily $p$ is true and $q$ is false. (This is the only situation in which $p \to q$ is false.) We construct the relevant row of the truth table for $\big(p \wedge (\neg q)\big) \vee \big((\neg p) \to q\big)$.

| $p$ | $q$ | $\neg q$ | $p \wedge (\neg q)$ | $\neg p$ | $(\neg p) \to q$ |
|---|---|---|---|---|---|
| $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |

| $\big(p \wedge (\neg q)\big) \vee \big((\neg p) \to q\big)$ |
|---|
| $T$ |

(b) [BB] There are three situations in which $p \to q$ is true. The question then is whether or not the truth value of $\big(p \wedge (\neg q)\big) \vee \big((\neg p) \to q\big)$ is the same in each of these cases. We construct a partial truth table.

| $p$ | $q$ | $\neg q$ | $p \wedge (\neg q)$ | $\neg p$ | $(\neg p) \to q$ | $\big(p \wedge (\neg q)\big) \vee \big((\neg p) \to q\big)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ |

As shown, $\big(p \wedge (\neg q)\big) \vee \big((\neg p) \to q\big)$ has different truth values on two occasions where $p \to q$ is true, so it is **not** possible to answer the question in this case.

3. [BB]

| $p$ | $q$ | $r$ | $s$ | $\neg r$ | $q \wedge (\neg r)$ | $p \to [q \wedge (\neg r)]$ | $\neg s$ | $(\neg s) \vee q$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |

| $r \leftrightarrow [(\neg s) \vee q]$ | $[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$ |
|---|---|
| $T$ | $T$ |

4.

| $p$ | $q$ | $r$ | $s$ | $\neg r$ | $q \wedge (\neg r)$ | $p \to [q \wedge (\neg r)]$ | $\neg s$ | $(\neg s) \vee q$ |
|---|---|---|---|---|---|---|---|---|
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |

| $r \leftrightarrow [(\neg s) \vee q]$ | $[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$ |
|---|---|
| $F$ | $T$ |

5. (a) [BB]

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $(p \wedge q) \to (p \vee q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

The final column shows that $(p \wedge q) \to (p \vee q)$ is true for all values of $p$ and $q$, so this statement is a tautology.

(b) [BB]

| $p$ | $q$ | $\neg p$ | $(\neg p) \wedge q$ | $\neg q$ | $p \vee (\neg q)$ | $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ |

The final column shows that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is false for all values of $p$ and $q$, so this statement is a contradiction.

6. (a)

| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow (p \rightarrow q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ |

Since $q \rightarrow (p \rightarrow q)$ is true for all values of $p$ and $q$, this statement is a tautology.

(b)

| $p$ | $q$ | $p \wedge q$ | $\neg p$ | $\neg q$ | $((\neg p) \vee (\neg q))$ | $(p \wedge q) \wedge [(\neg p) \vee (\neg q)]$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

Since $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$ is false for all values of $p$ and $q$, this statement is a contradiction.

7. (a) [BB]

| $p$ | $q$ | $r$ | $p \rightarrow q$ | $q \rightarrow r$ | $(p \rightarrow q) \wedge (q \rightarrow r)$ | $p \rightarrow r$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

| $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$ |
|---|
| $T$ |
| $T$ |
| $T$ |
| $T$ |
| $T$ |
| $T$ |
| $T$ |
| $T$ |

Since $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$ is true for all values of $p$, $q$, and $r$, this statement is a tautology.

(b) [BB] If $p$ implies $q$ which, in turn, implies $r$, then certainly $p$ implies $r$.

8. We must show that the given "or" statement can be both true and false. We construct truth tables for each part of the "or" and show that certain identical values for the variables make both parts $T$ (so that the "or" is true) and other certain identical values for the variables make both parts $F$ (so that the "or" is false).

| $p$ | $r$ | $s$ | $\neg r$ | $\neg s$ | $(\neg r) \rightarrow (\neg s)$ | $p \vee [(\neg r) \rightarrow (\neg s)]$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |

| $p$ | $q$ | $r$ | $s$ | $t$ | $\neg t$ | $(\neg t) \vee p$ | $s \rightarrow [(\neg t) \vee p]$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |

| $\neg q$ | $(\neg q) \rightarrow r$ | $[s \rightarrow ((\neg t) \vee p)] \vee [(\neg q) \rightarrow r]$ |
|---|---|---|
| $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |

9. We are given that $\mathcal{A}$ is false for any values of its variables.

   (a) [BB] An implication $p \rightarrow q$ is false only if $p$ is true and $q$ is false. Since $\mathcal{A}$ is always false, $\mathcal{A} \rightarrow \mathcal{B}$ is always true. So it is a tautology.

   (b) An implication $p \rightarrow q$ is false only if $p$ is true and $q$ is false. Since $\mathcal{A}$ is false and the tautology $\mathcal{B}$ is true for any values of the variables they contain, $\mathcal{B} \rightarrow \mathcal{A}$ is always false. So it is a contradiction.

10. (a) The tables below show that when all three variables $p$, $q$ and $r$ are false, $p \rightarrow (q \rightarrow r)$ is true, whereas $(p \rightarrow q) \rightarrow r$ is false. Thus these statements have different truth tables and hence are not logically equivalent.

| $p$ | $q$ | $r$ | $q \rightarrow r$ | $p \rightarrow (q \rightarrow r)$ |
|---|---|---|---|---|
| $F$ | $F$ | $F$ | $T$ | $T$ |

| $p$ | $q$ | $r$ | $p \rightarrow q$ | $(p \rightarrow q) \rightarrow r$ |
|---|---|---|---|---|
| $F$ | $F$ | $F$ | $T$ | $F$ |

   (b) The compound statement is false.

11. (a) [BB]

| $p$ | $q$ | $p \veebar q$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

   (b)

| $p$ | $q$ | $\neg p$ | $(\neg p) \wedge q$ | $p \veebar ((\neg p) \wedge q)$ | $(p \veebar ((\neg p) \wedge q)) \vee q$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |

   (c) [BB]

| $p$ | $q$ | $p \veebar q$ | $p \vee q$ | $(p \veebar q) \rightarrow (p \vee q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

The truth table shows that $(p \veebar q) \rightarrow (p \vee q)$ is true for all values of $p$ and $q$, so it is a tautology.

   (d)

| $p$ | $q$ | $p \veebar q$ | $p \leftrightarrow q$ | $\neg (p \leftrightarrow q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ |

Columns three and five are the same. So the truth values of $p \veebar q$ and $\neg (p \leftrightarrow q)$ are the same for all values of $p$ and $q$. Thus these statements are logically equivalent.

## Exercises 1.2

1. 1. [BB] (Idempotence) The truth tables at the right show that $p \vee p \iff p$ and $p \wedge p \iff p$.

| $p$ | $p \vee p$ |
|---|---|
| $T$ | $T$ |
| $F$ | $F$ |

| $p$ | $p \wedge p$ |
|---|---|
| $T$ | $T$ |
| $F$ | $F$ |

2. (Commutativity) The truth tables show that $p \vee q \iff q \vee p$ and $p \wedge q \iff q \wedge p$.

| $p$ | $q$ | $p \vee q$ | $q \vee p$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

| $p$ | $q$ | $p \wedge q$ | $q \wedge p$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

3. [BB] (Associativity) The equality of the fifth and seventh columns in the truth table shows that $((p \vee q) \vee r) \iff (p \vee (q \vee r))$.

| $p$ | $q$ | $r$ | $p \vee q$ | $(p \vee q) \vee r$ | $q \vee r$ | $p \vee (q \vee r)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

The equality of the fifth and seventh columns in the truth table shows that $((p \wedge q) \wedge r) \iff (p \wedge (q \wedge r))$.

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \wedge r$ | $q \wedge r$ | $p \wedge (q \wedge r)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

4. (Distributivity) The equality of the fifth and eighth columns in the truth table shows that $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r))$.

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

The equality of the fifth and eighth columns in the truth table shows that $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$.

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

5. [BB] (Double negation) The equality of the first and third columns in the truth table shows that $p \iff \neg(\neg p)$.

| $p$ | $\neg p$ | $\neg(\neg p)$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |

6. (The Laws of De Morgan) The equality of the fourth and seventh columns of the truth table shows that $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$.

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $(\neg p) \wedge (\neg q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

The equality of the fourth and seventh columns of the truth table shows that $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $(\neg p) \vee (\neg q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

7. [BB] The two truth tables show, respectively, that $p \vee \mathbf{1} \iff \mathbf{1}$ and $p \wedge \mathbf{1} \iff p$.

| $p$ | $\mathbf{1}$ | $p \vee \mathbf{1}$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ |

| $p$ | $\mathbf{1}$ | $p \wedge \mathbf{1}$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ |

8. The two truth tables show, respectively, that $p \vee \mathbf{0} \iff p$ and $p \wedge \mathbf{0} \iff \mathbf{0}$.

| $p$ | $\mathbf{0}$ | $p \vee \mathbf{0}$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ |

| $p$ | $\mathbf{0}$ | $p \wedge \mathbf{0}$ |
|---|---|---|
| $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ |

9. [BB] The two truth tables show, respectively, that $(p \vee (\neg p)) \iff \mathbf{1}$ and $(p \wedge (\neg p)) \iff \mathbf{0}$.

| $p$ | $\neg p$ | $p \vee (\neg p)$ | $\mathbf{1}$ |
|---|---|---|---|
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |

| $p$ | $\neg p$ | $p \wedge (\neg p)$ | $\mathbf{0}$ |
|---|---|---|---|
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ |

10. The truth tables show, respectively, that $(\neg \mathbf{1}) \iff \mathbf{0}$ and $(\neg \mathbf{0}) \iff \mathbf{1}$.

| $\mathbf{1}$ | $\neg \mathbf{1}$ | $\mathbf{0}$ |
|---|---|---|
| $T$ | $F$ | $F$ |

| $\mathbf{0}$ | $\neg \mathbf{0}$ | $\mathbf{1}$ |
|---|---|---|
| $F$ | $T$ | $T$ |

11. [BB] The third and sixth columns of the truth table show that $(p \to q) \iff ((\neg q) \to (\neg p))$.

| $p$ | $q$ | $p \to q$ | $\neg q$ | $\neg p$ | $(\neg q) \to (\neg p)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

12. The third and sixth columns of the truth table show that $(p \leftrightarrow q) \iff ((p \to q) \wedge (q \to p))$.

| $p$ | $q$ | $p \leftrightarrow q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

13. [BB] The third and fifth columns of the truth table show that $(p \to q) \iff ((\neg p) \vee q)$.

| $p$ | $q$ | $p \to q$ | $\neg p$ | $(\neg p) \vee q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

2. (a) We construct a truth table. Since $p \vee [\neg(p \wedge q)]$ is true for all values of $p$ and $q$, this statement is a tautology.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $p \vee [\neg(p \wedge q)]$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ |

(b) By one of the laws of DeMorgan, the negation is $(\neg p) \wedge (p \wedge q)$. By associativity, this is logically equivalent to $[(\neg p) \wedge p] \wedge q \iff \mathbf{0} \wedge q \iff 0$. So the negation is a contradiction.

3. (a) [BB] Using one of the laws of De Morgan and one distributive property, we obtain

$$[(p \wedge q) \vee (\neg((\neg p) \vee q))] \iff [(p \wedge q) \vee (p \wedge (\neg q))]$$
$$\iff [p \wedge (q \vee (\neg q))] \iff (p \wedge \mathbf{1}) \iff p.$$

(b) The given compound statement is of the form $x \to (y \to z)$ which is equivalent to $x \to ((\neg y) \vee z) \iff (\neg x) \vee (\neg y) \vee z$. (By associativity, no further parentheses are required here.) So the given statement is equivalent to $(\neg(p \vee r)) \vee ((\neg q) \wedge r) \vee (p \vee r)$. By commutativity, this is $(\neg(p \vee r)) \vee (p \vee r) \vee ((\neg q) \wedge r) \iff \mathbf{1} \vee ((\neg q \wedge r) \iff \mathbf{1}$. The given statement is a tautology!

(c) Using associativity to avoid extra parentheses, the left side of the given statement is

$$[(p \to q) \vee (q \to r)] \iff (\neg p) \vee q \vee (\neg q) \vee r$$
$$\iff (\neg p) \vee \mathbf{1} \vee r \iff \mathbf{1}.$$

The given statement is equivalent to $[1 \wedge (r \to s)]$ which is logically equivalent to $r \to s$.

4. (a) [BB]

| $p$ | $q$ | $p \wedge q$ | $p \vee (p \wedge q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

(b)

| $p$ | $q$ | $p \vee q$ | $p \wedge (p \vee q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

5. (a) [BB] Distributivity gives $[(p \vee q) \wedge (\neg p)] \iff [(p \wedge (\neg p)) \vee (q \wedge (\neg p))] \iff [\mathbf{0} \vee ((\neg p) \wedge q)] \iff [(\neg p) \wedge q]$.

(b) We have $(p \to (q \to r)) \iff (p \to ((\neg q) \vee r)) \iff (\neg p) \vee (\neg q) \vee r$
$\iff (\neg p) \vee r \vee (\neg q) \iff \neg(p \wedge (\neg r)) \vee (\neg q) \iff (p \wedge (\neg r)) \to (\neg q)$.

(c) $(\neg(p \leftrightarrow q)) \iff (\neg((p \to q) \wedge (q \to p))) \iff (\neg(((\neg p) \vee q) \wedge ((\neg q) \vee p)))$

$$\begin{aligned}
&\iff ((p \wedge (\neg q)) \vee (q \wedge (\neg p))) \\
&\iff ((p \wedge (\neg q)) \vee q) \wedge ((p \wedge (\neg q)) \vee (\neg p)) \\
&\iff ((p \vee q) \wedge ((\neg q) \vee q)) \wedge ((p \vee (\neg p)) \wedge ((\neg q) \vee (\neg p))) \\
&\iff ((p \vee q) \wedge \mathbf{1}) \wedge (\mathbf{1} \wedge ((\neg q) \vee (\neg p))) \\
&\iff (p \vee q) \wedge ((\neg q) \vee (\neg p)) \\
&\iff ((\neg p) \vee (\neg q)) \wedge (q \vee p) \\
&\iff (p \to (\neg q)) \wedge ((\neg q) \to p) \iff (p \leftrightarrow (\neg q)).
\end{aligned}$$

(d) [BB] $\neg[(p \leftrightarrow q) \vee (p \wedge (\neg q))] \iff [\neg(p \leftrightarrow q) \wedge \neg(p \wedge (\neg q))]$
$\iff [(p \leftrightarrow (\neg q)) \wedge ((\neg p) \vee q)]$, using Exercise 5(c).

(e) This is an immediate application of absorption law **4(b)** with $p \wedge (\neg q)$ in place of $p$ and $q \wedge (\neg r)$ in place of $q$.

(f) Using property 12 and associativity, $[p \to (q \vee r)] \iff [(\neg p) \vee q \vee r] \iff [\neg(p \wedge (\neg q))] \vee r$ (by De Morgan) $\iff [p \wedge (\neg q) \to r]$ using 12 again.

(g) $\neg(p \vee q) \vee [(\neg p) \wedge q] \iff [(\neg p) \wedge (\neg q)] \vee [(\neg p) \wedge q]$ (DeMorgan) $\iff (\neg p) \wedge [(\neg q) \vee q]$ (distributivity) $\iff (\neg p) \vee \mathbf{1} \iff \neg p$

6. $[(p \wedge (\neg q)) \to q] \iff [(\neg(p \wedge (\neg q))) \vee q] \iff [((\neg p) \vee q) \vee q] \iff [(\neg p) \vee q]$.
$[(p \wedge (\neg q)) \to (\neg p)] \iff [(\neg(p \wedge (\neg q))) \vee (\neg p)] \iff [(\neg p) \vee q \vee (\neg p)] \iff [(\neg p) \vee q]$.
So these are both logically equivalent to $(\neg p) \vee q$.

7. (a) We must show that $\mathcal{A} \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ have the same truth tables, given that $\mathcal{A}$ and $\mathcal{B}$ have the same truth tables. This requires four rows of a truth table.

| $\mathcal{A}$ | $\mathcal{B}$ | $\mathcal{C}$ | $\mathcal{A} \vee \mathcal{C}$ | $\mathcal{B} \vee \mathcal{C}$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

The last two columns establish our claim.

(b) We must show that $\mathcal{A} \wedge \mathcal{C}$ and $\mathcal{B} \wedge \mathcal{C}$ have the same truth tables, given that $\mathcal{A}$ and $\mathcal{B}$ have the same truth tables. This requires four rows of a truth table.

| $\mathcal{A}$ | $\mathcal{B}$ | $\mathcal{C}$ | $\mathcal{A} \wedge \mathcal{C}$ | $\mathcal{B} \wedge \mathcal{C}$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

The last two columns establish our claim.

8. 1. [BB] The truth table shows that idempotence fails.

| $p$ | $p \veebar p$ |
|---|---|
| $T$ | $F$ |
| $F$ | $F$ |

2. Commutativity still holds.

| $p$ | $q$ | $p \veebar q$ | $q \veebar p$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

3. [BB] Associativity holds.

| $p$ | $q$ | $r$ | $p \veebar q$ | $(p \veebar q) \veebar r$ | $q \veebar r$ | $p \veebar (q \veebar r)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

4. Just one distributive law holds. The table which follows shows that $p \veebar (q \wedge r)$ is not logically equivalent to $(p \veebar q) \wedge (p \veebar r)$,

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \veebar (q \wedge r)$ | $p \veebar q$ | $p \veebar r$ | $(p \veebar q) \wedge (p \veebar r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |

On the other hand, $(p \wedge (q \veebar r)) \iff (p \wedge q) \veebar (p \wedge r)$ as shown.

| $p$ | $q$ | $r$ | $q \veebar r$ | $p \wedge (q \veebar r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \veebar (p \wedge r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

6. Both laws of De Morgan fail. The table shows that $\neg(p \veebar q)$ is not logically equivalent to $(\neg p) \wedge (\neg q)$.

| $p$ | $q$ | $p \veebar q$ | $\neg(p \veebar q)$ | $\neg p$ | $\neg q$ | $(\neg p) \wedge (\neg q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |

The next table shows that $\neg(p \wedge q)$ is not logically equivalent to $(\neg p) \veebar (\neg q)$.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $(\neg p) \veebar (\neg q)$ |
|---|---|---|---|---|---|---|
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

7. [BB] The truth table shows that $p \veebar 1$ is no longer 1:

| $p$ | 1 | $p \veebar 1$ |
|---|---|---|
| $T$ | $T$ | $F$ |

8. The truth table shows that $p \veebar \mathbf{0} \iff p$:

| $p$ | $\mathbf{0}$ | $p \veebar \mathbf{0}$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ |

9. [BB] The truth table shows that $(p \veebar (\neg p)) \iff \mathbf{1}$:

| $p$ | $\neg p$ | $p \veebar (\neg p)$ | $\mathbf{1}$ |
|---|---|---|---|
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |

12. [BB] This is no longer true:

| $p$ | $q$ | $p \to q$ | $\neg p$ | $(\neg p) \veebar q$ |
|---|---|---|---|---|
| $F$ | $T$ | $T$ | $T$ | $F$ |

Finally, neither law of absorption holds: $p \veebar (p \wedge q)$ is not logically equivalent to $p$ and $p \wedge (p \veebar q)$ is not logically equivalent to $p$.

| $p$ | $q$ | $p \wedge q$ | $p \veebar (p \wedge q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |

| $p$ | $q$ | $p \veebar q$ | $p \wedge (p \veebar q)$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ |

9. (a) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$ is not in disjunctive normal form because the terms are not joined by $\vee$.

(b) [BB] $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$ is in disjunctive normal form.

(c) [BB] $p \vee ((\neg p) \wedge q)$ is not in disjunctive normal form: not all variables are included in the first term.

(d) $(p \wedge q) \vee ((\neg p) \wedge (\neg q) \wedge r)$ is not in disjunctive normal form: not all variables are included in the first term.

(e) $(p \wedge q \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$ is in disjunctive normal form.

10. (a) [BB] This is already in disjunctive normal form! The definition permits just a single midterm.

(b) [BB] One of the laws of De Morgan gives immediately that $(p \wedge q) \vee (\neg((\neg p) \vee q)) \iff (p \wedge q) \vee (p \wedge (\neg q))$.

(c) $(p \to q) \iff ((\neg p) \vee q) \iff [((\neg p) \wedge q) \vee ((\neg p) \wedge (\neg q)) \vee (q \wedge p) \vee (q \wedge (\neg p))]$.

(d) $[(p \vee q) \wedge ((\neg p) \vee (\neg q))] \iff [(p \wedge ((\neg p) \vee (\neg q))] \vee [q \wedge ((\neg p) \vee (\neg q))] \iff [(p \wedge (\neg p)) \vee (p \wedge ((\neg q))] \vee [(q \wedge ((\neg p)) \vee (q \wedge ((\neg q))] \iff [(p \wedge (\neg q)) \vee ((\neg p) \wedge q))]$

(e) $[(p \to q) \wedge (q \wedge r)] \iff [((\neg p) \vee q) \wedge (q \wedge r)] \iff [((\neg p) \wedge q \wedge r) \vee (q \wedge q \wedge r)] \iff [((\neg p) \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r))] \iff [((\neg p) \wedge q \wedge r) \vee (p \wedge q \wedge r)]$, omitting the repeated minterm at the final step.

(f) $\Big(p \vee [q \wedge (p \vee (\neg r))]\Big) \iff \Big(p \vee [(q \wedge p) \vee (q \wedge (\neg r))]\Big) \iff \Big(p \vee (q \wedge p) \vee (q \wedge (\neg r))\Big) \iff \Big((p \wedge q) \vee (p \wedge (\neg q)) \vee (p \wedge q) \vee (q \wedge (\neg r))\Big) \iff \Big((p \wedge q) \vee (p \wedge (\neg q)) \vee (q \wedge (\neg r))\Big) \iff \Big((p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \vee (\neg r)) \vee ((\neg p) \wedge q \wedge (\neg r))\Big) \iff \Big((p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee ((\neg p) \wedge q \wedge (\neg r))\Big)$, omitting the repeated minterm at the final step.

11. August De Morgan was born in India in 1806, died in England in 1871 and lived his life without the sight of his right eye, which was damaged at birth. He was the first Professor of Mathematics at University College, London, founded the London Mathematical Society and was its first president. He was apparently quite a man of principle. He refused to study for the MA degree because of a required theological exam and twice resigned his chair at University College, on matters of principle. He was

never a Fellow of the Royal Society because he refused to let his name be put forward and, similarly, refused an honorary degree from the University of Edinburgh.

De Morgan's mathematical contributions include the definition and introduction of "mathematical induction", the most important method of proof in mathematics today. See Section 5.1. His definition of a limit was the first attempt to define the idea in precise mathematical terms. Nonetheless, it is the area of mathematical logic with which De Morgan's name is most closely associated. The "Laws of De Morgan" introduced in this section, together with their set theoretical analogues—$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ (see Section 2.2)—are used extensively and of fundamental importance. De Morgan also developed a system of notation for symbolic logic that could denote converses and contradictions.

## Exercises 1.3

1. (a) [BB] Since $[p \to (q \to r)] \iff [p \to ((\neg q) \vee r)] \iff [(\neg p) \vee (\neg q) \vee r]$, the given argument can be rewritten

$$\frac{\begin{array}{c}[(\neg p) \vee r] \vee (\neg q) \\ q\end{array}}{(\neg p) \vee r}$$

which is valid by disjunctive syllogism.

(b) [BB] We analyze with a truth table. In row one, the premises are true but the conclusion is not. The argument is not valid.

| $p$ | $q$ | $r$ | $p \to q$ | $q \vee r$ | $\neg q$ | $r \to (\neg q)$ | |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $\star$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $\star$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $\star$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $\star$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | |

(c) We analyze with a truth table. In row three, the premises are true but the conclusion is not. The argument is not valid.

| $p$ | $q$ | $r$ | $p \to q$ | $r \to q$ | $r \to p$ | |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $\star$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $\star$ |

(d) This can be solved using a truth table with 16 rows. Alternatively, we can proceed as follows.

Assume that the argument is not valid. This means that we can find truth values for $p$, $q$, $r$ and $s$ such that the premises are true but the conclusion is false. Since $s \to (r \vee q)$ is false, we must have $s$ true and $r \vee q$ false. But this means both $r$ and $q$ are false. Since $p \to q$ is true and $q$ is false, $p$ must be false. But then $q \vee (\neg r)$ is true and $p \wedge s$ is false, contradicting the truth of $(q \vee (\neg r)) \to (p \wedge s)$. Hence we have a contradiction, so the argument is valid.

(e) This argument is valid. The second and third premises give $q$ by Modus Ponens. Together with the first premise (double negation and dijunctive syllogism), we get $\neg p$.

(f) This argument is valid. The first and third premises give $\neg p$ by modus tollens. Together with the second premise (double negation and a second application of modus tollens), we get $r$.

2. (a) If $p$ is true and $p$ implies $q$, then $q$ is true.

(b) This was PAUSE 5.

(c) If $p$ is true or $q$ is true and $p$ is not true, then $q$ must be true.

(d) We do this by contradiction. So assume the premises are true but the conclusion is false. Since $p \to r$ is false, we must have $p$ true and $r$ false. Since $q \to r$ is true and $r$ is false, $q$ must be false. Then, since $p \to q$ is true, $p$ must be false, giving a contradiction.

(e) We prove this by contradiction. Assume $p \vee q$ is false. This means both $p$ and $q$ are false. Since $p \vee r$ is true and $p$ is false, $r$ must be true. Since $q \vee (\neg r)$ is true and $q$ is false, $r$ must be false, giving a contradiction.

3. (a) [BB] We analyze with a truth table. There are five rows when the premises are all true and in each case the conclusion is also true. The argument is valid.

| $p$ | $q$ | $r$ | $p \vee q$ | $p \to r$ | $q \to r$ | $(p \vee q) \to r$ | |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $\star$ |

(b) We analyze with a partial truth table showing the nine situations in which both the premises are true. In every case, the conclusion is true. The argument is valid.

| $p$ | $r$ | $q$ | $s$ | $p \wedge q$ | $r \wedge s$ | $(p \wedge q) \to (r \wedge s)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $T$ |

(c) The second and third premises are $p \to r$ and $r \to s$ which together imply $p \to s$ by the chain rule. Thus the argument becomes

$$\frac{\begin{array}{c} p \vee q \\ p \to s \end{array}}{q \vee s}$$

which we check with a truth table.

| $p$ | $q$ | $s$ | $p \vee q$ | $p \to s$ | $q \vee s$ | |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | |
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ | |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | |

There are four rows when the premises are true and, in each case, the conclusion is also true. The argument is valid.

(d) Since $((\neg q) \wedge r) \iff \neg(q \vee (\neg r))$ and $\neg(p \wedge s) \iff [(\neg p) \vee (\neg s)] \iff (p \to (\neg s))$, the premises become

$$(q \vee (\neg r)) \to p$$
$$p \to (\neg s)$$

so the chain rule gives $(q \vee (\neg r)) \to (\neg s)$, which is logically equivalent to $(\neg(q \vee (\neg r))) \vee (\neg s)$, which is the desired conclusion by a law of De Morgan.

4. (a) [BB] Since $[(\neg r) \vee (\neg q)] \iff [q \to (\neg r)]$, the first two premises give $p \to (\neg r)$ by the chain rule. Now $\neg p$ follows by modus tollens.

(b) This argument is not valid. If $q$ and $r$ are false, $p$ and $s$ are true, and $t$ takes on any truth value, then all premises are true, yet the conclusion is false.

(c) [BB] This argument is valid. Since $p \leftrightarrow (t \vee s)$, we can replace $t \vee s$ with $p$ so that the premises become $p \vee (\neg q)$, $p \to (p \vee r)$, $(\neg r) \vee p$. The first of these is logically equivalent to $q \to p$ while the third is $r \to p$. Using Exercise 3**(a)**, we get $q \vee r \to p$, so certainly $q \vee r \to p \vee r$.

(d) This argument is valid. To see this, note that the second premise is $r \to (t \vee s)$, so the chain rule gives $r \to p$. Also, the first premise is $q \to p$. Exercise 3**(a)** tells us that $(q \vee r) \to p$, so certainly $(q \vee r) \to (p \vee r)$.

(e) This argument is not valid. If $p$ and $r$ are false while $q$ and $s$ are true, all premises are true, yet the conclusion is false.

(f) [BB] This is valid. The first premise is $q \vee [(\neg p) \vee s]$ while the second is $(\neg q) \vee r$. Now $(\neg p) \vee s \vee r$ follows by resolution, and this is the conclusion.

(g) This argument is valid. The first premise is $p \vee [(\neg q) \vee r]$, which is $(\neg q) \vee p \vee r$. Since $q \vee r$, resolution gives $p \vee r$. Using $r \to p$ and the result of part (f), we get $p \vee p$, which is $p$.

(h) This argument is valid. Using (g) twice, we have $p \to [s \vee (\neg p)]$ which is $(\neg p) \vee s \vee (\neg p)$, which is $(\neg p) \vee s$, which is the conclusion.

5. (a) [BB] Let $p$ and $q$ be the statements $p$: I stay up late at night
$q$: I am tired in the morning.

The given argument is $\dfrac{\begin{array}{l} p \to q \\ p \end{array}}{q}$.

This is valid by modus ponens.

(b) [BB] Let $p$ and $q$ be the statements
$p$: I stay up late at night
$q$: I am tired in the morning.

The given argument is
$$\begin{array}{l} p \to q \\ \underline{\quad q \quad} \\ p \end{array}.$$

This is not valid, as the truth table shows. In row three, the two premises are true but the conclusion is false.

| $p$ | $q$ | $p \to q$ | |
|---|---|---|---|
| $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $F$ | |
| $F$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $T$ | |

(c) Let $p$ and $q$ be the statements
$p$: I stay up late at night
$q$: I am tired in the morning.

The given argument is
$$\begin{array}{l} p \to q \\ \underline{\quad \neg q \quad} \\ \neg p \end{array}.$$

This is valid by modus tollens.

(d) Let $p$ and $q$ be the statements
$p$: I stay up late at night
$q$: I am tired in the morning.

The given argument is
$$\begin{array}{l} p \to q \\ \underline{\quad \neg p \quad} \\ \neg q \end{array}.$$

This is not valid, as the truth table shows. In row three, the two premises are true but the conclusion is false.

| $p$ | $q$ | $p \to q$ | $\neg p$ | $\neg q$ | |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | |
| $T$ | $F$ | $F$ | $F$ | $T$ | |
| $F$ | $T$ | $T$ | $T$ | $F$ | $\star$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $\star$ |

(e) [BB] Let $p$ and $q$ be the statements
$p$: I wear a red tie
$q$: I wear blue socks.

The given argument is
$$\begin{array}{l} p \vee q \\ \underline{\quad \neg q \quad} \\ p \end{array}.$$

This is valid by disjunctive syllogism.

(f) Let $p$ and $q$ be the statements
$p$: I wear a red tie
$q$: I wear blue socks.

The given argument is
$$\begin{array}{l} p \vee q \\ \underline{\quad q \quad} \\ \neg p \end{array}.$$

This is not valid, as the truth table shows. In row one, the two premises are true but the conclusion is false.

| $p$ | $q$ | $p \vee q$ | $\neg p$ | |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $\star$ |
| $T$ | $F$ | $T$ | $F$ | |
| $F$ | $T$ | $T$ | $T$ | $\star$ |
| $F$ | $F$ | $F$ | $T$ | |

(g) [BB] Let $p$, $q$, and $r$ be the statements
$p$: I work hard
$q$: I earn lots of money
$r$: I pay high taxes.

The given argument is $\begin{array}{l} p \to q \\ q \to r \\ \hline r \to p \end{array}$.

This is not valid, as the truth table shows. In row five, the two premises are true but the conclusion is false.

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $r \to p$ | |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $\star$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $\star$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $\star$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ | |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $\star$ |

(h) Let $p$, $q$, and $r$ be the statements
$p$: I work hard
$q$: I earn lots of money
$r$: I pay high taxes.

The given argument is $\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$.

This is valid by the chain rule.

(i) Let $p$, $q$, and $r$ be the statements
$p$: I work hard
$q$: I earn lots of money
$r$: I pay high taxes.

The given argument is $\begin{array}{l} p \to q \\ q \to r \\ \hline \neg p \to \neg r \end{array}$.

The conclusion is logically equivalent to $r \to p$, so this is the same as Exercise 5(g), hence not valid.

(j) Let $p$, $q$, and $r$ be the statements
$p$: I like mathematics
$q$: I study
$r$: I like football.

The given argument is $\begin{array}{l} p \to q \\ \neg q \\ p \vee r \\ \hline r \end{array}$.

The first two premises give $\neg p$ by modus tollens, so, since $p \vee r$ is true, the conclusion follows by disjunctive syllogism.

(k) Let $p$, $q$, and $r$ be the statements
$p$: I like mathematics
$q$: I study
$r$: I like football.

The given argument is $\begin{array}{l} q \vee r \\ r \to p \\ \hline (\neg q) \to p \end{array}$.

This is the same as $\begin{array}{l} q \vee r \\ (\neg r) \vee p \\ \hline q \vee p \end{array}$ and hence valid by resolution.

(l) [BB] Let $p$, $q$, and $r$ be the statements

$p$: I like mathematics
$q$: I study
$r$: I pass mathematics
$s$: I graduate.

The given argument is $\dfrac{\begin{array}{c}p \to q \\ (\neg q) \vee r \\ (\neg s) \to (\neg r)\end{array}}{s \to q}$.

This is the same as $\dfrac{\begin{array}{c}p \to q \\ q \to r \\ r \to s\end{array}}{s \to q}$

which is certainly not valid, as the following partial truth table shows.

| $p$ | $q$ | $r$ | $s$ | $p \to q$ | $q \to r$ | $r \to s$ | $s \to q$ |
|---|---|---|---|---|---|---|---|
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ |

(m) Let $p$, $q$, and $r$ be the statements

$p$: I like mathematics
$q$: I study
$r$: I pass mathematics
$s$: I graduate.

The given argument is $\dfrac{\begin{array}{c}p \to q \\ (\neg q) \vee r \\ (\neg s) \to (\neg r)\end{array}}{p \to s}$

which is the same as $\dfrac{\begin{array}{c}p \to q \\ q \to r \\ r \to s\end{array}}{p \to s}$

which is valid by two applications of the chain rule.

6. [BB] $r \vee q$ is logically equivalent to $[\neg(\neg r) \vee q] \iff [(\neg r) \to q]$ so, with $p \to \neg r$, we get $p \to q$ by the chain rule.

7. We will prove by contradiction that no such conclusion is possible. Say to the contrary that there is such a conclusion $\mathcal{C}$. Since $\mathcal{C}$ is not a tautology, some set of truth values for $p$ and $q$ must make $\mathcal{C}$ false. But if $r$ is true, then both the premises $(\neg p) \to r$ and $r \vee q$ are true regardless of the values of $p$ and $q$. This contradicts $\mathcal{C}$ being a valid conclusion for this argument.

8. (a) [BB] $p \wedge q$ is true precisely when $p$ and $q$ are both true.

   (b) By 8(a), we can replace $p \wedge q$ by the two premises $p$ and $q$. Using modus ponens, $p$ and $p \to r$ lead to the conclusion $r$. Using modus tollens, $q$ and $s \to (\neg q))$ lead to the conclusion $\neg s$. Finally, 8(a) says we can replace $\neg s$ and $r$ with $(\neg s) \wedge r$.

9. By Exercise 8(a), the final premise is equivalent to the list of premises $q_1, q_2, \ldots, q_n$. Now

$$[p_1 \to (q_1 \to r_1)] \iff [p_1 \to ((\neg q_1) \vee r_1)] \iff [(\neg p_1) \vee (\neg q_1) \vee r_1].$$

Together with $q_1$, disjunctive syllogism gives $(\neg p_1) \vee r_1$ which is logically equivalent to $p_1 \rightarrow r_1$. Thus the given premises imply

$$\begin{array}{c} p_1 \rightarrow r_1 \\ p_2 \rightarrow r_2 \\ \vdots \\ p_n \rightarrow r_n \end{array}$$

which, again using **8(a)**, are logically equivalent to the single premise

$$(p_1 \rightarrow r_1) \wedge (p_2 \rightarrow r_2) \wedge \cdots \wedge (p_n \rightarrow r_n).$$

10. [BB] In Latin, *modus ponens* means "method of affirming" and *modus tollens* means "method of denying". This is a reflection of the fact that *modus tollens* has a negative $\neg p$ as its conclusion, while *modus ponens* affirms the truth of a statement $q$.

# Chapter 1 Review

1.

| $p$ | $q$ | $r$ | $\neg r$ | $q \rightarrow (\neg r)$ | $p \wedge (q \rightarrow (\neg r))$ | $\neg q$ | $(\neg q) \vee r$ | $(p \wedge (q \rightarrow (\neg r))) \rightarrow ((\neg q) \vee r)$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |

2. We have $\neg r$ is $F$, so $(\neg r) \wedge s$ is $F$. Since $q$ is $T$, this means $q \rightarrow ((\neg r) \wedge s)$ is $F$. But $p$ is $T$, so $p \vee (q \rightarrow ((\neg r) \wedge s))$ is $T$. Also $r \wedge t$ is $T$. Hence the entire statement has truth value $T$.

3. (a) A truth table shows this is a contradiction.

| $p$ | $q$ | $\neg q$ | $\neg p$ | $p \wedge (\neg q)$ | $(\neg p) \vee q$ | $[p \wedge (\neg q)] \wedge [(\neg p) \vee q]$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |

(b) A truth table shows that this is neither a tautology nor a contradiction.

| $p$ | $q$ | $p \to q$ | $p \vee q$ | $(p \to q) \to (p \vee q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |

(c) A truth table shows this is a tautology.

| $p$ | $q$ | $r$ | $\neg q$ | $p \wedge (\neg q)$ | $[p \wedge (\neg q)] \to r$ | $p \vee [(p \wedge (\neg q)) \to r]$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |

(d) This is neither a tautology nor a contradiction.

| $p$ | $q$ | $r$ | $\neg q$ | $(\neg q) \wedge r$ | $p \vee q$ | $[(\neg q) \wedge r] \to (p \vee q)$ | $(p \vee q) \to [(\neg q) \wedge r]$ | $(p \vee q) \leftrightarrow [(\neg q) \wedge r]$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |

4. Assume that some set of truth values on the variables makes $\mathcal{A}$ true. If $\mathcal{B}$ were false, this would make $\mathcal{A} \to \mathcal{B}$ false and $\mathcal{B} \to \mathcal{A}$ true, contradicting logical equivalence. So $\mathcal{B}$ must be true also. Similarly, if $\mathcal{B}$ is true, then $\mathcal{A}$ must also be true. We conclude that $\mathcal{A}$ is true if and only if $\mathcal{B}$ is true. This means $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent.

5. (a) Since $\mathcal{A} \iff \mathcal{B}$, we know that $\mathcal{A}$ is true precisely when $\mathcal{B}$ is true. Since $\mathcal{B} \iff \mathcal{C}$, $\mathcal{B}$ is true precisely when $\mathcal{C}$ is true. Hence $\mathcal{A}$ is true if and only if $\mathcal{C}$ is true, that is, $\mathcal{A} \iff \mathcal{C}$.

   (b) Property 12 says $(p \to q) \iff ((\neg p) \vee q)$. Clearly, $((\neg p) \vee q) \iff (q \vee (\neg p))$, so part (a) tells us $(p \to q) \iff (q \vee (\neg p))$. But Property 12 also says $((\neg q) \to (\neg p)) \iff (\neg(\neg q) \vee (\neg p))$ and clearly $(\neg(\neg q) \vee (\neg p)) \iff (q \vee (\neg p))$. So we have $(p \to q) \iff (q \vee (\neg p))$ and also

$(q \vee (\neg p)) \iff ((\neg q) \to (\neg p))$. Hence part (a) again gives $(p \to q) \iff ((\neg q) \to (\neg p))$, which is Property 11.

6. (a) $((p \to q) \to r) \iff (((\neg p) \vee q) \to r) \iff ((\neg((\neg p) \vee q)) \vee r) \iff ((p \wedge (\neg q)) \vee r) \iff ((p \vee r) \wedge ((\neg q) \vee r)) \iff ((p \vee r) \wedge (\neg(q \wedge (\neg r))))$.

   (b) $[p \to (q \vee r)] \iff (\neg p) \vee (q \vee r) \iff \neg p \vee q \vee r$ (associativity) $\iff [\neg(p \wedge (\neg q)] \vee r$ (De Morgan) $\iff [p \wedge (\neg q)] \to r$.

7. (a) $((p \vee q) \wedge r) \vee ((p \vee q) \wedge (\neg p))$
$\iff ((p \wedge r) \vee (q \wedge r)) \vee ((p \wedge (\neg p)) \vee (q \wedge (\neg p)))$
$\iff ((p \wedge r) \vee (q \wedge r)) \vee (\mathbf{0} \vee (q \wedge (\neg p)))$
$\iff ((p \wedge r) \vee (q \wedge r)) \vee (q \wedge (\neg p))$
$\iff (p \wedge r \wedge q) \vee (p \wedge r \wedge (\neg q)) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge (\neg p)) \vee (q \wedge (\neg p) \wedge (\neg r))$
$\iff (p \wedge q \wedge r) \vee (p \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge (\neg r))$

   (b) $[p \vee (q \wedge (\neg r))] \wedge \neg(q \wedge r)$
$\iff [p \vee (q \wedge (\neg r))] \wedge [(\neg q) \vee (\neg r)]$
$\iff \Big([p \vee (q \wedge (\neg r))] \wedge (\neg q)\Big) \vee \Big([p \vee (q \wedge (\neg r))] \wedge (\neg r)\Big)$
$\iff (p \wedge (\neg q)) \vee (q \wedge (\neg r) \wedge (\neg q)) \vee (p \wedge (\neg r)) \vee (q \wedge (\neg r) \wedge (\neg r))$
$\iff (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee \mathbf{0} \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r) \vee ((\neg p) \wedge q \wedge (\neg r))$
$\iff (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r) \vee ((\neg p) \wedge q \wedge (\neg r))$

8. (a) This is not valid. If $p$ is false and $q$ is true, the premises are true but the conclusion is not.

   (b) The first hypothesis can be rewritten as $p \vee (\neg q)$, which is the same as $q \to p$. The second hypothesis is $(\neg p) \vee (\neg r)$, which is $p \to (\neg r)$. The third hypothesis is $(\neg r) \to s$. So the given argument is

$$\begin{array}{c} q \to p \\ p \to (\neg r) \\ \hline (\neg r) \to s \\ \hline q \to s \end{array}$$

   Two applications of the chain rule tell us this is valid.

   (c) This argument is not valid. If $p$, $r$ and $t$ are true while $q$ is false (and $s$ takes on either truth value), the hypotheses are true while the conclusion is false.

9. This argument is valid. The hypotheses can never both be true at the same time, so there can be no case when the hypotheses are true while the conclusion is false.

10. (a) This argument, an example of resolution, is valid.

    (b) This argument is not valid. We can write it as shown. When $s$ and $r$ are true and $p$ is false, the hypotheses are true, while the conclusion is false.

$$\begin{array}{c} p \to q \\ (\neg q) \vee r \\ \hline (\neg r) \to (\neg s) \\ \hline s \to p. \end{array}$$

## Exercises 2.1

1. (a) [BB] $\{-\sqrt{5}, \sqrt{5}\}$
   (b) $\{1, 3, 5, 15, -1, -3, -5, -15\}$
   (c) [BB] $\{0, -\frac{3}{2}\}$ (Although $\pm\sqrt{2}$ are solutions to the equation, they are not rational.)
   (d) $\{-1, 0, 1, 2, 3\}$
   (e) This is the empty set. There are no numbers less than $-4$ **and** bigger than $+4$.

2. (a) [BB] For example, $1 + i$, $1 + 2i$, $1 + 3i$, $-8 - 5i$ and $17 - 43i$.
   (b) For example, $\{1 - 2\sqrt{2}, 1 - 5\sqrt{2}, 1 - 7\sqrt{2}, 316 - 2\sqrt{2}$ and $394 - 7\sqrt{2}\}$.
   (c) If $x = 0$, $y = \pm 5$ and $x/y = 0$. If $x = 1$, $y = \pm\sqrt{24}$, and $x/y = \pm 1/\sqrt{24} = \pm\sqrt{24}/24$. If $x = 2$, $y = \pm\sqrt{21}$ and $x/y = \pm 2\sqrt{21}/21$. Five elements of the given set are $0$, $\sqrt{24}/24$, $-\sqrt{24}/24$, $2\sqrt{21}/21$ and $-2\sqrt{21}/21$.
   (d) $\{2, 3, 5, 6, 8\}$.

3. (a) [BB] $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$
   (b) $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$
   (c) $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$
   (d) $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$
   (e) $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$
   (f) $\emptyset$, $\{1\}$, $\{2\}$.

4. [BB] Only (c) is true. The set $A$ contains one element, $\{a, b\}$.

5. (a) [BB] True. 3 belongs to the set $\{1, 3, 5\}$.
   (b) False. $\{3\}$ is a subset of $\{1, 3, 5\}$ but not a member of this set.
   (c) True. $\{3\}$ is a proper subset of $\{1, 3, 5\}$.
   (d) [BB] False. $\{3, 5\}$ is a subset of $\{1, 3, 5\}$.
   (e) False. Although $\{1, 3, 5\}$ is a subset of itself, it is not a **proper** subset.
   (f) False. If $a + 2b$ is in the given set, $a$ is even, so $a + 2b$ is even and can't equal 1.
   (g) False. If $a + b\sqrt{2} = 0$ and $b \neq 0$, then $\sqrt{2} = -\frac{a}{b}$ is the quotient of rational numbers and, hence, rational. But this is not true.

6. (a) [BB] $\{\emptyset\}$; (b) $\{\emptyset, \{\emptyset\}\}$; (c) $\{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \{\,\{\emptyset\}\,\}\}$.

7. (a) [BB] True. The empty set is a subset of every set.
   (b) True. The empty set is a subset of every set.
   (c) False. The empty set does not contain any elements.
   (d) True. $\{\emptyset\}$ is a set containing one element, namely, $\emptyset$.
   (e) [BB] False. $\{1, 2\}$ is a subset of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (f) False. $\{1, 2\}$ is not an element of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (g) True. $\{1, 2\}$ is a proper subset of $\{1, 2, \{\{1, 2\}\}\}$.
   (h) [BB] False. $\{1, 2\}$ is not an element of $\{1, 2, \{\{1, 2\}\}\}$.

(i) True. $\{\{1,2\}\}$ contains just one element, $\{1,2\}$, and this is an element of $\{1,2,\{1,2\}\}$.

8. [BB] Yes it is; for example, let $x = \{1\}$ and $A = \{1,\{1\}\}$.

9. (a) i. $\{a,b,c,d\}$ ii. [BB] $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$
iii. $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$ iv. $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ v. $\emptyset$

(b) 16

10. (a) If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ is a set containing one element, so **its** power set contains two elements.

(b) $\mathcal{P}(A)$ contains two elements; $\mathcal{P}(\mathcal{P}(A))$ has four elements.

11. (a) [BB] 4; (b) [BB] 8.

(c) [BB] There are $2^n$ subsets of a set of $n$ elements. (See Exercise 15 in Section 5.1 for a proof.)

12. (a) [BB] False. Let $A = \{2\}$, $B = \{\{2\}\}$, $C = \{\{\{2\}\}\}$. Then $A$ is an element of $B$ (that is, $A \in B$) and $B$ is an element of $C$ ($B \in C$), but $A$ is not an element of $C$ (since $B$ is $C$'s only element).

(b) True. If $x \in A$, then $x \in B$ since $A \subseteq B$. But since $x \in B$, then $x \in C$ since $B \subseteq C$.

(c) True. As in the previous part, we know that $A \subseteq C$. To prove $A \neq C$, we note that there is some $x \in C$ such that $x \notin B$ (since $B \subsetneq C$). Then, since $x \notin B$, $x \notin A$. Therefore, $x$ is an element of $C$ which is not in $A$, proving $A \neq C$.

(d) [BB] True. $A \in B$ means that $A$ belongs to the set $B$. Since $B$ is a subset of $C$, any element of $B$ also belongs to $C$. Hence, $A \in C$.

(e) False. For example, let $A = \{1\}$, $B = \{\{1\},2\}$ and $C = \{\{1\},2,3\}$. Then $A \in B$, $B \subseteq C$, but $A \not\subseteq C$.

(f) False. Let $A = \{1\}$, $B = \{1,2\}$, $C = \{\{1,2\},3\}$. Then $A \subseteq B$ and $B \in C$, but $A \notin C$.

(g) False. Same example as 12(f) where $A \not\subseteq C$.

13. (a) This is false. As a counter-example, consider $A = \{1\}$, $B = \{2\}$. Then $A$ is not a subset of $B$ and $B$ is not a proper subset of $A$.

(b) The converse of the implication in (a) is the implication $B \subsetneq A \rightarrow A \not\subseteq B$. This is true. Since $B \subsetneq A$, there exists some element $a \in A$ which is not in $B$. Thus $A$ is not a subset of $B$.

14. (a) [BB] True. ($\longrightarrow$) If $C \in \mathcal{P}(A)$, then by definition of "power set," $C$ is a subset of $A$; that is, $C \subseteq A$.

($\longleftarrow$) If $C \subseteq A$, then $C$ is a subset of $A$ and so, again by definition of "power set," $C \in \mathcal{P}(A)$.

(b) True. ($\longrightarrow$) Suppose $A \subseteq B$. We prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. For this, let $X \in \mathcal{P}(A)$. Therefore, $X$ is a subset of $A$; that is, every element of $X$ is an element of $B$. Since $A \subseteq B$, every element of $X$ must be an element of $B$. So $X \subseteq B$; hence, $X \in \mathcal{P}(B)$.

($\longleftarrow$) Conversely, assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We must prove $A \subseteq B$. For any set $A$, we know that $A \subseteq A$ and, hence, $A \in \mathcal{P}(A)$. Here, with $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have, therefore, $A \in \mathcal{P}(B)$; that is, $A \subseteq B$, as desired.

(c) The double implication here is false because the implication $\longrightarrow$ is false. If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ and $\{\emptyset\} \neq \emptyset$.

## Exercises 2.2

1. (a) [BB] $A = \{1,2,3,4,5,6\}$, $B = \{-1,0,1,2,3,4,5\}$, $C = \{0,2,-2\}$.
   (b) $A \cup C = \{-2,0,1,2,3,4,5,6\}$, $B \cap C = \{0,2\}$,
   $B \setminus C = \{-1,1,3,4,5\}$, $A \oplus B = \{-1,6,0\}$,
   $C \times (B \cap C) = \{(0,0),(0,2),(2,0),(2,2),(-2,0),(-2,2)\}$,
   $(A \setminus B) \setminus C = \{6\}$, $A \setminus (B \setminus C) = \{2,6\}$,
   $(B \cup \emptyset) \cap \{\emptyset\} = \emptyset$.
   (c) $S = \{(1,-1),(2,0),(3,1),(4,2),(5,3),(6,4)\}$; $T = \{(1,2),(2,2)\}$.

2. (a) [BB] $S \cap T = \{\sqrt{2}, 25\}$, $S \cup T = \{2,5,\sqrt{2},25,\pi,\frac{5}{2},4,6,\frac{3}{2}\}$,
   $T \times (S \cap T) = \{(4,\sqrt{2}),(4,25),(25,\sqrt{2}),(25,25),(\sqrt{2},\sqrt{2}),$
   $(\sqrt{2},25),(6,\sqrt{2}),(6,25),(\frac{3}{2},\sqrt{2}),(\frac{3}{2},25)\}$.
   (b) [BB] $\mathsf{Z} \cup S = \{\sqrt{2},\pi,\frac{5}{2},0,1,-1,2,-2,\ldots\}$; $\mathsf{Z} \cap S = \{2,5,25\}$;
   $\mathsf{Z} \cup T = \{\sqrt{2},\frac{3}{2},0,1,-1,2,-2,\ldots\}$; $\mathsf{Z} \cap T = \{4,25,6\}$.
   (c) $\mathsf{Z} \cap (S \cup T) = \{2,5,25,4,6\} = (\mathsf{Z} \cap S) \cup (\mathsf{Z} \cap T)$. The two sets are equal.
   (d) $\mathsf{Z} \cup (S \cap T) = \{\sqrt{2},0,1,-1,2,-2,\ldots\} = \mathsf{Z} \cup \{\sqrt{2}\} = (\mathsf{Z} \cup S) \cap (\mathsf{Z} \cup T)$.
   The two sets are equal.

3. (a) [BB] $\{1,9,0,6,7\}$; (b) $\{4,6,5\}$; (c) $\{0,1\}$.

4. $A = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ and $B = \{\pm\frac{1}{2},\pm1,\pm2\}$.

5. (a) [BB] $\{c,\{a,b\}\}$; (b) $\{\emptyset\}$; (c) $A$; (d) $\emptyset$;
   (e) [BB] $\emptyset$; (f) $\{A\}$.

6. (a) [BB] $A^c = (-2,1]$; (b) $A^c = (-\infty,-3] \cup (4,\infty)$; (c) $A^c = \mathsf{R}$.

7. (a) $Y \cap Z = \{3,4,5\}$, so $X \oplus (Y \cap Z) = \{1,2,5\}$.
   (b) $(X^c \cup Y)^c = X \cap Y^c = X \setminus Y = \{1\}$.

8. (a) [BB] The subsets of $A$ containing $\{1,2\}$ are obtained by taking the union of $\{1,2\}$ with a subset of $\{3,4,5,\ldots,n\}$. Their number is the number of subsets of $\{3,4,5,\ldots,n\}$ which is $2^{n-2}$. (See Exercise 11 of Section 2.1.)
   (b) The subsets $B$ which have the property that $B \cap \{1,2\} = \emptyset$ are exactly the subsets of $\{3,4,5,\ldots,n\}$ and these number $2^{n-2}$.
   (c) The subsets $B$ which have the property that $B \cup \{1,2\} = A$ are precisely those subsets which contain $\{3,4,5,\ldots,n\}$ and these correspond, as in (a), to the subsets of $\{1,2\}$. There are four.

9. [BB] $(a,b)^c = (-\infty,a] \cup [b,\infty)$, $[a,b)^c = (-\infty,a) \cup [b,\infty)$, $(a,\infty)^c = (-\infty,a]$,
   $(-\infty,b]^c = (b,\infty)$.

10. (a) [BB] $CS \subseteq T$; (b) [BB] $M \cap P = \emptyset$; (c) $M \not\subseteq P$; (d) $CS \setminus T \subseteq P$;
    (e) $(M \cup CS) \cap P \subseteq T^c$.

11. (a) Negation: $CS \not\subseteq T$; Converse: $T \subseteq CS$.
    (b) Negation: $M \cap P \neq \emptyset$; Converse: $P \subseteq M^c$ or $P \cap M = \emptyset$,

(c) Negation: $M \subseteq P$; No converse since the statement is not an implication.

(d) Negation: $(M \cup CS) \cap P \not\subseteq T^c$; Converse: $T^c \subseteq (M \cup CS) \cap P$.

(e) Negation: $(M \cup CS) \cap P \cap T \neq \emptyset$; Converse $T^c \subseteq (M \cup CS) \cap P$.

12. (a) [BB] $\mathsf{E} \cap \mathsf{P} \neq \emptyset$ (b) $0 \in \mathsf{Z} \setminus \mathsf{N}$ (c) $\mathsf{N} \subseteq \mathsf{Z}$
(d) $\mathsf{Z} \not\subseteq \mathsf{N}$ (e) $(\mathsf{P} \setminus \{2\}) \subseteq \mathsf{E}^c$ (f) $2 \in \mathsf{E} \cap \mathsf{P}$
(g) $\mathsf{E} \cap \mathsf{P} = \{2\}$

13. (a) [BB] Since $A_{-3} \subseteq A_3$, $A_3 \cup A_{-3} = A_3$.

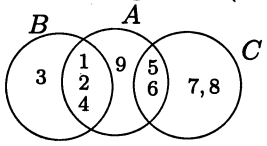
(b) Since $A_{-3} \subseteq A_3$, $A_3 \cap A_{-3} = A_{-3}$.

(c) $A_3 \cap (A_{-3})^c = \{a \in \mathsf{Z} \mid -3 < a \leq 3\} = \{-2, -1, 0, 1, 2, 3\}$.

(d) Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$, we have $\bigcap_{i=0}^{4} A_i = A_0$.

14. [BB] Region 2 represents $(A \cap C) \setminus B$. Region 3 represents $A \cap B \cap C$; region 4 represents $(A \cap B) \setminus C$.

15. (a)

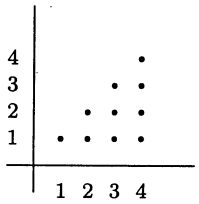



(b) i. $(A \cup B) \cap C = \{5, 6\}$
ii. $A \setminus (B \setminus A) = A = \{1, 2, 4, 7, 8, 9\}$
iii. $(A \cup B) \setminus (A \cap C) = \{1, 2, 3, 4, 9\}$
iv. $A \oplus C = \{1, 2, 4, 7, 8, 9\}$
v. $(A \cap C) \times (A \cap B) = \{(5, 1), (5, 2), (5, 4), (6, 1), (6, 2), (6, 4)\}$

16. (a) [BB] $A \subseteq B$, by Problem 7; (b) $B \subseteq A$, by PAUSE 4 with $A$ and $B$ reversed.

17. [BB] Think of listing the elements of the given set. There are $n$ pairs of the form $(1, b)$, $n - 1$ pairs of the form $(2, b)$, $n - 2$ pairs of the form $(3, b)$, and so on until finally we list the only pair of the form $(n, b)$. The answer is $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

18. Since $(1, 1) \in A$, $(2, 1)$ and $(2, 2)$ are in $A$. Since $(2, 1) \in A$, we get $(3, 1)$ and $(3, 2)$ in $A$ and since $(2, 2) \in A$, $(3, 3) \in A$. Now $(3, 1) \in A \rightarrow (4, 1)$ and $(4, 2) \in A$; $(3, 2) \in A \rightarrow (4, 3) \in A$ and $(3, 3) \in A \rightarrow (4, 4) \in A$. The points shown so far which belong to $A$ are plotted in the picture to the right and this makes it seem very plausible that $A$ contains the set $\{(m, n) \in \mathsf{N} \times \mathsf{N} \mid m \geq n\}$.




19. (a) Let $x \in B$. Certainly $x$ is also in $A$ or in $A^c$. This suggests cases.

**Case 1:** If $x \in A$, then $x \in A \cap B$, so $x \in C$.

**Case 2:** If $x \notin A$, then $x \in A^c \cap B$, so $x \in C$.

In either case, $x \in C$, so $B \subseteq C$.

(b) [BB] Yes. Given $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, certainly we have $A \cap B \subseteq C$ and $A^c \cap B \subseteq C$ so, from (a), we have that $B \subseteq C$. Reversing the roles of $B$ and $C$ in (a), we can also conclude that $C \subseteq B$; hence, $B = C$.

20. (a) The Venn diagram shown in Fig. 2.1 suggests the following counterexample: Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 3, 5, 7\}$. Then $A \cup (B \cap C) = A \cup \{3, 5\} = \{1, 2, 3, 4, 5\}$ whereas $(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6\} \cap C = \{2, 3, 5\}$.

(b) First, we prove $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

So let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$ so $x \in A \cup B$ and $x \in A \cup C$; that is, $x \in (A \cup B) \cap (A \cup C)$. In either case, $x \in (A \cup B) \cap (A \cup C)$ giving the desired inclusion. Second, we prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

So let $x \in (A \cup B) \cap (A \cup C)$. Thus, $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then we must have $x \in B$ and $x \in C$; that is, $x \in B \cap C$, so $x \in A \cup (B \cap C)$. In either case, $x \in A \cup (B \cap C)$ giving the desired inclusion and equality.

21. We use the fact that $(X^c)^c = X$ for any set $X$.

Let $X = A^c$ and $Y = B^c$. Then $A = X^c$ and $B = Y^c$, so $(A \cap B)^c = [X^c \cap Y^c]^c = [(X \cup Y)^c]^c$ (by the first law of De Morgan) $= X \cup Y = A^c \cup B^c$, as required.

22. [BB] Using the fact that $X \setminus Y = X \cap Y^c$, we have

$$(A \setminus B) \setminus C = (A \cap B^c) \cap C^c = A \cap (B^c \cap C^c) = A \cap (B \cup C)^c = A \setminus (B \cup C).$$

23. We use the laws of De Morgan and the facts that $(X^c)^c = X$ and $X \cap X^c = \emptyset$ for any set $X$. We have $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c = [(A^c \cap B^c) \cap (A \cap C^c)]^c \setminus D^c = \emptyset^c \setminus D^c = U \setminus D^c = U \cap (D^c)^c = U \cap D = D$.

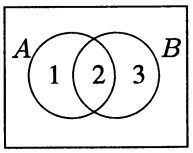
24. $A \setminus (B \setminus C) = A \setminus (B \cap C^c) = A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap (C^c)^c) = (A \setminus B) \cup (A \setminus C^c)$.

25. (a) [BB] $(A \cup B \cup C)^c = [A \cup (B \cup C)]^c = A^c \cap (B \cup C)^c = A^c \cap (B^c \cap C^c) = A^c \cap B^c \cap C^c$.

$(A \cap B \cap C)^c = [A \cap (B \cap C)]^c = A^c \cup (B \cap C)^c = A^c \cup (B^c \cup C^c) = A^c \cup B^c \cup C^c$.

(b)
$$\begin{aligned}(A \cap (B \setminus C))^c \cap A &= (A \cap B \cap C^c)^c \cap A = (A^c \cup B^c \cup C) \cap A \\ &= (A \cap (A^c \cup B^c)) \cup (A \cap C) = (A \cap A^c) \cup (A \cap B^c) \cup (A \cap C) \\ &= \emptyset \cup (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap C) \\ &= (A \setminus B) \cup (A \cap C)\end{aligned}$$

26. (a) [BB] Looking at the Venn diagram at the right, $A \oplus B$ consists of the points in regions 1 and 3. To have $A \oplus B = A$, we must have both regions 2 and 3 empty; that is, $B = \emptyset$. On the other hand, since $A \oplus \emptyset = A$, this condition is necessary and sufficient.




(b) Looking at the Venn diagram, $A \cap B$ is the set of points in region 2 while $A \cup B$ is the set of points in regions 1, 2 and 3. Hence, $A \cap B = A \cup B$ if and only if regions 1 and 3 are both empty; that is, if and only if $A = B$.

27. (a) [BB] This does not imply $B = C$. For example, let $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$. Then $A \cup B = A \cup C$, but $B \neq C$.

(b) This does not imply $B = C$. For example, let $A = \{1\}$, $B = \{1,2\}$, $C = \{1,3\}$. Then $A \cap B = A \cap C = A$, but $B \neq C$.

(c) This does imply $B = C$, and here is a proof. First let $b \in B$. Then, in addition, either $b \in A$ or $b \notin A$.

**Case 1:** $b \notin A$

In this case, $b \in A \oplus B$, so $b \in A \oplus C$ and since $b \notin A$ it follows that $b \in C$.

**Case 2:** $b \in A$. Here we have $b \in B \cap A$ and, hence, $b \notin A \oplus B$, so $b \notin A \oplus C$. Since $b \in A$, we must have $b \in C$ (otherwise, $b \in A \setminus C \subseteq A \oplus C$).

In either case, we obtain $b \in C$. It follows that $B \subseteq C$. A similar argument shows $C \subseteq B$ and, hence, $C = B$.

(d) This is false since for $A = \emptyset$, $A \times B = A \times C = \emptyset$ regardless of $B$ and $C$.

28. (a) True. Let $(a, b) \in A \times B$. Since $a \in A$ and $A \subseteq C$, we have $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. Thus, $(a, b) \in C \times D$ and $A \times B \subseteq C \times D$.

(b) False: Consider $A = \{1\}$, $B = \{2, 3\}$, $C = \{1, 2, 3\}$.

(c) False. Let $A = \{1\}$, $B = \emptyset$, $C = \{2\}$, $D = \{3\}$. Then $A \times B = \emptyset \subseteq \{(2, 3)\} = C \times D$, but $A \not\subseteq C$.

(d) False since, by (b), the implication $\leftarrow$ is false.

(e) [BB] True. Let $x \in A$. Then $x \in A \cup B$, so $x \in A \cap B$ and, in particular, $x \in B$. Thus, $A \subseteq B$. Similarly, we have $B \subseteq A$, so $A = B$.

29. Let $(x, y) \in (A \cap B) \times C$. This means $x \in A \cap B$ and $y \in C$. Hence, $x \in A$, $x \in B$, $y \in C$. Thus, $(x, y) \in A \times C$ and $(x, y) \in B \times C$; i.e., $(x, y) \in (A \times C) \cap (B \times C)$. Therefore, $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

Now let $(x, y) \in (A \times C) \cap (B \times C)$. This means that $(x, y) \in A \times C$ and $(x, y) \in B \times C$); that is, $x \in A$, $x \in B$, $y \in C$, so $x \in A \cap B$ and $y \in C$. Hence, $(x, y) \in (A \cap B) \times C$. Therefore, $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$ and we have equality, as desired.

30. (a) False. For example, let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$. Then $A \setminus (B \cup C) = \{1, 2\} \setminus \{1, 2\} = \emptyset$, but $(A \setminus B) \cup (A \setminus C) = \{2\} \cup \{1\} = \{1, 2\}$.

(b) True. Let $(x, y) \in (A \setminus B) \times C$. This means that $x \in A \setminus B$ and $y \in C$; that is, $x \in A$, $x \notin B$, $y \in C$. Hence, $(x, y) \in A \times C$, but $(x, y) \notin B \times C$, so $(x, y) \in (A \times C) \setminus (B \times C)$. Therefore, $(A \setminus B) \times C \subseteq (A \times C) \setminus (B \times C)$.

Now let $(x, y) \in (A \times C) \setminus (B \times C)$. This means that $(x, y) \in A \times C$, but $(x, y) \notin B \times C$. Since $(x, y) \in A \times C$, we have $x \in A$, $y \in C$. Since $y \in C$ and $(x, y) \notin B \times C$, we must have $x \notin B$; that is, $x \in A \setminus B$, $y \in C$, so $(x, y) \in (A \setminus B) \times C$. Therefore, $(A \times C) \setminus (B \times C) \subseteq (A \setminus B) \times C$ and we have equality as claimed.

(c) [BB] True. Let $(x, y) \in (A \oplus B) \times C$. This means that $x \in A \oplus B$ and $y \in C$; that is, $x \in A \cup B$, $x \notin A \cap B$, $y \in C$. If $x \in A$, then $x \notin B$, so $(x, y) \in (A \times C) \setminus (B \times C)$. If $x \in B$, then $x \notin A$, so $(x, y) \in (B \times C) \setminus (A \times C)$. In either case, $(x, y) \in (A \times C) \oplus (B \times C)$. So $(A \oplus B) \times C \subseteq (A \times C) \oplus (B \times C)$.

Now, let $(x, y) \in (A \times C) \oplus (B \times C)$. This means that $(x, y) \in (A \times C) \cup (B \times C)$, but $(x, y) \notin (A \times C) \cap (B \times C)$. If $(x, y) \in A \times C$, then $(x, y) \notin B \times C$, so $x \in A$, $y \in C$ and, therefore, $x \notin B$. If $(x, y) \in B \times C$, then $(x, y) \notin A \times C$, so $x \in B$, $y \in C$ and, therefore, $x \notin A$. In either case, $x \in A \oplus B$ and $y \in C$, so $(x, y) \in (A \oplus B) \times C$. Therefore, $(A \times C) \oplus (B \times C) \subseteq (A \oplus B) \times C$ and we have equality, as claimed.

(d) False. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then $1 \in A \cup B$, $4 \in C \cup D$, so $(1, 4) \in (A \cup B) \times (C \cup D)$. But $(1, 4) \notin A \times C$ and $(1, 4) \notin B \times D$, so $(1, 4) \notin (A \times C) \cup (B \times D)$.

(e) False. Let $A = \{1, 2\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then, since $3 \notin D$, $(2, 3) \in (A \times C) \setminus (B \times D)$. However, because $2 \in B$, $2 \notin A \setminus B$, so $(2, 3) \notin (A \setminus B) \times (C \setminus D)$.

31. George Boole (1815-1864) was one of the greatest mathematicians of the nineteenth century. He was the first Professor of Mathematics at University College Cork (then called Queen's College) and is best known today as the inventor of a subject called *mathematical logic*. Indeed he introduced much of the symbolic language and notation we use today. Like Charles Babbage and Alan Turing, Boole also had a great impact in computer science, long before the computer was even a dream. He invented an *algebra of logic* known as Boolean Algebra, which is used widely today and forms the basis of much of the internal logic of computers. His books, "The Mathematical Analysis of Logic" and "An Investigation of the Laws of Thought" form the basis of present-day computer science.

## Exercises 2.3

1. [BB] $S \times B$ is the set of ordered pairs $(s, b)$, where $s$ is a student and $b$ is a book; thus, $S \times B$ represents all possible pairs of students and books. One sensible example of a binary relation is $\{(s, b) \mid s \text{ has used book } b\}$.

2. $A \times B$ is the set of all ordered pairs $(a, b)$ where $a$ is a street and $b$ is a person. One binary relation would be $\{(a, b) \mid b \text{ lives on street } a\}$.

3. (a) [BB] not reflexive, not symmetric, not transitive.

   (b) (in most cases) reflexive, (in somewhat fewer cases) symmetric, certainly not transitive!

   (c) [BB] not reflexive, not symmetric, but it is transitive.

   (d) reflexive, symmetric, transitive.

   (e) not reflexive, not symmetric, not transitive

4. (a) [BB]

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | × | × | | |
| $b$ | × | × | | |
| $c$ | | | × | |
| $d$ | | | | × |

(b)

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | × | × | × | |
| $b$ | | × | | |
| $c$ | × | | × | |
| $d$ | | | | |

(c)

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | × | × | × | |
| $b$ | | | | |
| $c$ | | | × | × |
| $d$ | | | | × |

(d)

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | × | × | × | × |
| $b$ | | × | × | × |
| $c$ | | | × | |
| $d$ | | | × | × |

5. (a) [BB] $\{(1,1),(1,2),(2,3)\}$; (b) $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$;
   (c) $\{(1,2),(2,3),(2,1),(3,2)\}$; (d) $\{(1,2),(1,3),(2,3)\}$;
   (e) $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$;
   (f) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$; (g) [BB] $\{(1,2),(2,1),(1,1),(2,2)\}$;
   (h) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3),(2,1),(3,2),(3,1)\}$.

6. The answer is yes and the only such binary relations are subsets of the equality binary relation. To see why, let $\mathcal{R}$ be a binary relation on a set $A$ which is both symmetric and antisymmetric. Let $(a, b) \in \mathcal{R}$. Then $(b, a) \in \mathcal{R}$ by symmetry, so $a = b$ by antisymmetry.

7. [BB] The argument assumes that for $a \in \mathcal{R}$ there exists a $b$ such that $(a, b) \in \mathcal{R}$. This need not be the case: See Exercise 5(g).

8. (a) [BB] **Reflexive**: Every word has at least one letter in common with itself.
   **Symmetric**: If $a$ and $b$ have at least one letter in common, then so do $b$ and $a$.
   **Not antisymmetric**: (cat, dot) and (dot, cat) are both in the relation but dot $\neq$ cat!!
   **Not transitive**: (cat, dot) and (dot, mouse) are both in the relation but (cat, mouse) is not.

   (b) **Reflexive**: Let $a$ be a person. If $a$ is not enrolled at Miskatonic University, then $(a, a) \in \mathcal{R}$. On the other hand, if $a$ is enrolled at MU, then $a$ is taking at least one course with himself, so again $(a, a) \in \mathcal{R}$.
   **Symmetric**: If $(a, b) \in \mathcal{R}$, then either it is the case that neither $a$ nor $b$ is enrolled at MU (so neither is $b$ or $a$, hence, $(b, a) \in \mathcal{R}$) or it is the case that $a$ and $b$ are both enrolled and are taking at least one course together (in which case $b$ and $a$ are enrolled and taking a common course, so $(b, a) \in \mathcal{R}$). In any case, if $(a, b) \in \mathcal{R}$, then $(b, a) \in \mathcal{R}$.
   **Not antisymmetric**: If $a$ and $b$ are two different students in the same class at Miskatonic University, then $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, but $a \neq b$.
   At most universities, this is not a transitive relation. Let $a, b$ and $c$ be three students enrolled at MU such that $a$ and $b$ are enrolled in some course together and $b$ and $c$ are enrolled in some (other) course together, but $a$ and $c$ are taking no courses together. Then $(a, b)$ and $(b, c)$ are in $\mathcal{R}$ but $(a, c) \notin \mathcal{R}$.

9. (a) **Not reflexive**: $(1, 1) \notin \mathcal{R}$.
   **Not symmetric**: $(1, 2) \in \mathcal{R}$ but $(2, 1) \notin \mathcal{R}$.
   **Antisymmetric**: It is never the case that for two different elements $a$ and $b$ in $A$ we have both $(a, b)$ and $(b, a)$ in $\mathcal{R}$.
   **Transitive vacuously**; that is, there exists no counterexample to disprove transitivity: The situation $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ never occurs.

   (b) [BB] **Not reflexive**: $(2, 2) \notin \mathcal{R}$.
   **Not symmetric**: $(3, 4) \in \mathcal{R}$ but $(4, 3) \notin \mathcal{R}$.
   **Not antisymmetric**: $(1, 2)$ and $(2, 1)$ are both in $\mathcal{R}$ but $1 \neq 2$.
   **Not transitive**: $(2, 1)$ and $(1, 2)$ are in $\mathcal{R}$ but $(2, 2)$ is not.

   (c) [BB] **Reflexive**: For any $a \in \mathsf{Z}$, it is true that $a^2 \geq 0$. Thus, $(a, a) \in \mathcal{R}$.
   **Symmetric**: If $(a, b) \in \mathcal{R}$, then $ab \geq 0$, so $ba \geq 0$ and hence, $(b, a) \in \mathcal{R}$.
   **Not antisymmetric**: $(5, 2) \in \mathcal{R}$ because $5(2) = 10 \geq 0$ and similarly $(2, 5) \in \mathcal{R}$, but $5 \neq 2$.
   **Not transitive**: $(5, 0) \in \mathcal{R}$ because $5(0) = 0 \geq 0$ and similarly, $(0, -6) \in \mathcal{R}$; however, $(5, -6) \notin \mathcal{R}$ because $5(-6) \not\geq 0$.

   (d) **Reflexive**: For any $a \in \mathsf{R}$, $a^2 = a^2$, so $(a, a) \in \mathcal{R}$.
   **Symmetric**: If $(a, b) \in \mathcal{R}$ then $a^2 = b^2$, so $b^2 = a^2$ which says that $(b, a) \in \mathcal{R}$.
   **Not antisymmetric**: $(1, -1) \in \mathcal{R}$ and $(-1, 1) \in \mathcal{R}$ but $1 \neq -1$.
   **Transitive**: If $(a, b)$ and $(b, c)$ are both in $\mathcal{R}$, then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$ which says $(a, c) \in \mathcal{R}$.

   (e) **Reflexive**: For any $a \in \mathsf{R}$, $a - a = 0 \leq 3$ and so $(a, a) \in \mathcal{R}$.
   **Not symmetric**: For example, $(0, 7) \in \mathcal{R}$ because $0 - 7 = -7 \leq 3$, but $(7, 0) \notin \mathcal{R}$ because $7 - 0 = 7 \not\leq 3$.

**Not antisymmetric**: $(2,1) \in \mathcal{R}$ because $2-1=1 \leq 3$ and $(1,2) \in \mathcal{R}$ because $1-2=-1 \leq 3$, but $1 \neq 2$.

**Not transitive**: $(5,3) \in \mathcal{R}$ because $5-3=2 \leq 3$ and $(3,1) \in \mathcal{R}$ because $3-1=2 \leq 3$, but $(5,1) \notin \mathcal{R}$ because $5-1=4 \not\leq 3$.

(f) **Reflexive**: For any $(a,b) \in A$, $a-a=b-b$; thus, $((a,b),(a,b)) \in \mathcal{R}$.

**Symmetric**: If $((a,b),(c,d)) \in \mathcal{R}$, then $a-c=b-d$, so $c-a=d-b$ and, hence, $((c,d),(a,b)) \in \mathcal{R}$.

**Not antisymmetric**: $((5,2),(15,12)) \in \mathcal{R}$ because $5-15=2-12$ and similarly, $((15,12),(5,2)) \in \mathcal{R}$; however, $(15,12) \neq (5,2)$.

If $((a,b),(c,d)) \in \mathcal{R}$ and $((c,d),(e,f)) \in \mathcal{R}$ then $a-c=b-d$ and $c-e=d-f$. Thus, $a-e=(a-c)+(c-e)=(b-d)+(d-f)=b-f$ and so $((a,b),(e,f)) \in \mathcal{R}$.

(g) **Not reflexive**: If $n \in \mathsf{N}$, then $n \neq n$ is not true.

**Symmetric**: If $n_1 \neq n_2$, then $n_2 \neq n_1$.

**Not antisymmetric**: $1 \neq 2$ and $2 \neq 1$ so both $(1,2)$ and $(2,1)$ are in $\mathcal{R}$, yet $1 \neq 2$.

**Not transitive**: $1 \neq 2$, $2 \neq 1$, but $1=1$.

(h) **Not reflexive**: $(2,2) \notin \mathcal{R}$ because $2+2 \neq 10$.

**Symmetric**: If $(x,y) \in \mathcal{R}$, then $x+y=10$, so $y+x=10$, and hence, $(y,x) \in \mathcal{R}$.

**Not antisymmetric**: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $6 \neq 4$.

**Not transitive**: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $(6,6) \notin \mathcal{R}$ because $6+6 \neq 10$.

(i) [BB] **Reflexive**: If $(x,y) \in \mathsf{R}^2$, then $x+y \leq x+y$, so $((x,y),(x,y)) \in \mathcal{R}$.

**Not symmetric**: $((1,2),(3,4)) \in \mathcal{R}$ since $1+2 \leq 3+4$, but

$((3,4),(1,2)) \notin \mathcal{R}$ since $3+4 \not\leq 1+2$.

**Not antisymmetric**: $((1,2),(0,3)) \in \mathcal{R}$ since $1+2 \leq 0+3$

and $((0,3),(1,2)) \in \mathcal{R}$ since $0+3 \leq 1+2$, but $(1,2) \neq (0,3)$.

**Transitive**: If $((a,b),(c,d))$ and $((c,d),(e,f))$ are both in $\mathcal{R}$, then $a+b \leq c+d$ and $c+d \leq e+f$, so $a+b \leq e+f$ (by transitivity of $\leq$) which says $((a,b),(e,f)) \in \mathcal{R}$.

(j) **Reflexive**: $\frac{a}{a}=1 \in \mathsf{N}$ for any $a \in \mathsf{N}$.

**Not symmetric**: $(4,2) \in \mathcal{R}$ but $(2,4) \notin \mathcal{R}$.

**Antisymmetric**: If $\frac{a}{b}=n$ and $\frac{b}{a}=m$ are integers then $nm=1$ so $n,m \in \{\pm 1\}$. Since $a$ and $b$ are positive, so are $n$ and $m$. Therefore, $n=m=1$ and $a=b$.

**Transitive**: The argument given in Example 24 for Z works the same way for N.

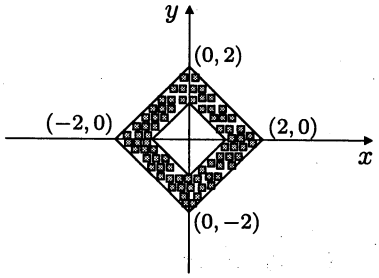
(k) **Not reflexive**: $\frac{0}{0}$ is not defined, let alone an integer!

**Not symmetric**: As before.

**Not antisymmetric**: $(4,-4)$ and $(-4,4)$ are both in $\mathcal{R}$.

**Transitive**: As shown in Example 24.

10. (a)




(b) The relation is not reflexive because, for example, $(2,2) \notin \mathcal{R}$. It is not transitive because, for example, $(2,0) \in \mathcal{R}$ and $(0,1) \in \mathcal{R}$ but $(2,1) \notin \mathcal{R}$.

(c) The relation is symmetric since if $(x,y) \in \mathcal{R}$, then $1 \leq |x| + |y| \leq 2$, so $1 \leq |y| + |x| \leq 2$, so $(y,x) \in \mathcal{R}$. It is not antisymmetric since, for example, $(0,1) \in \mathcal{R}$ and $(1,0) \in \mathcal{R}$, but $0 \neq 1$.

11. (a) [BB] **Reflexive**: For any set $X$, we have $X \subseteq X$.
**Not symmetric**: Let $a, b \in S$. Then $\{a\} \subseteq \{a,b\}$ but $\{a,b\} \not\subseteq \{a\}$.
**Antisymmetric**: If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
**Transitive**: If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(b) **Not reflexive**: For no set $X$ is it true that $X \subsetneq X$.
**Not symmetric**: As before.
**Antisymmetric "vacuously"**: It is impossible for $X \subsetneq Y$ and $Y \subsetneq X$. (Recall that an implication is false only when the hypothesis is true and the conclusion is false.)
**Transitive**: As before.

(c) **Not reflexive**: Since $S \neq \emptyset$, there is some element $a \in S$, and so some set $X = \{a\} \neq \emptyset \in \mathcal{P}(S)$. For this $X$, however, $X \cap X = X \neq \emptyset$, so $(X,X) \notin \mathcal{R}$.
**Symmetric**: If $(X,Y) \in \mathcal{R}$, then $X \cap Y = \emptyset$, so $Y \cap X = \emptyset$, hence, $(Y,X) \in \mathcal{R}$.
**Not antisymmetric**: Let $a, b$ be two elements in $S$ and let $X = \{a\}$, $Y = \{b\}$. Then $(X,Y) \in \mathcal{R}$ and $(Y,X) \in \mathcal{R}$, but $X \neq Y$.
**Not transitive**: Let $a, b$ be two elements in $S$ and let $X = \{a\}$, $Y = \{b\}$, $Z = \{a\}$. Then $(X,Y) \in \mathcal{R}$, $(Y,Z) \in \mathcal{R}$, but $(X,Z) \notin \mathcal{R}$.

12. (a) [BB] **Reflexive**: Any book has price $\geq$ its own price and length $\geq$ its own length, so $(a,a) \in \mathcal{R}$ for any book $a$.
**Not symmetric**: $(Y,Z) \in \mathcal{R}$ because the price of $Y$ is greater than the price of $Z$ and the length of $Y$ is greater than the length of $Z$, but for these same reasons, $(Z,Y) \notin \mathcal{R}$.
**Antisymmetric**: If $(a,b)$ and $(b,a)$ are both in $\mathcal{R}$, then $a$ and $b$ must have the same price and length. This is not the case here unless $a = b$.
**Transitive**: If $(a,b)$ and $(b,c)$ are in $\mathcal{R}$, then the price of $a$ is $\geq$ the price of $b$ and the price of $b$ is $\geq$ the price of $c$, so the price of $a$ is $\geq$ the price of $c$. Also the length of $a$ is $\geq$ the length of $b$ and the length of $b$ is $\geq$ the length of $c$, so the length of $a$ is $\geq$ the length of $c$. Hence, $(a,c) \in \mathcal{R}$.

(b) **Reflexive**: For any book $a$, the price of $a$ is $\geq$ the price of $a$ so $(a,a) \in \mathcal{R}$. (One could also use a similar argument concerning length.)
**Not symmetric**: As in part (a), $(Y,Z) \in \mathcal{R}$, but $(Z,Y) \notin \mathcal{R}$.
**Not antisymmetric**: $(W,X) \in \mathcal{R}$ because the price of $W$ is greater than or equal to the price of $X$, and $(X,W) \in \mathcal{R}$ because the length of $W$ is greater than or equal to the length of $X$, but $W \neq X$.
**Not transitive**: $(Z,U) \in \mathcal{R}$ because the length of $Z$ is $\geq$ the length of $U$ and $(U,Y) \in \mathcal{R}$ because the price of $U$ is $\geq$ the price of $Y$, but $(Z,Y) \notin \mathcal{R}$ because neither is the price of $Z \geq$ the price of $Y$ nor is the length of $Z \geq$ the length of $Y$.

13. Now the second binary relation would have an extra term, $\{\text{Mike}, 120\}$, and the third would have the extra term, $\{\text{Pippy Park}, 120\}$. But, in addition, the entry $\{\text{Pippy Park}, 74\}$ would be deleted. So Mike is now clearly identified as the one who shot 120, and Pippy Park is where that occurred. Hence, Mike's round of 74 was at Clovelly. Since Edgar has only one entry in binary relation two, he must have shot 72 at both courses. Finally, Bruce's 74 must have been at Clovelly and hence his 72 was at Pippy Park. All information has been retrieved in this case.

## Exercises 2.4

1. **Reflexive:** For any citizen $a$ of New York City, either $a$ does not own a cell phone (in which case $a \sim a$) or $a$ has a cell phone and $a$'s exchange is the same as $a$'s exchange (in which case again $a \sim a$).

   **Symmetric:** If $a \sim b$ and $a$ does not have a cell phone, then neither does $b$, so $b \sim a$; on the other hand, if $a$ does have a cell phone, then so does $b$ and their exchanges are the same, so again, $b \sim a$.

   **Transitive:** Suppose $a \sim b$ and $b \sim c$. If $a$ does not have a cell phone, then neither does $b$ and, since $b \sim c$, neither does $c$, so $a \sim c$. On the other hand, if $a$ does have a cell phone then so does $b$ and $a$'s and $b$'s exchanges are the same. Since $b \sim c$, $c$ has a cell phone with the same exchange as $b$. It follows that $a$ and $c$ have the same exchange and so, in this case as well, $a \sim c$.

   There is one equivalence class consisting of all residents of New York who do not own a cell phone and one equivalence class for each New York City exchange consisting of all residents who have cell phones in that exchange.

2. (a) [BB] This is not reflexive: $(2,2) \notin \mathcal{R}$.
   (b) This is not symmetric: $(2,3) \in \mathcal{R}$ but $(3,2) \notin \mathcal{R}$.
   It would also be acceptable to not that $\mathcal{R}$ is not transitive. $(3,1) \in \mathcal{R}$ and $(1,2) \in \mathcal{R}$, but $(3,2) \notin \mathcal{R}$.
   (c) This is not symmetric: $(1,3)$ is in the relation but $(3,1)$ is not.
   It would also be acceptable to note that this relation is not transitive: $(2,1) \in \mathcal{R}$, $(1,3) \in \mathcal{R}$, but $(2,3) \notin \mathcal{R}$.

3. [BB] Equality! The equivalence classes specify that $x \sim y$ if and only if $x = y$.

4. (a) **Reflexive**: If $a \in S$, then $a$ and $a$ have the same number of elements, so $a \sim a$.
   **Symmetric**: If $a \sim b$, then $a$ and $b$ have the same number of elements, so $b$ and $a$ have the same number of elements. Thus $b \sim a$.
   **Transitive**: If $a \sim b$ and $b \sim c$, then $a$ and $b$ have the same number of elements, and $b$ and $c$ have the same number of elements, so $a$ and $c$ have the same number of elements. Thus $a \sim c$.
   (b) There are **seven** equivalence classes, represented by $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$, $\{1,2,3,4,5\}$, $\{1,2,3,4,5,6\}$.

5. (a) [BB] **Reflexive:** If $a \in \mathsf{R} \setminus \{0\}$, then $a \sim a$ because $\frac{a}{a} = 1 \in \mathsf{Q}$.
   **Symmetric:** If $a \sim b$, then $\frac{a}{b} \in \mathsf{Q}$ and this fraction is not zero (because $0 \notin A$). So it can be inverted and we see that $\frac{b}{a} = 1/\frac{a}{b} \in \mathsf{Q}$ too. Therefore, $b \sim a$.
   **Transitive:** If $a \sim b$ and $b \sim c$, then $\frac{a}{b} \in \mathsf{Q}$ and $\frac{b}{c} \in \mathsf{Q}$. Since the product of rational numbers is rational, $\frac{a}{c} = \frac{a}{b}\frac{b}{c}$ is in $\mathsf{Q}$, so $a \sim c$.

(b) [BB] $\overline{1} = \{a \mid a \sim 1\} = \{a \mid \frac{a}{1} \in \mathsf{Q}\} = \{a \mid a \in \mathsf{Q}\} = \mathsf{Q} \setminus \{0\}$.

(c) [BB] $\frac{\sqrt{12}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \in \mathsf{Q}$, so $\sqrt{3} \sim \sqrt{12}$ and hence $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.

6. **Reflexive:** For any $a \in \mathsf{N}$, $a \sim a$ since $a^2 + a = a(a+1)$ is even, as the product of consecutive natural numbers.

**Symmetric:** If $a \sim b$, then $a^2 + b$ is even. It follows that either $a$ and $b$ are both even or both are odd. If they are both even, $b^2 + a$ is the sum of even numbers, hence, even. If they are both odd, $b^2 + a$ is the sum of odd numbers and, hence, again, even. In both cases $b^2 + a$ is even, so $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $a^2 + b$ and $b^2 + c$ are even, so $(a^2 + b) + (b^2 + c)$ is even; in other words, $(a^2 + c) + (b^2 + b)$ is even. Since $b^2 + b$ is even, $a^2 + c$ is even too; therefore, $a \sim c$.

The quotient set is the set of equivalence classes. Now

$$\overline{a} = \{x \mid x^2 + a \text{ is even}\} = \begin{cases} \text{evens} & \text{if } a \text{ is even} \\ \text{odds} & \text{if } a \text{ is odd} \end{cases}$$

So $A/\!\sim = \{2\mathsf{Z}, 2\mathsf{Z}+1\}$.

7. (a) [BB] **Reflexive:** For any $a \in \mathsf{R}$, $a \sim a$ because $a - a = 0 \in \mathsf{Z}$.
**Symmetric:** If $a \sim b$, then $a - b \in \mathsf{Z}$, so $b - a \in \mathsf{Z}$ (because $b - a = -(a - b)$) and, hence, $b \sim a$.
**Transitive:** If $a \sim b$ and $b \sim c$, then both $a - b$ and $b - c$ are integers; hence, so is their sum, $(a - b) + (b - c) = a - c$. Thus, $a \sim c$.

(b) [BB] The equivalence class of 5 is $\overline{5} = \{x \in \mathsf{R} \mid x \sim 5\} = \{x \mid x - 5 \in \mathsf{Z}\} = \mathsf{Z}$, because $x - 5 \in \mathsf{Z}$ implies $x \in \mathsf{Z}$.

$$\begin{aligned} \overline{5\tfrac{1}{2}} &= \{x \in \mathsf{R} \mid x \sim 5\tfrac{1}{2}\} \\ &= \{x \mid x - 5\tfrac{1}{2} \in \mathsf{Z}\} \\ &= \{x \mid x = 5\tfrac{1}{2} + k, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = 5 + k + \tfrac{1}{2}, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = n + \tfrac{1}{2}, \text{ for some } n \in \mathsf{Z}\} \end{aligned}$$

(c) [BB] For each $a \in \mathsf{R}$, $0 \le a < 1$, there is one equivalence class,

$$\overline{a} = \{x \in \mathsf{R} \mid x = a + n \text{ for some integer } n\}.$$

The quotient set is $\{\overline{a} \mid 0 \le a < 1\}$.

8. [BB] **Reflexive:** For any $a \in \mathsf{Z}$, $a \sim a$ because $2a + 3a = 5a$.

**Symmetric:** If $a \sim b$, then $2a + 3b = 5n$ for some integer $n$. So $2b + 3a = (5a + 5b) - (2a + 3b) = 5(a + b) - 5n = 5(a + b - n)$. Since $a + b - n$ is an integer, $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $2a + 3b = 5n$ and $2b + 3c = 5m$ for integers $n$ and $m$. Therefore, $(2a + 3b) + (2b + 3c) = 5(n + m)$ and $2a + 3c = 5(n + m) - 5b = 5(n + m - b)$. Since $n + m - b$ is an integer, $a \sim c$.

9. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $3a + a = 4a$ is a multiple of 4, so $a \sim a$.
**Symmetric:** If $a \sim b$, then $3a + b = 4k$ for some integer $k$. Since $(3a + b) + (3b + a) = 4(a + b)$, we see that $3b + a = 4(a + b) - 4k$ is a multiple of 4, so $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $3a + b = 4k$ for some integer $k$ and $3b + c = 4\ell$ for some integer $\ell$. Since $4(k+\ell) = (3a+b)+(3b+c) = (3a+c)+4b$, we see that $3a+c = 4(k+\ell)-4b$ is a multiple of 4 and, hence, that $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{Z} \mid x \sim 0\} = \{x \mid 3x = 4k \text{ for some integer } k\}$. Now if $3x = 4k$, $k$ must be a multiple of 3. So $3x = 12\ell$ for some $\ell \in \mathsf{Z}$ and $x = 4\ell$. $\overline{0} = 4\mathsf{Z}$.

(c) $\overline{2} = \{x \in \mathsf{Z} \mid x \sim 2\} = \{x \mid 3x + 2 = 4k \text{ for some integer } k\} = \{x \mid 3x = 4k - 2 \text{ for some integer } k\}$ Now if $3x = 4k - 2$, then $3x = 3k + k - 2$ and so $k - 2$ is a multiple of 3. Therefore, $k = 3\ell + 2$ for some integer $\ell$, $3x = 4(3\ell + 2) - 2 = 12\ell + 6$ and $x = 4\ell + 2$. So $\overline{2} = 4\mathsf{Z} + 2$.

(d) The quotient set is $\{4\mathsf{Z}, 4\mathsf{Z} + 1, 4\mathsf{Z} + 2, 4\mathsf{Z} + 3\}$.

10. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $a \sim a$ because $3a + 4a = 7a$ and $a$ is an integer.

**Symmetric:** If $a, b \in \mathsf{Z}$ and $a \sim b$, then $3a + 4b = 7n$ for some integer $n$. Then $3b + 4a = (7a + 7b) - (3a + 4b) = 7a + 7b - 7n = 7(a + b - n)$ and $a + b - n$ is an integer. Thus $b \sim a$.

**Transitive:** Suppose $a, b, c \in \mathsf{Z}$ with $a \sim b$ and $b \sim c$. Then $3a + 4b = 7n$ and $3b + 4c = 7m$ for some integers $n$ and $m$. Then $7n + 7m = (3a + 4b) + (3b + 4c) = (3a + 4c) + 7b$, so $3a + 4c = 7n + 7m - 7b = 7(n + m - b)$ and $n + m - b$ is an integer. Thus $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{Z} \mid x \sim 0\} = \{x \in \mathsf{Z} \mid 3x = 7n \text{ for some integer } n\}$. Now if $3x = 7n$, $n$ must be a multiple of 3. So $3x = 21k$ for some $k \in \mathsf{Z}$ and $x = 7k$. We conclude that $\overline{0} = 7\mathsf{Z}$.

11. (a) [BB] **Reflexive:** If $a \in \mathsf{Z} \setminus \{0\}$, then $aa = a^2 > 0$, so $a \sim a$.

**Symmetric:** If $a \sim b$, then $ab > 0$. So $ba > 0$ and $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $ab > 0$ and $bc > 0$. Also $b^2 > 0$ since $b \neq 0$. Hence,

$$ac = \frac{(ac)b^2}{b^2} = \frac{(ab)(bc)}{b^2} > 0$$

since $ab > 0$, $bc > 0$. Hence, $a \sim c$.

(b) [BB] $\begin{aligned} \overline{5} &= \{x \in \mathsf{Z} \setminus \{0\} \mid x \sim 5\} = \{x \mid 5x > 0\} = \{x \mid x > 0\} \\ \overline{-5} &= \{x \in \mathsf{Z} \setminus \{0\} \mid x \sim -5\} = \{x \mid -5x > 0\} = \{x \mid x < 0\} \end{aligned}$

(c) [BB] This equivalence relation partitions $\mathsf{Z} \setminus \{0\}$ into the positive and the negative integers.

12. (a) [BB] **Reflexive:** For any $a \in \mathsf{Z}$, $a^2 - a^2 = 0$ is divisible by 3, so $a \sim a$.

**Symmetric:** If $a \sim b$, then $a^2 - b^2$ is divisible by 3, so $b^2 - a^2$ is divisible by 3, so $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $a^2 - b^2$ is divisible by 3 and $b^2 - c^2$ is divisible by 3, so $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2)$ is divisible by 3.

(b) $\begin{aligned} \overline{0} &= \{x \in \mathsf{Z} \mid x \sim 0\} \\ &= \{x \in \mathsf{Z} \mid x^2 \text{ is divisible by } 3\} \\ &= \{x \in \mathsf{Z} \mid x \text{ is divisible by } 3\} = 3\mathsf{Z} \end{aligned}$

$\begin{aligned} \overline{1} &= \{x \in \mathsf{Z} \mid x \sim 1\} \\ &= \{x \mid x^2 - 1 \text{ is divisible by } 3\} \\ &= \{x \mid (x-1)(x+1) \text{ is divisible by } 3\} \\ &= \{x \mid x - 1 \text{ or } x + 1 \text{ is divisible by } 3\} \\ &= 3\mathsf{Z} + 1 \cup 3\mathsf{Z} + 2 \end{aligned}$

(c) This equivalence relation partitions the integers into the two disjoint sets $3\mathsf{Z}$ and $(3\mathsf{Z}+1) \cup (3\mathsf{Z}+2)$.

13. (a) [BB] Yes, this is an equivalence relation.

**Reflexive:** Note that if $a$ is any triangle, $a \sim a$ because $a$ is congruent to itself.

**Symmetric:** Assume $a \sim b$. Then $a$ and $b$ are congruent. Therefore, $b$ and $a$ are congruent, so $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $a$ and $b$ are congruent and $b$ and $c$ are congruent, so $a$ and $c$ are congruent. Thus, $a \sim c$.

(b) Yes, this is an equivalence relation.

**Reflexive:** If $a$ is a circle, then $a \sim a$ because $a$ has the same center as itself.

**Symmetric:** Assume $a \sim b$. Then $a$ and $b$ have the same center. Thus, $b$ and $a$ have the same center, so $b \sim a$.

**Transitive:** Assume $a \sim b$ and $b \sim c$. Then $a$ and $b$ have the same center and $b$ and $c$ have the same center, so $a$ and $c$ have the same center. Thus, $a \sim c$.

(c) Yes, this is an equivalence relation.

**Reflexive:** If $a$ is a line, then $a$ is parallel to itself, so $a \sim a$.

**Symmetric:** If $a \sim b$, then $a$ is parallel to $b$. Thus, $b$ is parallel to $a$. Hence, $b \sim a$.

**Transitive:** If $a \sim b$ and $b \sim c$, then $a$ is parallel to $b$ and $b$ is parallel to $c$, so $a$ is parallel to $c$. Thus, $a \sim c$.

(d) No, this is not an equivalence relation. The reflexive property does not hold because no line is perpendicular to itself. Neither is this relation transitive; if $\ell_1$ is perpendicular to $\ell_2$ and $\ell_2$ is perpendicular to $\ell_3$, then $\ell_1$ and $\ell_3$ are **parallel**, not perpendicular to one another.

14. (a) [BB] $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5),(4,3),(5,3),(5,4)\}$

(b) $\mathcal{R} = \{((1,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$

(c) $\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

15. (a) As suggested in the text, we list the partitions of $\{a\}$. There is only one; namely, $\{a\}$.

(b) As suggested in the text, we list the partitions of $\{a,b\}$. There are two; namely, $\{a,b\}$ and $\{a\},\{b\}$.

(c) [BB] As suggested in the text, a good way to list the equivalence relations on $\{a,b,c\}$ is to list the partitions of this set. Here they are:

$\{\ \{a\},\{b\},\{c\}\ \}$;

$\{\ \{a,b,c\}\ \}$;

$\{\ \{a,b\},\{c\}\ \}$; $\{\ \{a,c\},\{b\}\ \}$; $\{\ \{b,c\},\{a\}\ \}$

There are five in all.

(d) As suggested in the text, we list the partitions of $\{a,b,c,d\}$.

$\{\ \{a\},\{b\},\{c\},\{d\}\ \}$

$\{\ \{a,b,c,d\}\ \}$

$\{\ \{a,b\},\{c,d\}\ \}$; $\{\ \{a,c\},\{b,d\}\ \}$; $\{\ \{a,d\},\{b,c\}\ \}$

$\{\ \{a,b\},\{c\},\{d\}\ \}$; $\{\ \{a,c\},\{b\},\{d\}\ \}$; $\{\ \{a,d\},\{b\},\{c\}\ \}$; $\{\ \{b,c\},\{a\},\{d\}\ \}$; $\{\ \{b,d\},\{a\},\{c\}\ \}$; $\{\ \{c,d\},\{a\},\{b\}\ \}$

$\{\ \{a,b,c\},\{d\}\ \}$; $\{\ \{a,b,d\},\{c\}\ \}$; $\{\ \{a,c,d\},\{b\}\ \}$; $\{\ \{b,c,d\},\{a\}\ \}$

There are 15 in all.

16. (a) [BB] The given statement is an implication which concludes "$x - y = x - y$," whereas what is required is a logical argument which concludes "so $\sim$ is reflexive."

    A correct argument is this: For any $(x, y) \in \mathsf{R}^2$, $x - y = x - y$; thus, $(x, y) \sim (x, y)$. Therefore, $\sim$ is reflexive.

    (b) There is confusion between the elements of a binary relation on a set $A$ (which are ordered pairs) and the elements of $A$ which are themselves ordered pairs in this situation. The given statement is correct *provided* each of $x$ and $y$ is understood to be an ordered pair of real numbers, and we understand $\mathcal{R} = \{(x, y) \mid x \sim y\}$ but this is very misleading. Much better is to state symmetry like this:

    if $(x, y) \sim (u, v)$, then $(u, v) \sim (x, y)$.

    (c) The first statement asserts the implication "$x - y = u - v \rightarrow (x, y) \sim (u, v)$" which is the converse of what should have been said. Here is the correct argument:

    If $(x, y) \sim (u, v)$, then $x - y = u - v$, so $u - v = x - y$ and, hence, $(u, v) \sim (x, y)$.

    (d) This suggested answer is utterly confusing. Logical arguments consist of a sequence of implications but here it is not clear where these implications start. Certainly the first sentence is not an implication.

    If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$ then $x - y = u - v$ and $u - v = w - z$. So $x - y = w - z$ and, hence, $(x, y) \sim (w, z)$.

    (e) $\sim$ defines an equivalence relation on $\mathsf{R}^2$ because it is a reflexive, symmetric and transitive binary relation on $\mathsf{R}^2$.

    (f) The equivalence class of $(0, 0)$ is

$$\{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x - y = 0 - 0\} = \{(x, y) \mid y = x\}$$

    which is a straight line of slope 1 in the Cartesian plane passing through the origin. The equivalence class of $(2, 3)$ is

$$\{(x, y) \mid (x, y) \sim (2, 3)\} = \{(x, y) \mid x - y = 2 - 3 = -1\} = \{(x, y) \mid y = x + 1\}$$

    which is a straight line of slope 1 passing through the point $(2, 3)$.

17. [BB] **Reflexive:** If $(x, y) \in \mathsf{R}^2$, then $x^2 - y^2 = x^2 - y^2$, so $(x, y) \sim (x, y)$.

    **Symmetric:** If $(x, y) \sim (u, v)$, then $x^2 - y^2 = u^2 - v^2$, so $u^2 - v^2 = x^2 - y^2$ and $(u, v) \sim (x, y)$.

    **Transitive:** If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$, then $x^2 - y^2 = u^2 - v^2$ and $u^2 - v^2 = w^2 - z^2$, so $x^2 - y^2 = u^2 - v^2 = w^2 - z^2$; $x^2 - y^2 = w^2 - z^2$ and $(x, y) \sim (w, z)$.

$$\overline{(0,0)} = \{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x^2 - y^2 = 0^2 - 0^2 = 0\} = \{(x, y) \mid y = \pm x\}$$

    Thus, the equivalence class of $(0, 0)$ is the pair of lines with equations $y = x$, $y = -x$.

$$\overline{(1,0)} = \{(x, y) \mid (x, y) \sim (1, 0)\} = \{(x, y) \mid x^2 - y^2 = 1^2 - 0^2 = 1\}.$$

    Thus, the equivalence class of $(1, 0)$ is the hyperbola whose equation is $x^2 - y^2 = 1$.

18. (a) This is an equivalence relation.

    **Reflexive:** If $(a, b) \in \mathsf{R}^2$, then $a + 2b = a + 2b$, so $(a, b) \sim (a, b)$.

    **Symmetric:** If $(a, b) \sim (c, d)$, then $a + 2b = c + 2d$, so $c + 2d = a + 2b$ and $(c, d) \sim (a, b)$.

**Transitive:** If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $a+2b = c+2d$ and $c+2d = e+2f$, so $a+2b = e+2f$ and $(a,b) \sim (e,f)$.

The quotient set is the set of equivalence classes. We have

$$\begin{aligned}\overline{(a,b)} &= \{(x,y) \mid (x,y) \sim (a,b)\} = \{(x,y) \mid x+2y = a+2b\} \\ &= \{(x,y) \mid y-b = -\tfrac{1}{2}(x-a)\}\end{aligned}$$

which describes the line through $(a,b)$ with slope $-\frac{1}{2}$. The quotient set is the set of lines with slope $-\frac{1}{2}$.

(b) This is an equivalence relation.

**Reflexive:** If $(a,b) \in \mathsf{R}^2$, then $ab = ab$, so $(a,b) \sim (a,b)$.

**Symmetric:** If $(a,b) \sim (c,d)$, then $ab = cd$ so $cd = ab$ and $(c,d) \sim (a,b)$.

**Transitive:** If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ab = cd$ and $cd = ef$, so $ab = cd = ef$, $ab = ef$ and $(a,b) \sim (e,f)$.

The quotient set is the set of equivalence classes. We have

$$\overline{(a,b)} = \{(x,y) \mid (x,y) \sim (a,b)\} = \{(x,y) \mid xy = ab\}$$

and consider two cases. If either $a = 0$ or $b = 0$, then $\overline{(a,b)} = \{(x,y) \mid xy = 0\}$; that is, $\{(x,y) \mid x = 0 \text{ or } y = 0\}$. Hence, $\overline{(a,b)}$ is the union of the $x$-axis and the $y$-axis. On the other hand, if $a \neq 0$ and $b \neq 0$, then

$$\overline{(a,b)} = \{(x,y) \mid xy = ab\} = \{(x,y) \mid y = \frac{ab}{x}\}$$

since $x \neq 0$ in this case. This time, $\overline{(a,b)}$ is the hyperbola whose equation is $y = ab/x$.

(c) This is not an equivalence relation. We have $(0,2) \sim (1,1)$ because $0^2 + 2 = 2 = 1 + 1^2$; however, $(1,1) \not\sim (0,2)$ because $1^2 + 1 = 2 \neq 4 = 0 + 2^2$. The relation is not symmetric.

(d) **Reflexive:** For any $(a,b) \in \mathsf{R}^2$, $a = a$, so $(a,b) \sim (a,b)$.

**Symmetric:** If $(a,b) \sim (c,d)$, then $a = c$, so $c = a$ and, hence, $(c,d) \sim (a,b)$.

**Transitive:** If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $a = c$ and $c = e$; hence, $a = e$ and so $(a,b) \sim (e,f)$.

Since the relation is reflexive, symmetric and transitive, it is an equivalence relation. The quotient set is the set of equivalence classes. The equivalence class of $(a,b)$ is

$$\{(x,y) \in \mathsf{R}^2 \mid (x,y) \sim (a,b)\} = \{(x,y) \in \mathsf{R}^2 \mid x = a\}.$$

Geometrically, this set is the vertical line with equation $x = a$. The quotient set is the set of vertical lines.

(e) This is not an equivalence relation. For example, it is not reflexive: $(1,2) \not\sim (1,2)$ because $1(2) = 2 \neq 1 = 1^2$.

19. (a) "If $\overline{a} \cap \overline{b} = \emptyset$, then $\overline{a} \neq \overline{b}$."

(b) The converse is true. If $\overline{a} \cap \overline{b} = \emptyset$, then $a \in \overline{a}$ but $a \notin \overline{b}$, so $\overline{a} \neq \overline{b}$.

20. Remembering that $\overline{x}$ is just the set of elements equivalent to $x$, we are given that $a \sim b$, $c \sim d$ and $d \sim b$. By Proposition 2.4.3, $\overline{a} = \overline{b}$, $\overline{c} = \overline{d}$ and $\overline{d} = \overline{b}$. Thus $\overline{a} = \overline{b} = \overline{d} = \overline{c}$.

21. (a) [BB] The ordered pairs defined by $\sim$ are $(1,1)$, $(1,4)$, $(1,9)$, $(2,2)$, $(2,8)$, $(3,3)$, $(4,1)$, $(4,4)$, $(4,9)$, $(5,5)$, $(6,6)$, $(7,7)$, $(8,2)$, $(8,8)$, $(9,1)$, $(9,4)$, $(9,9)$.

    (b) [BB] $\overline{1} = \{1,4,9\} = \overline{4} = \overline{9}$; $\overline{2} = \{2,8\} = \overline{8}$; $\overline{3} = \{3\}$; $\overline{5} = \{5\}$; $\overline{6} = \{6\}$; $\overline{7} = \{7\}$.

    (c) [BB] Since the sets $\{1,4,9\}$, $\{2,8\}$, $\{3\}$, $\{5\}$, $\{6\}$ and $\{7\}$ partition $A$, they determine an equivalence relation, namely, that equivalence relation in which $a \sim b$ if and only if $a$ and $b$ belong to the same one of these sets. This is the given relation.

22. [BB] **Reflexive:** If $a \in A$, then $a^2$ is a perfect square, so $a \sim a$.

    **Symmetric:** If $a \sim b$, then $ab$ is a perfect square. Since $ba = ab$, $ba$ is also a perfect square, so $b \sim a$.

    **Transitive:** If $a \sim b$ and $b \sim c$, then $ab$ and $bc$ are each perfect squares. Thus $ab = x^2$ and $bc = y^2$ for integers $x$ and $y$. Now $ab^2c = x^2y^2$, so $ac = \dfrac{x^2y^2}{b^2} = \left(\dfrac{xy}{b}\right)^2$. Because $ac$ is an integer, so also $\dfrac{xy}{b}$ is an integer. Therefore, $a \sim c$.

23. (a) The order pairs of $\sim$ are $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$, $(7,7)$, $(1,2)$, $(2,1)$, $(1,4)$, $(4,1)$, $(2,4)$, $(4,2)$, $(3,6)$, $(6,3)$.

    (b) $\overline{1} = \{1,2,4\} = \overline{2} = \overline{4}$; $\overline{3} = \{3,6\} = \overline{6}$; $\overline{5} = \{5\}$; $\overline{7} = \{7\}$.

    (c) The sets $\{1,2,4\}$, $\{3,6\}$, $\{5\}$, $\{7\}$ partition $A$, so the given relation is an equivalence relation.

24. **Reflexive:** If $a \in A$, then $\dfrac{a}{a} = 1 = 2^0$ is a power of 2, so $a \sim a$.

    **Symmetric:** If $a \sim b$, then $\dfrac{a}{b} = 2^t$, so $\dfrac{b}{a} = 2^{-t}$. Since $-t$ is an integer, $\dfrac{b}{a}$ is also a power of 2, so $b \sim a$.

    **Transitive:** If $a \sim b$ and $b \sim c$, then $\dfrac{a}{b} = 2^t$ and $\dfrac{b}{c} = 2^s$ for integers $t$ and $s$. Thus $\dfrac{a}{c} = \dfrac{a}{b}\dfrac{b}{c} = 2^{t+s}$, showing that $a \sim c$.

25. We have to prove that the given sets are disjoint and have union $S$. For the latter, we note that since $\mathcal{R}$ is reflexive, for any $a \in S$, $(a,a) \in \mathcal{R}$ and so $a$ and $a$ are elements of the same set $S_i$; that is, $a \in S_i$ for some $i$. To prove that the sets are disjoint, suppose there is some $x \in S_k \cap S_\ell$. Since $S_k \not\subseteq \bigcup_{j \neq k} S_j$, there exists $y \in S_k$ such that $y \notin S_j$ for any $j \neq k$. Similarly, there exists $z \in S_\ell$ such that $z \notin S_j$ if $j \neq \ell$. Now if $y, x \in S_k$, then $(y,x) \in \mathcal{R}$ and $x, z \in S_\ell$ implies $(x,z) \in \mathcal{R}$. By transitivity, $(y,z) \in \mathcal{R}$, hence, $y$ and $z$ belong to the same set. But the only set to which $y$ belongs is $S_k$. Since $z$ does not belong to $S_k$, we have a contradiction: No $x \in S_k \cap S_\ell$ exists.

## Exercises 2.5

1. (a) [BB] This defines a partial order.

    **Reflexive:** For any $a \in \mathsf{R}$, $a \geq a$.

    **Antisymmetric:** If $a, b \in \mathsf{R}$, $a \geq b$ and $b \geq a$, then $a = b$.

    **Transitive:** If $a, b, c \in \mathsf{R}$, $a \geq b$ and $b \geq c$, then $a \geq c$.

    This partial order is a total order because for any $a, b \in \mathsf{R}$, either $a \geq b$ or $b \geq a$.

    (b) [BB] This is not a partial order because the relation is not reflexive; for example, $1 < 1$ is not true.

(c) This is not a partial order because the relation is not antisymmetric; for example, $-3 \preceq 3$ because $(-3)^2 \leq 3^2$ and $3 \preceq -3$ because $3^2 \leq (-3)^2$ but $-3 \neq 3$.

(d) This is not a partial order because the relation is not antisymmetric; for example, $(1,4) \preceq (1,8)$ because $1 \leq 1$ and similarly, $(1,8) \preceq (1,4)$, but $(1,4) \neq (1,8)$.

(e) This is a partial order.

**Reflexive:** For any $(a,b) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (a,b)$ because $a \leq a$ and $b \geq b$.

**Antisymmetric:** If $(a,b),(c,d) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$, then $a \leq c$, $b \geq d$, $c \leq a$ and $d \geq b$. So $a = c$, $b = d$ and, hence, $(a,b) = (c,d)$.

**Transitive:** If $(a,b),(c,d),(e,f) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, then $a \leq c$, $b \geq d$, $c \leq e$ and $d \geq f$. So $a \leq e$ (because $a \leq c \leq e$) and $b \geq f$ (because $b \geq d \geq f$) and, therefore, $(a,b) \preceq (e,f)$.

This is not a total order; for example, $(1,4)$ and $(2,5)$ are incomparable.

(f) This is reflexive and transitive but not antisymmetric and, hence, not a partial order. For example, $cat \preceq dog$ and $dog \preceq cat$ but $dog \neq cat$.

2. (a) [BB] 1, 10, 100, 1000, 1001, 101, 1010, 11, 110, 111

(b) 1, 11, 111, 110, 10, 101, 1010, 100, 1001, 1000

3. (a) [BB] $(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)$; (b) [BB] $(a,b),(c,d)$;
(c) $(a,b),(a,d),(c,d)$; (d) $(a,b),(a,c),(a,d)$;
(e) $(a,d),(a,e),(b,e),(b,c),(b,f),(c,f),(d,e)$;
(f) $(a,f),(b,c),(d,b),(d,c),(d,h),(d,i),(e,c),(e,i),(g,f),(h,i)$.

4. (a) [BB] $a$ is minimal and minimum; $d$ is maximal and maximum.

(b) [BB] $a$ and $c$ are minimal; $b$ and $d$ are maximal; there are no minimum nor maximum elements.

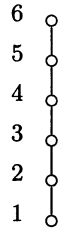
(c) $a$ and $c$ are minimal; $b$ and $d$ are maximal; there are no minimum nor maximum elements.

(d) $a$ is minimal and minimum; $b$, $c$ and $d$ and maximal; there is no maximum element.

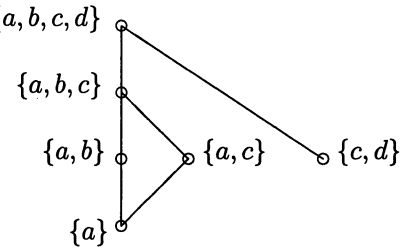
(e) $a$ and $b$ are minimal; $e$ and $f$ and maximal; there are no minimum nor maximum elements.

(f) $a$, $d$, $e$ and $g$ are minimal; $c$, $f$ and $i$ are maximal; there are no minimum nor maximum elements.

5. (a)




(b)




6. (a) 1 is minimal and minimum; 6 is maximal and maximum.

(b) $\{a\}$ and $\{c,d\}$ are minimal; there is no minimum.
The set $\{a,b,c,d\}$ is maximal and maximum.

7. [BB] $A \subsetneqq B$ and the set $B$ contains exactly one more element than $A$.

8. Helmut Hasse (1898–1979) was one of the more important mathematicians of the twentieth century. He grew up in Berlin and was a member of Germany's navy during the first World War. He received his PhD from the University of Göttingen in 1921 for a thesis in number theory, which was to be the subject of his life's work. He is known for his research with Richard Brauer and Emmy Noether on simple algebras, his proof of the Riemann Hypothesis (one of today's most famous open problems) for zeta functions on elliptic curves, and his work on the arithmetical properties of abelian number fields. Hasse's career started at Kiel and continued at Halle and Marburg. When the Nazis came to power in 1933, all Jewish mathematicians, including eighteen at the University of Göttingen, were summarily dismissed from their jobs. It is hard to know the degree of ambivalence Hasse may have had when he received an offer of employment at Göttingen around this time, but he accepted the position. While some of Hasse's closest research collaborators were Jewish, he nonetheless made no secret of his support for Hitler's policies. In 1945, he was dismissed by the British, lost his right to teach and eventually moved to Berlin. In May 1949, he was appointed professor at the Humboldt University in East Berlin but he moved to Hamburg the next year and worked there until his retirement in 1966.

9. (a) [BB] Let $(A, \preceq)$ be a finite poset and let $a \in A$. If $a$ is not maximal, there is an element $a_1$ such that $a_1 \succ a$. If $a_1$ is not maximal, there is an element $a_2$ such that $a_2 \succ a_1$. Continue. Since $A$ is finite, eventually this process must stop, and it stops at a maximal element. A similar argument shows that $(A, \preceq)$ must also contain minimal elements.

   (b) The result of (a) is not true in general. For example, $(\mathsf{R}, \leq)$ is a poset without maximal elements and without minimal elements.

10. (a) **Reflexive:** For any $\mathsf{a} = (a_1, a_2) \in A$, $\mathsf{a} \leq \mathsf{a}$ because $a_1 \leq a_1$ and $a_1 + a_2 \leq a_1 + a_2$.

    **Antisymmetric:** If $\mathsf{a} = (a_1, a_2)$ and $\mathsf{b} = (b_1, b_2)$ are in $A$ with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{a}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and $b_1 \leq a_1$, $b_1 + b_2 \leq a_1 + a_2$. Since $a_1 \leq b_1$ and $b_1 \leq a_1$, we have $a_1 = b_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $a_1 = b_1$, we have $a_2 \leq b_2$. Similarly, $b_2 \leq a_2$, so $a_2 = b_2$. Thus $\mathsf{a} = \mathsf{b}$.

    **Transitive:** If $\mathsf{a} = (a_1, a_2)$, $\mathsf{b} = (b_1, b_2)$ and $\mathsf{c} = (c_1, c_2)$ are elements of $A$ with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{c}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and also $b_1 \leq c_1$, $b_1 + b_2 \leq c_1 + c_2$. Since $a_1 \leq b_1$ and $b_1 \leq c_1$, we have $a_1 \leq c_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $b_1 + b_2 \leq c_1 + c_2$, we have $a_1 + a_2 \leq c_1 + c_2$, so $\mathsf{a} \preceq \mathsf{c}$.

    This partial order is not a total order: for example, $\mathsf{a} = (0, 0)$ and $\mathsf{b} = (-1, -2)$ are not comparable.

   (b) Let $A = \mathsf{Z}^n$ and for $\mathsf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathsf{b} = (b_1, b_2, \ldots, b_n)$ in $A$, define $\mathsf{a} \preceq \mathsf{b}$ if and only if $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, $a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$, $a_1 + a_2 + a_3 + a_4 \leq b_1 + b_2 + b_3 + b_4$, $\ldots$, $a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n$. Then $\preceq$ is a partial order on $A$.

11. (a) [BB] Suppose that $a$ and $b$ are two maximum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because $b$ is maximum and $b \preceq a$ because $a$ is maximum, so $a = b$ by antisymmetry.

    (b) Suppose that $a$ and $b$ are two minimum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because $a$ is minimum and $b \preceq a$ because $b$ is minimum, so $a = b$ by antisymmetry.

12. (a) [BB] Assuming it exists, the greatest lower bound $G$ of $A$ and $B$ has two properties:
    (1) $G \subseteq A, G \subseteq B$;
    (2) if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq G$.

    We must prove that $A \cap B$ has these properties. Note first that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $A \cap B$ satisfies (1). Also, if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$, so $A \cap B$ satisfies (2) and $A \cap B = A \wedge B$.

(b) Assuming it exists, the least upper bound of $A$ and $B$ has two properties:
(1) $A \subseteq L, B \subseteq L$;
(2) if $A \subseteq C$ and $B \subseteq C$, then $L \subseteq C$.
We must prove that $A \cup B$ has these properties. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup B$ satisfies (1). Also, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ satisfies (2) and $A \cup B = A \vee B$.

13. (a) [BB] $a \vee b = b$ and here is why. We are given $a \preceq b$ and have $b \preceq b$ by reflexivity. Thus $b$ is an upper bound for $a$ and $b$. It is least because if $c$ is any other upper bound, then $a \preceq c$, $b \preceq c$; in particular, $b \preceq c$.

(b) $a \wedge b = a$ and here is why. We are given $a \preceq b$ and have $a \preceq a$ by reflexivity. Thus $a$ is a lower bound for $a$ and $b$. It is greatest because if $c$ is any other lower bound, then $c \preceq a$, $c \preceq b$; in particular, $c \preceq a$.

14. (a) [BB] Suppose $x$ and $y$ are each glbs of two elements $a$ and $b$. Then $x \preceq a$, $x \preceq b$ implies $x \preceq y$ because $y$ is a **greatest** lower bound, and $y \preceq a$, $y \preceq b$ implies $y \preceq x$ because $x$ is greatest. So, by antisymmetry, $x = y$.

(b) Suppose $x$ and $y$ are each lubs of two elements $a$ and $b$. Then $a \preceq x$, $b \preceq x$ implies $y \preceq x$ because $y$ is a **least** upper bound, and $a \preceq y$, $b \preceq y$ implies $x \preceq y$ because $x$ is least. So, by antisymmetry, $x = y$.

15. (a) In a totally ordered set, every two elements are comparable. So given $a$ and $b$, either $a \preceq b$ or $b \preceq a$; hence, the elements $\max(a, b)$ and $\min(a, b)$ always exist. In a poset which is not totally ordered, they don't necessarily, however. In the two element poset $\{a\}, \{b\}$ with the relation $\subseteq$, for example, $\max(\{a\}, \{b\})$ does not exist because there is no element in the poset containing both $\{a\}$ and $\{b\}$. (Similarly, $\min(\{a\}, \{b\})$ does not exist.)

(b) To prove that a totally ordered set $(A, \preceq)$ is a lattice we must prove that every pair of elements has a glb and lub. We claim that $\text{glb}(a, b) = \min(a, b)$ and $\text{lub}(a, b) = \max(a, b)$.
We show that $\text{glb}(a, b) = \min(a, b)$. (The argument to show that $\text{lub}(a, b) = \max(a, b)$ is very similar.) Let $m = \min(a, b)$. (Note that $m = a$ or $m = b$). Certainly we have $m \preceq a$ and $m \preceq b$ so $m$ is a lower bound. Also, if for some element $c$ we have $c \preceq a$ and $c \preceq b$, then $c \preceq m$ if $m = a$, and $c \preceq m$ if $m = b$. In either case, we have $c \preceq m$, so $\text{glb}(a, b)$ is $\min(a, b)$ as required.

16. (a) [BB] $(\mathcal{P}(S), \subseteq)$ is not totally ordered provided $|S| \geq 2$ (since $\{a\}$ and $\{b\}$ are not comparable if $a \neq b$). But $\emptyset$ is a minimum because $\emptyset$ is a subset of any set and the set $S$ itself is a maximum because any of its subsets is contained in it.

(b) $(\mathsf{Z}, \leq)$ or $(\mathsf{R}, \leq)$ are obvious examples.

17. Suppose $a$ is maximal in a totally ordered set $(A, \preceq)$ and let $b$ be any other element of $A$. Since $A$ is totally ordered, either $a \preceq b$ or $b \preceq a$. In the first case, $a = b$ because $a$ is maximal so in either case, $b \preceq a$. Thus, $a$ is a maximum.

18. (a) [BB] We have to prove that if $b \preceq a$, then $b = a$. So suppose $b \preceq a$. Since $a$ is minimum, we have also $a \preceq b$. By antisymmetry, $b = a$.

(b) Let $b$ be a minimal element. We claim $b = a$. To see why, note that $a$ minimum implies $a \preceq b$. Then minimality of $b$ says $a = b$.

# Chapter 2 Review

1. Since $A = \{1,2,3,4,5,6\}$ and $B = \{3,4,5,6,7\}$, we have $A \oplus B = \{1,2,7\}$ and $(A \oplus B) \setminus C = \{1,7\}$.

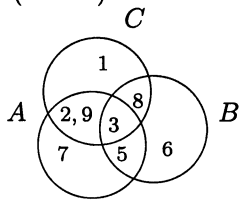
2. (a) $A = \{-1,0,1,2\}$, $B = \{-5,-3,-1,1\}$, $C = \{-\frac{2}{3}, -\frac{2}{5}, 0, \pm 1, \pm 2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$;
   (b) $A \cap B = \{\pm 1\}$, so $(A \cap B) \times B = \{(1,-5),(1,-3),(1,-1),(1,1),(-1,-5),(-1,-3),(-1,-1),(-1,1)\}$;
   (c) $B \setminus C = \{-5,-3\}$; (d) $A \oplus C = \{-\frac{2}{3}, -\frac{2}{5}, -2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$.

3. (a) True. ($\longrightarrow$) Suppose $A \cap B = A$ and let $a \in A$. Then $a \in A \cap B$, so $a \in B$. Thus $A \subseteq B$.
   ($\longleftarrow$) Suppose $A \subseteq B$. To prove $A \cap B = A$, we prove each of the two sets, $A \cap B$ and $A$, is a subset of the other. Let $x \in A \cap B$. By definition of $\cap$, $x$ is in both $A$ and $B$, in particular, $x \in A$. Thus $A \cap B \subseteq A$. Conversely, let $x \in A$. Since $A \subseteq B$, $x \in B$. Since $x$ is in both $A$ and $B$, $x \in A \cap B$. Thus $A \subseteq A \cap B$.
   (b) This is false. If $A = \emptyset$, $(A \cap B) \cup C = C$ while $A \cap (B \cup C) = \emptyset$, so any $C \neq \emptyset$, any $B$, and $A = \emptyset$ provides a counterexample.
   (c) False. Take $A = B = \emptyset$.

4. Let $b \in B$ and let $a$ be any element of $A$. Then $(a,b) \in A \times B$, so $(a,b) \in A \times C$. Thus $b \in C$. This shows tht $B \subseteq C$ and a similar argument shows $C \subseteq B$, so $B = C$.

5. (a) Region 2: $(A \cap C) \setminus B$ Region 3: $A \cap B \cap C$ Region 4: $(A \cap B) \setminus C$
   Region 5: $(B \cap C) \setminus A$ Region 6: $B \setminus (A \cup C)$ Region 7: $C \setminus (A \cup B)$
   (b) Region 2, 3, 4, 5, 7 is $(A \cap B) \cup C$; region 2, 3, 4 is $A \cap (B \cup C)$
   (c) $B \setminus (C \setminus A)$ consists of regions 3, 4, and 6. $(B \setminus C) \setminus A$ consists of region 6.

6. (a)




   (b) i. $(A \cup B) \cap C = \{2,3,8,9\}$
   ii. $A \setminus (B \setminus C) = \{2,3,7,9\}$
   iii. $A \oplus B = \{2,6,7,8,9\}$
   iv. $(A \setminus B) \times (B \cap C) = \{(2,3),(2,8),(7,3),(7,8),(9,3),(9,8)\}$

7. $\mathcal{P}(A) = \{\emptyset, A\}$, so $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{A\}, \{\emptyset, A\}\}$.

8. (a) Take $A = B = C = \{3\}$. Then $B \setminus C = \emptyset$ so $A \oplus (B \setminus C) = A$. On the other hand, $A \oplus B = \emptyset$ so $(A \oplus B) \setminus C = \emptyset \neq A$.
   (b) Let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$. Since $a \in A$ and $A \subseteq C$, $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. So $(a,b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

9. Take $B = C = \emptyset$, $A = \{1\} = D$. Then $A \times B = \emptyset = C \times D$, so $A \times B \subseteq C \times D$. On the other hand, $A \not\subseteq C$.

10. This follows quickly from one of the laws of De Morgan and the identity $X \setminus Y = X \cap Y^c$.

$$A \setminus (B \cap C) = A \cap (B \cap C)^c = A \cap (B^c \cup C^c) \overset{\downarrow}{=} (A \cap B^c) \cup (A \cap C^c) = (A \setminus B) \cup (A \setminus C),$$

using (3), p. 62 at the spot marked with the arrow.

11. (a) A *binary relation* on $A$ is a subset of $A \times A$.

(b) If $A$ has 10 elements, $A \times A$ has 100 elements, so there are $2^{100}$ binary relations on $A$.

12. **Reflexive**: For any $a$ with $|a| \leq 1$, we have $a^2 = |a^2| = |a||a| \leq |a|$, thus $(a, a) \in \mathcal{R}$.

**Symmetric** by definition.

**Not antisymmetric** because $(\frac{1}{2}, \frac{1}{4})$ is in $\mathcal{R}$ ($(\frac{1}{2})^2 \leq \frac{1}{4}$ and $(\frac{1}{4})^2 \leq \frac{1}{2}$) but $\frac{1}{2} \neq \frac{1}{4}$.

**Not transitive**. We have $(\frac{1}{2}, \frac{1}{3}) \in \mathcal{R}$ because $(\frac{1}{2})^2 \leq \frac{1}{3}$ and $(\frac{1}{3})^2 \leq \frac{1}{2}$ and $(\frac{1}{3}, \frac{1}{5}) \in \mathcal{R}$ because $(\frac{1}{3})^2 \leq \frac{1}{5}$ and $(\frac{1}{5})^2 \leq \frac{1}{3}$, but $(\frac{1}{2}, \frac{1}{5})$ is not in $\mathcal{R}$ because $(\frac{1}{2})^2 \not\leq \frac{1}{5}$.

13. (a) **Reflexive:** For any natural number $a$, we have $a \leq 2a$, so $a \sim a$.

**Not symmetric:** $2 \sim 5$ because $2 \leq 2(5)$, but $5 \not\sim 2$ because $5 \not\leq 2(2)$.

**Not antisymmetric:** Let $a = 1$ and $b = 2$. Then $a \sim b$ because $1 \leq 2(2)$ and also $b \sim a$ because $2 \leq 2(1)$.

**Not transitive:** Let $a = 3$, $b = 2$, $c = 1$. Then $a \sim b$ because $3 \leq 2(2)$ and $b \sim c$ because $2 \leq 2(1)$. However, $a \not\sim c$ because $3 \not\leq 2(1)$.

Since the relation is not transitive, it is not an equivalence relation and it is not a partial order.

(b) **Not reflexive:** $(1, 2) \not\sim (1, 2)$ because $1 \leq 2$ and $2 \not\leq 1$.

**Not symmetric:** $(1, 2) \sim (4, 3)$ because $1 \leq 2$ and $3 \leq 4$, but $(4, 3) \not\sim (1, 2)$ because $4 \not\leq 3$.

**Not antisymmetric:** $(1, 1) \sim (2, 2)$ because $1 \leq 1$ and $2 \leq 2$, and $(2, 2) \sim (1, 1)$ for the same reason, but $(1, 1) \neq (2, 2)$.

**Transitive:** if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq b$, $d \leq c$, $c \leq d$ and $f \leq e$, so $a \leq b$ and $f \leq e$ which implies $(a, b) \sim (e, f)$.

This is not an equivalence relation because it's not reflexive (or symmetric).

This is not a partial order because it's not reflexive (or antisymmetric).

14. We must determine whether or not $a < b$ and $b < a$ implies $a = b$. Since the hypothesis is always false, this implication is true. The relation is antisymmetric.

15. **Reflexive:** For any $a \in \mathbb{Z}$, $4a + a = 5a$ is a multiple of 5.

**Symmetric:** If $a\mathcal{R}b$, then $4a + b$ is a multiple of 5, so $4b + a = 5(a + b) - (4a + b)$ is also a multiple of 5, that is, $b\mathcal{R}a$.

**Transitive:** If $a\mathcal{R}b$ and $b\mathcal{R}c$, then both $4a + b$ and $4b + c$ are multiples of 5, hence so is their sum, $4a + 5b + c$. It follows that $(4a + 5b + c) - 5b = 4a + c$ is also a multiple of 5, so $a\mathcal{R}c$.

16. (a) **Reflexive:** For any $a \in \mathbb{Z}$, $a\mathcal{R}a$ because $2a + 5a = 7a$ is a multiple of 7.

**Symmetric:** If $a, b \in \mathbb{Z}$ and $a\mathcal{R}b$, then $2a + 5b = 7k$ for some integer $k$, so $5a + 2b = 7(a + b) - (2a + 5b)$ is the difference of multiples of 7, hence also a multiple of 7. Thus $b\mathcal{R}a$.

**Transitive:** If $a, b, c \in \mathbb{Z}$ with $a\mathcal{R}b$ and $b\mathcal{R}c$, then $2a + 5b = 7k$ for some integer $k$ and $2b + 5c = 7\ell$ for some integer $\ell$. Thus $(2a + 5b) + (2b + 5c) = 2a + 7b + 5c = 7(k + \ell)$ and $2a + 5c = 7(k + \ell) - 7b$ is the difference of multiples of 7, hence a multiple of 7. Thus $a\mathcal{R}c$.

(b) We have $0 \mathcal{R} 7$ and $7 \mathcal{R} 0$, yet $0 \neq 7$. The relation is not antisymmetric, so it is not a partial order.

17. ($\longrightarrow$) Assume $x \sim a$. Let $t \in \overline{x}$. Then $t \sim x$ so $t \sim a$ by transitivity. Thus $t \in \overline{a}$. This proves $\overline{x} \subseteq \overline{a}$. Similarly, $\overline{a} \subseteq \overline{x}$, so $\overline{x} = \overline{a}$.

($\longleftarrow$) Assume $\overline{x} = \overline{a}$. Since $x \in \overline{x}$, by symmetry, $x \in \overline{a}$. Thus $x \sim a$.

18. Since $a \in \overline{b}$, $a \sim b$ and hence $\overline{a} = \overline{b}$ by Proposition 2.4.3. Similarly $d \in \overline{b}$ implies $\overline{d} = \overline{b}$, hence $\overline{a} = \overline{d}$. Now $d \notin \overline{c}$ implies $\overline{d} \neq \overline{c}$, so $\overline{c} \cap \overline{d} = \emptyset$ by Proposition 2.4.4. Since $\overline{a} = \overline{d}$, so also $\overline{a} \cap \overline{c} = \emptyset$.

19. (a) We must show that $\preceq$ is reflexive, antisymmetric and transitive on $A$. The relation is reflexive: For any $(a, b) \in A$, $(a, b) \sim (a, b)$ because $a \leq a$ and $b \leq b$.

It is antisymmetric: If $(a, b), (c, d) \in A$, with $(a, b) \sim (c, d)$ and $(c, d) \sim (a, b)$, then $a \leq c$, $d \leq b$, $c \leq a$ and $b \leq d$. So $a = c$, $b = d$, hence, $(a, b) = (c, d)$.

It is transitive: If $(a, b), (c, d), (e, f) \in A$ with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq c$, $d \leq b$, $c \leq e$ and $f \leq d$. So $a \leq e$ (because $a \leq c \leq e$) and $f \leq b$ (because $f \leq d \leq b$), so $(a, b) \sim (e, f)$.

(b) $(A, \preceq)$ is not totally ordered since, for example, $(1, 4)$ and $(2, 5)$ are not comparable: $(1, 4) \not\sim (2, 5)$ because $5 \not\leq 4$ and $(2, 5) \not\sim (1, 4)$ because $2 \not\leq 1$.

20. (a) **Reflexive:** For any $p \in A$, $p \sim p$ since $p = p$.

**Symmetric:** If $p \sim q$ then either $p = q$ (so $q = p$) or the line through $p$ and $q$ passes through the origin (in which case the line through $q$ and $p$ passes through the origin). Thus $q \sim p$.

**Transitive:** Suppose $p \sim q$ and $q \sim r$. If the points $p, q, r$ are different, then the line through $p$ and $q$ passes through the origin, as does the line through $q$ and $r$. Since the line through the origin and $q$ is unique, $p$ and $r$ lie on this line, $p \sim r$. If $p = r$, then $p \sim r$. If $p = q \neq r$, then the line through $q$ and $r$ passes through the origin, so the line through $p$ and $r$ passes through the origin; thus $p \sim r$. The remaining case, $p \neq q = r$ is similar.

(b) The equivalence class of a point $p$ is the line through the origin and $p$. The equivalence classes are lines through the origin.

21. **Reflexive:** For any $A \in \mathcal{P}(\mathsf{Z})$, $A \subseteq A$.

**Antisymmetric:** If $A, B \subseteq \mathcal{P}(\mathsf{Z})$ with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

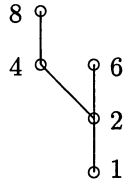
**Transitive:** If $A, B, C \subseteq \mathcal{P}(\mathsf{Z})$ with $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

22. (a) **Reflexive:** For any $a \in A$, $\dfrac{a}{a} = 1$ is an integer, so $a \preceq a$.

**Antisymmetric:** If $a \preceq b$ and $b \preceq a$, then both $\dfrac{b}{a}$ and $\dfrac{a}{b}$ are (necessarily positive) integers. The only positive integer whose reciprocal is also an integer is 1, so $a = b$.

**Transitive:** If $a \preceq b$ and $b \preceq c$, then $\dfrac{b}{a}$ and $\dfrac{c}{b}$ are both integers. Thus $\dfrac{b}{a}\,\dfrac{c}{b} = \dfrac{c}{a}$ is an integer. So $a \preceq c$.

(b)




(c) 1 is minimal and minimum; 6 and 8 are maximal. There is no maximum element.

(d) $(A, \preceq)$ is not totally ordered; for example, 4 and 6 are not comparable.

23. Two elements of a poset can have at most one least upper bound and here's why. Let $\ell_1, \ell_2$ each be least upper bounds for elements $a$ and $b$. Then $\ell_1$ is an upper bound for $a$ and $b$, so $\ell_2 \preceq \ell_1$ because $\ell_2$ is **least**. Interchanging $\ell_1$, $\ell_2$ in the preceding statement gives $\ell_1 \preceq \ell_2$. So $\ell_1 = \ell_2$ by antisymmetry.

## Exercises 3.1

1. (a) [BB] Not a function; $f$ contains two different pairs of the form $(3, -)$.
   (b) Not a function **with domain** $\{1, 2, 3, 4\}$. There's no pair of the form $(3, -)$.
   (c) [BB] This is a function.
   (d) Certainly not a function. There is more than one pair of the form $(1, -)$: In fact, there are four!
   (e) This is a function.

2. (a) [BB] This is not a function unless each student at the University of Calgary has just one professor, for if student $a$ is taking courses from professors $b_1$ and $b_2$, the given set contains $(a, b_1)$ and $(a, b_2)$.
   (b) Assuming that each student currently registered at the University of Calgary is taking at least one course, then this is a function.
   (c) Assuming some student $a$ has no classes on Friday afternoon, then this will not be a function since the set would contain no pair of the form $(a, -)$.

3. [BB] $A \times B$ is a function $A \to B$ if and only if $B$ contains exactly one element.

   To see why, first note that if $B = \{b\}$, then $A \times B = \{(a, b) \mid a \in A\}$ is certainly a function.

   Conversely, if $A \times B$ is a function but $B$ contains two elements $b_1, b_2$, then for any $a \in A$, $(a, b_1)$ and $(a, b_2)$ are both in $A \times B$, so $A \times B$ is not a function.

4. (a) [BB] the function defined by $f(n) = 2n$, for example.
   (b) the function defined by $f(1) = 1$ and for $n > 1$, $f(n) = n - 1$, for example.
   (c) the constant function $f(n) = 107$ for all $n$, for example.
   (d) the identify function, $\iota_{\mathbb{N}}$, for example: $\iota(n) = n$, for all $n$.

5. (a) [BB] If $x \in X$, then $x$ is a country in the British Commonwealth with a uniquely determined Prime Minister $y$ who lives in that country; that is, $y \in Y$.

   If $y \in Y$, then $y$ is a person living in one of the countries in the British Commonwealth. Thus, the domicile of $y$ is a uniquely determined element $x \in X$.
   (b) If $x_1 \neq x_2$ are two different countries, then their Prime Ministers are different individuals, so Prime Minister is one-to-one. On the other hand, Prime Minister is not onto since there are people in a country who are not the Prime Minister.
   (c) If $x \in X$, then $x$ is a country. If $y$ is any person living in that country, then Domicile: $y \mapsto x$, so Domicile is onto. On the other hand, if $y_1$ and $y_2$ are different people living in country $x$, then Domicile: $y_1 \mapsto x$ and Domicile: $y_2 \mapsto x$, so Domicile is not one-to-one.

6. [BB] Parking rates, bus fares, admission prices are several common examples.

7. (a) Jumps in the graph of $y = \lfloor 2x - 3 \rfloor$ occur whenever $2x - 3$ is an integer, that is, at $\{\frac{n}{2} \mid n \in \mathsf{Z}\}$.
   (b) Jumps in the graph of $y = \lceil \frac{1}{4}x + 7 \rceil$ occur whenever $\frac{1}{4}x + 7$ is an integer, that is, at $\{4n \mid n \in \mathsf{Z}\}$.
   (c) Jumps in the graph of $y = \lfloor \frac{x+3}{5} \rfloor$ occur whenever $\frac{1}{5}(x + 3)$ is an integer, that is, at $\{x \mid x = 5n - 3, n \in \mathsf{Z}\}$.

8. (a) [BB] The answer is "yes." By definition of "floor", we know that $kx$ is the unique integer satisfying $n - 1 < kx \leq n$. Thus $\frac{n}{k} - \frac{1}{k} < x \leq \frac{n}{k}$. Since $\frac{n}{k} - 1 \leq \frac{n}{k} - \frac{1}{k}$, we have $\frac{n}{k} - 1 < x \leq \frac{n}{k}$. Thus $x = \lfloor \frac{n}{k} \rfloor$.
   (b) The multiples of 3 in the indicated interval are $3, 6, 9, \ldots, 3k$, where $3k = \lfloor 50000 \rfloor$. Thus $k = \lfloor \frac{50000}{3} \rfloor = 16666$.

9. [BB] We know that $\lfloor x \rfloor$ is an integer less than or equal to $x$ and that $\lfloor y \rfloor$ is an integer less than or equal to $y$. Thus $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer less than or equal to $x + y$. Since $\lfloor x + y \rfloor$ is the **greatest** such integer, we get the desired result.

10. We know that $\lceil x \rceil$ is an integer greater than or equal to $x$ and that $\lceil y \rceil$ is an integer greater than or equal to $y$. Thus $\lceil x \rceil + \lceil y \rceil$ is an integer greater than or equal to $x + y$. Since $\lceil x + y \rceil$ is the **smallest** such integer, we get the desired result.

11. $\lfloor x \rfloor$ is the unique integer which satisfies $x - 1 < \lfloor x \rfloor \leq x$. This implies $(x+n) - 1 < \lfloor x \rfloor + n \leq x + n$. But there is exactly one integer $a$ in the range $(x+n) - 1 < a \leq x + n$ and that is $\lfloor x + n \rfloor$. This gives the result.

12. We have $f(1) = 2(1) - 5 = -3$, $f(2) = 2^2 + 1 = 5$, $f(3) = 1$, $f(4) = 17$, $f(5) = 5$, and so, as a subset of $S \times \mathsf{Z}$, $f = \{(1, -3), (2, 5), (3, 1), (4, 17), (5, 5)\}$.

    No, $f$ is not 1–1 because $f(2) = f(5)$ but $2 \neq 5$ (equivalently, $(2, 5)$ and $(5, 5)$ are both in $f$).

13. (a) [BB] add is not one-to-one since, for example, $\mathsf{add}(1, 1) = \mathsf{add}(0, 2)$ while $(1, 1) \neq (0, 2)$. It is onto, however, because, for any $y \in \mathsf{R}$, the equation $y = \mathsf{add}(x)$ has the solution $x = (y, 0)$.
    (b) mult is not one-to-one since, for example, $\mathsf{mult}(1, 4) = \mathsf{mult}(2, 2)$ while $(1, 4) \neq (2, 2)$. It is onto, though, since for any $y \in \mathsf{R}$, the equation $y = \mathsf{mult}(x)$ has the solution $x = (y, 1)$.

14. (a) [BB] $g$ is not one-to-one since, for example, $g(1) = g(-1) = 2$. Neither is $g$ onto: For any $x \in \mathsf{Z}$, $|x| \geq 0$, so $|x| + 1 \geq 1$. Thus, for example, $0 \notin \operatorname{rng} g$.
    (b) [BB] $g$ is not one-to-one as in (a). This time it is onto, however, because for $n \in \mathsf{N}$, the equation $n = g(x)$ has the solution $x = n - 1$. (Note that $n > 0$ for $n \in \mathsf{N}$, so $|n - 1| = n - 1$.)

15. (a) [BB] If $f(x_1) = f(x_2)$, then $3x_1 + 5 = 3x_2 + 5$, so $x_1 = x_2$ which proves that $f$ is one-to-one. Also $f$ is onto, since given $y \in \mathsf{Q}$, $y = f(\frac{1}{3}(y - 5))$ with $\frac{1}{3}(y - 5) \in \mathsf{Q}$.
    (b) This function is one-to-one, as in (a), but it is not onto because $f(n) = 3n + 5 > 8$ for any $n \in \mathsf{N}$ and so, for example, 1 is not in the range of $f$.

16. (a) $h$ is not one-to-one since, for example, $h(1) = 3 = h(-1)$ and $1 \neq -1$.
    $h$ is not onto since $1 \notin \operatorname{rng} h$: $1 \in \mathsf{Z}$, but if we try to solve $x^2 + 2 = 1$, we obtain $x^2 = -1$ which has no solutions in Z.
    (b) $h$ is one-to-one since if $h(x_1) = h(x_2)$, then $x_1^2 + 2 = x_2^2 + 2$, so $x_1^2 = x_2^2$ and $x_1 = \pm x_2$. Since $x_1$ and $x_2$ are both in N, we must have $x_1 = x_2$.
    $h$ is not onto; as in (a), 1 is not in the range of $h$.

17. (a) The function is not onto since, for example, $g(x) = 0$ has no integer solutions (by the quadratic formula), but it is one-to-one. To see why, suppose $g(x_1) = g(x_2)$. Then $3x_1^2 + 14x_1 - 51 = 3x_2^2 + 14x_2 - 51$, so $3(x_1 - x_2)(x_1 + x_2) = 14(x_2 - x_1)$. Thus $x_1 = x_2$ or $x_1 + x_2 = -\frac{14}{3}$. Since $x_1$ and $x_2$ are integers, $x_1 + x_2 = -\frac{14}{3}$ is impossible. Thus, $x_1 = x_2$, so $g$ is one-to-one.

    (b) Since the graph of $g$ is a parabola, $g$ is neither one-to-one nor onto.

18. (a) [BB] Note that $f(x) = (x + 7)^2 - 100$. If $f(x_1) = f(x_2)$ it follows that $(x_1 + 7)^2 - 100 = (x_2 + 7)^2 - 100$, so $(x_1 + 7)^2 = (x_2 + 7)^2$ and, taking square roots, $|x_1 + 7| = |x_2 + 7|$. Since $x_1, x_2 \in \mathsf{N}$, we know $x_1 + 7 > 0$ and $x_2 + 7 > 0$. Thus, $x_1 + 7 = x_2 + 7$ and $x_1 = x_2$. Thus, $f$ is one-to-one, but it is not onto: For example, $1 \in B$, but there is no $x \in \mathsf{N}$ with $f(x) = 1$ since $(x + 7)^2 - 100 = 1$ implies $(x + 7)^2 = 101$ and this equation has no solution in the natural numbers.

    (b) This function is not one-to-one since, for example, $f(0) = f(-14)$ $(= -51)$ and it is not onto, as in (a).

    (c) This function is not one-to-one; as in (b), $f(0) = f(-14)$. But it is onto since for any $y \geq -100$, $x = \sqrt{100 + y} - 7$ is a solution to $y = f(x)$.

19. (a) [BB] The domain of $f$ is R. Its range is also R because every $y \in \mathsf{R}$ can be written $y = f(x)$ for some $x$; namely, $x = y^{1/3}$. The function is, therefore, onto. It is also one-to-one: If $f(x_1) = f(x_2)$, then $x_1^3 = x_2^3$ and this implies $x_1 = x_2$.

    (b) The domain of $g$ is R and so is the range. For the latter, note that

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

    so that if $y \geq 0$, then $y = g(\sqrt{y})$ while if $y < 0$, $y = g(-\sqrt{-y})$. Since $\operatorname{rng} g = \mathsf{R}$, $g$ is onto. To see that $g$ is 1–1, either inspect the graph or assume $g(x_1) = g(x_2)$ and consider the cases.

    **Case 1:** Both $x_1$ and $x_2$ are nonnegative.
    In this case, $x_1^2 = x_2^2$, so $\sqrt{x_1^2} = \sqrt{x_2^2}$; hence, $|x_1| = |x_2|$ and $x_1 = x_2$.

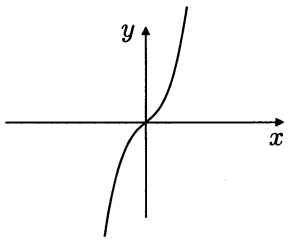
    **Case 2:** Both $x_1$ and $x_2$ are negative.
    In this case, $-x_1^2 = -x_2^2$, so $x_1^2 = x_2^2$, $\sqrt{x_1^2} = \sqrt{x_2^2}$, $|x_1| = |x_2|$, $-x_1 = -x_2$ and $x_1 = x_2$.

    **Case 3:** One of $x_1, x_2$ is nonnegative and the other is negative.
    Here we may assume that $x_1$ is nonnegative, so $x_1^2 = -x_2^2$. Since the left side is at least 0 and the right side is less than 0, the equation cannot be true.
    In all cases $x_1 = x_2$, so $g$ is 1–1.

    (c) The domain of $\beta$ is $(\frac{4}{3}, \infty)$ as given. Its range is R because for any $y \in \mathsf{R}$, $y = \beta(x)$ for $x = \frac{1}{3}(2^y + 4)$. Since $\operatorname{rng} \beta = \mathsf{R}$, $\beta$ is onto. It is also one-to-one: If $\beta(x_1) = \beta(x_2)$, then $2^{\log_2(3x_1 - 4)} = 2^{\log_2(3x_2 - 4)}$, so $3x_1 - 4 = 3x_2 - 4$ and $x_1 = x_2$.

    (d) The domain of $f$ is R as given. To find the range, we must remember that $2^t > 0$ for all $t$ and thus, $f(x) > 3$ for all $x$. In fact $\operatorname{rng} f = (3, \infty)$ since, for any $y > 3$, $y = f(x)$ for $x = 1 + \log_2(y - 3)$. Since $\operatorname{rng} f \neq \mathsf{R}$, $f$ is not onto. It is one-to-one, however, for if $f(x_1) = f(x_2)$, then $2^{x_1 - 1} + 3 = 2^{x_2 - 1} + 3$, so $2^{x_1 - 1} = 2^{x_2 - 1}$. Taking $\log_2$ of each side, $x_1 - 1 = x_2 - 1$, so $x_1 = x_2$.

20. (a) [BB] The graph of $f$ shown at the right makes it clear that $f$ is one-to-one and onto.



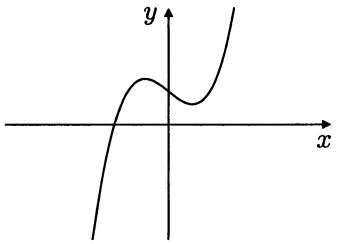

(b) [BB] **Solution 1.** Since $f\colon \mathsf{R} \to \mathsf{R}$ is one-to-one by part (a), it is also one-to-one as a function with domain Z. Here, however, $f$ is not onto for we note that $f(0) = 0$, $f(1) = 4$ and $f$ is increasing, so 1 is not in the range of $f$.

**Solution 2.** (This solution mimics that given in Problem 8 in our discussion of discrete functions in this section.)

If $f(x_1) = f(x_2)$, then $3x_1^3+x_1 = 3x_2^3+x_2$, so $3(x_1^3-x_2^3) = x_2-x_1$ and $3(x_1-x_2)(x_1^2+x_1x_2+x_2^2) = x_2 - x_1$. If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = -\frac{1}{3}$, which is impossible for integers $x_1, x_2$. Thus, $x_1 = x_2$ and $f$ is one-to-one.

On the other hand, $f$ is not onto. In particular, 1 is not in the range of $f$ since $1 = f(a)$ for some $a$ implies $3a^3 + a = 1$; that is, $a(3a^2 + 1) = 1$. But the only pairs of integers whose product is 1 are the pairs $1, 1$ and $-1, -1$. So here, we'd have to have $a = 3a^2 + 1 = 1$ or $a = 3a^2 + 1 = -1$, neither of which is possible.

21. (a) The graph of $g$ shown at the right makes it clear that $g$ is onto but not one-to-one.




(b) Since $g(0) = g(1)$, $g$ is not one-to-one. Furthermore, $g$ is not onto; for example, $2 = g(x)$ has no **integer** solution since $2 = x^3 - x + 1$ implies $x(x^2 - 1) = 1$ (for an integer $x$), so $x = x^2 - 1 = 1$ or $x = x^2 - 1 = -1$, neither of which is possible.

(c) The argument given in (b) applies to this case as well, so $g$ is not onto. To determine whether or not $g$ is one-to-one, suppose $g(x_1) = g(x_2)$. Then $x_1^3 - x_1 + 1 = x_2^3 - x_2 + 1$, so $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2$. If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = 1$. For natural numbers $x_1, x_2$, this is not possible because $x_1^2 + x_1x_2 + x_2^2 \geq 3$. Thus, $x_1 = x_2$ and $g$ is one-to-one.

22. (a) [BB] This is not one-to-one since, for example, $f(1,3) = f(4,1) = 11$ while $(1,3) \neq (4,1)$. The function is not onto since, for example, the equation $f(x) = 1$ has no solution $x = (n, m)$ (because $2n + 3m \geq 5$ for every $n, m \in \mathsf{N}$).

(b) [BB] This is not one-to-one since, for example, $f(1,3) = f(4,1) = 11$ while $(1,3) \neq (4,1)$. The function is onto, however, since for $k \in \mathsf{Z}$, the equation $k = f(x)$ has the solution $x = (-k, k)$.

(c) This is not one-to-one since, for example, $f(12,1) = f(1,8)$ $(= 190)$ but $(12,1) \neq (1,8)$. It's not onto because $f(n,m)$ is even for any $(n,m)$ so, for example, the equation $f(n,m) = 1$ cannot be solved.

(d) This is not one-to-one since, for example, $f(246,0) = f(0,89)$. It's onto since $(-17)(246) + 47(89) = 1$ implies that for any $k \in \mathsf{Z}$, $k = (-17k)(246) + (47k)(89) = f(47k, -17k)$.

(e) This is not one-to-one because, for example, $f(7,8) = f(8,7)$. It's not onto because, for example, the equation $f(x) = 4$ has no solution $x = (n,m)$: $n^2 + m^2 + 1 \neq 4$ for $n, m \in \mathsf{Z}$.

(f) This is not one-to-one because, for example, $f(2,2) = f(3,2)$ $(= 2)$. It is onto, however, because $1 = f(1,2)$ and, for $n \geq 2$, we have $n = f(n-1, 1)$.

23. ($\rightarrow$) Suppose $f$ is one-to-one and $x_1 \neq x_2$. We must prove that $x_1^2 + x_1x_2 + x_2^2 + a(x_1 + x_2) + b \neq 0$. Since $f$ is one-to-one and $x_1 \neq x_2$, we know that $f(x_1) \neq f(x_2)$; thus, $x_1^3 + ax_1^2 + bx_1 + c \neq x_2^3 + ax_2^2 + bx_2 + c$; equivalently,

$$(x_1^3 - x_2^3) + a(x_1^2 - x_2^2) + b(x_1 - x_2) \neq 0.$$

Since $x_1 - x_2 \neq 0$, we may divide this last inequality by $x_1 - x_2$. Since $x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)$ and $x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$, we obtain $(x_1^2 + x_1x_2 + x_2^2) + a(x_1 + x_2) + b \neq 0$ as required.

($\leftarrow$) Suppose that $x_1 \neq x_2$ implies $x_1^2 + x_1x_2 + x_2^2 + a(x_1 + x_2) + b \neq 0$. We must prove that $f$ is one-to-one. Thus, we assume $f(x_1) = f(x_2)$ and prove $x_1 = x_2$. The equation $f(x_1) = f(x_2)$ says $x_1^3 + ax_1^2 + bx_1 + c = x_2^3 + ax_2^2 + bx_2 + c$; or, equivalently, as before,

$$(x_1^3 - x_2^3) + a(x_1^2 - x_2^2) + b(x_1 - x_2) = 0.$$

If $x_1 \neq x_2$, then we divide by $x_1 - x_2$ as before, obtaining $x_1^2 + x_1x_2 + x_2^2 + a(x_1 + x_2) + b = 0$, a contradiction. Thus $x_1 = x_2$.

24. (a) [BB] We require $x \neq 3$, so we take $A = \{x \in \mathsf{R} \mid x \neq 3\}$. Then $\operatorname{rng} f = \{y \mid y \neq 0\}$ because if $y \neq 0$, $y = f(x)$ for $x = 3 + \frac{1}{y} \in \operatorname{dom} f$.

(b) We require $1 - x > 0$, so we take $A = \{x \in \mathsf{R} \mid x < 1\}$. Then $\operatorname{rng} f = \{y \mid y > 0\}$ because for any $y > 0$, $y = f(x)$ for $x = 1 - \frac{1}{y^2} \in \operatorname{dom} f$.

25. (a) The function is not one-to-one because $(a, \alpha)$ and $(d, \alpha)$ are both in $f$, but $a \neq d$. Restricting the domain to $\{a, b, c\}$, for instance, we obtain the one-to-one function $\{(a, \alpha), (b, \beta), (c, \gamma)\}$.

(b) [BB] Note that $f(x) = -(2x - 3)^2$. This function is not one-to-one since, for example, $f(1) = f(2)$ $(= -1)$. Restrict the domain to $\{x \mid x \geq 3/2\}$ (or to $\{x \mid x \leq \frac{3}{2}\}$).

(c) The sine function is not one-to-one since, for example, $\sin 0 = \sin \pi$. The largest interval on which a one-to-one function can be defined has length $\pi$; for example, $\{x \mid -\pi/2 \leq x \leq \pi/2\}$ is an interval on which sine is one-to-one.

26. (a) **Reflexive:** If $f \in A$, then $f(5) = f(5)$, so $f \sim f$.
**Symmetric:** If $f \sim g$, then $f(5) = g(5)$, so $g(5) = f(5)$ and hence, $g \sim f$.
**Transitive:** If $f \sim g$ and $g \sim h$, then $f(5) = g(5)$ and $g(5) = h(5)$, so $f(5) = g(5) = h(5)$ and $f \sim h$.

(b) $\overline{f} = \{g \colon S \to S \mid g(5) = f(5) = a\}$. Since there are three possible images for each of $g(a)$ and $g(b)$, there are altogether 9 functions in $\overline{f}$; namely,

$$\begin{aligned}
f_1 &= \{(5,a),(a,5),(b,5)\} & f_2 &= \{(5,a),(a,5),(b,a)\} \\
f_3 &= \{(5,a),(a,5),(b,b)\} & f_4 &= \{(5,a),(a,a),(b,5)\} \\
f_5 &= \{(5,a),(a,a),(b,a)\} & f_6 &= \{(5,a),(a,a),(b,b)\} \\
f_7 &= \{(5,a),(a,b),(b,5)\} & f_8 &= \{(5,a),(a,b),(b,a)\} \\
f_9 &= \{(5,a),(a,b),(b,b)\}
\end{aligned}$$

27. (a) **Reflexive:** If $a \in A$, then $a \sim a$ because $f(a) = f(a)$.
**Symmetric:** If $a, b \in A$ and $a \sim b$, then $f(a) = f(b)$, so $f(b) = f(a)$; hence, $b \sim a$.
**Transitive:** If $a, b, c \in A$, $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$, implying $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{R} \mid f(x) = f(0) = 0\} = [0,1)$; $\overline{7/5} = [1,2)$ since $\lfloor 7/5 \rfloor = 1$; $\overline{-3/4} = [-1,0)$ since $\lfloor -3/4 \rfloor = -1$.

(c) $\overline{1} = \{1\}$; $\overline{2} = \{2,3,6\}$; $\overline{4} = \{4\}$; $\overline{5} = \{5\}$.

28. (a) [BB] Here are the functions $X \to Y$:

$$\begin{array}{lll} \{(a,1),(b,1)\} & \{(a,1),(b,2)\} & \{(a,1),(b,3)\} \\ \{(a,2),(b,1)\} & \{(a,2),(b,2)\} & \{(a,2),(b,3)\} \\ \{(a,3),(b,1)\} & \{(a,3),(b,2)\} & \{(a,3),(b,3)\}. \end{array}$$

Here are the functions $Y \to X$:

$$\begin{array}{ll} \{(1,a),(2,a),(3,a)\} & \{(1,a),(2,a),(3,b)\} \\ \{(1,a),(2,b),(3,a)\} & \{(1,a),(2,b),(3,b)\} \\ \{(1,b),(2,a),(3,a)\} & \{(1,b),(2,a),(3,b)\} \\ \{(1,b),(2,b),(3,a)\} & \{(1,b),(2,b),(3,b)\}. \end{array}$$

(b) [BB] There are no one-to-one functions $Y \to X$. The one-to-one functions from $X \to Y$ are

$$\begin{array}{lll} \{(a,1),(b,2)\} & \{(a,1),(b,3)\} & \{(a,2),(b,1)\} \\ \{(a,2),(b,3)\} & \{(a,3),(b,1)\} & \{(a,3),(b,2)\}. \end{array}$$

(c) [BB] There are no onto functions $X \to Y$. The onto functions $Y \to X$ are

$$\begin{array}{ll} \{(1,a),(2,a),(3,b)\} & \{(1,a),(2,b),(3,a)\} \\ \{(1,a),(2,b),(3,b)\} & \{(1,b),(2,a),(3,a)\} \\ \{(1,b),(2,a),(3,b)\} & \{(1,b),(2,b),(3,a)\}. \end{array}$$

29. (a) There are no one-to-one functions $Y \to X$. Here are the 24 one-to-one functions $X \to Y$.

$$\begin{array}{lll} \{(a,1),(b,2),(c,3)\} & \{(a,1),(b,2),(c,4)\} & \{(a,1),(b,3),(c,2)\} \\ \{(a,1),(b,3),(c,4)\} & \{(a,1),(b,4),(c,2)\} & \{(a,1),(b,4),(c,3)\} \\ \{(a,2),(b,3),(c,1)\} & \{(a,2),(b,3),(c,4)\} & \{(a,2),(b,4),(c,1)\} \\ \{(a,2),(b,4),(c,3)\} & \{(a,2),(b,1),(c,3)\} & \{(a,2),(b,1),(c,4)\} \\ \{(a,3),(b,1),(c,2)\} & \{(a,3),(b,1),(c,4)\} & \{(a,4),(b,1),(c,2)\} \\ \{(a,4),(b,1),(c,3)\} & \{(a,3),(b,2),(c,1)\} & \{(a,3),(b,2),(c,4)\} \\ \{(a,4),(b,2),(c,1)\} & \{(a,4),(b,2),(c,3)\} & \{(a,3),(b,4),(c,1)\} \\ \{(a,3),(b,4),(c,2)\} & \{(a,4),(b,3),(c,1)\} & \{(a,4),(b,3),(c,2)\} \end{array}$$

(b) There are no onto functions $X \to Y$. Here are the 36 onto functions $Y \to X$.

$$\begin{array}{lll} \{(1,a),(2,a),(3,b),(4,c)\} & \{(1,a),(2,a),(3,c),(4,b)\} & \{(1,b),(2,b),(3,a),(4,c)\} \\ \{(1,b),(2,b),(3,c),(4,a)\} & \{(1,c),(2,c),(3,a),(4,b)\} & \{(1,c),(2,c),(3,b),(4,a)\} \\ \{(1,a),(2,b),(3,a),(4,c)\} & \{(1,a),(2,c),(3,a),(4,b)\} & \{(1,b),(2,a),(3,b),(4,c)\} \\ \{(1,b),(2,c),(3,b),(4,a)\} & \{(1,c),(2,a),(3,c),(4,b)\} & \{(1,c),(2,b),(3,c),(4,a)\} \\ \{(1,a),(2,b),(3,c),(4,a)\} & \{(1,a),(2,c),(3,b),(4,a)\} & \{(1,b),(2,a),(3,c),(4,b)\} \\ \{(1,b),(2,c),(3,a),(4,b)\} & \{(1,c),(2,a),(3,b),(4,c)\} & \{(1,c),(2,b),(3,a),(4,c)\} \\ \{(1,b),(2,a),(3,a),(4,c)\} & \{(1,c),(2,a),(3,a),(4,b)\} & \{(1,a),(2,b),(3,b),(4,c)\} \\ \{(1,c),(2,b),(3,b),(4,a)\} & \{(1,a),(2,c),(3,c),(4,b)\} & \{(1,b),(2,c),(3,c),(4,a)\} \\ \{(1,b),(2,a),(3,c),(4,a)\} & \{(1,c),(2,a),(3,b),(4,a)\} & \{(1,a),(2,b),(3,c),(4,b)\} \\ \{(1,c),(2,b),(3,a),(4,b)\} & \{(1,a),(2,c),(3,b),(4,c)\} & \{(1,b),(2,c),(3,a),(4,c)\} \\ \{(1,b),(2,c),(3,a),(4,a)\} & \{(1,c),(2,b),(3,a),(4,a)\} & \{(1,a),(2,c),(3,b),(4,b)\} \\ \{(1,c),(2,a),(3,b),(4,b)\} & \{(1,a),(2,b),(3,c),(4,c)\} & \{(1,b),(2,a),(3,c),(4,c)\} \end{array}$$

30. [BB]

| $m$ \ $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 9 | 16 |
| 3 | 1 | 8 | 27 | 64 |
| 4 | 1 | 16 | 81 | 256 |

We guess that the number of functions $X \to Y$ is $n^m$.

31. (a) Let $A = \{a_1, a_2, \ldots, a_n\}$.
If $f\colon A \to B$ were one-to-one, then $f(a_1), f(a_2), \ldots, f(a_n)$ would be $n$ different elements in $B$ contradicting the fact that $B$ has only $m < n$ elements.

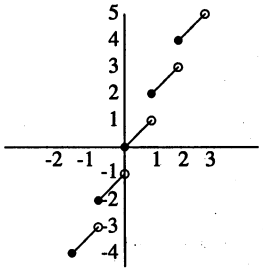
(b) Let $B = \{b_1, b_2, \ldots, b_m\}$. If $f\colon A \to B$ were onto, then there would exist elements $a_1, a_2, \ldots, a_m$ in $A$ such that $f(a_i) = b_i$, distinct by definition of "function." But this contradicts the fact that $A$ has $n < m$ elements.

32. (a) [BB ($\longrightarrow$)] Suppose $A$ and $B$ each contain $n$ elements. Assume that $f\colon A \to B$ is one-to-one and let $C = \{f(a) \mid a \in A\}$. Since $f(a_1) \neq f(a_2)$ if $a_1 \neq a_2$, $C$ is a subset of $B$ containing $n$ elements; so $C = B$. Therefore, $f$ is onto.
Conversely, suppose $f$ is onto. Then $\{f(a) \mid a \in A\} = B$ and so the set on the left here contains $n$ distinct elements. Since $A$ contains only $n$ elements, we cannot have $f(a_1) = f(a_2)$ for distinct $a_1, a_2$; thus, $f$ is one-to-one.

(b) $f(a) = 2a$ for all $a \in \mathsf{N}$. This does not contradict (a) because N is not finite.

(c) $f(1) = 1$, $f(n) = n - 1$ for $n \geq 2$. Again there is no contradiction since N is not finite.

33. (a) [BB]




(b) [BB] The domain of $f$ is R. The range is the set of all real numbers of the form $2n + a$, where $n \in \mathsf{Z}$ and $0 \leq a < 1$. To prove this algebraically, we first note that any $y = 2n + a$ is $f(x)$ for $x = n + a$. On the other hand, suppose $y = f(x)$ for some $x$. Writing $x = n + a$, $n = \lfloor x \rfloor$, $0 \leq a < 1$, then $y = x + \lfloor x \rfloor = (n + a) + n = 2n + a$ as required.

34. $s$ is not one-to-one since, for example, $s(\frac{1}{2}) = s(1\frac{1}{2})$ $(= \frac{1}{2})$. Neither is $s$ onto since, for any $x$, $x - \lfloor x \rfloor$ is in the interval $[0, 1) = \{x \in \mathsf{R} \mid 0 \leq x < 1\}$, so, for example, $1 \neq s(x)$ for any $x$.

## Exercises 3.2

1. [BB]

(a) $f^{-1} = \{(1,5), (2,1), (3,2), (4,3), (5,4)\}$ (b) $f^{-1} = \{(1,4), (2,1), (3,3), (4,2), (5,5)\}$

2. [BB] Let $y = f^{-1}(x)$. Then $x = f(y) = -y^2$, so $y^2 = -x$ and $y = \pm\sqrt{-x}$. Since $y \in \operatorname{dom} f$, $y \geq 0$, thus $y = +\sqrt{-x} = f^{-1}(x)$.

3. Let $y = f^{-1}(x)$. Then $x = f(y) = y^2$, so $y = \pm\sqrt{x}$. Since $y \in \operatorname{dom} f$, $y \leq 0$, so $y = -\sqrt{x} = f^{-1}(x)$.

4. Let $y = f^{-1}(x)$. Then $x = f(y) = -\sqrt{y}$, so $y = x^2 = f^{-1}(x)$.

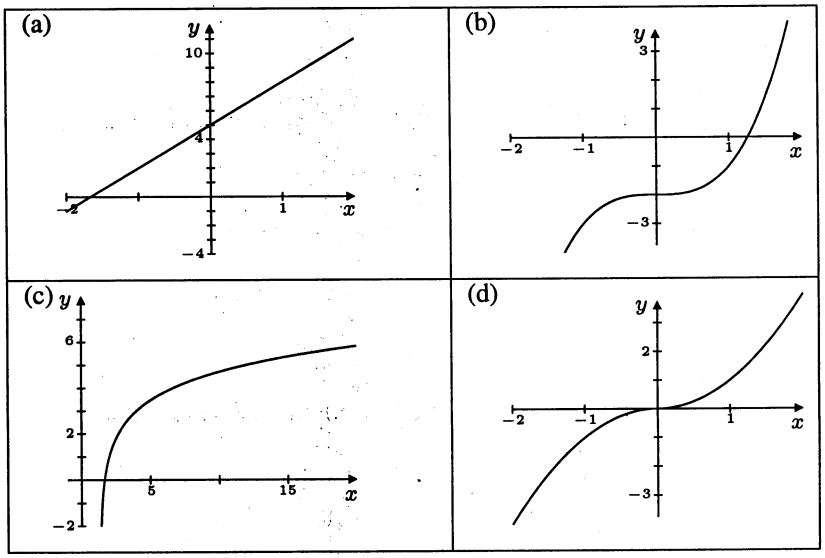
5. Since $g(x)$ is an integer, $f \circ g(x) = g(x)$. Similarly, $g \circ f(x) = f(x)$.

6. (a) [BB] $f^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $f^{-1}(x) = \frac{1}{3}(x-5)$.

   (b) $f^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $f^{-1}(x) = (x+2)^{1/3}$.

   (c) $\beta^{-1}\colon \mathsf{R} \to (\frac{4}{3}, \infty)$ is given by $\beta^{-1}(x) = \frac{1}{3}(2^x + 4)$.

   (d) $g^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $g^{-1}(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0. \end{cases}$




7. (a) [BB] If $f(x_1) = f(x_2)$, then $1 + \dfrac{1}{x_1 - 4} = 1 + \dfrac{1}{x_2 - 4}$, $\dfrac{1}{x_1 - 4} = \dfrac{1}{x_2 - 4}$ and $x_1 - 4 = x_2 - 4$. Thus $x_1 = x_2$ and $f$ is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = 1 + \frac{1}{x-4} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y - 1 = \frac{1}{x-4} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } (y-1)(x-4) = 1 \\ &\leftrightarrow y \neq 1. \end{aligned}$$

   Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 1\}$ and $f$ has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = 1 + \dfrac{1}{y-4}$, so $x - 1 = \dfrac{1}{y-4}$, $(x-1)(y-4) = 1$ and, since $x \neq 1$, $y - 4 = \dfrac{1}{x-1}$ and $f^{-1}(x) = y = 4 + \dfrac{1}{x-1}$.

   (b) Suppose $f(x_1) = f(x_2)$. Then $5 - \dfrac{1}{1+x_1} = 5 - \dfrac{1}{1+x_2}$, so $\dfrac{1}{1+x_1} = \dfrac{1}{1+x_2}$, $1 + x_1 = 1 + x_2$

and $x_1 = x_2$. Thus $f$ is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = 5 - \frac{1}{1+x} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y - 5 = -\frac{1}{1+x} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } (y-5)(1+x) = -1 \\ &\leftrightarrow y \neq 5. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 5\}$ and $f$ has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = 5 - \frac{1}{1+y}$, so $x - 5 = -\frac{1}{1+y}$, $(x-5)(1+y) = -1$ and, since $x \neq 5$, $1 + y = -\frac{1}{x-5}$ and $f^{-1}(x) = y = -1 - \frac{1}{x-5}$.

(c) Suppose $f(x_1) = f(x_2)$. Then $\frac{3x_1}{2x_1+1} = \frac{3x_2}{2x_2+1}$, so $6x_1x_2 + 3x_1 = 6x_1x_2 + 3x_2$ and $x_1 = x_2$. Thus $f$ is one-to-one. Now

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{3x}{2x+1} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } 2xy + y = 3x \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x(2y-3) = -y \\ &\leftrightarrow y \neq \frac{3}{2}. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq \frac{3}{2}\}$ and $f$ has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = \frac{3y}{2y+1}$, so $2xy + x = 3y$ and $y(2x-3) = -x$. Since $x \in B$, we know $2x - 3 \neq 0$; thus, $y = \frac{-x}{2x-3} = f^{-1}(x)$.

(d) Suppose $f(x_1) = f(x_2)$. Then $\frac{x_1-3}{x_1+3} = \frac{x_2-3}{x_2+3}$, so $x_1x_2+3x_1-3x_2-9 = x_1x_2-3x_1+3x_2-9$, $6x_1 = 6x_2$ and $x_1 = x_2$. Thus $f$ is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{x-3}{x+3} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y(x+3) = x - 3 \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x - yx = 3y + 3 \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x(1-y) = 3(y+1) \\ &\leftrightarrow y \neq 1. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 1\}$ and $f$ has an inverse $B \to A$. Let $y = f^{-1}(x)$ with $x \in B$. Then $x = f(y) = \frac{y-3}{y+3}$, so $x(y+3) = y - 3$, $xy + 3x = y - 3$, $y(1-x) = 3(x+1)$. Since $x \neq 1$, $f^{-1}(x) = y = \frac{3(1+x)}{1-x}$.

8. (a) First we show that $f$ is one-to-one. Suppose then that $f(x_1) = f(x_2)$. If $x_1$ and $x_2$ are both negative, then $2|x_1| = 2|x_2|$. Since $|x| = -x$ for $x < 0$, we have $-2x_1 = -2x_2$ and hence, $x_1 = x_2$. If $x_1$ and $x_2$ are both $\geq 0$, we have $2x_1 + 1 = 2x_2 + 1$, so $2x_1 = 2x_2$ and again, $x_1 = x_2$. Finally, we note that it is impossible for $f(x_1) = f(x_2)$ with $x_1 < 0$ and $x_2 \geq 0$ (or vice versa) since in this case, one of $f(x_1), f(x_2)$ is an even integer while the other is odd.

   Next we show that $f$ is onto. Suppose that $n \in \mathsf{N}$. If $n = 2k$ is even, then $n = 2|-k| = f(-k)$ since $-k < 0$, while if $n = 2k + 1$ is odd, then $n = f(k)$ since $k \geq 0$. Since $f$ is one-to-one and onto, it has an inverse.

   (b) Let $f^{-1}(2586) = a$. We must solve the equation $2586 = f(a)$. Since 2586 is even, we see that $2586 = f(-2586/2) = f(-1293)$.

9. (a) [BB] maternal grandmother (b) paternal grandmother (c) maternal grandfather
   (d) mother-in-law (e) father (f) father-in-law
   (g) you (h) maternal grandmother (i) maternal grandmother

10. (a) [BB] $f \circ g = \{(1,1),(3,8),(2,1),(4,9),(5,1)\}$;
    $g \circ f$ is not defined because $\operatorname{rng} f = \{1,2,3,8,9\} \not\subseteq \operatorname{dom} g = \{1,2,3,4,5\}$.
    $f \circ f$ is not defined because $\operatorname{rng} f = \{1,2,3,8,9\} \not\subseteq \operatorname{dom} f = \{1,2,3,4,5\}$.
    $g \circ g = \{(1,2),(2,2),(3,2),(4,1),(5,2)\}$.

    (b) $f$ is one-to-one and onto;
    $g$ is not one-to-one: It contains both $(1,2)$ and $(2,2)$;
    $g$ is not onto: 4 is not in the range of $g$.

    (c) Since $f$ is one-to-one and onto, it has an inverse: $f^{-1} = \{(8,1),(9,3),(3,4),(1,2),(2,5)\}$.

    (d) Since $g$ is not one-to-one (or since $g$ is not onto), $g$ does not have an inverse.

11. (a) [BB] $g^{-1} \circ f \circ g = \{(1,3),(2,1),(3,2),(4,4)\}$; (b) $f \circ g^{-1} \circ g = f$;
    (c) $g \circ f \circ g^{-1} = \{(1,2),(2,4),(3,3),(4,1)\}$; (d) $g \circ g^{-1} \circ f = f$;
    (e) $f^{-1} \circ g^{-1} \circ f \circ g = \{(1,1),(2,4),(3,2),(4,3)\}$.

12. [BB] $g \circ f(x) = \dfrac{1}{(x+2)^2+1} = \dfrac{1}{x^2+4x+5}$; $f \circ g(x) = \dfrac{1}{x^2+1} + 2 = \dfrac{1+2(x^2+1)}{x^2+1} = \dfrac{2x^2+3}{x^2+1}$;
    $h \circ g \circ f(x) = 3$; $g \circ h \circ f(x) = g(3) = \frac{1}{10}$.

    Since $f^{-1} \circ f(x) = x$, we have $g \circ f^{-1} \circ f(x) = g(x) = \dfrac{1}{x^2+2}$.

    Since $f^{-1}(x) = x - 2$, we have

$$f^{-1} \circ g \circ f(x) = f^{-1}\Big(\frac{1}{(x+2)^2+1}\Big) = \frac{1}{(x+2)^2+1} - 2 = \frac{-2(x+2)^2-1}{(x+2)^2+1}.$$

13. $g \circ f(x) = \dfrac{1}{\frac{x}{x+1}} = \dfrac{x+1}{x}$; $f \circ g(x) = \dfrac{\frac{1}{x}}{\frac{1}{x}+1} = \dfrac{1}{x+1}$;

    $h \circ g \circ f(x) = h\Big(\dfrac{x+1}{x}\Big) = \dfrac{x+1}{x} + 1 = \dfrac{2x+1}{x}$; $f \circ g \circ h(x) = f\Big(\dfrac{1}{x+1}\Big) = \dfrac{\frac{1}{x+1}}{\frac{1}{x+1}+1} = \dfrac{1}{x+2}$.

14. [BB] $(g \circ f)(x) = g(f(x)) = f(x) - c$. Thus the graph of $g \circ f$ is the graph of $f$ translated vertically $c$ units down if $c > 0$ and $-c$ units up if $c < 0$. The graphs are identical if $c = 0$.

15. $(f \circ g)(x) = f(g(x)) = f(x-c)$. Thus the graph of $f \circ g$ is the graph of $f$, but translated horizontally $c$ units to the right if $c > 0$ and $-c$ units to the left if $c < 0$. The graphs are identical if $c = 0$.

16. (a) [BB] Since $-|x| = \begin{cases} -x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ we have $f \circ g(x) = f(-|x|) = \begin{cases} f(-x) & \text{if } x \geq 0 \\ f(x) & \text{if } x < 0. \end{cases}$

So the graph of $f \circ g$ is the same as the graph of $f$ to the left of the $y$-axis (where $x < 0$) while to the right of the $y$-axis, the graph of $f \circ g$ is the reflection (mirror image) of the left half of the graph of $f$ in the $y$-axis. We call $f \circ g$ an *even* function since it is symmetric with respect to the $y$-axis: $f \circ g(-x) = f \circ g(x)$.

(b) Since $g \circ f(x) = -|f(x)| = \begin{cases} -f(x) & \text{if } f(x) \geq 0 \\ f(x) & \text{if } f(x) < 0 \end{cases}$ the graph of $g \circ f$ is the same as the graph of $f$ wherever the graph of $f$ is below the $x$-axis ($y < 0$) while it is the reflection (mirror image) of the graph of $f$ in the $x$-axis wherever the graph of $f$ is above the $x$-axis. (In particular, the graph of $g \circ f$ lies entirely on or below the $x$-axis.)

17. (a) $f \circ g(x) = f(g(x)) = f(\frac{1}{1-x}) = 1 - \dfrac{1}{\frac{1}{1-x}} = 1 - (1-x) = \iota(x)$.

$g \circ r(x) = g(r(x)) = g(\frac{x}{x-1}) = \dfrac{1}{1 - \frac{x}{x-1}} = \frac{x-1}{(x-1)-x} = \frac{x-1}{-1} = 1 - x = s(x)$.

| $\circ$ | $\iota$ | $f$ | $g$ | $h$ | $r$ | $s$ |
|---|---|---|---|---|---|---|
| $\iota$ | $\iota$ | $f$ | $g$ | $h$ | $r$ | $s$ |
| $f$ | $f$ | $g$ | $\iota$ | $s$ | $h$ | $r$ |
| $g$ | $g$ | $\iota$ | $f$ | $r$ | $s$ | $h$ |
| $h$ | $h$ | $r$ | $s$ | $\iota$ | $f$ | $g$ |
| $r$ | $r$ | $s$ | $h$ | $g$ | $\iota$ | $f$ |
| $s$ | $s$ | $h$ | $r$ | $f$ | $g$ | $\iota$ |

(b) All these functions have inverses.

| function | $\iota$ | $f$ | $g$ | $h$ | $r$ | $s$ |
|---|---|---|---|---|---|---|
| inverse | $\iota$ | $g$ | $f$ | $h$ | $r$ | $s$ |

18. (a)

| $\circ$ | $\iota$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
|---|---|---|---|---|---|---|
| $\iota$ | $\iota$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
| $f_1$ | $f_1$ | $\iota$ | $f_4$ | $f_5$ | $f_2$ | $f_3$ |
| $f_2$ | $f_2$ | $f_5$ | $\iota$ | $f_4$ | $f_3$ | $f_1$ |
| $f_3$ | $f_3$ | $f_4$ | $f_5$ | $\iota$ | $f_1$ | $f_2$ |
| $f_4$ | $f_4$ | $f_3$ | $f_1$ | $f_2$ | $f_5$ | $\iota$ |
| $f_5$ | $f_5$ | $f_2$ | $f_3$ | $f_1$ | $\iota$ | $f_4$ |

(b) All these functions have inverses.

| function | $\iota$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
|---|---|---|---|---|---|---|
| inverse | $\iota$ | $f_1$ | $f_2$ | $f_3$ | $f_5$ | $f_4$ |

19. (a) [BB] $f \circ g = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$; $g \circ f = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$
Clearly, $f \circ g \neq g \circ f$.

(b) $f^{-1} = \{(1,2),(2,1),(3,5),(4,3),(5,4)\}$; $g^{-1} = \{(1,3),(2,4),(3,1),(4,5),(5,2)\}$
Functions $f$ and $g$ have inverses because they are one-to-one and onto while $h$ does not have an inverse because it is not one-to-one (equally because it is not onto).

(c) $(f \circ g)^{-1} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$
$g^{-1} \circ f^{-1} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\} = (f \circ g)^{-1}$.
$f^{-1} \circ g^{-1} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\} \neq (f \circ g)^{-1}$.

20. (a) [BB] For $x \in B$, $(f \circ g)(x) = f(\frac{2x}{x-1}) = \dfrac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = \dfrac{2x}{2x - 2(x-1)} = \dfrac{2x}{2} = x$.

(b) [BB] For $x \in A$, $(g \circ f)(x) = g(\frac{x}{x-2}) = \dfrac{2(\frac{x}{x-2})}{\frac{x}{x-2} - 1} = \dfrac{2x}{x-(x-2)} = \dfrac{2x}{2} = x$ and so, by Proposition 3.2.7, $f$ and $g$ are inverses.

21. (a) Suppose $0 \in A$ and let $a = f(0)$. Then $0 = f^{-1}(a) = \dfrac{1}{f(a)}$ which is not possible for any real number $f(a)$.

(b) Since $f$ and $f^4$ each have domain $A$, we have only to prove that $f^4(a) = a$ for all $a \in A$. Let then $a$ be some element of $A$. Let $a_1 = f(a)$, $a_2 = f(a_1) = f^2(a)$, $a_3 = f(a_2) = f^3(a)$ and $a_4 = f(a_3) = f^4(a)$. We must show that $a_4 = a$. From $a_1 = f(a)$, we have $a = f^{-1}(a_1) = \dfrac{1}{f(a_1)}$ and so $a_2 = f(a_1) = \dfrac{1}{a}$. From $a_3 = f(a_2)$, we obtain $a_2 = f^{-1}(a_3) = \dfrac{1}{f(a_3)}$ and so $a_4 = f(a_3) = \dfrac{1}{a_2} = \dfrac{1}{1/a} = a$ as desired.

22. (a) [BB] Suppose $g(b_1) = g(b_2)$ for $b_1, b_2 \in B$. Since $f$ is onto, $b_1 = f(a_1)$ and $b_2 = f(a_2)$ for some $a_1, a_2 \in A$. Thus, $g(f(a_1)) = g(f(a_2))$; that is, $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$. Therefore, $b_1 = f(a_1) = f(a_2) = b_2$ proving that $g$ is one-to-one.

(b) Given $b \in B$, we must find $a \in A$ such that $f(a) = b$. Consider $g(b) \in C$. Since $g \circ f : A \to C$ is onto, there is some $a \in A$ with $g \circ f(a) = g(b)$; that is, $g(f(a)) = g(b)$. But $g$ one-to-one implies $f(a) = b$, so we have the desired element $a$.

23. (a) [BB] Suppose $f : A \to B$ and $g : B \to C$ are one-to-one. We prove that $g \circ f : A \to C$ is one-to-one. For this, suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$ for some $a_1, a_2 \in A$. Then $g(f(a_1)) = g(f(a_2))$ [an equation of the form $g(b_1) = g(b_2)$]. Since $g$ is one-to-one, we conclude that $f(a_1) = f(a_2)$, and then, since $f$ is one-to-one, that $a_1 = a_2$.

(b) Let $f = \{(1,1),(2,2)\}$ and $g = \{(-1,5),(1,5),(2,10)\}$. Then $g$ is not 1–1, but the composition $g \circ f = \{(1,5),(2,10)\}$ is.

(c) Let $f$ and $g$ be functions, $f : A \to B$, $g : B \to C$ and suppose $g \circ f : A \to C$ is 1–1. Then $f$ must be 1–1 since $f(a_1) = f(a_2)$ implies $g(f(a_1)) = g(f(a_2))$, that is, $g \circ f(a_1) = g \circ f(a_2)$, from which we conclude that $a_1 = a_2$, because $g \circ f$ is 1–1.

24. (a) Suppose $f : A \to B$ and $g : B \to C$ are onto. We prove that $g \circ f : A \to C$ is onto. Let then $c \in C$. We must find $a \in A$ such that $g \circ f(a) = c$; that is, $g(f(a)) = c$ [an equation of the form $g(b) = c$]. Since $g$ is onto, there is $b \in B$ such that $g(b) = c$. Also, since $f$ is onto, we know there is $a \in A$ such that $b = f(a)$. Thus, $g(f(a)) = g(b) = c$ as desired.

(b) [BB] With $A = \{1,2\}$, $B = \{-1,1,2\}$, $C = \{5,10\}$, $f = \{(1,1),(2,2)\}$ and $g = \{(-1,5),(1,5),(2,10)\}$, we have $g \circ f = \{(1,5),(2,10\}$. Then $g \circ f : A \to C$ is onto but $f$ is not.

(c) Suppose $f : A \to B$, $g : B \to C$ and $g \circ f : A \to C$ is onto. Then $g$ must be onto because if $c \in C$, there exists $a \in A$ with $g \circ f(a) = c$ ($g \circ f$ is onto), so $g(f(a)) = c$. Thus, there exists $b \in B$ with $g(b) = c$ ($b = f(a)$).

25. [BB] Since a bijective function is, by definition, a one-to-one onto function, we conclude, by the results of part (a) of the previous two exercises, that indeed the composition of bijective functions is bijective.

26. (a) $f(1000) = 998$; $f(999) = f(f(1003)) = f(1001) = 999$; $f(998) = f(f(1002)) = f(1000) = 998$; $f(997) = f(f(1001)) = f(999) = 999$.

(b) We guess that $f(n) = \begin{cases} 998 & \text{if } n \text{ is even} \\ 999 & \text{if } n \text{ is odd.} \end{cases}$ (c) We guess $\operatorname{rng} f = \{998, 999\}$.

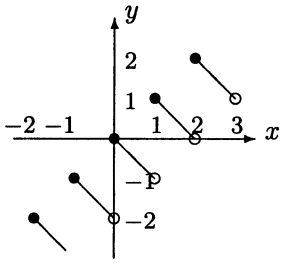
27. Suppose $f(x_1) = f(x_2)$. Then $\dfrac{x_1}{\sqrt{x_1^2+2}} = \dfrac{x_2}{\sqrt{x_2^2+2}}$, so $\dfrac{x_1^2}{x_1^2+2} = \dfrac{x_2^2}{x_2^2+2}$, $x_1^2x_2^2 + 2x_1^2 = x_1^2x_2^2 + 2x_2^2$, $2x_1^2 = 2x^2$ and $x_1 = \pm x_2$. Note that $x_2 = -x_1$ is not possible (unless $x_1 = x_2 = 0$) since $f(-x_1) = \dfrac{-x_1}{\sqrt{x_1^2+2}} \neq \dfrac{x_1}{\sqrt{x_1^2+2}} = f(x_1)$. Thus $f$ is one-to-one. Next we note that

$$y \in \operatorname{rng} f \leftrightarrow y = f(x) \text{ for some } x \in \mathsf{R} \leftrightarrow y = \frac{x}{\sqrt{x^2+2}} \text{ for some } x \in \mathsf{R}.$$

Now, if $y = \dfrac{x}{\sqrt{x^2+2}}$, then $y^2(x^2+2) = x^2$, so $x^2(y^2-1) = -2y^2$ and this implies that $y^2 \neq 1$ (otherwise we have the equation $0 = 1$) and also $y^2 - 1 \leq 0$ because $x^2 \geq 0$, $-2y^2 \leq 0$ and the product of two nonnegative numbers is nonnegative. If $y \in \operatorname{rng} f$, then $y^2 - 1 < 0$; that is, $-1 < y < 1$. On the other hand, if $-1 < y < 1$, then $y^2 - 1 < 0$, $\dfrac{-2y^2}{y^2-1} \geq 0$, and so this element is $x^2$ for some $x$; hence $y \in \operatorname{rng} f$. This shows that $\operatorname{rng} f = B = (-1, 1)$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$ for $x \in B$. Then $x = f(y) = \dfrac{y}{\sqrt{y^2+2}}$, so $x^2(y^2+2) = y^2$ and $y^2(x^2-1) = -2x^2$. Since $x^2 - 1 \neq 0$, $y^2 = \dfrac{2x^2}{1-x^2}$ and since the fraction on the right here is nonnegative, there are (apparently) two solutions $y = \pm\sqrt{\dfrac{2x^2}{1-x^2}}$. The equation $x = \dfrac{y}{\sqrt{y^2+2}}$, however, shows that $x$ and $y$ have the same sign. Thus

$$y = f^{-1}(x) = \begin{cases} \sqrt{\dfrac{2x^2}{1-x^2}} & \text{if } x > 0 \\ -\sqrt{\dfrac{2x^2}{1-x^2}} & \text{if } x < 0. \end{cases}$$

28. (a) From the graph, we see immediately that $t$ is one-to-one and onto, so $t$ has an inverse.




(b) [BB] Writing $x = \lfloor x \rfloor + a$, $0 \leq a < 1$, we have $t(x) = \lfloor x \rfloor - a$. Now it is straightforward to see that $t$ is one-to-one: Suppose $t(x_1) = t(x_2)$, where $x_1 = \lfloor x_1 \rfloor + a_1$, $x_2 = \lfloor x_2 \rfloor + a_2$ and $0 \leq a_1, a_2 < 1$. Then $\lfloor x_1 \rfloor - a_1 = \lfloor x_2 \rfloor - a_2$, so $\lfloor x_1 \rfloor - \lfloor x_2 \rfloor = a_1 - a_2$. The left side is an integer; hence, so is the right. Because of the restrictions on $a_1$ and $a_2$, the only possibility is $a_1 = a_2$. Hence, also $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$, so $x_1 = x_2$.

(c) To see that $t$ is onto, let $y \in \mathsf{R}$. Then we have $y = \lfloor y \rfloor + b$ for $0 \leq b < 1$. If $b = 0$, then $t(y) = y$. Otherwise, $0 < 1 - b < 1$. So, with $x = \lfloor y \rfloor + 1 + (1 - b)$, we have $\lfloor x \rfloor = \lfloor y \rfloor + 1$. Thus, $t(x) = \lfloor x \rfloor - (1 - b) = (\lfloor y \rfloor + 1) - (1 - b) = \lfloor y \rfloor + b = y$. In any case, we have an $x$ such that $t(x) = y$.

(d) $t^{-1}\colon \mathsf{R} \to \mathsf{R}$ is given by $t^{-1}(x) = \begin{cases} x & \text{if } x \in \mathsf{Z} \\ n + 1 + (1 - b) & \text{if } x = n + b, n \in \mathsf{Z}, 0 < b < 1 \end{cases}$

## Exercises 3.3

1. [BB] Ask everyone to find a seat.

2. [BB] The two lists $1^2, 2^2, 3^2, 4^2, \ldots$ and $1, 2, 3, 4, \ldots$ obviously have the same length; $a^2 \mapsto a$ is a one-to-one correspondence between the set of perfect squares and N.

3. (a) $x \leftrightarrow 14, y \leftrightarrow -3, \{a, b, c\} \leftrightarrow t$.

   (b) The function $f\colon 2\mathsf{Z} \to 17\mathsf{Z}$ defined by $f(2k) = 17k$ for $2k \in 2\mathsf{Z}$.

   (c) [BB] The function $f\colon \mathsf{N} \times \mathsf{N} \to \mathsf{C}$ defined by $f(m, n) = m + ni$ for all $m, n \in \mathsf{N}$.

   (d) The function $f\colon \mathsf{N} \to \mathsf{Q}$ defined by $f(n) = n/2$.

4. $f^{-1}(z) = \begin{cases} 2z & \text{if } z > 0 \\ 1 - 2z & \text{if } z \leq 0 \end{cases}$

5. (a) If $g(m_1, n_1) = g(m_2, n_2)$, then $(m_1, f(n_1)) = (m_2, f(n_2))$, so $m_1 = m_2$ and $f(n_1) = f(n_2)$. Since $f$ is one-to-one, $n_1 = n_2$. Thus $(m_1, n_1) = (m_2, n_2)$, so $g$ is one-to-one. To show that $g$ is onto, let $(a, b) \in \mathsf{N} \times \mathsf{Z}$. Since $f$ is onto, there exists $n \in \mathsf{N}$ such that $f(n) = b$. Then $g(a, n) = (a, f(n)) = (a, b)$, so $g$ is onto.

   (b) Let $f\colon \mathsf{N} \to \mathsf{Z}$ be any one-to-one onto function (for example, the function defined in Problem 29, of this section). By part (a), the function $g\colon \mathsf{N} \times \mathsf{N} \to \mathsf{N} \times \mathsf{Z}$ defined by $g(m, n) = (m, f(n))$ is a one-to-one correspondence $\mathsf{N} \times \mathsf{N} \to \mathsf{N} \times \mathsf{Z}$.

6. [BB] This is false. For example, $|\mathsf{N}| = |\mathsf{N} \cup \{0\}|$, as shown in the text.

7. This is not a partial order because it is not antisymmetric. If $a, b \in S$ and $a \neq b$, then $\{a\} \preceq \{b\}$ because $|\{a\}| \leq |\{b\}|$ and for the same reason, $\{b\} \preceq \{a\}$; however $\{a\} \neq \{b\}$.

8. [BB] In this case, either $S = \emptyset$ or $|S| = 1$. This time, in each case, $(S, \preceq)$ is a partial order.

   **Case 1:** If $S = \emptyset$, then $\mathcal{P}(S)$ contains a single element, $\emptyset$.

   **Reflexive:** Certainly $A \preceq A$ for all $A \in \mathcal{P}(S)$ since $0 = |\emptyset| \leq |\emptyset|$.

   **Antisymmetric:** If $A \preceq B$ and $B \preceq A$, then $A = B$ since there is only one set in $\mathcal{P}(S)$.

   **Transitive:** If $A \preceq B$ and $B \preceq C$, then $A \preceq C$ since necessarily $A = B = C$ and for the single set $A$ in $\mathcal{P}(S)$, $A \preceq A$.

   **Case 2:** If $S$ contains one element, then $\mathcal{P}(S) = \{\emptyset, S\}$ contains two elements.

   **Reflexive:** $A \preceq A$ for each $A \in \mathcal{P}(S)$ because $|A| \leq |A|$.

**Antisymmetric:** If $A, B \in \mathcal{P}(S)$, $A \preceq B$ and $B \preceq A$, then we have $|A| \leq |B|$ and $|B| \leq |A|$, so $|A| = |B|$. Since $\mathcal{P}(S)$ does not contain different sets of the same cardinality, it follows that $A = B$.

**Transitive:** Suppose $A \preceq B$ and $B \preceq C$. If $A = \emptyset$, then $|A| = 0 \leq |C|$ no matter what $C$ is, so we'd have $A \preceq C$. If $A = S$, then $A \preceq B$ means $B = S$ and $B \preceq C$ means $C = S$, so $A = B = C = S$ and $A \preceq C$.

9. $f\colon A \times B \to B \times A$ defined by $f(a, b) = (b, a)$ is a one-to-one onto function.

10. (a) [BB] False. Let $X = \{1\}, Y = \{2\}, Z = \{3\}$. Then $((1, 2), 3) \in (X \times Y) \times Z$ but $((1, 2), 3) \notin X \times (Y \times Z)$ (a set whose **second** coordinates are ordered pairs).

    (b) [BB] Define $f\colon (X \times Y) \times Z \to X \times (Y \times Z)$ by $f((x, y), z) = (x, (y, z))$.

11. (a) **Reflexivity:** For any set $A$, $\iota_A$ is a one-to-one onto function $A \to A$, so $A$ has the same cardinality as itself.

    **Symmetry:** If $A$ and $B$ have the same cardinality, then there is a one-to-one onto function $f\colon A \to B$. Such a function has an inverse $f^{-1}\colon B \to A$ which is one-to-one and onto (because it has an inverse), so $B$ and $A$ have the same cardinality.

    **Transitivity:** Suppose $A$, $B$ and $C$ are sets such that $A$ and $B$ have the same cardinality and $B$ and $C$ have the same cardinality. Then there is a one-to-one onto function $f\colon A \to B$ and a one-to-one onto function $g\colon B \to C$. Since the composition of one-to-one functions is one-to-one and the composition of onto functions is onto (Exercises 23 and 24 of Section 3.2), $g \circ f\colon A \to C$ is one-to-one and onto. Thus, $A$ and $C$ have the same cardinality.

    (b) [BB] By Problem 27, $(0, 1)$ and $(1, \infty)$ have the same cardinality and $(0, 1)$ and $(3, \infty)$ have the same cardinality. By transitivity of "same cardinality", $(1, \infty)$ and $(3, \infty)$ have the same cardinality too.

    (c) [BB] To find an explicit one-to-one onto function $(1, \infty) \to (3, \infty)$, we use the function $f$ defined by $f(x) = \dfrac{1}{x} - 1 + 3 = \dfrac{1}{x} + 2$ from $(0, 1)$ to $(3, \infty)$ exhibited in Problem 27 and the function $g$ defined by $g(x) = \dfrac{1}{x + 1 - 1} = \dfrac{1}{x}$ from $(1, \infty)$ to $(0, 1)$, which is the inverse of the function exhibited in Problem 27—see line (3). The function we seek is the composition $f \circ g$, which is defined by $f \circ g(x) = f(g(x)) = \dfrac{1}{\frac{1}{x}} + 2 = x + 2$, a rather obvious choice!

    (d) $f(x) = x + b - a$

12. (a) [BB] As suggested just before Problem 28, we look for a function defined with a rule like $f(x) = kx + \ell$ and discover (the rather obvious) $f(x) = x + 1$.

    (b) The function defined by $f(x) = 2x + 4$ is a one-to-one correspondence between $(0, 1)$ and $(4, 6)$.

    (c) The function $f\colon (0, 1) \to (a, b)$ defined by $f(x) = a + (b - a)x$ is a one-to-one correspondence, and the function $g\colon (0, 1) \to (c, d)$ given by $g(x) = c + (d - c)x$ is also a one-to-one correspondence. Thus $g \circ f^{-1}$ provides a one-to-one correspondence $(a, b) \to (c, d)$. The reader may check that $g \circ f^{-1}(x) = c + \dfrac{d - c}{b - a}(x - a)$.

13. (a) [BB] Using the result of Problem 27, we obtain the function defined by $g(x) = \dfrac{1}{x} + 9$ as a one-to-one correspondence between $(0, 1)$ and $(10, \infty)$.

(b) [BB] The function $f\colon (2,5) \to (0,1)$ defined by $f(x) = \frac{1}{3}x - \frac{2}{3}$ is a one-to-one correspondence, as is the function $g\colon (0,1) \to (10,\infty)$ defined by $g(x) = \dfrac{1}{x} + 9$. Thus, the composition $g \circ f\colon (2,5) \to (10,\infty)$ is also a one-to-one correspondence. Note that $(g \circ f)(x) = \dfrac{3}{x-2} + 9$.

(c) The function $f\colon (a,b) \to (0,1)$ defined by $f(x) = \dfrac{x-a}{b-a}$ is a one-to-one correspondence, as is the function $g\colon (0,1) \to (c,\infty)$ defined by $g(x) = \dfrac{1}{x} - 1 + c$. Thus a one-to-one correspondence $(a,b) \to (c,\infty)$ is the composition $g \circ f$. Note that $(g \circ f)(x) = \dfrac{b-a}{x-a} - 1 + c$.

14. In Problem 27, p. 129, we saw that the function $g\colon (0,1) \to (0,\infty)$ defined by $g(x) = \dfrac{1}{x} - 1$ is a one-to-one correspondence. Furthermore, the function $f\colon (a,b) \to (0,1)$ defined by $f(x) = \dfrac{x-a}{b-a}$ is a one-to-one correspondence. (See the remarks preceding Problem 28.) Thus $g \circ f\colon (a,b) \to (0,\infty)$ is a one-to-one correspondence. Note that $(g \circ f)(x) = \dfrac{b-a}{x-a} - 1$.

15. [BB] $f$ is certainly a function from R to $\mathsf{R}^+$ since $2^x > 0$ for all $x \in \mathsf{R}$. If $2^x = 2^y$, then $x \log 2 = y \log 2$ (any base), so $x = y$. Thus, $f$ is one-to-one. If $r \in \mathsf{R}^+$, then $2^{\log_2 r} = r$, so $f$ is onto. We conclude that R and $\mathsf{R}^+$ have the same cardinality.

16. [BB] Since $(a,b)$ has the same cardinality as $\mathsf{R}^+$ by Exercise 14 and since $\mathsf{R}^+$ and R have the same cardinality by Exercise 15, the result follows by transitivity of the notion of "same cardinality"—see Exercise 11.

17. $f$ is certainly defined on $(0,1)$ and takes values in R. To show that $f$ is one-to-one, assume $f(x_1) = f(x_2)$; that is, assume

$$\frac{x_1 - \frac{1}{2}}{x_1(x_1-1)} = \frac{x_2 - \frac{1}{2}}{x_2(x_2-1)}.$$

Then $x_2(x_2-1)(x_1 - \frac{1}{2}) = x_1(x_1-1)(x_2 - \frac{1}{2})$, so

$$x_1x_2^2 - \tfrac{1}{2}x_2^2 - x_2x_1 + \tfrac{1}{2}x_2 = x_2x_1^2 - \tfrac{1}{2}x_1^2 - x_1x_2 + \tfrac{1}{2}x_1.$$

Therefore,

$$x_1x_2^2 - x_2x_1^2 - \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1 = 0$$

$$x_1x_2(x_2 - x_1) - \tfrac{1}{2}(x_2 + x_1)(x_2 - x_1) + \tfrac{1}{2}(x_2 - x_1) = 0$$

$$(x_2 - x_1)(2x_1x_2 - x_2 - x_1 + 1) = 0$$

and so $x_2 = x_1$ or $2x_1x_2 - x_2 - x_1 + 1 = 0$. The second possibility is the same as $x_1x_2 + (x_1 - 1)(x_2 - 1) = 0$ and, since for $0 < x_1, x_2 < 1$ both $x_1x_2$ and $(x_1 - 1)(x_2 - 1)$ are nonnegative, this case is impossible. We conclude that $x_1 = x_2$; thus, $f$ is one-to-one.

To show that $f$ is onto, let $r \in \mathsf{R}$. We wish to find an $x$ such that $f(x) = r$; that is, we wish to find an $x$ such that $\dfrac{x - \frac{1}{2}}{x(x-1)} = r$. Hence, we wish to solve $rx^2 - (r+1)x + \frac{1}{2} = 0$. If $r = 0$, the solution is

$x = \frac{1}{2}$. If $r \neq 0$, the quadratic formula tells us that this equation has solutions

$$x = \frac{r+1 \pm \sqrt{(r+1)^2 - 2r}}{2r} = \frac{r+1 \pm \sqrt{r^2+1}}{2r} .$$

We claim that the solution

$$x = \frac{r+1-\sqrt{r^2+1}}{2r}$$

is always in $(0,1)$. First note that if $r > 0$, then $r+1 > \sqrt{r^2+1}$ since $(r+1)^2 > r^2+1$. On the other hand, if $r < 0$, then $r+1 < 1 < \sqrt{r^2+1}$. In either case, we have shown that

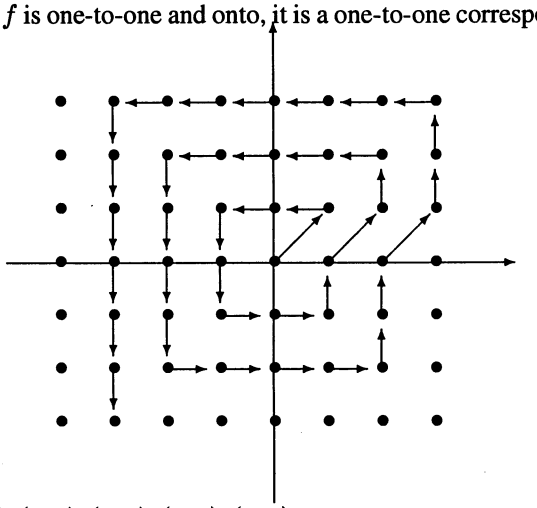
$$\frac{r+1-\sqrt{r^2+1}}{2r} > 0 .$$

To show that this expression is also less than 1, we will prove the equivalent inequality

$$\frac{1-r-\sqrt{r^2+1}}{2r} < 0 .$$

If $r > 0$, then $1-r < 1 < \sqrt{r^2+1}$ giving the result. If $r < 0$, then $1-r > \sqrt{r^2+1}$ because $(1-r)^2 > r^2+1$, again giving the result. Since $f$ is one-to-one and onto, it is a one-to-one correspondence.

18. Start at $(0,0)$ and move as illustrated.



19. (a) $2, -2, 4, -4, 8, -8, 16, -16, \ldots$

(b) $1 = 2^0, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, \ldots$

(c) $1, 4, 7, 10, 13, 16, 19, \ldots$

(d) $(1,1), (1,2), (1,3), (2,1), (2,2), (3,3), (3,1), (3,2), (3,3), (4,1), \ldots$

(e) [BB] Follow the procedure given in the text for all rational numbers and omit those with even denominators. The listing starts $1, 2, \frac{1}{3}, 3, 4, \frac{2}{3}, \frac{1}{5}, 5, 6, \frac{4}{3}, \frac{2}{5}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7, \ldots$.

(f) Follow the procedure given in the text for $\mathsf{N} \times \mathsf{N}$ but with two modifications. We include an extra row across the bottom as follows: $(1,0)$ $(2,0)$ $(3,0)$ $(4,0)$ $\ldots$ . Then, when doing the listing, follow immediately every pair $(a,b)$, $b > 0$, with the pair $(a,-b)$. The first few terms would be $(1,0), (2,0), (1,1), (1,-1), (1,2), (1,-2), (2,1), (2,-1), (3,0), (4,0), (3,1), (3,-1), (2,2), (2,-2), \ldots$.

(g) First of all, enumerate $\mathsf{N} \cup \{0\} \times \mathsf{N} \cup \{0\}$ by the diagonal procedure given in the text for $\mathsf{N} \times \mathsf{N}$. The first few terms would be $(0,0), (1,0), (0,1), (0,2), (1,1), (2,0), (3,0), (2,1), (1,2), (0,3)$. Then enumerate $\mathsf{Z} \times \mathsf{Z}$ by taking this list, replacing every element $(a,0)$, $a > 0$, with $(a,0), (-a,0)$; every element $(0,a)$, $a > 0$, with $(0,a), (0,-a)$, and every pair $(a,b)$, $a,b > 0$ with $(a,b)$, $(a,-b)$, $(-a,b)$, $(-a,-b)$.

20. (a) [BB] This set is uncountable. The function defined by $f(x) = x - 1$ gives a one-to-one correspondence between it and $(0, 1)$, which we showed in the text to be uncountable.

    (b) This set is countably infinite. Just follow the sequence given in the text for the set of all positive rationals, but omit any rational number not in $(1, 2)$.

    (c) This set is finite. In fact, it contains at most $99^2$ elements since there are 99 possible numerators and, for each numerator, 99 possible denominators.

    (d) This set is countably infinite. List the elements as follows (deleting any repetitions such as $5/100 = 1/20$): $\frac{99}{6}, \frac{99}{7}, \ldots, \frac{99}{104}, \frac{98}{6}, \frac{98}{7}, \ldots, \frac{98}{104}, \cdots$ .

    (e) [BB] This set is countably infinite. In Exercise 3(c) we showed it is in one-to-one correspondence with $\mathsf{N} \times \mathsf{N}$.

    (f) This set is countably infinite because it is in one-to-one correspondence with Q via the function $f$ defined by $f(a, b) = a$.

    (g) This set is uncountable. It is in one-to-one correspondence with $[-1, 1] = \{x \mid -1 \leq x \leq 1\}$ via the function $f$ defined by $f(a, b) = a$. The interval $[-1, 1]$ contains $(0, 1)$ which we showed in the text to be uncountable.

21. (a) Finite! There is some minimum volume $V$ which a grain of sand must occupy. On the other hand, there is a finite value $M$ for the volume of all the sand. So the number of grains is $\leq M/V$.

    (b) Countably infinite. Here is a listing: $3^0, 3^1, 3^{-1}, 3^2, 3^{-2}, \ldots$.

    (c) The set of sentences in the English language is certainly not finite for if it were, and $S$ were the sentence with the most words, then "$S$ and roses are red." would be a longer sentence. So we have to decide if the set of sentences is countably infinite or uncountable. In fact, it is countably infinite. To see this, first note that since there are only finitely many words (see part (c)) there are only finitely many sentences $k$ words long for each $k = 1, 2, \ldots$. If $a_{k,1}, a_{k,2}, a_{k,3}, \ldots$ is a list of the sentences which are $k$ words long (extended to a countably infinite list by defining all $a_{k,i} =$ "Jqrzx" after we run out of sentences), then the set of all sentences can be listed by the diagonal scheme that we used for $\mathsf{N} \times \mathsf{N}$: $a_{1,1}, a_{2,1}, a_{1,2}, a_{1,3}, a_{2,2} \ldots$, where we omit all occurrences of "Jqrzx."

22. (a) [BB] Impossible. To the contrary, suppose that the union were a finite set $S$. Since $S$ has only finitely many subsets (the precise number is $2^{|S|}$), there could not have been infinitely many sets at the outset.

    (b) Impossible. The maximum number of elements in the union of sets $A_1, \ldots, A_n$ occurs when each intersection $A_i \cap A_j$ is empty. Thus, $|A_1 \cup A_2 \cup \cdots \cup A_n| \leq |A_1| + |A_2| + \cdots + |A_n|$, which is finite.

    (c) [BB] Impossible. If even one infinite set is contained in a **union**, then the union must be infinite.

23. [BB] Imagine $S_1$ sitting inside $S_2$, both spheres with the same center. Rays emanating from this common point establish a one-to-one correspondence between the points on $S_1$ and the points on $S_2$.

24. (a) [BB] Let $s_1, s_2, s_3, \ldots$ be a countably infinite subset of $S$. Define $f: S \to S \cup \{x\}$ by

$$\begin{aligned} f(s_1) &= x \\ f(s_{k+1}) &= s_k \quad \text{for } k \geq 1 \\ f(s) &= s \quad \text{if } s \notin \{s_1, s_2, s_3, \ldots\}. \end{aligned}$$

    Then $f$ is a one-to-one correspondence.

(b) This follows immediately from part (a) since $(0,1] = (0,1) \cup \{1\}$.

25. We employ a concept known as *stereographic projection.* Imagine the sphere sitting on the Cartesian plane with south pole at the origin. Any line from the north pole to the plane punctures the sphere at a unique point and the collection of such lines establishes a one-to-one correspondence between the points of the plane and the sphere except for the north pole. A small modification of this correspondence finishes the job. Suppose $p_0, p_1, p_2, \ldots$ are the points of the sphere which correspond to the points $(0,0), (1,0), (2,0), \ldots$ in the plane; thus, the line from the north pole to $(n,0)$ punctures the sphere at $p_n$ (in particular, $p_0 = (0,0)$). Map the north pole to $(0,0)$, the origin to $(1,0)$, $p_1$ to $(2,0)$, and so forth and let all other points of the sphere go to the same points as before.

26. [BB] We are given that $A = \{a_1, a_2, \ldots, a_n\}$ for some $n \in \mathsf{N}$ and that $B = \{b_1, b_2, b_3, \ldots\}$. Then $A \cup B$ is countably infinite because it is infinite and its elements can be listed as follows:

$$a_1, a_2, \ldots, a_n, b_1, b_2, b_3, \ldots$$

The function $f\colon \mathsf{N} \to A \cup B$ corresponding to this listing is defined by $f(i) = \begin{cases} a_i & \text{if } i \leq n \\ b_{i-n} & \text{if } i > n. \end{cases}$

27. (a) We will do this by exhibiting a one-to-one correspondence between $A \times B$ and $\mathsf{N} \times \mathsf{N}$, which was shown in the text to be countable. Since $A$ is countable, we can list its elements: $a_1, a_2, a_3, \ldots$. Since $B$ is countable, we can list its elements: $b_1, b_2, b_3, \ldots$. Define $f\colon A \times B \to \mathsf{N} \times \mathsf{N}$ by $f(a_i, b_j) = (i,j)$. If $f(a_i, b_j) = f(a_k, b_\ell)$, then $(i,j) = (k,\ell)$ so $i = k, j = \ell$ and $(a_i, b_j) = (a_k, b_\ell)$. Thus, $f$ is one-to-one. Also, for any $(m,n) \in \mathsf{N} \times \mathsf{N}$, we have $f(a_m, b_n) = (m,n)$, so $f$ is onto. Thus, $f$ is a one-to-one correspondence.

(b) A polynomial of degree at most one with integer coefficients is an expression of the form $a + bx$, where $a, b \in \mathsf{Z}$. The function $a + bx \mapsto (a,b)$ is a one-to-one correspondence between these polynomials and the set $\mathsf{Z} \times \mathsf{Z}$, which is countable by Exercise 27(a). Thus the given set of polynomials is countable too.

28. We offer a proof by contradiction. If $|X| = |\mathcal{P}(X)|$, then there is a one-to-one onto function $f\colon X \to \mathcal{P}(X)$. Thus, for each $x \in X$, $f(x)$ is a subset of $X$. Define a subset $Y$ of $X$ as follows: for each $x \in X$, put $x \in Y$ if and only if $x \notin f(x)$. Since $f$ is onto, $Y = f(y)$ for some $y \in X$. Notice that if $y \in Y$, then $y \in f(y)$, so $y \notin Y = f(y)$, by definition. On the other hand, if $y \notin f(y) = Y$, then $y \in Y$, again by definition. We reach the absurd situation that $y$ is neither in or not in $Y$. It follows that there can exist no onto function $f\colon X \to \mathcal{P}(X)$.

29. Assume, to the contrary, that $S$ is countable and, as in Problem 31, write each of its elements in a list as

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ &\vdots \end{aligned}$$

where each $a_i$ is 3 or 4. Define the sequence $b_1, b_2, b_3, \ldots$ by $b_i = \begin{cases} 4 & \text{if } a_{ii} = 3 \\ 3 & \text{if } a_{ii} = 4. \end{cases}$

Then $b = 0.b_1b_2b_3\ldots$ is in $S$, yet it is different from each $a_i$ because $b_i \neq a_{ii}$ for each $i$. This contradiction gives the result.

30. Assume, to the contrary, that $S$ is countable and that its elements can be listed $a_1, a_2, a_3, \ldots$. By moving this element to the start of the list, if necessary, we may assume that $a_1 = 0.01000\ldots$. Now the same argument as in Exercise 29 yields the result. (Note that the choice of $a_1$ guarantees that $b \neq 0$.)

31. The Fields Medal is the highest honour which can be bestowed upon a mathematician, there being no Nobel Prize in mathematics. John Charles Fields (1863-1932) was born in Hamilton, Ontario, obtained a PhD from John Hopkins University in 1887 and served for over thirty years on the faculty of the University of Toronto. His will established the Fields Medal (under the name "International Medal for Outstanding Discoveries in Mathematics"), which was officially adopted at the International Congress of Mathematicians in Zürich in 1932. One of the three Canadian mathematics institutes is also named after Fields. Michael Monastyrsky wrote an informative article on the history of the Fields Medal for the March 2001 *Notes of the Canadian Mathematical Society* **33**, no. 2, pp. 3–5. More information can be also obtained from `http://www.math.toronto.edu/fields.html`.

## Chapter 3 Review

1. This function is not one-to-one since, for example, $f(2,2) = 4 = f(3,1)$, yet $(2,2) \neq (3,1)$.

2. $f^{-1} = \{(2,1),(3,2),(4,3),(1,4)\}$; $g \circ f = \{(1,1),(2,4),(3,2),(4,3)\}$. The function $g$ is not one-to-one since $g(2) = g(5) = 1$, but $2 \neq 5$.

3. This function is not one-to-one since $f(1) = f(-1) = 4$, but $1 \neq -1$. Neither is it onto since, for instance, there is no $m$ with $f(m) = 0$: $f(m) = 0 \rightarrow 3|m| = -1$.

4. Suppose $f(x_1) = f(x_2)$. Then $2x_1^3 + x_1 = 2x_2^3 + x_2$, so $2(x_1^3 - x_2^3) = x_2 - x_1$. If $x_1 \neq x_2$, we may divide both sides by $x_1 - x_2$, obtaining $2(x_1^2 + x_1x_2 + x_2^2) = -1$ which is impossible because the left side is even and the right side odd. Thus $x_1 = x_2$, so $f$ is one-to-one.

   This function is not onto because 1 is not in the range of $f$. To see why, suppose $f(x) = 1$ for some $x$. Then $x(2x^2 + 1) = 1$ for some $x$. Over Z, this implies that both $x$ and $2x^2 + 1$ are 1 or both are $-1$. Since $2x^2 + 1 > 0$, $x$ must be positive too, so $x = 2x^2 + 1 = 1$. This is not true.

5. (a) i. Yes, $f$ is one-to-one. If $f(a_1) = f(a_2)$, then $\dfrac{3a_1 - 4}{2} = \dfrac{3a_2 - 4}{2}$, so, (multiplying by 2) $3a_1 - 4 = 3a_2 - 4$, hence $a_1 = a_2$.

   ii. $f$ is not onto since, for example, there is no even integer $a$ which satisfies $f(a) = 0$.

   iii. $f$ is not bijective since it is not onto.

   (b) i. Yes, $f$ is one-to-one. If $f(a_1) = f(a_2)$, then $\dfrac{5 - 2a_1}{3} = \dfrac{5 - 2a_2}{3}$, so, (multiplying by 3) $5 - 2a_1 = 5 - 2a_2$, hence $a_1 = a_2$.

   ii. $f$ is onto because if $b$ is any rational number, we have $b = f(a)$ with $a = \dfrac{5 - 3b}{2}$.

   iii. $f$ is bijective since it is one-to-one and onto: $f^{-1}(x) = \dfrac{5 - 3x}{2}$.

6. (a) The domain of $f$ is $\mathbb{R} \setminus \{\frac{1}{2}\}$. To find the range, suppose $y$ is in the range. Then $y = f(x)$ for some $x$. This leads to $x(2y - 4) = y$, an equation with a solution if and only if $y \neq 2$. The range of $f$ is $\mathbb{R} \setminus \{2\}$.

(b) $f^{-1}$ can be defined on the range of $f$, that is, $\mathsf{R} \setminus \{2\}$.

(c) Let $y = f^{-1}(x)$, $x \neq 2$. Then $x = f(y) = \dfrac{4y}{2y-1}$. This implies $y(2x-4) = x$. Since $x \neq 2$, we have $y = \dfrac{x}{2x-4}$. Thus $f^{-1}(x) = \dfrac{x}{2x-4}$.

7. (a) The values are $7, 5, 3, 1, 2, 4, 6$.

(b) Let $n \in \mathsf{N}$. We must show that $n = f(a)$ for some integer $a$. If $n$ is even, let $a = \frac{1}{2}n$. If $n$ is odd, let $a = -\frac{1}{2}(n-1)$.

8. A function $f$ on a set $A$ is a reflexive relation if and only if $f = \iota$ is the identity on $A$. Certainly, the identity function is reflexive. Conversely, if $f$ is reflexive, then for any $a \in A$, $(a, a) \in f$. Since $(a, x), (a, y) \in f$ imply $x = y$, it follows that for any $a \in A$, $(a, x) \in f$ if and only if $x = a$. Thus $f = \iota$.

9. Lots of functions $\mathsf{R} \to \mathsf{R}$ are symmetric, for example, the function defined by $f(x) = -x$.

10. The greatest integer function is one example. For any $a, b, c \in \mathsf{R}$, if $b = \lfloor a \rfloor$ and $c = \lfloor b \rfloor$, then $c = b$ because $b$ is an integer, so $c = \lfloor a \rfloor$.

11. (a) This is false: $x = -4$, $y = 2$ gives a counterexample since $|x| = 4 = y^2$, but $y|y| = 4 \neq x$.

(b) This is true. If $x = y|y|$, then $|x| = |y||y| = |y|^2 = y^2$.

(c) This is false because the implication $\to$ is false.

12. (a) If $f(x_1) = f(x_2)$, then $\dfrac{ax_1}{x_1+b} = \dfrac{ax_2}{x_2+b}$. Cross-multiplication gives $ax_1x_2 + abx_1 = ax_1x_2 + abx_2$, so $abx_1 = abx_2$. Since $ab \neq 0$, $x_1 = x_2$. Therefore $f$ is one-to-one.

(b) If $f(x) = a$, then $\dfrac{ax}{x+b} = a$. Cross multiplication yields $ax = ax + ab$, so $ab = 0$, a contradiction. There is no $x$. The equation $f(x) = a$ has no solution. If $f(x) = b$, then $\dfrac{ax}{x+b} = b$ implies $ax = bx + b^2$, so $x(a-b) = b^2$. Since $b \neq 0$, there is a solution if and only if $a \neq b$.

(c) $f$ is not onto since there is no $x$ with $f(x) = a$.

13. The graph of $y = f(x)$ is a parabola opening up with vertex $(-2, -11)$. So we could take $A = [-2, \infty) = \{x \in \mathsf{R} \mid x \geq -2\}$ and then $B = [-11, \infty) = \{y \in \mathsf{R} \mid y \geq -11\}$.

14. Suppose $g(b_1) = g(b_2)$. Since $f$ is onto, there exist $a_1$ and $a_2$ in $A$ such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Thus $g(f(a_1)) = g(f(a_2))$; that is, $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is one-to-one, we have $a_1 = a_2$. Thus $b_1 = b_2$. So $g$ is one-to-one.

15. (a) finite (b) uncountable (c) countably infinite
(d) countably infinite (e) finite (f) finite

16. (a) **Reflexive.** We have $(a, b) \sim (a, b)$ because $ab = ba$.
**Symmetric.** If $(a, b) \sim (c, d)$, then $ad = bc$, so $cb = da$. Hence $(c, d) \sim (a, b)$.
**Transitive.** If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ad = bc$ and $cf = de$, so $adf = bcf = bde$. Thus $d(af - be) = 0$. Since $d \neq 0$, this means $af = be$, so $(a, b) \sim (e, f)$.

(b) Define $f\colon \mathsf{Q} \to A/\!\sim \to \mathsf{Q}$ by $f(\frac{a}{b}) = \overline{(a, b)}$. This function is well defined because if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, so $(a, b) \sim (c, d)$ and $\overline{(a, b)} = \overline{(c, d)}$, hence $f(\frac{a}{b}) = f(\frac{c}{d})$. The function is one-to-one because if $f(\frac{a}{b}) = f(\frac{c}{d})$, then $\overline{(a, b)} = \overline{(c, d)}$, so $(a, b) \sim (c, d)$, $ad = bc$ and $\frac{a}{b} = \frac{c}{d}$. The function is onto because $\overline{(a, b)} = f(\frac{a}{b})$.

17. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since $A \cap B = \emptyset$, $a_i \neq b_j$ for any $i, j$. Let $C = \{1, 2, \ldots, n+m\}$. If we can establish a one-to-one correspondence between $A \cup B$ and $C$, then we will have $|A \cup B| = |C| = n + m$. Define $f\colon A \cup B \to C$ by $f(a_1) = 1$, $f(a_2) = 2$, ..., $f(a_n) = n$, $f(b_1) = n + 1$, $f(b_2) = n + 2$, ..., $f(b_m) = n + m$. Since $f(x) = f(y) \to x = y$, $f$ is one-to-one. Clearly $f$ is onto, so it is a one-to-one correspondence.

18. The function $f\colon (-2, 2) \to (1, 9)$ defined by $f(x) = 2x + 5$ is a bijection.

19. Define $f\colon (1, 3) \to \mathrm{R}$ by $f(x) = \dfrac{1}{x-1} - \dfrac{1}{2}$ and suppose $f(x_1) = f(x_2)$.

    Then $\dfrac{1}{x_1 - 1} - \dfrac{1}{2} = \dfrac{1}{x_2 - 1} - \dfrac{1}{2}$, so $\dfrac{1}{x_1 - 1} = \dfrac{1}{x_2 - 1}$, $x_1 - 1 = x_2 - 1$ and $x_1 = x_2$. Therefore, $f$ is one-to-one. Given $y \in (0, \infty)$, let $x = \dfrac{2y+3}{2y+1}$.

    We leave it to the student to verify that $f(x) = y$. Also $1 < \dfrac{2y+3}{2y+1} < 3$. Finally, $\operatorname{rng} f = (0, \infty)$, so $f$ is a one-to-one correspondence between $(1, 3)$ and $(0, \infty)$.

20. (a) Assume to the contrary that $S$ is countable. In that case, we can list the elements of $S$ as $a_1, a_2, a_3, \ldots$. Writing each of these using their decimal expansions, we have

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ &\vdots \end{aligned}$$

    where each $a_{ij}$ is 0, 2 or 7. For each $i \geq 1$, define $b_i$ by $b_i = \begin{cases} 2 & \text{if } a_{ii} = 0 \text{ or } 7 \\ 7 & \text{if } a_{ii} = 2. \end{cases}$

    Now consider the number $b = 0.b_1b_2b_3\ldots$. Since each $b_i$ is 2 or 7, $b$ is in $S$. But for any $i$, we have $b \neq a_i$ since $b$ and $a_i$ differ in the $i$th decimal place. This contradicts $S$ being countable.

    (b) Note that if $x$ belongs to $S'$, then $x$ has a terminating decimal expansion (that is, all decimal places 0 after finitely many places). It follows that $x$ belongs to Q, so $S' \subseteq \mathrm{Q}$. We conclude from the first remark at the end of Section 3.3 that $S'$ is countable. Since $S'$ contains the infinite sequence $0.2, 0.02, 0.002, \ldots$, it must be countably infinite: $|S'| = \aleph_0$.

21. First assume $S$ is finite with $n$ elements $x_1, x_2, \ldots, x_n$. Then $\mathcal{P}(S)$ contains the $n + 1$ elements $\emptyset, \{x_1\}, \{x_2\}, \ldots, \{x_n\}$ so $|\mathcal{P}(S)| > |S|$ and $f$ cannot be onto. In the case that $S$ is infinite, we offer a proof by contradiction. Suppose then that $f$ is onto. Thus every subset of $S$ is of the form $f(x)$, for some $x \in S$. Following the hint, we let $T = \{s \in S \mid s \notin f(s)\}$. Then $T = f(x)$ for some $x \in S$. Reflecting on the question "is $x \in T$?" leads to the contradiction that $x \in T$ if and only if $x \notin T$.

## Exercises 4.1

1. [BB] $(a + b)c = c(a + b)$ (by commutativity) $= ca + cb$ (by the first distributive law) $= ac + bc$ (by commutativity again).

2. (a) [BB] True. If $a$ and $b$ are real numbers, certainly $a - b$ is a real number.
   (b) False. If $a = 1, b = 2$, then $a - b = -1$ but $b - a = 1$.
   (c) False. If $a = 1, b = 2, c = 3$, then $(a-b)-c = -1-3 = -4$, while $a-(b-c) = 1-(-1) = 2$.

3. Let $a = 3$, $b = 4$, $c = -2$. Then $3 \leq 4$ but $3(-2) = -6 \not\leq -8 = 4(-2)$.

4. (a) [BB] $q = 29, r = 7$; (b) [BB] $q = -30, r = 10$; (c) [BB] $q = -29, r = 7$;
   (d) [BB] $q = 30, r = 10$.

5. (a) $q = 278, r = 4$; (b) $q = -279, r = 15$; (c) $q = -278, r = 4$;
   (d) $q = 279, r = 15$; (e) $q = 0, r = 19$; (f) $q = -1, r = 5267$.

6. (a) $q = -316, r = 21$; (b) $q = 29, r = 972$; (c) $q = 70, r = 613$;
   (d) $q = 20, r = 2984$; (e) $q = 55, r = 134$;

   (f) $\begin{aligned} 555{,}555{,}123 &= 555{,}555{,}555 - 432 = 5(111{,}111{,}111) - 432 \\ &= 4(111{,}111{,}111) + 111{,}111{,}111 - 432 = 4(111{,}111{,}111) + 111{,}110{,}679 \end{aligned}$

   So $q = 4$; $r = 111{,}110{,}679$

   (g) $q = 2{,}104{,}000$; $r = 0$

7. [BB] Write $a = 3q + r$ with $r = 0, 1, 2$.

   **Case 1: $r = 0$.** In this case, $a = 3q$, so $a^2 = 9q^2 = 3k$ with $k = 3q^2$.

   **Case 2: $r = 1$.** Here, $a = 3q + 1$, so $a^2 = 9q^2 + 6q + 1 = 3k + 1$ with $k = 3q^2 + 2q$.

   **Case 3: $r = 2$.** In this case, $a = 3q + 2$ and $a^2 = 9q^2 + 12q + 4 = 3k + 1$ with $k = 3q^2 + 4q + 1$.

8. (a) [BB] The domain of $f$ is Z; its range is Z as well, because given $q \in \mathsf{Z}$, we have $q = f(qn)$.
   (b) [BB] $f$ is not one-to-one since $f(0) = f(1)$. (Note that $n > 1$ guarantees that the quotient is 0 when either 0 or 1 is divided by $n$.)
   (c) [BB] $f$ is onto, as shown in (a).

9. (a) The domain of $f$ is Z; the range is $\{0, 1, 2, \ldots, n-1\}$.
   (b) $f$ is not one-to-one since, for example, $f(0) = f(n)$ $(= 0)$ but $0 \neq n$.
   (c) Since the range is a proper subset of $\mathsf{N} \cup \{0\}$, $f$ is not onto.

10. [BB] By definition, $x \leq \lceil x \rceil < x+1$ for any $x$. Letting $k = \lceil \frac{a}{b} \rceil$, we have $\frac{a}{b} \leq k < \frac{a}{b} + 1$. Multiplying by the negative number $b$, we obtain $a \geq kb > a + b$. Thus, $0 \leq a - kb < -b$. Letting $r = a - kb$, we have $a = kb + r$ with $0 \leq r < |b|$, so, by uniqueness, $k = q$ as asserted.

11. (a) [BB] $4034 = (111{,}111{,}000{,}010)_2 = (7702)_8 = (\mathrm{FC2})_{16}$.
   (b) $57{,}483 = (1{,}110{,}000{,}010{,}001{,}011)_2 = (160{,}213)_8 = (\mathrm{E08B})_{16}$.
   (c) $185{,}178 = (101{,}101{,}001{,}101{,}011{,}010)_2 = (551{,}532)_8 = (\mathrm{2D35A})_{16}$.

12. (a) Since $b^{n-1} \leq N < b^n$ and $\log_b$ is an increasing function, we have $\log_b b^{n-1} \leq \log_b N < \log_b b^n$; that is, $n - 1 \leq \log_b N < n$. But this says that $n - 1$ is the largest integer $\leq \log_b N$ or that $n - 1 = \lfloor \log_b N \rfloor$, as required. The parenthetical comment follows from the fact that $N$ has $n$ digits in base $b$.
   (b) By the result of part (a), $7^{254}$ has $1 + \lfloor \log_{10} 7^{254} \rfloor$ digits. Since $\log_{10} 7^{254} = 254 \log_{10} 7 \approx 214.66$, the integer $7^{254}$ has 215 digits.

(c) By the result of part (a), $319^{566}$ has $1+\lfloor \log_{10} 319^{566} \rfloor$ digits. Since $\log_{10} 319^{566} = 566 \log_{10} 319 \approx 1417.15$, the integer $319^{566}$ has 1418 digits.

## Exercises 4.2

1. (a) [BB] Not totally ordered; for example, 6 and 21 are not comparable.

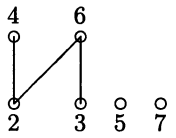
   (b) [BB] 1 is a minimum element since $1 \mid n$ for all $n \in \mathsf{N}$, but there is no maximum element since given any $n \in \mathsf{N}$, $n \mid 2n$, so $n$ can't be maximum.

2. (a) The greatest lower bound of natural numbers is their greatest common divisor. Since the gcd of any two elements of this poset is, in each case, still an element of the poset, every pair of elements has a glb.

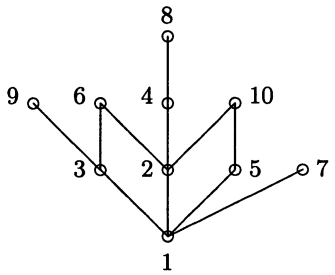
   (b) The least upper bound of natural numbers is their least common multiple. The lcm of 4 and 6 is 12. Since 12 is not in the poset, 4 and 6 have no lub.

   (c) This is not a lattice (because not all pairs of elements have a lub.)

3. (a) [BB]




(b)




4. (a) 2, 3, 5 and 7 are minimal; 4,5, 6 and 7 are maximal. There are no minimum nor maximum elements.

   (b) 1 is minimal and minimum; 6, 7, 8, 9 and 10 are maximal. There is no maximum.

5. [BB] As in Problem 6, write $a = nq + r$, where $0 \leq r < n$. Thus, $r = 0, 1, 2, \ldots,$ or $n - 1$. If $r = 0$, then $a = qn$ is divisible by $n$. If $r = 1$, then $a = nq + 1$, so $a + n - 1$ is divisible by $n$. If $r = 2$, then $a = nq + 2$ and $a + n - 2$ is divisible by $n$. In general, if $r = k$, then $a + n - k$ is divisible by $n$.

6. [BB] Assume, to the contrary, that for some integer $n$, we have $4 \mid (n^2 - 2)$; that is, $n^2 - 2 = 4x$ for some integer $x$. We derive a contradiction. First consider the case that $n$ is even; that is, $n = 2y$ for some integer $y$. Then $(2y)^2 - 2 = 4x$ says $4(y^2 - x) = 2$. Since $y^2 - x$ is an integer, this says $4 \mid 2$, which is impossible.

   Next suppose $n$ is odd; that is, $n = 2y + 1$ for some integer $y$. Then $(2y + 1)^2 - 2 = 4x$ says $4(y^2 + y - x) = 1$. Since $y^2 + y - x$ is an integer, this says $4 \mid 1$, another impossibility. In either case we reach a contradiction, so $4 \mid (n^2 - 2)$ can never be true.

7. Since $a \mid (11x + 3)$, we know $11x + 3 = am$ for some integer $m$. Since $a \mid (55x + 52)$, we have also $55x + 52 = an$ for some integer $n$. Now $11x + 3 = am$ implies $55x + 15 = 5am$. Thus, $(55x + 52) - (55x + 15) = an - 5am$ and so $37 = a(n - 5m)$. But the only $a > 1$ dividing 37 is 37 itself. Therefore, $a = 37$.

8. [BB] We have $a = q_1 n + r$ and $b = q_2 n + r$ for some integers $q_1$, $q_2$ and $r$. Subtracting, $a - b = (q_1 n + r) - (q_2 n + r) = (q_1 - q_2)n$. Thus, $n \mid (a - b)$ as required.

9. [BB] ($\rightarrow$) If $10a + b = 7k$ for some integer $k$, then $5(10a + b) = 35k$, so $49a + a + 5b = 35k$ and $a + 5b = 35k - 49a$ is divisible by 7.

   ($\leftarrow$) Conversely, if $a + 5b = 7k$, then $10(a + 5b) = 70k$, so $10a + 50b = 10a + b + 49b = 70k$ and $10a + b = 70k - 49b$ is divisible by 7.

10. (a) [BB] True. Since $a \mid b$, $b = ax$ for some integer $x$. Since $b \mid (-c)$, $-c = by$ for some integer $y$. Thus, $c = -by = -axy = a(-xy)$. Since $-xy$ is an integer, $a \mid c$.

    (b) False. Let $a = 3$, $b = 15$, $c = 3$. Then $3 \mid 15$, so $a \mid b$ and $c \mid b$, but $ac = 9 \nmid 15$.

    (c) True. Since $a \mid b$, $b = ax$ for some integer $x$. Thus, $bc = axc = a(xc)$. Since $xc$ is an integer, $a \mid bc$.

    (d) True. Since $a \mid b$, $b = ax$ for some integer $x$. Since $c \mid d$, $d = cy$ for some integer $y$. Thus, $bd = axcy = ac(xy)$. Since $xy$ is an integer, $ac \mid bd$.

    (e) True. Since $a \mid b$, $b = ax$ for some integer $x$. Since $c \mid \frac{b}{a}$, $\frac{b}{a} = cy$ for some integer $y$. Hence, $b = a(cy) = c(ay)$, with $ay$ an integer, so $c \mid b$. Also, $\frac{b}{c} = ay$ says $a \mid \frac{b}{c}$ since $y$ is an integer.

11. We have $ra + sb = 1$ for some integers $r$ and $s$. Multiplying by $c$, $rac + sbc = c$. Now, since $b \mid c$, $ab \mid rac$ and since $a \mid c$, $ab \mid sbc$. Thus, $ab$ divides the sum $rac + sbc$; in other words, $ab \mid c$, as required.

12. (a) [BB]

| | | |
|---|---|---|
| 119 | 1 | 0 |
| 93 | 0 | 1 |
| 26 | 1 | $-1$ |
| 15 | $-3$ | 4 |
| 11 | 4 | $-5$ |
| 4 | $-7$ | 9 |
| 3 | 18 | $-23$ |
| 1 | $-25$ | 32 |

So $\gcd(93, 119) = 1 = (-25)(119) + 32(93)$.

    (b) [BB] $\gcd(-93, 119) = \gcd(93, 119) = 1 = (-25)(119) + 32(93)$ by part (a), so $\gcd(-93, 119) = 1 = (-25)(119) + (-32)(-93)$.

    (c) [BB] $\gcd(-93, -119) = \gcd(93, 119) = 1 = (-25)(119) + 32(93)$, by part (a), so $\gcd(-93, -119) = 1 = (25)(-119) + (-32)(-93)$.

    (d)

| | | |
|---|---|---|
| 1575 | 1 | 0 |
| 231 | 0 | 1 |
| 189 | 1 | $-6$ |
| 42 | $-1$ | 7 |
| 21 | 5 | $-34$ |

So $\gcd(1575, 231) = 21 = 5(1575) + (-34)(231)$.

    (e) $\gcd(1575, -231) = \gcd(1575, 231) = 21 = 5(1575) + (-34)(231)$ by part 12(d), so $\gcd(1575, -231) = 21 = 5(1575) + (34)(-231)$.

    (f) $\gcd(-1575, -231) = \gcd(1575, 231) = 21 = 5(1575) + (-34)(231)$, by part 12(d), so $\gcd(-1575, -231) = 21 = (-5)(-1575) + 34(-231)$.

(g)

| | | |
|---|---|---|
| 8416 | 1 | 0 |
| 3719 | 0 | 1 |
| 978 | 1 | −2 |
| 785 | −3 | 7 |
| 193 | 4 | −9 |
| 13 | −19 | 43 |
| 11 | 270 | −611 |
| 2 | −289 | 654 |
| 1 | 1715 | −3881 |

$\gcd(8416, 3719) = 1 = 1715(8416) + (-3881)(3719)$, so $\gcd(-3719, 8416) = 1 = 3881(-3719) + 1715(8416)$.

(h)

| | | |
|---|---|---|
| 100996 | 1 | 0 |
| 20048 | 0 | 1 |
| 756 | 1 | −5 |
| 392 | −26 | 131 |
| 364 | 27 | −136 |
| 28 | −53 | 267 |

So $\gcd(100996, 20048) = 28 = -53(100996) + 267(20048)$.

(i)

| | | |
|---|---|---|
| 28844 | 1 | 0 |
| 15712 | 0 | 1 |
| 13132 | 1 | −1 |
| 2580 | −1 | 2 |
| 232 | 6 | −11 |
| 28 | −67 | 123 |
| 8 | 542 | −995 |
| 4 | −1693 | 3108 |

$\gcd(28844, 15712) = 4 = (-1693)(28844) + 3108(15712)$, so $\gcd(28844, -15712) = 4 = -1693(28844) + (-3108)(-15712)$.

(j)

| | | |
|---|---|---|
| 54321 | 1 | 0 |
| 12345 | 0 | 1 |
| 4941 | 1 | −4 |
| 2463 | −2 | 9 |
| 15 | 5 | −22 |
| 3 | −822 | 3617 |

So, $\gcd(12345, 54321) = 3 = -822(54321) + 3617(12345)$.

13. The pairs $(93, 119)$, $(-93, 119)$, $(-93, -119)$ and $(3719, 8416)$ are each relatively prime.

14. [BB] Since $a$ and $b$ are relatively prime, we have $ma + nb = 1$ for some integers $m$ and $n$, so $2 = 2ma + 2nb$. Now suppose $x \mid (a+b)$ and $x \mid (a-b)$. Then, by Proposition 4.2.3, $x \mid [(a+b)+(a-b)]$; that is, $x \mid 2a$. Also $x \mid [(a+b)-(a-b)]$; that is, $x \mid 2b$. Again, by Proposition 4.2.3, $x \mid (2ma+2nb)$, so $x \mid 2$ and we conclude that $x = \pm 1$ or $x = \pm 2$. The result follows.

15. [BB] If $g_1$ and $g_2$ are each greatest common divisors of $a$ and $b$, then $g_1 \leq g_2$ (because $g_2$ is greatest) and $g_2 \leq g_1$ (because $g_1$ is greatest), so $g_1 = g_2$.

16. First note that $a = b = 0 \iff a = a + b = 0$, so $\gcd(a, b)$ exists if and only if $\gcd(a, a + b)$ exists. Suppose this is the case. As in the proof of Lemma 4.2.5, let $g_1 = \gcd(a, a + b)$ and $g_2 = \gcd(a, b)$. We prove that $g_1 \leq g_2$ and $g_2 \leq g_1$. First, since $g_2 \mid a$ and $g_2 \mid b$, then $g_2 \mid (a + b)$. Thus, $g_2$ divides both $a$ and $a + b$, so $g_2 \leq g_1$, the greatest common divisor of $a$ and $a + b$. Also, since $g_1$ divides both $a$ and $a + b$, $g_1$ must divide their difference, $(a + b) - a = b$. Thus, $g_1$ divides both $a$ and $b$, hence, $g_1 \leq g_2$, the greatest common divisor of $a$ and $b$.

17. (a) [BB] Multiplying $-2647(17369) + 8402(5472) = 1$ by 4, we have
$17369(-10588) + (5472)(33608) = 4$.

(b) We have

| | | |
|---|---|---|
| 260 | 1 | 0 |
| 154 | 0 | 1 |
| 106 | 1 | −1 |
| 48 | −1 | 2 |
| 10 | 3 | −5 |
| 8 | −13 | 22 |
| 2 | 16 | −27 |

and so $\gcd(154, 260) = 2 = 16(260) + (-27)(154)$. Multiplying by 2, $4 = 32(260) + (-54)(154)$: $x = -54$, $y = 32$.

(c) [BB] Consider the possibility that $154x + 260y = 3$ for certain integers $x$ and $y$. The left side of this equation is even while the right side is odd, a contradiction.

(d) Suppose we have $196x + 260y = 14$ for some integers $x$ and $y$. Since the left side is divisible by 4, the right side would have to be divisible by 4, which is not the case.

18. (a) [BB] We know that $\gcd(x, y) \mid x$ and $\gcd(x, y) \mid y$ and so $\gcd(x, y) \mid (mx + ny)$ (Proposition 4.2.3).

(b) Yes, it is true. Assume $\gcd(x, y) \mid d$ so that $d = \ell(\gcd(x, y))$ for some integer $\ell$. By Theorem 4.2.9, there are integers $s$ and $t$ such that $\gcd(x, y) = sx + ty$. Hence, $d = \ell sx + \ell ty$, and $m = \ell s$, $n = \ell t$ will work.

19. (a) Take $x = -6$, $y = 9$, for example.

(b) Suppose $7x + 5y = 3$. Since $7(-6) + 5(9) = 3$ as well, subtracting gives $7(x + 6) + 5(y - 9) = 0$. Since $7 \mid 7(x + 6)$, so also $7 \mid 5(y - 9)$. Since the integers 5 and 7 are relatively prime, we must have $7 \mid y - 9$, so $y - 9 = 7k$ for some integer $k$. Thus $y = 9 + 7k$ and $7(x + 6) = -5(y - 9) = -5(7k)$, so $x + 6 = -5k$ and $x = -6 - 5k$.

20. Say $x \mid (3k + 2)$ and $x \mid (5k + 3)$. Then $x \mid (15k + 10)$ and $x \mid (15k + 9)$, but then $x \mid [(15k + 10) - (15k + 9)]$. Thus, $x \mid 1$, so $\gcd(2k + 2, 5k + 3) = 1$.

21. [BB] Let $g_1 = \gcd(a, b)$ and $g_2 = \gcd(ac, bc)$. We have to show that $g_2 = cg_1$. First note that $cg_1 \mid ac$ since $g_1 \mid a$, and, similarly, $cg_1 \mid bc$. Thus, $cg_1$ is a common divisor of $ac$ and $bc$, hence, $cg_1 \leq g_2$, the largest of the common divisors of $ac, bc$. On the other hand, since $g_1 = am + bn$ for some integers $m, n$, $cg_1 = acm + bcn$. Since $g_2 \mid acm$ and $g_2 \mid bcn$, it must be that $g_2 \mid cg_1$, and so, since $c, g_1, g_2$ are all positive, $g_2 \leq cg_1$. We conclude that $g_2 = cg_1$, as required.

22. Suppose $x \mid a$ and $x \mid (a + 2)$. Then $x \mid (a + 2 - a)$, $x \mid 2$. Therefore, $x = \pm 1$ or $x = \pm 2$. If $a$ is odd, then we must have $x = 1$, so $\gcd(a, a + 2) = 1$. However, if $a$ is even, then $2 \mid a$ and $2 \mid (a + 2)$, so $\gcd(a, a + 2) = 2$.

23. [BB] Say $x \mid n$ and $x \mid (n + 1)$. Then $x \mid [(n + 1) - n]$, so $x \mid 1$ and $\gcd(n, n + 1) = 1$. We have $(-1)n + (1)(n + 1) = 1$.

24. Say $x \mid a$ and $x \mid b$. Since $x \mid b$ and $b \mid c$, we have $x \mid c$ by transitivity of $\mid$. Thus, $x \mid a$ and $x \mid c$, so $x \mid \gcd(a, c)$ and thus, $x = \pm 1$. We conclude $\gcd(a, b) = 1$.

25. First assume that $\gcd(n, s) = 1$. Thus, there exist integers $a$ and $b$ so that $an + bs = 1$. So $k = ank + bsk$ and since $n$ divides the right side, it also divides the left; that is, $n \mid k$. It follows that the smallest $k$ is $n = \dfrac{n}{\gcd(n, s)}$. In general, let $g = \gcd(n, s)$ and write $n = n_1 g$, $s = s_1 g$ with $\gcd(n_1, s_1) = 1$. Note that $n \mid ks$ if and only if $n_1 \mid ks_1$ since $ks = tn$ if and only if $ks_1 = tn_1$. Thus the smallest $k$ such that $n \mid ks$ is the smallest $k$ such that $n_1 \mid ks_1$. By the first step, $k = n_1 = \dfrac{n}{g} = \dfrac{n}{\gcd(n, s)}$.

26. [BB] As suggested by the hint, consider the set $S$ of all positive linear combinations of $a$ and $b$. Since $S$ contains $a = 1a + 0b$ if $a > 0$ and $-a$ otherwise, $S$ is not empty so, by the Well-Ordering Principle, $S$ contains a smallest element $g$. Since $g \in S$, we know that $g = ma + nb$ for integers $a$ and $b$; hence, we have only to prove that $g$ is the greatest common divisor of $a$ and $b$. First we prove that $g \mid a$. Write $a = qg + r$ with $0 \leq r < g$. Note that $r$ is a linear combination of $a$ and $b$ since $r = a - qg = a - q(ma + nb) = (1 - qa)m + (-qn)b$. Since $g$ is the smallest positive linear combination of $a$ and $b$, we have $r = 0$, so $g \mid a$ as desired. Similarly, $g \mid b$. Finally, if $c \mid a$ and $c \mid b$, then $c \mid (ma + nb)$, so $c \mid g$, so $c \leq g$.

27. [BB] Since $\gcd(63, 273) = 21$, $\operatorname{lcm}(63, 273) = \frac{63(273)}{21} = 819$ by formula (2) of this section.

    Since $\gcd(56, 200) = 8$, $\operatorname{lcm}(56, 200) = \frac{56(200)}{8} = 1400$ by formula (2) of this section.

28. (a) [BB] $\operatorname{lcm}(93, 119) = \frac{93 \cdot 119}{1} = 11{,}067$; (b) $\operatorname{lcm}(-93, 119) = 11{,}067$;
    (c) $\operatorname{lcm}(-93, -119) = 11{,}067$;
    (d) $\operatorname{lcm}(1575, 231) = (1575)(231)/21 = 17{,}325$;
    (e) $\operatorname{lcm}(1575, -231) = 17{,}325$; (f) $\operatorname{lcm}(-1575, -231) = 17{,}325$;
    (g) $\operatorname{lcm}(-3719, 8416) = (3719)(8416)/1 = 31{,}299{,}104$;
    (h) $\operatorname{lcm}(100996, 20048) = (100996 \cdot 20048)/28 = 72{,}313{,}136$;
    (i) $\operatorname{lcm}(-28844, -15712) = (28844)(5712)/4 = 113{,}299{,}232$;
    (j) $\operatorname{lcm}(12345, 54321) = (12345)(54321)/3 = 223{,}530{,}915$.

29. Write $m = q\ell + r$, $0 \leq r < \ell$. We must show that $r = 0$. For this, note that since $a \mid m$ and $a \mid \ell$, it follows from Proposition 4.2.3 that $a \mid r$. Similarly, $b \mid r$ so that $r$ is a common multiple of $a$ and $b$. By minimality of $\ell$, $r = 0$.

30. [BB] $\gcd(a, b) \mid a$ and $a \mid \operatorname{lcm}(a, b)$ so $\gcd(a, b) \mid \operatorname{lcm}(a, b)$ by transitivity of $\mid$. (See Proposition 4.2.2.)

31. Let $g = \gcd(a, b)$. First note that $\dfrac{|ab|}{\gcd(a, b)}$ is a positive integer because $\gcd(a, b) \geq 1$. Next we note that $g \mid a$ and $g \mid b$. Since

$$\frac{|ab|}{g} = \frac{|b|}{g}|a| = \pm\frac{|b|}{g}a$$

    we see that $\frac{|ab|}{g}$ is a multiple of $a$. Similarly, it is a multiple of $b$. It remains only to show that $\frac{|ab|}{g}$ is the least of the positive common multiples of $a$ and $b$. For this, suppose $\ell > 0$ and $a \mid \ell$, $b \mid \ell$. Since $g = sa + tb$ for some integers $a$ and $b$, it follows that $\ell g = s\ell a + t\ell b$ is a multiple of $ab$, hence, a multiple of $|ab|$. Since $\ell g > 0$, $\ell g \geq |ab|$. Therefore, $\ell \geq \frac{|ab|}{g}$ implying that $\frac{|ab|}{g}$ is "least," as required.

32. (a) [BB] Let $x = (a_n a_{n-1} \ldots a_0)_{10}$ be the integer. Thus, $x = a_0 + 10a_1 + \cdots + 10^n a_n$.

$$\begin{aligned} \text{Now } x &= (a_0 + a_1 + \cdots + a_n) + (9a_1 + 99a_2 + \cdots + \underbrace{99\cdots9}_{n \text{ times}} a_n) \\ &= (a_0 + a_1 + \cdots + a_n) + 3(3a_1 + 33a_2 + \cdots + \underbrace{33\cdots3}_{n \text{ times}} a_n). \end{aligned}$$

    Since $a_0 + a_1 + \cdots + a_n$ is divisible by 3 and 3 clearly divides the second term on the right, $3 \mid x$ as required.

    (b) [BB] Reverse the argument in (a); that is, note that if $3 \mid x$, then we can conclude from the preceding that $3 \mid (a_0 + a_1 + \cdots + a_n)$.

(c) Repeat the preceding arguments, but instead, writing the second term as $9(a_1 + 11a_2 + \cdots + \underbrace{11\cdots1}_{n \text{ times}} a_n)$.

33. (a) Let $g_1 = \gcd(a, b)$ and $g = \gcd(g_1, c)$. We have to show that $g = \gcd(a, b, c)$. Since $g \mid g_1$ and $g_1 \mid a$ we see that $g \mid a$. Similarly, $g \mid b$. Since also $g \mid c$, we see that $g$ is a common divisor of $a, b, c$. Next suppose that $d$ is any common divisor of $a, b, c$. Since in particular, $d$ is a common divisor of $a$ and $b$, $d \mid g_1$ (by Corollary 4.2.11). Since also $d \mid c$, we see that $d$ is a common divisor of $g_1$ and $c$, so $d \leq g$. This proves that $g$ is the largest of the common divisors of $a$, $b$ and $c$.

(b) There exist integers $m$ and $n$ such that $\gcd(a, b) = ma + nb$ and integers $r$ and $s$ such that $\gcd(\gcd(a, b), c) = r\gcd(a, b) + sc$. Since $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$, we get $\gcd(a, b, c) = r(ma + nb) + sc = (rm)a + (rn)b + sc$ which expresses $\gcd(a, b, c)$ as an integral linear combination of $a$, $b$ and $c$.

(c) $\gcd(231, 165, 105) = \gcd(\gcd(231, 165), 105) = \gcd(33, 105) = 3$.
Since $33 = -2(231) + 3(165)$ and $3 = -5(105) + 16(33)$, we have
$3 = -5(105) + 16[-2(231) + 3(165)] = -5(105) - 32(231) + 48(165)$.

(d) $\gcd(8580, 6279) = 39 = -30(8580) + 41(6279)$ and $\gcd(2873, 39) = 13 = -1(2873) + 74(39)$.
So $13 = \gcd(6279, 8580, 2873) = -1(2873) + 74[-30(8580) + 41(6279)]$
$= -1(2873) - 2220(8580) + 3034(6279)$.

(e) $\gcd(5577, 18837, 25740) = \gcd(\gcd(25740, 18837), 5577) = \gcd(117, 5577) = 39$.
Since $117 = -30(25740) + 41(18837)$ and $39 = -1(5577) + 48(117)$, we have
$39 = -1(5577) + 48[-30(25740) + 41(18837)]$
$= -1(5577) - 1440(25740) + 1968(18837)$.

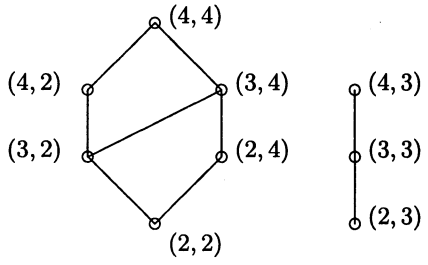
34. (a) **Reflexive:** If $(x_1, x_2) \in A_1 \times A_2$, then $x_1 \in A_1$ so $x_1 \preceq_1 x_1$ (because $\preceq_1$ is a reflexive relation on $A_1$) and $x_2 \preceq_2 x_2$ (because $\preceq_2$ is a reflexive relation on $A_2$). Thus, $(x_1, x_2) \preceq (x_1, x_2)$.

**Antisymmetric:** Suppose $(x_1, x_2)$ and $(y_1, y_2) \in A_1 \times A_2$ and that $(x_1, x_2) \preceq (y_1, y_2)$ and $(y_1, y_2) \preceq (x_1, x_2)$. Then $x_1 \preceq_1 y_1$, $x_2 \preceq_2 y_2$, $y_1 \preceq_1 x_1$ and $y_2 \preceq_2 x_2$. So $x_1 = y_1$ (because $x_1 \preceq_1 y_1$, $y_1 \preceq_1 x_1$ and $\preceq_1$ is antisymmetric) and $x_2 = y_2$ (because $x_2 \preceq_2 y_2$, $y_2 \preceq_2 x_2$ and $\preceq_2$ is antisymmetric). Therefore, $(x_1, x_2) = (y_1, y_2)$.

**Transitive:** Suppose $(x_1, x_2)$, $(y_1, y_2)$, and $(z_1, z_2)$ are in $A_1 \times A_2$, $(x_1, x_2) \preceq (y_1, y_2)$ and $(y_1, y_2) \preceq (z_1, z_2)$. Then $x_1 \preceq_1 y_1$ and $y_1 \preceq_1 z_1$, so $x_1 \preceq_1 z_1$ (because $\preceq_1$ is transitive) and $x_2 \preceq_2 y_2$ and $y_2 \preceq_2 z_2$, so $x_2 \preceq_2 z_2$ (because $\preceq_2$ is transitive). Thus, $(x_1, x_2) \preceq (z_1, z_2)$.

(b) $(2,2) \preceq (2,4)$, $(2,2) \preceq (3,2)$, $(2,2) \preceq (3,4)$, $(2,2) \preceq (4,2)$, $(2,2) \preceq (4,4)$, $(2,3) \preceq (3,3)$, $(2,3) \preceq (4,3)$, $(2,4) \preceq (3,4)$, $(2,4) \preceq (4,4)$, $(3,2) \preceq (3,4)$, $(3,2) \preceq (4,2)$, $(3,2) \preceq (4,4)$, $(3,3) \preceq (4,3)$, $(3,4) \preceq (4,4)$, $(4,2) \preceq (4,4)$

(c)



(d) $(2,2)$ and $(2,3)$ are minimal; $(4,3)$ and $(4,4)$ are maximal; there are no minimum nor maximum elements.

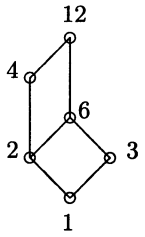
(e)

| | $a$ | $b$ | glb$(a,b)$ | lub$(a,b)$ |
|---|---|---|---|---|
| *i.* | $(2,2)$ | $(3,3)$ | * | * |
| *ii.* | $(4,2)$ | $(3,4)$ | $(3,2)$ | $(4,4)$ |
| *iii.* | $(3,2)$ | $(2,4)$ | $(2,2)$ | $(3,4)$ |
| *iv.* | $(3,2)$ | $(3,4)$ | $(3,2)$ | $(3,4)$ |

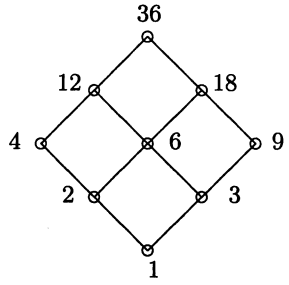
(f) Let $A_1 = A_2 = \mathsf{Z}$ and let $\preceq_1$ and $\preceq_2$ be the usual (total) order relation on the integers. Then $\mathsf{Z} \times \mathsf{Z}$ is not totally ordered since, for example, $(1,2)$ and $(0,3)$ are not comparable.

35. (a) First we note that $(A, |\,)$ is a poset, in the same way that $(\mathsf{N}, |\,)$ is a poset. Now we show that any two elements $a, b \in A$ have a glb and a lub. Certainly $\text{lcm}(a,b)$ has the properties required of a least upper bound and $\gcd(a,b)$ of a greatest lower bound. The key point is to show that these elements lie in $A$, and this follows from the definitions of gcd and lcm. If $a, b \in A$, then $a \mid n$ and $b \mid n$, hence, $\text{lcm}(a,b) \mid n$ by Exercise 29, so $\text{lcm}(a,b) \in A$. Furthermore, since $\gcd(a,b) \mid a$ and $a \mid n$, we have $\gcd(a,b) \mid n$ and so $\gcd(a,b) \in A$ too.

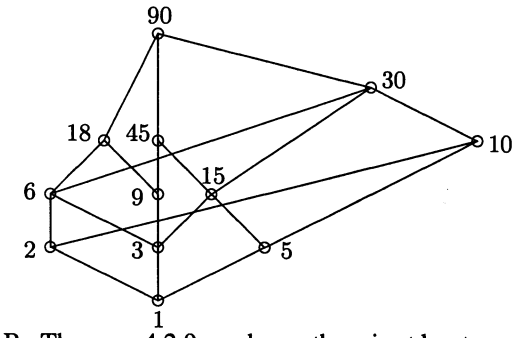
(b) [BB]




(c)




(d)




36. (a) [BB] By Theorem 4.2.9, we know there is at least one pair of integers $a, b$ such that $g = am + bn$. Now observe that for any integer $k$, it is also true that $g = (a + kn)m + (b - km)n$.

(b) i. By Theorem 4.2.9, there is at least one pair of integers $a, b$ such that $ma + nb = 1$. As in part (a), this means that $(a+kn)m+(b-km)n = 1$ for any integer $k$. By the Division Algorithm, select $k$ such that $b = km + r$ with $0 \leq r < m$. Thus $a'm + b'n = 1$ with $a' = a + kn$, $b' = r = b - km$ and $0 \leq b' < m$. Note that $b' = 0$ is not possible since this would imply $a'm = 1$, contradicting $m > 1$. Finally we must show that $a'$ and $b'$ are unique. Assume then that we also have $a''m + b''n = 1$, with $0 < b'' < m$. Then $(a' - a'')m + (b' - b'')n = 0$ and so $m \mid (b' - b'')n$. Since $\gcd(m,n) = 1$, $m \mid (b' - b'')$ by Corollary 4.2.10. Since both $b'$ and $b''$ are in the range $0 < b', b'' < m$, the only way for $b' - b''$ to be a multiple of $m$ is if the multiple is 0. Thus $b' - b'' = 0 = a' - a''$.

ii. Part (i) says we can find unique integers $a, b$ with $0 < b < m$ such that $am + bn = 1$. Since $b > 0$, $n > 0$ and $m > 0$, we must have $a < 0$. Also $a > -n$ because $b < m$. Now

$am + bn = 1$ implies $am = -bn + 1$, so $(a + n)m = (-b + m)n + 1$. Set $a' = a + n$ and $b' = -b + m$. Because $0 < b < m$, $0 < b' < m$ also. Because $-n < a < 0$, $0 < a' < n$. The uniqueness of $a'$ and $b'$ follows from the uniqueness of $a$ and $b$. (Uniqueness can also be proved directly as in (i).)

## Exercises 4.3

1. (a) [BB] 157 is prime;. (b) [BB] 9831 is not prime; $3 \mid 9831$;
   (c) 9833 is prime; (d) 55,551,111 is not prime; $11 \mid 55{,}551{,}111$.
   (e) $2^{216,090} - 1$ is not prime: $2^{216,090} - 1 = (2^{108,045} + 1)(2^{108,045} - 1)$.

2. No. For example, take $n = 9$. Then $n$ has no prime factor smaller than $\sqrt{9}$.

3. (a) [BB] Note that $n = p(n/p)$. Now if $n/p$ is not prime, then it has a prime factor $q < \sqrt{n/p}$ by Lemma 4.3.4. Since a prime factor of $n/p$ is a prime factor of $n$, $q$ would then be a prime factor of $n$ not exceeding $\sqrt{n/p}$, contradicting the fact that the smallest prime factor of $n$ is bigger than this.

   (b) [BB] Note that $16{,}773{,}121 = 433(38{,}737)$ and that $\sqrt{38{,}737} \approx 197$. Since 433, the smallest prime dividing 38,737, is larger than 197, 38,737 is prime, by part (a). Thus, $16{,}773{,}121 = 433(38{,}737)$ is the representation of 16,773,121 as the product of primes.

4. (a) [BB] $856 = 2 \cdot 2 \cdot 2 \cdot 107 = 2^3 \cdot 107$; (b) $2323 = 23 \cdot 101$;
   (c) $6647 = 17 \cdot 17 \cdot 23$; (d) $9970 = 2 \cdot 5 \cdot 997$;
   (e) [BB] $(2^8 - 1)^{20} = 3^{20} \cdot 5^{20} \cdot 17^{20}$; (f) $55{,}551{,}111 = 3 \cdot 7 \cdot 11 \cdot 101 \cdot 2381$.

5. [BB] If $14^n$ terminates in 0, then it is divisible by 10 and, hence, by the prime 5; that is, there is a 5 in the factorization of $14^n$. This contradicts unique factorization because $14^n = 2^n 7^n$ says that the only primes dividing $14^n$ are 2 and 7.

6. (a) For example, $0, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$.

   (b) [BB] We have to prove that $a, b \in A$ implies $a + b \in A$. Let $a, b \in A$. Then $a = \frac{m}{n}$ and $b = \frac{k}{\ell}$, where $3 \nmid n$, $3 \nmid \ell$. Then

   $$a + b = \frac{m\ell + nk}{n\ell}.$$

   Since 3 is prime, if $3 \mid n\ell$, then $3 \mid n$ or $3 \mid \ell$, neither of which is true. Thus, $3 \nmid n\ell$, so $a + b \in A$.

   (c) We have to prove that $a, b \in A$ implies $ab \in A$. Let $a, b \in A$. Then $a = \frac{m}{n}$ and $b = \frac{k}{\ell}$, where $3 \nmid n$, $3 \nmid \ell$. Then $ab = mk/n\ell$. Since 3 is prime, if $3 \mid n\ell$, then $3 \mid n$ or $3 \mid \ell$, neither of which is true. Thus, $3 \nmid n\ell$, so $ab \in A$.

7. We must prove that $\sim$ is reflexive, symmetric and transitive.

   **Reflexive:** If $a \in A$, $aa = a^2$ is a perfect square, so $a \sim a$.

   **Symmetric:** If $a, b \in A$ and $a \sim b$, then $ab$ is a perfect square, so $ba = ab$ is a perfect square and $b \sim a$.

   **Transitive:** If $a, b, c \in A$, $a \sim b$ and $b \sim c$, then $ab = n^2$ and $bc = m^2$ are perfect squares. So $(ab)(bc) = (nm)^2$. To prove that $ac$ is a perfect square, we prove that the prime powers occurring

in the decomposition of $ac$ as the product of primes are all even, that is, that $ac = p_1^{2\alpha_1} p_2^{2\alpha_2} \cdots p_r^{2\alpha_r}$. It would follow that $ac$ is the perfect square $\ell^2$, where $\ell = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Suppose then that $p^\alpha$ is the largest power of a prime $p$ dividing $ac$ and that $p^{2\beta}$ is the largest power of $p$ dividing $b^2$, then the largest power of $p$ dividing $(nm)^2$ is $p^{\alpha+2\beta}$. Since this power is even, $\alpha$ is even. It follows that $ac$ is a perfect square, hence, $a \sim c$.

8. (a) True. This is precisely Lemma 4.3.2.

   (b) [BB] False. The prime described here has the miraculous property that it divides all natural numbers greater than 1. Certainly this prime cannot be 2 since $2 \nmid 3$, neither can it be odd since if $p$ is an odd prime, $p \nmid 2$.

9. (a) The range of $f$ is the set of all prime numbers since any prime $p$ is $f(n)$ for some $n$: Take $n = p$, for example.

   (b) $f$ is not one-to-one. For example, $f(2) = 2 = f(4)$ but $2 \neq 4$.

   (c) Since the range is a proper subset of $\mathbb{N}$, $f$ is not onto.

   (d) Since 1 is not prime, $f(1)$ would not be defined. Remember our agreement that when we write $f\colon A \to B$, it is understood that $\operatorname{dom} f = A$.

10. (a) [BB] This function is one-to-one. If $f(n_1, m_1) = f(n_2, m_2)$, then $2^{m_1} 6^{n_1} = 2^{m_2} 6^{n_2}$, and so $2^{m_1} 2^{n_1} 3^{n_1} = 2^{m_2} 2^{n_2} 3^{n_2}$. By the Fundamental Theorem of Arithmetic, the powers of 3 and 2 on each side of this equation are the same. So $n_1 = n_2$, hence $m_1 = m_2$ too. Therefore, $(n_1, m_1) = (n_2, m_2)$.

    (b) This function is not one-to-one since $f(1,0) = 36 = f(0,2)$ but $(1,0) \neq (0,2)$.

11. (a) [BB] $\pi(10) = 4$, $10/\ln 10 \approx 4.343$ and $\frac{\pi(10)}{10/\ln 10} \approx \frac{4}{4.343} \approx 0.921$.

    (b) $\pi(50) = 15$, $50/\ln 50 \approx 12.781$ and $\frac{\pi(50)}{50/\ln 50} \approx \frac{15}{12.781} \approx 1.174$.

    (c) $\pi(95) = 24$, $95/\ln 95 \approx 20.861$ and $\frac{\pi(95)}{95/\ln 95} \approx \frac{24}{20.861} \approx 1.150$.

12. (a) Here are the primes less than 200.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 |
| 179 | 181 | 191 | 193 | 197 | 199 | | | | |

    $\pi(200) = 46$, $\frac{200}{\ln 200} \approx 37.748$ and $\frac{\pi(200)}{200/\ln 200} \approx 1.219$.

    (b) Here are the primes less than 500.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 |
| 179 | 181 | 191 | 193 | 197 | 199 | 211 | 223 | 227 | 229 |
| 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 |
| 353 | 359 | 367 | 373 | 379 | 383 | 389 | 397 | 401 | 409 |
| 419 | 421 | 431 | 433 | 439 | 443 | 449 | 457 | 461 | 463 |
| 467 | 479 | 487 | 491 | 499 | | | | | |

$\pi(500) = 95$, $\frac{500}{\ln 500} \approx 80.456$ and $\frac{\pi(500)}{500/\ln 500} \approx 1.181$.

13. There are approximately

    $5000/\ln 5000 \approx 587$ primes less than 5000,
    $50{,}000/\ln 50{,}000 \approx 4621$ primes less than 50,000,
    $500{,}000/\ln 500{,}000 \approx 38{,}103$ primes less than 500,000 and
    $5{,}000{,}000/\ln 5{,}000{,}000 \approx 324{,}150$ primes less than 5,000,000.

14. [BB] This is a special case of Exercise 11, Section 4.2 since if $p$ and $q$ are distinct primes, then $p$ and $q$ are relatively prime. The result also follows quickly from the Fundamental Theorem of Arithmetic since, writing $n = p_1 p_2 \cdots p_r$, the hypotheses say that one of the $p_i$ is $p$ and some other $p_j$ is $q$.

15. (a) Since $2x_0 + 5y_0 = 2x + 5y$, we have $5(y_0 - y) = 2(x - x_0)$. Since 5 divides the left side, $5 \mid 2(x - x_0)$, so $5 \mid (x - x_0)$ by Proposition 4.3.7. Thus $x - x_0 = 5k$ (hence $x = x_0 + 5k$) for some integer $k$. From $5(y_0 - y) = 2(x - x_0)$, we conclude that $5(y_0 - y) = 10k$, so $y_0 - y = 2k$ and $y = y_0 - 2k$ as desired.

    (b) This follows from part (a) since $x_0 = 2$, $y_0 = 3$ satisfies $2x_0 + 5y_0 = 19$.

16. [BB] Without loss of generality, we may assume that $a < b$. Since there are infinitely many primes, there exists an integer $n \geq 0$ such that $b + n$ is prime. Since $a + n < b + n$, $b + n$ cannot divide $a + n$, so $a + n$ and $b + n$ are relatively prime.

17. (a) If $n$ is prime, then $n$ is divisible only by 1 and $n$, so $d(n) = 2$. On the other hand, if $n$ is not prime, let $p$ be a prime dividing it. Then $n$ is divisible by the distinct integers 1, $p$ and $n$, so $d(n) \geq 3$. Also $d(1) = 1$. Thus, the $n$ with $d(n) = 2$ are precisely those $n$ which are prime.

    (b) [BB] First note that, if $p$ is prime, $d(p^2) = 3$ since the only divisors of $p^2$ are 1, $p$ and $p^2$. We claim that only integers of the form $p^2$ satisfy the condition. To see this, notice that if $n$ is divisible by distinct primes $p$ and $q$, then $n$ is divisible by each of 1, $p$, $q$ and $pq$ (see Exercise 14) and so $d(n) \geq 4$. Thus, $n$ has just one prime factor. Also, if $n$ is divisible by the cube of a prime, $p^3$, then $n$ is divisible by each of 1, $p$, $p^2$, $p^3$ so again, $d(n) \geq 4$. We conclude that $n = p^2$ for some prime $p$.

    (c) First consider the case $n = p^\alpha$ for some prime $p$. Then $d(n) = \alpha + 1$, so $n = p^4$ satisfies $d(n) = 5$. Next assume that $n$ is divisible by three distinct primes $p, q, r$. In this case, $d(n) \geq 7$ since $1, p, q, r, pq, pr, qr, pqr$ all divide $n$. Thus, $n$ cannot have three distinct prime factors. Finally, assume $n = p^\alpha q^\beta$ for distinct primes $p$ and $q$. If $\alpha = \beta = 1$, then $d(n) = 4$. If $\alpha > 1$ (or, equivalently, $\beta > 1$), then $n$ is divisible by $1, p, q, pq, p^2, p^2q$, so $d(n) \geq 6$. We conclude that $d(n) = 5$ if and only if $n = p^4$.

18. (a) [BB] No. $2^{15} - 1 = (2^5 - 1)(2^{10} + 2^5 + 1)$.

    (b) No. $2^{91} - 1 = (2^{13} - 1)(2^{78} + 2^{65} + 2^{52} + 2^{39} + 2^{26} + 2^{13} + 1)$

    (c) If $n$ is not prime, write $n = rs$ for $1 < r, s < n$. Then $2^n - 1 = 2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 1)$ is not prime, a contradiction.

    (d) The converse of (c) is not true. See the section on Mersenne primes.

19. (a) [BB] $2^6 + 1 = (2^2 + 1)(2^4 - 2^2 + 1)$.

    (b) $2^{20} + 1 = (2^4 + 1)(2^{16} - 2^{12} + 2^8 - 2^4 + 1)$

(c) If $n$ is not a power of 2, we can write $n = ab$ with $b$ odd. Then $2^{ab} + 1 = (2^a + 1)(2^{a(b-1)} - 2^{a(b-2)} + \cdots + 1)$. Note the last term in the second factor is "+"1 because $b$ is odd.

20. (a) If $p = 2k + 1$ and $q = 2\ell + 1$, then $p + q = 2(k + \ell + 1)$, so $2 \mid (p + q)$. Also $p + q > 2$ since $p > 2$ and $q > 2$. Thus, $p + q > 2$ is divisible by 2 and so not prime.

(b) No. Consider $2 + 3 = 5$.

21. [BB] Suppose, to the contrary, that $p$ and $q$ are consecutive primes (in that order) such that $p + q = 2r$, where $r$ is a prime. Then

$$p = \frac{p+p}{2} < \frac{p+q}{2} = r < \frac{q+q}{2} = q;$$

that is, $p < r < q$. This contradicts the fact that $p$ and $q$ are **consecutive** primes.

22. Suppose there exist integers $x$ and $y$ such that $x^2 - y^2 = 2n$. Then $2 \mid (x + y)(x - y)$ so $2 \mid (x + y)$ or $2 \mid (x - y)$. But $x + y = (x - y) + 2y$ shows that $x + y$ and $x - y$ are both even or both odd. Thus, 2 divides both $x + y$ and $x - y$ so $4 \mid (x^2 - y^2)$, $4 \mid 2n$ and $2 \mid n$, a contradiction.

23. [BB] This is true, and the proof has given Andrew Wiles a place in history. This is Fermat's Last Theorem (4.3.14)!

24. [BB] The condition tells us that in the prime decompositions of $a$, $b$ and $c$, no prime appears in two of the decompositions. Hence, $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $b = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$, $c = r_1^{\gamma_1} r_2^{\gamma_2} \cdots r_t^{\gamma_t}$ with the primes $p_i$, $q_i$, $r_i$ all different. Thus,

$$\begin{aligned} ab &= p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_s^{\beta_s} \\ ac &= p_1^{\alpha_1} \cdots p_r^{\alpha_r} r_1^{\gamma_1} \cdots r_t^{\gamma_t} \\ bc &= q_1^{\beta_1} \cdots q_s^{\beta_s} r_1^{\gamma_1} \cdots r_t^{\gamma_t}. \end{aligned}$$

Suppose $\gcd(ab, bc, ac) \neq 1$. Then there must exist a prime $s$ such that $s \mid ab$, $s \mid bc$, $s \mid ac$. But there is no prime common to the decompositions of $ab$, $ac$, $bc$. Hence, no such $s$ exists and $\gcd(ab, bc, ac) = 1$.

25. If $abc$ and $n$ are not relatively prime, then there exists a prime $p$ such that $p \mid abc$ and $p \mid n$. By Corollary 4.3.8, $p \mid a$ or $p \mid b$ or $p \mid c$. In the first case, $p \mid a$ and $p \mid n$ contradicting $\gcd(a, n) = 1$. The other two cases are similar.

26. Since $\gcd(a, p^2) = p$, we know that $a = pr$ where $p \nmid r$. Since $\gcd(b, p^3) = p^2$, we know that $b = p^2 s$ where $p \nmid s$.

(a) [BB] $ab = p^3 rs$ and $p \nmid rs$ by Proposition 4.3.7. Thus, $\gcd(ab, p^4) = p^3$.

(b) $a + b = pr + p^2 s = p(r + ps)$ and $p \nmid (r + ps)$ since $p \nmid r$. Thus, $\gcd(a + b, p^4) = p$.

27. Let $x$ be such that $p_1 p_2 \cdots p_n + 2 \leq x \leq p_1 p_2 \cdots p_n + p_{n+1} - 1$. Thus, $x = p_1 p_2 \cdots p_n + k$ with $2 \leq k \leq p_{n+1} - 1$. Let $p$ be a prime dividing $k$. It follows that $p = p_i$ for some $i$, $1 \leq i \leq n$, because $k < p_{n+1}$. But then, $p_i \mid (p_1 p_2 \cdots p_n + k)$; that is, $p_i \mid x$. So $x$ is composite.

The first part of this exercise provides a sequence of $p_{n+1} - 2$ consecutive composite integers. Since there are primes $p_{n+1}$ of arbitrarily large size, we obtain in this way gaps of arbitrary length.

28. Writing $a = p^\alpha r$ and $b = p^\beta s$ with $\gcd(p, r) = 1$ and $\gcd(p, s) = 1$, we are given that $\gcd(r, s) = 1$, $\alpha \geq 1$, $\beta \geq 1$ and either $\alpha = 1$ or $\beta = 1$.

(a) [BB] We have $a^2 = p^{2\alpha}r^2$ with $\gcd(p, r^2) = 1$ and $b = p^\beta s$. Also $\gcd(r^2, s) = 1$ since $\gcd(r, s) = 1$. We conclude that there are two possibilities: If $\beta = 1$, then $\gcd(a^2, b) = p$, while if $\beta > 1$, then $\gcd(a^2, b) = p^2$.

(b) Here, $a^3 = p^{3\alpha}r^3$ and $b = p^\beta s$. Now there are three possibilities. If $\beta = 1$, then $\gcd(a^3, b) = p$. If $\beta = 2$, then $\gcd(a^3, b) = p^2$, and if $\beta \geq 3$, then $\gcd(a^3, b) = p^3$.

(c) Here we have $a^2 = p^{2\alpha}r^2$ and $b^3 = p^{3\beta}s^3$. There are two possibilities. If $\alpha = 1$, then $\gcd(a^2, b^3) = p^2$. If $\alpha > 1$, then $\gcd(a^2, b^3) = p^3$.

29. [BB] Let the prime decompositions of $a$ and $b$ be $a = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ and $b = q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s}$. Since $\gcd(a, b) = 1$, we know that $p_i \neq q_j$ for any $i$ and $j$. Thus, $ab = p_1^{\alpha_1}\cdots p_r^{\alpha_r}q_1^{\beta_1}\cdots q_s^{\beta_s}$ with no simplification possible. Since $ab = x^2$, it follows that each $\alpha_i$ and each $\beta_i$ must be even. But this means that $a$ and $b$ are perfect squares.

30. (a) We proceed exactly as in the solution to Problem 17. We write $a$ and $b$ in the form

$$a = \pm p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}, \quad b = \pm p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$$

for certain primes $p_1, p_2, \ldots, p_r$ and integers $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_r$. (Some $\alpha_i$ or $\beta_i$ may well be 0 since we assume that the same primes appear in the decompositions of both $a$ and $b$.) We claim that the least common multiple of $a$ and $b$ is

$$\ell = p_1^{\max(\alpha_1,\beta_1)}p_2^{\max(\alpha_2,\beta_2)}\cdots p_r^{\max(\alpha_r,\beta_r)}$$

where $\max(\alpha_i, \beta_i)$ denotes the larger of the two integers $\alpha_i$ and $\beta_i$.
Now $a \mid \ell$ and $b \mid \ell$ since the exponent $\max(\alpha_i, \beta_i)$ is at least as big as the exponent of the prime $p_i$ in each of $a$ and $b$. Next assume that $a \mid m$ and $b \mid m$. Since $p_i^{\alpha_i} \mid a$ $p_i^{\alpha_i} \mid m$ and since $p_i^{\beta_i} \mid b$, $p_i^{\beta_i} \mid m$. Thus, $p_i^{\max(\alpha_i,\beta_i)} \mid m$. Thus, $m = p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}$ for some integers $\gamma_i \geq \max(\alpha_i, \beta_i)$. So $\ell \mid m$, proving that $\ell = \operatorname{lcm}(a, b)$ as claimed.

(b) Using (a) and the result of Problem 17, we have

$$\gcd(a, b)\operatorname{lcm}(a, b) = p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)}\cdots p_r^{\min(\alpha_r,\beta_r)+\max(\alpha_r,\beta_r)}.$$

The result now follows from the fact that $\min(\alpha, \beta) + \max(\alpha, \beta) = \alpha + \beta$ for any integers $\alpha$ and $\beta$.

31. [BB] Let $a = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Since $a$ is a square, all the $\alpha_i$ are even. Since $a$ is a cube, all the $\alpha_i$ are divisible by 3. It follows from Exercise 14 that $6 \mid \alpha_i$ for all $i$. This means that $a$ is a 6th power.

32. (a) [BB] True. Write $a = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ as the product of powers of primes.
Then $a^{11} = p_1^{11\alpha_1}\cdots p_r^{11\alpha_r}$. If $p \mid a^{11}$, then $p$ must be one of the $p_i$, so $p \mid a$.

(b) True. Since $p \mid a$, $p \mid a^2$. Now $p \mid a^2$ and $p \mid (a^2 + b^2)$ forces $p \mid b^2$. Then, by Proposition 4.3.7, we conclude that $p \mid b$.

(c) False. Let $a = 2$. Then $a^9 + a^{17} = 131,584$. Now 257 is prime and $257 \mid 131,584$, but $257 \nmid 2$.

33. [BB] Whenever $a, b, c$ satisfy $x^2 + y^2 = z^2$, so also do $ka, kb, kc$ satisfy the equation, for any integer $k$. Thus, for example, all triples of the form $3k, 4k, 5k$ satisfy $a^2 + b^2 = c^2$.

34. (a) [BB] Write $3n+2 = p_1p_2\cdots p_m$ as the product of (necessarily odd) primes. Each of the $p_i$ is either 3 or of the form $3k+1$ or $3k+2$. Now we notice that the product of integers of the form $3k+1$ is of the same form and the product of 3 with such an integer is a multiple of 3. So if none of the $p_i$ is of the form $3k+2$, the product cannot be either. If we drop the word odd, the result is false. For example, $3(2)+2 = 8$ is divisible by only the prime 2, which is not of the form $3n+2$ for $n \in \mathsf{N}$.

    (b) Write $4n+3 = p_1p_2\cdots p_m$ as the product of (necessarily odd) primes. Now any odd prime is of the form $4k+1$ or $4k+3$. The product of any two integers of the form $4k+1$ is of the same form, so if all the $p_i$ were of this form so would be the product, a contradiction.

    (c) Write $6n+5 = p_1p_2\cdots p_m$ as the product of primes. Each of the $p_i$ is of the form $6k+1$ or $6k+5$. The product of two integers of the form $6k+1$ is also of this form, so at least one of the $p_i$ has to be of the form $6k+5$, as asserted.

    (d) Adopting the idea of Euclid's proof that there are infinitely many primes (Theorem 4.3.3), we suppose that there are just finitely many primes of the form $6n+5$, say $p_1, p_2, \ldots, p_m$, and form the product $N = p_1p_2\cdots p_m$. It is easily checked that the product of two integers of the form $6k+5$ is an integer of the form $6k+1$, that the product of two integers of the form $6k+1$ is of the same form, and that the product of an integer of the form $6k+5$ and another of the form $6k+1$ is an integer of the form $6k+5$. Therefore, $N$ is of the form $6k+1$ or $6k+5$ depending on whether $m$ is even or odd, respectively.

    If $m$ is even (so that $N = 6k+1$), we consider the integer $N+4$ which is of the form $6k+5$. By part (c) of this question, $6k+5$ has a prime factor of the same form; that is, $N+4$ is divisible by one of the $p_i$. Since $p_i \mid N$, however, this would imply $p_i \mid 4$, which is untrue.

    If $k$ is odd, the argument is similar. This time $N = 6k+5$, hence, $N+6$ is of the same form, hence, divisible by some $p_i$. Since $N$ is also divisible by $p_i$, we get $p_i \mid 6$, which is not true (since 5 is the only prime $p_i$ of the form $6k+5$ which is less than 6 and $5 \nmid 6$).

35. Write $p+1 = 6q+r$ with $0 \le r < 6$.

    If $r = 1$, then $p = 6q$, contradicting $p$ being prime.

    If $r = 2$, then $p = 6q+1$, so $p+2 = 6q+3$, $3 \mid (p+2)$, contradicting $p+2 > 3$ being prime.

    If $r = 3$, then $p = 6q+2$ is divisible by 2, contradicting $p$ being a prime greater than 2.

    If $r = 4$, then $p = 6q+3$, so $3 \mid p$, a contradiction since $p > 3$.

    If $r = 5$, then $p = 6q+4$ and $2 \mid p$, so again, we have a contradiction.

    The only possibility is that $r = 0$; that is, $6 \mid (p+1)$.

36. (a) [BB; $f(8)$ only] $f(8) = 2^{2(3)} = 2^6 = 64$ since $n = 1, m = 8 = 2^3$.
    $f(\frac{1}{8}) = 2^{2(3)-1} = 2^5 = 32$ since $m = 1, n = 1/8 = 1/2^3$.
    $f(100) = 2^{2(2)}5^{2(2)} = 2^45^4 = 10,000$ since $100 = 2^25^2$.
    $f(\frac{40}{63}) = 2^{2(3)}5^{2(1)}3^{2(2)-1}7^{2(1)-1} = 2^65^23^37^1 = (64)(25)(27)(7) = 302{,}400$ since $m = 40 = 2^3 \cdot 5$ and $n = 3^2 \cdot 7$.

    (b) [BB; 1000000 only] Since $1000000 = 10^6 = 2^65^6$, $x = 2^35^3 = 1000$.
    Since $10000000 = 10^7 = 2^75^7 = 2^{2(4)-1}5^{2(4)-1}$, $t = 1/(2^45^4) = 1/10,000$.
    Since $365,040 = (2^4)(3^3)(5)(13^2)$, $m = 2^2(13) = 52$, $n = 3^2(5^1) = 45$ and $r = \frac{52}{45}$.

(c) We must prove that $f$ is one-to-one and onto.

One-to-one: Suppose $f(m_1/n_1) = f(m_2/n_2)$. Assume first that $m_1 = m_2 = 1$, $n_1 = p_1^{e_1} \cdots p_\ell^{e_\ell}$ and $n_2 = q_1^{f_1} \cdots q_k^{f_k}$, where $p_1, \ldots, p_\ell$ are distinct primes, $q_1, \ldots, q_k$ are distinct primes, $e_i > 0$ and $f_i > 0$ for all $i$. Then $p_1^{2e_1-1} \cdots p_\ell^{2e_\ell-1} = q_1^{2f_1-1} \cdots q_k^{2f_k-1}$. By the Fundamental Theorem of Arithmetic, we conclude that $n_1 = n_2$, so $m_1/n_1 = m_2/n_2$. The same approach can be applied to the other eight possible cases for the pair $(m_1/n_1, m_2/n_2)$, showing that $f$ is one-to-one.

Onto: Given $y \in \mathsf{N}$, if $y = 1$, then $y = f(1)$. Otherwise, $y$ is a natural number $> 1$ and so, by the Fundamental Theorem of Arithmetic, is the unique product of powers of distinct primes. Separating the powers into even and odd powers, we can write $y = p_1^{2e_1} \cdots p_k^{2e_k}$ if there are no odd powers, $y = q_1^{2f_1-1} \cdots q_\ell^{2f_\ell-1}$ if there are no even powers, and $y = p_1^{2e_1} \cdots p_k^{2e_k} q_1^{2f_1-1} \cdots q_\ell^{2f_\ell-1}$ if there are both odd and even powers. In the first case, $y = f(m)$ where $m = p_1^{e_1} \cdots p_k^{e_k}$, in the second case, $y = f(\frac{1}{n})$ where $n = q_1^{f_1} \cdots q_\ell^{f_\ell}$ and, in the third case, $y = f(\frac{m}{n})$ where $m = p_1^{e_1} \cdots p_k^{e_k}$ and $n = q_1^{f_1} \cdots q_\ell^{f_\ell}$. Thus, $f$ is onto.

37. (a) **Reflexive.** $a \sim a$ because $a^1 = a^1$, so $m = n = 1$ works.

**Symmetric.** if $a \sim b$, then $a^m = b^n$ for certain integers $m$ and $n$ each $\geq 1$. Then $b^n = a^m$, so $b \sim a$.

**Transitive.** if $a \sim b$ and $b \sim c$, then $a^m = b^n$ for certain $m \geq 1$ and $n \geq 1$, and $b^r = c^s$ for certain $r \geq 1$ and $s \geq 1$. Thus, $a^{mr} = (a^m)^r = (b^n)^r = (b^r)^n = (c^s)^n = c^{sn}$ with $mr \geq 1$ and $sn \geq 1$. So $a \sim c$.

(b) $\overline{3} = \{3^n \mid n \in \mathsf{N}\}$ is the set of powers of 3; $\overline{4} = \{2^n \mid n \in \mathsf{N}\}$ is the set of powers of 2; $\overline{144} = \{12^n \mid n \in \mathsf{N}\}$ is the set of powers of 12.

(c) As suggested, we write $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and claim that $\overline{a}$ is the set of powers of $b$, where

$$b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \text{ where } \beta_i = \alpha_i/g, \quad g = \gcd(\alpha_1, \ldots, \alpha_k).$$

(For example, the equivalence class of $144 = 2^4 3^2$ is the set of powers of $12 = 2^2 3$.)

To establish our general claim, we first note that $b^t \sim a$ for any power $b^t$ of $b$ because $b^g = a$ (and so $(b^t)^g = (b^g)^t = a^t$). We prove that, conversely, any natural number equivalent to $a$ is a power of $b$. So suppose $x \sim a$. Then $x \sim b$ by transitivity. So $x^n = b^m$ for certain $n, m \geq 1$. Write $n = tn_1$, $m = tm_1$ where $\gcd(n_1, m_1) = 1$. Then (taking $t$th roots) $x^{n_1} = b^{m_1}$. Writing $x = p_1^{\psi_1} p_2^{\psi_2} \cdots p_k^{\psi_k}$, we have $n_1\psi_i = m_1\beta_i$ for all $i$. Now any prime divisor of $n_1$ divides $\beta_i$, for all $i$ (since $n_1$ and $m_1$ are relatively prime). Since the $\beta$'s are relatively prime, it follows that $n_1 = 1$. Hence, $x = b^{m_1}$ is a power of $b$ as desired.

38. (a) Here is some MAPLE code which does the trick. Crossing out and circling is implemented by means of an one-dimensional array called `p` whose entries initially are all 1. When integer `I` is divisible by a prime, `p[I]` is set equal to 0 (this corresponds to crossing out). When the program terminates, those integers `I` with `p[I]=1` are the desired primes.

```
An implementation of the Sieve of Eratosthenes.
Find all the primes not exceeding a given integer N.
> restart:
> N := 1000:
> M := ceil(sqrt(N)):  #take advantage of Lemma 4.3.4
Initialize a vector p which keeps track of integers being
```

```
          "crossed out"
> for i to N do
>    p[i] := 1:   #p[i] = 0 when integer i is "crossed out"
> od:
> for n from 2 to M do
>    if (p[n]=1) then
>       i := 1:
>       while (i*n<=N) do
>          p[i*n] := 0:   # "cross out" all multiples of of n
>          i := i+1:
>       od:
>    fi:
> od:
Print the results
> j := 0:
> for i to N do
>    if (p[i] = 1) then
>       printf("%5d", i):
>       j := j+1:  #j = no of primes printed
>       if (j mod 10 = 0) then printf("\n")
                    #print 10 primes to a line
>       fi:
>    fi:
> od:
```

(b) Here are the primes less than 1000.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 |
| 179 | 181 | 191 | 193 | 197 | 199 | 211 | 223 | 227 | 229 |
| 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 |
| 353 | 359 | 367 | 373 | 379 | 383 | 389 | 397 | 401 | 409 |
| 419 | 421 | 431 | 433 | 439 | 443 | 449 | 457 | 461 | 463 |
| 467 | 479 | 487 | 491 | 499 | 503 | 509 | 521 | 523 | 541 |
| 547 | 557 | 563 | 569 | 571 | 577 | 587 | 593 | 599 | 601 |
| 607 | 613 | 617 | 619 | 631 | 641 | 643 | 647 | 653 | 659 |
| 661 | 673 | 677 | 683 | 691 | 701 | 709 | 719 | 727 | 733 |
| 739 | 743 | 751 | 757 | 761 | 769 | 773 | 787 | 797 | 809 |
| 811 | 821 | 823 | 827 | 829 | 839 | 853 | 857 | 859 | 863 |
| 877 | 881 | 883 | 887 | 907 | 911 | 919 | 929 | 937 | 941 |
| 947 | 953 | 967 | 971 | 977 | 983 | 991 | 997 | | |

(c) $\pi(1000)$ is the number of primes not exceeding 1000 which, by (b), is 168; $1000/\ln 1000 \approx 144.765$ and $\frac{\pi(1000)}{1000/\ln 1000} \approx 168/144.765 \approx 1.161$.

## Exercises 4.4

1. (a) [BB] $5, 12, 19, -2, -9, -16$ are in $\overline{5}$. $4, 11, 18, -3, -10, -17$ are in $\overline{-3}$.
   (b) [BB] The general element of $\overline{5}$ is an integer of the form $7k + 5$ for some integer $k$. The general form of an element in $\overline{-3}$ is an integer of the form $7k - 3$ for some integer $k$.

2. (a) $3, 16, 29, 42, -10, -23, -36, -49, -62$ are elements of $\overline{3}$.
   $11, 24, 37, 50, -2, -15, -28, -41, -54$ are elements of $\overline{-2}$.
   (b) The general integer in $\overline{3}$ is of the form $13k + 3$ for some integer $k$. The general integer in $\overline{-2}$ is of the form $13k - 2$ for some integer $k$.

3. (a) [BB] $1286 = 32(39) + 38$, so $1286 \pmod{39} = 38$.
   (b) $43{,}197 = 129(333) + 240$, so $43{,}197 \pmod{333} = 240$.
   (c) [BB] $-545{,}608 = -10{,}699(51) + 41$, so $-545{,}608 \pmod{51} = 41$.
   (d) $-125{,}617 = -399(315) + 68$, so $-125{,}617 \pmod{315} = 68$.
   (e) $11{,}111{,}111{,}111 = 10{,}001{,}000(1111) + 111$, so $11{,}111{,}111{,}111 \pmod{1111} = 111$.

4. (a) [BB] False: $18 - 2 = 16$ is not divisible by 10. (b) True: $-13 - 7 = -20$ is divisible by 5.
   (c) [BB] True: $44 - (-8) = 52$ is divisible by 13.
   (d) False: $423 - 17 = 406$ is divisible by 29, so $\overline{17} = \overline{423}$.
   (e) False: $400 - (-18) = 418$ is divisible by 19.

5. (a) [BB] $\overline{0} = \overline{3}, \overline{1} = \overline{4}, \overline{2} = \overline{5}$.
   (b) $\overline{0} = \overline{5}, \overline{1} = \overline{6}, \overline{2} = \overline{-3}, \overline{3} = \overline{-7}, \overline{4} = \overline{9}$.
   (c) $\overline{0} = \overline{8}, \overline{1} = \overline{9}, \overline{2} = \overline{18}, \overline{3} = \overline{-5}, \overline{4} = \overline{-4}, \overline{5} = \overline{-11}, \overline{6} = \overline{14}, \overline{7} = \overline{-1}$.

6. (a) [BB] Since $21{,}758{,}623 \equiv 1 \pmod 6$ and $17{,}123{,}055 \equiv 3 \pmod 6$,
   $21{,}758{,}623 + 17{,}123{,}055 \equiv 1 + 3 = 4 \pmod 6$.
   (b) Since $21{,}758{,}623 \equiv 1 \pmod 6$ and $17{,}123{,}055 \equiv 3 \pmod 6$,
   $(21{,}758{,}623)(17{,}123{,}055) \equiv 1(3) = 3 \pmod 6$.
   (c) Since $17{,}123 \equiv 5 \pmod 6$, $17{,}123^{50} \equiv 5^{50} \pmod 6$.
   Since $(5^2) = 25 \equiv 1 \pmod 6$, $5^{50} = (5^2)^{25} \equiv 1^{25} = 1 \pmod 6$.
   (d) $10^4 \equiv 3^4 = 9(9) \equiv 2(2) = 4 \pmod 7$ $\quad 10^8 = (10^4)^2 \equiv 4^2 \equiv 2 \pmod 7$
   $10^{12} = (10^4)(10^8) \equiv 4(2) \equiv 1 \pmod 7$ $\quad 10^{20} = (10^8)(10^{12}) \equiv 2(1) = 2 \pmod 7$
   $10^{24} = (10^{20})(10^4) \equiv 2(4) \equiv 1 \pmod 7$
   (e) [BB] $2, 4, 8, 5, 10, 9, 7, 3, 6, 1$
   (f) $4, 5, 9, 3, 1, 4, 5, 9, 3, 1$

7. (a) $a \equiv 8$ and $b \equiv 43 \pmod{85}$, so $a+b \equiv 51$, $ab \equiv 344 \equiv 4$ and $(a+b)^2 \equiv 2601 \equiv 51 \pmod{85}$.
   (b) $a \equiv 16$ and $b \equiv 42 \pmod{143}$, so $a + b \equiv 58$, $ab \equiv 672 \equiv 100$ and $(a + b)^2 \equiv 3364 \equiv 75 \pmod{143}$.
   (c) $a \equiv -9221 \equiv 779$ and $b \equiv 3450 \pmod{10{,}000}$, so $a + b \equiv 4229$ and $ab \equiv 2{,}687{,}550 \equiv 7550$ and $(a + b)^2 \equiv 17{,}884{,}441 \equiv 4441 \pmod{10{,}000}$.

8. [BB] Observe that $5^0 \equiv 1 \pmod 7$, $5^1 \equiv 5 \pmod 7$, $5^2 \equiv 4 \pmod 7$, $5^3 \equiv 6 \pmod 7$, $5^4 \equiv 2 \pmod 7$ and $5^5 \equiv 3 \pmod 7$. Thus, each of the integers $1, 2, 3, 4, 5, 6$ is congruent mod 7 to some power of 5. By Proposition 4.4.5, any integer $a$ is congruent mod 7 to one of $0, 1, 2, 3, 4, 5, 6$. So the result follows.

9. (a) [BB] No $x$ exists. The values of $3x$ mod 6 are 0 and 3.

   (b) $x = 2$, $x = 5$; (c) $x = 6$.

   (d) No $x$ exists. The values of $4x$ mod 6 are 0, 4 and 2.

   (e) [BB] If $50 \mid (2x - 18)$, then $2x - 18 = 50k$, so $x = 9 + 25k$ for some integer $k$. We obtain $x = 9$, $x = 34$.

   (f) [BB] By trial and error, $x = 9$, or using the method of Problem 25, write $11 - 2(5) = 1$, deducing that $-2(5) \equiv 1 \pmod{11}$ and so $x = -2(1) \equiv 9 \pmod{11}$.

   (g) If $25 \mid (5x - 5)$, then $5x - 5 = 25k$ so $x = 1 + 5k$ for some integer $k$. So $x = 1, 6, 11, 16, 21$.

   (h) No $x$ exists. If $592 \mid (4x - 301)$, then $4x - 301$ is even. This is impossible since, for any $x$, $4x$ is even.

   (i) No $x$ exists. If $65x \equiv 27 \pmod{169}$, then $169 \mid (65x - 27)$, so $13 \mid (65x - 27)$. Since, for any $x$, $13 \mid 65x$, this is impossible because $13 \nmid 27$.

   (j) If $4x \equiv 320 \pmod{592}$, then $592 \mid (4x - 320)$, so $4x = 320 + 592k$ for some $k$; that is, $x = 80 + 148k$. The values of $x$ are 80, 228, 376 and 524.

   (k) [BB] We use the method of Problem 25. Write $-5(595) + 186(16) = 1$, deduce $186(16) \equiv 1 \pmod{595}$ and obtain $x = 186(301) = 55{,}986 \equiv 56 \pmod{595}$.

   (l) We use the method of Problem 25. Write $36(722) - 329(79) = 1$, deduce $-329(79) \equiv 1 \pmod{722}$ and obtain $x = -329(15) = -4935 \equiv 119 \pmod{722}$.

   (m) Following the method outlined in Problem 25, we write $69(1404) - 625(155) = 1$. From this, we deduce that $-625(155) \equiv 1 \pmod{1404}$ and obtain $x = -625(1185) = -740{,}625 \equiv 687 \pmod{1404}$.

   (n) We want to find $x \pmod{78}$ such that $78 \mid (58x - 6)$, that is, $58x - 6 = 78k$ for some integer $k$. This says $29x - 3 = 39k$, so we first solve $29x \equiv 3 \pmod{39}$. We have $39(3) + 29(-4) = 1$, so $39(9) + 29(-12) = 3$. Thus $29x \equiv 3 \pmod{39}$ has the unique solution $x = -12 \equiv 27 \pmod{39}$. So our congruence has the two solutions $x \equiv 27 \pmod{78}$ and $x \equiv 66 \pmod{78}$ $(66 = 27 + 39)$.

10. (a) $(ax+b)(cx+d) \equiv 0 \pmod p$ implies $p \mid (ax+b)(cx+d)$. Since $p$ is prime, by Proposition 4.3.7, we conclude that $p \mid (ax + b)$ or $p \mid (cx + d)$. The first case says $ax + b \equiv 0 \pmod p$ and the second that $cx + d \equiv 0 \pmod p$, giving the desired result.

    (b) i. [BB] $(x + 2)(x - 2) \equiv 0 \pmod{13}$ implies $x + 2 \equiv 0 \pmod{13}$ or $x - 2 \equiv 0 \pmod{13}$ (using (a)). So $x \equiv \pm 2 \pmod{13}$. The solutions are $x = 2$, $x = 11$.

    ii. $(2x + 1)(3x + 4) \equiv 0 \pmod{17}$ implies $2x + 1 \equiv 0 \pmod{17}$ or $3x + 4 \equiv 0 \pmod{17}$ (using (a)). So $2x \equiv -1 \pmod{17}$ or $3x \equiv -4 \pmod{17}$. The first congruence yields $x = 8$ and the second $x = 10$.

    iii. $3x^2 + 14x - 5 = (3x - 1)(x + 5) \equiv 0 \pmod{97}$ implies $3x - 1 \equiv 0 \pmod{97}$ or $x + 5 \equiv 0 \pmod{97}$ (by (a)). The first congruence yields $x = 65$ and the second $x = 92$.

    iv. [BB] No $x$ exists. The values of $x^2 \pmod 6$ are 0, 1, 3 and 4.

    v. $x = 2$ or $x = 4$.

vi. No $x$ exists. The values of $4x^2 + 3x + 7 \pmod 5$ are 2, 3 and 4.

11. (a) [BB] Multiplying the second congruence by 2 gives $2x \equiv 0 \pmod 6$ and, hence, $x \equiv 0$ or $x \equiv 3 \pmod 6$. If $x \equiv 0$, then the first congruence says $y \equiv 1$ and it is easily checked that $x = 0, y = 1$ satisfies both congruences. If $x \equiv 3$, then the first congruence says $y \equiv 1$, but these values for $x$ and $y$ do not satisfy the second congruence. There is just one solution mod 6; namely, $x = 0$, $y = 1$.

    (b) [BB] Subtracting the first congruence from the second gives $3x \equiv -2 \equiv 7 \pmod 9$. But the values of $3x$ mod 9 are just 0, 3 and 6. There is no solution.

    (c) Subtracting the first congruence from the second gives $3x \equiv -2 \equiv 6 \pmod 8$ and, hence, $x \equiv 2$. The first congruence then says $2 + 5y \equiv 3$; that is, $5y \equiv 1 \pmod 8$. So $y \equiv 5 \pmod 8$. One checks easily that $x = 2$, $y = 5$ is a solution to the given congruences.

    (d) Multiply the first congruence by 2 to get $14x + 4y \equiv 6 \pmod{15}$. Subtracting the second congruence gives $5x \equiv 0 \pmod{15}$ so $x = 0, 3, 6, 9$ or 12.

    If $x = 0$, $2y \equiv 3$, so $y = 9$.

    If $x = 3$, $2y \equiv 3 - 21 = -18 \equiv -3 \equiv 12$ so $y = 6$.

    If $x = 6$, $2y \equiv 3 - 42 = -39 \equiv -9 \equiv 6$, so $y = 3$.

    If $x = 9$, $2y \equiv 3 - 63 = -60$, so $y = 0$.

    If $x = 12$, $2y = 3 - 84 = -81 \equiv 9$, so $y = 12$.

    (e) Multiplying the first congruence by 5 and the second by 3 gives

$$\begin{aligned} 15x + 25y &\equiv 70 \equiv 14 \pmod{28} \\ 15x + 27y &\equiv 18 \pmod{28} \end{aligned}$$

    and so, subtracting, $2y \equiv 4 \pmod{28}$. Thus, $y = 2$ or $y = 16$.

    If $y = 2$, $3x \equiv 14 - 10 = 4$, so $x = 20$.

    If $y = 16$, $3x \equiv 14 - 80 = -66 \equiv -10 \equiv 18$, so $x = 6$.

12. [BB] Since $\gcd(c, n) = 1$, we have $cx + ny = 1$ for some integers $x$ and $y$. We also know that $ac - bc = nk$ for some integer $k$. So $(a - b)cx = nkx$; that is, $(a - b)(1 - ny) = nkx$, or, $a - b = n(kx + (a - b)y)$. Thus, $n \mid (a - b)$ and $a \equiv b \pmod n$, as desired.

13. [BB] Since $n \mid (a - b)$, we have $a - b = qn$ for some integer $q$, hence, $a - qn = b$. Let $g = \gcd(a, n)$. Then $g \mid a$ and $g \mid n$. Therefore, $g \mid b$ (and $g \mid n$), so $g \mid \gcd(b, n)$. Similarly $\gcd(b, n) \mid g$ and so these natural numbers are equal.

14. (a) [BB] False! For example, with $a = b = 1$ and $n = 4$, we have $a \equiv b \pmod 4$, but $3a = 3$, $b^2 = 1$ so $3 \not\equiv 1 \pmod 4$.

    (b) True. Either apply Proposition 4.4.7: if $a \equiv a \pmod n$ and $b \equiv b \pmod n$, then $a(a) \equiv b(b) \pmod n$ or give a direct proof such as the following. Since $n \mid (a - b)$, $n \mid (a - b)(a + b)$; that is, $n \mid (a^2 - b^2)$.

    (c) False! For example, with $a = 5$, $b = 2$ and $n = 3$, we have $a \equiv b \pmod 3$, but $a^2 = 25 \equiv 1 \pmod 3$ and $b^3 = 8 \equiv 2 \pmod 3$ so $a^2 \not\equiv b^3 \pmod 3$.

    (d) False! For example, with $n = 3$, $a = 4$, $b = 1$, we have $4 \equiv 1 \pmod 3$ but $16 \not\equiv 1 \pmod 9$.

15. (a) [BB] This is false. Consider $r = 5$, $s = 1$, $a = 2$, $n = 4$. Then $5 \equiv 1 \pmod 4$, but $2^5 \not\equiv 2^1 \pmod 4$ because $2^5 - 2^1 = 30$ and $4 \nmid 30$.

(b) This is true. From the identity $r^a - s^a = (r-s)(r^{a-1} + r^{a-2}s + r^{a-3}s^2 + \cdots rs^{a-2} + s^{a-1})$, we see that $n \mid (r^a - s^a)$.

16. Following the hint, we note that $10 \equiv -1 \pmod{11}$ and so $10^k \equiv (-1)^k \pmod{11}$ for any natural number $k$. Since $(a_{n-1}a_{n-2}\ldots a_3a_2a_1a_0)_{10} = 10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \cdots + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$, we have

$$(a_{n-1}a_{n-2}\ldots a_3a_2a_1a_0)_{10} \equiv (-1)^{n-1}a_{n-1} + (-1)^{n-2}a_{n-2} + \cdots - a_3 + a_2 - a_1 + a_0 \pmod{11}$$

and this is $0 \pmod{11}$ if and only if $a_0 + a_2 + \cdots \equiv a_1 + a_3 + \cdots \pmod{11}$.

17. (a) [BB] Suppose $\sqrt{2} = \frac{a}{b}$ for some integers $a$ and $b$. If $a$ and $b$ have any factors in common, these can be canceled leaving us with an equation of the form $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ have no common factors (except $\pm 1$). So we now make this assumption. Then $a^2 = 2b^2$, so $a^2 + b^2 = 3b^2 \equiv 0 \pmod{3}$. By Problem 23, $a$ and $b$ have 3 as a common factor, which is not true.

(b) As in (a), we assume $\sqrt{n} = a/b$ where $a$ and $b$ are integers with no common factors except $\pm 1$. Then $a^2 = nb^2 \equiv 2b^2 \pmod{3}$, so $a^2 + b^2 \equiv 3b^2 \equiv 0 \pmod{3}$. By Problem 23, $3 \mid a$ and $3 \mid b$, which is not true.

18. (a) i. Testing values gives $x = 1$ or $x = 4$.
ii. Testing values gives $x = 1$ or $x = 6$.
iii. Testing values gives $x = 1$ or $x = 12$.

(b) [BB] If $x^2 \equiv 1 \pmod{p}$, then $p \mid (x^2 - 1)$; that is, $p \mid (x+1)(x-1)$. By Proposition 4.3.7, $p \mid (x+1)$ or $p \mid (x-1)$; that is, $x \equiv -1 \pmod{p}$ or $x \equiv 1 \pmod{p}$. Thus, $x = 1$ and $x = p - 1$ are the desired solutions.

19. (a) i. [BB] We have $25 \mid (x^2 - 1)$, so $25 \mid (x-1)(x+1)$. Note that it is not possible to have $5 \mid (x-1)$ and $5 \mid (x+1)$, for then we would have $5 \mid [(x+1) - (x-1)]$ which is not true. Thus, $25 \mid (x+1)$ or $25 \mid (x-1)$. The solutions are $x = 1$ and $x = 24$.
ii. The same argument as in i. gives $x = 1$ and $x = 124$.
iii. The same argument as in i. gives $x = 1$ and $x = 48$.

(b) If $x^2 \equiv 1 \pmod{p^k}$, then $p^k \mid (x^2 - 1)$; so $p^k \mid (x+1)(x-1)$. We claim that $p^k \mid (x+1)$ or $p^k \mid (x-1)$, hence, that $x \equiv \pm 1 \pmod{p^k}$. But if $p^k$ divides neither $x+1$ nor $x-1$, then $p \mid (x+1)$ and $p \mid (x-1)$, so $p \mid [(x+1) - (x-1)]$. Thus, $p \mid 2$, so $p = 2$, a contradiction. The answers are $x = 1$ and $x = p^k - 1$.

20. (a) i. $x = 1$
ii. Testing values gives $x = 1$ and $x = 3$.
iii. Testing values gives $x = 1$, $x = 3$, $x = 5$ and $x = 7$.
iv. Testing values gives $x = 1$, $x = 7$, $x = 9$ and $x = 15$.

(b) If $x^2 \equiv 1 \pmod{2^k}$, then $2^k \mid (x+1)(x-1)$. When $k = 1$, we must have $2 \mid (x+1)$ or $2 \mid (x-1)$ by Proposition 4.3.7; hence, $x \equiv \pm 1$, $x = 1$.

When $k = 2$, we have $4 \mid (x+1)(x-1)$. Clearly $x$ is not even, for otherwise, both $x+1$ and $x-1$ and, hence, $(x+1)(x-1)$ would be odd (and not divisible by 4). So $x$ is odd, that is, $x \equiv \pm 1 \pmod{4}$, and again we have only the obvious solutions $x \equiv \pm 1 \pmod{4}$; that is, $x = 1$, $x = 3$.

The most complex situation occurs when $k \geq 3$, which we assume henceforth. Since $2 \mid (x+1)(x-1)$, we must have $x$ odd, as before. Since $k \geq 3$ and $2^k \mid (x+1)(x-1)$, some power $2^\ell$ of 2, with $\ell \geq 2$, must divide either $x+1$ or $x-1$. Suppose $2^\ell \mid (x-1)$ with $\ell \geq 2$. Then $x+1 = 2^\ell a + 2 = 2(1+2^\ell a)$, and since $\ell \geq 2$, this means that $2 \mid (x+1)$, but $4 \nmid (x+1)$. It follows that $\ell \geq k-1$. If $\ell \geq k$, we again have the obvious solution $x \equiv 1 \pmod{2^k}$. However, we have a new possibility; namely, $\ell = k-1$; that is, $x \equiv 1 \pmod{2^{k-1}}$. In this case,

$$(x+1)(x-1) = (2^{k-1}m)(2^{k-1}m+2) = 2^k(m(1+2^{k-2}m))$$

is divisible by $2^k$, so we have a solution. Arguing similarly if $2^\ell \mid (x+1)$, we obtain the solutions

$$x = 1, \quad x = 2^{k-1}-1, \quad x = 2^{k-1}+1, \quad x = 2^k - 1.$$

21. (a) Since $7(8) = 56 \equiv 1 \pmod{11}$, the inverse of 7 (mod 11) is 8. The solution to $7x \equiv 3 \pmod{11}$ is $x = 7^{-1}(3) = 8(3) = 24 \equiv 2 \pmod{11}$.

    (b) Since $15(3) = 45 \equiv 1 \pmod{22}$, the inverse of 15 (mod 22) is 3. The solution to $15x \equiv 9 \pmod{22}$ is $x = 15^{-1}(9) = 3(9) = 27 \equiv 5 \pmod{22}$.

    (c) Since $32(5) = 160 \equiv 1 \pmod{53}$, the inverse of 32 (mod 53) is 5. The solution to $32x \equiv 19 \pmod{53}$ is $x = 32^{-1}(19) = 5(19) = 95 \equiv 42 \pmod{53}$.

    (d) Since $8(14) = 112 \equiv 1 \pmod{111}$, the inverse of 8 (mod 111) is 14. The solution to $8x \equiv 28 \pmod{111}$ is $x = 8^{-1}(28) = 14(28) = 392 \equiv 59 \pmod{111}$.

22. (a) [BB] By Fermat's Little Theorem, $18^{8970} \equiv 1 \pmod{8971}$. So $18^{8971} \equiv 18$ and $18^{8972} \equiv 18^2 = 324 \pmod{8971}$.

    (b) By Fermat's Little Theorem, $53^{20,592} \equiv 1 \pmod{20,593}$.
    So $53^{20,593} \equiv 53$ and $53^{20,594} \equiv 53^2 = 2809 \pmod{20,593}$.

    (c) By Fermat's Little Theorem, $3^{508} \equiv 1 \pmod{509}$. So $3^{509} \equiv 3$ and $3^{512} \equiv 3^4 = 81 \pmod{509}$.

    (d) By Fermat's Little Theorem, $6^{10,588} \equiv 1 \pmod{10,589}$. So $6^{10,589} \equiv 6$ and $6^{10,594} \equiv 6^6 = 46,656 \equiv 4300 \pmod{10,589}$.

    (e) By Fermat's Little Theorem, $8^{4056} \equiv 1 \pmod{4057}$. So $8^{4058} \equiv 8^2 = 64$ and $8^{4060} \equiv 8^4 = 4096 \equiv 39 \pmod{4057}$.

    (f) By Fermat's Little Theorem, $2^{3942} \equiv 1 \pmod{3943}$. So $2^{3948} \equiv 2^6 = 64 \pmod{3943}$. Since $(2^{3941})2 \equiv 1$ and $2(1971) = 3942 \equiv 1 \pmod{3943}$, $2(2^{3941} - 1971) \equiv 0$. Since $\gcd(2, 3943) = 1$, $2^{3941} - 1971 \equiv 0$; that is, $2^{3941} \equiv 1971 \pmod{3943}$.

23. [BB] The key to this problem is the observation that 97 is a prime. Thus, by Fermat's Little Theorem, if $x \not\equiv 0 \pmod{97}$, then $x^{97} \equiv x \pmod{97}$, so $x^{97} - x + 1 \equiv 1 \not\equiv 0 \pmod{97}$. Since $x \equiv 0 \pmod{97}$ is also not a solution, there are no solutions at all.

24. (a) In this case, $\preceq$ is a partial order.

    **Reflexivity:** For any $\overline{a} \in A$, $\overline{a} \preceq \overline{a}$ because $a(a) \equiv a^2 \pmod{p}$.

    **Antisymmetry:** If $\overline{a}, \overline{b} \in A$, $\overline{a} \preceq \overline{b}$ and $\overline{b} \preceq \overline{a}$, then $ab \equiv a^2 \pmod{p}$ and $ba \equiv b^2 \pmod{p}$. Thus, $a(a-b) \equiv 0 \pmod{p}$ and $b(a-b) \equiv 0 \pmod{p}$. It follows that $(a-b)(a-b) = a(a-b) - b(a-b) \equiv 0 \pmod{p}$ and, hence, that $p \mid (a-b)^2$. So, by Proposition 4.3.7, $p \mid (a-b)$; that is, $a \equiv b \pmod{p}$ and $\overline{a} = \overline{b}$ as required.

**Transitivity:** Suppose $\overline{a}, \overline{b}, \overline{c} \in A$, $\overline{a} \preceq \overline{b}$ and $\overline{b} \preceq \overline{c}$; that is, $ab \equiv a^2 \pmod{p}$ and $bc \equiv b^2 \pmod{p}$. We want to prove that $\overline{a} \preceq \overline{c}$, that is, that $ac \equiv a^2 \pmod{p}$ or, equivalently, that $p \mid (ac - a^2)$. Now we have that $(ab)(bc) \equiv a^2b^2 \pmod{p}$ and, hence, that $p \mid b^2(ac - a^2)$. Again, by Proposition 4.3.7, $p \mid b^2$ or $p \mid (ac - a^2)$. The latter case is what we want. In the former case, we must have $p \mid b$, hence, that $ab \equiv 0 \pmod{p}$. Therefore, $a^2 \equiv ab \equiv 0$, so $a \equiv 0$ and again $ac - a^2 \equiv 0$, the desired result.

(b) This is a partial order.

**Reflexivity:** For any $\overline{a} \in A$, $\overline{a} \preceq \overline{a}$ because $a(a) \equiv a^2 \pmod{pq}$.

**Antisymmetry:** If $\overline{a}, \overline{b} \in A$, $\overline{a} \preceq \overline{b}$ and $\overline{b} \preceq \overline{a}$, then $ab \equiv a^2 \pmod{pq}$ and $ba \equiv b^2 \pmod{pq}$. Thus, $ab \equiv a^2 \pmod{p}$ and $ba \equiv b^2 \pmod{p}$ and, as in (a), we get $a \equiv b \pmod{p}$. Similarly, we obtain $a \equiv b \pmod{q}$. So we have $a - b$ divisible by both $p$ and $q$. By Exercise 14 of Section 4.3, $pq \mid (a - b)$, giving $a \equiv b \pmod{pq}$ as desired.

**Transitivity:** Suppose $\overline{a}, \overline{b}, \overline{c} \in A$, $\overline{a} \preceq \overline{b}$ and $\overline{b} \preceq \overline{c}$; that is, $ab \equiv a^2 \pmod{pq}$ and $bc \equiv b^2 \pmod{pq}$. Then $ab \equiv a^2 \pmod{p}$ and $bc \equiv b^2 \pmod{p}$ so, as in (a), $ac \equiv a^2 \pmod{p}$. Similarly, $ac \equiv a^2 \pmod{q}$ and so, by Exercise 14 of Section 4.3 again, we get $ac \equiv a^2 \pmod{pq}$ as desired.

(c) This is **not** a partial order because it is not antisymmetric. In the case $n = 12$, $\overline{0} \preceq \overline{6}$ because $0(6) \equiv 0^2 \pmod{12}$ and $\overline{6} \preceq \overline{0}$ because $6(0) \equiv 6^2 \pmod{12}$, but $\overline{0} \neq \overline{6}$. In general, if $n = p^2m$ for some prime $p$, then $\overline{0} \preceq \overline{pm}$ because $0(pm) \equiv 0^2 \pmod{n}$ and $\overline{pm} \preceq \overline{0}$ because $(pm)0 \equiv (pm)^2 \pmod{n}$ $((pm)^2 = nm)$, but $\overline{0} \neq \overline{pm}$.

## Exercises 4.5

1. (a) [BB] This number is valid because $1(0) + 2(1) + 3(2) + 4(3) + 5(4) + 6(5) + 7(6) + 8(7) + 9(8) + 10(9) = 330 \equiv 0 \pmod{11}$.

   (b) [BB] This number is not valid because $1(0) + 2(4) + 3(3) + 4(2) + 5(0) + 6(9) + 7(1) + 8(0) + 9(5) + 10(5) = 181 \equiv 5 \not\equiv 0 \pmod{11}$.

   (c) This number is not valid because $1(1) + 2(6) + 3(6) + 4(7) + 5(1) + 6(3) + 7(2) + 8(4) + 9(2) + 10(6) = 206 \equiv 8 \not\equiv 0 \pmod{11}$.

   (d) This number is valid because $1(9)+2(4)+3(1)+4(2)+5(8)+6(8)+7(3)+8(1)+9(9)+10(6) = 286 \equiv 0 \pmod{11}$.

   (e) This number is valid because $1(3)+2(4)+3(9)+4(2)+5(1)+6(6)+7(6)+8(2)+9(7)+10(10) = 308 \equiv 0 \pmod{11}$.

2. (a) [BB] We must have

$$1(3) + 2(4) + 3(1) + 4(6) + 5(1) + 6(0) + 7(9) + 8(2) + 9(7) + 10A \equiv 0 \pmod{11};$$

   hence, $185 + 10A \equiv 0 \pmod{11}$, so $A = 9$.

   (b) We must have

$$1(A) + 2(4) + 3(6) + 4(1) + 5(2) + 6(2) + 7(8) + 8(3) + 9(7) + 10(4) \equiv 0 \pmod{11};$$

   hence, $235 + A \equiv 0 \pmod{11}$, $4 + A \equiv 0 \pmod{11}$, $A = 7$.

(c) We must have

$$1(9) + 2(1) + 3(2) + 4(3) + 5(A) + 6(4) + 7(5) + 8(5) + 9(1) + 10(10) \equiv 0 \pmod{11};$$

hence, $5A + 237 \equiv 0 \pmod{11}$, $5A + 6 \equiv 0 \pmod{11}$, $5A \equiv 5 \pmod{11}$ and $A = 1$.

(d) [BB] We must have

$$1(2) + 2(7) + 3(2) + 4(9) + 5(1) + 6(8) + 7(8) + 8(A) + 9(6) + 10(2) \equiv 0 \pmod{11};$$

hence, $8A + 241 \equiv 0 \pmod{11}$, $8A + 10 \equiv 0 \pmod{11}$, $8A \equiv 1 \pmod{11}$ and $A = 7$.

(e) We must have

$$1(8) + 2(9) + 3(A) + 4(5) + 5(3) + 6(9) + 7(8) + 8(2) + 9(8) + 10(4) \equiv 0 \pmod{11};$$

hence, $3A + 299 \equiv 0 \pmod{11}$, $3A + 2 \equiv 0 \pmod{11}$, $3A \equiv -2 \equiv 9 \pmod{11}$ and $A = 3$.

3. The first digit $A$ would have to satisfy

$$1(A) + 2(3) + 3(1) + 4(5) + 5(2) + 6(6) + 7(6) + 8(7) + 9(8) + 10(2) \equiv 0 \pmod{11};$$

hence, $A + 265 \equiv 0 \pmod{11}$, $A + 1 \equiv 0 \pmod{11}$, and there is no $A$ in the range 0–9 with this property, so no such ISBN number exists.

4. [BB] $a_1 + 2a_2 + \cdots + 9a_9 + 10a_{10} \equiv 0 \pmod{11}$

$$\leftrightarrow a_1 + 2a_2 + \cdots + 9a_9 \equiv -10a_{10} \leftrightarrow a_1 + 2a_2 + \cdots + 9a_9 \equiv a_{10}$$

since $-10a_{10} \equiv a_{10} \pmod{11}$.

5. (a) Suppose the ISBN number $a_1a_2 \cdots a_{10}$ is correct. Thus $a = a_1 + 2a_2 + \cdots + ia_i + \cdots + 10a_{10} \equiv 0 \pmod{11}$. If digit $a_i$ is miscopied as $x$, then the test computes $b = a_1 + 2a_2 + \cdots + ix + \cdots + 10a_{10}$. The difference $a - b = ia_i - ix = i(a_i - x)$, so $b \equiv i(x - a_i) \pmod{11}$. Now $11 \mid i(a_i - x)$ implies $11 \mid i$ or $11 \mid (a_i - x)$ (because 11 is prime). Since $1 \leq i \leq 10$, the first possibility cannot occur. Since the numbers $a_i$ and $x$ are in the range 0–10, $a_i \not\equiv x \pmod{11}$ and $b \not\equiv 0 \pmod{11}$ (so the test picks up the error) unless $x = a_i$, in which case there was no error.

   (b) The answer is no. From part (a), if an error is made in digit $a_i$, the test computes $b \equiv i(x - a_i)$ and this could be the same for different values of $i$; for example, 11 . . . could be miscopied as 21 . . . or 17 . . .. In the first case, $b = 1(2 - 1) = 1 \pmod{11}$ while, in the second, $b = 2(7 - 1) = 12 \equiv 1 \pmod{11}$ too.

6. (a) [BB] For example, 2-41971<u>3</u>-29-0 is not valid because $a_1 + 2a_2 + \cdots + 10a_{10} = 208 \not\equiv 0 \pmod{11}$.

   (b) For example, 2-41973<u>3</u>-29-0 is valid because $a_1 + 2a_2 + \cdots + 10a_{10} = 220 \equiv 0 \pmod{11}$.

   (c) Given an ISBN with digits $a_1, a_2, \ldots, a_{10}$, we show that any two digits $a_i$, $a_j$ can be changed so that the result is still a valid ISBN.

   Let $x$ be any integer such that $a_i + x \not\equiv 10 \pmod{11}$. (This ensures that $a_i + x \pmod{11}$ is a (single) digit.) Replacing $a_i$ by the digit $a_i + x \pmod{11}$ adds $ix \pmod{11}$ to the sum $a_1 + 2a_2 + \cdots + 10a_{10}$. Since 11 is prime, there exists an integer $y$ such that $jy \equiv -ix \pmod{11}$. Note that different values of $x \pmod{11}$ lead to different values of $y \pmod{11}$. Hence, we can find $x$ and $y$ such that $a_i + x \not\equiv 10 \pmod{11}$, $a_j + y \not\equiv 10 \pmod{11}$ and $jy \equiv -ix \pmod{11}$. Replacing $a_i$ by $a_i + x$ and $a_j$ by $a_j + y$ (both mod 11) gives the desired result.

7. (a) [BB] The given rule is $\mathsf{w} \cdot \mathsf{a} \equiv 0 \pmod{11}$ for $\mathsf{a} = (a_1, a_2, \ldots, a_{10})$ and $\mathsf{w} = (1,1,1,1,1,1,1,1,1,1)$.

   (b) Suppose an error is made in the $i$th digit of an ISBN number so that instead of $a_1 a_2 \cdots a_i \cdots a_{10}$, the number appears as $a_1 a_2 \cdots a_i' \cdots a_{10}$, with $a_i' \neq a_i$. Then $0 \equiv a_1 + a_2 \cdots + a_i + \cdots + a_{10} \not\equiv a_1 + a_2 + \cdots + a_i' + \cdots + a_{10}$ because

$$[a_1 + a_2 + \cdots + a_i + \cdots + a_{10}] - [a_1 + a_2 + \cdots + a_i' + \cdots + a_{10}] = a_i - a_i' \not\equiv 0 \pmod{11}$$

   since $a_i$ and $a_i'$ are digits between 0 and 9.

   (c) Suppose $a_1 a_2 \cdots a_{10}$ is a correct ISBN number. The transposition of $a_i$ and $a_{i+1}$ cannot be detected because when we apply our test to the incorrect number $a_1 \cdots a_{i+1} a_i \cdots a_{10}$, we compute $a_1 + \cdots + a_{i+1} + a_i + \cdots + a_{10}$ which is, of course, congruent to 0 (mod 11) because it's the exactly the same as $a_1 + \cdots + a_i + a_{i+1} + \cdots + a_{10}$.

8. (a) [BB] 1-13579-02468-8; (b) 1-02468-13579-6; (c) 0-62608-39349-6;
   (d) [BB] 0-63042-00635-5; (e) 0-63000-16425-7; (f) 8-29447-10493-1.

9. (a) [BB] Not valid: $3(0+2+4+6+8+0)+(1+3+5+7+9+1) = 60+26 = 86 \not\equiv 0 \pmod{10}$.

   (b) Not valid: $3(1+6+2+7+5+1)+(6+3+6+3+5+5) = 66+28 = 94 \not\equiv 0 \pmod{10}$.

   (c) [BB] Valid: $3(2+2+9+1+3+4)+(5+9+8+7+9+9) = 63+47 = 110 \equiv 0 \pmod{10}$.

   (d) Valid: $3(9+3+8+2+6+7)+(5+8+9+2+8+3) = 105+35 = 140 \equiv 0 \pmod{10}$.

   (e) Not valid: $3(8+1+8+1+3+4)+(4+2+5+9+8+2) = 75+30 = 105 \not\equiv 0 \pmod{10}$.

10. (a) [BB] We need $3(0+2+8+2+1+9)+(1+x+9+9+0+4) \equiv 0 \pmod{10}$; that is, $66+(23+x) \equiv 0$, so $x = 1$.

    (b) We need $3(1+9+4+4+x+6)+(2+3+7+9+2+8) \equiv 0 \pmod{10}$; that is, $72+3x+31 \equiv 0$, so $3x \equiv -103 \equiv 7$ and $x = 9$.

    (c) We need $3(4+9+1+1+2+8)+(2+2+7+0+5+x) \equiv 0 \pmod{10}$; that is, $75+(16+x) \equiv 0$, so $x = 9$.

    (d) We need $3(5+1+5+x+3+2)+(9+0+7+9+3+2) \equiv 0 \pmod{10}$; that is, $48+3x+30 \equiv 0$, so $3x \equiv -78 \equiv 2$ and $x = 4$.

    (e) We need $3(x+9+5+9+9+7)+(7+1+4+1+3+6) \equiv 0 \pmod{10}$; that is, $3x+117+22 \equiv 0$, so $3x \equiv -139 \equiv 1$ and $x = 7$.

11. (a) [BB] Suppose the sum of the digits in the odd positions is $a$ and the sum of the digits in the even positions is $b$. If the check digit in our bar code is correct, $3a + b \equiv 0 \pmod{10}$. An error in an even position digit changes $b$ to $c \not\equiv b \pmod{10}$. For the new number, we compute $3a + c \not\equiv 3a + b$ and thus, $3a + c \not\equiv 0 \pmod{10}$. So we know an error has been made.

    (b) [BB] Here is a valid number: 3-12498-66132-9. Changing the 1 and 4 in positions two and four to 8 and 7 respectively produces the number 3-82798-66132-9 which is also valid.

12. (a) Suppose the sum of the digits in the odd positions is $a$ and the sum of the digits in the even positions is $b$. If the check digit in our bar code is correct, $3a + b \equiv 0 \pmod{10}$. An error in an odd position digit changes $a$ to $c \not\equiv a \pmod{10}$. Note that $3c \not\equiv 3a \pmod{10}$ since 3 and 10 are relatively prime. Thus, when we compute $3c + b$ for the new number, we have $3c + b \not\equiv 3a + b$ and, thus, $3c + b \not\equiv 0 \pmod{10}$. So we know an error has been made.

(b) Here is a valid number: 3-12498-66132-9. Changing the 3 and 2 in positions one and three to 8 and 7 respectively produces the number 8-17498-66132-9 which is also valid.

13. (a) The test will not identify the digit that is wrong: An error of the same magnitude in two even or two odd digits will yield the same $3a+b$. For example, if a universal product number begins 1234, either incorrect number 3234 or 1254 will yield the same $3a + b$.

(b) The test will not even identify whether the error is in an odd position or in an even position since, for example, a difference of 1 in an odd position will yield the same $3a + b$ as a difference of 3 in an even position. For example, if a universal product number begins 1234, the number $3a + b$ is the same for the incorrect numbers 2234 and 1534.

14. (a) [BB] 1-23586-98732-6; (b) [BB] ] 1-23006-98732-6.

15. Suppose that $a_1$-$a_2a_3a_4a_5a_6$-$a_7a_8a_9a_{10}a_{11}$-$x$ is a valid universal product code. Thus,

$$A = 3(a_1 + a_3 + a_5 + a_7 + a_9 + a_{11}) + (a_2 + a_4 + a_6 + a_8 + a_{10} + x) \equiv 0 \pmod{10}$$

If $a_2$ and $a_3$ are transposed, the error will not be detected provided

$$B = 3(a_1 + a_2 + a_5 + a_7 + a_9 + a_{11}) + (a_3 + a_4 + a_6 + a_8 + a_{10} + x) \equiv 0 \pmod{10}$$

Now $A - B = 3(a_3 - a_2) + (a_2 - a_3) = 2(a_3 - a_2)$. So, if $a_3 - a_2 = 5$, we'll have $A - B \equiv 0$, hence, $B \equiv A \equiv 0 \pmod{10}$ and the transposition will not be detected. We conclude that the test given by (7) does **not** detect errors due to the transposition of adjacent digits.

16. (a) Let $\mathsf{a} = (a_1, a_2, \ldots, a_{12})$ and $\mathsf{w} = (12, 11, 10, \ldots, 2, 1)$.

(b) [BB] Single-digit errors are not detected. Since $10a_3 \equiv 0 \pmod{10}$, the third digit can in fact be changed arbitrarily and the new number will still pass the test.

(c) Errors due to transposition of consecutive digits will be detected. To see why, suppose $a_1a_2\cdots a_{12}$ satisfies the test. Hence,

$$A = 12a_1 + 11a_2 + \cdots + na_i + (n-1)a_{i+1} + \cdots + 2a_{11} + a_{12} \equiv 0 \pmod{10}$$

Suppose $a_i$ and $a_{i+1}$ are transposed. The test of the new number requires the calculation of

$$B = 12a_1 + 11a_2 + \cdots + na_{i+1} + (n-1)a_i + \cdots + 2a_{11} + a_{12} \equiv 0 \pmod{10}$$

Now $A - B = n(a_i - a_{i+1}) + (n-1)(a_{i+1} - a_i) = a_i - a_{i+1}$ and so, assuming $a_i \neq a_{i+1}$, $B \not\equiv 0 \pmod{10}$ and the error will be detected.

(d) Generalizing the calculation of part (c), we see that if digits $a_i$ and $a_j$ are transposed, then the difference between the correct number and incorrect numbers is $(j-i)(a_i - a_j)$. Such an error will, therefore, go unnoticed whenever $(j-i)(a_i - a_j) \equiv 0 \pmod{10}$, for instance, $i = 1, j = 3, a_i = 5, a_j = 0$, that is, numbers of the form 5-a0bcd-efghi-j.

17. (a) [BB] If $n \leq 9$, then any $a_i$ in the range $0 \leq a_i \leq 9 - n$ could be replaced by $n + a_i$ and the test would miss the error.

(b) Suppose $a_1 \ldots a_i \ldots a_k$ is miscopied and written as $a_1 \ldots x \ldots a_k$. Let $\mathsf{a} = (a_1, \ldots, a_k)$ and $\mathsf{a}' = (a_1, \ldots, x, \ldots, a_k)$. Suppose $\mathsf{w} \cdot \mathsf{a}' = \mathsf{w} \cdot \mathsf{a}$. This means that $w_i(a_i - x) \equiv 0 \pmod{n}$. Since $w_i$ is relatively prime to $n$, it follows that $a_i \equiv x \pmod{n}$ forcing $a_i = x$ since $n > 9$.

(c) Suppose the digits $a_i$ and $a_j$ of $a_1 \dots a_k$ are transposed. (Without loss of generality, assume $i < j$.) Let $\mathsf{a} = (a_1, \dots, a_i, \dots, a_j, \dots a_k)$ and $\mathsf{a}' = (a_1, \dots, a_j, \dots, a_i, \dots a_k)$. If $\mathsf{w} \cdot \mathsf{a}' = \mathsf{w} \cdot \mathsf{a}$, then $w_i a_j + w_j a_i = w_i a_i + w_j a_j$, so $(w_i - w_j)(a_j - a_i) \equiv 0 \pmod{n}$. The assumption tells us that $a_j \equiv a_i \pmod{n}$ and, since $n > 9$, we conclude that $a_j = a_i$.

18. (a) [BB] $1 = (-4)5 + 3(7)$, so $x = 4(-4)(5) + 3(3)(7) = -17 \equiv 18 \pmod{35}$. Therefore, $x = 18$.

(b) $1 = (-2)4 + 1(9)$, so $x = 8(-2)(4) + 1(1)(9) = -55 \equiv 17 \pmod{36}$. Therefore, $x = 17$.

(c) $1 = (-3)5 + 2(8)$, so $x = 7(-3)(5) + 3(2)(8) = -57 \equiv 23 \pmod{40}$. Therefore, $x = 23$.

(d) [BB] $1 = (-3)8 + 1(25)$, so $x = 17(-3)(8) + 6(1)(25) = -258 \equiv 142 \pmod{200}$. Therefore, $x = 142$.

(e) $1 = 48(1917) - 239(385)$, so $x = 75(48)(1917) - 3(239)(385) = 6{,}625{,}155 \equiv 720{,}795 \pmod{738{,}045}$. Therefore, $x = 720{,}795$.

(f) $1 = -2647(17369) + 8402(5472)$, so

$$\begin{aligned} x &= 2974(-2647)(17369) + 1003(8402)(5472) \\ &= 46{,}113{,}671{,}232 - 136{,}731{,}859{,}682 = -90{,}618{,}188{,}450 \end{aligned}$$

Then, since $-90{,}618{,}188{,}450 = -954(95{,}043{,}168) + 52{,}993{,}822$, we have $x = 52{,}993{,}822$.

(g) First we solve $x \equiv 1 \pmod{4}$, $x \equiv 8 \pmod{9}$. Since $1 = 9 - 2(4)$, $x = 1(9) - 8(2)(4) = -55 \equiv 17 \pmod{36}$. Then we solve $x \equiv 17 \pmod{36}$, $x \equiv 10 \pmod{25}$. Since $-9(36) + 13(25) = -324 + 325 = 1$, $x = 10(-324) + 17(325) = 2285 \equiv 485 \pmod{900}$. Therefore, $x = 485$.

19. (a) [BB] We wish to solve the system of congruences

$$\begin{aligned} x &\equiv 1 \pmod{4} \\ x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{25}. \end{aligned}$$

Solving the first two congruences, we obtain $x \equiv 5 \pmod{12}$. Since $1(25) - 2(12) = 1$, $x = 5(25) - 3(2)(12) = 53 \pmod{300}$.

(b) We wish to solve the system of congruences

$$\begin{aligned} x &\equiv 2 \pmod{4} \\ x &\equiv 0 \pmod{3} \\ x &\equiv 6 \pmod{25}. \end{aligned}$$

Solving the first two congruences, we obtain $x \equiv 6 \pmod{12}$. Since $1(25) - 2(12) = 1$, $x = 6(25) - 6(2)(12) = 6 \pmod{300}$.

(c) We wish to solve the system of congruences

$$\begin{aligned} x &\equiv 3 \pmod{4} \\ x &\equiv 1 \pmod{3} \\ x &\equiv 17 \pmod{25}. \end{aligned}$$

Solving the first two congruences, we obtain $x \equiv 7 \pmod{12}$. Since $1(25) - 2(12) = 1$, $x = 7(25) - 17(2)(12) = -233 \equiv 67 \pmod{300}$.

(d) [BB] We wish to solve the system of congruences

$$\begin{aligned} x &\equiv 0 \pmod 4 \\ x &\equiv 2 \pmod 3 \\ x &\equiv 10 \pmod{25}. \end{aligned}$$

Solving the first two congruences, we obtain $x \equiv 8 \pmod{12}$. Since $1(25) - 2(12) = 1$, $x = 8(25) - 10(2)(12) = -40 \equiv 260 \pmod{300}$.

(e) We wish to solve the system of congruences

$$\begin{aligned} x &\equiv 3 \pmod 4 \\ x &\equiv 2 \pmod 3 \\ x &\equiv 24 \pmod{25}. \end{aligned}$$

Solving the first two congruences, we obtain $x \equiv 11 \pmod{12}$. Since $1(25) - 2(12) = 1$, $x = 11(25) - 24(2)(12) = -301 \equiv 299 \pmod{300}$.

20. [BB] Call the number $x$. Then $x \equiv 4 \pmod{12}$ and $x \equiv 15 \pmod{25}$. By the Chinese Remainder Theorem, this system has a solution which is unique mod $12(25) = 300$. There is exactly one such number in the interval $200 < x < 500$ so the given congruences determine the number uniquely. Since $1(25) - 2(12) = 1$, the method suggested in the text gives $x \equiv 4(1)(25) - 15(2)(12) = -260 \equiv 40 \pmod{300}$. The number was 340.

21. (a) [BB] $ab \equiv 3(3) = 1 \pmod 4$
$ab \equiv 0(8) = 0 \pmod 9$
$ab \equiv 4(18) = 22 \pmod{25}$.

Since $1 = 9 - 2(4)$, we get $ab \equiv 1(9) - 0(2)(4) = 9 \pmod{36}$ as the solution to the first two congruences. Then, since $1 = -9(36) + 13(25)$, we get $x = 22(-9)(36) + 9(13)(25) = -4203 \equiv 297 \pmod{900}$. Since there are many integers less than 50,000 which are congruent to 297 (mod 900), it is impossible to determine $ab$ exactly.

(b) $ab \equiv 7(7) = 1 \pmod 8$
$ab \equiv 9(8) = 18 \pmod{27}$
$ab \equiv 29(18) \equiv 22 \pmod{125}$.

Since $1 = 3(27) - 10(8)$, we get $ab \equiv 1(3)(27) - 18(10)(8) = -1359 \equiv 153 \pmod{216}$. Then, since $1 = 11(216) - 19(125)$, we get $ab \equiv 22(11)(216) - 153(19)(125) = -311{,}103 \equiv 12{,}897 \pmod{27{,}000}$. Since there are two integers less than 50,000 which are congruent to 12,897 (mod 27,000), we cannot determine $ab$ exactly.

(c) $ab \equiv 2 \pmod 8$
$ab \equiv 8 \pmod{27}$
$ab \equiv 10 \pmod{125}$.

Since $1 = 3(27) - 10(8)$, we get $ab \equiv 2(3)(27) - 8(10)(8) = -478 \equiv 170 \pmod{216}$. Then, since $1 = 11(216) - 19(125)$, we get $ab \equiv 10(11)(216) - 170(19)(125) = -379{,}990 \equiv 25{,}010 \pmod{27{,}000}$. Since 25,010 is the only integer less than 50,000 which is congruent to 25,010 (mod 27,000), it must be that $ab = 25{,}010$.

(d) $ab \equiv 6 \pmod 8$
$ab \equiv 12 \pmod{27}$
$ab \equiv 39 \pmod{125}$.

Since $1 = 3(27) - 10(8)$, we get $ab \equiv 6(3)(27) - 12(10)(8) = -474 \equiv 174 \pmod{216}$. Then, since $1 = 11(216) - 19(125)$, we get $ab \equiv 39(11)(216) - 174(19)(125) = -320{,}586 \equiv 3414 \pmod{27{,}000}$. Since there are two integers less than 50,000 which are congruent to 3414 (mod 27,000), we cannot determine $ab$ exactly.

(e) [BB] $ab \equiv 3 \pmod{4}$
$ab \equiv 12 \pmod{25}$
$ab \equiv 104 \pmod{343}$.

Since $1 = 25 - 6(4)$, we get $ab \equiv 3(25) - 12(6)(4) = -213 \equiv 87 \pmod{100}$. Since $1 = 7(343) - 24(100)$, we get $ab \equiv 87(7)(343) - 104(24)(100) = -40{,}713 \equiv 27{,}887 \pmod{34{,}300}$. Since 27,887 is the only integer less than 50,000 which is congruent to 27,887 (mod 34,300), it must be that $ab = 27{,}887$.

(f) $ab \equiv 3 \pmod{4}$
$ab \equiv 2 \pmod{25}$
$ab \equiv 255 \pmod{343}$.

Since $1 = 25 - 6(4)$, we get $ab \equiv 3(25) - 2(6)(4) = 27 \pmod{100}$. Then, since $1 = 7(343) - 24(100)$, we get $ab \equiv 27(7)(343) - 255(24)(100) = -547{,}173 \equiv 1627 \pmod{34{,}300}$. Since there are two integers less than 50,000 which are congruent to 1627 (mod 34,300), we cannot determine $ab$ exactly.

(g) $ab \equiv 2 \pmod{5}$
$ab \equiv 42 \pmod{49}$
$ab \equiv 1 \pmod{11}$
$ab \equiv 3 \pmod{13}$

Since $1 = 10(5) - 49$, we get $ab \equiv 42(10)(5) - 2(49) = 2002 \equiv 42 \pmod{245}$. Then, since $1 = 4(245) - 89(11)$, we get $ab \equiv 1(4)(245) - 42(89)(11) = -40{,}138 \equiv 287 \pmod{2695}$. Finally, since $1 = -3(2695) + 622(13)$, $ab = 3(-3)(2695) + 287(622)(13) = 2{,}296{,}427 \equiv 19{,}152 \pmod{35{,}035}$. Since 19,152 is the only integer less than 50,000 which is congruent to 19,152 (mod 35,035), it must be that $ab = 19{,}152$.

(h) $ab \equiv 3 \pmod{5}$
$ab \equiv 14 \pmod{49}$
$ab \equiv 10 \pmod{11}$
$ab \equiv 12 \pmod{13}$

Since $1 = 10(5) - 49$, we get $ab \equiv 14(10)(5) - 3(49) = 553 \equiv 63 \pmod{245}$. Then, since $1 = 4(245) - 89(11)$, we get $ab \equiv 10(4)(245) - 63(89)(11) = -51{,}877 \equiv 2023 \pmod{2695}$. Finally, since $1 = -3(2695) + 622(13)$, $ab = 12(-3)(2695) + 2023(622)(13) = 16{,}260{,}958 \equiv 4718 \pmod{35{,}035}$. Since there are two integers less than 50,000 which are congruent to 4718 (mod 35,035), we cannot determine $ab$ exactly.

22. [BB] First we address the hint. Suppose you want to write $7s + 22t = 1$. The "obvious" numbers are $s = -3$, $t = 1$: $(-3)7 + (1)22 = 1$. But suppose you want $s > 0$. This can be achieved by adding and subtracting 22(7) in the equation $(-3)7 + 1(22)$: $(-3 + 22)7 + (1 - 7)22 = 1$; that is, $19(7) + (-6)22 = 1$. In general, we know that $1 = sm + tn$ for some integers $s$ and $t$ by Theorem 4.2.9. Now if $s \not> 0$, choose $k$ such that $s + kn > 0$ and note that $(s + kn)m + (t - km)n = 1$.

23. (a) [BB] X corresponds to the number $24 = M$. We encode this as $M^s = M^5 \pmod{391}$ (since $r = pq = 391$). $E = 300$.

(b) HELP corresponds to the number 8051216. $E = M^5 \pmod{391} = 33$.

(c) [BB] AIR corresponds to the number 10918. $E = M^5 \pmod{391} = 259$.

(d) BYE corresponds to the number 22505. $E = M^5 \pmod{391} = 97$.

(e) NOW corresponds to the number 141523. $E = M^5 \pmod{391} = 104$.

24. Since $1 = 1(5) - 1(4)$ and $1 = 5(5) - 4(6)$ we have $a = 1$, $b = 5$ in the notation of the text.

(a) [BB] $E^a \pmod{p}$ is $31^1 \pmod{5} = 1$; $E^b \pmod{q}$ is $31^5 \pmod{7} = 5$. To solve $M \equiv 1 \pmod{5}$, $M \equiv 5 \pmod{7}$, we write $1 = 10(5) - 7(7)$ and get $M = 5(10)(5) - 1(7)(7) = 201 \equiv 26 \pmod{35}$. So the message was the letter Z.

(b) $E^a \pmod{p}$ is $24^1 \pmod{5} = 4$; $E^b \pmod{q}$ is $24^5 \pmod{7} = 5$. To solve $M \equiv 4 \pmod{5}$, $M \equiv 5 \pmod{7}$, we write $1 = 10(5) - 7(7)$ and get $M = 5(10)(5) - 4(7)(7) = 54 \equiv 19 \pmod{35}$. So the message was the letter S.

(c) [BB] $E^a \pmod{p}$ is $7^1 \pmod{5} = 2$; $E^b \pmod{q}$ is $7^5 \pmod{7} = 0$. To solve $M \equiv 2 \pmod{5}$, $M \equiv 0 \pmod{7}$, we write $1 = 10(5) - 7(7)$ and get $M = 0(10)(5) - 2(7)(7) = -98 \equiv 7 \pmod{35}$. So the message was the letter G.

(d) $E^a \pmod{p}$ is $11^1 \pmod{5} = 1$; $E^b \pmod{q}$ is $11^5 \pmod{7} = 2$. To solve $M \equiv 1 \pmod{5}$, $M \equiv 2 \pmod{7}$, we write $1 = 10(5) - 7(7)$ and get $M = 2(10)(5) - 1(7)(7) = 51 \equiv 16 \pmod{35}$. So the message was the letter P.

(e) $E^a \pmod{p}$ is $23^1 \pmod{5} = 3$; $E^b \pmod{q}$ is $23^5 \pmod{7} = 4$. To solve $M \equiv 3 \pmod{5}$, $M \equiv 4 \pmod{7}$, we write $1 = 10(5) - 7(7)$ and get $M = 4(10)(5) - 3(7)(7) = 53 \equiv 18 \pmod{35}$. So the message was the letter R.

25. We have $r = 1003$ and, in the notation of the text, $a = 11$, $b = 39$.

(a) [BB] $E^a = 456^{11} \equiv 10 \pmod{17}$, while $E^b = 456^{39} \equiv 42 \pmod{59}$. Hence, we wish to solve the system $x \equiv 10 \pmod{17}$, $x \equiv 42 \pmod{59}$. Since $7(17) - 2(59) = 119 - 118 = 1$, $x = 42(119) - 10(118) = 4998 - 1180 = 3818 \equiv 809 \pmod{1003}$. The message is HI.

(b) $E^a = 926^{11} \equiv 2 \pmod{17}$, while $E^b = 926^{39} \equiv 35 \pmod{59}$. Hence, we wish to solve the system

$$\begin{aligned} x &\equiv 2 \pmod{17} \\ x &\equiv 35 \pmod{59} \end{aligned}$$

Since $1 = 7(17) - 2(59)$, $x = 35(7)(17) + 2(-2)(59) = 3929 \equiv 920 \pmod{1003}$. The message is IT.

## Chapter 4 Review

1. $$\begin{aligned} 11{,}109{,}999{,}999{,}997 &= 11{,}110{,}000{,}000 - 3 \\ &= (10{,}000{,}000)(1111) - 3 \\ &= (9{,}999{,}999)(1111) + (1111 - 3) = (9{,}999{,}999)(1111) + 1008. \end{aligned}$$

The quotient is 9,999,999 and the remainder is 1008.

2. (a) $(1100101)_2 = 1 + 1(2^2) + 1(2^5) + 1(2^6) = 101$.

(b) We have $32145 = 8(4018) + 1$, $4018 = 8(502) + 2$, $502 = 8(62) + 6$, $62 = 8(7) + 6$, $7 = 8(0) + 7$, so $32145 = (76621)_8$.

3. In base $b$, $31 = 1+3b$, $156 = 6+5b+b^2$ and $220 = 2b+2b^2$. The equation $(1+3b)+(6+5b+b^2) = 2b+2b^2$ implies $7+8b+b^2 = 2b+2b^2$, so $b^2-6b-7 = 0 = (b-7)(b+1)$. Buddy is correct in base $b = 7$.

4. The integer $n$ has prime divisors 2, 3 and 7, so $n = 2(3)(7)k$ for some integer $k$. Since $2(3)(7)$ has eight factors—$1, 2, 3, 7, 6, 14, 21, 42$—$k = 1$ and $n = 42$.

5. The given integer is

$$1+2b+3b^2+4b^3+5b^4+6b^5+7b^6+8b^7+9b^8$$
$$= (1+2b+4b^3+6b^5+8b^7)+(3b^2+5b^4+7b^6+9b^8),$$

the sum of an odd and an even integer. The given integer is never even.

6. (a) Since $8215 = 5+1(10)+2(100)+8(1000)$, the stated "observation"—popularly called "casting out 9's"—follows from Proposition 4.4.7 and the fact that any power of 10 is $\equiv 1 \pmod 9$. This holds for any integer, because $(a_{n-1}a_{n-2}\ldots a_1a_0)_{10}$ is just $a_0+10a_1+10^2a_2+\cdots+10^{n-1}a_{n-1}$.

   (b) This follows immediately from part (a).

   (c) This is true and again, Proposition 4.4.7 comes to the rescue. If $a \equiv x \pmod 9$ and $b \equiv y \pmod 9$, then $ab \equiv xy \pmod 9$. Part (a) shows that one can take $x$ and $y$ to be the sum of the digits of the numbers $a$ and $b$.

7.

| | | |
|---|---|---|
| 2700 | 1 | 0 |
| 504 | 0 | 1 |
| 180 | 1 | −5 |
| 144 | −2 | 11 |
| 36 | 3 | −16 |

Thus $36 = 3(2700) - 16(504)$. We have $\gcd(2700, -504) = 36 = 3(2700) + 16(-504)$.

8. We have $ab = kx$ for some integer $k$. Also, since $x$ and $a$ are relatively prime, $tx + sa = 1$ for certain integers $t$ and $s$. Multiplying by $b$, we get $b = tbx + sab = tbx + skx = x(tb + sk)$, so $x \mid b$.

9. (a)

| | | |
|---|---|---|
| 571 | 1 | 0 |
| 386 | 0 | 1 |
| 185 | 1 | −1 |
| 16 | −2 | 3 |
| 9 | 23 | −34 |
| 7 | −25 | 37 |
| 2 | 48 | −71 |
| 1 | −169 | 250 |

   (b) The Euclidean algorithm gives $1 = -169(571) + 250(386)$, so $10 = 571(-1690) + 386(2500)$: Take $a = -1690$ and $b = 2500$.

   (c) Suppose $571a+386b = 10$. Subtracting $571(-1690)+386(2500) = 10$, we get $571(a+1690)+386(b-2500) = 0$. Now $386 \mid 386(b-2500)$ so $386 \mid 571(a+1690)$. Since 386 and 571 are relatively prime, Exercise **??** implies $386 \mid (a+1690)$. The smallest positive $a$ for which this occurs is $a = 240$. Now $0 = 571(a+1690)+386(b-2500) = 571(1930)+386(b-2500)$, so $571(5)+(b-2500) = 0$, giving $b = -355$.

10. (a) We use the Euclidean algorithm:
$$\begin{array}{rrr} 97 & 1 & 0 \\ 17 & 0 & 1 \\ 12 & 1 & -5 \\ 5 & -1 & 6 \\ 2 & 3 & -17 \\ 1 & -7 & 40 \end{array}.$$
Thus $1 = -7(97) + 40(17)$.

(b) By part (a), $10 = -70(97) + 400(17)$. Mod 97, this says $400(17) \equiv 10$, so $x = 400 \equiv 12 \pmod{97}$.

11. (a) We know the solution is unique since $\gcd 8, 27 = 1$. A patient person might use trial and error to get $x = 17$. Alternatively, using the method suggested by Problem 25, writing $1 = 3(27) - 10(8)$ shows that $x = -10 \equiv 17 \pmod{27}$.

(b) As in part (a), we know the solution is unique. Since $3(17) \equiv 1 \pmod{27}$, $3(15)(17) \equiv 15 \pmod{27}$, so $x = 3(15) = 45 \equiv 18 \pmod{27}$.

(c) We want $16 \mid (4x - 8)$, so $4 \mid (x - 2)$; that is, $x - 2 = 4k$ for some $k$. The values of $x \pmod{16}$ are $x = 2, 6, 14$.

(d) There is no solution because the values of $4x \pmod{16}$ are 0, 4, and 8.

(e) We want $380 \mid (6x - 7)$, so $6x - 7 = 380k$ for some integer $k$. This says $7 = 6x - 380k$ is even, which is not true. There is no solution.

(f) We want $380 \mid (6x - 8)$, so $6x - 8 = 380k$ for some integer $k$. This is equivalent to $3x = 190k + 4 = 4, 194, 384, 574, 764, 954, \ldots$ so $x = 128$ or $x = 318$.

12. (a) Since $\gcd(6, 25) = 1$, we know there exist integers $a$ and $b$ so that $6a + 25b = 1$. In fact, we find that $6(-4) + 25(1) = 1$. Multiplying by 13 gives $6(-52) + 25(13) = 13$, so take $x_0 = -52$ and $y_0 = 13$.

(b) Suppose $6x + 25y = 13$. Then $6x + 25y = 6x_0 + 25y_0$, so $6(x - x_0) = 25(y_0 - y)$. Since 6 divides the left side, it divides the right side too. Since $\gcd(6, 25) = 1$, $6 \mid (y_0 - y)$, so $y_0 - y = 6k$ for some integer $k$. Thus $6(x - x_0) = 150k$ and so $x - x_0 = 25k$. Hence $x = x_0 + 25k$ and $y = y_0 - 6k$ as desired.

13. The squares of integers modulo 8 are $0^2, 1^2, \ldots, 7^2$, that is, 0, 1, and 4. No three of these sum to 7.

14. Suppose $p$ is a prime which divides both $4k + 3$ and $7k + 5$. Since $7(4k + 3) - 4(7k + 5) = 1$, and $p$ divides the left side, so also $p$ must divide 1. This is a contradiction. Thus $\gcd(4k + 3, 7k + 5) = 1$.

15. (a) **Reflexive:** For any $a \in \mathsf{Z}$, we have $a \sim a$ because $a^2 - a^2 = 0$ is divisible by 3.

**Symmetric:** If $a, b \in \mathsf{Z}$ and $a \sim b$, then $a^2 - b^2 = 3k$ for some integer $k$. Then $b^2 - a^2 = 3(-k)$ says that $b^2 - a^2$ is also divisible by 3, so $b \sim a$.

**Transitive:** If $a, b, c \in \mathsf{Z}$ with $a \sim b$ and $b \sim c$, then $a^2 - b^2 = 3k$ and $b^2 - c^2 = 3\ell$ for some integers $k$ and $\ell$. Then $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 3(k + \ell)$ says that $a^2 - c^2$ is divisible by 3. So $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{Z} \mid x^2 - 0^2 \text{ is divisible by } 3\}$
$= \{x \in \mathsf{Z} \mid x^2 = 3k \text{ for some integer } k\} = 3\mathsf{Z}$

(Here we use the fact that 3 is prime to observe that if $3 \mid x^2$, then $3 \mid x$.)

(c) $\overline{1} = \{x \in \mathsf{Z} \mid x^2 - 1^2 \text{ is divisible by } 3\}$
$= \{x \in \mathsf{Z} \mid x^2 - 1 = 3k \text{ for some integer } k\}$
$= \{x \in \mathsf{Z} \mid x - 1 = 3\ell \text{ for some } \ell \in \mathsf{Z} \text{ or } x + 1 = 3\ell \text{ for some } \ell \in \mathsf{Z}\}$
$= (3\mathsf{Z} + 1) \cup (3\mathsf{Z} - 1)$

(Since $x^2 - 1 = (x + 1)(x - 1)$, if $3 \mid (x^2 - 1)$, then $3 \mid (x - 1)$ or $3 \mid (x + 1)$.)

16. (a) Suppose there are just a finite number of primes, $p_1, p_2, \ldots, p_n$. Then look at the number $p_1 p_2 \cdots p_n + 1$. This can't be prime since it is bigger than each $p_i$. So it is divisible by some prime, hence by some $p_i$. But this implies $p_i \mid 1$, which is not true.

(b) Every integer $n \geq 2$ can be written in the form $n = p_1 p_2 \cdots p_r$ for unique primes $p_1, p_2, \ldots, p_r$; equivalently, every integer $n \geq 2$ can be written $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ as the product of powers of distinct prime numbers $q_1, q_2, \ldots, q_s$. These primes and the exponents $\alpha_1, \alpha_2, \ldots, \alpha_s$ are unique.

17. $2^{119} - 1 = 2^{(17)(7)} - 1 = (2^{17} - 1)(2^{102} + 2^{85} + 2^{68} + 2^{51} + 2^{34} + 2^{17} + 1)$ is not prime; $3^{109} - 1$ is not prime because it is even; $4^{109} - 1 = (4 - 1)(4^{108} + 4^{107} + 4^{106} + \cdots + 16 + 4 + 1)$ is not prime.

18. Since $7^2 = 49 \equiv 9 \pmod{10}$, we have $7^4 \equiv 81 \equiv 1 \pmod{10}$. Since $355 = 4(88) + 3$, $7^{355} = (7^4)^{88} 7^3 \equiv 7^3 = 7^2(7) \equiv 9(7) = 63 \equiv 3 \pmod{10}$. The last digit of $7^{355}$ is 3.

19. (a) False: Take $a = 2$, $b = 1$, $c = 4$. Then $1 \mid 2$ and $2 \mid 4$, but $4 \nmid 2$.

(b) True: We are given that $a = bx$ and $c = \dfrac{a}{b} y = xy$ for some integers $x$ and $y$. Thus $bc = bxy = ay$, so $a \mid bc$.

20. There exist integers $q_1$ and $q_2$ such that $b = q_1 a$ and $b + 1 = q_2(a + 1)$. Thus $q_1 a + 1 = q_2 a + q_2$, so $a(q_1 - q_2) = q_2 - 1 > 0$. This implies $q_1 - q_2 \geq 1$ and $q_1 > q_2$. Now $a^2 \leq a^2(q_1 - q_2) = a(q_2 - 1) \leq aq_2 < aq_1 = b$.

21. (a) Since 7 is prime, we have $3^6 \equiv 1 \pmod{7}$ by Fermat's Little Theorem. Thus $3^{80} = 3^{78+2} = 3^{6(13)+2} \equiv 1^{13} 9 \equiv 2 \pmod{7}$.

(b) If $5x \equiv 1 \pmod{100}$, then $5x - 1 = 100k$ for some integer $k$, so $5x - 100k = 1$. The left side is divisible by 5 while the right is not. There is no such $x$.

22. Letting $x = 0, 1, 2, \ldots, 11$, we have $9x \equiv 0, 9, 6, 3, 0, 9, 6, 3, 0, 9, 6, 3$, respectively. The answer is $x \equiv 3, 7, 11$.

23. We want $243 \mid (36x - 9)$, hence $36x - 9 = 243k$ for some integer $k$. This is equivalent to $4x - 1 = 27k$, which has the unique solution $x \equiv 7 \pmod{27}$. Our solutions have the form $x = 7 + 27k$, so $x = 7, 34, 61, 88, 115, 142, 169, 196, 223$.

24. Multiplying the first equation by 2 gives $2x \equiv 0 \pmod{10}$, so $x \equiv 0, 5 \pmod{10}$. Substituting $x \equiv 0$ in the second equation gives $3y \equiv 1 \pmod{10}$, which has the unique solution $y \equiv 7$. Since $x = 0$, $y = 7$ satisfy the first equation, we have one solution.

Substituting $x \equiv 5$ in the second equation gives $3y \equiv 6 \pmod{10}$ which has the unique solution $y \equiv 2 \pmod{10}$. Since $x = 5$, $y = 2$ also satisfy the first equation, we have a second solution.

25. (a) $A + 2(2) + 3(5) + 4(3) + 5(2) + 6(2) + 7(8) + 8(4) + 9(6) + 10(7) \equiv 0 \pmod{11}$ implies $A \equiv 10 \pmod{11}$, so $\mathsf{A} = \mathsf{X}$.

(b) $3(3 + 5 + 1 + 3 + x + 5) + (2 + 8 + 4 + 9 + 7 + 6) \equiv 0 \pmod{10}$ implies $51 + 3x + 36 \equiv 0 \pmod{10}$, so $3x \equiv 1 \pmod{10}$ and $x = 1$.

26.

| | | |
|---|---|---|
| 341 | 1 | 0 |
| 189 | 0 | 1 |
| 152 | 1 | −1 |
| 37 | −1 | 2 |
| 4 | 5 | −9 |
| 1 | −46 | 83 |

Thus $1 = \gcd(341, 189) = -46(341) + 83(189)$. So $(5)(83)(189) + 11(-46)(341) = -94111$. Since $(341)(189) = 64449$, the answer is $-94111 + 2(64449) = 34787$.

27. We use the method described in Problem 29. First, we solve the first two congruences. We have $3(5) - 2(7) = 1$, so $x = 5(3)5 - 3(2)7 = 33$. Now we solve $x \equiv 33 \pmod{35}$ and $x \equiv 7 \pmod{11}$. We have $-5(35) + 16(11) = 1$, so $x = 7(-5)(35) + 33(16)(11) = 4583$. Since $5(7)(11) = 385$, the smallest positive integer is $4583 \pmod{385} = 348$.

## Exercises 5.1

1. (a) [BB] $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

   (b) [BB] $\sum_{i=1}^{4} 2^i = 2^1 + 2^2 + 2^3 + 2^4 = 2 + 4 + 8 + 16 = 30$

   (c) [BB] $\sum_{t=1}^{1} \sin \pi t = \sin \pi(1) = \sin \pi = 0$

   (d) $\sum_{j=0}^{2} 3^{j+2} = 3^{0+2} + 3^{1+2} + 3^{2+2} = 3^2 + 3^3 + 3^4 = 9 + 27 + 81 = 117$

   (e) $\sum_{k=-1}^{4}(2k^2 - k + 1) = [2(-1)^2 - (-1) + 1] + [(2(0^2) - 0 + 1]$
   $+[2(1^2) - 1 + 1] + [2(2^2) - 2 + 1] + [2(3^2) - 3 + 1] + [2(4^2) - 4 + 1]$
   $= 4 + 1 + 2 + 7 + 16 + 29 = 59$

   (f) $\sum_{k=0}^{n}(-1)^k = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^n$
   $= 1 - 1 + 1 - 1 + \cdots + (-1)^n = 0$ if $n$ is odd and 1 if $n$ is even.

2. (a) [BB] The values of $\sum_{i=1}^{n}(-1)^i$ for $n = 0, 1, 2, 3$ are $1, 1-1 = 0, 1-1+1 = 1$ and $1-1+1-1 = 0$. The answer is $0, 1$.

   (b) The values of $\sum_{i=1}^{n} 2^i$ for $n = 1, 2, 3, 4, 5$ are $2, 2+4 = 6, 2+4+8 = 14, 2+4+8+16 = 30$ and $2 + 4 + 8 + 16 + 32 = 62$. The answer is $2, 6, 14, 30, 62$.

3. If the order is for 32 pounds of fish, we can fill it with three 9-pound and one 5-pound fish. Now suppose we know the result is true for an order of $k \geq 32$ pounds of fish and suppose we have an order for $k+1$ pounds. By the induction hypothesis we can fill an order for $k$ pounds. If this requires at least one nine-pound fish, then replacing it with two five-pound fish gives us a total weight of $k+1$ pounds. On the other hand, if no nine-pound fish are used to fill an order for $k$ pounds, then only five-pound fish are needed and since $k \geq 32$, at least seven five-pound fish. Now replacing seven five-pound fish by four nine-pounders gives us fish with a total weight of $k + 1$ pounds.

4. (a) [BB] If $n = 1$, $n^3 + 2n = 1^3 + 2 \cdot 1 = 3$, which is divisible by 3. Suppose $k \geq 1$ and the result is true for $n = k$; that is, $3 \mid (k^3 + 2k)$. We wish to prove that $3 \mid [(k+1)^3 + 2(k+1)]$. But $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + (3k^2 + 3k + 3)$. Since $3 \mid (k^3 + 2k)$ (by the induction hypothesis) and since $3 \mid (3k^2 + 3k + 3)$, we have the desired result. We conclude that $3 \mid (n^3 + 2n)$ for all $n \geq 1$, by the Principle of Mathematical Induction.

(b) If $n = 1$, $n^3 + (n+1)^3 + (n+2)^3 = 1^3 + 2^3 + 3^3 = 36$ and $9 \mid 36$. Suppose the result is true for $n = k \geq 1$; that is, $9 \mid [k^3 + (k+1)^3 + (k+2)^3]$. Then $(k+1)^3 + (k+2)^3 + (k+3)^3 = (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27 = [k^3 + (k+1)^3 + (k+2)^3] + [9k^2 + 27k + 27]$. Since 9 divides the first term by the induction hypothesis and since 9 clearly divides the second term, 9 divides $(k+1)^3 + (k+2)^3 + (k+3)^3$, as desired. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

(c) [BB] If $n = 1$, $5^n - 1 = 5 - 1 = 4$ is divisible by 4. Suppose the result is true for $n = k \geq 1$; that is, $4 \mid (5^k - 1)$. Thus $5^k - 1 = 4N$ for some integer $N$. Now $5^{k+1} - 1 = 5(5^k) - 1 = 5(4N + 1) - 1 = 20N + 4 = 4(5N + 1)$, so $4 \mid (5^{k+1} - 1)$ as desired. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

(d) If $n = 1$, $8^n - 3^n = 8 - 3 = 5$ and $5 \mid 5$. Suppose the result is true for $n = k \geq 1$; that is, $5 \mid (8^k - 3^k)$ so that $8^k - 3^k = 5N$ for some integer $N$. Then

$$\begin{aligned} 8^{k+1} - 3^{k+1} &= 8(8^k) - 3^{k+1} \\ &= 8(5N + 3^k) - 3^{k+1} \\ &= 40N + 8(3^k) - 3^{k+1} = 40N + 3^k(8 - 3) = 40N + 5(3^k) = 5(8N + 3^k) \end{aligned}$$

so that $5 \mid (8^{k+1} - 3^{k+1})$. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

(e) If $n = 1$, $5^{2n} - 2^{5n} = 5^2 - 2^5 = 25 - 32 = -7$ is divisible by 7. Suppose $k \geq 1$ and the result is true for $k$, that is, assume $5^{2k} - 2^{5k}$ is divisible by 7. Thus $5^{2k} - 2^{5k} = 7N$ for some integer $N$. Then

$$\begin{aligned} 5^{2(k+1)} - 2^{5(k+1)} &= 25(5^k) - 2^{5k+5} \\ &= 25(2^{5k} + 7N) - 32(2^{5k}) = -7(2^{5k}) + 25(7N) = 7(25N - 2^{5k}) \end{aligned}$$

showing that $5^{2(k+1)} - 2^{5(k+1)}$ is divisible by 7. By the Principle of Mathematical Induction, the result holds for all $n \geq 1$.

(f) [BB] If $n = 1$, $10^{n+1} + 10^n + 1 = 100 + 10 + 1 = 111$ is divisible by 3 (because $111 = 3(37)$). Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose that $3 \mid (10^{k+1} + 10^k + 1)$. Then

$$\begin{aligned} 10^{k+2} + 10^{k+1} + 1 &= 10(10^{k+1} + 10^k) + 1 \\ &= 10(10^{k+1} + 10^k + 1 - 1) + 1 \\ &= 10(10^{k+1} + 10^k + 1) - 10 + 1 = 10(10^{k+1} + 10^k + 1) - 9 \end{aligned}$$

Since the first term here is divisible by 3, by the induction hypothesis, and, since 9 is divisible by 3, $10^{k+2} + 10^{k+1} + 1$ is divisible by 3, as desired. By the Principle of Mathematical Induction, $10^{n+1} + 10^n + 1$ is divisible by 3 for all $n \geq 1$.

(g) For $n = 1$, $1^3 + 5(1) = 6$ is divisible by 6. Now assume that $k \geq 1$ and that $k^3 + 5k$ is divisible by 6; that is, assume $k^3 + 5k = 6N$ for some integer $N$. We have

$$\begin{aligned} (k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5(k+1) \\ &= (k^3 + 5k) + 3k^2 + 3k + 6 = 6N + 3k(k+1) + 6 \end{aligned}$$

(using the induction hypothesis), and this is divisible by 6 because $k(k+1)$ is even. Thus, we obtain the desired divisibility with $n = k + 1$. By the Principle of Mathematical Induction, we conclude that $n^3 + 5n$ is divisible by 6 for all $n \geq 1$.

(h) For $n = 1$, $2^n + 3^n - 5^n = 0$ is divisible by 6. Now assume that $k \geq 1$ and that $2^k + 3^k - 5^k$ is divisible by 6; that is, assume that $5^k = 2^k + 3^k + 6N$ for some integer $N$. We have

$$\begin{aligned} 2^{k+1} + 3^{k+1} - 5^{k+1} &= 2(2^k) + 3(3^k) - 5(5^k) \\ &= 2(2^k) + 3(3^k) - 5(2^k + 3^k + 6N) \quad \text{using the induction hypothesis,} \\ &= -3(2^k) - 2(3^k) - 30N \\ &= -3(2)(2^{k-1}) - 2(3)(3^{k-1}) - 30N, \end{aligned}$$

which is divisible by 6 because each term is divisible by 6. By the Principle of Mathematical Induction, we conclude that $2^n + 3^n - 5^n$ is divisible by 6 for all $n \geq 1$.

(i) For $n = 1$, $16^n + 10n - 1 = 25$ is divisible by 25. Now assume that $k \geq 1$ and that $16^k + 10k - 1$ is divisible by 25; that is, assume that $16^k = 1 - 10k + 25N$ for some integer $N$. We have

$$\begin{aligned} 16^{k+1} + 10(k+1) - 1 &= 16(16^k) + 10k + 9 \\ &= 16(1 - 10k + 25N) + 10k + 9 \quad \text{using the induction hypothesis,} \\ &= 25 - 150k + 16(25N), \end{aligned}$$

which is divisible by 25 because each term is divisible by 25. By the Principle of Mathematical Induction, we conclude that $16^n + 10n - 1$ is divisible by 25 for all $n \geq 1$.

(j) [BB] For $n = 1$, $2! = 2$ is divisible by $2^1$, so the result is true. Now assume that $k \geq 1$ and that $(2k)!$ is divisible by $2^k$. We have $[2(k+1)]! = (2k+2)! = (2k+2)(2k+1)(2k)!$. Since $(2k)!$ is divisible by $2^k$ and $2k + 2$ is divisible by 2, $[2(k+1)]!$ is divisible by $2^{k+1}$, the desired result with $n = k + 1$. By the Principle of Mathematical Induction, we conclude that $(2n)!$ is divisible by $2^n$ for all $n \geq 1$.

(k) For $n = 1$, $a^1 - b^1 = a - b$ is certainly divisible by $a - b$. Now suppose that $k \geq 1$ and $a^k - b^k$ is divisible by $a - b$. Then

$$a^{k+1} - b^{k+1} = a(a^k) - a(b^k) + a(b^k) - b(b^k) = a(a^k - b^k) + b^k(a - b)$$

The first term here is divisible by $a-b$ by the induction hypothesis and the second term is obviously divisible by $a-b$, hence, the sum is divisible by $a-b$. By the Principle of Mathematical Induction, the given statement is true for all $n \geq 1$.

5. (a) [BB] For $n = 1$, $1 + 2 + 3 \cdots + n$ is 1, by definition, and $\frac{n(n+1)}{2} = \frac{2}{2} = 1$ also. Thus, the result is true for $n = 1$. Now suppose that $k \geq 1$ and the result is true for $k$; that is, assume that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ for some integer $k \geq 1$.
Then $1 + 2 + \cdots + (k+1) = (1 + 2 + \cdots + k) + (k+1)$

$$\begin{aligned} &= \frac{k(k+1)}{2} + (k+1) \quad \text{(by the induction hypothesis)} \\ &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}, \end{aligned}$$

which is the given statement with $n = k + 1$. So, by the Principle of Mathematical Induction, we conclude that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \geq 1$.

(b) For $n = 1$, $1^3 + 2^3 + \cdots + n^3 = 1^3 = 1$, by definition, and $n^2(n+1)^2/4 = 1^2(2^2)/4 = 1$ as well, so the result is true for $n = 1$. Now suppose that $k \geq 1$ and

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Then

$$1^3+2^3+\cdots+(k+1)^3 = (1^3+2^3+\cdots+k^3)+(k+1)^3$$

$$= \frac{k^2(k+1)^2}{4}+(k+1)^3 \quad \text{(by the induction hypothesis)}$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2+4(k+1)}{4} = \frac{(k+1)^2(k+2)^2}{4},$$

which is the given statement with $n = k+1$. So by the Principle of Mathematical Induction, we conclude that $1^3+2^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

(c) This follows directly from (a) and (b) and the fact that $(\frac{n(n+1)}{2})^2 = \frac{n^2(n+1)^2}{4}$.

6. (a) [BB] If $n = 1$, $1+2^1+2^2+2^3+\cdots+2^n = 1+2 = 3$, while $2^{1+1}-1 = 4-1 = 3$, so the result is true if $n = 1$. Suppose the result is true for $n = k \geq 1$; that is, suppose $1+2+2^2+2^3+\cdots+2^k = 2^{k+1}-1$. We wish to prove the result for $n = k+1$; that is, we wish to prove that $1+2+2^2+2^3+\cdots+2^{k+1} = 2^{(k+1)+1}-1$. Now $1+2+2^2+2^3+\cdots+2^{k+1} = (1+2+2^2+2^3+\cdots+2^k)+2^{k+1} = (2^{k+1}-1)+2^{k+1}$ (by the induction hypothesis) $= 2\cdot 2^{k+1}-1 = 2^{k+2}-1 = 2^{(k+1)+1}-1$ as required. By the Principle of Mathematical Induction, we conclude that the given assertion is true for all $n \geq 1$.

(b) [BB] If $n = 1$, $1^2-2^2+3^2-4^2+\cdots+(-1)^{n-1}n^2 = 1^2 = 1$, by definition, while $(-1)^{n-1}\frac{n(n+1)}{2} = (-1)^{1-1}\frac{1(1+1)}{2} = 1$, and so the result is true for $n = 1$. Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose that

$$1^2-2^2+3^2-4^2+\cdots+(-1)^{k-1}k^2 = (-1)^{k-1}\frac{k(k+1)}{2}.$$

We wish to prove that $1^2-2^2+3^2-4^2+\cdots+(-1)^{(k+1)-1}(k+1)^2$
$= (-1)^{(k+1)-1}\dfrac{(k+1)[(k+1)+1]}{2}$.

Now $1^2-2^2+3^2-4^2+\cdots+(-1)^{(k+1)-1}(k+1)^2$

$$= [1^2-2^2+3^2-4^2+\cdots+(-1)^{k-1}k^2]+(-1)^{(k+1)-1}(k+1)^2$$

$$= (-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2 \qquad \text{(using the induction hypothesis)}$$

$$= (-1)^{k-1}\frac{k^2+k}{2}+(-1)^{k-1}(-1)(k^2+2k+1)$$

$$= (-1)^{k-1}\left(\frac{k^2+k}{2}-k^2-2k-1\right)$$

$$= (-1)^{k-1}\frac{k^2+k-2k^2-4k-2}{2}$$

$$= (-1)^{k-1}\left(\frac{-k^2-3k-2}{2}\right) = (-1)^k\frac{(k+1)(k+2)}{2}$$

as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

(c) When $n = 1$, $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = 1^2 = 1$, by definition, while $\frac{n(2n-1)(2n+1)}{3} = \frac{1(2-1)(2+1)}{3} = 1$ and so the result holds. Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}.$$

We wish to prove that the result is true for $n = k+1$, that is, that

$$1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 = \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}.$$

Now

$$\begin{aligned}
&1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 \\
&\qquad = [1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2] + [2(k+1)-1]^2 \\
&\qquad = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \text{(by the induction hypothesis)} \\
&\qquad = \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} = \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} \\
&\qquad = \frac{(2k+1)(2k^2 - k + 6k + 3)}{3} = \frac{(2k+1)(2k^2 + 5k + 3)}{3} \\
&\qquad = \frac{(2k+1)(2k+3)(k+1)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3}
\end{aligned}$$

as desired. By the Principle of Mathematical Induction, the given statement is true for all $n \geq 1$.

(d) When $n = 1$, $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = 1 \cdot 2 \cdot 3 = 6$, by definition, while $\frac{1(1+1)(1+2)(1+3)}{4} = \frac{1(2)(3)(4)}{4} = 6$, so the result holds. Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}.$$

We must prove that the result is true for $n = k+1$, that is, that

$$\begin{aligned}
&1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (k+1)[(k+1)+1][(k+1)+2] \\
&\qquad = \frac{(k+1)[(k+1)+1][(k+1)+2][(k+1)+3]}{4}.
\end{aligned}$$

Now the left hand side of this equation is

$$\begin{aligned}
&1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (k+1)(k+2)(k+3) \\
&= [1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2)] + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \qquad \text{(using the induction hypothesis)} \\
&= (k+1)(k+2)(k+3)\left(\frac{k}{4} + 1\right) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

as required. By the Principle of Mathematical Induction, the given statement is true for all $n \geq 1$.

(e) When $n = 1$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} = \frac{1}{2}$$

and so the result holds. Suppose the result is true for $n = k \geq 1$; that is, suppose

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We must prove that the result is true for $n = k + 1$, that is, that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)[(k+1)+1]} = \frac{k+1}{(k+1)+1}.$$

Now

$$\begin{aligned}
&\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)[(k+1)+1]} \\
&\qquad = \left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)[(k+1)+1]} \\
&\qquad = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{(by the induction hypothesis)} \\
&\qquad = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}
\end{aligned}$$

as desired. By the Principle of Mathematical Induction, the assertion is true for all $n \geq 1$.

(f) [BB] For $n = 1$, the left side is $\frac{2}{3} = 1 - \frac{1}{3^1}$, so the formula is correct. Now assume $k \geq 1$ and that the formula holds for $n = k$; that is, assume that

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} = 1 - \frac{1}{3^k}.$$

With $n = k + 1$, the left side of the sum in question is

$$\begin{aligned}
\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{k+1}} &= \left(1 - \frac{1}{3^k}\right) + \frac{2}{3^{k+1}} \\
&= \frac{3^{k+1} - 3 + 2}{3^{k+1}} = \frac{3^{k+1} - 1}{3^{k+1}} = 1 - \frac{1}{3^{k+1}},
\end{aligned}$$

which is the desired formula. By the Principle of Mathematical Induction, the formula holds for all $n \geq 1$.

(g) [BB] When $n = 1$, the left side is $6 - 2 = 4$ and the right side is $1(4)$, so the formula holds. Now assume that $k \geq 1$ and that $4 + 10 + 16 + \cdots + (6k - 2) = k(3k + 1)$. We have

$$\begin{aligned}
4 + 10 + 16 + \cdots + [6(k+1) - 2] &= 4 + 10 + 16 + \cdots + (6k + 4) \\
&= \big(4 + 10 + 16 + \cdots + (6k - 2)\big) + (6k + 4) \\
&= k(3k + 1) + (6k + 4) \qquad \text{using the induction hypothesis} \\
&= 3k^2 + 7k + 4 = (k + 1)(3k + 4)
\end{aligned}$$

which is the given formula with $n = k + 1$. By the Principle of Mathematical Induction, we conclude that $4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1)$ for all $n \geq 1$.

(h) When $n = 1$, the left side is $\frac{1}{1\cdot 5}$ and the right side is $\frac{1}{5}$, so the formula holds. Now assume that $k \geq 1$ and that $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$. We have

$$\begin{aligned}
&\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{[4(k+1)-3)][4(k+1)+1)]} \\
&\quad = \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4k+1)(4k+5)} \\
&\quad = \left(\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4k-3)(4k+1)}\right) + \frac{1}{(4k+1)(4k+5)} \\
&\quad = \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} \qquad \text{using the induction hypothesis} \\
&\quad = \frac{k(4k+5)+1}{(4k+1)(4k+5)} \\
&\quad = \frac{4k^2+5k+1}{(4k+1)(4k+5)} = \frac{(k+1)(4k+1)}{(4k+1)(4k+5)} = \frac{k+1}{4k+5}
\end{aligned}$$

which is the given formula with $n = k+1$. By the Principle of Mathematical Induction, we conclude that $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$ for all $n \geq 1$.

7. (a) [BB] $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = \sum_{k=0}^{n} 2^k$

(b) $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = \sum_{k=1}^{n}(-1)^{k-1}k^2$

(c) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \sum_{k=1}^{n}(2k-1)^2$

(d) $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \cdots + n(n+1)(n+2) = \sum_{k=1}^{n} k(k+1)(k+2)$

(e) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)}$.

(Note that the last term on each of the left sides above provides a description of the general term required on the right. See the remarks following the expression (1).)

8. (a) [BB] For $n = 1$, $\sum_{i=1}^{1}(i+1)2^i = 2(2^1) = 4$, while $n2^{n+1} = 1(2^2) = 4$. Thus, the formula is correct for $n = 1$. Now suppose that $k \geq 1$ and the formula is correct for $n = k$; thus, we suppose that

$$\sum_{i=1}^{k}(i+1)2^i = k2^{k+1}.$$

We must prove that the formula is correct for $n = k+1$; that is, we must prove that

$$\sum_{i=1}^{k+1}(i+1)2^i = (k+1)2^{k+2}.$$

$$\text{Now } \sum_{i=1}^{k+1}(i+1)2^i = \sum_{i=1}^{k}(i+1)2^i + (k+2)2^{k+1}$$
$$= k2^{k+1} + (k+2)2^{k+1} \text{(using the induction hypothesis)}$$
$$= (k+k+2)2^{k+1} = (2k+2)2^{k+1} = 2(k+1)2^{k+1} = (k+1)2^{k+2}$$

as required. By the Principle of Mathematical Induction, the given assertion is true for all $n \geq 1$.

(b) When $n = 1$,

$$\sum_{i=1}^{n} \frac{i^2}{(2i-1)(2i+1)} = \frac{1^2}{[2(1)-1][2(1)+1]} = \frac{1}{1(3)} = \frac{1}{3},$$

while $\frac{n(n+1)}{2(2n+1)} = \frac{1(2)}{2(3)} = \frac{1}{3}$ and so the result holds. Now suppose that $k \geq 1$ and the result is true for $n = k$; thus, we suppose that

$$\sum_{i=1}^{k} \frac{i^2}{(2i-1)(2i+1)} = \frac{k(k+1)}{2(2k+1)}.$$

We have to show that the result holds for $n = k+1$, that is, that

$$\sum_{i=1}^{k+1} \frac{i^2}{(2i-1)(2i+1)} = \frac{(k+1)[(k+1)+1]}{2[2(k+1)+1]}.$$

Now

$$\sum_{i=1}^{k+1} \frac{i^2}{(2i-1)(2i+1)} = \sum_{i=1}^{k} \frac{i^2}{(2i-1)(2i+1)} + \frac{(k+1)^2}{[2(k+1)-1][2(k+1)+1]}$$

$$= \sum_{i=1}^{k} \frac{i^2}{(2i-1)(2i+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \text{(by the induction hypothesis)}$$

$$= \frac{(k+1)[k(2k+3)+2(k+1)]}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

as required. By the Principle of Mathematical Induction, the given assertion is true for all $n \geq 1$.

(c) When $n = 1$, $\sum_{i=1}^{1}(2i-1)(2i) = (2(1)-1)(2) = 2$, while $\frac{n(n+1)(4n-1)}{3} = \frac{1(2)(3)}{3} = 2$ too, so

the formula is correct. Now suppose that $k \geq 1$ and the formula is correct for $n = k$. Then

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1)(2i) &= \sum_{i=1}^{k}(2i-1)(2i) + [2(k+1)-1][2(k+1)] \\ &= \frac{k(k+1)(4k-1)}{3} + (2k+1)(2k+2) \quad \text{(by the induction hypothesis)} \\ &= \frac{k(k+1)(4k-1)+6(2k+1)(k+1)}{3} \\ &= \frac{(k+1)(4k^2-k+12k+6)}{3} \\ &= \frac{(k+1)(k+2)(4k+3)}{3} = \frac{(k+1)[(k+1)+1][4(k+1)-1]}{3}\end{aligned}$$

as required. By the Principle of Mathematical Induction, the formula holds for all $n \geq 1$.

9. (a) [BB] If $n = 5$, $2^5 = 32$, $5^2 = 25$. Since $32 > 25$, the result is true for $n = 5$. Now suppose that $k \geq 5$ and the result is true for $n = k$; that is, suppose that $2^k > k^2$. We must prove that the result is true for $n = k + 1$; that is, we must prove that $2^{k+1} > (k+1)^2$. Now $2^{k+1} = 2 \cdot 2^k > 2k^2$ by the induction hypothesis, and $2k^2 = k^2 + k^2 \geq k^2 + 5k$ (since $k \geq 5$), and $k^2 + 5k = k^2 + 4k + k > k^2 + 2k + 1$ (since $k \geq 1$), and $k^2 + 2k + 1 = (k+1)^2$. By the Principle of Mathematical Induction, the result is true for all $n \geq 5$.

   (b) Since $2^{-3} = \frac{1}{8}$, the inequality holds when $n = -3$. Now assume that $k \geq -3$ and $2^k \geq \frac{1}{8}$. We have

$$\begin{aligned}2^{k+1} &= 2(2^k) \geq 2(\frac{1}{8}) \qquad \text{by the induction hypothesis} \\ &= \frac{1}{4} > \frac{1}{8},\end{aligned}$$

   which is the desired result with $n = k + 1$. By the Principle of Mathematical Induction, we conclude that $2^n > \frac{1}{8}$ for all $n \geq -3$.

   (c) [BB] Since $6! = 720 > 216 = 6^3$, the inequality holds for $n = 6$. Now assume that $k \geq 6$ and that $k! > k^3$. We have

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)k^3 && \text{using the induction hypothesis} \\ &= k^4 + k^3 \\ &\geq 6k^3 + k^3 && \text{since } k \geq 6 \\ &= k^3 + 3k^3 + 3k^3 \\ &\geq k^3 + 3k^2 + 3(36)k && \text{since } k^2 \geq 36 \\ &\geq k^3 + 3k^2 + 3k + 1 = (k+1)^3\end{aligned}$$

   which is the desired inequality with $n = k + 1$. By the Principle of Mathematical Induction, we conclude that $n! > n^3$ for all $n \geq 6$.

(d) When $n = 1$, $(1 + \frac{1}{2})^1 = \frac{3}{2}$, while $1 + \frac{1}{2} = \frac{3}{2}$ so the result is true. Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose $(1+\frac{1}{2})^k \geq 1+\frac{k}{2}$. We must suppose that the statement is true for $n = k + 1$; that is, we must prove that $(1 + \frac{1}{2})^{k+1} \geq 1 + \frac{k+1}{2}$. Now

$$\begin{aligned}\left(1+\frac{1}{2}\right)^{k+1} &= \left(1+\frac{1}{2}\right)^k\left(1+\frac{1}{2}\right)\\ &\geq \left(1+\frac{k}{2}\right)\left(1+\frac{1}{2}\right) \quad \text{(by the induction hypothesis)}\\ &= 1+\frac{k}{2}+\frac{1}{2}+\frac{k}{4} > 1+\frac{k+1}{2}\end{aligned}$$

since $k > 0$. This is what we wanted to show. Thus, by the Principle of Mathematical Induction, the result is true for all natural numbers.

(e) When $n = 1$, $(1 + x)^n = 1 + x$, while $1 + nx = 1 + x$. Thus, the result is true for $n = 1$. Now suppose that $k \geq 1$ and the result is true for $n = k$; thus, we suppose that $(1 + x)^k \geq 1 + kx$. We must prove it's true for $n = k + 1$; that is, we must prove that $(1 + x)^{k+1} \geq 1 + (k + 1)x$. Now

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x) \quad \text{(by the induction hypothesis)}\\ &= 1+kx+x+kx^2 > 1+(k+1)x\end{aligned}$$

since $kx^2 \geq 0$, and this is what we wanted. So, by the Principle of Mathematical Induction, the given statement is true for all $n \geq 1$.

(f) For $n = 2$,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12} > \frac{13}{24}.$$

Now suppose $k \geq 2$ and the inequality holds for $k$; that is, suppose that

$$\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k} > \frac{13}{24}.$$

We must prove that the inequality holds for $k + 1$; that is, we must prove that

$$\frac{1}{k+2}+\frac{1}{k+3}+\frac{1}{k+4}+\cdots+\frac{1}{2k+2} > \frac{13}{24}.$$

Now,

$$\begin{aligned}\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k+2} &= \frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}\\ &= \frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{1}{k+1}\end{aligned}$$

adding and subtracting $\frac{1}{k+1}$

$$> \frac{13}{24}+\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}.$$

using the induction hypothesis. Now the sum of the last three terms is

$$\frac{1}{2k+1}+\frac{1}{2}\left(\frac{1}{k+1}\right)-\frac{1}{k+1}=\frac{1}{2k+1}-\frac{1}{2(k+1)}>0$$

since $2k+1<2(k+1)$. Hence,

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2(k+1)}>\frac{13}{24},$$

as desired. Thus, by the Principle of Mathematical Induction, the given inequality is true for all $n \geq 2$.

(g) For $n=2$, $\frac{1}{1^2}+\frac{1}{2^2}=1+\frac{1}{4}=\frac{5}{4}<\frac{3}{2}=2-\frac{1}{2}$, so the inequality holds. Now assume that $k \geq 2$ and that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}<2-\frac{1}{k}$. We have

$$\begin{aligned}
&\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2} \\
&\qquad=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}\right)+\frac{1}{(k+1)^2} \\
&\qquad<\left(2-\frac{1}{k}\right)+\frac{1}{(k+1)^2} \qquad \text{using the induction hypothesis} \\
&\qquad<2-\frac{(k+1)^2-k}{k(k+1)^2} \\
&\qquad=2-\frac{k^2+k+1}{k(k+1)^2}<2-\frac{k^2+k}{k(k+1)^2}=2-\frac{1}{k+1},
\end{aligned}$$

which is the desired inequality with $n=k+1$. By the Principle of Mathematical Induction, we conclude that the inequality holds for all $n \geq 2$.

(h) When $n=2$, $\sum_{i=1}^{n}\frac{1}{\sqrt{i}}=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}=1+\frac{1}{\sqrt{2}}$. Now $1+\frac{1}{\sqrt{2}}=1+\frac{\sqrt{2}}{2}>\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$ (since $\sqrt{2}\approx 1.4$) and $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=2\frac{\sqrt{2}}{2}=\sqrt{2}$. Thus, $\sum_{i=1}^{n}\frac{1}{\sqrt{i}}>\sqrt{n}$ when $n=2$. Now suppose that $k \geq 2$ and the given assertion is true for $n=k$; that is, suppose $\sum_{i=1}^{k}\frac{1}{\sqrt{i}}>\sqrt{k}$. We have to prove that the assertion is true for $n=k+1$; that is, we have to prove that $\sum_{i=1}^{k+1}\frac{1}{\sqrt{i}}>\sqrt{k+1}$. Now

$$\begin{aligned}
\sum_{i=1}^{k+1}\frac{1}{\sqrt{i}} &= \sum_{i=1}^{k}\frac{1}{\sqrt{i}}+\frac{1}{\sqrt{k+1}} \\
&> \sqrt{k}+\frac{1}{\sqrt{k+1}} \quad \text{(by the induction hypothesis)} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}=\frac{\sqrt{k^2+k}+1}{\sqrt{k+1}}>\frac{\sqrt{k^2}+1}{\sqrt{k+1}}=\frac{k+1}{\sqrt{k+1}}=\sqrt{k+1}
\end{aligned}$$

as desired. By the Principle of Mathematical Induction, the given assertion is true for all $n \geq 2$.

(i) [BB] With $n=2$, $2(2)-1=3 \geq 2(2)-2=2$, so the inequality holds. Now assume $k \geq 2$ and

that $1(3)(5)\cdots(2k-1) \geq 2(4)(6)\cdots(2k-2)$. We have

$$\begin{aligned} 1(3)(5)\cdots[2(k+1)-1] &= 1(3)(5)\cdots(2k+1) \\ &= \big(1(3)(5)\cdots(2k-1)\big)(2k+1) \\ &\geq \big(2(4)(6)\cdots(2k-2)\big)(2k+1) \quad \text{using the induction hypothesis,} \\ &\geq 2(4)(6)\cdots(2k-2)(2k), \end{aligned}$$

which is the desired inequality with $n = k+1$ (since $2k = 2(k+1)-2$). By the Principle of Mathematical Induction, $1(3)(5)\cdots(2n-1) \geq 2(4)(6)\cdots(2n-2)$ for every integer $n \geq 2$.

10. (a) [BB] For $n = 1$, $\sum_{i=1}^{1}(x_i + y_i)$ is, by convention, just the single term $x_1 + y_1$. Since this equals $\sum_{i=1}^{1} x_1 + \sum_{i=1}^{1} y_1$ the formula holds for $n = 1$. Now suppose that $k \geq 1$ and the formula holds for $n = k$. Then

$$\begin{aligned} \sum_{i=1}^{k+1}(x_i + y_i) &= \Big[\sum_{i=1}^{k}(x_i + y_i)\Big] + (x_{k+1} + y_{k+1}) \\ &= \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + (x_{k+1} + y_{k+1}) \quad \text{(by the induction hypothesis)} \\ &= \Big[\Big(\sum_{i=1}^{k} x_i\Big) + x_{k+1}\Big] + \Big[\Big(\sum_{i=1}^{k} y_i\Big) + y_{k+1}\Big] = \sum_{i=1}^{k+1} x_i + \sum_{i=1}^{k+1} y_i, \end{aligned}$$

which is the formula with $n = k+1$. Thus, the result holds for all $n \geq 1$ by the Principle of Mathematical Induction.

(b) For $n = 1$, $\sum_{i=1}^{n} cx_i = \sum_{i=1}^{1} cx_i = cx_1 = c\sum_{i=1}^{1} x_i$ and the formula is true. Now suppose that $k \geq 1$ and the formula is true for $n = k$. Then

$$\begin{aligned} \sum_{i=1}^{k+1} cx_i &= \Big(\sum_{i=1}^{k} cx_i\Big) + cx_{k+1} = \Big(c\sum_{i=1}^{k} x_i\Big) + cx_{k+1}, \quad \text{(by the induction hypothesis)} \\ &= c\Big[\Big(\sum_{i=1}^{k} x_i\Big) + x_{k+1}\Big] = c\sum_{i=1}^{k+1} x_i, \end{aligned}$$

which is the formula with $n = k+1$. Thus, the result holds for all $n \geq 1$ by the Principle of Mathematical Induction.

(c) For $n = 2$, $\sum_{i=2}^{n}(x_i - x_{i-1}) = \sum_{i=2}^{2}(x_i - x_{i-1})$ which is, by convention, the single term $x_2 - x_{2-1} = x_2 - x_1 = x_n - x_1$. Thus, the formula is true for $n = 2$. Now suppose that $k \geq 2$ and the formula

is true for $n = k$. Then

$$\sum_{i=2}^{k+1}(x_i - x_{i-1}) = \left(\sum_{i=2}^{k}(x_i - x_{i-1})\right) + (x_{k+1} - x_{(k+1)-1})$$

$$= (x_k - x_1) + (x_{k+1} - x_k) \quad \text{(by the induction hypothesis)}$$

$$= x_{k+1} - x_1,$$

which is the formula with $n = k + 1$. Thus, the result holds for all $n \geq 2$ by the Principle of Mathematical Induction.

11. [BB] The $k$ to $k + 1$ step does not apply to the case $k = 1$. When $k = 1$, $G = \{a_1, a_2\}$. Observe that the groups $\{a_1\}$, $\{a_2\}$ have no member in common.

12. [BB] The induction was not started properly. When $n = 1$, the left side is 1, while the right side is $\frac{9}{8}$. The statement is not true when $n = 1$.

13. As with the example at the end of this section, the problem is that the induction hypothesis has been applied to an integer to which it does not necessarily apply. The induction hypothesis applies only to integers $\ell$ in the range $10 \leq \ell < k$, $10 \leq \ell$ being implicit because all integers in this problem are at least $n_0 = 10$. The induction hypothesis was applied to the integer $k - 5$, but this is not valid for $k < 15$.

14. The argument presented is flawed by its failure to specify $n_0$ and to verify the statement for $n = n_0$. Obviously one cannot achieve a total score of 1 or 2 points with field goals and touchdowns although one can achieve any score above 11 this way. Twelve points can be achieved with four field goals. Then the argument given in the exercise can be easily modified to show that for $k \geq 12$, $k$ points possible implies $k + 1$ points possible. If two field goals were involved in achieving $k$ points, then subtracting the corresponding six points and adding two touchdowns gives a total of $k + 1$ points. On the other hand, if two field goals were not required to achieve $k$ then at most three points of the $k$ came via field goals. Since $k \geq 12$ and at least 9 points did not come with field goals, at least two touchdowns were required. Now continue with the given argument.

15. [BB] If a set $A$ contains $n = 0$ elements, then $A = \emptyset$ and $A$ has exactly 1 subset, namely, $\emptyset$. Since $1 = 2^0$, the statement is true for $n = 0$. Now assume that a set of $k$ elements has $2^k$ subsets. Let $A = \{a_1, a_2, \ldots, a_{k+1}\}$ be a set of $k + 1$ elements. We show that $A$ must have $2^{k+1}$ subsets. This follows because every subset of $A$ either contains $a_{k+1}$ or it doesn't. By the induction hypothesis there are $2^k$ sets which do not contain $a_{k+1}$; but, by the induction hypothesis, there are also $2^k$ subsets which **do** contain $a_{k+1}$ since these are precisely those sets obtained by forming the union of a subset of $\{a_1, a_2, \ldots, a_k\}$ and $\{a_{k+1}\}$. Altogether $A$ has $2^k + 2^k = 2(2^k) = 2^{k+1}$ subsets. So, by the Principle of Mathematical Induction, we conclude that a set of $n$ elements contains $2^n$ subsets.

16. [BB] Let $S$ be a nonempty set of natural numbers. Let $a$ be any element of $S$. Then the smallest element of $S$ is the smallest element of the nonempty set $\{x \in S \mid 1 \leq x \leq a\}$ (which is finite).

17. (a) We use induction on $n$. If $n = 1$, the set has a single element which is its smallest element. Next, assume that $k \geq 1$ and the result holds for $n = k$; that is, assume that any set of $k \geq 1$ real numbers has a smallest element. We must prove that the statement is true for $n = k + 1$; that is, we must prove that any set of $k + 1$ real numbers has a smallest element. Let $S$ be a set of $k + 1$ real numbers. Since $k + 1 > 1$, $S \neq \emptyset$. Let $x \in S$. Since $|S \setminus \{x\}| = k$, we know that $S \setminus \{x\}$

has a smallest element, say $y$, by the induction hypothesis. If $y > x$, then $x$ is the smallest element in $S$, while, if $x > y$, then $y$ is the smallest element in $S$. In either case, $S$ has a smallest element, as required. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

(b) The real numbers is an infinite set without a smallest element: For any $r \in \mathsf{R}$, $r - 1$ is always smaller than $r$. On the other hand, the natural numbers does have a smallest element.

(c) The Well-Ordering Principle!

18. [BB] For $n = 1$, $\bigcap_{i=1}^{n} B_i = B_1$, and so $A \cup \left(\bigcap_{i=1}^{n} B_i\right) = A \cup B_1 = \bigcap_{i=1}^{n} (A \cup B_i)$ and the result holds for $n = 1$. Now assume that $A \cup \left(\bigcap_{i=1}^{k} B_i\right) = \bigcap_{i=1}^{k} (A \cup B_i)$ for $k \geq 1$. Then, given a set $A$ and $k+1$ sets $B_1, B_2, \ldots, B_{k+1}$, we have

$$\begin{aligned} A \cup \left(\bigcap_{i=1}^{k+1} B_i\right) &= A \cup \left[\left(\bigcap_{i=1}^{k} B_i\right) \cap B_{k+1}\right] \\ &= \left[A \cup \left(\bigcap_{i=1}^{k} B_i\right)\right] \cap \left(A \cup B_{k+1}\right) \quad \text{since } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ &= \left[\bigcap_{i=1}^{k} (A \cup B_i)\right] \cap (A \cup B_{k+1}) \quad \text{(by the induction hypothesis)} \\ &= \bigcap_{i=1}^{k+1} \left(A \cup B_i\right), \end{aligned}$$

giving the result for $k + 1$. By the Principle of Mathematical Induction, we conclude that $A \cup \left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n} \left(A \cup B_i\right)$, for all $n \geq 1$.

19. For $n = 1$, $\bigcup_{i=1}^{n} B_i = B_1$ and so

$$A \cap \left(\bigcup_{i=1}^{n} B_i\right) = A \cap B_1 = \bigcup_{i=1}^{n} (A \cap B_i)$$

and the result holds for $n = 1$. Now assume that

$$A \cap \left(\bigcup_{i=1}^{k} B_i\right) = \bigcup_{i=1}^{k} \left(A \cap B_i\right)$$

for $k \geq 1$. Then, given a set $A$ and $k+1$ sets $B_1, B_2, \ldots, B_{k+1}$, we have

$$\begin{aligned} A \cap \left(\bigcup_{i=1}^{k+1} B_i\right) &= A \cap \left[\left(\bigcup_{i=1}^{k} B_i\right) \cup B_{k+1}\right] \\ &= \left[A \cap \left(\bigcup_{i=1}^{k} B_i\right)\right] \cup (A \cap B_{k+1}) \quad \text{since } A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ &= \left[\bigcup_{i=1}^{k} (A \cap B_i)\right] \cup (A \cap B_{k+1}) \quad \text{(by the induction hypothesis)} \\ &= \bigcup_{i=1}^{k+1} (A \cap B_i), \end{aligned}$$

giving the result for $k + 1$. By the Principle of Mathematical Induction, we conclude that $A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} \left(A \cap B_i\right)$, for all $n \geq 1$.

20. We proceed by induction on $n$. For $n = 2$, $A_1 \oplus A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ consists of those elements of $A_1 \cup A_2$ which belong to exactly one of $A_1$, $A_2$. Since 1 is odd, the result holds. Now assume that $k \geq 2$ and the result is true for $n = k$. We must prove it is true for $n = k + 1$. In this case, the set of interest is

$$A_1 \oplus A_2 \oplus \cdots \oplus A_{k+1} = (A_1 \oplus A_2 \oplus \cdots \oplus A_k) \oplus A_{k+1}$$

Putting $S = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, we are concerned with the set $S \oplus A_{k+1} = (S \setminus A_{k+1}) \cup (A_{k+1} \setminus S)$. Now, if $x \in S \setminus A_{k+1}$, then since $x \in S$, $x$ belongs to an odd number of the sets $A_1, A_2, \ldots, A_k$ (by

the induction hypothesis). Since $x \notin A_{k+1}$, $x$ belongs to an odd number of the sets $A_1, A_2, \ldots, A_{k+1}$. On the other hand, if $x \in A_{k+1} \setminus S$, then, since $x \notin S$, we know that $x$ belongs to an **even** number (possibly zero) of the sets $A_1, A_2, \ldots, A_k$, again by the induction hypothesis. Since $x \in A_{k+1}$, we conclude again that $x$ belongs to an odd number of the sets $A_1, A_2, \ldots, A_{k+1}$. This proves that the symmetric difference of $A_1, \ldots, A_{k+1}$ consists of elements which belong to an odd number of these sets.

Conversely, suppose $x$ belongs to an odd number of the sets $A_1, \ldots, A_{k+1}$. We must show $x$ belongs to the symmetric difference of $A_1, \ldots, A_{k+1}$. If $x$ belongs to $A_{k+1}$, then $x$ belongs to an even number of the sets $A_1, \ldots, A_k$ and, hence, $x \in A_{k+1} \setminus S$ by the induction hypothesis. However, if $x$ does not belong to $A_{k+1}$, then clearly $x \in S \setminus A_{k+1}$. Thus, $x$ belongs to the symmetric difference of $A_1, \ldots, A_{k+1}$.

By the Principle of Mathematical Induction, we conclude that for all $n \geq 2$, $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ consists of those elements which are members of an odd number of the sets $A_1, \ldots, A_n$.

21. We prove this by induction on $t$, the number of congruences. The case $t = 1$ is obvious. Assume that $k \geq 1$ and the result is true for some integer $t = k$. With $t = k+1$, consider the system of congruences

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_{k+1} \pmod{m_{k+1}}. \end{aligned}$$

By the induction hypothesis, the first $k$ of these congruences has a solution $x \equiv a \pmod{m_1 m_2 \cdots m_k}$, which is unique mod $m_1 m_2 \cdots m_k$. Now consider the system of two congruences

$$\begin{aligned} x &\equiv a \pmod{m_1 m_2 \cdots m_k} \\ x &\equiv a_{k+1} \pmod{m_{k+1}} \end{aligned}$$

Since $\gcd(m_i, m_{k+1}) = 1$ for all $i \leq k$, it follows that

$$\gcd(m_1 m_2 \cdots m_k, m_{k+1}) = 1$$

and the case $t = 2$ proven in the text (see 8) says that this system has a unique solution modulo $(m_1 m_2 \cdots m_k) m_{k+1}$, as required. By the Principle of Mathematical Induction, the result is proved for all $t \geq 1$.

22. [BB] When $n = 2$, $\gcd(a_1, a_2) = s_1 a_1 + s_2 a_2$ by Theorem 4.2.9. Now suppose that $k \geq 2$ and the statement is true for $n = k$; that is, suppose that the gcd of any $k \geq 2$ integers is an integral linear combination of them. We have to show the statement is true for $n = k + 1$, that is, that the gcd of $k + 1$ integers is an integral linear combination of them. Now $\gcd(a_1, a_2, \ldots, a_{k+1}) = \gcd(a_1, \gcd(a_2, \ldots, a_{k+1}))$. By the induction hypothesis, there are $k$ integers $r_2$, $r_3$, ..., $r_{k+1}$ such that $\gcd(a_2, \ldots, a_{k+1}) = \sum_{i=2}^{k+1} r_i a_i$. Then Theorem 4.2.9 tells us that

$$\gcd\left(a_1, \textstyle\sum_2^{k+1} r_i a_i\right) = x a_1 + y\left(\sum r_i a_i\right) = x a_1 + (y r_2) a_2 + \cdots (y r_{k+1}) a_{k+1}$$

as desired ($s_1 = x$, $s_2 = y r_2$, ..., $s_{k+1} = y r_{k+1}$). By the Principle of Mathematical Induction, the statement is true for all $n \geq 1$.

23. Let $g = \gcd(a_1, \ldots, a_n)$, as defined in Exercise 22. We first prove by induction on $n$ that $g$ divides each $a_i$.

    If $n = 3$, we know $g = \gcd(a_1, \gcd(a_2, a_3))$ divides both $a_1$ and $\gcd(a_2, a_3)$. Since $\gcd(a_2, a_3)$ divides $a_2$ and $a_3$, by definition (see Definition 4.2.4), we conclude that $g$ divides each of $a_1$, $a_2$ and $a_3$. Now assume that $k \geq 3$ and that the result is true for $n = k$. Given $k + 1$ nonzero integers $a_1, \ldots, a_{k+1}$, we know that $g = \gcd(a_1, \gcd(a_2, \ldots, a_{k+1})$ divides both $a_1$ and $\gcd(a_2, \ldots, a_{k+1})$. By the induction hypothesis, $\gcd(a_2, \ldots, a_{k+1})$ divides each of $a_2, \ldots, a_{k+1}$, hence, $g$ divides each of $a_1, \ldots, a_{k+1}$. By the Principle of Mathematical Induction, the result is proved.

    Now let $c$ be any common divisor of $a_1, \ldots, a_n$. Since $\gcd(a_2, \ldots, a_n)$ is an integral linear combination of $a_2, \ldots, a_n$ by Exercise 22, $c$ is a divisor of $\gcd(a_2, \ldots, a_n)$. Since $c$ also divides $a_1$ and $g$ is an integral linear combination of $a_1$ and $\gcd(a_2, \ldots, a_n)$, $c$ is a divisor of $g$. Certainly then $c \leq g$ and we have shown that all the properties in Definition 4.2.14 are satisfied.

24. The result is trivially true when $t = 1$. Now suppose that $k \geq 1$ and the result is true for $t = k$, that is, that if $k$ pairwise relatively prime integers divide an integer $n$ then so does their product. We must prove the result for $t = k + 1$; that is, we must prove that if $m_1, m_2, \ldots, m_{k+1}$ are relatively prime integers, each of which divides an integer $n$, then the product $m_1 \cdots m_{k+1}$ divides $n$. Let $m = m_1 \cdots m_k$. Any prime which divides $m$ and $m_{k+1}$ divides some $m_i$ and $m_{k+1}$ (Corollary 4.3.8). This is impossible because $m_i$ and $m_{k+1}$ are relatively prime. Thus, $m$ and $m_{k+1}$ are relatively prime. By the induction hypothesis $m \mid n$. Also $m_{k+1} \mid n$; thus, $mm_{k+1} \mid n$. (See Exercise 11 of Section 4.1.) We conclude that the product $m_1 \cdots m_{k+1} \mid n$, as required. By the Principle of Mathematical Induction, the given assertion is true for all $t \geq 1$.

25. (a) $f(1000) = 998$, $f(999) = f(f(1003)) = f(1001) = 999$,
    $f(998) = f(f(1002)) = f(1000) = 998$,
    $f(997) = f(f(1001)) = f(999) = 999$,
    $f(996) = f(f(1000)) = f(998) = 998$,
    $f(995) = f(f(999)) = 999$.

    (b) We conjecture that

$$f(1000 - n) = \begin{cases} 998 & \text{if } n \text{ is even} \\ 999 & \text{if } n \text{ is odd} \end{cases}$$

    and prove this by induction (strong form) on $n \geq 0$.

    For $n = 0$, $f(1000) = 998$ agreeing with the conjecture because 0 is even. Similarly, the conjecture is true for $n = 1$, 2 and 3. Now assume that $k > 3$ and the conjecture is true for all $n$, $0 \leq n < k$. Then $f(1000 - k) = f(f(1004 - k)) = f(f(1000 - (k - 4)))$. Since $k - 4 \geq 0$, we may apply the induction hypothesis to $n = k - 4$, obtaining

$$\begin{aligned} f(1000 - k) &= \begin{cases} f(998) & \text{if } k - 4 \text{ is even} \\ f(999) & \text{if } k - 4 \text{ is odd} \end{cases} \\ &= \begin{cases} 998 & \text{if } k \text{ is even} \\ 999 & \text{if } k \text{ is odd,} \end{cases} \end{aligned}$$

    which is the desired result for $n = k$. By the Principle of Mathematical Induction, we conclude that the conjecture is true for all $n \geq 0$.

    (c) $f(5) = f(1000 - 995) = 999$ because 995 is odd.
    $f(20) = f(1000 - 980) = 998$ because 980 is even.

(d) $\operatorname{rng} f = \{n \in \mathbb{Z} \mid n \geq 998\}$.

26. [BB]

(a) i. $0 \in n\mathbb{Z}$ because $0 = 0(n)$ is a multiple of $n$.

ii. If $a \in n\mathbb{Z}$, then $a = kn$ for some $k$, so $-a = (-k)n \in n\mathbb{Z}$ because it is also a multiple of $n$.

iii. If $a, b \in n\mathbb{Z}$, then $a = k_1 n$ and $b = k_2 n$ for some integers $k_1, k_2$, so $a+b = (k_1+k_2)n \in n\mathbb{Z}$ because it is also a multiple of $n$.

(b) i. $0 \in A$ by definition of ideal, so if $A$ contains just one element, that element must be 0, in which case $A = 0\mathbb{Z}$ is of the form $n\mathbb{Z}$.

ii. If $A$ contains more than element, it contains a nonzero element $a$. By definition of ideal, $A$ contains both $a$ and $-a$, one of which is positive.

iii. The set of positive integers in $A$ is not empty (by ii.) and, hence, contains a smallest element by the Well-Ordering Principle (4.1.2).

iv. First, $1n = n \in A$. Then, if $kn \in A$ for some $k > 0$, so also is $(k+1)n \in A$, by definition, because $(k+1)n = kn + n$. By the Principle of Mathematical Induction, we conclude that $kn \in A$ for all $k > 0$. Since $(-k)n = -kn$ is the negative of $kn$, all negative multiples of $n$ are in $A$. Finally, $0n = 0 \in A$, by definition of ideal. Thus, $kn \in A$ for any $k \in \mathbb{Z}$; that is, $n\mathbb{Z} \subseteq A$.

v. Let $a \in A$ and write $a = kn+r$, $0 \leq r < n$. We have $(-k)n \in A$ by iv, so $r = a+(-kn) \in A$, by definition of ideal. Since $r$ is smaller than the smallest positive integer in $A$, it cannot be that $r > 0$. So $r = 0$ and $a = kn \in n\mathbb{Z}$. Therefore, $A \subseteq n\mathbb{Z}$, as desired.

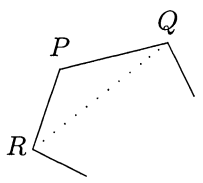
27. [BB] When $n = 2$, we have two distinct points in the plane joined by exactly one line. Since $\frac{2(2-1)}{2} = 1$, the result holds in this case. Now suppose the result is true for $n = k \geq 2$; that is, suppose that the number of lines obtained by joining $k$ distinct points in the plane, no three of which are collinear, is $\frac{1}{2}k(k-1)$. We must prove the result for $n = k+1$; that is, we must prove that the number of lines obtained by joining $k+1$ distinct points, no three of which are collinear, is $\frac{1}{2}(k+1)[(k+1)-1] = \frac{1}{2}k(k+1)$. Let $P$ be one of the $k+1$ points. By the induction hypothesis, the remaining $k$ points are joined by $\frac{1}{2}k(k-1)$ lines. On the other hand, $P$ can be joined to each of those remaining $k$ points, giving $k$ additional lines. (Since no three of the $k+1$ points are collinear, these additional lines are different from the lines determined by the other $k$ points.) The total number of lines is

$$\frac{k(k-1)}{2} + k = \frac{k(k-1)+2k}{2} = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$$

as desired. We conclude that the given assertion is true for all $n \geq 2$, by the Principle of Mathematical Induction.

28. If $n = 3$, we have a triangle. In this case, Euclidean geometry tells us that the sum of the interior angles is $180° = (3-2)180°$. Thus, the statement is true when $n = 3$. Now suppose that $k \geq 3$ and the result is true for $n = k$; that is, suppose that the sum of the interior angles of a $k$-gon is $(k-2)180°$. We must prove the statement for $n = k+1$; that is, we must prove that the sum of the interior angles of a $(k+1)$-gon is $[(k+1)-2]180° = (k-1)180°$.

Consider a $(k+1)$-gon. Let $P$ be a vertex and $Q$, $R$ the vertices adjacent to $P$. Join $QR$ as shown on the right. Since $k \geq 3$, the line $QR$ separates the figure into two distinct pieces, triangle $PQR$ and the $k$-gon formed by $QR$ and all sides of the original $(k+1)$-gon except $PQ$ and $PR$. By the induction hypothesis, the sum of the interior angles of this new $k$-gon equals $(k-2)180°$. Also, triangle $PQR$ has sum of interior angles equal to 180°.




The sum of the angles of the $(k+1)$-gon is the sum of the angles in the $k$-gon and in the triangle, that is, $(k-2)180° + 180° = (k-1)180°$, as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 3$.

29. We prove this by induction on $n$, the number of straight lines drawn. If $n = 1$, we have two regions which, if colored with different colors, gives a proper coloring using just two colors. (By "proper," we mean that bordering countries have different colors.) Now suppose that $k \geq 1$ and the statement is true for $n = k$; that is, suppose that a map made by drawing $k$ straight lines through a rectangle can be properly colored with just two colors. We must prove the statement is true for $n = k+1$; that is, we must prove that a map made by drawing $k+1$ straight lines through a rectangle can be properly colored with just two colors. Suppose we have such a map, drawn with $k+1$ lines. Remove one of the lines, call it $\ell$, and, using the induction hypothesis, properly color the resulting map. Now reinsert $\ell$ and reverse the colors on one side of it; that is, choose either side of $\ell$, call this side $S$, reverse the color on any country lying entirely within $S$, and for any country divided into two by $\ell$, change the color of just that portion lying within $S$. This provides a proper coloring of the original map. By the Principle of Mathematical Induction, we conclude that the result is true for all $n \geq 1$.

30. (a) [BB] First set aside two coins; then separate the remaining six into two piles of three and compare weights. If the weights are equal, the only possible light coin is one of the two set aside and this can be found in a second weighing. If one set of three is lighter than the other, then there is indeed a light coin and it is one of a known three. Remove one of these three and compare the weights of the remaining two coins (with a second weighing). Either one of these coins will be found to be lighter or the one set aside is the light one.

    (b) If $n = 1$, we have two coins and it is clear that a light one, if any, can be identified with a single weighing. Now let $k \geq 1$ and assume the result for $3^k - 1$ coins, at most one of which is light. Given $3^{k+1} - 1$ coins, at most one light, first set aside $3^k - 1$ of these in a pile we'll call $A$. This leaves $(3^{k+1} - 1) - (3^k - 1) = 3^{k+1} - 3^k = 2(3^k)$ coins; separate these into two equal piles $B$ and $C$ of $3^k$ coins each and compare their weights (with the first weighing). If the weights are the same, then the light coin, if any, is in $A$ and, using the induction hypothesis, can be identified with an additional $k$ weighings, giving $k+1$ weighings in all. If the weights of $B$ and $C$ differ, then there is indeed a light coin and, without loss of generality, we may assume that this coin is in pile $B$. Remove one coin from $B$. Using the induction hypothesis applied to the remaining $3^k - 1$ coins, an additional $k$ weighings will either turn up a light coin or confirm that the one first removed from $B$ was the light one.

31. (a) [BB] False. If $n = 3$, $5^n + n + 1 = 129$ and $7 \nmid 129$.

    (b) False. If $n = 1$, $\sum_{k=0}^{n}(k+1) = 1 + 2 = 3$, while $n(n+3)/2 = (1 \cdot 4)/2 = 2$.

    (c) False. If $n = 3$, $\gcd(\frac{(n+2)!}{3}, \frac{(n+3)!}{2}) = \gcd(\frac{5!}{3}, \frac{6!}{2}) = \gcd(40, 360) = 40$, while $(n+2)!/6 = 5!/6 = 20$.

    (d) False. Using L'Hôpital's Rule, $\lim_{n\to\infty} 2^n/n^{15} = +\infty$, so $2^n > n^{15}$ for all sufficiently large $n$.

32. We use induction on $n$ with $n_0 = 2$. For $n = 2$, certainly $\dfrac{p_1 \to p_2}{p_1 \to p_2}$ is a valid argument. Now assume that $k \geq 2$ and the result is true for $k$. Given the premises

$$\begin{aligned} p_1 &\to p_2 \\ p_2 &\to p_3 \\ &\vdots \\ p_{k-1} &\to p_k \\ p_k &\to p_{k+1} \end{aligned}$$

we note that the first $k-1$ yield $p_1 \to p_k$ by the induction hypothesis. Together with $p_k \to p_{k+1}$, we obtain the conclusion $p_1 \to p_{k+1}$ by the chain rule. So the result is true by the Principle of Mathematical Induction.

33. If $n = 1$, $x^1 = x$, $\frac{d}{dx}x = 1 = 1x^0$ and the formula is correct. Now suppose that $k \geq 1$ and the formula is correct for $n = k$; that is, suppose that $\frac{d}{dx}x^k = kx^{k-1}$. We must prove that the formula is correct for $n = k+1$; that is, we must prove that $\frac{d}{dx}x^{k+1} = (k+1)x^{(k+1)-1} = (k+1)x^k$. Using the product rule,

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x^k \cdot x) = \left(\frac{d}{dx}x^k\right) \cdot x + x^k \frac{d}{dx}x = kx^{k-1} \cdot x + x^k \cdot 1 = kx^k + x^k = (k+1)x^k$$

as required. By the Principle of Mathematical Induction, we conclude that the formula is correct for all $n \geq 1$.

34. When $n = 1$, $\prod_{r=1}^{n}(1 + x^{2^r}) = 1 + x^2$ and $\dfrac{1 - x^{2^{r+1}}}{1 - x^2} = \dfrac{1 - x^4}{1 - x^2} = 1 + x^2$ and so the formula is correct. Now suppose that $k \geq 1$ and the result is true for $n = k$; that is, suppose

$$\prod_{r=1}^{k}(1 + x^{2^r}) = \frac{1 - x^{2^{k+1}}}{1 - x^2}.$$

We must prove that it is correct for $n = k + 1$; that is, we must prove that

$$\prod_{r=1}^{k+1}(1 + x^{2^r}) = \frac{1 - x^{2^{k+2}}}{1 - x^2}.$$

Now $\displaystyle\prod_{r=1}^{k+1}(1 + x^{2^r})$

$$= \left(\prod_{r=1}^{k}(1 + x^{2^r})\right)(1 + x^{2^{k+1}}) \quad = \left(\frac{1 - x^{2^{k+1}}}{1 - x^2}\right)(1 + x^{2^{k+1}}) \quad \text{(by the induction hypothesis)}$$

$$= \frac{1 - (x^{2^{k+1}})^2}{1 - x^2} = \frac{1 - x^{2^{k+2}}}{1 - x^2}$$

as required. By the Principal of Mathematical Induction, we conclude that the formula is correct for all $n \geq 1$.

35. [BB] $\lim_{x\to\pm1}\dfrac{1-x^{2^{r+1}}}{1-x^2} = \lim_{x\to\pm1}\prod_{r=1}^{n}(1+x^{2^r})$ (using the result of Exercise 34)

$$= \prod_{r=1}^{n}(1+(\pm1)^{2^r}) = 2^n.$$

36. $\displaystyle\prod_{r=2}^{n}\frac{2r-1}{2r-3} = \frac{3}{1}\cdot\frac{5}{3}\cdots\frac{2n-1}{2n-3} = 2n-1.$

**Proof.** If $n=2$, $\prod_{r=2}^{n}\frac{2r-1}{2r-3} = \frac{3}{1} = 3 = 2(2)-1$ and so the result is true. Now suppose that $k \geq 1$ and the result is true for $n=k$; that is, suppose $\prod_{r=2}^{k}\frac{2r-1}{2r-3} = 2k-1$. We must prove it is true for $n=k+1$; that is, we must prove $\prod_{r=2}^{k+1}\frac{2r-1}{2r-3} = 2(k+1)-1 = 2k+1$. We have

$$\prod_{r=2}^{k+1}\frac{2r-1}{2r-3} = \Big(\prod_{r=2}^{k}\frac{2r-1}{2r-3}\Big)\Big(\frac{2(k+1)-1}{2(k+1)-3}\Big) = (2k-1)\Big(\frac{2k+1}{2k-1}\Big) = 2k+1$$

as claimed. By the Principle of Mathematical Induction, the result is true for all $n \geq 2$. ■

37. (a) For $n=1$, $\displaystyle\prod_{r=0}^{n-1}F_r = F_0 = 3 = 5-2 = F_1-2$, so the formula is correct.

Now let $k \geq 1$ and assume that $\displaystyle\prod_{r=0}^{k-1}F_r = F_k - 2$. Then

$$\prod_{r=0}^{k}F_r = F_k\prod_{r=0}^{k-1}F_r = F_k(F_k-2) \quad \text{(by the induction hypothesis)}$$

$$= F_k^2 - 2F_k = (2^{2^k}+1)^2 - 2(2^{2^k}+1)$$

$$= 2^{2^{k+1}} + 2(2^{2^k}) + 1 - 2(2^{2^k}) - 2 = 2^{2^{k+1}} + 1 - 2 = F_{k+1} - 2$$

By the Principle of Mathematical Induction, the desired result follows.

(b) Without loss of generality, we may assume $m < n$. By part (a), for some integer $s$ (a product of suitable $F_k$'s), we have $sF_m = F_n - 2$. Thus, if $p$ is prime, $p \mid F_m$ and $p \mid F_n$, then $p \mid 2$. Thus $p=2$. This is not possible since each $F_k$ is odd.

(c) Consider the infinite sequence $F_1, F_2, F_3, \ldots$. By part (b), there is a prime $p_m$ which divides $F_m$ but no other $F_n$. So the set $\{p_1, p_2, \ldots,\}$ is an infinite set of primes.

38. As suggested, let $P_n$ be the set of all polynomials of degree at most $n$ with integer coefficients. An element of $P_n$ is an expression of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in \mathrm{Z}$. By Exercise 27(b) of Section 3.3, $P_1$ is countable. Let $k > 0$ and assume that $P_k$ is countable. The function $a_0 + a_1x + \cdots + a_kx^k + a_{k+1}x^{k+1} \mapsto (a_0 + a_1x + \cdots + a_kx^k, a_{k+1})$ is a one-to-one correspondence between $P_{k+1}$ and $P_k \times \mathrm{Z}$. Since $P_k$ and Z are each countable, so is their direct product, by Exercise 27(a) of Section 3.3. Thus $P_{k+1}$ is countable and the result for $P_n$ follows from the Principle of Mathematical Induction.

39. (a) We prove that a set of $n \geq 1$ natural numbers has a smallest element. If $n = 1$, this is true because the single element of a set of size 1 is that set's smallest element. Now assume that $k > 1$ and that the statement is true for all $\ell$ in the range $1 \leq \ell < k$. We must prove it is true for $n = k$. Consider then a set $S$ of $k$ elements. Since $k > 1$, we can remove a single element $a$ from $S$. The remaining $k - 1$ elements have a smallest element $b$ (by the induction hypothesis) and the smaller of $a$ and $b$ is the smallest element of $S$. This proves that any nonempty **finite** set of natural numbers has a smallest element. It then follows by Exercise 16 that **any** nonempty set of natural numbers has a smallest element.

(b) Given a statement $\mathcal{P}$ involving the integer $n$, suppose

1. $\mathcal{P}$ is true for $n = n_0$, and
2. if $k > n_0$ and $\mathcal{P}$ is true for all $n$ in the range $n_0 \leq n < k$, then it's true also for $k$.

We must prove $\mathcal{P}$ is true for all integers $n \geq n_0$.

**Proof.** If $\mathcal{P}$ is not true for all integers $n \geq n_0$, then the set $S$ of such integers is a nonempty set of natural numbers and so, by the Well-Ordering Principle, has a smallest element $a$. Certainly $a > n_0$ and, by minimality of $a$, $\mathcal{P}$ is true for all $n$ in the range $n_0 \leq n < a$. By (2), $\mathcal{P}$ is true also for $a$, a contradiction. ■

40. (a) For $n = 1$, it is given that $P(1)$ is true. Now assume that $k \geq 1$ and $P(k)$ is true. We must prove that $P(k+1)$ is true. By (2), we know that $P(2k)$ is true and so, by (3), that $P(2k-1)$ is true (since $2k \geq 2$). Continuing to use (3), we then obtain that $P(2k-2)$, $P(2k-3)$ and finally, $P(2k-(k-1))$ (that is, $P(k+1)$) are true.

(b) $$\frac{x+y}{2} \geq \sqrt{xy} \iff \frac{x^2+2xy+y^2}{4} \geq xy, \quad \text{since } x \geq 0 \text{ and } y \geq 0$$
$$\iff x^2+2xy+y^2 \geq 4xy \iff x^2-2xy+y^2 \geq 0,$$

which is true, because $x^2 - 2xy + y^2 = (x-y)^2$ is a perfect square.

(c) Let $P(n)$ be the statement $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$.

1) For $n = 1$, the left side is $a_1$ and the right side is also $a_1$, so $P(1)$ is true.

2) Assume $k \geq 1$ and $P(k)$ is true; that is, assume

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \cdots a_k}$$

for any $k$ nonnegative numbers $a_1, \ldots, a_k$. We must prove that $P(2k)$ is true. Assume then that $a_1, a_2, \ldots, a_{2k}$ are $2k$ nonnegative numbers. Then

$$\frac{a_1 + a_2 + \cdots + a_{2k}}{2k} = \frac{1}{2}\left(\frac{a_1 + \cdots + a_k}{k} + \frac{a_{k+1} + \cdots + a_{2k}}{k}\right)$$
$$\geq \frac{1}{2}\left(\sqrt[k]{a_1 \cdots a_k} + \sqrt[k]{a_{k+1} \cdots a_{2k}}\right) \quad \text{using } P(k) \text{ twice}$$
$$\geq \left((a_1 \cdots a_k)^{1/k}(a_{k+1} \cdots a_{2k})^{1/k}\right)^{1/2} \quad \text{[using the result of part (b)]}$$
$$= (a_1 \cdots a_{2k})^{1/2k} = \sqrt[2k]{a_1 \cdots a_{2k}},$$

which is $P(2k)$.

3) Let $k \geq 2$ and assume $P(k)$ is true. We must prove that $P(k-1)$ is true. So we suppose that $a_1, \ldots, a_{k-1}$ are $k-1$ nonnegative numbers. The idea is now to apply $P(k)$ to the $k$ numbers $a_1, a_2, \ldots, a_{k-1}, \sqrt[k-1]{a_1 \cdots a_{k-1}}$ since the $k$th root of the product of these numbers is

$$\begin{aligned}\sqrt[k]{a_1 \cdots a_{k-1}(a_1 \cdots a_{k-1})^{\frac{1}{k-1}}} &= \sqrt[k]{(a_1 \cdots a_{k-1})^{1+\frac{1}{k-1}}} \\ &= \sqrt[k]{(a_1 \cdots a_{k-1})^{\frac{k}{k-1}}} \\ &= (a_1 \cdots a_{k-1})^{1/k-1} = \sqrt[k-1]{a_1 \cdots a_{k-1}}.\end{aligned}$$

Thus,

$$\begin{aligned}\sqrt[k-1]{a_1 \cdots a_{k-1}} &= \sqrt[k]{(a_1 \cdots a_{k-1})\sqrt[k-1]{a_1 \cdots a_{k-1}}} \\ &\leq \frac{a_1 + a_2 + \cdots + a_{k-1} + \sqrt[k-1]{a_1 \cdots a_{k-1}}}{k}, \quad \text{using } P(k) \\ &= \frac{a_1 + \cdots + a_{k-1}}{k} + \frac{\sqrt[k-1]{a_1 \cdots a_{k-1}}}{k}\end{aligned}$$

and so

$$\begin{aligned}\sqrt[k-1]{a_1 \cdots a_{k-1}}\left(1 - \frac{1}{k}\right) &\leq \frac{a_1 + \cdots + a_{k-1}}{k} \\ \sqrt[k-1]{a_1 \cdots a_{k-1}}\left(\frac{k-1}{k}\right) &\leq \frac{a_1 + \cdots + a_{k-1}}{k} \\ \sqrt[k-1]{a_1 \cdots a_{k-1}} \leq \left(\frac{k}{k-1}\right)\frac{a_1 + \cdots + a_{k-1}}{k} &= \frac{a_1 + \cdots + a_{k-1}}{k-1},\end{aligned}$$

which is $P(k-1)$, as desired.

41. (With thanks to M. Oleson) (a) Assume to the contrary that $mn - m - n = am + bn$ where $a$ and $b$ are nonnegative integers. Then $m(n-1-a) = n(1+b)$, so $n \mid m(n-1-a)$. Since $m$ and $n$ are relatively prime, $n \mid (n-1-a)$ by Corollary 4.2.10, which is impossible since $0 < n-1-a < n$.

(b) There exist integers $a$ and $b$, $0 < a < n$, $0 < b < m$, such that $bn = am - 1$ and hence $(n-a)m = (m-b)n - 1$. Suppose we have achieved postage $r$ with $m$- and $n$-cent stamps. As long as we used at least $a$ $m$-cent stamps or $m-b$ $n$-cent stamps, we can make postage $r-1$. If we had $a$ $m$-cent stamps, remove these and replace with $b$ $n$-cent stamps (and note that $bn = am-1$). If we had $m - b$ $n$-cent stamps, remove these and replace with $n - a$ $m$-cent stamps (and note that $(n-a)m = (m-b)n - 1$). Thus the first time we have possible problems is when we have fewer than $a$ $m$-cent stamps and fewer than $m-b$ $n$-cent stamps, that is, when the current postage is $(a-1)m + (m-b-1)n$ cents. Note that this critical number is $mn - m - n + 1$. By part (a), we cannot achieve $mn - m - n$ cents. For any $r > mn - m - n + 1$, we must have at least $a$ $m$-cent stamps or $m - b$ $n$-cent stamps and we have argued we may achieve $r - 1$. (In particular, we can achieve $[(mn - m - n + 1) + 1] - 1 = mn - m - n + 1$.) If you can go from $r$ to $r-1$, you can go from $r-1$ to $r$. So we can achieve every postage $r \geq mn - m - n + 1$.

## Exercises 5.2

1. (a) [BB] $a_1 = 1$; $a_{k+1} = 5a_k$ for $k \geq 1$.

   (b) $a_1 = 5$; $a_{k+1} = a_k - 2$ for $k \geq 1$.

   (c) $a_1 = 4$; $a_2 = 1$; $a_{k+2} = a_k - a_{k+1}$ for $k \geq 1$.

   (d) There are two possibilities:
   $a_1 = 1$; $a_{k+1} = a_k + (-1)^{k+1}k$ for $k \geq 1$.
   Also, $a_1 = 1$, $a_2 = 2$, $a_k = a_{k-2} + (-1)^k$ for $k \geq 3$.

2. (a) [BB] $16, 8, 4, 2, 1, 1, 1$; (b) $17, 8, 4, 2, 1, 1, 1$;
   (c) $18, 9, 4, 2, 1, 1, 1$; (d) $100, 50, 25, 12, 6, 3, 1$.

3. When $n = 1$, $3^{n-1} = 3^0 = 1$, which agrees with $a_1$. Now assume the result is true for $n = k$; that is, assume $a_k = 3^{k-1}$. We must prove the result for $n = k + 1$; that is, we must prove that $a_{k+1} = 3^{(k+1)-1} = 3^k$. Now $a_{k+1} = 3a_k = 3(3^{k-1})$ (by the induction hypothesis) $= 3^k$, as required. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

4. [BB] When $n = 1$, the formula becomes $\frac{(1-1)(1+2)(1^2+1+2)}{4} = 0$, which is $a_1$. Now assume that the formula is correct when $n = k$. When $n = k + 1$,

$$\begin{aligned} a_{k+1} &= (k+1)^3 + a_k = (k+1)^3 + \frac{(k-1)(k+2)(k^2+k+2)}{4} \\ &= \frac{4k^3 + 12k^2 + 12k + 4 + k^4 + 2k^3 + k^2 - 4}{4} \\ &= \frac{k^4 + 6k^3 + 13k^2 + 12k}{4} \\ &= \frac{k(k+3)(k^2+3k+4)}{4} = \frac{k(k+3)[(k+1)^2 + (k+1) + 2]}{4} \end{aligned}$$

   as desired. By the Principle of Mathematical Induction, we conclude that the formula is correct for all $n \geq 1$.

5. The first seven terms are $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \frac{21}{64}$.

   When $n = 1$, $\frac{1}{3}(1 - (-\frac{1}{2})^{n-1}) = 0$, agreeing with $a_1$.

   When $n = 2$, $\frac{1}{3}(1 - (-\frac{1}{2})^{2-1}) = \frac{1}{3}(\frac{3}{2}) = \frac{1}{2}$, agreeing with $a_2$. Now let $k > 2$ and assume the result is true for all $n$ in the interval $1 \leq n < k$. We wish to prove the result for $n = k$; that is, we wish to prove that $a_k = \frac{1}{3}(1 - (-\frac{1}{2})^{k-1})$. Now, $a_k = \frac{1}{2}(a_{k-2} + a_{k-1})$ since $k > 2$ and, applying the induction hypothesis to $k - 2$ and $k - 1$, we have

$$\begin{aligned} a_k &= \tfrac{1}{2}(\, a_{k-2} + a_{k-1}\,) \\ &= \tfrac{1}{2}\,[\,\tfrac{1}{3}(\,1 - (-\tfrac{1}{2})^{k-3}\,) + \tfrac{1}{3}(\,1 - (-\tfrac{1}{2})^{k-2}\,)\,] \\ &= \tfrac{1}{3}\,[\,\tfrac{1}{2} - \tfrac{1}{2}(-\tfrac{1}{2})^{k-3} + \tfrac{1}{2} - \tfrac{1}{2}(-\tfrac{1}{2})^{k-2}\,] \\ &= \tfrac{1}{3}\,[1 - (-\tfrac{1}{2})^{k-3}(\tfrac{1}{2} - \tfrac{1}{4})\,] = \tfrac{1}{3}\,[1 - (-\tfrac{1}{2})^{k-3}(-\tfrac{1}{2})^2)\,] = \tfrac{1}{3}(1 - (-\tfrac{1}{2})^{k-1}) \end{aligned}$$

   as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

6. [BB] The first six terms are $1, 3, 7, 15, 31, 63$. Our guess is that $a_n = 2^n - 1$. When $n = 1$, $2^1 - 1 = 1$, agreeing with $a_1$. Now assume that $k \geq 1$ and the result is true for $n = k$; that is, assume that $a_k = 2^k - 1$. We must prove the result is true for $n = k+1$; that is, we must prove that $a_{k+1} = 2^{k+1} - 1$. Now $a_{k+1} = 2a_k + 1 = 2(2^k - 1) + 1$ (by the induction hypothesis) $= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$, as required. By the Principle of Mathematical Induction, we conclude that the result is true for all $n \geq 1$.

7. The first six terms are $\frac{3}{2}, \frac{13}{2}, \frac{63}{2}, \frac{313}{2}, \frac{1563}{2}$ and $\frac{7813}{2}$. We guess that $a_n = \frac{5^n+1}{4}$. Note that when $n = 1$, the formula gives $\frac{5+1}{4} = \frac{3}{2}$, agreeing with $a_1$. Now assume that $a_k = \frac{5^k+1}{4}$ for some $k \geq 1$. We must prove that $a_{k+1} = \frac{5^{k+1}+1}{4}$. Now

$$\begin{aligned} a_{k+1} &= 5a_k - 1 \\ &= 5\left(\frac{5^k + 1}{4}\right) - 1 \qquad \text{by the induction hypothesis} \\ &= \frac{5^{k+1} + 5}{4} - 1 = \frac{5^{k+1} + 1}{4}, \end{aligned}$$

as desired.

8. (a) [BB] $a_2 = 1(a_1 + a_0) = 1(1 + 1) = 2$; $a_3 = 2(a_2 + a_1) = 2(2 + 1) = 2(3) = 6$; $a_4 = 3(a_3 + a_2) = 3(6 + 2) = 3(8) = 24$; $a_5 = 4(a_4 + a_3) = 4(24 + 6) = 4(30) = 120$.

   (b) We guess that $a_n = n!$ and prove this using the strong form of mathematical induction. Certainly the formula is correct for $n = 0$ since $0! = 1 = a_0$. We note also that it is correct for $n = 1$ since $1! = 1 = a_1$. Now suppose $k > 1$ and assume that the formula is correct for all $n$, $1 \leq n < k$; that is, if $n < k$, then $a_n = n!$. We must prove that the formula is correct for $n = k$; that is, we must prove that $a_k = k!$. Since $k \geq 2$, the given relation $a_{n+1} = n(a_n + a_{n-1})$ can be employed to give an expression for $a_k$. We have

$$\begin{aligned} a_k &= (k-1)(a_{k-2} + a_{k-1}) \\ &= (k-1)[(k-2)! + (k-1)!] \\ &= (k-1)(k-2)! + (k-1)(k-1)! \\ &= (k-1)! + (k-1)(k-1)! = (k-1)![1 + (k-1)] = (k-1)!(k) = k! \end{aligned}$$

   as required. We conclude, by the Principle of Mathematical Induction, that $a_n = n!$ for all $n \geq 0$.

9. [BB] $a_1 = 1$, $a_2 = 2^2 - a_1 = 4 - 1 = 3$, $a_3 = 3^2 - a_2 = 9 - 3 = 6$. $a_4 = 4^2 - a_3 = 16 - 6 = 10$, $a_5 = 5^2 - a_4 = 25 - 10 = 15$, $a_6 = 6^2 - a_5 = 36 - 15 = 21$. Thus, the first six terms are $1, 3, 6, 10, 15, 21$. We recognize these numbers as the six sums $1$, $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, $1 + 2 + 3 + 4 + 5$ and $1 + 2 + 3 + 4 + 5 + 6$ and so guess that $a_n = n(n + 1)/2$. (See Exercise 5(a) of Section 5.1.)

   For $n = 1$, $n(n + 1)/2 = 1(2)/2 = 1$, which agrees with $a_1$. Now assume the formula holds for $n = k$; that is, assume $a_k = \frac{k(k+1)}{2}$. We must prove the formula holds for $n = k + 1$; that is, we must prove that $a_{k+1} = \frac{(k+1)(k+2)}{2}$. Now

$$\begin{aligned} a_{k+1} &= (k+1)^2 - a_k = (k+1)^2 - \frac{k(k+1)}{2} \qquad \text{using the induction hypothesis} \\ &= \frac{2(k+1)^2 - k(k+1)}{2} = \frac{(k+1)[2(k+1) - k]}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

   as required. By the Principle of Mathematical Induction, we conclude that the result is true for all $n \geq 1$.

10. The first few terms are $1, 2, 3\cdot 2 = 6, 4(3\cdot 2) = 24, \ldots$. Our guess is that $a_n = n!$. When $n = 1$, $1! = 1$, in agreement with $a_1$. Now assume the formula is correct for $n = k$; that is, assume that $a_k = k!$. We must prove the formula is correct for $n = k + 1$; that is, we must prove that $a_{k+1} = (k + 1)!$. But $a_{k+1} = (k + 1)a_k = (k + 1)k! = (k + 1)!$, as desired. By the Principle of Mathematical Induction, the formula is true for all $n \geq 1$.

11. [BB] The first few terms are $0, 1, 0, 4, 0, 16, \ldots$. Our guess is that

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 4^{\frac{n}{2}-1} & \text{if } n \text{ is even.} \end{cases}$$

We will prove this using the strong form of mathematical induction. Note first that $a_1 = 0$ and $4^{\frac{2}{2}-1} = 4^0 = 1 = a_2$ so that our guess is correct for $a_1$ and $a_2$. Now let $k > 2$ and assume the result is true for all $n$, $1 \leq n < k$. We must show that the result is true if $n = k$. If $k$ is odd, then $a_k = 4a_{k-2} = 4(0) = 0$ since $a_{k-2} = 0$ by the induction hypothesis, $k - 2$ being odd. If $k$ is even, then

$$a_k = 4a_{k-2} = 4(4^{\frac{k-2}{2}-1}) \text{ (using the induction hypothesis) } = 4^{\frac{k-2}{2}} = 4^{\frac{k}{2}-1}$$

as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

12. (a) $a_0 = 2$, $a_1 = 3$, $a_2 = 3a_1 - 2a_0 = 5$, $a_3 = 3a_2 - 2a_1 = 9$, $a_4 = 3a_3 - 2a_2 = 17$ and $a_5 = 3a_4 - 2a_3 = 33$.

    (b) We guess $a_n = 2^n + 1$.

    (c) We use the strong form of mathematical induction (with $n_0 = 0$). When $n = 0$, we have $2^0 + 1 = 1 + 1 = 2$, in agreement with the given value for $a_0$. When $n = 1$, $2^1 + 1 = 3 = a_1$. Now that the formula has been verified for $k = 0$ and $k = 1$, we may assume that $k > 1$ and that $a_n = 2^n + 1$ for all $n$ in the interval $0 \leq n < k$. We wish to prove the formula is valid for $n = k$; that is, we wish to prove that $a_k = 2^k + 1$. Since $k \geq 2$, we know that $a_k = 3a_{k-1} - 2a_{k-2}$. Applying the induction hypothesis to $k - 1$ and to $k - 2$ (each of which are in the range $0 \leq n < k$), we have $a_{k-1} = 2^{k-1} + 1$ and $a_{k-2} = 2^{k-2} + 1$. Thus,

$$\begin{aligned} a_k &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) \\ &= 3(2^{k-1}) + 3 - 2^{k-1} - 2 = 2(2^{k-1}) + 1 = 2^k + 1 \end{aligned}$$

    as required. By the Principle of Mathematical Induction, the formula is valid for all $n \geq 0$.

    (d) Observe that $a_n = 2a_{n-1} - 1$ for all $n \geq 1$. We verify this is correct by observing that $2(2^{n-1} + 1) - 1 = 2^n + 1$.

13. [BB] We use the strong form of mathematical induction. When $n = 1$, the formula gives $2^1(1-1/2) = 1$, agreeing with $a_1$. When $n = 2$, the formula gives $2^2(1 - 1) = 0$, agreeing with $a_2$. Now let $k > 2$ and assume the formula is correct for all $1 \leq n < k$. We must prove it is correct for $n = k$. Since $k > 2$, we have $a_k = 4a_{k-1} - 4a_{k-2}$. Applying the induction hypothesis to both $k - 1$ and $k - 2$, we have

$$\begin{aligned} a_k &= 4\left[2^{k-1}(1 - \frac{k-1}{2})\right] - 4\left[2^{k-2}(1 - \frac{k-2}{2})\right] \\ &= 2^{k+1}\left(1 - \frac{k-1}{2}\right) - 2^k\left(1 - \frac{k-2}{2}\right) \\ &= 2^{k+1} - 2^k(k-1) - 2^k + 2^{k-1}(k-2) \\ &= 2(2^k) - k2^k + 2^k - 2^k + k2^{k-1} - 2^k = 2^k - k2^k + \frac{k}{2}2^k = 2^k\left(1 - k + \frac{k}{2}\right) = 2^k\left(1 - \frac{k}{2}\right) \end{aligned}$$

as desired. By the Principle of Mathematical Induction, we conclude that the formula is correct for all $n \geq 1$.

14. We have $a_2 = 1^2 a_1 = 1 \cdot 1 = 1$, $a_3 = 2^2 a_2 = 4 \cdot 1 = 4 = 2^2$, $a_4 = 3^2 a_3 = 3^2 \cdot 2^2 = 36$, $a_5 = 4^2 a_4 = 4^2 \cdot 3^2 \cdot 2^2 = 16 \cdot 9 \cdot 4 = 576$, and $a_6 = 5^2 a_5 = 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 = 25 \cdot 16 \cdot 9 \cdot 4 = 14400$. It seems that $a_n = (n-1)^2(n-2)^2 \cdots 3^2 \cdot 2^2 = [(n-1)!]^2$. So we make the conjecture that $a_n = [(n-1)!]^2$ for $n \geq 1$.

For $n = 1$, $[(1-1)!]^2 = 0!^2 = 1^2 = 1$, in agreement with the conjectured value of $a_1$. Now we assume the conjecture is true for $n = k$; that is, we assume that $a_k = [(k-1)!]^2$. We must prove the conjecture is true for $n = k+1$; that is, we must prove that $a_{k+1} = [((k+1)-1)!]^2$, that is, that $a_{k+1} = (k!)^2$. Now

$$\begin{aligned} a_{k+1} &= k^2 a_k = k^2[(k-1)!]^2 \quad \text{(using the induction hypothesis)} \\ &= [k(k-1)!]^2 = (k!)^2 \end{aligned}$$

as desired. We conclude that our conjecture is correct for all $n \geq 1$ by the Principle of Mathematical Induction.

15. (a) With five points, the circle is divided into $2^4 = 16$ regions.

    (b) We might guess $2^{n-1}$ regions for $n$ points, but this is incorrect. When $n = 6$, fewer than $2^5 = 32$ regions are formed (at most 31 depending on how the lines intersect).

16. (a) First we note that two integers in the same congruence class $\pmod 3$ go to different integers under $f$. If $m_1 = 3q_1$ and $m_2 = 3q_2$, then $f(m_1) = 2(3q_1)/3 = 2q_1$ and $f(m_2) = 2(3q_2)/3 = 2q_2$, so $m_1 \neq m_2$ implies $f(m_1) \neq f(m_2)$. If $m_1 = 3q_1 + 1$ and $m_2 = 3q_2 + 1$, then $f(m_1) = [4(3q_1+1)-1]/3 = 4q_1 + 1$ and $f(m_2) = 4q_2 + 1$ and again $m_1 \neq m_2$ implies $f(m_1) \neq f(m_2)$. Finally, if $m_1 = 3q_1 + 2$ and $m_2 = 3q_2 + 2$, then $f(m_1) = [4(3q_1 + 2) + 1]/3 = 4q_1 + 3$ and $f(m_2) = 4q_2 + 3$ and again, $m_1 \neq m_2$ implies $f(m_1) \neq f(m_2)$. Note in the above that $f(m)$ is even if and only if $m \equiv 0 \pmod 3$, $f(m) \equiv 1 \pmod 4$ if and only if $m \equiv 1 \pmod 3$ and $f(m) \equiv 3 \pmod 4$ if and only if $m \equiv 2 \pmod 3$. Thus, if $m_1$ and $m_2$ are in different congruence classes, then $f(m_1) \neq f(m_2)$. We have shown that $f$ is one-to-one.

    To show that $f$ is onto, let $n \in \mathbb{N}$. If $n = 2k$ is even, then $f(3k) = 2k = n$. If $n \equiv 1 \pmod 4$ (that is, $n = 4k + 1$), then $f(3k + 1) = [4(3k + 1) - 1]/3 = 4k + 1 = n$. Finally, if $n \equiv 3 \pmod 4$ (that is, $n = 4k + 3$), then $f(3k + 2) = [4(3k + 2) + 1]/3 = 4k + 3 = n$. So $f$ is onto.

    (b) [BB] $f^1(1) = f(1) = [4(1) - 1]/3 = 1$. $f^2(1) = f \circ f(1) = f(f(1)) = f(1) = 1$. This continues: $f^n(1) = 1$ for all $n$.

    (c) $f^1(2) = f(2) = [4(2) + 1]/3 = 3$.
    $f^2(2) = f \circ f(2) = f(f(2)) = f(3) = 2(3)/3 = 2$.
    $f^3(2) = f(f^2(2)) = f(2) = 3$. The values of $f^n(2)$ alternate. The sequence is $2, 3, 2, 3, \ldots$.

    (d) $f^1(4) = f(4) = [4(4) - 1]/3 = 5$; $f^2(4) = f(5) = [4(5) + 1]/3 = 7$;
    $f^3(4) = f(7) = [4(7) - 1]/3 = 9$; $f^4(4) = f(9) = 2(9)/3 = 6$;
    $f^5(4) = f(6) = 2(6)/3 = 4$.
    At this point, the terms repeat: $f^6(4) = f(4) = 5$, $f^7(4) = f(5) = 7$, and so on. The sequence is $5, 7, 9, 6, 4, 5, 7, 9, 6, 4, \ldots$.

    (e) $f^1(8) = f(8) = [4(8) + 1]/3 = 11$; $f^2(8) = f(11) = [4(11) + 1]/3 = 15$;
    $f^3(8) = f(15) = 2(15)/3 = 10$; $f^4(8) = f(10) = [4(10) - 1]/3 = 13$;

$f^5(8) = f(13) = [4(13) - 1]/3 = 17$; $f^6(8) = f(17) = [4(17) + 1]/3 = 23$;
$f^7(8) = f(23) = [4(23) + 1]/3 = 31$; $f^8(8) = f(31) = [4(31) - 1]/3 = 41$;
$f^9(8) = f(41) = [4(41) + 1]/3 = 55$; $f^{10}(8) = f(55) = [4(55) - 1]/3 = 73$.

17. For $n = 341$, the values are $g^1(341) = 512$, $g^2(341) = g(512) = 256$, $g^3(341) = 128$, $g^4(341) = 64$, $g^5(341) = 32$, $g^6(341) = 16$, $g^7(341) = 8$, $g^8(341) = 4$, $g^9(341) = 2$, $g^{10}(341) = 1$.

    For $m = 96$, the values of $g^n(m)$ are $g^1(96) = g(96) = 96/2 = 48$, $g^2(96) = g(48) = 24$,
    $g^3(96) = g(24) = 12$, $g^4(96) = g(12) = 6$,
    $g^5(96) = g(6) = 3$, $g^6(96) = g(3) = [3(3) + 1]/2 = 5$,
    $g^7(96) = g(5) = [3(5) + 1]/2 = 8$, $g^8(96) = g(8) = 4$,
    $g^9(96) = g(4) = 2$, $g^{10}(96) = g(2) = 1$.

    Similarly, for $m = 104$, the values of $g^n(m)$ are $52, 26, 13, 20, 10, 5, 8, 4, 2, 1$;
    for $m = 336$, the values of $g^n(m)$ are $168, 84, 42, 21, 32, 16, 8, 4, 2, 1$;
    and for $m = 133$, the values are $200, 100, 50, 25, 38, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1$.

18. (a) [BB] The first ten terms are $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$. The 123rd term is $2 + 122(3) = 368$.
    (b) [BB] We attempt to solve $752 = 2 + (n - 1)3$ and discover $n - 1 = 250$. So 752 does belong to the sequence; it is the 251st term.
    (c) Since there is no integer solution to $1023 = 2 + (n - 1)3$, we conclude that 1023 does not belong to the sequence. Solving $4127 = 2 + (n - 1)3$, we obtain $n - 1 = 1375$, so 4127 belongs to the sequence. It is the 1376th term.
    (d) [BB] The sum of 75 terms is $\frac{75}{2}[2(2) + (74)3] = 8475$.

19. (a) [BB] $a_{17} = 7 + 16(-\frac{1}{2}) = -1$; $a_{92} = 7 + 91(-\frac{1}{2}) = -38\frac{1}{2}$.
    (b) [BB] $S = \frac{38}{2}\left[2(7) + 37\left(\frac{-1}{2}\right)\right] = 19\left[14 - \frac{37}{2}\right] = -\frac{171}{2}$.

20. We have $a = 116$, $d = -7$.
    (a) $a_{300} = 116 + (299)(-7) = -1977$.
    (b) We attempt to solve $-480 = 116 + (n - 1)(-7)$ obtaining $-596 = -7(n - 1)$. Since $7 \nmid 596$, there is no (integer) solution. We conclude that $-480$ does not belong to the sequence.
    (c) The sum of 300 terms is $\frac{300}{2}[2(116) + (299)(-7)] = 150(232 - 2093) = -279150$.

21. We have $a = 4$ and $d = 11$, so $a_n = 4 + 11(n - 1)$. We want $a_n = k^2$, so $11(n - 1) = k^2 - 4 = (k + 2)(k - 2)$. Since 11 is prime, and $11 \mid (k + 2)(k - 2)$, either $11 \mid (k + 2)$ or $11 \mid (k - 2)$—see Proposition 4.3.7. The next three perfect squares correspond to $k+2 = 11$, $k-2 = 11$ and $k+2 = 22$, respectively, namely, $9^2 = 81$, $13^2 = 169$ and $20^2 = 400$.

22. [BB; first half] First we establish (6). For $n = 1$, $a + (1 - 1)d = a$, which agrees with $a_1 = a$. Now assume that the formula is correct for the integer $k$, that is, that $a_k = a + (k - 1)d$. Then $a_{k+1} = a_k + d = a + (k - 1)d + d = a + kd$ shows that the formula is true also for $k + 1$. So, by the Principle of Mathematical Induction, the formula is correct for all $n \geq 1$.

    Now we establish (7). When $n = 1$, $\frac{n}{2}(2a + (n - 1)d) = \frac{n}{2}(2a) = a$, the sum of one term of the sequence. Now assume the result is true for $n = k$; that is, assume that the sum of the first $k$ terms is $\frac{k}{2}[2a + (k - 1)d]$. We wish to show that the sum of the first $k + 1$ terms is $\frac{k+1}{2}(2a + kd)$. But the sum

of $k+1$ terms equals the sum of $k$ terms plus the $(k+1)$st term. The induction hypothesis, together with the fact that $a_{k+1} = a + kd$ says that this is equal to

$$\frac{k}{2}[2a + (k-1)d] + a + kd = ak + a + \left(\frac{k(k-1)}{2} + k\right)d$$
$$= ak + a + \left(\frac{k(k+1)}{2}\right)d = \left(\frac{k+1}{2}\right)(2a + kd)$$

as required. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

23. Let the sequence be $a, a+d, a+2d, \ldots$. We are given $\frac{10}{2}(2a+9d) = 2\,\frac{6}{2}(2a+5d)$, that is, $10a + 45d = 12a + 30d$, or $2a = 15d$. Also, $\frac{20}{2}(2a+19d) = 1360$, so $20a + 190d = 1360$. Substituting $20a = 10(2a) = 150d$, we obtain $340d = 1360$, so $d = 4$ and $a = \frac{15}{2}d = 30$. The sum of $n$ terms of the sequence is $\frac{n}{2}[60 + 4(n-1)] = \frac{n}{2}(4n + 56) = 2n^2 + 28n$.

24. Let $x, y, z, w$ be $a, a+d, a+2d, a+3d$, respectively. We are given that $4a+6d = 8$, so $d = \frac{2}{3}(2-a)$. We are also given that $a(a+3d) + (a+d)(a+2d) = -2$, which is $a^2 + 3ad + d^2 = -1$. Substituting $d = \frac{2}{3}(2-a)$, we obtain $a^2 + 2(2-a)a + \frac{4}{9}(4 - 4a + a^2) = -1$, so $a^2 - 4a - 5 = 0 = (a-5)(a+1)$. This gives $a = -1$ or $a = 5$, so $d = 2$ or $d = -2$, respectively. There are two possible values for $x, y, z, w$, namely, $-1, 1, 3, 5$ and $5, 3, 1, -1$.

25. (a) [BB] The first ten terms are $59049, -19683, 6561, -2187, 729, -243, 81, -27, 9, -3$. The 33rd term is $59049(-\frac{1}{3})^{32} = \frac{3^{10}}{3^{32}} = (\frac{1}{3})^{22}$.

    (b) [BB] The sum of the first 12 terms is

$$59049\left(\frac{1 - (-\frac{1}{3})^{12}}{1 - (-\frac{1}{3})}\right) = (3^{10})\frac{3}{4}(1 - (\tfrac{1}{3})^{12}) = \frac{1}{4}(3^{11}) - \frac{1}{12} = \frac{3^{12} - 1}{12}.$$

26. Note that $a = -3072 = -3(2^{10})$ and $r = -\frac{1}{2}$.

    (a) $a_{13} = -3072(-\frac{1}{2})^{12} = -\frac{3}{4}$; $\quad a_{20} = -3072(-\frac{1}{2})^{19} = \frac{3}{2^9} = \frac{3}{512}$;

    (b) The sum of nine terms is $-3072\dfrac{1 - (-\frac{1}{2})^9}{1 - (-\frac{1}{2})} = -3(2^{10})\dfrac{1 + \frac{1}{2^9}}{\frac{3}{2}} = -2(2^{10} + 2) = -2052.$

27. Since $a_6 = -\frac{3}{2}$, $-\frac{3}{2} = 48r^5$, so $r^5 = -\frac{1}{32}$, $r = -\frac{1}{2}$. Then the sum of the first ten terms is

$$\frac{48(1 - (-\frac{1}{2})^{10})}{1 - (-\frac{1}{2})} = 48\left(\frac{2}{3}\right)\left(1 - \frac{1}{2^{10}}\right) = \frac{1023}{32}.$$

28. (a) [BB] The geometric sequence with first term $a$ and common ratio $r$ has $n$th term $a_n = ar^{n-1}$. Here we have $a = -0.0000124$ and $r = -.00001364/.00001240 = -1.1$, so

$$a_{129} = (-.00001240)(-1.1)^{128} \approx -2.4643.$$

    (b) [BB] Using formula (8), $S = \frac{-.00001240(1-(-1.1)^{129})}{2.1} \approx -1.2908$.

29. [BB] This is straightforward to prove by mathematical induction but we give an alternative proof. Letting $S = a + ar + ar^2 + \cdots + ar^{n-1}$, we have $rS = ar + ar^2 + \cdots + ar^n$. Subtraction gives $(1-r)S = S - rS = a - ar^n = a(1 - r^n)$ from which the formula follows.

30. $a_1 = 1$, $a_2 = a_1 = 1$, $a_3 = a_1 + a_2 = 2$, $a_4 = a_1 + a_2 + a_3 = 4$, $a_5 = a_1 + a_2 + a_3 + a_4 = 8$, $a_6 = a_1 + a_2 + a_3 + a_4 + a_5 = 16$. We guess $a_n = 2^{n-2}$ for $n \geq 2$ and prove this result by mathematical induction (strong form).

For $n = 2$, $a_2 = 1$ while $2^{2-2} = 2^0 = 1$ also, so the formula is correct. Now assume that $k > 2$ and the formula is true for all $n$, $2 \leq n < k$. Then

$$\begin{aligned} a_k &= \sum_{i=1}^{k-1} a_i = a_1 + \sum_{i=2}^{k-1} a_i \\ &= 1 + \sum_{i=2}^{k-1} 2^{i-2} \quad \text{using } a_1 = 1 \text{ and the induction hypothesis} \\ &= 1 + 1 + 2 + 4 + \cdots + 2^{k-3} = 1 + \frac{1(1-2^{k-2})}{1-2} = 1 + (2^{k-2} - 1) = 2^{k-2} \end{aligned}$$

which is the desired formula with $n = k$. (Note that we used formula (8) to sum $k - 2$ terms of the geometric sequence with $a = 1$, $r = 2$ at the second last step.) We conclude that the formula is true for all $n \geq 2$, by the Principle of Mathematical Induction.

31. (a) The sum is $\frac{7}{1024}\left(\frac{1-8^{18}}{1-8}\right) = \frac{7}{2^{10}}\left(\frac{2^{54}-1}{7}\right) = 2^{44} - 2^{-10} \approx 2^{44}$.

(b) [BB] If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Hence, $\lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}$.

(c) [BB] This is the sum of 101 terms of a geometric sequence with $a = 3$ and $r = 1/2$. Using the result of (b), the sum is approximately $\frac{3}{1-\frac{1}{2}} = 6$.

(d) Here $a = 144$ and $r = 1/3 < 1$. Since 1,000,000 is rather large, we have (by (b)) that the sum is $\approx \frac{144}{1-\frac{1}{3}} = \frac{144}{\frac{2}{3}} = 216$.

32. Note that $a = 98415 = 5(3^9)$ and $r = \frac{1}{3}$.

(a) $a_{19} = 98415(\frac{1}{3})^{18} = 5(3^{-9})$; $\quad a_{100} = 98415(\frac{1}{3})^{99} = 5(3^{-90})$.

(b) The sum of the first 15 terms is $(98415)\frac{1-(\frac{1}{3})^{15}}{1-\frac{1}{3}} = \frac{5}{2}(3^{10})(1 - 3^{-15}) = \frac{5}{2}(3^{10} - 3^{-5})$ ($\approx$ 147622.49).

(c) Using the result of Exercise 31(b), the sum of the first $10,000$ terms is approximately $\frac{98415}{1-1/3} =$ 147622.5.

33. (a) [BB] This is the sum of an arithmetic sequence with $a = 75$, $d = -4$. Solving $-61 = a + (n-1)d = 75 + (n-1)(-4)$, we obtain $n = 35$. So the sum is $\frac{35}{2}[2(75) + 34(-4)] = 245$.

(b) [BB] This is the sum of a geometric sequence with $a = 75$, $r = \frac{1}{5}$. Solving $\frac{3}{5^7} = ar^{n-1} = 75(\frac{1}{5})^{n-1}$, we get $n = 10$. So the sum is

$$75\frac{1-\frac{1}{5^{10}}}{1-\frac{1}{5}} = 3(5^2)\frac{5}{4}\left(1 - \frac{1}{5^{10}}\right) = \frac{3}{4}\left(5^3 - \frac{1}{5^7}\right) \approx 93.75.$$

(c) This is the sum of an arithmetic sequence with $a = -52$, $d = 11$. Solving $949 = a + (n-1)d = -52 + (n-1)11$, we get $n = 92$. Thus, the sum is $\frac{92}{2}[2(-52) + 91(11)] = 41262$.

(d) This is the sum of a geometric sequence with $a = 1$, $r = -\frac{1}{2}$. Solving $\frac{1}{2^{60}} = ar^{n-1}$, we get $n = 61$, so the sum is

$$\frac{1-(-\frac{1}{2})^{61}}{1-(-\frac{1}{2})} = \frac{2}{3}\Big(1+\frac{1}{2^{61}}\Big).$$

(e) This is the sum of a geometric sequence with $a = 2$, $r = 3$. Noting that $354294 = 2(3^{11})$ and solving $ar^{n-1} = 354294$, we get $n = 12$. So the sum is

$$2\Big(\frac{1-3^{12}}{1-3}\Big) = 3^{12} - 1.$$

34. (a) [BB] This is the sum of an arithmetic sequence with $a = 1004$ and $d = -3$. With $a_n = -397 = a + (n-1)d$, we have $1004 - 3(n-1) = -397$, so $3(n-1) = 1401$, $n - 1 = 467$, so $n = 468$. The sum is $S = \frac{n}{2}[2a + (n-1)d] = \frac{468}{2}[2008 + 467(-3)] = 234(607) = 142038$.

(b) This is the sum of a geometric sequence with $a = 324$ and $r = -\frac{2}{3}$.
Solving $ar^{n-1} = -\frac{131072}{177147}$ gives $(-\frac{2}{3})^{n-1} = \frac{32768}{14348907}$, so $n - 1 = 15$, $n = 16$.
The sum is

$$\frac{a(1-r^n)}{1-r} = \frac{324(1-(-\frac{2}{3})^{16})}{1-(-\frac{2}{3})} = \frac{324(1-(\frac{2}{3})^{16})}{\frac{5}{3}} = \frac{3(324)(1-(\frac{2}{3})^{16})}{5} = \frac{34384948}{177147}.$$

35. [BB] Suppose the arithmetic sequence $a_0, a_0 + d, a_0 + 2d, \ldots$ is also the geometric sequence $a_1, a_1 r, a_1 r^2, \ldots$ . Then $a_0 = a_1$. Let's call this number $a$. Note that one and only one possibility occurs when $a = 0$ (namely, $0, 0, 0, \ldots$) so assume henceforth that $a \neq 0$. Since the sequence is both arithmetic and geometric, it follows that all terms of this sequence are nonzero. We have $a + d = ar$ and $a + 2d = ar^2$, so

$$\frac{a+d}{a} = \frac{ar}{a} = r = \frac{a+2d}{a+d}.$$

So $(a+d)^2 = a(a+2d)$, giving $a^2 + 2ad + d^2 = a^2 + 2ad$ and hence $d = 0$. Since $ar = a + d = a$, $r = 1$. The only arithmetic sequences which are also geometric are constant, of the form $a, a, a, \ldots$ for some number $a$.

36. (a) [BB] We want $b$ to be the $n + 2^{\text{nd}}$ term of an arithmetic sequence, so $b = a + (n+1)d$. Thus $d = \frac{b-a}{n+1}$ and $x_i = a + id = a + \frac{(b-a)i}{n+1}$.

(b) We want $b$ to be the $n + 2$nd term of a geometric sequence, so $b = ar^{n+1}$. Thus $r = \sqrt[n+1]{b/a}$ and $x_i = ar^i = a(\sqrt[n+1]{b/a})^i$.

37. We wish to find $S = a_0 + a_1 + a_2 + \ldots + a_{n-1}$ with $a_k = 1 + r + r^2 + \cdots + r^k$, the sum of a geometric sequence with first term 1 and common ratio $r$. Thus $a_k = \frac{1-r^{k+1}}{1-r}$ and $S = \frac{1}{1-r}[(1-r) + (1-r^2) + \cdots + (1-r^n)] = \frac{1}{1-r}[n - (r + r^2 + \cdots + r^n)]$. The sequence $r, r^2, \ldots, r^n$ is geometric with first term $r$ and common ratio $r$, so $r + r^2 + \cdots + r^n = \frac{r(1-r^n)}{1-r}$ and

$$S = \frac{1}{1-r}\Big[n - \frac{r(1-r^n)}{1-r}\Big] = \frac{1}{(1-r)^2}\big[n(1-r) - r + r^{n+1}\big],$$

which is the desired formula.

38. (a) Let $f(x) = x+x^2+x^3+\cdots+x^n$. The desired sum is $f'(x)$. Since $f(x)$ is the sum of a geometric sequence with $a = x$ and $r = x$, we have $f(x) = \dfrac{x(1-x^n)}{1-x} = \dfrac{x-x^{n+1}}{1-x}$. Thus

$$f'(x) = \frac{[1-(n+1)x^n](1-x)-(x-x^{n+1})(-1)}{(1-x)^2} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$

(b) As $n \to \infty$, $|x|^n \to 0$, so the sum in (a) approaches $\dfrac{1}{(1-x)^2}$.

(c) Let $S = 1+3x+5x^2+7x^3+\cdots$ be the desired sum and let $S_1 = 1+2x+3x^2+4x^3+\cdots$ be the sum found in (b). We have $S-S_1 = x+2x^2+3x^3+4x^4+\cdots = x(1+2x+3x^2+4x^3+\cdots) = xS_1$. Thus $S = S_1 + xS_1 = (1+x)S_1 = \dfrac{1+x}{(1-x)^2}$.

(d) The desired sum is $1 + 3x(1 + 2x + 3x^2 + 4x^3 + \cdots) = 1 + 3xS_1$, where $S_1$ is the sum found in part (b). So the sum here is $1 + \dfrac{3x}{(1-x)^2} = \dfrac{1+x+x^2}{(1-x)^2}$.

39. We have $a_k = \dfrac{2^k-1}{4^{k-1}} = \dfrac{2^k}{4^{k-1}} - \dfrac{1}{4^{k-1}} = \dfrac{1}{2^{k-2}} - \dfrac{1}{4^{k-1}}$. Thus the sum in question is the difference between the sums of two geometric sequences. The first has $a = \dfrac{1}{2^{-1}} = 2$, $r = \frac{1}{2}$ and the second has $a = \dfrac{1}{4^0}$ and $r = \frac{1}{4}$. The sum is $\dfrac{2}{1-\frac{1}{2}} - \dfrac{1}{1-\frac{1}{4}} = 4 - \frac{4}{3} = \frac{8}{3}$.

40. (a) $s_0 = P$ and for $n > 0$, $s_n = (1+i)s_{n-1}$.

(b) We have $s_1 = (1+i)s_0 = (1+i)P$, $s_2 = (1+i)s_1 = (1+i)^2P$, $s_3 = (1+i)^3P$ and generally, $s_n = (1+i)^nP$.

41. (a) [BB] $\$1000(1.15)^2 = \$1322.50$.

(b) [BB] We seek $t$ such that $1000(1.15)^t = 2000$. Thus, $1.15^t = 2$, $t = \log 2/\log 1.15 \approx 4.9595$ years.

42. The original \$50 will accumulate to $50(1.01)^{12} = \$56.34$. The January 31 deposit of \$56 will accumulate to $\$56(1.01)^{11}$; the February 28 deposit of \$56 will accumulate to $\$56(1.01)^{10}$; and so on. The monthly deposits will accumulate to

$$\$56[(1.01)^{11} + (1.01)^{10} + (1.01)^9 + (1.01)^8 + \cdots + 1.01 + 1]$$
$$= \$56\left(\frac{1-(1.01)^{12}}{1-1.01}\right) = \$56\frac{.12682502}{.01} = \$710.22.$$

The balance on January 1, 2002 is \$766.56.

43. (a) You receive $2^{20}$ cents on June 21, that is, \$10,485.76.

(b) In all, over the first 21 days of June you receive $1+2+2^2+2^3+\cdots+2^{20} = \frac{2^{21}-1}{2-1} = 2{,}097{,}151$ cents; that is, \$20,971.51. It would not be wise to take this deal!

(c) After 30 payments, you would get $1 + 2 + 2^2 + 2^3 + \cdots + 2^{29} = \frac{2^{30}-1}{2-1} = 1{,}073{,}741{,}823$ cents, that is, \$10,737,418.23. Who wouldn't take this deal?

44. We use the strong form of the Principle of Mathematical Induction. When $n = 0$, $a_0 = 0 \leq 20(0) = 0$, so the result holds. Now let $k > 0$ and assume the result holds for all $0 \leq n < k$. We wish to prove the result for $n = k$; that is, we wish to prove that $a_k \leq 20k$. Now,

$$\begin{aligned} a_k &= a_{\lfloor k/5 \rfloor} + a_{\lfloor 3k/5 \rfloor} + k \\ &\leq 20\lfloor k/5 \rfloor + 20\lfloor 3k/5 \rfloor + k \quad \text{applying the induction hypothesis to } \lfloor k/5 \rfloor \text{ and } \lfloor 3k/5 \rfloor \\ &\leq 20\left(\frac{k}{5}\right) + 20\left(\frac{3k}{5}\right) + k = 4k + 12k + k = 17k < 20k \end{aligned}$$

as desired. By the Principle of Mathematical Induction, we conclude that the inequality is true for all $n \geq 0$.

45. Observe that $\Phi^2 = 1 + \Phi$, hence, $\Phi^3 = \Phi + \Phi^2$, $\Phi^4 = \Phi^2 + \Phi^3$ and, in general, $\Phi^{n+1} = \Phi^n + \Phi^{n-1}$ for all $n \geq 1$. This says that $a_n = \Phi^n$ satisfies the Fibonacci recurrence relation.

46. [BB] Rewriting $f_{k+1} = f_k + f_{k-1}$ as $f_{k-1} = f_{k+1} - f_k$, we see that $f_0 = f_2 - f_1 = 1 - 1 = 0$. Similarly, $f_{-1} = 1$, $f_{-2} = -1$, $f_{-3} = 2$, $f_{-4} = -3$, $f_{-5} = 5$, $f_{-6} = -8$. In general, $f_{-n} = (-1)^{n+1} f_n$.

47. [BB] The Fibonacci sequence is the sequence $1, 1, 2, 3, 5, 8 \ldots$ , each term after the first two being the sum of the two previous terms. For $n = 1$, $f_2 f_1 = 1(1) = 1^1 = f_1^2$. So the assertion is correct for $n = 1$. Let $k \geq 1$ and assume the assertion is correct for $n = k$; that is, assume that $f_{k+1} f_k = \sum_{i=1}^{k} f_i^2$. We wish to prove the assertion is correct for $n = k + 1$; that is, we wish to prove that $f_{k+2} f_{k+1} = \sum_{i=1}^{k+1} f_i^2$. But $f_{k+2} = f_{k+1} + f_k$, so, $f_{k+2} f_{k+1} = (f_{k+1} + f_k) f_{k+1} = f_{k+1}^2 + f_{k+1} f_k = f_{k+1}^2 + \sum_{i=1}^{k} f_i^2$ (using the induction hypothesis) $= \sum_{i=1}^{k+1} f_i^2$ as desired. By the Principle of Mathematical Induction, we conclude that the assertion is correct for all $n \geq 1$.

48. (a) The first eight terms of the Fibonacci sequence are

$$f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21.$$

With $n = 4$, $f_1 + f_2 + f_3 + f_4 = 7 = 8 - 1 = f_6 - 1$.
With $n = 5$, $f_1 + f_2 + \cdots + f_5 = 12 = 13 - 1 = f_7 - 1$.
With $n = 6$, $f_1 + f_2 + \cdots + f_6 = 20 = 21 - 1 = f_8 - 1$.

(b) With $n = 1$, $f_1 + f_2 + \cdots + f_n = f_1 = 1 = 2 - 1 = f_3 - 1$, so the formula holds. Now assume that $k > 1$ and that the formula holds for $k$. Then

$$\begin{aligned} f_1 + f_2 + \cdots + f_{k+1} &= (f_1 + f_2 + \cdots + f_k) + f_{k+1} \\ &= (f_{k+2} - 1) + f_{k+1} \quad \text{applying the induction hypothesis} \\ &= f_{k+3} - 1 \end{aligned}$$

(since $f_{k+3} = f_{k+2} + f_{k+1}$) and this is the formula with $n = k+1$. We conclude, by the Principle of Mathematical Induction, that the formula holds for all $n \geq 1$.

49. For $n = 1$, $f_1 = 1 < 2^1$, so the inequality holds. For $n = 2$, $f_2 = 1 < 2^2$, so again the inequality holds. Now assume that $k > 2$ and $f_j < 2^j$ for all positive integers $j < k$. We have $f_k = f_{k-1} + f_{k-2} < 2^{k-1} + 2^{k-2}$ by the induction hypothesis (since $k-1 < k$ and $k-2 < k$). Since $2^{k-1} + 2^{k-2} = 2^{k-2}(2 + 1) = 3(2^{k-2}) < 4(2^{k-2}) = 2^k$, the inequality holds for all $n \geq 1$ by the Principle of Mathematical Induction.

50. For $n = 3$, $f_3 = 2 > \frac{125}{64} = (\frac{5}{4})^3$, so the inequality holds. For $n = 4$, $f_4 = 3 > \frac{625}{256} = (\frac{5}{4})^4$, so again the inequality holds. Now assume that $k > 4$ and that $f_j > (\frac{5}{4})^j$ for all positive integers $j$ satisfying $3 \leq j < k$. Applying the induction hypothesis to $j = k - 1$ and $j = k - 2$, we find $f_k = f_{k-1} + f_{k-2} > (\frac{5}{4})^{k-1} + (\frac{5}{4})^{k-2} = (\frac{5}{4})^{k-2}(\frac{5}{4} + 1) = (\frac{5}{4})^{k-2}(\frac{9}{4}) > (\frac{5}{4})^{k-2}(\frac{25}{16}) = (\frac{5}{4})^k$, which is the desired inequality for $n = k$. We conclude that the inequality holds for all $n \geq 3$ by the Principle of Mathematical Induction.

51. [BB] The induction hypothesis is valid only for $n$ in the interval $3 \leq n < k$ because all integers in this problem are at least as large as $n_0 = 3$. But the induction hypothesis is applied to the integer $k - 2$; this is not valid if $k = 4$.

52. The term $f_n$ is even if and only if $n \equiv 0 \pmod 3$.

    **Proof.** We use the strong form of mathematical induction. For $n = 1$, $a_1 = 1$ is odd and the statement is correct, and similarly if $n = 2$. Now assume that $k > 2$ and the statement is true for all $n < k$. We must prove it is true for $n = k$.

    **Case 1:** $k \equiv 0 \pmod 3$.

    In this case, neither $k - 1$ nor $k - 2$ is $0 \pmod 3$, so by the induction hypothesis, both $f_{k-1}$ and $f_{k-2}$ are odd. Therefore, $f_k = f_{k-1} + f_{k-2}$ is even.

    **Case 2:** $k \not\equiv 0 \pmod 3$.

    In this case, exactly one of $f_{k-1}, f_{k-2}$ is odd, so $f_k$ is the sum of an even and an odd integer, hence, odd. ■

53. (a) [BB] $a_1 = 1$, $a_2 = 2$ (2 and $1 + 1$), $a_3 = 4$, as given. We can write $4 = 4 = 3 + 1 = 1 + 3 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 1 + 1 + 1$ so $a_4 = 8$. Careful counting shows that $a_5 = 16$.

    (b) We guess that $a_n = 2^{n-1}$ and we prove this using the strong form of the Principle of Mathematical Induction. For $n = 1$, $2^{1-1} = 1 = a_1$. Now let $k > 1$ and assume the result is true for all $n < k$; that is, we assume that $a_n = 2^{n-1}$ for all $n < k$. We wish to prove the result for $n = k$; that is, we wish to prove that $a_k = 2^{k-1}$. Consider any sequence of natural numbers adding to $k$. This sequence could consist of $k$ itself; otherwise, it has at least two terms. Removing the last term leaves us with a sequence whose sum is less than $k$. Conversely, given any sequence with sum $S < k$, we can form a sequence with sum $k$ by adding the single natural number $k - S$ to the end of the given sequence as a final term. In this way we see that $a_k = a_{k-1} + a_{k-2} + \cdots + a_1 + 1$. Applying the induction hypothesis to $k - 1, k - 2, \ldots, 1$ we see that

    $$a_k = 2^{k-2} + 2^{k-3} + \cdots + 1 + 1 = \frac{1 - 2^{k-1}}{1 - 2} + 1 = 2^{k-1}$$

    since we are summing a geometric sequence [see formula (8) in this section]. Thus, the result holds for $n = k$. By the Principle of Mathematical Induction, the result is true for all $n \geq 1$.

54. (a) $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, $b_4 = 5$ and $b_5 = 8$.

    (b) This sequence looks like the Fibonacci sequence, but starting with the second term. To see that this is correct, we show why the recursive definition $b_{n+1} = b_n + b_{n-1}$ applies. We use the same argument as in Exercise 53. If we have a sequence with sum $n + 1$, then that sequence ends in a 1 or a 2. Removing the final term gives a sequence adding either to $n$ or to $n - 1$. Conversely,

any sequence adding to $n$ or to $n-1$ can be turned into a sequence adding to $n+1$ by adding a 1 or a 2 (respectively) at the end. Thus, $b_{n+1} = b_n + b_{n-1}$. Adding the initial conditions $b_1 = 1, b_2 = 2$ gives the desired recursive definition and proves that the sequence is the Fibonacci sequence, starting with the second term.

55. (a) [BB] Each line of $n-1$ people determines two lines of $n$ people—the first person in the new line can be a man or a woman. So $a_n = 2a_{n-1}$. The sequence is the sequence $2, 4, 8, \ldots$ of powers of 2.

(b) A line starts with a male or a female. If it starts with a female, there are $a_{n-1}$ ways to complete the line. On the other hand, if it starts with a male, then the second person must be female. There are $a_{n-2}$ ways then to complete such a line. Thus, $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 2$ (M or F) and $a_2 = 3$ (FF, MF, or FM), the next few terms are $5, 8, 13, \ldots$. This is that part of the Fibonacci sequence from 2 onward.

56. (a) We use induction on $n$. For $n = 1$, $\gcd(f_2, f_1) = \gcd(1, 1) = 1$ as desired. Assume $k \geq 1$ and that $\gcd(f_{k+1}, f_k) = 1$. Since $f_{k+2} = f_{k+1} + f_k$, we have $\gcd(f_{k+2}, f_{k+1}) = \gcd(f_{k+1}, f_k) = 1$. (See Lemma 4.2.5.) By the Principle of Mathematical Induction, $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 1$.

(b) We use induction on $n$, the strong form. The smallest possible value for $n$ is $n_0 = 3$. In this case, $m = 2$ and since $f_3 = f_2 f_2 + f_1 f_1$ the result is true for $n_0$. Now assume $k \geq 4$ and that the result is true for all integers $\ell$ in the range $3 \leq \ell < k$. We must prove that the result is true for $k$; that is, we must prove that

$$(*) \qquad f_k = f_{k-m+1} f_m + f_{k-m} f_{m-1}$$

for any $m$ in the range $k > m > 1$. There are three possibilities: $k = m+1$, $k = m+2$ and $k > m+2$.

If $k = m+1$, then $f_k = f_{k-1} + f_{k-2} = 1f_m + 1f_{m-1} = f_2 f_m + f_1 f_{m-1}$, the desired equation.

If $k = m+2$, then $3 \leq k-1 < k$ and $k-1 > m$ so we can apply the induction hypothesis to $k-1$. We obtain $f_k = f_{k-1} + f_{k-2} = f_{k-m} f_m + f_{k-m-1} f_{m-1} + f_{k-2} = f_2 f_m + f_1 f_{m-1} + f_m = 2f_m + f_{m-1} = f_3 f_m + f_2 f_{m-1}$, which is (*).

If $k > m+2$, then $k-1 > m$ and $k-2 > m$. Also, Since $m \geq 2$, $k \geq 5$. Thus, $3 \leq k-1 < k$ and $3 \leq k-2 < k$. It follow that we may apply the induction hypothesis both to $k-1$ and to $k-2$. We obtain

$$\begin{aligned} f_k &= f_{k-1} + f_{k-2} \\ &= f_{k-m} f_m + f_{k-m-1} f_{m-1} + f_{k-2-m+1} f_m + f_{k-2-m} f_{m-1} \\ &= (f_{k-m} + f_{k-m-1}) f_m + (f_{k-m-1} + f_{k-m-2}) f_{m-1} \\ &= f_{k-m+1} f_m + f_{k-m} f_{m-1} \end{aligned}$$

which is (*). By the Principle of Mathematical Induction, $f_n = f_{n-m+1} f_m + f_{n-m} f_{m-1}$ is true for all $n \geq 3$ with $n > m > 1$.

(c) We have to prove $\gcd(f_n, f_m) = f_{\gcd(n,m)}$ for all positive integers $n$ and $m$. Without loss of generality, we may assume $n \geq m$. We use the strong form of induction on $n$. The smallest value for $n$ is $n_0 = 1$. If $n = 1$, then $m = 1$ and since $\gcd(f_1, f_1) = 1 = f_1 = f_{\gcd(1,1)}$ the result is true. Now assume that $k > 1$ and the result is true for all $\ell$ in the range $1 \leq \ell < k$. We must prove that

$$(*) \qquad \gcd(f_k, f_m) = f_{\gcd(k,m)}$$

for all $m$, $1 \le m \le k$. There are two possibilities: $m = k$ and $m < k$. In the first case, $\gcd(f_k, f_m) = \gcd(f_m, f_m) = f_m = f_{\gcd(m,m)} = f_{\gcd(k,m)}$ which is the desired result. In the second case, we have

$$f_k = f_{k-m+1} f_m + f_{k-m} f_{m-1}$$

by part (b), hence,

$$\gcd(f_k, f_m) = \gcd(f_m, f_{k-m} f_{m-1})$$

by Lemma 4.2.5. By part (a), we know that $\gcd(f_m, f_{m-1}) = 1$. Thus, $\gcd(f_m, f_{k-m} f_{m-1}) = \gcd(f_m, f_{k-m})$. (See Problem 11 in Chapter 4.) Since $m < k$ and $k - m < k$, we may apply our induction hypothesis to whichever of these is the larger. Since $\gcd(m, k-m) = \gcd(m, k)$, we obtain $\gcd(f_m, f_{k-m}) = f_{\gcd(m,k-m)} = f_{\gcd(m,k)}$ as desired.

57. (a) For $n = 1$, $v_n - u_n = 1 - 0 = 1 = 1/4^0$, so the result is true. Now assume the result for $n = k$. Then

$$v_{k+1} - u_{k+1} = \frac{u_k + 3v_k}{4} - \frac{u_k + v_k}{2} = \frac{u_k + 3v_k - 2u_k - 2v_k}{4}$$
$$= \frac{v_k - u_k}{4} = \frac{1}{4}\frac{1}{4^{k-1}} = \frac{1}{4^k}$$

which verifies the statement for $n = k + 1$. Thus, $v_n - u_n = 1/4^{n-1}$ for all $n \ge 1$.

(b) $u_{n+1} - u_n = \dfrac{u_n + v_n}{2} - u_n = \dfrac{u_n + v_n - 2u_n}{2} = \dfrac{v_n - u_n}{2} = \dfrac{1}{2}\dfrac{1}{4^{n-1}} > 0$, so $u_{n+1} > u_n$, for all $n$.

Therefore, $u_n$ is an increasing sequence.

(c) $v_{n+1} - v_n = \dfrac{u_n + 3v_n}{4} - v_n = \dfrac{u_n + 3v_n - 4v_n}{4} = \dfrac{u_n - v_n}{4} = -\dfrac{1}{4}\dfrac{1}{4^{n-1}} < 0$, so $v_n$ is a decreasing sequence.

(d) $u_1 = 0 = \frac{2}{3} - \frac{1}{6}\frac{1}{4^{-1}}$, so the result is true for $n = 1$. Now assume it's true for $n = k$. Then

$$\begin{aligned} u_{k+1} &= \frac{u_k + v_k}{2} = \frac{u_k}{2} + \frac{1}{2}\left(u_k + \frac{1}{4^{k-1}}\right) \\ &= u_k + \frac{1}{2}\frac{1}{4^{k-1}} = \frac{2}{3} - \frac{1}{6}\frac{1}{4^{k-2}} + \frac{1}{2}\frac{1}{4^{k-1}} \\ &= \frac{2}{3} - \frac{1}{6}\left(\frac{1}{4^{k-2}} - \frac{3}{4^{k-1}}\right) = \frac{2}{3} - \frac{1}{6}\left(\frac{4-3}{4^{k-1}}\right) = \frac{2}{3} - \frac{1}{6}\frac{1}{4^{k-1}} \end{aligned}$$

which is the result for $n = k + 1$. By the Principle of Mathematical Induction, $u_n = \frac{2}{3} - \frac{1}{6}\frac{1}{4^{n-2}}$ for all $n \ge 1$.

58. If $A = [a]$ is a $1 \times 1$ matrix, $\det A = a$. Now suppose $A$ is an $n \times n$ matrix whose $(i, j)$ entry is $a_{ij}$. Let $M_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting row $i$ and column $j$. Then

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det M_{i1}$$

(We have given the definition by "expansion of cofactors of the first column." Equally good definitions can be given for expansion of cofactors of any other row, or column, by making appropriate changes in the two occurrences of the subscript $i1$. See an appendix.)

## Exercises 5.3

1. [BB] The characteristic polynomial is $x^2 - x - 6$; the characteristic roots are $-2, 3$. We obtain $a_n = c_1(-2)^n + c_2(3^n)$. Since $c_1 + c_2 = 1$, $-2c_1 + 3c_2 = 3$, we find $c_1 = 0$, $c_2 = 1$. The solution is $a_n = 3^n$.

2. The characteristic polynomial is $x^2 + 6x - 7$; the characteristic roots are $-7, 1$. We obtain $a_n = c_1(-7)^n + c_2(1^n)$. Since $c_1 + c_2 = 32$, $-7c_1 + c_2 = -17$, we find $c_1 = 49/8$, $c_2 = 207/8$. The solution is
$$a_n = \tfrac{49}{8}(-7)^n + \tfrac{207}{8}(1^n) = \tfrac{1}{8}[(-7)^{n+2} + 207].$$

3. [BB] The characteristic polynomial here, $x^2 - 6x + 9$, has the repeated root $x = 3$ and so Theorem 5.3.1 tells us that the solution is $a_n = c_1(3^n) + c_2 n(3^n)$. From the given values of $a_0$ and $a_1$, we obtain $c_1(3^0) + c_2(0)(3^0) = -5$ (so $c_1 = -5$) and $c_1(3^1) + c_2(1)(3^1) = 3$; that is, $3c_1 + 3c_2 = 3$. So $c_1 + c_2 = 1$ and $c_2 = 1 - c_1 = 6$. The solution is $a_n = -5(3^n) + 6n(3^n)$.

4. To fit this problem to the notation of Theorem 5.3.1, we rewrite the given recurrence in the equivalent form $a_n = 7a_{n-1} - 10a_{n-2}$, $n \geq 3$. The characteristic polynomial is $x^2 - 7x + 10$ with characteristic roots $2, 5$. We obtain $a_n = c_1(2^n) + c_2(5^n)$. Since $a_1 = c_1(2) + c_2(5) = 10$ and $a_2 = c_1(2^2) + c_2(5^2) = 29$, we get $c_1 = 7/2$, $c_2 = 3/5$, and so $a_n = \frac{7}{2}(2^n) + \frac{3}{5}(5^n) = 7(2^{n-1}) + 3(5^{n-1})$.

5. [BB] The characteristic polynomial, $x^2 + 8x + 1$, has the distinct roots $-4 \pm \sqrt{15}$ and so the solution is $a_n = c_1(-4 + \sqrt{15})^n + c_2(-4 - \sqrt{15})^n$. The initial conditions give
$$\begin{array}{rcl} a_0 = 0 & = & c_1 + c_2 \\ a_1 = 1 & = & c_1(-4 + \sqrt{15}) + c_2(-4 - \sqrt{15}) \end{array}$$
Substituting $c_2 = -c_1$ in the second equation, we obtain $c_1 = \frac{1}{2\sqrt{15}}$, $c_2 = -\frac{1}{2\sqrt{15}}$ and so
$a_n = \frac{1}{2\sqrt{15}}(-4 + \sqrt{15})^n - \frac{1}{2\sqrt{15}}(-4 - \sqrt{15})^n$.

6. The characteristic polynomial, $x^2 + 5x - 6$, has the distinct roots $-6$ and $1$, so the solution is $a_n = c_1(1^n) + c_2(-6)^n = c_1 + c_2(-6)^n$. The initial conditions give $a_0 = 5 = c_1 + c_2$, $a_1 = 19 = c_1 - 6c_2$ and, hence, $c_1 = 7$, $c_2 = -2$. The solution is $a_n = 7 - 2(-6)^n$.

7. [BB] To fit this problem to the notation of Theorem 5.3.1, we rewrite the given recurrence in the equivalent form $a_n = 2a_{n-1} + 3a_{n-2}$, $n \geq 2$. The characteristic polynomial is $x^2 - 2x - 3$ with characteristic roots $-1$ and $3$. We obtain $a_n = c_1(-1)^n + c_2(3^n)$. Since $a_0 = c_1 + c_2 = 0$ and $a_1 = -c_1 + 3c_2 = 8$, we have $c_1 = -2$, $c_2 = 2$. The solution is $a_n = -2(-1)^n + 2(3^n) = 2[(-1)^{n+1} + 3^n]$.

8. The characteristic polynomial is $x^2 - 2x + 1$; this has a repeated root 1. We obtain $a_n = c_1(1^n) + nc_2(1^n) = c_1 + nc_2$. Since $a_0 = c_1 = 40$ and $a_1 = c_1 + c_2 = 37$, we have $c_1 = 40$, $c_2 = -3$ and $a_n = 40 - 3n$.

9. Rewrite this as $a_n = \frac{2}{3}a_{n-1} - \frac{1}{9}a_{n-2}$. The characteristic polynomial is $x^2 - \frac{2}{3}x + \frac{1}{9}$ with repeated characteristic root $\frac{1}{3}$. We obtain $a_n = c_1(\frac{1}{3})^n + c_2 n(\frac{1}{3})^n$. Since $c_1 = 3$, $\frac{1}{3}c_1 + \frac{1}{3}c_2 = -1$, we have $c_1 = 3$, $c_2 = -6$, so the solution is $a_n = 3(\frac{1}{3})^n - 6n(\frac{1}{3})^n = (\frac{1}{3})^{n-1} - 2n(\frac{1}{3})^{n-1} = (\frac{1}{3})^{n-1}(1 - 2n)$.

10. (a) [BB] The characteristic polynomial, $x^2 + 2x - 15$, has distinct roots 3 and $-5$, so the solution is $a_n = c_1(3^n) + c_2(-5)^n$. The initial conditions give $a_0 = 1 = c_1 + c_2$, $a_1 = -1 = 3c_1 - 5c_2$. We obtain $c_1 = c_2 = \frac{1}{2}$ and so $a_n = \frac{1}{2}(3^n) + \frac{1}{2}(-5)^n$.

(b) [BB] The function $f(n) = 24$ is constant, so we try $p_n = k$. This gives $k = -2k + 15k + 24$, so $k = -2 = p_n$. In part (a), we saw that the corresponding homogeneous recurrence relation has solution $q_n = c_1(3^n) + c_2(-5)^n$, so $a_n = p_n + q_n = c_1(3^n) + c_2(-5)^n - 2$. The initial conditions give $1 = a_0 = c_1 + c_2 - 2$ and $-1 = a_1 = 3c_1 - 5c_2 - 2$, so $c_1 = 2$, $c_2 = 1$ and our solution is $a_n = 2(3^n) + (-5)^n - 2$.

11. (a) To fit this problem to Theorem 5.3.1, we rewrite the given recurrence as $a_n = -8a_{n-1} - 16a_{n-2}$, $n \geq 2$. The characteristic polynomial is $x^2 + 8x + 16$ with repeated characteristic root $-4$. We obtain $a_n = c_1(-4)^n + c_2 n(-4)^n$. Since $c_1 = 5$, $-4c_1 - 4c_2 = 17$, we get $c_1 = 5$, $c_2 = -37/4$. The solution is $a_n = 5(-4)^n - \frac{37}{4}n(-4)^n = (-4)^n \frac{20-37n}{4} = (-4)^{n-1}(37n - 20)$.

(b) Since $f(n) = 5$ is constant, we try $p_n = k$ and obtain $k = -8k - 16k + 5$, so $k = \frac{1}{5} = p_n$. From part (a), we know that the corresponding homogeneous recurrence relation has solution $q_n = c_1(-4)^n + c_2 n(-4)^n$ so $a_n = p_n + q_n = c_1(-4)^n + c_2 n(-4)^n + \frac{1}{5}$. The initial conditions give $2 = a_0 = c_1 + \frac{1}{5}$ and $-1 = a_1 = -4c_1 - 4c_2 + \frac{1}{5}$, so $c_1 = \frac{9}{5}$ and $c_2 = -\frac{3}{2}$. Our solution is $a_n = \frac{9}{5}(-4)^n - \frac{3}{2}n(-4)^n + \frac{1}{5}$.

12. (a) [BB] Using Theorem 5.3.1 (or guessing and verifying as in Section 5.2), the solution is $a_n = 4^n$.

(b) [BB] Try $p_n = c(8^n)$. We obtain $c(8^n) = 4[c(8^{n-1})] + 8^n$; that is, $8^n c = 4(8^{n-1})c + 8^n$, so $2^{3n}c = 2^{3n-1}c + 8^n$. Thus, $2^{3n-1}c = 8^n$, so $c = 2$ and $p_n = 2(8^n)$ is a particular solution. By (a), the homogeneous recurrence relation has solution $q_n = c(4^n)$, so $p_n + q_n = 2(8^n) + c(4^n)$. The initial condition gives $2 + c = 1$, so $c = -1$ and $a_n = 2(8^n) - 4^n$.

(c) [BB] For $n = 0$, $2(8^0) - 4^0 = 2 - 1 = 1$, as required.

Let $k \geq 0$ and assume that $a_k = 2(8^k) - 4^k$. We wish to prove that $a_{k+1} = 2(8^{k+1}) - 4^{k+1}$. Now

$$\begin{aligned} a_{k+1} &= 4a_k + 8^{k+1} = 4[2(8^k) - 4^k] + 8^{k+1} \quad \text{(by the induction hypothesis)} \\ &= 8^{k+1} - 4^{k+1} + 8^{k+1} = 2(8^{k+1}) - 4^{k+1} \end{aligned}$$

as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 0$.

13. (a) The characteristic polynomial is $x^2 - 5x + 6$ with characteristic roots $2, 3$. Thus, $a_n = c_1(2^n) + c_2(3^n)$. Since $c_1 + c_2 = 2$, $2c_1 + 3c_2 = 11$, we obtain $c_1 = -5$, $c_2 = 7$ and $a_n = -5(2^n) + 7(3^n)$.

(b) We try $p_n = a + bn$ obtaining

$$a + bn = 5[a + b(n-1)] - 6[a + b(n-2)] + 3n = -a + 7b + (3 - b)n.$$

We obtain $a = \frac{21}{4}$, $b = \frac{3}{2}$, so $p_n = \frac{21}{4} + \frac{3}{2}n$. Using (a), we obtain $p_n + q_n = \frac{21}{4} + \frac{3}{2}n + c_1(2^n) + c_2(3^n)$. The initial conditions give $\frac{21}{4} + c_1 + c_2 = 2$, $\frac{21}{4} + \frac{3}{2} + 2c_1 + 3c_2 = 14$, whence, $c_1 = -17$, $c_2 = \frac{55}{4}$. Thus, $a_n = \frac{21}{4} + \frac{3}{2}n - 17(2^n) + \frac{55}{4}(3^n)$.

(c) We use the strong form of mathematical induction. If $n = 0$, $\frac{21}{4} + \frac{3}{2}(0) - 17(2^0) + \frac{55}{4}(3^0) = 2$, as required. If $n = 1$, $\frac{21}{4} + \frac{3}{2}(1) - 17(2^1) + \frac{55}{4}(3^1) = 14$, as required. Now let $k > 1$ and assume that for all $1 \leq n < k$, $a_n = \frac{21}{4} + \frac{3}{2}n - 17(2^n) + \frac{55}{4}(3^n)$. We have $a_k = 5a_{k-1} - 6a_{k-2} + 3k$, so

$$\begin{aligned} a_k &= 5[\tfrac{21}{4} + \tfrac{3}{2}(k-1) - 17(2)^{k-1} + \tfrac{55}{4}(3^{k-1})] \\ &\quad -6(\tfrac{21}{4} + \tfrac{3}{2}(k-2) - 17(2)^{k-2} + \tfrac{55}{4}(3^{k-2}) + 3k \\ &= -\tfrac{21}{4} + \tfrac{21}{2} - \tfrac{3}{2}k - 34(2^{k-1}) + 3(\tfrac{55}{4})(3^{k-1}) + 3k = \tfrac{21}{4} + \tfrac{3}{2}k - 17(2^k) + \tfrac{55}{4}(3^k) \end{aligned}$$

as desired. By the Principle of Mathematical Induction, the result holds for all $n \geq 0$.

14. (a) The characteristic polynomial is $x^2 + 6x + 9$ with repeated characteristic root $-3$. Hence, $a_n = c_1(-3)^n + c_2 n(-3)^n$. So $c_1 = 1$, $-3c_1 - 3c_2 = -4$; that is, $c_1 = 1$, $c_2 = 1/3$. Thus, $a_n = (-3)^n + \frac{1}{3}n(-3)^n = (-3)^n(1 + n/3) = -(n+3)(-3)^{n-1}$.

(b) Try $p_n = a + bn + cn^2$. We obtain

$$\begin{aligned} a + bn + cn^2 &= -6[a + b(n-1) + c(n-1)^2] - 9[a + b(n-2) + c(n-2)^2] + n^2 + 3n \\ &= -6[a - b + c + (b - 2c)n + cn^2] \\ &\quad - 9[a - 2b + 4c + (b - 4c)n + cn^2] + n^2 + 3n \\ &= -15a + 24b - 42c + (-15b + 48c + 3)n + (-15c + 1)n^2. \end{aligned}$$

Solving, we obtain $a = \frac{51}{128}$, $b = \frac{3}{8}$, $c = \frac{1}{16}$, so $p_n = \frac{51}{128} + \frac{3}{8}n + \frac{1}{16}n^2$. From part (a), $q_n = c_1(-3)^n + c_2 n(-3)^n$, so

$$p_n + q_n = \tfrac{51}{128} + \tfrac{3}{8}n + \tfrac{1}{16}n^2 + c_1(-3)^n + c_2 n(-3)^n.$$

Using the initial conditions, we obtain

$$\tfrac{51}{128} + c_1 = \tfrac{179}{128} \text{ and } \tfrac{51}{128} + \tfrac{3}{8} + \tfrac{1}{16} - 3c_1 - 3c_2 = -\tfrac{21}{128},$$

so $c_1 = 1$, $c_2 = -\frac{2}{3}$. Thus, $a_n = \frac{51}{128} + \frac{3}{8}n + \frac{1}{16}n^2 + (-3)^n(1 - \frac{2}{3}n)$.

(c) If $n = 0$, the above formula gives $\frac{51}{128} + 1 = \frac{179}{128}$, as desired. If $n = 1$, the above formula gives $\frac{51}{128} + \frac{3}{8} + \frac{1}{16} - 1 = -\frac{21}{128}$, as desired. Now assume that $k > 1$ and the formula is correct for all $1 \le n < k$. We have $a_k = -6a_{k-1} - 9a_{k-2} + k^2 + 3k$, so

$$\begin{aligned} a_k &= -6\big[\tfrac{51}{128} + \tfrac{3}{8}(k-1) + \tfrac{1}{16}(k-1)^2 + (-3)^{k-1}(1 - \tfrac{2}{3}(k-1)\big] \\ &\quad - 9\big[\tfrac{51}{128} + \tfrac{3}{8}(k-2) + \tfrac{1}{16}(k-2)^2 \\ &\qquad + (-3)^{k-2}(1 - \tfrac{2}{3}(k-2)\big] + k^2 + 3k \\ &= -15(\tfrac{51}{128}) + 9 - \tfrac{42}{16} - \tfrac{45}{8}k + \tfrac{48}{16}k + 3k - \tfrac{15}{16}k^2 + k^2 \\ &\quad + (-3)^k\big(2 - \tfrac{4}{3}(k-1) - 1 + \tfrac{2}{3}(k-2)\big) \\ &= \tfrac{51}{128} + \tfrac{3}{8}k + \tfrac{1}{16}k^2 + (-3)^k(1 - \tfrac{2}{3}k) \end{aligned}$$

as desired. By the Principle of Mathematical Induction, the result holds for all $n \ge 0$.

15. (a) [BB] With $p_n = k$, we obtain $k = 4k - 9$, so $k = 3$. A particular solution is $p_n = 3$. The homogeneous recurrence $a_n = 4a_{n-1}$ has solution $q_n = c4^n$, so $a_n = p_n + q_n = c4^n + 3$ The initial condition gives $c + 3 = 4$ so $c = 1$ and we obtain $a_n = 4^n + 3$ as our solution.

(b) [BB] Try $p_n = (an + b)2^n$. We obtain

$$\begin{aligned} (an + b)2^n &= 4[a(n-1) + b]2^{n-1} + 3n2^n = 2(an + b - a)2^n + 3n2^n \\ an + b &= 2(an + b - a) + 3n \end{aligned}$$

Thus, $a = 2a + 3$ (so $a = -3$) and $b = 2b - 2a$ (so $b = 2a = -6$). Hence, a particular solution is $p_n = -(3n + 6)2^n$. The homogeneous recurrence $a_n = 4a_{n-1}$ has solution $q_n = 4^n$, so

$$p_n + q_n = c4^n - (3n + 6)2^n.$$

The initial condition gives $c - 6 = 4$ so $c = 10$ and we obtain $a_n = 10(4^n) - (3n + 6)2^n$ as our solution.

16. We try $p_n = a + bn$ and obtain $a + bn = 4(a + b(n-1)) - 4(a + b(n-2)) + n$, so $a + bn = 4b + n$; that is, $a = 4$, $b = 1$. Thus, we have $p_n = 4 + n$ as a particular solution. As seen in Problem 21, the homogeneous recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ has solution $q_n = c_1(2^n) + c_2 n(2^n)$, so $p_n + q_n = 4 + n + c_1(2^n) + c_2 n(2^n)$. The initial conditions give $4 + c_1 = 5$, $5 + 2c_1 + 2c_2 = 9$, so $c_1 = 1$, $c_2 = 1$. Therefore, $a_n = 4 + n + 2^n + n2^n = 4 + n + 2^n(1 + n)$.

17. [BB] Try $p_n = a + bn + cn^2$. We obtain

$$\begin{aligned} a + bn + cn^2 &= 5[a + b(n-1) + c(n-1)^2] - 2[a + b(n-2) + c(n-2)^2] + 3n^2 \\ &= 5[a - b + c + (b - 2c)n + cn^2] - 2[a - 2b + 4c + (b - 4c)n + cn^2] + 3n^2 \\ &= 3a - b - 3c + (3b - 2c)n + (3c + 3)n^2. \end{aligned}$$

This gives the system

$$a = 3a - b - 3c; \quad b = 3b - 2c; \quad c = 3c + 3$$

with solution $a = -3$, $b = -\frac{3}{2}$, $c = -\frac{3}{2}$. Hence, a particular solution is $p_n = -3 - \frac{3}{2}n - \frac{3}{2}n^2$. The homogeneous recurrence $a_n = 5a_{n-1} - 2a_{n-2}$ has characteristic polynomial $x^2 - 5x + 2$ with characteristic roots $\frac{5 \pm \sqrt{17}}{2}$. Hence, $q_n = c_1(\frac{5-\sqrt{17}}{2})^n + c_2(\frac{5+\sqrt{17}}{2})^n$ and

$$p_n + q_n = -3 - \tfrac{3}{2}n - \tfrac{3}{2}n^2 + c_1\left(\tfrac{5-\sqrt{17}}{2}\right)^n + c_2\left(\tfrac{5+\sqrt{17}}{2}\right)^n.$$

The initial conditions give $-3 + c_1 + c_2 = 0$ and $-6 + c_1(\frac{5-\sqrt{17}}{2}) + c_2(\frac{5+\sqrt{17}}{2}) = 3$. This gives $c_1 = \frac{3}{2}(1 - \frac{1}{\sqrt{17}})$, $c_2 = \frac{3}{2}(1 + \frac{1}{\sqrt{17}})$ and, hence,

$$a_n = -3 - \tfrac{3}{2}n - \tfrac{3}{2}n^2 + \tfrac{3}{2}\left(1 - \tfrac{1}{\sqrt{17}}\right)\left(\tfrac{5-\sqrt{17}}{2}\right)^n + \tfrac{3}{2}\left(1 + \tfrac{1}{\sqrt{17}}\right)\left(\tfrac{5+\sqrt{17}}{2}\right)^n.$$

18. We look for a particular solution of the form $p_n = a(5^n)$. Substituting into the recurrence relation, we get

$$\begin{aligned} a(5^n) &= 2a(5^{n-1}) + 3a(5^{n-2}) + 5^n \\ 25a &= 10a + 3a + 25 \\ 12a &= 25, \end{aligned}$$

so $a = \frac{25}{12}$. The student should confirm that $p_n = \frac{25}{12}(5^n)$ is a particular solution. Next we solve the homogeneous recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$. The characteristic polynomial, $x^2 - 2x - 3$, has distinct roots $-1$ and 3, so the solution is $q_n = c_1(-1)^n + c_2(3^n)$ and the given recurrence relation has general solution $a_n = p_n + q_n = \frac{25}{12}(5^n) + c_1(-1)^n + c_2(3^n)$. The initial conditions give

$$a_0 = -2 = \tfrac{25}{12} + c_1 + c_2$$

$$a_1 = 1 = \tfrac{25}{12}(5) - c_1 + 3c_2,$$

hence, $c_1 = -\frac{17}{24}$, $c_2 = -\frac{27}{8}$. The solution is $a_n = \frac{25}{12}(5^n) - \frac{17}{24}(-1)^n - \frac{27}{8}(3^n)$.

19. (a) [BB] $1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots$.
The corresponding characteristic polynomial is $x^2 - x + 1$ whose roots are $\frac{1}{2}(1 \pm \sqrt{3}i)$. By Theorem 5.3.1, the solution is $a_n = c_1(\frac{1+\sqrt{3}i}{2})^n + c_2(\frac{1-\sqrt{3}i}{2})^n$. The initial conditions give $a_0 = 1 = c_1 + c_2$, $a_1 = 1 = c_1(\frac{1+\sqrt{3}i}{2}) + c_2(\frac{1-\sqrt{3}i}{2})$, so $c_1 = \frac{1}{2} - \frac{i}{2\sqrt{3}}$, $c_2 = \frac{1}{2} + \frac{i}{2\sqrt{3}}$ and $a_n = (\frac{1}{2} - \frac{i}{2\sqrt{3}})(\frac{1+\sqrt{3}i}{2})^n + (\frac{1}{2} + \frac{i}{2\sqrt{3}})(\frac{1-\sqrt{3}i}{2})^n$.

(b) $-1, 1, 4, 6, 4, -4, -16, -24, -16, 16, \ldots$.

The corresponding characteristic polynomial is $x^2 - 2x + 2$ whose roots are $1 \pm i$. By Theorem 5.3.1, the solution is $a_n = c_1(1+i)^n + c_2(1-i)^n$. The initial conditions give $a_0 = -1 = c_1 + c_2$, $a_1 = 1 = c_1(1+i) + c_2(1-i)$, so $c_1 = -\frac{1}{2}(1+2i)$, $c_2 = -\frac{1}{2}(1-2i)$ and $a_n = -\frac{1}{2}(1+2i)(1+i)^n - \frac{1}{2}(1-2i)(1-2i)^n$.

(c) $-2, 0, 20, -40, -120, 640, -80, -6240, 13280, 35840, \ldots$.

The corresponding characteristic polynomial is $x^2 + 2x + 10$ whose roots are $-1 \pm 3i$. By Theorem 5.3.1, the solution is $a_n = c_1(-1+3i)^n + c_2(-1-3i)^n$. The initial conditions give $a_0 = -2 = c_1 + c_2$, $a_1 = 0 = c_1(-1+3i) + c_2(-1-3i)$, so $c_1 = \frac{1}{3}(-3+i)$, $c_2 = -\frac{1}{3}(3+i)$ and $a_n = \frac{1}{3}(-3+i)(-1+3i)^n - \frac{1}{3}(3+i)(-1-3i)^n$.

20.

| $n$ | $f_n$ | $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ |
|---|---|---|
| 0 | 1 | 0.724 |
| 1 | 1 | 1.171 |
| 2 | 2 | 1.894 |
| 3 | 3 | 3.065 |
| 4 | 5 | 4.960 |
| 5 | 8 | 8.025 |
| 6 | 13 | 12.985 |
| 7 | 21 | 21.010 |
| 8 | 34 | 33.994 |
| 9 | 55 | 55.004 |

In each case, $f_n$ is the integer closest to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$.

21. [BB] In Problem 22, we showed that the $n$th term of the Fibonacci sequence is $a_{n-1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. Thus, $|a_{n-1} - \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n| = |\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n|$. Now $|\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n| \leq \frac{1}{\sqrt{5}} < \frac{1}{2}$ since $|\frac{1-\sqrt{5}}{2}| < 1$, so $|a_{n-1} - \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n| < \frac{1}{2}$. Remembering that $a_{n-1}$ is an integer, the result follows from the fact that there is precisely one integer within $\frac{1}{2}$ of any given real number.

22. (a) Consider all $n+1$-digit numbers of the required type. There are $a_n$ of these which end in 2, $a_n$ which end in 3 and $a_n$ which end in 4. To determine the number which end in 1, we note that in this case, the number comprised of the first $n$ digits is **not** of the required type; there are $4^n - a_n$ of these. Thus, $a_{n+1} = 3a_n + (4^n - a_n) = 2a_n + 4^n$.

(b) The homogeneous recurrence relation $a_n = 2a_{n-1}$ has solution $a_n = c2^n$, while a particular solution of $a_n = 2a_{n-1} + 4^{n-1}$ is $\frac{1}{2}(4^n)$. Thus, the given relation has a solution of the form $a_n = c2^n + \frac{1}{2}4^n$ and, since $a_1 = 3$, $c = \frac{1}{2}$. Thus, $a_n = \frac{1}{2}(2^n + 4^n)$.

23. (a) Labelling the pegs $A$, $B$ and $C$ and numbering the disks $1, 2, \ldots, n$ here is a solution for $n = 2$. ($*$ denotes no disks.)

| | $A$ | $B$ | $C$ | |
|---|---|---|---|---|
| Initial position | 1, 2 | $*$ | $*$ | |
| Move 1 | 2 | $*$ | 1 | Three moves are required. |
| Move 2 | $*$ | 2 | 1 | |
| Move 3 | $*$ | 1, 2 | $*$ | |

Here is a solution for $n = 3$.

| | $A$ | $B$ | $C$ | |
|---|---|---|---|---|
| Initial position | 1,2,3 | $*$ | $*$ | |
| Move 1 | 2,3 | 1 | $*$ | |
| Move 2 | 3 | 1 | 2 | |
| Move 3 | 3 | $*$ | 1,2 | Seven moves are required. |
| Move 4 | $*$ | 3 | 1,2 | |
| Move 5 | 1 | 3 | 2 | |
| Move 6 | 1 | 2,3 | $*$ | |
| Move 7 | $*$ | 1,2,3 | $*$ | |

(b) To transfer $n$ discs from peg $A$ to peg $B$, just mimic what we did for 2 and for 3, but with the top disk replaced by the top $n-1$ disks: following:

1. Transfer the top $n-1$ discs to peg $C$ in $a_{n-1}$ moves.
2. Transfer the remaining (largest) disc to peg $B$ in one move.
3. Transfer the $n-1$ smaller discs from peg $C$ to peg $B$ in another $a_{n-1}$ moves.

We obtain $a_n = 2a_{n-1} + 1$.

(c) We obtain easily that $p_n = -1$ is a particular solution to the recurrence relation $a_n = 2a_{n-1} + 1$. The characteristic polynomial associated with $a_n = 2a_{n-1}$ is $x^2 - 2x$ with characteristic roots $0, 2$. Thus, $q_n = c(2^n)$ and $p_n + q_n = c(2^n) - 1$. Since $a_1 = 1$, $c(2^1) - 1 = 1$ and $c = 1$. We obtain $a_n = 2^n - 1$.

(d) If $n = 8$, we require $2^8 - 1 = 255$ seconds $\approx 4$ minutes.

If $n = 16$, we require $2^{16} - 1 = 65535$ seconds $\approx 18$ hr

If $n = 32$, we require $2^{32} - 1 = 4294967295$ sec $\approx 136$ yrs.

If $n = 64$, we require $2^{64} - 1$ sec $\approx 5.8 \times 10^{11}$ yr.

24. (a) Labelling the pegs $A$, $B$ and $C$ and numbering the disks $1, 2, \ldots, n$ here is a solution for $n = 2$ with the tower initially on peg $A$ ($*$ denotes no disks).

| | $A$ | $B$ | $C$ | |
|---|---|---|---|---|
| Initial position | $1, 2$ | $*$ | $*$ | |
| Move 1 | 2 | 1 | $*$ | Four moves are required. By reversing these steps, we see that four moves also suffice when the tower is initially on the middle peg. |
| Move 2 | 2 | $*$ | 1 | |
| Move 3 | $*$ | 2 | 1 | |
| Move 4 | $*$ | $1, 2$ | $*$ | |

Here is a solution for $n = 3$, again assuming that the tower is initially on peg $A$.

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| Initial position | 1,2,3 | * | * |
| Move 1 | 2,3 | 1 | * |
| Move 2 | 2,3 | * | 1 |
| Move 3 | 3 | 2 | 1 |
| Move 4 | 3 | 1,2 | * |
| Move 5 | 1,3 | 2 | * |
| Move 6 | 1,3 | * | 2 |
| Move 7 | 3 | 1 | 2 |
| Move 8 | 3 | * | 1,2 |
| Move 9 | * | 3 | 1,2 |
| Move 10 | * | 1,3 | 2 |
| Move 11 | 1 | 3 | 2 |
| Move 12 | 1 | 2,3 | * |
| Move 13 | * | 1,2,3 | * |

Thirteen moves are required. By reversing the steps, we see that thirteen moves also suffice when the tower is initially on the middle peg.

(b) To transfer $n$ discs from peg $A$ to peg $B$, do the following:

1. Transfer the top $n-1$ discs to peg $B$ in $a_{n-1}$ moves.
2. Then transfer all $n-1$ discs from peg $B$ to peg $C$ in another $a_{n-1}$ moves.
3. Transfer the largest disc from $A$ to $B$ in one move.
4. Finally transfer the $n-1$ discs from peg $C$ to peg $B$ in another $a_{n-1}$ moves.

We obtain $a_n = 3a_{n-1} + 1$.

(c) We obtain easily that $p_n = -\frac{1}{2}$ is a particular solution to the recurrence relation $a_n = 3a_{n-1} + 1$. The characteristic polynomial associated with $a_n = 3a_{n-1}$ is $x^2 - 3x$ with characteristic roots $0, 3$. Thus, $q_n = c(3^n)$ and $p_n + q_n = c(3^n) - \frac{1}{2}$. Since $a_1 = 1$, $c(3^1) - \frac{1}{2} = 1$ and $c = \frac{1}{2}$. We obtain $a_n = \frac{1}{2}(3^n - 1)$.

(d) If $n = 8$, we require $\frac{1}{2}(3^8 - 1) = 3280$ seconds $\approx 55$ minutes.
If $n = 16$, we require $\frac{1}{2}(3^{16} - 1) = 21,523,360$ seconds $\approx 35\frac{1}{2}$ weeks.
If $n = 32$, we require $\frac{1}{2}(3^{32} - 1) \approx 9.3 \times 10^{14}$ secs. $\approx 300,000$ centuries.
If $n = 64$, we require $\frac{1}{2}(3^{64} - 1)$ secs. $\approx 5 \times 10^{20}$ centuries.

25. (a) [BB] Since $x$ is a root of the characteristic polynomial, we have $x^2 = rx + s$. Hence,

$$ra_{n-1} + sa_{n-2} = r(cx^{n-1}) + s(cx^{n-2}) = cx^{n-2}(rx + s) = cx^{n-2}(x^2) = cx^n = a_n$$

(b) [BB] We know that $p_n = rp_{n-1} + sp_{n-2}$ and $q_n = rq_{n-1} + sq_{n-2}$. Thus, $p_n + q_n = rp_{n-1} + sp_{n-2} + rq_{n-1} + sq_{n-2} = r(p_{n-1} + q_{n-1}) + s(p_{n-2} + q_{n-2})$.

(c) [BB] If $x_1$ and $x_2$ are the characteristic roots, part (a) tells us that $c_1x_1^n$ and $c_2x_2^n$ both satisfy the recurrence relation while part (b) says that $c_1x_1^n + c_2x_2^n$ is also a solution. Two initial conditions determine $c_1$ and $c_2$ for they determine two linear equations in the unknowns $c_1, c_2$ which, because $x_1 \neq x_2$, must have a unique solution.

26. (a) [BB] The characteristic polynomial is $x^2 - rx - s$ with characteristic roots $x = \dfrac{r \pm \sqrt{r^2 + 4s}}{2}$. Since there is only one root, we must have $r^2 = -4s$ in which case $x = r/2$. Hence, $r = 2x$ and $s = -r^2/4 = -4x^2/4 = -x^2$.

(b) We have

$$\begin{aligned} ra_{n-1} + sa_{n-2} &= 2x[c(n-1)x^{n-1}] - x^2[c(n-2)x^{n-2}] \\ &= 2c(n-1)x^n - c(n-2)x^n = cnx^n = a_n \end{aligned}$$

as desired.

(c) By part (b), $c_2 nx^n$ is one solution and by part (a) of Exercise 25 $c_1 x^n$ is another. Hence, by part (b) of Exercise 25, we conclude that $c_1 x^n + c_2 nx^n$ is also a solution. Finally, two initial conditions will determine $c_1$ and $c_2$ because they determine two linear equations in the two unknowns $c_1, c_2$ which are guaranteed to have a unique solution.

27. (a) We have $p_n = rp_{n-1} + sp_{n-2} + f(n)$ for $n \geq 2$ and also $q_n = rq_{n-1} + sq_{n-2}$ for $n \geq 2$. Hence,

$$\begin{aligned} p_n + q_n &= rp_{n-1} + sp_{n-2} + f(n) + rq_{n-1} + sq_{n-2} \\ &= r(p_{n-1} + q_{n-1}) + s(p_{n-2} + q_{n-2}) + f(n) \end{aligned}$$

which says that $p_n + q_n$ satisfies the given relation for $n \geq 2$.

(b) Part (a) shows that $p_n + q_n$ satisfies the given recurrence relation where $q_n = c_1 x_1^n + c_2 x_2^n$ or $q_n = c_1 x^n + c_2 nx^n$ according as the polynomial $x^2 - rx - s$ has two distinct roots or just one. Two initial conditions provide two equations which can be solved for $c_1, c_2$.

## Exercises 5.4

1. (a) [BB] Since $(2-3x)^2 = 4 - 12x + 9x^2$ the associated sequence is $4, -12, 9, 0, 0, \ldots$.

(b) Since $\dfrac{x^4}{1-x} = x^4(1 + x + x^2 + \cdots) = x^4 + x^5 + x^6 + \cdots$,
the associated sequence is $0, 0, 0, 0, 1, 1, 1, 1, \ldots$

(c) [BB] Since $\dfrac{1}{(1+3x)^2} = 1 + 2(-3x) + 3(-3x)^2 + 4(-3x)^3 = 1 - 6x + 27x^2 - 108x^3 + \cdots$,
the associated sequence is $1, -6, 27, -108, \ldots, (n+1)(-3)^n, \ldots$.

(d) Since $\dfrac{1}{(1-x^3)^2} = 1 + 2(x^3) + 3(x^3)^2 + 4(x^3)^3 + \cdots = 1 + 2x^3 + 3x^6 + 4x^9 + \cdots$,
the associated sequence is $1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, 5, \ldots$

(e) Since $\dfrac{x^2}{(1+x)^2} + \dfrac{1}{1+5x}$
$= x^2[1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \cdots] + [1 + (-5x) + (-5x)^2 + (-5x)^3 + \cdots]$
$= [x^2 - 2x^3 + 3x^4 - 4x^5 + \cdots] + [1 - 5x + 25x^2 - 125x^3 + 625x^4 + \cdots]$
the associated sequence is $1, -5, 26, \ldots, (-5)^n + (-1)^n(n-1), \ldots$

2. (a) [BB] The generating function is $1 + 2x + 5x^2 + 0x^3 + 0x^4 + \cdots$. This is just the polynomial $1 + 2x + 5x^2$.

(b) The generating function is $0 + x + 4x^2 + x^3$. This is just the polynomial $x + 4x^2 + x^3$.

(c) This is a geometric sequence with generating function $\frac{1}{1-2x}$ as in PAUSE 10.

(d) The generating function for this geometric sequence is $\frac{1}{1+x}$, as in PAUSE 10.

(e) The generating function is $3 + 3x + 3x^2 + 3x^3 + \cdots = 3(1 + x + x^2 + \cdots) = \frac{3}{1-x}$.

(f) [BB] The generating function is $1 + x^2 + x^4 + x^6 + \cdots = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots$. This is obtained by replacing $x$ by $x^2$ in (12). The generating function is $\frac{1}{1-x^2}$.

(g) The generating function is $1 - 2x + 3x^2 - 4x^3 + \cdots$. This is obtained by replacing $x$ by $-x$ in (13). The generating function is $\frac{1}{(1+x)^2}$.

3. [BB]
$$\begin{array}{rcl} f(x) & = & a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 2xf(x) & = & 2a_0x + 2a_1x^2 + \cdots + 2a_{n-1}x^n + \cdots \end{array}$$
Subtracting gives $f(x) - 2xf(x) = a_0 + (a_1 - 2a_0)x + \cdots + (a_n - 2a_{n-1})x^n + \cdots = 1$ since $a_0 = 1$ and $a_n - 2a_{n-1} = 0$ for $n \geq 1$. Thus,
$$\begin{aligned} f(x) &= \frac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots \\ &= 1 + 2x + 4x^2 + \cdots + 2^nx^n + \cdots \end{aligned}$$
We conclude that $a_n = 2^n$. Our solution is the same as the sequence defined in (5) of Section 5.2 except that it has one additional term, $a_0 = 1$.

4. We have
$$\begin{array}{rcl} f(x) & = & a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 3xf(x) & = & 3a_0x + 3a_1x^2 + \cdots + 3a_{n-1}x^n + \cdots . \end{array}$$
Subtracting gives
$$f(x) - 3xf(x) = a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)x^2 + \cdots + (a_n - 3a_{n-1})x^n + \cdots .$$
Since $a_0 = 1$ and $a_n - 3a_{n-1} = 1$ for $n \geq 1$, this gives $(1 - 3x)f(x) = 1 + x + x^2 + \cdots + x^n + \cdots$. Thus
$$\begin{aligned} f(x) &= \frac{1}{1-3x}(1 + x + x^2 + \cdots + x^n + \cdots) \\ &= (1 + 3x + 9x^2 + \cdots + 3^nx^n + \cdots)(1 + x + x^2 + \cdots + x^n + \cdots) \\ &= 1 + 4x + 13x^2 + \cdots + (3^n + 3^{n-1} + 3^{n-2} + \cdots + 1)x^n + \cdots \\ &= 1 + 4x + 13x^2 + \cdots + \tfrac{3^{n+1}-1}{2}x^n + \cdots \end{aligned}$$
We conclude that $a_n = \frac{3^{n+1}-1}{2}$. Our solution is the same as the sequence in Problem 11 except that it begins with $a_0 = 1$. Thus, the formula for $a_n$ which we have obtained here should be the formula for the $(n+1)$st term of the sequence in Problem 11, and it is.

5. [BB]
$$\begin{array}{rcl} f(x) & = & a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 5xf(x) & = & 5a_0x + 5a_1x^2 + \cdots + 5a_{n-1}x^n + \cdots \\ 6x^2f(x) & = & 6a_0x^2 + \cdots + 6a_{n-2}x^n + \cdots \end{array}$$
Therefore, $f(x) - 5xf(x) + 6x^2f(x)$
$= a_0 + (a_1 - 5a_0)x + (a_2 - 5a_1 + 6a_0)x^2 + \cdots + (a_n - 5a_{n-1} + 6a_{n-2})x^n + \cdots$.
Since $a_0 = -3$, $a_1 - 5a_0 = -2 - 5(-3) = 13$ and $a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$, we get $f(x)(1 - 5x + 6x^2) = -3 + 13x$. So,
$$f(x) = \frac{-3 + 13x}{(1-2x)(1-3x)}$$

Using partial fractions we set $\dfrac{-3+13x}{(1-2x)(1-3x)} = \dfrac{A}{1-2x} + \dfrac{B}{1-3x}$ and, equating numerators,

$$\begin{aligned} A(1-3x) + B(1-2x) &= -3+13x \\ (A+B) + (-3A-2B)x &= -3+13x. \end{aligned}$$

Hence, $A+B=-3$, $-3A-2B=13$ and $A=-7$, $B=4$. Therefore,

$$\begin{aligned} f(x) &= \frac{-7}{1-2x} + \frac{4}{1-3x} \\ &= (-7)(1+2x+4x^2+\cdots+2^nx^n+\cdots) + 4(1+3x+9x^2+\cdots+3^nx^n+\cdots) \\ &= -3-2x+8x^2+\cdots+(-7\cdot 2^n+4\cdot 3^n)x^n+\cdots \end{aligned}$$

and, hence, $a_n = -7(2^n)+4(3^n)$ is the solution, confirming the result of Problem 20.

6.
$$\begin{array}{rcccccccccc} f(x) &=& a_0 &+& a_1x &+& a_2x^2 &+\cdots+& a_nx^n &+\cdots \\ 4xf(x) &=& & & 4a_0x &+& 4a_1x^2 &+\cdots+& 4a_{n-1}x^n &+\cdots \\ 4x^2f(x) &=& & & & & 4a_0x^2 &+\cdots+& 4a_{n-2}x^n &+\cdots \end{array}$$

Therefore, $f(x)-4xf(x)+4x^2f(x)$

$= a_0+(a_1-4a_0)x+(a_2-4a_1+4a_0)x^2+\cdots+(a_n-4a_{n-1}+4a_{n-2})x^n+\cdots$.

Since $a_0=1$, $a_1-4a_0=4-4(1)=0$ and $a_n-4a_{n-1}+4a_{n-2}=0$ for $n\geq 2$, we get $f(x)(1-4x+4x^2)=1$. Therefore,

$$\begin{aligned} f(x) &= \frac{1}{(1-2x)^2} = 1+2(2x)+3(2x)^2+\cdots+(n+1)(2x)^n+\cdots \\ &= 1+4x+12x^2+\cdots+(n+1)2^nx^n+\cdots; \end{aligned}$$

hence, $a_n=(n+1)2^n$ is the solution. Our solution is exactly the same as the sequence in Problem 21 and so our answer here verifies the result obtained earlier.

7. [BB]
$$\begin{array}{rcccccccc} f(x) &=& a_0 &+& a_1x &+& a_2x^2 &+\cdots+& a_nx^n &+\cdots \\ xf(x) &=& & & a_0x &+& a_1x^2 &+\cdots+& a_{n-1}x^n &+\cdots \end{array}$$

Subtracting gives

$$\begin{aligned} f(x)-xf(x) &= a_0+(a_1-a_0)x+\cdots+(a_n-a_{n-1})x^n+\cdots \\ (1-x)f(x) &= 5+2x+4x^2+\cdots+2^nx^n+\cdots \\ &= (1+2x+4x^2+\cdots+2^nx^n+\cdots)+4 \end{aligned}$$

since $a_0=5$ and $a_n-a_{n-1}=2^n$. Thus,

$$\begin{aligned} f(x) &= \frac{1}{1-x}(1+2x+4x^2+\cdots+2^nx^n+\cdots)+4\left(\frac{1}{1-x}\right) \\ &= [1+(2+1)x+(4+2+1)x^2+\cdots+(2^n+2^{n-1}+2^{n-2}+\cdots+2+1)x^n+\cdots] \\ &\quad +4(1+x+x^2+\cdots+x^n+\cdots) \\ &= (1+3x+7x^2+\cdots+(2^{n+1}-1)x^n+\cdots)+(4+4x+4x^2+\cdots+4x^n+\cdots) \end{aligned}$$

Note that $1+2+2^2+\cdots+2^n=\frac{2^{n+1}-1}{2-1}=2^{n+1}-1$ as the sum of a geometric sequence. Thus, $a_n=2^{n+1}-1+4=2^{n+1}+3$.

8. (a) First we look for a particular solution and try $p_n = a(4^n)$. Substituting, we obtain $a(4^n) = 4a(4^{n-1}) - 4a(4^{n-2}) + 4^n$; that is, $4^n a = 4^n a - 4^{n-1}a + 4^n$. This gives $4^{n-1}a = 4^n$, so $a = 4$ and $p_n = 4^{n+1}$. In Problem 21, the homogeneous recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ was seen earlier to have general solution $a_n = c_1(2^n) + c_2 n(2^n)$, so $p_n + q_n = 4^{n+1} + c_1(2^n) + nc_2(2^n)$. Bringing in our initial conditions, we obtain $4^1 + c_1(2^0) + 0c_2(2^0) = 2$ and $4^2 + c_1(2^1) + 1c_2(2^1) = 8$; that is,

$$\begin{aligned} 4 + c_1 &= 2 \\ 16 + 2c_1 + 2c_2 &= 8. \end{aligned}$$

Solving, we obtain $c_1 = -2$, $c_2 = -2$, so

$$a_n = 4^{n+1} - 2^{n+1} - n2^{n+1} = 4^{n+1} - 2^{n+1}(1 + n).$$

(b)
$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 4xf(x) &= 4a_0x + 4a_1x^2 + \cdots + 4a_{n-1}x^n + \cdots \\ 4x^2f(x) &= 4a_0x^2 + \cdots + 4a_{n-2}x^n + \cdots \end{aligned}$$

Therefore, $f(x) - 4xf(x) + 4x^2f(x)$
$= a_0 + (a_1 - 4a_0)x + (a_2 - 4a_1 + 4a_0)x^2 + \cdots + (a_n - 4a_{n-1} + 4a_{n-2})x^n + \cdots$.
So $f(x)(1 - 4x + 4x^2) = 2 + (0)x + 4^2x^2 + \cdots + 4^nx^n + \cdots$ and

$$\begin{aligned} f(x) &= \frac{1}{(1-2x)^2}(2 + 16x^2 + \cdots + 4^nx^n + \cdots) \\ &= (1 + 2(2x) + 3(2x)^2 + \cdots + (n+1)(2x)^n + \cdots)(2 + 16x^2 + \cdots + 4^nx^n + \cdots) \\ &= 2 + 8x + 40x^2 + \cdots + [(1)4^n + 2(2)4^{n-1} + 3(2^2)4^{n-2} + \cdots \\ &\qquad + (n-1)(2^{n-2})4^2 + (n+1)2^n(2)]x^n + \cdots \end{aligned}$$

To find a formula for $a_n$, we must now sum the terms in the square brackets, but it is not immediately obvious how this can be done. An induction argument can be used to show that this sum equals $4^{n+1} - 2^{n+1}(1 + n)$ as in (a).

The authors prefer the method of the characteristic polynomial to generating functions in this instance.

9. (a) [BB]
$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 5xf(x) &= 5a_0x + 5a_1x^2 + \cdots + 5a_{n-1}x^n + \cdots \end{aligned}$$

Adding, we obtain

$$f(x) + 5xf(x) = a_0 + (a_1 + 5a_0)x + (a_2 + 5a_1)x^2 + \cdots + (a_n + 5a_{n-1})x^n + \cdots = 2$$

since $a_0 = 2$, $a_n + 5a_{n-1} = 0$ for $n \geq 1$. Thus,

$$f(x) = \frac{2}{1 + 5x} = 2(1 - 5x + 25x^2 + \cdots + (-5)^nx^n + \cdots)$$

and so $a_n = 2(-5)^n$. The characteristic polynomial is $x^2 + 5x$ with characteristic roots 0 and $-5$. We obtain $a_n = c_1(-5)^n$. Since $a_0 = 2$, we have $c_1(-5)^0 = 2$; that is, $c_1 = 2$, so $a_n = 2(-5)^n$ as proven before.

(b)
$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 5xf(x) &= 5a_0x + 5a_1x^2 + \cdots + 5a_{n-1}x^n + \cdots \end{aligned}$$

Adding, we obtain

$$\begin{aligned} f(x)+5xf(x) &= a_0+(a_1+5a_0)x+(a_2+5a_1)x^2+\cdots+(a_n+5a_{n-1})x^n+\cdots \\ f(x)(1+5x) &= 2+3x+3x^2+\cdots+3x^n+\cdots \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= (1-5x+25x^2+\cdots+(-5)^nx^n+\cdots)(2+3x+3x^2+\cdots+3x^n+\cdots) \\ &= 2-7x+38x^2+\cdots+[1(3)+(-5)(3)+\cdots+(-5)^{n-1}(3)+(-5)^n(2)]x^n+\cdots \end{aligned}$$

We simplify the term in square brackets.

$$\begin{aligned} &3[1+(-5)+(-5)^2+\cdots+(-5)^{n-1}]+2(-5)^n \\ &\qquad = 3\frac{(-5)^n-1}{-5-1}+2(-5)^n, \quad \text{summing a geometric sequence,} \\ &\qquad = 2(-5)^n+\frac{1-(-5)^n}{2}=\frac{1+3(-5)^n}{2} \end{aligned}$$

and so $a_n=\frac{1}{2}(1+3(-5)^n)$. According to Section 5.3, the solution has the form $p_n+q_n$, where $q_n$ is the general solution to the homogeneous relation $a_n=-5a_{n-1}$ and $p_n$ is a particular solution to $a_n=-5a_{n-1}+3$. As in (a), we obtain $q_n=c_1(-5)^n$. For $p_n$, we try a linear function $p_n=a+bn$. Substituting in $a_n=-5a_{n-1}+3$, we get $a+bn=(-5a+5b+3)-5bn$, hence, $a=-5a+5b+3$, $b=-5b$. These equations give $b=0$, $a=\frac{1}{2}$. Thus, $p_n+q_n=c_1(-5)^n+\frac{1}{2}$. Using $a_0=2$, we get $c_1=\frac{3}{2}$ and so $a_n=\frac{3}{2}(-5)^n+\frac{1}{2}$ in agreement with our previous solution.

10. (a)
$$\begin{array}{rcccccccc} f(x) &=& a_0+ & a_1x+ & a_2x^2+ & \cdots+ & a_nx^n+ & \cdots \\ 4xf(x) &=& & 4a_0x+ & 4a_1x^2+ & \cdots+ & 4a_{n-1}x^n+ & \cdots \\ 3x^2f(x) &=& & & 3a_0x^2+ & \cdots+ & 3a_{n-2}x^n+ & \cdots \end{array}$$

Therefore, $f(x)-4xf(x)+3x^2f(x)$
$=a_0+(a_1-4a_0)x+(a_2-4a_1+3a_0)x^2+\cdots+(a_n-4a_{n-1}+3a_{n-2})x^n+\cdots=2-3x.$
So, $f(x)(1-4x+3x^2)=2-3x$ and $f(x)=\dfrac{2-3x}{(1-x)(1-3x)}$. Now write

$$\frac{2-3x}{(1-x)(1-3x)}=\frac{A}{1-x}+\frac{B}{1-3x}=\frac{(A+B)+(-3A-B)x}{(1-x)(1-3x)}.$$

Then $A+B=2$, $-3A-B=-3$; $A=\frac{1}{2}$, $B=\frac{3}{2}$ and so

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\frac{1}{1-x}\right)+\frac{3}{2}\left(\frac{1}{1-3x}\right) \\ &= \tfrac{1}{2}(1+x+x^2+\cdots+x^n+\cdots)+\tfrac{3}{2}(1+3x+9x^2+\cdots+3^nx^n+\cdots) \\ &= 2+5x+\cdots+(\tfrac{1}{2}+\tfrac{3}{2}(3^n))x^n+\cdots \end{aligned}$$

and so $a_n=\frac{1}{2}+\frac{3}{2}(3^n)$.

(b)
$$\begin{array}{rcccccccc} f(x) &=& a_0+ & a_1x+ & a_2x^2+ & \cdots+ & a_nx^n+ & \cdots \\ 10xf(x) &=& & 10a_0x+ & 10a_1x^2+ & \cdots+ & 10a_{n-1}x^n+ & \cdots \\ 25x^2f(x) &=& & & 25a_0x^2+ & \cdots+ & 25a_{n-2}x^n+ & \cdots \end{array}$$

Therefore, $f(x) + 10xf(x) + 25x^2 f(x)$
$= a_0 + (a_1 + 10a_0)x + (a_2 + 10a_1 + 25a_0)x^2 + \cdots + (a_n + 10a_{n-1} + 25a_{n-2})x^n + \cdots$.
So $f(x)(1 + 10x + 25x^2) = 1 + 35x$. Therefore,

$$\begin{aligned} f(x) &= \frac{1}{(1+5x)^2}(1+35x) \\ &= (1 + 2(-5x) + \cdots + (n+1)(-5x)^n + \cdots)(1+35x) \\ &= 1 + 25x - 275x^2 + \cdots + [(n+1)(-5)^n + n(-5)^{n-1}(35)]x^n + \cdots \\ &= 1 + 25x - 275x^2 + \cdots + [(1-6n)(-5)^n]x^n + \cdots \end{aligned}$$

and so $a_n = (1-6n)(-5)^n$.

11. [BB]
$$\begin{array}{rcccccccccc} f(x) &=& a_0 + & a_1x + & a_2x^2 + & a_3x^3 + & \cdots + & a_nx^n & + & \cdots \\ 2xf(x) &=& & 2a_0x + & 2a_1x^2 + & 2a_2x^3 + & \cdots + & 2a_{n-1}x^n & + & \cdots \\ x^2f(x) &=& & & a_0x^2 + & a_1x^3 + & \cdots + & a_{n-2}x^n & + & \cdots \\ 2x^3f(x &=& & & & 2a_0x^3 + & \cdots + & 2a_{n-3}x^n & + & \cdots \end{array}$$

Therefore, $f(x) - 2xf(x) - x^2f(x) + 2x^3f(x)$

$$\begin{aligned} &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 - a_0)x^2 + (a_3 - 2a_2 - a_1 + 2a_0)x^3 \\ &\quad + \cdots + (a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3})x^n + \cdots = 1 + x - x^2 \end{aligned}$$

since $a_0 = 1$, $a_1 - 2a_0 = 3 - 2(1) = 1$, $a_2 - 2a_1 - a_0 = 6 - 2(3) - 1 = -1$ and $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$ for $n \geq 3$. So

$$(1 - 2x - x^2 + 2x^3)f(x) = 1 + x - x^2$$

and

$$f(x) = \frac{1+x-x^2}{1-2x-x^2+2x^3} = \frac{1+x-x^2}{(1+x)(1-x)(1-2x)}$$

Writing $\dfrac{1+x-x^2}{(1+x)(1-x)(1-2x)} = \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{C}{1-2x}$ and equating numerators, we have

$$\begin{aligned} 1 + x - x^2 &= A(1-x)(1-2x) + B(1+x)(1-2x) + C(1+x)(1-x) \\ &= (A + B + C) + (-3A - B)x + (2A - 2B - C)x^2. \end{aligned}$$

Thus, $A + B + C = 1$, $-3A - B = 1$, $2A - 2B - C = -1$ and so $A = -\frac{1}{6}$, $B = -\frac{1}{2}$, $C = \frac{5}{3}$. Therefore,

$$\begin{aligned} f(x) &= -\frac{1}{6}\left(\frac{1}{1+x}\right) - \frac{1}{2}\left(\frac{1}{1-x}\right) + \frac{5}{3}\left(\frac{1}{1-2x}\right) \\ &= -\tfrac{1}{6}(1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots) \\ &\qquad -\tfrac{1}{2}(1 + x + x^2 + \cdots + x^n + \cdots) + \tfrac{5}{3}(1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots) \\ &= 1 + 3x + 6x^2 + \cdots + [(-\tfrac{1}{6})(-1)^n - \tfrac{1}{2} + \tfrac{5}{3}(2^n)]x^n + \cdots \end{aligned}$$

and thus, $a_n = -\frac{1}{2} + (-\frac{1}{6})(-1)^n + \frac{5}{3}(2^n)$.

12. [BB]
$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \\ xf(x) &= a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-1}x^n + \cdots \\ x^2f(x) &= a_0x^2 + a_1x^3 + \cdots + a_{n-2}x^n + \cdots \\ x^3f(x) &= a_0x^3 + \cdots + a_{n-3}x^n + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) - xf(x) - x^2f(x) + x^3f(x) & \\ &= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1 + a_0)x^3 \\ &\quad + \cdots + (a_n - a_{n-1} - a_{n-2} + a_{n-3})x^n + \cdots \\ &= 2 - 3x + 2x^2 \end{aligned}$$

since $a_0 = 2$, $a_1 = -1$, $a_2 = 3$ and $a_n - a_{n-1} - a_{n-2} + a_{n-3} = 0$ for $n \geq 3$. So

$$(1 - x - x^2 + x^3)f(x) = 2 - 3x + 2x^2$$

and $f(x) = \dfrac{2 - 3x + 2x^2}{1 - x - x^2 + x^3} = \dfrac{2 - 3x + 2x^2}{(1-x)^2(1+x)}$. As the hint suggests, we write

$$f(x) = \frac{2 - 3x + 2x^2}{(1-x)^2(1+x)} = \frac{Ax + B}{(1-x)^2} + \frac{C}{1+x}$$

obtaining

$$\begin{aligned} (Ax + B)(1 + x) + C(1 - x)^2 &= 2 - 3x + 2x^2 \\ (B + C) + (A + B - 2C)x + (A + C)x^2 &= 2 - 3x + 2x^2 \end{aligned}$$

So $B + C = 2$, $A + B - 2C = -3$, $A + C = 2$. The solution to these equations is $A = B = \frac{1}{4}$, $C = \frac{7}{4}$. Therefore,

$$\begin{aligned} f(x) &= \tfrac{1}{4}\Big(\frac{1+x}{(1-x)^2}\Big) + \tfrac{7}{4}\Big(\frac{1}{1+x}\Big) \\ &= \tfrac{1}{4}(1 + x)(1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots) \\ &\quad + \tfrac{7}{4}(1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots) \\ &= \tfrac{1}{4}(1 + 2x + 3x^2 + 4x^3 + \cdots) + \tfrac{1}{4}(x + 2x^2 + 3x^3 + \cdots) + \tfrac{7}{4}(1 - x + x^2 - x^3 + \cdots) \\ &= \tfrac{1}{4}(1 + 3x + 5x^2 + 7x^3 + \cdots + (2n+1)x^n + \cdots) + \tfrac{7}{4}(1 - x + x^2 - x^3 + \cdots) \end{aligned}$$

Thus, $a_n = \frac{1}{4}(2n + 1) + \frac{7}{4}(-1)^n$.

13. (a) The characteristic polynomial of the Pell sequence is $x^2 - 2x - 1$ whose roots are $x = 1 \pm \sqrt{2}$. Thus $p_n = c_1(1 + \sqrt{2})^n + c_2(1 - \sqrt{2})^n$ for constants $c_1$ and $c_2$. Since $p_0 = 1$ and $p_1 = 2$, we have $c_1 + c_2 = 1$ and $c_1(1 + \sqrt{2}) + c_2(1 - \sqrt{2}) = 2$, so $c_1 = \frac{2+\sqrt{2}}{4}$ and $c_2 = \frac{2-\sqrt{2}}{4}$. Thus $p_n = \big(\frac{2+\sqrt{2}}{4}\big)(1 + \sqrt{2})^n + \big(\frac{2-\sqrt{2}}{4}\big)(1 - \sqrt{2})^n$.

(b) Note that $|1 - \sqrt{2}| < \frac{1}{2}$, so $|1 - \sqrt{2}|^n < \frac{1}{2}$ for all $n \geq 1$. Also $\sqrt{2} < 2 < 2 + \sqrt{2}$ implies $0 < 2 - \sqrt{2} < 2$, so $0 < \frac{2-\sqrt{2}}{4} < \frac{2}{4} = \frac{1}{2}$. Thus the second term in the formula for $p_n$ obtained in Exercise 13(a) is less than $\frac{1}{2}$ for any $n$, that is, that $|p_n - \big(\frac{2+\sqrt{2}}{4}\big)(1 + \sqrt{2})^n| < \frac{1}{2}$ for any $n$. Since $p_n$ is an integer, this part follows from the fact that there is precisely one integer within $\frac{1}{2}$ of any given real number.

(c) Let $f(x) = p_0 + p_1x + p_2x^2 + \cdots$ be the generating function. Then

$$\begin{array}{rcccccccc} f(x) & = & p_0 & + & p_1x & + & p_2x^2 & + & p_3x^3 & + \cdots \\ 2xf(x) & = & & & 2p_0x & + & 2p_1x^2 & + & 2p_2x^3 & + \cdots \\ x^2f(x) & = & & & & & p_0x^2 & + & p_1x^3 & + \cdots \end{array}$$

So $(1 - 2x - x^2)f(x) = p_0 = 1$ and

$$\begin{aligned} f(x) &= \frac{1}{1-2x-x^2} = \frac{1}{1-x(2+x)} \\ &= 1 + x(2+x) + x^2(2+x)^2 + x^3(2+x)^3 + \cdots \\ &= 1 + 2x + x^2 + x^2(4 + 2x + x^2) + x^3(8 + 12x + 6x^2 + x^3) + \cdots \\ &= 1 + 2x + 5x^2 + 12x^3 + \cdots \end{aligned}$$

14. (a)
$$\begin{array}{rcccccccc} f(x) & = & a_0 & + & a_1x & + & a_2x^2 & + \cdots + & a_nx^n & + \cdots \\ xf(x) & = & & & a_0x & + & a_1x^2 & + \cdots + & a_{n-1}x^n & + \cdots \\ x^2f(x) & = & & & & & a_0x^2 & + \cdots + & a_{n-2}x^n & + \cdots \end{array}$$

Therefore, $f(x) - xf(x) - x^2f(x)$

$= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \cdots + (a_n - a_{n-1} - a_{n-2})x^n + \cdots = 1,$

since $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

Therefore, $(1 - x - x^2)f(x) = 1$ and $f(x) = \dfrac{1}{1-x-x^2}$.

(b) The roots of $1 - x - x^2 = 0$ are, evidently, the roots of $1 - \alpha x = 0$ and $1 - \beta x = 0$; that is,

$$\frac{1}{\alpha}, \frac{1}{\beta} = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{-2} = \frac{-1 \pm \sqrt{5}}{2}$$

Thus, $\alpha = \dfrac{1}{\frac{1}{2}(-1+\sqrt{5})} = \dfrac{2}{\sqrt{5}-1} = \dfrac{1+\sqrt{5}}{2}$,

$\beta = \dfrac{1}{\frac{1}{2}(-1-\sqrt{5})} = \dfrac{-2}{\sqrt{5}+1} = \dfrac{1-\sqrt{5}}{2}$.

(c) Set $\dfrac{1}{1-x-x^2} = \dfrac{A}{1-\alpha x} + \dfrac{B}{1-\beta x} = \dfrac{(A+B) - (A\beta + B\alpha)x}{1-x-x^2}$.

Thus, $A + B = 1$, $A\beta + B\alpha = 0$, so

$$A = \frac{\alpha}{\alpha - \beta} = \frac{\alpha}{\sqrt{5}} = \frac{1}{\sqrt{5}}\alpha$$

$$B = -\frac{\beta}{\alpha}A = -\frac{\beta}{\alpha}\frac{1}{\sqrt{5}}\alpha = -\frac{1}{\sqrt{5}}\beta$$

(d) From the previous parts, it follows that

$$\begin{aligned} f(x) &= \frac{1}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A(1 + \alpha x + \alpha^2x^2 + \cdots + \alpha^nx^n + \cdots) + B(1 + \beta x + \beta^2x^2 + \cdots + \beta^nx^n + \cdots) \end{aligned}$$

and so $a_n = A(\alpha)^n + B(\beta)^n = \frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1}$, in agreement with the formula derived in Problem 22. The method of Section 5.3 seems preferable to the method of generating functions.

# Chapter 5 Review

1. When $n = 1$, the sum is $1(1!) = 1$, while the right hand side is $2! - 1 = 1$, so the result holds. Now assume that $k \geq 1$ and that the result is true for $n = k$; that is, assume $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. We must prove that $\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$. We have

$$\begin{aligned}\sum_{i=1}^{k+1} i(i!) &= \left[\sum_{i=1}^{k} i(i!)\right] + (k+1)(k+1)! \\ &= [(k+1)! - 1] + (k+1)(k+1)! \qquad \text{by the induction hypothesis} \\ &= (k+1)!(1+k+1) - 1 \\ &= (k+1)!(k+2) - 1 = (k+2)! - 1\end{aligned}$$

as desired. So the result is true, by the Principle of Mathematical Induction.

2. For $n = 1$, $\sum_{i=1}^{n} 3^{i-1} = 3^0 = 1$ while $\frac{3^n - 1}{2} = \frac{3^1 - 1}{2} = 1$, so the result holds. Now suppose $k \geq 1$ and the result is true for $n = k$; that is, assume that $\sum_{i=1}^{k} 3^{i-1} = \frac{3^k - 1}{2}$. We wish to prove that the result is true for $n = k + 1$; that is, we wish to prove that $\sum_{i=1}^{k+1} 3^{i-1} = \frac{3^{k+1} - 1}{2}$. We have

$$\begin{aligned}\sum_{i=1}^{k+1} 3^{i-1} &= \sum_{i=1}^{k} 3^{i-1} + 3^{(k+1)-1} = \frac{3^k - 1}{2} + 3^k \qquad \text{by the induction hypothesis} \\ &= \frac{3^k + 2(3^k) - 1}{2} = \frac{3(3^k) - 1}{2} = \frac{3^{k+1} - 1}{2}\end{aligned}$$

the desired answer. By the Principle of Mathematical Induction, we conclude that the result is true for all $n \geq 1$.

3. Since $\left(1 - \frac{1}{2}\right)^1 = 1 - \frac{1}{2} \geq 1 - \frac{1}{2}$, the result is true for $n = 1$.

Now assume that $k \geq 1$ and $\left(1 - \frac{1}{2}\right)^k \geq 1 - \frac{k}{2}$. Then

$$\left(1 - \frac{1}{2}\right)^{k+1} = \left(1 - \frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right) \geq \left(1 - \frac{k}{2}\right)\frac{1}{2}$$

by the induction hypothesis and the induction step is complete provided $\left(1 - \frac{k}{2}\right)\frac{1}{2} \geq 1 - \frac{k+1}{2}$. This is true if and only if

$$\begin{aligned}\frac{1}{2} - \frac{k}{4} \geq 1 - \frac{k+1}{2} &\leftrightarrow \frac{k+1}{2} - \frac{k}{4} \geq 1 - \frac{1}{2} \leftrightarrow \frac{2(k+1) - k}{4} \geq \frac{1}{2} \\ &\leftrightarrow \frac{k+2}{4} \geq \frac{1}{2} \leftrightarrow 2k + 4 \geq 4\end{aligned}$$

which is indeed true. The induction step is complete; we conclude that $\left(1-\frac{1}{2}\right)^n \geq 1-\frac{n}{2}$ for all $n \geq 1$.

4. When $n = 1$, the sum is $1 > 2(\sqrt{2}-1)$ because $\sqrt{2} \approx 1.4$. Now assume that $k \geq 1$ and that $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > 2(\sqrt{k+1}-1)$. We have

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\ &> 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}} \qquad \text{by the induction hypothesis} \\ &= \frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}}.\end{aligned}$$

We want to show that this quantity is at least as large as $2\sqrt{k+2}-2$, in order to get the desired inequality with $n = k+1$. Now

$$\begin{aligned}\frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}} &\geq 2\sqrt{k+2} - 2 \\ \text{if and only if} \quad & 2(k+1) - 2\sqrt{k+1} + 1 \geq 2\sqrt{k+1}\sqrt{k+2} - 2\sqrt{k+1} \\ \text{if and only if} \quad & 2(k+1) + 1 \geq 2\sqrt{k^2+3k+2} \\ \text{if and only if} \quad & 2k+3 \geq 2\sqrt{k^2+3k+2}\end{aligned}$$

and this is true because the numbers on each side of this inequality are positive and $(2k+3)^2 = 4k^2+12k+9 > 4(k^2+3k+2)$. We have established the inequality for $n = k+1$ and conclude, by the Principle of Mathematical Induction, that it holds for all $n \geq 1$.

5. When $n = 1$, $10^{n+2}+10^n+1 = 10^3+10+1 = 1011 = 3(337)$ is divisible by 3. Assume that $k \geq 1$ and the result is true for $n = k$; that is, assume that $10^{k+2} + 10^k + 1 = 3x$ for some integer $x$. We wish to show that the result is true when $n = k+1$; that is, we wish to show that $10^{k+3} + 10^{k+1} + 1$ is divisible by 3. Now

$$10^{k+3} + 10^{k+1} + 1 = 10(10^{k+2} + 10^k) + 1 = 10(3x-1) + 1,$$

by the induction hypothesis. Thus $10^{k+3} + 10^{k+1} + 1 = 30x - 9 = 3(10x-3)$. Since $10x-3$ is an integer, we have shown that $3 \mid (10^{k+3} + 10^{k+1} + 1)$, the desired result. By the Principle of Mathematical Induction, $10^{n+2} + 10^n + 1$ is divisible by 3 for all $n \geq 1$.

6. The even factors of 18! are $18, 16, 14, 12, 10, 8, 6, 4, 2$. The powers of 2 dividing these are, respectively, $2, 2^4, 2, 2^2, 2, 2^3, 2, 2^2, 2$, the product of which is $2^{16}$. Thus (18)! is divisible by $2^{15} = 8^5$, so the result is true when $n = 5$. Now assume that $k \geq 1$ and that $8^k \mid (4k-2)!$. We have

$$\begin{aligned}[4(k+1)-2]! &= (4k+2)! \\ &= (4k+2)(4k+1)(4k)(4k-1)(4k-2)! \\ &= 2(4)(2k+1)(4k+1)k(4k-1)(4k-2)!\end{aligned}$$

Since $8^k \mid (4k-2)!$ by the induction hypothesis, we see that $8^{k+1} \mid [4(k+1)-2]!$, giving the result with $n = k+1$. By the Principle of Mathematical Induction, we conclude that $8^n \mid (4n-2)!$ for all $n \geq 5$.

7. Let $S = \{a, b\}$ be a set of two elements. The only 2-element subset of $S$ is $S$ itself. Also, for $n = 2$, $\frac{1}{2}n(n-1) = 1$, so the result is true. Now assume that $k \geq 2$ and the result is true for sets of $k$ elements. Let $S$ be a set of $k+1$ elements and let $a$ be an element of $S$. By the induction hypothesis, there are $\frac{1}{2}k(k-1)$ 2-element subsets of $S \setminus \{a\}$. The only other 2-element subsets of $S$ contain $a$, and there are $k$ of these. Altogether, the number of 2-element subsets of $S$ is $\frac{1}{2}k(k-1)+k = \frac{1}{2}k(k+1)$, which is the desired formula for $n = k+1$. By the Principle of Mathematical Induction, we conclude that the result is true for all $n \geq 2$.

8. (a) Any logarithm $(0, \infty) \to \mathbb{R}$ is an example.

   (b) For $n = 1$, $f(x_1) = f(x_1)$, which is the desired result. Now assume that $k \geq 1$ and that $f(x_1x_2 \cdots x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for any real numbers $x_1, x_2, \ldots, x_k \in A$. Given $x_1, x_2, \ldots, x_{k+1} \in A$, we have

$$
\begin{aligned}
& f(x_1x_2 \cdots x_{k+1}) \\
&\quad = f\big((x_1x_2 \cdots x_k)x_{k+1}\big) \\
&\quad = f(x_1x_2 \cdots x_k) + f(x_{k+1}) && \text{since we are given that } f(ab) = f(a) + f(b) \text{ for any } a, b \text{ (here take } a = x_1x_2 \cdots x_k \text{ and } b = x_{k+1}) \\
&\quad = \big(f(x_1) + f(x_2) + \cdots + f(x_k)\big) + f(x_{k+1}) && \text{by the induction hypothesis} \\
&\quad = f(x_1) + f(x_2) + \cdots + f(x_{k+1})
\end{aligned}
$$

   which is the result with $n = k+1$. By the Principle of Mathematical Induction, we conclude that $f(x_1x_2 \cdots x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$ for any $n \geq 1$ and any $x_1, x_2, \ldots, x_n \in A$.

9. (a) $a_1 = 1$ and, for $k \geq 1$, $a_{k+1} = 6a_k - 1$.

   (b) $a_1 = 3$ and, for $k \geq 1$, $a_{k+1} = \frac{1}{2}(a_k^2 + 1)$.

   (c) $a_1 = 1$ and, for $k \geq 1$, $a_{k+1} = (\sqrt{a_k} + k + 1)^2$.

10. (a) The values of the given product for $n = 2, 3, 4, 5$ are, respectively, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, so we guess that

$$(*) \qquad (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{n}) = \frac{1}{n}$$

   for $n \geq 2$. For $n = 2$, the left side is $1 - \frac{1}{2} = \frac{1}{2}$, which is the right side, so the formula is correct. Now assume that $k \geq 2$ and

$$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{k}) = \frac{1}{k}.$$

   With $n = k+1$, the product is

$$
\begin{aligned}
& (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{k+1}) \\
&\qquad = \big[(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{k})\big](1 - \frac{1}{k+1}) \\
&\qquad = \frac{1}{k}(1 - \frac{1}{k+1}), && \text{using the induction hypothesis,} \\
&\qquad = \frac{1}{k} - \frac{1}{k(k+1)} = \frac{(k+1)-1}{k(k+1)} = \frac{k}{k(k+1)} = \frac{1}{k+1},
\end{aligned}
$$

which is formula $(*)$ with $n = k + 1$. By the Principle of Mathematical Induction, the formula given in $(*)$ holds for all $n \geq 2$.

(b) The values of the given product, for $n = 2, 3, 4, 5$ are, respectively, $\frac{3}{4}, \frac{2}{3} = \frac{4}{6}, \frac{5}{8}, \frac{3}{5} = \frac{6}{10}$, so we guess that

$$(**) \qquad (1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

for $n \geq 2$. For $n = 2$, the left side is $1 - \frac{1}{4} = \frac{3}{4}$, which is the right side, so the formula is correct. Now assume that $k \geq 2$ and

$$(1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{k^2}) = \frac{k+1}{2k}.$$

With $n = k + 1$, the product is

$$\begin{aligned}
&(1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{(k+1)^2}) \\
&\qquad = [(1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{k^2})](1 - \frac{1}{(k+1)^2}) \\
&\qquad = \frac{k+1}{2k}(1 - \frac{1}{(k+1)^2}), \qquad \text{using the induction hypothesis,} \\
&\qquad = \frac{k+1}{2k} - \frac{1}{2k(k+1)} \\
&\qquad = \frac{(k+1)^2 - 1}{2k(k+1)} = \frac{k^2 + 2k}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)},
\end{aligned}$$

which is formula $(**)$ with $n = k + 1$. By the Principle of Mathematical Induction, the formula given in $(**)$ holds for all $n \geq 2$.

11. The sequence is $47, 41, 35, 29, \ldots$, which is arithmetic with $a = 47$ and $d = -6$. So $S = \frac{n}{2}[2a + (n - 1)d] = 50[94 + 99(-6)] = -25000$.

12. This is the sum of an arithmetic sequence with $a = 529$, $d = -4$ and $a_n = -459$. Since $a_n = a + (n - 1)d$, we obtain $n = 498$. The sum is

$$S = \frac{n}{2}[2a + (n - 1)d] = \frac{498}{2}[1058 - 4(497] = 249(-930) = -231{,}570.$$

13. For $n = 1$, the formula gives $\frac{2}{9}(1 - (-\frac{1}{2})^0) = \frac{2}{9}(1 - 1) = 0 = a_1$. For $n = 2$, the formula gives $\frac{2}{9}(1 - (-\frac{1}{2})^1) = \frac{2}{9}(1 + \frac{1}{2}) = \frac{1}{3} = a_2$. Thus the result is true for $n = 1$ and $n = 2$. Now assume that $k > 2$ and that the formula is correct for all $n < k$. We wish to prove that the result is true when $n = k$. We have

$$\begin{aligned}
a_k &= \tfrac{1}{2}(a_{k-2} + a_{k-1}) && \text{(given)} \\
&= \tfrac{1}{2}\left[\tfrac{2}{9}\left(1 - (-\tfrac{1}{2})^{k-3}\right) + \tfrac{2}{9}\left(1 - (-\tfrac{1}{2})^{k-2}\right)\right] && \text{(the induction hypothesis)} \\
&= \tfrac{1}{2}\left[\tfrac{2}{9} - \tfrac{2}{9}(-\tfrac{1}{2})^{k-3} + \tfrac{2}{9} - \tfrac{2}{9}(-\tfrac{1}{2})^{k-2}\right] \\
&= \tfrac{1}{2}(\tfrac{2}{9})\left[2 - (-\tfrac{1}{2})^{k-3}(1 - \tfrac{1}{2})\right] \\
&= \tfrac{1}{2}(\tfrac{2}{9})\left[2 + (-\tfrac{1}{2})^{k-3}(-\tfrac{1}{2})\right] = \tfrac{2}{9}\left(1 - (-\tfrac{1}{2})^{k-1}\right)
\end{aligned}$$

which is the desired formula for $n = k$. By the Principle of Mathematical Induction, the formula holds for all $n \geq 1$.

14. (a) We have $a_1 = f(1) = -1$, $a_2 = f(a_1) = f(-1) = 7$, $a_3 = f(7) = -25$, $a_4 = f(a_3) = 103$ and $a_5 = -409$.

    (b) $a_2 = a_1$ implies $f(a_1) = a_1$, so $f(f(t)) = f(t)$. This yields $3 - 4f(t) = 3 - 4t$, which gives $3 - 4(3 - 4t) = 3 - 4t$ or $3 - 12 + 16t = 3 - 4t$, $20t = 12$, $t = \frac{3}{5}$. Since $f(\frac{3}{5}) = 3 - 4(\frac{3}{5}) = 3 - \frac{12}{5} = \frac{3}{5}$, the sequence is constant.

    (c) The answer is no. $a_1 < a_2$ implies $f(t) < f(f(t))$, which is $3 - 4t < -9 + 16t$, so that $t > \frac{3}{5}$. But $a_2 < a_3$ implies $-9 + 16t < f(-9 + 16t) = 3 - 4(-9 + 16t) = 39 - 64t$, so $80t < 48$, that is, $t < \frac{3}{5}$. No such $t$ exists.

15. We have $a = 5$, $d = 4$.

    (a) $a_{32} = 5 + 31(4) = 129$; $a_{100} = 5 + 99(4) = 401$.

    (b) Attempting to solve $125 = 5 + (n - 1)4$, we get $n - 1 = 30$. Thus, 125 is the 31st term of this sequence.

    (c) Attempting to solve $429 = 5 + (n - 1)4$, we get $n - 1 = 106$. Thus, 429 is the 107th term of the sequence. Attempting to solve $1000 = 5 + (n - 1)4$, we discover that there is no integral solution. So 1000 does not occur in the sequence.

    (d) The sum of 18 terms of this sequence is $\frac{18}{2}[10 + 17(4)] = 702$.

16. (a) The 27th term is $6 + (26)(-4) = -98$.
    The sum of the first 30 terms is $\frac{30}{2}[2(6) + (29)(-4)] = -1560$.

    (b) The 27th term is $6(\frac{1}{3})^{26} = \dfrac{2}{3^{25}}$. The sum of the first 30 terms is $\dfrac{6(1 - (\frac{1}{3})^{30})}{1 - \frac{1}{3}} = 9(1 - (\frac{1}{3})^{30})$.

17. There are numbers $a$ and $d$ such that $a_1 = a$, $a_2 = a + d$, $a_3 = a + 2d$ and $a_4 = a + 3d$. The question thus asks for verification that $a(a + 3d) + 3(a + d)(a + 2d) = 2[a(a + 2d) + (a + d)(a + 3d)]$. It is straightforward to verify that each side is $4a^2 + 12ad + 6d^2$. Hence $f(x) = (3x - a_1a_4)(x - a_2a_3)$.

18. This is the sum of a geometric series with $a = (\frac{3}{2})^{742}$ and $r = -\frac{4}{9}$. The sum of $n = 500$ terms is

$$\frac{a(1 - r^n)}{1 - r} = \frac{(\frac{3}{2})^{742}(1 - (-\frac{4}{9})^{500})}{1 - (-\frac{4}{9})}$$

$$= \frac{(\frac{3}{2})^{742}(1 - (\frac{4}{9})^{500})}{\frac{13}{9}} = \frac{(\frac{3}{2})^{742}(1 - (\frac{2}{3})^{1000})}{\frac{13}{9}} = \frac{9}{13}[(\frac{3}{2})^{742} - (\frac{2}{3})^{258}].$$

19. (a) $f_1 = 1$, $f_2 = 1$ and, for $k \geq 2$, $f_{k+1} = f_k + f_{k-1}$.

    (b) No. If $f_k$ and $f_{k-1}$ are odd, then the next term, which is the sum of these numbers, must be even.

    (c) No. If $f_k$ and $f_{k+1}$ are even, then $k > 2$, so $f_{k-1} = f_{k+1} - f_k$ is also even. Repeating this argument shows that $f_i$ is even for all $2 \leq i \leq k - 1$. This is not true since $f_2 = 1$.

20. If $n = 2$, we have $f_2 = 1 < \frac{7}{4} = (\frac{7}{4})^{2-1}$ and if $n = 3$, $f_3 = 2 < \frac{49}{16} = (\frac{7}{4})^{3-1}$. Thus, the result is true for $n = 2$ and $n = 3$. Now let $k > 3$ and assume the result holds for all $\ell$, $3 \leq \ell < k$. We wish to

prove that $f_k < (\frac{7}{4})^{k-1}$. Now

$$\begin{aligned} f_k &= f_{k-1} + f_{k-2} \\ &< \left(\tfrac{7}{4}\right)^{(k-1)-1} + \left(\tfrac{7}{4}\right)^{(k-2)-1} && \text{(applying the induction hypothesis to } k-1 \text{ and to} \\ & && k-2, \text{ both of which are in the range } 2 \le \ell < k \\ & && \text{where the result holds)} \\ &= \left(\tfrac{7}{4}\right)^{k-3}\left(\tfrac{7}{4}+1\right) = \left(\tfrac{7}{4}\right)^{k-3}\left(\tfrac{11}{4}\right) \\ &< \left(\tfrac{7}{4}\right)^{k-3}\left(\tfrac{49}{16}\right) = \left(\tfrac{7}{4}\right)^{k-1} \end{aligned}$$

as required. By the Principle of Mathematical Induction (strong form), we conclude that the result holds for all $n \ge 1$.

21. For $n = 1$, $f_2 = 1 = 2 - 1 = f_3 - 1$, so the formula holds. Now assume $k \ge 1$ and that the formula holds for $n = k$. We have

$$\begin{aligned} f_2 + f_4 + \cdots + f_{2(k+1)} &= f_2 + f_4 + \cdots + f_{2k+2} \\ &= (f_2 + f_4 + \cdots + f_{2k}) + f_{2k+2} \\ &= (f_{2k+1} - 1) + f_{2k+2} && \text{by the induction hypothesis} \\ &= f_{2k+3} - 1 && \text{using the definition of } f_{2k+3} \end{aligned}$$

and this is the desired formula with $n = k + 1$ since $2(k + 1) + 1 = 2k + 3$. By the Principle of Mathematical Induction, the formula is correct for all $n \ge 1$.

22. Let $a_i$ be the number of ways in which you can reach stair $i$. Then $a_1 = 1$, $a_2 = 2$ (one single step or one double step) and, in general, you can reach stair $n$ in one step from stair $n - 1$ or a double step from stair $n - 2$; thus, $a_n = a_{n-1} + a_{n-2}$. The sequence $a_1, a_2, a_3, \ldots$ is the Fibonacci sequence $f_2, f_3, f_4, \ldots$, so $a_n = f_{n+1}$ is the integer closest to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$.

23. The characteristic polynomial is $x^2 - 5x + 4$. This has distinct roots $x_1 = 1$, $x_2 = 4$, so $a_n = c_1(1)^n + c_2(4)^n$ for certain $c_1$ and $c_2$. The initial conditions say that $-3 = c_1 + c_2$ and $6 = c_1 + 4c_2$, so $c_2 = 3$, $c_1 = -6$ and $a_n = -6 + 3(4)^n$.

24. Letting $f(x)$ be the generating function for the sequence in question, we have

$$\begin{array}{rcccccccccc} f(x) & = & a_0 + & a_1x + & a_2x^2 + & \cdots + & a_nx^n + & \cdots \\ 5xf(x) & = & & 5a_0x + & 5a_1x^2 + & \cdots + & 5a_{n-1}x^n + & \cdots \\ 4x^2f(x) & = & & & 4a_0x^2 + & \cdots + & 4a_{n-2}x^n + & \cdots . \end{array}$$

Therefore, $f(x) - 5xf(x) + 4x^2f(x)$

$= a_0 + (a_1 - 5a_0)x + (a_2 - 5a_1 + 4a_0)x^2 + \cdots + (a_n - 5a_{n-1} + 4a_{n-2})x^n + \cdots = -3 + 21x$,

since $a_0 = -3$, $a_1 = 6$ and $a_n - 5a_{n-1} + 4a_{n-2} = 0$ for $n \ge 2$.

So $(1 - 5x + 4x^2)f(x) = -3 + 21x$ and $f(x) = \dfrac{-3 + 21x}{(1 - x)(1 - 4x)}$.

Using the method of partial fractions, we set

$$\frac{1}{(1-x)(1-4x)} = \frac{A}{1-x} + \frac{B}{1-4x} = \frac{A(1-4x) + B(1-x)}{(1-x)(1-4x)} = \frac{(-4A - B)x + (A + B)}{(1-x)(1-4x)}.$$

Solving $-4A - B = 0$, $A + B = 1$ gives $A = -\frac{1}{3}$, $B = \frac{4}{3}$, so

$$\frac{1}{(1-x)(1-4x)} = -\frac{1}{3}\left(\frac{1}{1-x}\right) + \frac{4}{3}\left(\frac{1}{1-4x}\right)$$

and we obtain

$$\begin{aligned}
f(x) &= -\frac{1}{3}\left(\frac{1}{1-x}\right)(-3+21x) + \frac{4}{3}\left(\frac{1}{1-4x}\right)(-3+21x) \\
&= -\tfrac{1}{3}(1 + x + x^2 + \cdots)(-3+21x) \\
&\quad + \tfrac{4}{3}(1 + 4x + 16x^2 + \cdots)(-3+21x) \\
&= -\tfrac{1}{3}(-3 + 18x + \cdots + 18x^n + \cdots +) \\
&\quad + \tfrac{4}{3}(-3 + 9x + 36x^2 + \cdots + [(-3)(4^n) + 21(4^{n-1})]x^n + \cdots +) \\
&= -\tfrac{1}{3}(-3 + 18x + \cdots + 18x^n + \cdots +) \\
&\quad + \tfrac{4}{3}(-3 + 9x + 36x^2 + \cdots + 9(4^{n-1})x^n + \cdots +) \\
&= -3 + 6x + 42x^2 + \cdots + (-6 + 3(4^n))x^n + \cdots +
\end{aligned}$$

so we obtain $a_n = -6 + 3(4^n)$ as before.

25. First we find a particular solution $p_n$. Since $f(n) = 5$ is linear, we try a linear function $p_n = a + bn$ for $p_n$. Substituting in the recurrence relation, we have

$$a + bn = 3[a + b(n-1)] - 2[a + b(n-2)] + 5 = a + bn + b + 5,$$

for which $b = -5$ and any $a$ is a solution. We take $a = 0$ and $p_n = -5n$. The corresponding homogeneous recurrence relation is $a_n = 3a_{n-1} - 2a_{n-2}$, whose characteristic polynomial is $x^2 - 3x + 2 = (x-1)(x-2)$, with roots $x_1 = 1$ and $x_2 = 2$. The solution to the homogeneous recurrence relation is

$$q_n = c_1(1^n) + c_2(2^n) = c_1 + c_2 2^n.$$

Thus, the solution to the given recurrence is

$$a_n = p_n + q_n = -5n + c_1 + c_2(2^n).$$

The initial conditions give $1 = c_1 + c_2$ and $1 = c_1 + 2c_2 - 5$, so $c_1 = -4$, $c_2 = 5$ and $a_n = -4 + 5(2^n) - 5n$.

26. (a) The first five terms are $-2, 1, 12, -81, 378$.

    (b) The characteristic polynomial $x^2 + 6x + 9 = (x+3)^2$ has a repeated root, so the solution takes the form $a_n = c_1(-3)^n + c_2 n(-3)^n$. The initial conditions give $a_0 = -2 = c_1$ and $a_1 = 1 = -3c_1 - 3c_2$, so $3c_2 = -1 - 3c_1 = 5$ and $c_2 = \frac{5}{3}$. The solution is $a_n = (-2)3^n + \frac{5}{3}n(-3)^n = (-2)3^n - 5n(-3)^{n-1}$.

27. We first try $p_n = a(3^n)$. Substituting gives

$$\begin{aligned}
a(3^n) &= 4a(3^{n-1}) + 5a(3^{n-2}) + 3^n \\
9a &= 12a + 5a + 9,
\end{aligned}$$

so $a = -\frac{9}{8}$ and $p_n = -\frac{9}{8}(3^n)$ is a particular solution. Next we solve $a_n = 4a_{n-1} + 5a_{n-2}$. The characteristic polynomial, $x^2 - 4x - 5$, has distinct roots $x_1 = -1$, $x_2 = 5$, so the solution is $q_n =$

$c_1(-1)^n + c_2(5^n)$ for certain $c_1, c_2$. Therefore, the given recurrence relation has solution $-\frac{9}{8}(3^n) + c_1(-1)^n + c_2(5^n)$. The initial conditions give

$$-\tfrac{9}{8} + c_1 + c_2 = 4$$
$$-\tfrac{27}{8} - c_1 + 5c_2 = -1$$

so $c_2 = \frac{5}{4}$, $c_1 = \frac{31}{8}$ and the solution is $a_n = -\frac{9}{8}(3^n) + \frac{31}{8}(-1)^n + \frac{5}{4}(5^n)$.

28. (a) Note that $\tau^2 = \tau + 1$. Thus $n\tau^2 - n\tau = n$.

    (b) Let $\lfloor n\tau \rfloor = k_1$ and $\lfloor n\tau^2 \rfloor = k_2$ so that $n\tau = k_1 + \epsilon_1$ and $n\tau^2 = k_2 + \epsilon_2$ with $0 \le \epsilon_1, \epsilon_2 < 1$. Then

    $$\lfloor n\tau^2 \rfloor - \lfloor n\tau \rfloor = k_2 - k_1 = n\tau^2 - \epsilon_2 - (n\tau - \epsilon_1) = n\tau^2 - n\tau + (\epsilon_1 - \epsilon_2).$$

    The left side of this equation is an integer, hence so is the right side. In part (a), we showed that $n\tau^2 - n\tau$ is an integer, so $\epsilon_1 - \epsilon_2$ must be an integer too. Since, $-1 < \epsilon_1 - \epsilon_2 < 1$, it must be that $\epsilon_1 - \epsilon_2 = 0$, so $\lfloor n\tau^2 \rfloor - \lfloor n\tau \rfloor = n\tau^2 - n\tau = n$, as desired.

29. Dividing by $a_n$ each side of the recurrence relation $a_{n+1} = a_n + a_{n-1}$, we obtain $\frac{a_{n+1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$ As $n \to \infty$, the left side has limit $k$ while the right side has limit $1 + \frac{1}{k}$. The solutions to $k = 1 + \frac{1}{k}$ are the solutions to $k^2 - k - 1 = 0$, namely, $k = \frac{1 \pm \sqrt{5}}{2}$. Since $k > 0$, $k = \frac{1+\sqrt{5}}{2}$ is the golden mean.

30. Let $S = \sum_1^\infty \frac{f_k}{100^k}$. Then

$$\begin{aligned} S &= \frac{1}{100} + \frac{1}{10000} + \sum_3^\infty \frac{f_{k-1} + f_{k-2}}{100^k} \\ &= \frac{1}{100} + \frac{1}{10000} + \frac{1}{100}\sum_3^\infty \frac{f_{k-1}}{100^{k-1}} + \frac{1}{10000}\sum_3^\infty \frac{f_{k-2}}{100^{k-2}} \\ &= \frac{1}{100} + \frac{1}{10000} + \frac{1}{100}\left(S - \frac{1}{100}\right) + \frac{1}{10000}S \\ &= \frac{1}{100} + \frac{1}{10000} + \frac{1}{100}S - \frac{1}{10000} + \frac{1}{10000}S = \frac{1}{100} + \frac{1}{100}S + \frac{1}{10000}S. \end{aligned}$$

Thus $S\left(1 - \frac{1}{100} - \frac{1}{10000}\right) = \frac{1}{100}$, $\frac{9899}{10000}S = \frac{1}{100}$ and $S = \frac{100}{9899} = .0101020305081321 3456$. The first ten terms of the Fibonacci sequence are clearly visible.

## Exercises 6.1

1. [BB] Let $B$ be the set of connoisseurs of Canadian bacon, and $A$ the set of those who like anchovies.

    (a) This asks for $|B \cup A| = |B| + |A| - |B \cap A| = 10 + 7 - 6 = 11$.

    (b) This asks for $|B \setminus A| = |B| - |B \cap A| = 10 - 6 = 4$.

(c) Here we want $|B \oplus A| = |B \cup A| - |B \cap A| = 11 - 6 = 5$.

(d) This asks for the number of elements in $B^c \cap A^c = (B \cup A)^c$. This is $|U| - |B \cup A| = 15 - 11 = 4$.

2. Suppose $n$ people were surveyed. Let $E$ be the set of those who had been emotionally abused and $P$ the set of those who had been physically abused. We are given that $|E| = .98n$ and $|P| = .89n$. We have $n \geq |E \cup P| = |E| + |P| - |E \cap P| = 1.87n - |E \cap P|$ and so $|E \cap P| \geq .87n$. Thus, at least 87% of those surveyed had experienced both types of abuse.

3. Let $U$, $J$, $F$ be the sets of all students in the class, those knowing JAVA and those knowing HTML, respectively.

(a) $|J \cup H| = |J| + |H| - |J \cap H| = 15 + 12 - 5 = 22$.

(b) $|J \setminus H| = |J| - |J \cap H| = 15 - 5 = 10$.

(c) $|H \setminus J| = |H| - |H \cap J| = 12 - 5 = 7$.

(d) $|H \oplus J| = |H \cup J| - |H \cap J| = 22 - 5 = 17$.

(e) $|U \setminus (J \cup H)| = |U| - |J \cup H| = 30 - 22 = 8$.

4. [BB] Let $A$ be the set of people with undergraduate degrees in arts, $S$ those with undergraduate degrees in science, and $G$ those with graduate degrees.

(a) This part asks for $|A \cup S \cup G|$.

$$\begin{aligned} |A \cup S \cup G| &= |A| + |S| + |G| - |A \cap S| - |A \cap G| - |S \cap G| + |A \cap S \cap G| \\ &= 200 + 95 + 120 - 25 - 55 - 40 + 5 = 300 \end{aligned}$$

(b) This part asks for $|S \setminus (A \cup G)| = |S| - |S \cap (A \cup G|$.
Now $S \cap (A \cup G) = (S \cap A) \cup (S \cap G)$, so

$$\begin{aligned} |S \cap (A \cup G)| &= |S \cap A| + |S \cap G| - |(S \cap A) \cap (S \cap G)| \\ &= |S \cap A| + |S \cap G| - |S \cap A \cap G| = 25 + 40 - 5 = 60. \end{aligned}$$

Therefore, $|S \setminus (A \cup G| = 95 - 60 = 35$.

5. Let $P$, $G$, and $C$ denote the sets of children who purchased popsicles, gum, and candy bars, respectively. The number of children present was at least

$$\begin{aligned} |P \cup G \cup C| &= |P| + |G| + |C| - |P \cap G| - |P \cap C| - |G \cap C| + |P \cap G \cap C| \\ &= 10 + 7 + 12 - 5 - 6 - 2 + |P \cap G \cap C| \\ &= 16 + |P \cap G \cap C| \geq 16. \end{aligned}$$

Since there were only 15 children in all, the numbers don't tally.

6. Let $E$ be the set of students taking English and $M$ the set of students taking Math.

(a) We want $|E \cap M|$. Now $|(E \cup M)^c| = 12$, so $|E \cup M| = 82 - 12 = 70$. Since $|E \cup M| = |E| + |M| - |E \cap M|$, $|E \cap M| = |E| + |M| - |E \cup M| = 59 + 46 - 70 = 35$.

(b) We are given $|E| = 2|M|$, $|E \oplus M| = 53$, $|(E \cup M)^c| = 15$. Hence, $|E \cup M| = 97 - 15 = 82$. So

$$(*) \qquad 82 = |E| + |M| - |E \cap M|.$$

But $53 = |E| + |M| - 2|E \cap M|$ (by Proposition 6.1.1(e)).
Subtracting, we obtain $29 = |E \cap M|$; 29 students are taking both English and Math. Substituting in $(*)$, we have $82 = |E| + |M| - 29 = 3|M| - 29$, so $3|M| = 111$, $|M| = 37$ and $|E| = 74$.

7. [BB] Let $D$ be the set of delegates who voted to decrease the deficit, $E$ be the set of delegates who voted in favor of the motion concerning environmental issues, and $T$ be the set of delegates who voted in favor of not increasing taxes.

(a) This part asks for $|(D \cup E \cup T)^c|$. Since

$$|D \cup E \cup T| = 1542 + 569 + 1197 - 327 - 92 - 839 + 50 = 2100,$$

we have $\left|(D \cup E \cup T)^c\right| = 2300 - 2100 = 200$.

(b) The set of delegates who voted against increasing taxes but who did not vote in favor of either of the other motions is $T \setminus (D \cup E)$. Now

$$\begin{aligned} |T \setminus (D \cup E)| &= |T| - |T \cap (D \cup E)| \\ &= |T| - |(T \cap D) \cup (T \cap E)| \\ &= |T| - |T \cap D| - |T \cap E| + |(T \cap D) \cap (T \cap E)| \\ &= |T| - |T \cap D| - |T \cap E| + |T \cap D \cap E| \\ &= 1197 - 839 - 92 + 50 = 316 \end{aligned}$$

8. Let $A$ be the people who dislike the NDP, $B$ the people who dislike the Liberals, $C$ the people who dislike the Conservatives and $D$ the people who dislike the Green Party. We are asked for $\left|(A \cup B \cup C \cup D)^c\right|$. We have

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\ &\quad - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ &\quad - |A \cap B \cap C \cap D| \\ &= 7 + 10 + 11 + 6 - 5 - 5 - 3 - 6 - 4 - 5 + 3 + 2 + 3 + 4 - 2 = 16 \end{aligned}$$

and so $\left|(A \cup B \cup C \cup D)^c\right| = |U| - |A \cup B \cup C \cup D|$ (by Proposition 6.1.1) $= 3$.

9. Let $O$, $A$, $G$, and $C$ denote the sets of people who bought orange juice, apple juice, grapefruit juice, and citrus punch, respectively.

(a) [BB] We want $|O \cap A \cap G \cap C|$. Now

$$\begin{aligned} |O \cup A \cup G \cup C| &= |O| + |A| + |G| + |C| - |O \cap A| - |O \cap G| - |O \cap C| \\ &\quad - |A \cap G| - |A \cap C| - |G \cap C| + |O \cap A \cap G| + |O \cap A \cap C| \\ &\quad + |O \cap G \cap C| + |A \cap G \cap C| - |O \cap A \cap G \cap C| \end{aligned}$$

and so

$$890 = 750 + 400 + 100 + 50 - 328 - 25 - 12 - 35 - 8 - 33 + 17 + 4 + 9 + 2 - |O \cap A \cap G \cap C|.$$

Thus, $|O \cap A \cap G \cap C| = 891 - 890 = 1$.

(b) We want $|G \setminus (O \cup A \cup C)|$, and we know

$$|C \setminus (O \cup A \cup C)| = |G| - |G \cap (O \cup A \cup C)| = 100 - |G \cap (O \cup A \cup C)|.$$

Now $G \cap (O \cup A \cup C) = (G \cap O) \cup (G \cap A) \cup (G \cap C)$ and so

$$\begin{aligned} |G \cap (O \cup A \cup C)| &= |G \cap O| + |G \cap A| + |G \cap C| - |G \cap O \cap A| - |G \cap O \cap C| \\ &\quad - |G \cap B \cap C| + |G \cap O \cap A \cap C| \\ &= 25 + 35 + 33 - 17 - 9 - 2 + 1 = 66 \end{aligned}$$

So the number we want is $100 - 66 = 34$.

(c) There are six possibilities to consider. The number who bought orange juice and apple juice but no other kind of juice is

$$\begin{aligned} |(O \cap A) \setminus (G \cup C)| &= |O \cap A| - |(O \cap A) \cap (G \cup C)| \\ &= |O \cap A| - |(O \cap A \cap G) \cup (O \cap A \cap C)| \\ &= |O \cap A| - |O \cap A \cap G| - |O \cap A \cap C| + |O \cap A \cap G \cap C| \\ &= 328 - 17 - 4 + 1 = 308. \end{aligned}$$

Similarly, the number who bought orange juice and grapefruit juice but no other kind of juice is $25-17-9+1 = 0$. The number who bought only orange juice and citrus punch is $12-4-9+1 = 0$. The number who bought only apple juice and grapefruit juice is $35 - 17 - 2 + 1 = 17$. The number who bought only apple juice and citrus punch is $8 - 4 - 2 + 1 = 3$. The number who bought only grapefruit juice and citrus punch is $33 - 2 - 9 + 1 = 23$. Thus, we conclude that $308 + 0 + 0 + 17 + 3 + 23 = 351$ people bought exactly two kinds of juice.

(d) First we determine the number of people who bought exactly three kinds of juice. There are four possibilities. The number of people who bought orange juice, apple juice and grapefruit juice but not citrus punch is

$$|(O \cap A \cap G) \setminus C| = |O \cap A \cap G| - |O \cap A \cap G \cap C| = 17 - 1 = 16.$$

Similarly, the number who bought all but grapefruit juice is $4 - 1 = 3$, the number who bought all but apple juice is $9 - 1 = 8$ and the number who bought all but orange juice is $2 - 1 = 1$. Thus, the number buying exactly three kinds of juice is $16 + 3 + 8 + 1 = 28$. Since one person bought all four kinds, $28 + 1 = 29$ people bought three or four (that is, more than two) kinds.

10. (a) [BB] First we determine the number of elements which belong to at least one of the subsets. This number is $|A_1 \cup A_2 \cup A_3 \cup A_4|$, which we can find by applying the Principle of Inclusion-Exclusion. There are six pairs of sets and four triples, so

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4(28) - 6(12) + 4(5) - 1 = 59.$$

Thus, there are $75 - 59 = 16$ elements in none of the subsets.

(b) The number of elements which belong to $A_1$ and $A_2$ but to no other $A_i$ is $|A_1 \cap A_2 \setminus (A_3 \cup A_4)|$. We determine this number and then multiply by 6 because there are six ways for elements to belong to exactly two of the sets. Since $(A_1 \cap A_2) \cap (A_3 \cup A_4) = (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_4)$,

$$\begin{aligned}|(A_1 \cap A_2) \cap (A_3 \cup A_4)| \\ &= |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_1 \cap A_2 \cap A_4| \\ &= 5 + 5 - 1 = 9.\end{aligned}$$

Thus $|(A_1 \cap A_2) \setminus (A_3 \cup A_4)| = |A_1 \cap A_2| - |(A_1 \cap A_2) \cap (A_3 \cup A_4)| = 12 - 9 = 3$. There are $6 \times 3 = 18$ elements in exactly two of the four subsets.

11. (a) [BB] Let $A$ and $B$ be the set of integers between 1 and 500 which are divisible by 3 and 5, respectively. The question asks for $|A \cup B|$. This number is $|A| + |B| - |A \cap B| = \lfloor \frac{500}{3} \rfloor + \lfloor \frac{500}{5} \rfloor - \lfloor \frac{500}{15} \rfloor = 166 + 100 - 33 = 233$.

(b) [BB] Let $A$ and $B$ be as in (a) and $C$ be the set of integers between 1 and 500 which are divisible by 6. The question asks for $A \setminus (B \cup C)|$ and so the number we want is $|A| - |A \cap (B \cup C)|$. Now $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, so the number of elements here is $\lfloor \frac{500}{15} \rfloor + \lfloor \frac{500}{6} \rfloor - \lfloor \frac{500}{30} \rfloor = 33 + 83 - 16 = 100$. Finally, we obtain $|A \setminus (B \cup C)| = 166 - 100 = 66$.

12. (a) Since $500 = 2^2 5^3$, an integer is relatively prime to 500 if and only if it is not divisible by 2 or by 5. Let $A$ and $B$ denote the sets of integers less than or equal to 500 which are divisible by 2 and divisible by 5, respectively. We want $|(A \cup B)^c|$.
Now $|A| = \frac{500}{2} = 250$, $|B| = \frac{500}{5} = 100$ and $|A \cap B| = \frac{500}{10} = 50$. So $|A \cup B| = 250 + 100 - 50 = 300$ and $|(A \cup B)^c| = 500 - 300 = 200$.

(b) Since $9975 = 3(5^2)(7)(19)$, an integer is relatively prime to 9975 if and only if it is not divisible by 3, by 5, by 7 or by 19. Let $A$, $B$, $C$, $D$ be the sets of integers less than or equal to 9975 which are divisible by 3, by 5, by 7, and by 19, respectively. Then

$$|A| = \frac{9975}{3} = 3325, \quad |B| = \frac{9975}{5} = 1995, \quad |C| = \frac{9975}{7} = 1425, \quad |D| = \frac{9975}{19} = 525$$

$$|A \cap B| = \frac{9975}{15} = 665, \quad |A \cap C| = \frac{9975}{21} = 475, \quad |A \cap D| = \frac{9975}{57} = 175,$$

$$|B \cap C| = \frac{9975}{35} = 285, \quad |B \cap D| = \frac{9975}{95} = 105, \quad |C \cap D| = \frac{9975}{133} = 75$$

$$|A \cap B \cap C| = \frac{9975}{105} = 95, \quad |A \cap B \cap D| = \frac{9975}{285} = 35,$$

$$|A \cap C \cap D| = \frac{9975}{399} = 25, \quad |B \cap C \cap D| = \frac{9975}{665} = 15$$

$$|A \cap B \cap C \cap D| = \frac{9975}{1995} = 5$$

and so

$$\begin{aligned}|A \cup B \cup C \cup D| = 3325 + 1995 + 1425 + 525 - 665 - 475 - 175 - 285 - 105 - 75 \\ + 95 + 35 + 25 + 15 - 5 = 5655\end{aligned}$$

and so the number of integers relatively prime to 9975 is the number of integers not in $A \cup B \cup C \cup D$; that is, $9975 - 5655 = 4320$.

13. [BB] Let $A$, $B$, and $C$ be the sets of integers between 1 and 250 which are divisible by 4, by 6, and by 15, respectively. We want $|A \cup B \cup C|$. Now

$$|A| = \lfloor \tfrac{250}{4} \rfloor = 62, \quad |B| = \lfloor \tfrac{250}{6} \rfloor = 41, \quad |C| = \lfloor \tfrac{250}{15} \rfloor = 16$$

$$|A \cap B| = \lfloor \tfrac{250}{12} \rfloor = 20, \quad |A \cap C| = \lfloor \tfrac{250}{60} \rfloor = 4, \quad |B \cap C| = \lfloor \tfrac{250}{30} \rfloor = 8$$

$$|A \cap B \cap C| = \lfloor 250/60 \rfloor = 4.$$

So $|A \cup B \cup C| = 62 + 41 + 16 - 20 - 4 - 8 + 4 = 91$.

14. (a) [BB] Let $A$, $B$, $C$, $D$ be the sets of integers between 1 and 1000 (inclusive) which are divisible by 2, by 3, by 5, and by 7, respectively. We want $|(A \cup B \cup C \cup D)^c|$. Now

$$\begin{aligned} |A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| \\ & \quad + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|. \end{aligned}$$

We have

$$|A| = \lfloor \tfrac{1000}{2} \rfloor = 500, \quad |B| = \lfloor \tfrac{1000}{3} \rfloor = 333, \quad |C| = \lfloor \tfrac{1000}{5} \rfloor = 200, \quad |D| = \lfloor \tfrac{1000}{7} \rfloor = 142,$$

$$|A \cap B| = \lfloor \tfrac{1000}{6} \rfloor = 166, \quad |A \cap C| = \lfloor \tfrac{1000}{10} \rfloor = 100, \quad |A \cap D| = \lfloor \tfrac{1000}{14} \rfloor = 71,$$

$$|B \cap C| = \lfloor \tfrac{1000}{15} \rfloor = 66, \quad |B \cap D| = \lfloor \tfrac{1000}{21} \rfloor = 47, \quad |C \cap D| = \lfloor \tfrac{1000}{35} \rfloor = 28,$$

$$|A \cap B \cap C| = \lfloor \tfrac{1000}{30} \rfloor = 33, \quad |A \cap B \cap D| = \lfloor \tfrac{1000}{42} \rfloor = 23,$$

$$|A \cap C \cap D| = \lfloor \tfrac{1000}{70} \rfloor = 14, \quad |B \cap C \cap D| = \lfloor \tfrac{1000}{105} \rfloor = 9,$$

$$|A \cap B \cap C \cap D| = \lfloor \tfrac{1000}{210} \rfloor = 4.$$

So $|A \cup B \cup C \cup D| = 772$. Our answer is $1000 - 772 = 228$.

(b) Now 1 and 1000 must be excluded. Since 1 is not divisible by 2, 3, 4 or 7 while 1000 is divisible by 5, the answer is $228 - 1 = 227$.

15. Let $A$, $B$, $C$, $D$ be the sets of integers between 1 and 10,000 which are divisible by 3, by 5, by 7, and by 11, respectively. We have

$$|A| = \lfloor \tfrac{10000}{3} \rfloor = 3333, \quad |B| = \lfloor \tfrac{10000}{5} \rfloor = 2000, \quad |C| = \lfloor \tfrac{10000}{7} \rfloor = 1428, \quad |D| = \lfloor \tfrac{10000}{11} \rfloor = 909,$$

$$|A \cap B| = \lfloor \tfrac{10000}{15} \rfloor = 666, \quad |A \cap C| = \lfloor \tfrac{10000}{21} \rfloor = 476, \quad |A \cap D| = \lfloor \tfrac{10000}{33} \rfloor = 303,$$

$$|B \cap C| = \lfloor \tfrac{10000}{35} \rfloor = 285, \quad |B \cap D| = \lfloor \tfrac{10000}{55} \rfloor = 181, \quad |C \cap D| = \lfloor \tfrac{10000}{77} \rfloor = 129,$$

$$|A \cap B \cap C| = \lfloor \tfrac{10000}{105} \rfloor = 95, \quad |A \cap B \cap D| = \lfloor \tfrac{10000}{165} \rfloor = 60,$$

$$|A \cap C \cap D| = \lfloor \tfrac{10000}{231} \rfloor = 43, \quad |B \cap C \cap D| = \lfloor \tfrac{10000}{385} \rfloor = 25,$$

$$|A \cap B \cap C \cap D| = \lfloor \tfrac{10000}{1155} \rfloor = 8.$$

(a) This part asks us for $|A \cup B \cup C \cup D|$ and we have

$$\begin{aligned}|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\ &\quad - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ &\quad - |A \cap B \cap C \cap D| \\ &= 3333 + 2000 + 1428 + 909 - 666 - 476 - 303 - 285 - 181 - 129 \\ &\quad + 95 + 60 + 43 + 25 - 8 = 5845.\end{aligned}$$

(b) This part asks for $|(A \cap B) \setminus (C \cup D)| = |A \cap B| - |(A \cap B) \cap (C \cup D)|$.
Now $(A \cap B) \cap (C \cup D) = (A \cap B \cap C) \cup (A \cap B \cap D)$, so

$$\begin{aligned}|(A \cap B) \cap (C \cup D)| &= |A \cap B \cap C| + |(A \cap B \cap D)| - |(A \cap B \cap C) \cap (A \cap B \cap D)| \\ &= |A \cap B \cap C| + |(A \cap B \cap D)| - |(A \cap B \cap C \cap D)| \\ &= 95 + 60 - 8 = 147.\end{aligned}$$

The answer is $666 - 147 = 519$.

(c) This question asks for the number of elements in

$$\big((A \cap B \cap C) \setminus D\big) \cup \big((A \cap B \cap D) \setminus C\big) \cup \big((A \cap C \cap D) \setminus B\big) \cup \big((B \cap C \cap D) \setminus A\big).$$

Since these sets are *pairwise disjoint*, that is, the intersection of two of the four is empty, the number of elements in the union is the sum of the elements in each set. Since

$$\begin{aligned}|(A \cap B \cap C| \setminus D| &= |A \cap B \cap C| - |A \cap B \cap C \cap D| = 95 - 8 = 87, \\ |(A \cap B \cap D| \setminus C| &= |A \cap B \cap D| - |A \cap B \cap C \cap D| = 60 - 8 = 52, \\ |(A \cap C \cap D| \setminus B| &= |A \cap C \cap D| - |A \cap B \cap C \cap D| = 43 - 8 = 35, \\ |(B \cap C \cap D| \setminus A| &= |B \cap C \cap D| - |A \cap B \cap C \cap D| = 25 - 8 = 17,\end{aligned}$$

the answer is $87 + 52 + 35 + 17 = 191$.

(d) This part asks for the number of elements in $(A \cap B \cap C \cap D)^c$, which is $10000 - |A \cap B \cap C \cap D| = 10000 - 8 = 99992$.

16. Let $A$ be the set of squares and $B$ the set of cubes between 1 and $10^6$ (inclusive). We wish to calculate $|(A \cup B)^c|$ and recall that $|A \cup B| = |A| + |B| - |A \cap B|$. Now $|A|$ is the number of squares between 1 and $10^6$. Since $1 = 1^2$ and $10^6 = (10^3)^2$, $|A| = 10^3 = 1000$. Also, $|B|$ is the number of cubes between 1 and $10^6$. Since $1 = 1^3$ and $10^6 = (10^2)^3$, $|B| = 10^2 = 100$. Also, $|A \cap B|$ is the number of sixth powers between 1 and $10^6$ (by Exercise 31 of Section 4.3). Thus, $|A \cap B| = 10$. Therefore, $|A \cup B| = 1000 + 100 - 10 = 1090$ and so $|(A \cup B)^c| = 10^6 - 1090 = 998,910$.

17. [BB] Using results from Chapter 4, we know that a natural number less than 200 (and greater than 1) is prime if and only if it is equal to or not divisible by any of 2, 3, 5, 7, 11, 13 ($13 < \sqrt{200} < 17$). Using the Principle of Inclusion–Exclusion, we obtain

$$\begin{aligned}&|A \cup B \cup C \cup D \cup E \cup F| \\ &\quad = 98 + 65 + 38 + 27 + 17 + 14 - 33 - 19 - 14 - 9 - 7 - 13 - 9 - 6 - 5 - 5 \\ &\qquad -3 - 3 - 2 - 2 - 1 + 6 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 = 152.\end{aligned}$$

Thus, the number of primes less than 200 is $198 - 152 = 46$.

18. [BB] Write $a = qb + r$ with $0 \leq r < b$. Then $\lfloor \frac{a}{b} \rfloor = q$. Clearly the positive integers $b, 2b, \ldots, qb$ are all less than or equal to $a$ and are all divisible by $b$. On the other hand, if $sb \leq a$, then $s$ must belong to the set $\{1, 2, \ldots, q\}$. Hence, there are exactly $q$ such natural numbers, as required.

19. Since $|A \cup B| = |A| + |B| - |A \cap B| < |A| + |B|$, $|A \cap B| > 0$. Thus $A \cap B \neq \emptyset$.

20. [BB] Recall that $A \times B = \{(a, b) \mid a \in A, b \in B\}$. If $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$, the ordered pairs of $A \times B$ can be enumerated with $n$ rows of $m$ elements each

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots & (a_1, b_m) \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots & (a_2, b_m) \\ \vdots & \vdots & \vdots & & \vdots \\ (a_n, b_1) & (a_n, b_2) & (a_n, b_3) & \ldots & (a_n, b_m) \end{array}$$

giving $nm = |A| \times |B|$ elements in all.

21. $|A \cup B^c| = |(A^c \cap B)^c|$ by one of the laws of De Morgan. So

$$|A \cup B^c| = |U| - |A^c \cap B| = |U| - |B \setminus A| = |U| - (|B| - |B \cap A|) = |U| - |B| + |A \cap B|$$

22. (a) [BB]
$$\begin{aligned} |(A \oplus B) \cap C| &= \big|\big((A \setminus B) \cup (B \setminus A)\big) \cap C\big| \\ &= \big|\big((A \setminus B) \cap C\big) \cup \big((B \setminus A) \cap C\big)\big| \\ &= |(A \cap C) \setminus B| + |(B \cap C) \setminus A| \quad \text{since } (A \setminus B) \cap (B \setminus A) = \emptyset \\ &= |A \cap C| - |A \cap B \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= |A \cap C| + |B \cap C| - 2|A \cap B \cap C| \end{aligned}$$

(b) $|A \oplus B \oplus C| = |(A \oplus B) \oplus C|$

$$\begin{aligned} &= |A \oplus B| + |C| - 2|(A \oplus B) \cap C| && \text{by Proposition 6.1.1} \\ &= |A| + |B| - 2|A \cap B| + |C| - 2|(A \oplus B) \cap C| && \text{again using Proposition 6.1.1} \\ &= |A| + |B| + |C| \\ &\quad -2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 4|A \cap B \cap C| && \text{by part (a)} \end{aligned}$$

23. The weak form of the Principle of Mathematical Induction can be used, but we shall also need the special case $n = 2$ derived in the text. For $n = 1$ we have to prove that $|A_1| = \sum |A_i|$. Since, in this case, $\sum |A_i| = |A_1|$, the result holds. Now assume the result is true for $n = k$. Then given $k + 1$ sets $A_1, A_2, \ldots, A_{k+1}$,

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_{k+1}| &= \big|(A_1 \cup \cdots \cup A_k) \cup A_{k+1}\big| \\ &= |A_1 \cup \cdots \cup A_k| + |A_{k+1}| - \big|(A_1 \cup \cdots \cup A_k) \cap A_{k+1}\big| \\ &\qquad \text{using the result for } n = 2 \\ &= |A_1 \cup A_2 \cup \cdots \cup A_k| + |A_{k+1}| \\ &\quad - \big|(A_1 \cap A_{k+1}) \cup (A_2 \cap A_{k+1}) \cup \cdots \cup (A_k \cap A_{k+1})\big| \end{aligned}$$

using (3) of Section 2.2 and its extension in Exercise 19 of Section 5.1. Applying the induction hypothesis to both of the unions of $k$ sets above, we obtain

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_{k+1}| = &\sum_{i\le k}|A_i| - \sum_{i<j\le k}|A_i \cap A_j| + \sum_{i<j<\ell\le k}|A_i \cap A_j \cap A_\ell| - \cdots \\ &+ (-1)^{k+1}|A_1 \cap A_2 \cap \cdots \cap A_k| + |A_{k+1}| - \sum_{i\le k}|A_i \cap A_{k+1}| \\ &+ \sum_{i<j\le k}|(A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1})| + \cdots \\ &- (-1)^{k+1}\big|(A_1 \cap A_{k+1}) \cap (A_2 \cap A_{k+1}) \cap \cdots \cap (A_k \cap A_{k+1})\big|.\end{aligned}$$

Note that $(A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1}) = A_i \cap A_j \cap A_{k+1}$ with similar simplifications in other terms above. Hence,

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_{k+1}| = &\Big(\sum_{i\le k}|A_i|\Big) + |A_{k+1}| - \sum_{i<j\le k}|A_i \cap A_j| - \sum_{i\le k}|A_i \cap A_{k+1}| \\ &+ \sum_{i<j<\ell\le k}|A_i \cap A_j \cap A_\ell| + \sum_{i<j\le k}|A_i \cap A_j \cap A_{k+1}| + \cdots \\ &+ (-1)^{(k+1)+1}|A_1 \cap A_2 \cap \cdots \cap A_{k+1}|.\end{aligned}$$

This is the desired result for $n = k + 1$. By the Principle of Mathematical Induction, the Principle of Inclusion–Exclusion is true for all $n \ge 1$.

## Exercises 6.2

1. (a) [BB] There are three choices for the first course and four for the second. By the multiplication rule, the answer is $3 \times 4 = 12$.

   (b) [BB] This time she must choose either a morning or an afternoon course. There are $4 + 3 = 7$ choices.

2. (a) By the addition rule, there are $2 + 2 + 6 = 10$ choices of molding.

   (b) By the multiplication rule, there are $2 \times 4 = 8$ choices of metal molding, $2 \times 4 = 8$ choices of wood molding and $6 \times 4 = 24$ choices of plastic molding. By the addition rule, the consumer can choose any of $8 + 8 + 24 = 40$ moldings.

3. [BB] $10 \times 9 \times 8 = 720$.

4. (a) There are nine choices for the first digit, nine for the second and eight for the third; in all $9 \times 9 \times 8 = 648$ numbers.

   (b) There are five choices for the last digit, then eight choices for the first and eight choices for the middle; in all, $5 \times 8 \times 8 = 320$ numbers.

5. (a) [BB] $9 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 9 \cdot 26^3 \cdot 10^3 = 158{,}184{,}000$.

   (b) [BB] $9 \times 26 \times 10^5 = 23{,}400{,}000$.

   (c) [BB] $158{,}184{,}000 + 23{,}400{,}000 = 181{,}584{,}000$.

6. (a) The first three letters can be chosen arbitrarily, but then last two are determined. So the number of palindromes is $26^3 = 17{,}576$.

   (b) $26^4 = 456{,}976$.

   (c) $26 + 26 + 26^2 + 26^2 + 26^3 + 26^3 + 26^4 + 26^4 = 950{,}508$

   (d) $26^{10}$

7. (a) [BB] $13 \times 6 \times 2 \times 4 = 624$; (b) [BB] $13 + 4 = 17$; (c) [BB] 25.

8. (a) 26; (b) 8; (c) 36; (d) 40.

9. [BB] (a) $4 \times 4 = 16$; (b) $4 \times 4 \times 4 = 64$; (c) $16 + 64 = 80$.

10. $8 \times 8 \times 9 \times 10 \times 10 \times 10 \times 10 = 5{,}760{,}000$

11. [BB] $60 \times 60 \times 60 = 216{,}000$

12. Choose a row and then one of the seven adjacent pairs of squares in that row, or choose a column and then one of the seven adjacent pairs of squares in that column. There are $8 \times 7 + 8 \times 7 = 112$ adjacent pairs of squares.

13. [BB] $500 + (500)(499) + (500)(499)(498) = 1.24501 \times 10^8$

14. (a) $3 \times 5 = 15$; (b) $15 \times 15 = 225$; (c) $3 \times 5 \times 4 \times 2 = 120$.

15. The numbers required are of the form $25x$, $52x$, $2x5$, $5x2$, $x25$ or $x52$. There are seven numbers of each type since there are seven digits 1, 2, 4, 5, 6, 8, 9 available. By the addition rule, there are $7(6) = 42$ numbers altogether. Not all of these are different, however. When $x = 2$ or $x = 5$, six numbers are counted twice. So the answer is $42 - 6 = 36$ numbers.

16. (a) [BB] $52 \times 3 \times 2 \times 1 = 312$.

   (b) There are $52 \times 3 \times 48 \times 3$ ways in which the first two and the last two cards match (but not each other), $52 \times 48 \times 3 \times 3$ ways in which the first and third match and the second and fourth match (but not each other), and $52 \times 48 \times 3 \times 3$ ways in which the first and fourth cards match and the second and third cards match, but not each other. By the addition rule, the number of ways you can get two pairs is $3(52 \times 48 \times 3 \times 3) = 67392$.

   (c) $312 + 67{,}392 = 67704$

   (d) There are $52 \times 3 \times 2 \times 48$ ways in which just the first three cards match, $52 \times 3 \times 48 \times 2$ ways in which the first two cards and the fourth (but not the third) card match, $52 \times 48 \times 3 \times 2$ ways in which the first, third and fourth cards (but not the second) match and $52 \times 48 \times 3 \times 2$ ways in which just the last three cards match. By the addition rule, there are $4(52 \times 48 \times 3 \times 2) = 59904$ ways to get three, but not four, of a kind.

   (e) The number of hands with no pairs is $52 \times 48 \times 44 \times 40 = 4{,}392{,}960$, while the total number of hands is $52 \times 51 \times 50 \times 49 = 6{,}497{,}400$. Thus, the number of hands with at least one pair is $6{,}497{,}400 - 4{,}392{,}960 = 2{,}104{,}440$.

17. (a) Here are the possibilities. There are five ways to get a total of eight.

| First die | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Second die | 6 | 5 | 4 | 3 | 2 |

(b) Here are the possibilities. There are six ways to get a total of seven.

| First die | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Second die | 6 | 5 | 4 | 3 | 2 | 1 |

(c) There are $5 + 6 = 11$ ways to get an eight or a seven.

(d) Getting doubles means two 1's, two 2's, two 3's, two 4's, two 5's or two 6's. There are six ways to get doubles.

18. [BB]

| Total | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| No. of ways | 1 | 2 | 3 | 4 | 5 | 6 | 5 | 4 | 3 | 2 | 1 |

19. (a) $6 \times 6 = 36$; (b) $6^5 = 7776$; (c) $6^n$.

(d) Since there are $6^n$ ways in which the dice can land and in precisely six of these ways the dice all land the same, the answer is $6^n - 6$.

20. (a) [BB] HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT.
There are 16 possibilities in all.

(b) 4 ways; (b) 6 ways; (c) 4 ways; (d) $4 + 6 + 4 + 1 = 15$ ways.

21. (a) $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90{,}000$; (b) $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27{,}216$; (c) $90000 - 27216 = 62{,}784$.

22. (a) [BB] $26 \times 26 \times 26 \times 10 \times 9 \times 8 = 12{,}654{,}720$

(b) $26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15{,}600{,}000$

(c) $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11{,}232{,}000$

23. (a) $4 \times 6 \times 3 \times 5 = 360$

(b) There are six ways of choosing two appetizers. Hence, the answer is $6 \times 6 \times 3 \times 5 = 540$.

(c) Mike can choose an appetizer and beverage (in $4 \times 6$ ways), an appetizer and entrée (in $4 \times 3$ ways), an appetizer and vegetable (in $4 \times 5$ ways), a beverage and entrée (in $6 \times 3$ ways), a beverage and vegetable (in $6 \times 5$ ways) or an entrée and a vegetable (in $3 \times 5$ ways). By the addition rule, the number of two-course meals is $24 + 12 + 20 + 18 + 30 + 15 = 119$.

(d) There are 18 ways of choosing exactly one course. In (c), we saw that the number of ways of choosing exactly two courses is 119. The number of ways of choosing exactly three courses is

$$(4 \times 6 \times 3) + (4 \times 6 \times 5) + (4 \times 3 \times 5) + (6 \times 3 \times 5) = 342.$$

We saw in (a) that there are 360 ways of choosing a four-course meal. Hence, the answer is $18 + 119 + 342 + 360 = 839$.

Alternatively, we note that if one is permitted to omit some course, then "no selection" is an additional choice within each group. The number of possible meals is, therefore, $5 \times 7 \times 4 \times 6 = 840$. Since this includes the possibility that nothing is ordered from any group, the number of meals in which at least one course is selected is $840 - 1 = 839$.

24. For a particular $i$, the entire term $p_i^{\alpha_i}$ must be a factor of $s$ or of $t$ (since $\gcd(s,t) = 1$). Hence, for each $i$, there are two possibilities—$p_i^{\alpha_i}$ is in $s$ or in $t$. Thus, there are $2^r$ ways of dividing the powers between $s$ and $t$. In this scheme, however, a given pair $s, t$ is counted twice, once in the order $s, t$ and once as $t, s$. So the answer is $\frac{1}{2}(2^r) = 2^{r-1}$.

25. (a) [BB] The easiest way to see that there are $2^n$ functions from $A$ to $B$ is to note that $\{a_i \mid (a_i, 0) \in f\} \leftrightarrow f$ is a one-to-one correspondence between the $(2^n)$ subsets of $A$ and the set of functions $f\colon A \to B$.
Alternatively, one can argue as follows. There are two choices for $j$ such that $(a_1, j)$ is part of a function. For each of these choices, there are two possibilities for $j$ such that $(a_2, j)$ is part of a function. Thus, there are four possibilities for both the pairs $(a_1, j_1)$, $(a_2, j_2)$ to be part of a function. Continuing this way, we see that there are $2^n$ functions from $A$ to $B$.

(b) [BB] Of the $2^n$ functions $A \to B$, precisely two are not onto,

$$\{(a_1, 0), (a_2, 0), \ldots, (a_n, 0)\} \quad \text{and} \quad \{(a_1, 1), (a_2, 1), \ldots, (a_n, 1)\}.$$

Thus, the number of onto functions is $2^n - 2$ as claimed.

26. (a) We use induction on $n$. If $n = 1$, then $A = \{a_1\}$ and there are three possible functions: $\{(1,1)\}$, $\{(1,2)\}$ and $\{(1,3)\}$. Since $3^1 = 3$, the result holds for $n = 1$.
Now assume that the result is true for $n = k$: Assume that there are $3^k$ functions from $\{a_1, \ldots, a_k\}$ to $B$. We must prove that there are $3^{k+1}$ functions from $A = \{a_1, \ldots, a_{k+1}\}$ to $B$. By the induction hypothesis, we have $3^k$ functions from $\{a_1, a_2, \ldots, a_k\}$ to $B$. For any of these, say $f$, there correspond three functions $A \to B$ obtained by adding any one of the three pairs $(a_{k+1}, 1)$, $(a_{k+1}, 2)$, $(a_{k+1}, 3)$ to the ordered pairs of $f$. Conversely, any function $f\colon A \to B$ is determined uniquely as a function $\{a_1, \ldots, a_k\} \to B$ —namely, $\{(a_i, j) \mid 1 \le i \le k, (i, j) \in f\}$— with an additional pair of the form $(a_{k+1}, j)$ added. It follows that each function $\{a_1, \ldots, a_k\} \to B$ gives rise to three functions $A \to B$, so, by the multiplication rule, the number of functions $A \to B$ is three times the number of functions $\{a_1, \ldots, a_k\} \to B$; that is, $3(3^k) = 3^{k+1}$ as desired. By the Principle of Mathematical Induction, the result holds for all $n \ge 1$.

(b) Let $A_1$ be the set of functions $A \to B$ for which 1 is not in the range, $A_2$, the set of functions $A \to B$ for which 2 is not in the range, and $A_3$, the set of functions $A \to B$ for which 3 is not in the range. The question asks for $|A_1 \cup A_2 \cup A_3|$. The set $A_1$ consists of the functions $A \to \{2, 3\}$. By Exercise 25, $|A_1| = 2^n$. Similarly, $|A_2| = |A_3| = 2^n$. The elements of $A_1 \cap A_2$ are the functions $A \to \{3\}$, thus $|A_1 \cap A_2| = 1$ and similarly, $|A_2 \cap A_3| = |A_1 \cap A_3| = 1$. There are no elements in $A_1 \cap A_2 \cap A_3$. By the Principle of Inclusion-Exclusion, the number of functions $A \to B$ which are not onto is $|A_1 \cup A_2 \cup A_3|$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 3(2^n) - 3.$$

(c) There are $3(2^n) - 3$ functions $A \to B$ which are not onto and $3^n$ functions $A \to B$ altogether, so the number of onto functions is $3^n - 3(2^n) + 3$.

## Exercises 6.3

1. [BB] There are seven days of the week. Hence, we wish to put eight objects (people) into seven boxes (days). By the Pigeon-Hole Principle, some day must have at least two people corresponding to it.

2. Let the integers be $a_1, a_2, \ldots, a_6$ and consider the six integers $a_1$, $a_1 + a_2$, ..., $a_1 + \ldots + a_6$. If one of these is divisible by 6, the result holds. Otherwise, there are only five possible remainders when each of these is divided by 6 (namely, 1, 2, 3, 4 or 5). The Pigeon-Hole Principle says that two of these must give the same remainder; so for some $s \neq t$,

$$a_1 + a_2 + \cdots + a_s = 6q + r$$
$$a_1 + a_2 + \cdots + a_t = 6q' + r$$

for the same remainder $r$. Assuming $s > t$ (without loss of any generality), subtracting gives $a_{t+1} + \cdots + a_s = 6(q - q')$ and the result holds.

3. Let the integers be $a_1, a_2, \ldots, a_n$ and consider the $n$ integers $a_1$, $a_1 + a_2$, ..., $a_1 + \ldots + a_n$. If one of these is divisible by $n$, the result holds. Otherwise, there are only $n - 1$ possible remainders when each of these is divided by $n$ (namely, $1, 2, \ldots,$ or $n - 1$). The Pigeon-Hole Principle says that two of these must give the same remainder; so for some $s \neq t$,

$$\begin{aligned} a_1 + a_2 + \cdots + a_s &= nq + r \\ a_1 + a_2 + \cdots + a_t &= nq' + r \end{aligned}$$

for the same remainder $r$. Assuming $s > t$ (without loss of any generality), subtracting gives $a_{t+1} + \cdots + a_s = n(q - q')$ and the result holds.

4. [BB] Applying the strong form of the Pigeon-Hole Principle with $k = 100$ and $n = 12$, we see that at least nine people must have their birthdays in the same month since $\lceil \frac{100}{12} \rceil = 9$.

5. (a) Divide the square into four squares of side 1 as in Problem 13. By the strong form of the Pigeon-Hole Principle, at least $\lceil \frac{102}{4} \rceil = 26$ points will lie in one square and hence be within $\sqrt{2}$ of each other, $\sqrt{2}$ being the length of a diagonal of the unit square.

   (b) Divide the square into $2^{2(n+1)}$ smaller squares each of side $\frac{1}{2^n}$. We want the squares as small as possible yet with the guarantee that one must contain two points. By the Pigeon-hole Principle, we want the largest $n$ so that $\lceil \frac{102}{2^{2(n+1)}} \rceil = 2$. We find $n = 2$. A square of side $\frac{1}{2^n}$ has diagonal of length $\sqrt{\frac{1}{2^{2n}} + \frac{1}{2^{2n}}} = \frac{\sqrt{2}}{2^n}$. With $n = 2$, this is $\frac{\sqrt{2}}{8} \approx 0.18$.

6. Consider the part about suits first. Place the $n$ cards into four boxes labelled club, diamond, heart and spade. By the strong form of the Pigeon-Hole Principle, one box must contain at least $\lceil \frac{n}{4} \rceil$ cards implying at least this many cards of a single suit. By the same sort of reasoning, at least $\lceil \frac{n}{13} \rceil$ cards of the same denomination.

7. (a) [BB] Any given processor is connected to at least one of 19 other processors. There are 20 processors, so the Pigeon-Hole Principle assures us that at least two are connected to the same number.

   (b) [BB] The result is still true, though for a somewhat more subtle reason. The number of processors to which a given processor is connected is in the range 0–19 (inclusive). On the other hand, if 0 occurs (that is, some processor is not connected to any other), then 19 cannot. So, as in part (a), there are at most 19 possibilities for the number of processors connected to a given processor. By the Pigeon-Hole Principle, at least two processors are connected to the same number.

8. (a) There are 2000 passengers (pigeons) to be distributed among 30 buses (pigeon holes). One bus must carry $\lceil 2000/30 \rceil = 67$ passengers.

   (b) There are 400 empty seats to be distributed among 30 buses, so some bus will have at least $\lceil 400/30 \rceil = 14$ empty seats.

9. (a) Make a box for each person and put into that box a checkmark for each acquaintance with that person. There are 208 checkmarks to be entered into these boxes, so some box must contain at least $\lceil 208/30 \rceil = 7$ checkmarks.

   (b) If this is wrong, then each person knows at least seven others. This would mean at least $30(7) = 210$ ordered pairs of people who know each other; that is, in the 210, Herb and Edgar are counted twice, once because Herb knows Edgar and once because Edgar knows Herb. So we get at least $\frac{1}{2}(210) = 105$ pairs of acquaintances, a contradiction.

10. [BB] Mimicking the solution to Problem 14, we let $a_i$ be the number of sets Martina plays on day $i$. Then we have $1 \leq a_1 < a_1 + a_2 < \cdots < a_1 + a_2 + \cdots + a_{77} \leq 132$. Now it is not true that the only integer in the range 1–77 which is divisible by 21 is 21 itself. Thus, while two of these sums must leave the same remainder upon division by 21, we can conclude this time only that the difference of these sums is divisible by 21, not that the difference **is** 21.

11. [BB] Suppose Brad and his mother drive $a_1$ quarter hours on day one, $a_2$ quarter hours on day two, and so on, $a_{35}$ quarter hours on day 35. The list

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots,\ a_1 + a_2 + \cdots + a_{35}$$

consists of 35 natural numbers between 1 and $15 \times 4 = 60$ (15 hours = 60 quarter hours) and so, if any of them is divisible by 35 ($8\frac{3}{4}$ hrs. $= 35$ quarter hrs.), it is 35 and we are done. Otherwise, each of these numbers leaves a remainder between 1 and 34 upon division by 35, and so two leave the same remainder. Suppose

$$\begin{aligned} a_1 + a_2 + \cdots + a_s &= 35q + r \\ a_1 + a_2 + \cdots + a_t &= 35q' + r. \end{aligned}$$

Then (assuming $s > t$) $a_{t+1} + a_{t+2} + \cdots + a_s$ is divisible by and, hence, equal to 35; that is, Brad drives $8\frac{3}{4}$ hours on days $t+1, t+2, \ldots, s$.

12. Suppose Linda studies $a_1$ hours on day one, $a_2$ hours on day two, and in general, $a_i$ hours on day $i$, $1 \leq i \leq 42$. Consider the sums

$$1 \leq a_1 < a_1 + a_2 < \cdots < a_1 + a_2 + \cdots + a_{42} \leq 50$$

Upon division by 33, two of these 42 numbers must leave the same remainder; thus, for some $s, t, q, q'$ and $r$,

$$\begin{aligned} a_1 + a_2 + \cdots + a_s &= 33q + r \\ a_1 + a_2 + \cdots + a_t &= 33q' + r \end{aligned}$$

Assuming $s > t$, subtracting gives $a_{t+1} + a_{t+2} + \cdots + a_s = 33(q - q')$. Since the sum here is a number between 1 and 50, since it is divisible by 33 it must be 33. But this says that Linda studies for 33 hours on days $t+1$ through $s$.

13. Let $b_i$ be the number of words George learns on days 1 through $i$ (inclusive) and consider the 106 numbers

$$b_1, b_2, \ldots, b_{53}, b_1 + 15, b_2 + 15, \ldots, b_{53} + 15.$$

The largest number here is $b_{53} + 15 = 90 + 15 = 105$. By the Pigeon-Hole Principle, we conclude that two numbers are the same. Since $b_i > b_j$ if $i > j$, the only way for two to be equal is for $b_i = b_j + 15$—that is, $b_i - b_j = 15$—for some $i$ and $j$. It follows that George learns 15 new words on days $j+1$ through $i$.

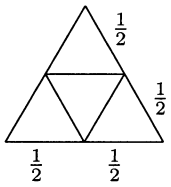
14. [BB] Let box 1 correspond to days 1, 2, and 3, box 2 to days 4, 5, and 6, box 3 to days 7, 8, and 9, and box 4 to days 10, 11, and 12. Putting each bill into the box corresponding to the day it was mailed, we see that one box must contain at least $\lceil \frac{195}{4} \rceil = 49$ bills, by the general form of the Pigeon-Hole Principle. This gives the desired result.

15. First note that $1+2+3+\cdots+36 = \frac{36(37)}{2} = 666$. Since there are twelve blocks of three consecutive sectors, there must be a sum of at least $\lceil \frac{666}{12} \rceil = 56$ in one of these blocks. (There are 666 pigeons shoved into 12 pigeon holes.)

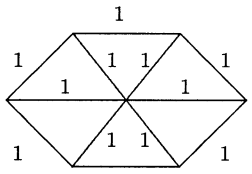
16. By the multiplication rule, there are $26^2$ possible pairs of initials. If $n$ seats are occupied, the Pigeon-Hole Principle (strong form) says that $\lceil \frac{n}{26^2} \rceil$ people seated will have the same first and last initials. The smallest $n$ such that $\lceil \frac{n}{26^2} \rceil = 3$ is $n = 2(26^2) + 1 = 1353$.

17. [BB] Divide the rectangle into 25 rectangles, each $3 \times 4$. By the Pigeon-Hole Principle, at least two points are within, or on the boundary, of one of these smaller rectangles. This gives the result since the maximum distance between two points of such a rectangle is $\sqrt{3^2+4^2} = 5$.

18. Subdivide the triangle into four equilateral triangles of side $\frac{1}{2}$, as shown. By the Pigeon-Hole Principle, at least two of the five chosen points must lie in, or on the boundary of, the same smaller triangle. But these points are at most $\frac{1}{2}$ apart (the distance between vertices of each small triangle).




19. Subdivide the hexagon into six equilateral triangles, as shown. (1 unit = 30 cms) By the Pigeon-Hole Principle, at least two flowers must lie in, or on the boundary of, the same triangle. Hence, these flowers are at most 30 cms. apart, the length of a side.




20. (a) Since $|S| = 8$, there are $2^8 - 1 = 255$ nonempty subsets of $S$. The sum of the elements of a nonempty subset of $S$ is not less than 2 and not more than $2+3+5+7+11+13+17+19 = 77$; thus at most 76 sums are possible. By the Pigeon-Hole Principle (strong form), at least $\lceil \frac{255}{77} \rceil = 4$ subsets have the same sum.

    (b) The elements of $S$ are primes. By the Fundamental Theorem of Arithmetic, the product associated with a nonempty subset of $S$ is uniquely determined by that subset. So it is not possible for two (different) subsets to have the same associated product.

21. [BB] As suggested, we consider the sequence of natural numbers $M_1 = 3$, $M_2 = 33$, ..., $M_n = 33\cdots3$ ($n$ 3's). If one of these is divisible by $n$, we have the desired result. Otherwise, at least two of these natural numbers leave the same (nonzero) remainder upon division by $n$; that is, for some $i \neq j$, $M_i = qn + r$, $M_j = q'n + r$ for the same $r$. Assuming, without loss of generality, that $i > j$, $M_i - M_j = (q - q')n$ is divisible by $n$ and has only 3's and 0's in its base 10 representation.

22. Consider the 2003 integers $a_0 = 1$, $a_1 = 11$, $a_2 = 111$, ..., $a_{2002} = \underbrace{111\cdots1}_{2003\ 1\text{'s}}$. By the Pigeon-Hole Principle, two of these are congruent modulo 2002. Thus, for some $i, j$ with $i > j$, we have $2002 \mid (a_i - a_j)$. Since $a_i - a_j$ consists of $(i-j)$ 1's followed by $j+1$ 0's, the result follows.

23. [BB] After finitely many steps in the long division process, we will be adjoining a "0" at each stage to the remainder from the division at the previous stage. Since only finitely many remainders are possible (each remainder is less than the divisor), two must be the same and from the first repetition, all steps (and corresponding decimal places) will repeat.

24. Let box 1 consist of the numbers $1, 2, 4, 8, \ldots, 128$; let box 3 consist of $3, 6, 12, \ldots, (64)3$, box 5 consist of $5, 10, 20, \ldots, 32(5)$, and so on; box 199 consists of 199 alone. (In general, for odd $i$, box $i$ consists of all numbers of the form $2^k i$ which do not exceed 200. There are 100 boxes in all.) If the 101 chosen numbers are $a_1, a_2, \ldots, a_{101}$ and $a_i$ is assigned to the box which contains it (by the hint, every number belongs to some box), by the Pigeon-Hole Principle, some box contains two numbers. Since, for any two numbers in a box, one is a multiple of the other, we have the desired result.

25. (a) [BB] Suppose no two people have the same age so that there are at least 51 different ages in the room. Let box 1 correspond to integers 1 and 2, box 2 correspond to integers 3 and 4, ..., box 50 to 99 and 100. Assigning ages to boxes, by the Pigeon-Hole Principle, some box contains two ages. This says that ages of two people are consecutive integers, as required.

    (b) Suppose no two people have the same age so that there are at least 51 different ages in the room. Let box 1 correspond to the integers 1, 2, 4, 8, ..., 64, box 3 to the integers 3, 6, 12, 24, ..., 96, ..., box 99 to the single integer 99. (In general, for odd $i$, box $i$ corresponds to $\{i, 2i, 4i, 8i, \ldots\}$, the point being that, for any two elements of box $i$, one is a multiple of the other.) Since there are 50 boxes, by the Pigeon-Hole Principle, some box contains two ages, and this means that two people in the room have ages one of which is a multiple of the other.

    (c) Suppose there are $n$ people in the room. The number of possible hands each person shakes is from the set $\{0, 1, 2, \ldots, n-1\}$. But if someone shakes 0 hands, no one can shake $n-1$ hands, and vice versa. So there are in fact just $n-1$ possible numbers for the number of handshakes in the room (either $0, 1, \ldots, n-2$ or $1, 2, \ldots, n-1$). It follows that two people shook the same number of hands.

    (d) Suppose that $k$ is the most number of hands shaken by anybody in the room and consider the group of $k+1$ people which consists of somebody who shook $k$ hands together with the $k$ people whose hands she shook. The numbers of hands shaken by this group is among the integers $1, 2, \ldots, k$. Thus, two people shook the same (positive) number.

26. (a) The number of nonempty subsets is $2^7 - 1 = 127$. The largest possible sum is $21 + 20 + 19 + 18 + 17 + 16 + 15 = 126$, while the smallest possible sum is 1. Hence, there are 126 possible sums for the 127 subsets. So at least two subsets must have the same sum.

    (b) Let $A = \{a_1, a_2, \ldots, a_7\}$ and order the $a_i$ so that $a_1 < a_2 < \cdots < a_7$. The subset of $A$ giving the largest sum is $A$ itself while the (nonempty) subset of $A$ giving the smallest sum is $\{a_1\}$. Thus, the sums of nonempty subsets of $A$ range from $a_1$ to $a_1 + a_2 + \cdots + a_7$, an interval containing $(a_1 + a_2 + \cdots + a_7) - a_1 + 1 = a_2 + a_3 + \cdots + a_7 + 1$ integers. The most this can be is $18 + 19 + 20 + 21 + 22 + 23 + 1 = 124$. Since there at most 124 numbers among 127 subsets, the Pigeon-Hole Principle says that two must be the same.

    (c) Using the argument of (a), we observe that the largest possible sum is $12+11+10+9+8+7+6 = 63$ while the smallest is still 1. Hence, at most 63 sums are possible for the 127 subsets. By the general Pigeon-Hole Principle, at least $\lceil 127/63 \rceil = 3$ subsets must have equal sums.

    (We leave it to the reader to prove that even fine tuning this argument as in (b), we can't improve on the result that three subsets must have equal sums.)

27. (a) [BB] Focus on one of the ten people, say Hilda. Suppose first that at least four of the remaining nine people are strangers to Hilda. If these four are mutual friends we are done; otherwise two of these are strangers and, hence, together with Hilda, we have three mutual strangers and again we are done. Thus, we suppose that less than four people are strangers to Hilda; so Hilda has at least six friends in the group. By Problem 17, this group of six contains either three mutual strangers (in which case we are done) or three mutual friends who, together with Hilda, give four mutual friends.

    (b) Focus attention on one of the 20 people, say Bruce. Bruce has at least 10 friends among the remaining 19 people or else at least 10 people are strangers to him. If 10 are strangers, then among these there are either four mutual friends or three mutual strangers (applying part (a)) and if three mutual strangers, then with Bruce we get a set of four mutual strangers. So we have the desired result in the case that 10 of the group are strangers to Bruce. If this is not so, then Bruce has 10 friends. An argument analogous to that in (a), shows that in any group of 10 people there are either four mutual strangers or three mutual friends. If there are four mutual strangers among Bruce's friends, we have the result; otherwise, we have three mutual friends which, together with Bruce, give us four mutual friends.

28. (a) [BB] The set $S = \{a_1, a_2, \ldots, a_{10}\}$ has $2^{10} = 1024$ subsets and the sum of the elements in each subset is between 0 and 1000, so the Pigeon-Hole Principle implies the existence of two subsets with the same sum.

    (b) Get two subsets with equal sums as in part (a) and remove the elements in their intersection.

29. If $m = 1$ and we have $n > 1$ objects, they all go into the lone box, so this box contains at least two objects. This establishes the result for $m = 1$. Now assume that $k \geq 1$ and the result is true for $k$ boxes. Given $k + 1$ boxes and $n$ objects in these boxes, $n > k + 1$, we must prove that some box contains at least two objects. Focus on one particular box, say box $X$. If $X$ contains two or more objects, we have we want. Otherwise, $X$ contains at most one object, to the remaining $k$ boxes contain at least $n - 1$ objects. Since $n - 1 > k$, by the induction hypothesis, one of these $k$ boxes contains at least two objects. By the Principle of Mathematical Induction, we conclude that the Pigeon-Hole Principle is true for any number $m$ of boxes.

30. **Case 1:** Assume that three of the integers are divisible by 50. Then these numbers are $50a$, $50b$ and $50c$ for certain integers $a, b, c$, two of which must have the same parity. If, without loss of generality, these are $a$ and $b$, then $50a - 50b = 50(a - b)$ is divisible by 100.

    **Case 2:** Assume that at most two of the given integers are divisible by 50. Then we can select 51 of these integers, $x_1, x_2, \ldots, x_{51}$, such that 50 does not divide any of $x_2, \ldots, x_{51}$. Let

    $$S = \{x_1 \pm x_i \mid 2 \leq i \leq 51\}.$$

    If one of the elements in $S$ is divisible by 100, we're done. Otherwise, two of the elements in $S$ must give the same remainder when divided by 100. If these are $x_1 + x_i$ and $x_1 + x_j$, then $x_i - x_j$ is divisible by 100. A similar argument works for the other three possibilities except in the case where it is $x_1 + x_i$ and $x_1 - x_i$ which give the same remainder. In this case, however, $2x_i$ would be divisible by 100, contradicting the fact that 50 does not divide $x_i$.

# Chapter 6 Review

1. By the Principle of Inclusion-Exclusion, $|A \cap B| = |A| + |B| - |A \cup B|$, so $|A \cap B| > 0$.

2. Let $A$, $B$, $C$, $D$ be the sets of integers between 1 and 2000 (inclusive) which are divisible by 2, by 3, by 5, and by 7, respectively. We want $|(A \cup B \cup C \cup D)|$. Now

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|.\end{aligned}$$

   We have

$$|A| = \lfloor \tfrac{2000}{2} \rfloor = 1000, \quad |B| = \lfloor \tfrac{2000}{3} \rfloor = 666, \quad |C| = \lfloor \tfrac{2000}{5} \rfloor = 400, \quad |D| = \lfloor \tfrac{2000}{7} \rfloor = 285,$$

$$|A \cap B| = \lfloor \tfrac{2000}{6} \rfloor = 333, \quad |A \cap C| = \lfloor \tfrac{2000}{10} \rfloor = 200, \quad |A \cap D| = \lfloor \tfrac{2000}{14} \rfloor = 142,$$

$$|B \cap C| = \lfloor \tfrac{2000}{15} \rfloor = 133, \quad |B \cap D| = \lfloor \tfrac{2000}{21} \rfloor = 95, \quad |C \cap D| = \lfloor \tfrac{2000}{35} \rfloor = 57,$$

$$|A \cap B \cap C| = \lfloor \tfrac{2000}{30} \rfloor = 66, \quad |A \cap B \cap D| = \lfloor \tfrac{2000}{42} \rfloor = 47,$$

$$|A \cap C \cap D| = \lfloor \tfrac{2000}{70} \rfloor = 28, \quad |B \cap C \cap D| = \lfloor \tfrac{2000}{105} \rfloor = 19,$$

$$|A \cap B \cap C \cap D| = \lfloor \tfrac{2000}{210} \rfloor = 9.$$

   So $|A \cup B \cup C \cup D| = 1542$.

3. (a) A palindrome is a word which is spelt the same way forwards as backwards, such as SUNUNUS.

   (b) We subtract from the total number of seven-letter palindromes which begin with S the number which use four different letters. The answer is $1 \times 26 \times 26 \times 26 - 1 \times 25 \times 24 \times 23 = 17{,}576 - 13{,}800 = 3776$.

   (c) $26 \times 26 \times 26 \times 26 = 456{,}976$.

   (d) Let $A$ be the set of seven-letter palindromes with four different letters and $B$ the set of seven-letter palindromes which do not contain S. We want $|(A \cup B)^c|$. Since $|A \cup B| = |A| + |B| - |A \cap B| = 26(25)(24)(23) + 25^4 - 25(24)(23)(22) = 445{,}825$ and 456,976 seven-letter palindromes in all (by Exercise 3(d)), the answer is $456{,}976 - 445{,}825 = 11{,}151$.

   (e) ABBA. Remember "Dancing queen" and "Take a chance on me"?

4. The number of nine-letter palindromes equals the number of 10-letter palindromes since, in each case, the first five letters can be chosen arbitrarily, after which the remaining ones are determined. The number in each case is $26^5$.

5. The problem asks for $|A_1 \cup A_2 \cup_3 \cup A_4|$ which, by the Principle of Inclusion-Exclusion, is

$$\begin{aligned}& |A_1| + |A_2| + |A_3| + |A_4| \\ & -|A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & +|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & -|A_1 \cap A_2 \cap A_3 \cap A_4| = 4(5) - 6(1) = 14.\end{aligned}$$

6. Let $S$ be the set of cars with stereo systems, $A$ the set of cars with air conditioners and $R$ the set with sun roofs. We are given that $|S| = 30$, $|A| = 30$, $|R| = 40$, $|(S \cap A) \cup (S \cap R) \cup (A \cap R)| = 30$ and $|S \cap A \cap R| = 10$. By the Principle of Inclusion–Exclusion,

$$\begin{aligned} 30 &= |(S \cap A) \cup (S \cap R) \cup (A \cap R)| \\ &= |S \cap A| + |S \cap R| + |A \cap R| - |(S \cap A) \cap (S \cap R)| - |(S \cap A) \cap (A \cap R)| \\ &\quad - |(S \cap R) \cap (A \cap R)| + |(S \cap A) \cap (S \cap R) \cap (A \cap R)| \\ &= |S \cap A| + |S \cap R| + |A \cap R| - 2|S \cap A \cap R|. \end{aligned}$$

Thus, $|S \cap A| + |S \cap R| + |A \cap R| = 30 + 2(10) = 50$. It follows that the number of car with at least one feature is

$$\begin{aligned} |S \cup A \cup R| &= |S| + |A| + |R| - |S \cap A| - |S \cap R| - |A \cap R| + |S \cap A \cap R| \\ &= 30 + 30 + 40 - 50 + 10 = 60. \end{aligned}$$

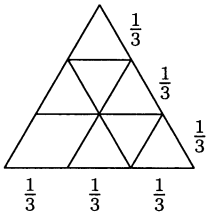
Since the number of cars with exactly one feature is the number with at least one feature less the number with at least two, the number with exactly one is $60 - 30 = 30$.

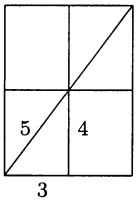
7. If $n$ objects are put into $m$ boxes and $n > m$, then some box must contain at least $\lceil \frac{n}{m} \rceil$ objects.

8. Label the 18 integers $a_1, a_2, \ldots, a_{18}$ and consider their congruence classes mod 17. Two of these are the same by the Pigeon-Hole Principle. If these are $\overline{a_i}$ and $\overline{a_j}$, then $\overline{a_i} = \overline{a_j}$ implies $a_i \equiv a_j \pmod{17}$, so $17 \mid (a_i - a_j)$.

9. We prove by contradiction. If Z is finite, there is a one-to-one onto function $f\colon \mathrm{Z} \to \{1, 2, \ldots, n\}$ for some positive integer $n$. For each $i$, $1 \leq i \leq n$, assign to Box $i$, all $k$ with $f(k) = n$. In particular, the $n + 1$ integers $f(1), f(2), \ldots, f(n+1)$ are assigned to $n$ boxes. By the Pigeon-Hole Principle, some box is assigned two of these numbers, so $f$ is not one-to-one. This contradiction shows that Z is not finite. By definition, this makes Z infinite.

10. As shown, subdivide the triangle into nine congruent equilateral triangles, each of side $\frac{1}{3}$. By the Pigeon-Hole Principle, at least two of the ten points must lie in, or on the boundary of, the same smaller triangle, so the result follows.




11. 


Divide the island into four smaller rectangles as shown. Each of these rectangles has diagonal of length 5 kilometres. By the Pigeon-Hole Principle, two hermits live within the same rectangle, so we have the desired result.

12. (a) A factor of $p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ is an integer of the form $p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, where $0 \leq k_i \leq a_i$. So there are $a_1 + 1$ choices for $k_1$, $a_2 + 1$ choices for $k_2$, and so on. By the multiplication rule, $n$ has $(a_1 + 1)(a_2 + 1) \cdots (a_s + 1)$ factors.

(b) Writing $n$ as the product of prime powers as in part (a), the product $(a_1+1)(a_2+1)\cdots(a_s+1)$ is odd if and only if each factor $a_i+1$ is odd, hence, if and only if each $a_i = 2b_i$ is even. And this occurs if and only if $n = (p_1^{b_1}p_2^{b_2}\cdots p_s^{b_s})^2$ is a perfect square.

(c) At the end of the day, a picket is white if and only if it has a number which is a perfect square. To see why, we need to determine those pickets whose color changes an even number of times. Picket $k$ changes color each time there is an integer $d$, $1 < d \leq k$, which divides $k$. The number of such $d$, together with 1, is the total number of factors of $d$. By (a), this number is odd (and the number of $d > 1$ is even) if and only if $k$ is a perfect square.

## Exercises 7.1

1. [BB] $13 \cdot 12 \cdot 11 \cdots 6 = P(13,8) = 51{,}891{,}840$
2. $8 \cdot 7 \cdot 6 = P(8,3) = 336$
3. [BB] $10 \cdot 9 \cdot 8 \cdot 7 = P(10,4) = 5040$
4. In $30 \cdot 29 \cdots 23 = P(30,8)$ ways.
5. [BB] $3 \times 5! = 360$
6. $5 \times 4 \times 3 \times 6! = 43,200$
7. (a) This question asks for the number of lines of 14 people. The answer is 14!.
   (b) [BB] Arrange the boys in a line in one of 10! ways, then arrange the girls in a line in one of 4! ways, and finally put either the boys or the girls at the start of the line. The answer is 10!4!2!.
   (c) Arrange the girls in one of 4! ways. Then the girls form one block and the boys ten other blocks. The 11 blocks can be arranged in one of 11! ways. The answer is 4!11!.
   (d) Let $A$ be the set of ways in which the girls sit together and $B$ the set of ways in which the boys and girls each sit together. The question asks for $|A \setminus B| = |A| - |B| = 4!11! - 10!4!2! = 9(4!10!)$.
8. (a) The question asks for the number of circles of 14 people. The answer is 13!.
   (b) Arrange the boys into one of 10! lines and then the girls into one of 4! lines. Then there is just one way to have them form a circle. The answer is 10!4!.
   (c) Arrange the girls into one of 4! lines and then form a circle from amongst the 11 blocks—10 boys, one line of girls. The answer is 4!10!.
   (d) It is impossible to arrange the girls and boys into a circle so that just the girls sit together. The answer is 0 ways.
9. (a) [BB] $3! \times 4! = 144$; (b) [BB] $P(4,2) \times 4! = 288$.
10. (a) [BB] There are $m!$ bijective functions $X \to X$.
    (b) [BB] If $m > n$, there are 0 injective functions $X \to Y$.
    If $m \leq n$, the number is $\underbrace{n(n-1)(n-2)\cdots(n-m+1)}_{m \text{ factors}} = P(n,m)$.

11. Let $A$ be the set of permutations which contain the pattern $bge$: $|A| = 5!$ (thinking of $bge$ as a single symbol). Similarly, if $B$ is the set of permutations which contain $eaf$, $|B| = 5!$. For a permutation to contain both $bge$ and $eaf$ it must contain $bgeaf$, so $|A \cap B| = 3!$. The number containing either $bge$ or $eaf$ is $|A \cup B| = |A| + |B| - |A \cap B| = 5! + 5! - 3! = 234$. Hence, the answer is $7! - 234 = 5040 - 234 = 4806$.

12. [BB] Let $N$ be the set of lines in which the Noseworthys are beside each other and $A$ the set of lines in which the Abbotts are beside each other.

    (a) For each of the two ways in which the Noseworthys can stand beside each other, there are 3! possible lines—the Noseworthys as one unit and the two Abbotts. Thus, $|N| = 2 \times 3! = 12$.

    (b) The number of possible lines altogether is $4! = 24$. Thus, $|N^c| = 24 - 12 = 12$.

    (c) There are 2! orders for the couples and, in each case, two arrangements for the Noseworthys and two for the Abbotts: $|N \cap A| = 2 \times 2 \times 2 = 8$.

    (d) The Noseworthys must be in positions 2 and 3 in either order; the Abbotts must be in positions 1 and 4 in either order. There are four lines.

    (e) $|N \cup A| = |N| + |A| - |N \cap A| = 12 + 12 - 8 = 16$.

    (f) $|N \oplus A| = |N| + |A| - 2|N \cap A| = 12 + 12 - 2(8) = 8$.

13. Let $N$ be the set of lines in which the Noseworthys are beside each other and $A$ the set of lines in which the Abbotts are beside each other.

    (a) For each of the two ways in which the Noseworthys can stand beside each other, there are 4! possible lines—the Noseworthys as one unit, two Abbotts and a dog. Thus, $N = 2 \times 4! = 48$.

    (b) The number of possible lines altogether is $5! = 120$. Thus, $|N^c| = 120 - 48 = 72$.

    (c) There are 3! lines of the three units—Noseworthys, Abbotts, dog—and, in each case, two arrangements for the Noseworthys and two for the Abbotts: $|N \cap A| = 3! \times 2 \times 2 = 24$.

    (d) This part asks for $|N \cap A^c|$. Now the set $N$ is the disjoint union of $N \cap A$ and $N \cap A^c$. Since $|N \cap A| = 24$ and $|N| = 48$, $|N \cap A^c| = 48 - 24 = 24$.

    (e) $|N \cup A| = |N| + |A| - |N \cap A| = 48 + 48 - 24 = 72$.

    (f) $|N \oplus A| = |N| + |A| - 2|N \cap A| = 48 + 48 - 2(24) = 48$.

14. (a) [BB] For each of the two possible orders for Mr. and Mrs. Jones, there are 5! lines which can be formed (thinking of the couple as a single unit to be ordered with the other four people). Thus, the number of lines in which the Joneses are together is $2 \times 5! = 240$.

    (b) For each of the $2 \times 2$ ways the Joneses can stand next to each other and the Murphys next to each other, there are 4! lines which can be formed (thinking of each couple as a single unit to be ordered with the two remaining people). Thus, the number of lines with each of the two couples, Jones and Murphy, next to each other is $4 \times 24 = 96$.

    (c) Let $A_i$ be the set of lines in which couple $i$ is together. The question asks for $|A_1 \cup A_2 \cup A_3|$. By part (a), $|A_i| = 240$ for each $i$. By part (b), each $|A_i \cap A_j| = 96$. The number of lines in which each couple is together is $2^3 \times 3! = 48$. Thus, $|A_1 \cup A_2 \cup A_3| = 3(240) - 3(96) + 48 = 480$.

15. First assume there are two letters between $a$ and $b$. As in Problem 6, the number of permutations here is $2 \times P(5, 2) \times 4! = 960$. Next assume there are three letters between $a$ and $b$. The number of such permutations is $2 \times P(5, 3) \times 3! = 720$. By the addition rule, the answer is $960 + 720 = 1680$.

16. (a) [BB] $9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 9 \times P(9,6) = 9 \times 60,480 = 544,320$.

    (b) The total number of 7-digit numbers which do not contain a 6 is $8 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 161{,}280$. The number which contain neither a 3 nor a 6 is $7 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 35{,}280$. Hence, the answer is $161{,}280 - 35{,}280 = 126{,}000$.

17. (a) $P(9,7) = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 181,440$.

    (b) [BB] There are seven available positions for the 3, then six for the 6, then five for the remaining seven digits. The answer is $7 \times 6 \times P(7,5) = 42 \times 2520 = 105{,}840$.

    (c) There are six pairs of consecutive positions into each of which we can place 36 or 63. After this, there remain five positions to be filled from the remaining seven digits. The answer is $6 \times 2 \times P(7,5) = 30{,}240$.

    (d) [BB] There are seven digits to be placed in seven positions. The answer is $P(7,7) = 5040$.

    (e) There are seven positions for the 3, then seven numbers to be placed in six positions. The answer is $7 \times P(7,6) = 35{,}280$.

    (f) Using the result of the previous part, $35280 + 35280 = 70560$.

    (g) If neither 36 nor 63 appear, then

    - neither 3 nor 6 appear (5040 of these by (d)),
    - or exactly one of 3, 6 appears (70,560 of these by (f)),
    - both digits appear, but not consecutively ($105{,}840 - 30{,}240 = 75{,}600$ of these, by (b) and (c)).

    So the desired number is $5040 + 70{,}560 + 75{,}600 = 151{,}200$.

18. Suppose $Y = \{t_1, t_2, \ldots, t_n\}$. Let $f\colon X \to Y$ be surjective. Then the sets $X_1 = \{a \in X \mid f(a) = t_1\}$, $X_2 = \{a \in X \mid f(a) = t_2\}$,..., $X_n = \{a \in X \mid f(a) = t_n\}$ form a partition of $X$ into $n$ subsets. (They are pairwise disjoint and their union is $X$ by the definition of function. The partition contains nonempty sets since $f$ is surjective.) It follows that a surjective function $X \to Y$ can be defined by first selecting a partition $X_1, X_2, \ldots, X_n$ of $X$ into $n$ subsets, then mapping the elements of $X_1$ to any of the $n$ elements of $Y$, then the elements of $X_2$ to any of the remaining $n-1$ elements of $Y$, and so on. By the multiplication rule, the number of surjective functions is $f(n,m) \cdot n(n-1)(n-2)\cdots 2 \cdot 1 = f(n,m)n!$ as required.

## Exercises 7.2

1. (a) [BB] $\binom{6}{2}\binom{7}{2}\binom{8}{2} = (15)(21)(28) = 8820$

    (b) [BB] We can choose three people from Nebraska, two from Idaho, and two from Louisiana, or two from Nebraska, three from Idaho, and two from Louisiana, or two from Nebraska, two from Idaho, and three from Louisiana. By the addition rule, the number of possible committees is

    $$\binom{6}{3}\binom{7}{2}\binom{8}{2} + \binom{6}{2}\binom{7}{3}\binom{8}{2} + \binom{6}{2}\binom{7}{2}\binom{8}{3} = (20)(21)(28) + (15)(35)(28) + (15)(21)(56) = 44{,}100.$$

2. (a) Each way corresponds to a selection of six players for the Good Guys, the remaining six forming the Bad Guys. The answer is $\binom{12}{6} = 924$.

(b) The answer is $\frac{924}{2} = 462$, one half the answer to (a) since, the unnamed teams $\{1,2,3,4,5,6\}$, $\{7,8,9,10,11,12\}$ give rise to two possibilities for the Good Guys.

3. [BB] Choose the five positions for the 1's and put the 0's in the remaining places. The answer is $\binom{12}{5} = \frac{12!}{7!5!} = 792$.

4. This is the number of ways to select three locations from seven for the red flags. The answer is $\binom{7}{3} = \frac{7!}{3!4!} = 35$.

5. (a) [BB] $\binom{15}{5}\binom{10}{5}$; (b) [BB] $\binom{15}{7}\binom{8}{4}$; (c) [BB] $3\binom{15}{7}\binom{8}{4}$.

6. [BB] $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = 1 + 10 + 45 + 120 = 176$

7. (a) A game card can be completed in $\binom{49}{6} = 13{,}983{,}816$ ways.

   (b) A player with one card has one chance in 13,983,816 of winning; a player with $n$ cards has $n$ chances in 13,983,816 . The smallest $n$ such that $n/13{,}983{,}816 \geq 10^{-6}$ is $n = 14$.

   (c)

| | |
|---|---|
| No numbers match | $\binom{43}{6} = 6{,}096{,}454$ |
| Exactly one number matches | $\binom{6}{1}\binom{43}{5} = 5{,}775{,}588$ |
| Exactly two numbers match | $\binom{6}{2}\binom{43}{4} = 1{,}851{,}150$ |
| Exactly three numbers match | $\binom{6}{3}\binom{43}{3} = 246{,}820$ |
| Exactly four numbers match | $\binom{6}{4}\binom{43}{2} = 13{,}545$ |
| Exactly five numbers match | $\binom{6}{5}\binom{43}{1} = 258$ |
| All six numbers match | 1 |

   (d) The sum of the answers to part (c) is the total number of ways of completing a game card. By part (a), this is 13,983,816 .

   (e) The number of ways in which at least one number matches (call this $x$) plus the number of ways in which no number matches is the total number of ways of completing a game card. So $x + \binom{43}{6} = \binom{49}{6}$; $x = \binom{49}{6} - \binom{43}{6} = 7{,}887{,}362$. (There's about a 56% chance of matching at least one of six chosen numbers.)

   (f) The number of ways in which at least two numbers match is $\binom{49}{6} - \binom{43}{6} - \binom{6}{1}\binom{43}{5} = 2{,}111{,}774$. (The chance of at least two matches is about 15%).

8. (a) [BB] Choose one of four suits and then five of the 13 cards of that suit. There are $4\binom{13}{5} = 5148$ possible hands.

   (b) Choose two aces in one of $\binom{4}{2}$ ways and then three of the remaining 48 "not ace" cards. The answer is $\binom{4}{2}\binom{48}{3} = 6(17296) = 103{,}776$.

9. $30 \cdot 29 \cdot 28 \cdot \binom{27}{2} = 30 \cdot 29 \cdot 28 \cdot 351 = 8{,}550{,}360$.

10. (a) [BB] There are two choices for each toss. Hence, there are $2 \times 2 \times \cdots \times 2 = 2^{10}$ possible sequences.

    (b) [BB] $\binom{10}{4} = 210$

11. (a) [BB] $\binom{25}{5} = 53{,}130$; (b) [BB] $\binom{15}{5} = 3003$.

(c) [BB] Choose the red balls in $\binom{15}{3}$ ways and then the white balls in $\binom{10}{2}$ ways. The answer is $\binom{15}{3}\binom{10}{2} = 20{,}475$.

12. (a) $\binom{8}{5} = 56$

(b) Choose the mathematicians in $\binom{5}{3}$ ways and then the geologists in $\binom{3}{2}$ ways. The answer is $\binom{5}{3}\binom{3}{2} = 10(3) = 30$.

13. (a) [BB] $\binom{100}{20}$; (b) [BB] $\binom{100}{20}$; (c) [BB] $\binom{100}{20} \times \binom{100}{20}$.

(d) [BB] Choose the five cars to undergo each of the tests, then choose 15 from the remaining 95 for the first test and 15 from the available 80 for the second test. The answer is $\binom{100}{5} \times \binom{95}{15} \times \binom{80}{15}$. This is also $\binom{100}{20}\binom{20}{5}\binom{80}{15}$. (Why?)

14. (a) $\binom{18}{4} = 3060$; (b) $\binom{18}{5} = 8568$; (c) $\binom{18}{6} = 18564$.

15. (a) $\binom{9}{6} = 84$

(b) The number of groups which include the divorced couple is $\binom{7}{4}$. Thus, the number in which not both of these people are present is $\binom{9}{6} - \binom{7}{4} = 84 - 35 = 49$.

(c) Our hostess must either invite two married couples and two singles or all three married couples. The answer is $\binom{3}{2}\binom{3}{2} + 1 = 10$.

16. (a) $\binom{22}{5} = 26334$

(b) The total number of committees is $\binom{22}{5}$. The number of committees which contain no statistician is $\binom{19}{5}$, so the number which contain at least one is $\binom{22}{5} - \binom{19}{5} = 26334 - 11628 = 14706$.

(c) The total number of committees is $\binom{22}{5}$. The number of committees which include both Dr. G and Dr. P is $\binom{20}{3}$. So the number which do not include both is $\binom{22}{5} - \binom{20}{3} = 26334 - 1140 = 25194$.

(d) There are just two possibilities for the numbers of mathematicians, computer scientists and statisticians which fit the stated requirements.

| Mathematicians | Computer Scientists | Statisticians |
|---|---|---|
| 4 | 1 | 0 |
| 3 | 2 | 0 |

There are $7\binom{12}{4}$ committees of the first type and $\binom{12}{3}\binom{7}{2}$ of the second. The answer is $7\binom{12}{4} + \binom{12}{3}\binom{7}{2} = 7(495) + (220)(21) = 8085$.

17. (a) [BB] $\binom{10}{8} = 45$; (b) $\binom{7}{5} = 21$.

(c) [BB] She can answer either three of the first five and all of the last five or four of the first five and four of the last five. The answer is $\binom{5}{3} + \binom{5}{4}\binom{5}{4} = 10 + 25 = 35$.

(d) The number where she answers three of the last five and all of the first five is $\binom{5}{3} = 10$. The number where she answers four of the last five is $\binom{5}{4}\binom{5}{4} = 25$. The number where she answers five of the last five is the number of ways to select three of the first five questions, $\binom{5}{3} = 10$. The answer is $10 + 25 + 10 = 45$.

18. By the Principle of Inclusion-Exclusion, the number of elements in at least one of the given sets is

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| &= 5(23) - \tbinom{5}{2}(10) + \tbinom{5}{3}(4) - \tbinom{5}{4}(1) + 0 \\ &= 115 - 100 + 40 - 5 = 50.\end{aligned}$$

Thus, $52 - 50 = 2$ elements belong to none.

19. (a) [BB] A line segment can be formed between any two points and if two pairs of points are different so are the lines they determine. The number of different line segments is $\binom{10}{2} = 45$.

(b) For any choice of three points, a triangle is determined. Also, if two triples are different, so are the triangles they determine. So the number of triangles is $\binom{10}{3} = 120$.

(c) A triangle with $A$ as a vertex can be formed for any two points different from $A$, so the number is $\binom{9}{2} = 36$.

20. (a) [BB] A triangle is determined by three of the 12 vertices. The answer is $\binom{12}{3} = 220$.

(b) It is impossible for three sides of the polygon to form a triangle. There are 12 ways in which a triangle can be formed involving exactly two (necessarily adjacent) sides of the polygon (because there are 12 choices for the vertex where these two sides meet).

We next count the number of triangles involving exactly one side of the polygon. There are 12 choices for that side and, for each of these choices, we have 8 possibilities for the third vertex (since only one side of the polygon is to be used). Hence, there are $12 \times 8 = 96$ triangles which use exactly one side of the polygon. The answer is $220 - 12 - 96 = 112$.

21. (a) [BB] There are $\binom{8}{2}$ possible line segments between vertices of an octagon, eight of which are the sides of the octagon. Hence, the number of diagonals is $\binom{8}{2} - 8 = 28 - 8 = 20$.

(b) There are $\binom{11}{2} = \frac{11(10)}{2} = 55$ line segments between pairs of vertices eleven of which are the sides. A loonie has $55 - 11 = 44$ diagonals.

22. (a) There are $\binom{n}{2} = \frac{n(n-1)}{2}$ line segments between pairs of vertices $n$ of which are sides. There are $\frac{n(n-1)}{2} - n = \frac{n^2-n-2n}{2} = \frac{n(n-3)}{2}$ diagonals.

(b) We must have $3n = \frac{1}{2}n(n-3)$, so $n^2 - 3n = 6n$, $n^2 - 9n = 0$, $n(n-9) = 0$. Since $n \neq 0$, we have $n = 9$.

23. [BB] The product of $n$ consecutive natural numbers starting at $m$ is

$$m(m+1)(m+2)\cdots(m+n-1) = P(m+n-1, n) = n!\tbinom{m+n-1}{n}.$$

So $\frac{m(m+1)(m+1)\cdots(m+n-1)}{n!}$ is the integer $\binom{m+n-1}{n}$.

24. (a) [BB] $\binom{n}{k}\binom{n-k}{\ell} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{\ell!(n-k-\ell)!} = \frac{n!}{k!\ell!(n-k-\ell)!}$;

$\binom{n}{\ell}\binom{n-\ell}{k} = \frac{n!}{\ell!(n-\ell)!}\frac{(n-\ell)!}{k!(n-\ell-k)!} = \frac{n!}{\ell!k!(n-\ell-k)!}$. The two expressions are equal.

(b) [BB] To choose two teams as suggested, we may choose either team first. Choosing the team of size $k$ first, the number of possibilities is $\binom{n}{k}\binom{n-k}{\ell}$; choosing the team of size $\ell$ first, the number is $\binom{n}{\ell}\binom{n-\ell}{k}$. Certainly these numbers must be the same.

25. (a) $\binom{n-1}{k-1}+\binom{n-1}{k} = \dfrac{(n-1)!}{[(n-1)-(k-1)]!(k-1)!}+\dfrac{(n-1)!}{(n-1-k)!k!}$

$= \dfrac{(n-1)!}{(n-k)!(k-1)!}+\dfrac{(n-1)!}{(n-1-k)!k!}$

$= \dfrac{(n-1)!k+(n-1)!(n-k)}{(n-k)!k!}$

$= \dfrac{(n-1)![k+(n-k)]}{(n-k)!k!} = \dfrac{(n-1)!n}{(n-k)!k!} = \dfrac{n!}{(n-k)!k!} = \binom{n}{k}$

(b) Suppose you want to select $r$ people from a group of $n$ men and one woman. You can either select $r$ men, or $r-1$ men and the woman. Now the identity follows from the addition rule.

26. (a) $\binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1) = 2n^2-n$ ; $2\binom{n}{2}+n^2 = 2\frac{n(n-1)}{2}+n^2 = n(n-1)+n^2 = 2n^2-n$.

(b) $\binom{2n}{2}$ is the number of ways of choosing 2 objects from $2n$. From a different point of view, however, suppose we divide the objects into two sets of $n$ objects each. To choose 2 objects we can choose from the first set (in $\binom{n}{2}$ ways), from the second set (also in $\binom{n}{2}$ ways) or one from each set (in $n(n)$ ways). So the number of ways of choosing 2 objects from $2n$ is $\binom{n}{2}+\binom{n}{2}+n^2 = 2\binom{n}{2}+n^2$.

## Exercises 7.3

1. [BB] (a) $\frac{1}{36}$; (b) $\frac{3}{36} = \frac{1}{12}$.

(c) Since the possibilities are $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $(5,6)$ and their reverses, the answer is $\frac{10}{36} = \frac{5}{18}$.

(d) There are $3 \times 3 = 9$ cases where the odd number is first and 9 more where the even number is first. The answer is $\frac{18}{36} = \frac{1}{2}$.

2. (a) $\frac{1}{6}$; (b) $\frac{2}{6} = \frac{1}{3}$; (c) $1 - P(6) = 1 - \frac{1}{6} = \frac{5}{6}$; (d) $\frac{3}{6} = \frac{1}{2}$;

(e) Since the only possible number is 3, the answer is $\frac{1}{6}$.

3. (a) $\frac{1}{36}$

(b) The complement of this event is $\{(5,6),(6,5),(6,6)\}$, so the answer is $1 - \frac{3}{36} = \frac{11}{12}$.

(c) $70 = 2 \times 5 \times 7$. The possibilities are $(1,1)$, $(1,2)$, $(1,5)$, $(2,5)$ and their reverses. The answer is $\frac{7}{36}$.

(d) The possible sums are 5 and 10. The answer is $\frac{4+3}{36} = \frac{7}{36}$.

4. (a) The only possibility is $(5,5,5)$, so the answer is $\frac{1}{216}$.

(b) The complement of the event is $\{(6,6,5),(6,5,6),(5,6,6),(6,6,6)\}$, so the answer is $1 - \frac{4}{216} = \frac{53}{54}$.

(c) The prime numbers which can appear are 2, 3, 5, and these can be arranged in 3! ways. The answer is $\frac{6}{216} = \frac{1}{36}$.

(d) $\frac{6\times5\times4}{6\times6\times6} = \frac{5}{9}$.

5. [BB] (a) $\frac{1}{64}$; (b) $\frac{\binom{6}{2}}{64} = \frac{15}{64}$; (c) $\frac{1+\binom{6}{1}+\binom{6}{2}}{64} = \frac{11}{32}$;

(d) $1 - \frac{1}{64} = \frac{63}{64}$; (e) $\frac{\binom{6}{1}+\binom{6}{3}+\binom{6}{5}}{64} = \frac{1}{2}$.

6. (a) $\frac{1}{128}$; (b) $\frac{\binom{7}{5}}{128} = \frac{21}{128}$ (c) $1 - \frac{1+\binom{7}{1}}{128} = \frac{15}{16}$;

(d) $\frac{1+\binom{7}{1}}{128} = \frac{1}{16}$ (e) $\frac{\binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7}}{128} = \frac{1}{2}$

7. [BB] (a) There are $\binom{49}{6}$ outcomes in the sample space. For an outcome to be in the event described, we need to choose three from the six selected numbers and three from the 43 other numbers. The answer is $\frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \approx 0.018$.

(b) $\frac{\binom{6}{3}\binom{43}{3}+\binom{6}{4}\binom{43}{2}+\binom{6}{5}\binom{43}{1}+1}{\binom{49}{6}} \approx 0.019$. (c) $1 - \frac{1}{\binom{49}{6}} \approx 1.000$.

8. (a) $\frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} \approx 0.001$; (b) $1 - \frac{\binom{6}{5}\binom{43}{1}+1}{\binom{49}{6}} \approx 1.000$; (c) $\frac{\binom{43}{6}+\binom{6}{1}\binom{43}{5}}{\binom{49}{6}} \approx 0.849$.

9. [BB] (a) $\frac{4(4)}{12(12)} = \frac{1}{9}$; (b) $1 - \frac{1}{9} = \frac{8}{9}$; (c) $1 - \frac{8(8)}{12(12)} = \frac{5}{9}$; (d) $\frac{2(5)(4)}{12(12)} = \frac{5}{18}$;

(e) We will use the formula given in part (3) of Theorem 7.3.2. Let $A$ be the event "at least one white ball" and $B$ be "exactly one red ball." We saw in (c) that $P(A) = \frac{5}{9}$. Also $P(B) = \frac{2(5)(7)}{12(12)} = \frac{35}{72}$. Since $A \cap B$ is the event described in (d), $P(A \cap B) = \frac{5}{18}$. The required probability is $P(A \cup B) = \frac{5}{9} + \frac{35}{72} - \frac{5}{18} = \frac{55}{72}$.

10. (a) $\frac{3(3)}{12(12)} = \frac{1}{16}$; (b) $1 - \frac{1}{16} = \frac{15}{16}$; (c) $1 - \frac{9(9)}{12(12)} = \frac{7}{16}$; (d) $\frac{2(3)(4)}{12(12)} = \frac{1}{6}$;

(e) Let $A$ be "at least one blue ball" and $B$ be "at least one white ball." We saw in (c) that $P(A) = \frac{7}{16}$. Also, Exercise 9(c) says $P(B) = \frac{5}{9}$. Since $A \cap B$ is the event "one blue ball and one white ball," part (d) says $P(A \cap B) = \frac{1}{6}$. The required probability is $P(A \cup B) = \frac{7}{16} + \frac{5}{9} - \frac{1}{6} = \frac{119}{144}$.

11. [BB] (a) $\frac{4(3)}{12(11)} = \frac{1}{11}$; (b) $1 - \frac{1}{11} = \frac{10}{11}$; (c) $1 - \frac{8(7)}{12(11)} = \frac{19}{33}$; (d) $\frac{2(5)(4)}{12(11)} = \frac{10}{33}$;

(e) Let $A$ be "at least one white ball" and $B$ be "exactly one red ball." We saw in (c) that $P(A) = \frac{19}{33}$. Also, $P(B) = \frac{2(5)(7)}{12(11)} = \frac{35}{66}$. Since $A \cap B$ is the event "one white ball and one red ball," part (d) says $P(A \cap B) = \frac{10}{33}$. The required probability is $P(A \cup B) = \frac{19}{33} + \frac{35}{66} - \frac{10}{33} = \frac{53}{66}$.

12. (a) $\frac{\binom{3}{2}}{\binom{12}{2}} = \frac{1}{22}$; (b) $1 - \frac{1}{22} = \frac{21}{22}$; (c) $1 - \frac{\binom{9}{2}}{\binom{12}{2}} = \frac{5}{11}$; (d) $\frac{\binom{3}{1}\binom{4}{1}}{\binom{12}{2}} = \frac{2}{11}$;

(e) Let $A$ be "at least one blue ball" and $B$ be "at least one white ball." We saw in (c) that $P(A) = \frac{5}{11}$. From Exercise 11(c), we know that $P(B) = \frac{19}{33}$. Since $A \cap B$ is the event "one blue ball and one white ball," part (d) says $P(A \cap B) = \frac{2}{11}$. The required probability is $P(A \cup B) = \frac{5}{11} + \frac{19}{33} - \frac{2}{11} = \frac{28}{33}$.

13. [BB] (a) $\frac{\binom{5}{4}\binom{7}{2}}{\binom{12}{6}} = \frac{5}{44}$ (b) $\frac{\binom{7}{4}\binom{5}{2} + \binom{7}{5}\binom{5}{1} + \binom{7}{6}}{\binom{12}{6}} = \frac{1}{2}$ (c) $1 - \frac{\binom{7}{3}\binom{5}{3}}{\binom{12}{6}} = \frac{41}{66}$

    (d) Let $A$ be "majority of Americans" and $B$ be "exactly four Canadians." From (a) and (b), we know $P(A) = \frac{1}{2}$ and $P(B) = \frac{5}{44}$. Since $A \cap B = \emptyset$, $A$ and $B$ are mutually exclusive, so the answer is $P(A \cup B) = \frac{1}{2} + \frac{5}{44} = \frac{27}{44}$.

14. (a) $\frac{\binom{5}{3}\binom{6}{2}}{\binom{11}{5}} = \frac{25}{77}$ (b) $\frac{\binom{6}{3}\binom{5}{2} + \binom{6}{4}\binom{5}{1} + \binom{6}{5}}{\binom{11}{5}} = \frac{281}{462}$ (c) $1 - \frac{\binom{6}{5} + \binom{6}{4}\binom{5}{1}}{\binom{11}{5}} = \frac{127}{154}$

    (d) Let $A$ be "exactly three Americans" and $B$ be "at most three Canadians." From (a) and (c), we know $P(A) = \frac{25}{77}$ and $P(B) = \frac{127}{154}$. The event "three Americans and two Canadians" is $A \cap B$, but this is just $A$. Thus $P(A \cup B) = \frac{25}{77} + \frac{127}{154} - \frac{25}{77} = \frac{127}{154}$.

15. [BB] (a) $\frac{11}{15}$ (b) $\frac{4}{15}$ (c) $\frac{1}{3}$ (d) $\frac{4}{15}$ (e) $\frac{3}{5}$

16. [BB] (a) $\frac{233}{500}$ (b) $\frac{33}{250}$ (c) $\frac{17}{500}$

17. Let $A$ be "divisible by 3," $B$ be "divisible by 5," and $C$ be "divisible by 7." We want

$$\begin{aligned} &P(A \cup B \cup C) - P(A \cap B \cap C) \\ &\qquad = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\qquad = \frac{1666}{5000} + \frac{1000}{5000} + \frac{714}{5000} - \frac{333}{5000} - \frac{238}{5000} - \frac{142}{5000} = \frac{2667}{5000}. \end{aligned}$$

18. (a) Using Exercise 15(a) of Section 6.1, we obtain $\frac{5845}{10{,}000} = \frac{1169}{2000}$.

    (b) This means divisible by $3(5)(7)(11) = 1155$, so the answer is $\frac{\left\lfloor \frac{10{,}000}{1155} \right\rfloor}{10{,}000} = \frac{8}{10{,}000} = \frac{1}{1250}$.

    (c) Using Exercise 15 and part (b), we obtain $\frac{191+8}{10{,}000} = \frac{199}{10{,}000}$.

19. [BB] $\frac{720}{1000} = \frac{18}{25}$

20. There are now 900 integers from which to choose and the number with distinct digits is $9(9)(8) = 648$, so the answer is $\frac{648}{900} = \frac{162}{225}$.

21. There are 15 such routes in total. The number which use only paved roads is $2 \times 4 = 8$. So the answer is $\frac{7}{15}$.

22. John has 360 possible meals. Of these, $4 \times 6 \times 1 \times 4 = 96$ fit the conditions. The answer is $\frac{96}{360} = \frac{4}{15}$.

23. [BB] The event mentioned has one choice for president, eight for treasure and then $8 \times 7$ choices for the other two positions. The answer is $\frac{8(8)(7)}{10(9)(8)(7)} = \frac{4}{45}$.

24. $\frac{5(4)(3)(6!)}{9!} = \frac{5}{42}$

25. (a) $\frac{61}{7!} = \frac{1}{7}$ (b) $1 - \frac{5!}{7!} = \frac{41}{42}$

    (c) The number which contain either $ab$ or $bgc$ is $6! + 5! - 4!$, so the answer is $1 - \frac{6!+5!-4!}{7!} = \frac{88}{105}$.

26. [BB] (a) $\frac{\binom{100}{20}\binom{20}{5}\binom{80}{15}}{\binom{100}{20}\binom{100}{20}} = \frac{\binom{20}{5}\binom{80}{15}}{\binom{100}{20}}$ (b) $\frac{\binom{100}{20}\binom{80}{20}}{\binom{100}{20}\binom{100}{20}} = \frac{\binom{80}{20}}{\binom{100}{20}}$

27. (a) $\dfrac{\binom{18}{4}}{\binom{20}{6}}$ (b) $\dfrac{\binom{18}{5}}{\binom{20}{6}}$ (c) $\dfrac{\binom{18}{6}}{\binom{20}{6}}$

28. [BB] $A$ and $B$ are not mutually exclusive: The ordered pair $(2,2)$ is in each.

[BB] $A$ and $C$ are mutually exclusive: $(2,2)$ is not in $C$ because $4 \not| 70$.

$A$ and $D$ are mutually exclusive because $2+2$ is not divisible by 5.

$B$ and $C$ are not mutually exclusive: The ordered pair $(2,5)$ is in each.

$B$ and $D$ are not mutually exclusive: The ordered pair $(2,3)$ is in each.

$C$ and $D$ are mutually exclusive: $D$ consists of the ordered pairs

$$(1,4),(2,3),(3,2),(4,1),(4,6),(5,5),(6,4)$$

and none of these is in $C$.

29. The result follows from the facts that $A = (A \cap B) \cup (A \cap B^c)$ and $(A \cap B) \cap (A \cap B^c) = \emptyset$.

30. (a) [BB] $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$. Note that when $n = 2$, $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = S$, so $B_2 = B_1^c$ and the identity here indeed generalizes that of Exercise 30.

(b) The result follows from the facts that $A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$, and that $(A \cap B_i) \cap (A \cap B_j) = \emptyset$ for $i \neq j$.

31. (a) [BB] Using Proposition 6.1.1(c), we have

$$P(A \setminus B) = \frac{|A \setminus B|}{|S|} = \frac{|A| - |A \cap B|}{|S|} = \frac{|A|}{|S|} - \frac{|A \cap B|}{|S|} = P(A) - P(A \cap B).$$

(b) From (a), $\begin{aligned} P(A \setminus B) &= P(A) - P(A \cap B) \\ &= P(A) - (P(A) + P(B) - P(A \cup B)) = P(A \cup B) - P(B). \end{aligned}$

(c) Using Proposition 6.1.1(d), we have

$$\begin{aligned} P(A \oplus B) &= \frac{|A \oplus B|}{|S|} = \frac{|A \cup B| - |A \cap B|}{|S|} = \frac{|A \cup B}{|S|} - \frac{|A \cap B|}{|S|} \\ &= P(A \cup B) - P(A \cap B) \\ &= (P(A) + P(B) - P(A \cap B)) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

(d) [BB] From (c), $\begin{aligned} P(A \oplus B) &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A) + P(B) - 2(P(A) + P(B) - P(A \cup B)) \\ &= 2P(A \cup B) - P(A) - P(B). \end{aligned}$

(e) [BB] $P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$.

(f) Using (e), $P(A \cap (B \cup C))$

$$\begin{aligned} &= (P(A) + P(B) - P(A \cup B)) + (P(A) + P(C) - P(A \cup C)) - P(A \cap B \cap C) \\ &= 2P(A) + P(B) + P(C) - P(A \cup B) - P(A \cup C) - P(A \cap B \cap C). \end{aligned}$$

(g) $\begin{aligned} P(A \cup (B \cap C) &= P((A \cup B) \cap (A \cup C)) \\ &= P(A \cup B) + P(A \cup C) - P(A \cup B \cup C). \end{aligned}$

(h) Using (g), $P(A \cup (B \cap C))$

$$= P(A \cup B) + P(A \cup C) - P(A \cup B \cup C)$$
$$= (P(A) + P(B) - P(A \cap B)) + (P(A) + P(C) - P(A \cap C)) - P(A \cup B \cup C)$$
$$= 2P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(A \cup B \cup C).$$

(i) [BB] $P((A \oplus B) \cap C) = \dfrac{|(A \oplus B) \cap C|}{|S|}$

$$= \frac{|A \cap C| + |B \cap C| - 2|A \cap B \cap C|}{|S|}, \quad \text{using Exercise 22(a)}$$
$$= P(A \cap C) + P(B \cap C) - 2P(A \cap B \cap C).$$

(j) $P(A \oplus B \oplus C) = \dfrac{|A \oplus B \oplus C|}{|S|}$

$$= \frac{|A| + |B| + |C| - 2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 4|A \cap B \cap C|}{|S|}$$

using Exercise 22(b)

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C) + 4P(A \cap B \cap C).$$

32. Let $A$ be the event "number of heads is greater than number of tails." Since $n$ is odd, it is not possible for the number of heads to equal the number of tails, so $A^c$ is "number of tails is greater than number of heads." Since $P(H) = P(T) = \frac{1}{2}$, we must have $P(A) = P(A^c)$. Since $P(A) + P(A^c) = 1$, $2P(A) = 1$ and $P(A) = \frac{1}{2}$, as required.

## Exercises 7.4

1. [BB] (a) $P(2) + P(4) + P(6) = \frac{1}{8} + \frac{1}{6} + \frac{5}{24} = \frac{1}{2}$ (b) $P(5) + P(6) = \frac{7}{24}$

2. (a) [BB] $P(2,6) + P(3,5) + P(4,4) + P(5,3) + P(6,2)$
   $= 2(\frac{1}{8})(\frac{5}{24}) + 2(\frac{1}{6})(\frac{1}{12}) + (\frac{1}{6})(\frac{1}{6}) = \frac{31}{288}$
   (b) $2(\frac{1}{12})(\frac{5}{24}) + (\frac{5}{24})^2 = \frac{5}{64}$
   (c) Using the result of Exercise 1(a), we obtain $(\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$.
   (d) [BB] $2(\frac{1}{4})(\frac{1}{12}) + 2(\frac{1}{8})(\frac{5}{24}) = \frac{3}{32}$
   (e) $2(\frac{1}{4})(\frac{1}{12}) + 2(\frac{1}{8})(\frac{5}{24}) + 2(\frac{1}{4})(\frac{5}{24}) = \frac{19}{96}$

3. [BB] We have $P(H) = 4P(T)$ and $P(H) + P(T) = 1$. So $5P(T) = 1$, giving $P(T) = \frac{1}{5}$ and $P(H) = \frac{4}{5}$.

4. We have $[P(T)]^2 = 2P(H)P(T)$. This means $P(T) = 0$ or $P(T) = 2P(H)$. If $P(T) = 0$, then $P(H) = 1$. If $P(T) = 2P(H)$, then $3P(H) = 1$, so $P(H) = \frac{1}{3}$ and $P(T) = \frac{2}{3}$.

5. [BB] We have $P(1)+2P(1)+3P(1)+4P(1)+5P(1)=1$, so $15P(1)=1$, $P(1)=\frac{1}{15}$, $P(2)=\frac{2}{15}$, $P(3)=\frac{1}{5}$, $P(4)=\frac{4}{15}$, and $P(5)=\frac{1}{3}$.

6. We have $P(1)+2P(1)+4P(1)+8P(1)+16P(1)=1$, so $31P(1)=1$, $P(1)=\frac{1}{31}$, $P(2)=\frac{2}{31}$, $P(3)=\frac{4}{31}$, $P(4)=\frac{8}{31}$, and $P(5)=\frac{16}{31}$.

7. [BB] (a) $\binom{6}{4}(\frac{1}{4})^4(\frac{3}{4})^2=\frac{135}{4096}$ (b) $\binom{6}{5}(\frac{1}{4})^5(\frac{3}{4})+(\frac{1}{4})^6=\frac{19}{4096}$ (c) $1-(\frac{3}{4})^6=\frac{3367}{4096}$

8. (a) $\binom{5}{2}(\frac{3}{5})^2(\frac{2}{5})^3=\frac{144}{625}$ (b) $(\frac{2}{5})^5+\binom{5}{1}(\frac{2}{5})^4(\frac{3}{5})=\frac{272}{3125}$ (c) $1-(\frac{2}{5})^5=\frac{3093}{3125}$

9. [BB] (a) $A\cap B$ is "three or four heads appear," so $P(A\cap B)=\left[\binom{5}{3}+\binom{5}{4}\right](\frac{1}{2})^5=\frac{15}{32}$. Also, $P(A)=\left[\binom{5}{3}+\binom{5}{4}+1\right](\frac{1}{2})^5=\frac{1}{2}$ and $P(B)=1-(\frac{1}{2})^5=\frac{31}{32}$. So $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}=\frac{15}{31}$.

   (b) $P(B\mid A)=\dfrac{P(B\cap A)}{P(A)}=\frac{15}{16}$

   (c) Since $A\cap C=\emptyset$, $P(A\cap C)=0$, so $P(A\mid C)=0$.

   (d) $P(C\mid A)=\dfrac{P(C\cap A)}{P(A)}=0$

   (e) Since $B\cap C=C$, $P(B\mid C)=\dfrac{P(B\cap C)}{P(C)}=1$.

   (f) $P(C\mid B)=\dfrac{P(C\cap B)}{P(B)}=\dfrac{P(C)}{P(B)}=\dfrac{\binom{5}{2}}{31}=\frac{10}{31}$.

10. (a) $A\cap B$ is the event "the total is eight or nine," so $P(A\cap B)=\frac{5+4}{36}=\frac{1}{4}$. Also, $P(A)=\frac{5+4+3+2+1}{36}=\frac{5}{12}$ and $P(B)=1-\frac{3+2+1}{36}=\frac{5}{6}$. So $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}=\frac{3}{10}$.

    (b) $P(B\mid A)=\dfrac{P(B\cap A)}{P(A)}=\frac{3}{5}$.

    (c) $A\cap C$ is the event "the total is nine or eleven," so $P(A\cap C)=\frac{4+2}{36}=\frac{1}{6}$. Also $P(C)=\frac{1}{2}$, so $P(A\mid C)=\dfrac{P(A\cap C)}{P(C)}=\frac{1}{3}$.

    (d) $P(C\mid A)=\dfrac{P(C\cap A)}{P(A)}=\frac{2}{5}$.

    (e) $B\cap C$ is the event "the total is 3, 5, 7, or 9, so $P(B\cap C)=\frac{16}{36}=\frac{4}{9}$ and $P(B\mid C)=\dfrac{(P(B\cap C)}{P(C)}=\frac{8}{9}$.

    (f) $P(C\mid B)=\dfrac{P(C\cap B)}{P(B)}=\frac{8}{15}$.

11. [BB] $A\cap B$ is the event "a 6 appears." So $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{\frac{5}{24}}{\frac{7}{24}}=\frac{5}{7}$, and $P(B\mid A)=\dfrac{\frac{5}{24}}{\frac{1}{2}}=\frac{5}{12}$. $A$ and $B$ are not independent because $P(A)P(B)=\frac{1}{2}(\frac{7}{24})=\frac{7}{48}$ while $P(A\cap B)=\frac{5}{24}$. $A$ and $B$ are not mutually exclusive because $A\cap B\neq\emptyset$.

12. Here $A \cap B = \emptyset$, so $P(A \cap B) = 0$. So $P(A \mid B) = 0$ and $P(B \mid A) = 0$. $A$ and $B$ are not independent because $P(A)P(B) = \frac{5}{64}(\frac{19}{96})$ while $P(A \cap B) = 0$. $A$ and $B$ are mutually exclusive because $A \cap B = \emptyset$.

13. [BB] Let $A$ be the event "a 1 appears on the first die" and $B$ the event "a 1 appears on the second die." $A$ and $B$ are not mutually exclusive because $(1,1)$ belongs to $A \cap B$. However, $P(A) = \frac{1}{6} = P(B)$ and $P(A \cap B) = \frac{1}{36} = P(A)P(B)$, so $A$ and $B$ are independent.

14. $A$ and $B$ are independent and mutually exclusive if and only if $A \cap B = \emptyset$ and either $P(A) = 0$ or $P(B) = 0$. To see this, note first that if these conditions hold, then $A$ and $B$ are certainly mutually exclusive and they are independent because $P(A \cap B) = 0 = P(A)P(B)$. Conversely, if $A$ and $B$ are mutually exclusive, then $A \cap B = \emptyset$, so $P(A)P(B) = P(A \cap B) = 0$, implying $P(A) = 0$ or $P(B) = 0$.

15. (a) [BB] Here $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{2}$. So $P(A \cap B) \neq P(A)P(B)$.
    (b) [BB] Here $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(A \cap B) = \frac{3}{8}$, so $P(A \cap B) = P(A)P(B)$.
    (c) In general, $A$ can occur in $\binom{n}{n-1} + 1 = n + 1$ ways, so $P(A) = \frac{n+1}{2^n}$. In general, $B$ can occur in $2^n - 2$ ways (the only outcomes not in $B$ are "all heads" and "all tails"), so $P(B) = \frac{2^n-2}{2^n}$. Finally, $A \cap B$ is the event "exactly $n-1$ heads appear," so $P(A \cap B) = \dfrac{\binom{n}{n-1}}{2^n} = \frac{n}{2^n}$. Consider

$$\begin{aligned} P(A)P(B) - P(A \cap B) &= \frac{(n+1)(2^n-2)}{2^{2n}} - \frac{n}{2^n} \\ &= \frac{(n+1)(2^n-2) - 2^n n}{2^{2n}} = \frac{2^n - 2(1+n)}{2^{2n}} = \frac{2^{n-1} - (1+n)}{2^{2n-1}}. \end{aligned}$$

    Since the graph of the function defined by $f(x) = 2^{x-1}$ lies above the graph of that defined by $g(x) = 1 + x$ for $x > 3$ (they intersect at $x = 3$ and the exponential function grows more rapidly), we have shown that $P(A)P(B) \neq P(A \cap B)$ for $n > 3$.

16. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .8 + .6 - .92 = .48$ and $P(A)P(B) = .8(.6) = .48$. Since $P(A \cap B) = P(A)P(B)$, $A$ and $B$ are independent.

17. [BB] $P(A^c)P(B^c) = (1 - P(A))(1 - P(B))$

$$\begin{aligned} &= 1 - P(A) - P(B) + P(A)P(B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \quad \text{since } A \text{ and } B \text{ are independent} \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - P(A \cup B) = P((A \cup B)^c) = P(A^c \cap B^c). \end{aligned}$$

18. Exercise 30 asserted that $P(A) = P(A \cap B) + P(A \cap B^c)$. So

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \quad \text{since } A \text{ and } B \text{ are independent} \\ &= P(A)(1 - P(B)) = P(A)P(B^c). \end{aligned}$$

19. Letting $A$ and $B$ be the events of Brian's and Jennifer's passing Math 2320, respectively, we have $P(A) = .93$, $P(B) = .95$ and $P(A \cap B^c) = .04$. Using Exercise 30 of Section 7.3, $P(A \cap B) = P(A) - P(A \cap B^c) = .89$. Since $P(A)P(B) = .8835 \neq P(A \cap B)$, $A$ and $B$ are not independent. Brian's performance depends on Jennifer's.

20. [BB] We use induction on $n$, the strong form. For $n = 1$, the result is clear. If $n = 2$, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$. Now assume $n > 2$ and the result holds for all integers $k$ with $1 \leq k < n$. We have

$$\begin{aligned} &P(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &\qquad = P((A_1 \cup A_2 \cup \cdot \cup A_{n-1}) \cup A_n) \\ &\qquad \leq P(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) + P(A_n) && \text{using the } k = 2 \text{ case} \\ &\qquad \leq (P(A_1) + P(A_2) + \cdots + P(A_{n-1}) + P(A_n) && \text{using the } k = n-1 \text{ case.} \end{aligned}$$

By the Principle of Mathematical Induction, the result follows.

21. We use induction on $n$, the strong form. For $n = 1$, the result is clear. If $n = 2$, $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) \geq P(A_1) + P(A_2) - 1$. Now assume $n > 2$ and the result holds for all integers $k$ with $1 \leq k < n$. We have

$$\begin{aligned} &P(A_1 \cap A_2 \cap \cdots \cap A_n) \\ &\quad = P((A_1 \cap A_2 \cap \cdot \cap A_{n-1}) \cap A_n) \\ &\quad \geq P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) + P(A_n) - 1 && \text{using the } k = 2 \text{ case} \\ &\quad \geq (P(A_1) + P(A_2) + \cdots + P(A_{n-1}) - (n-2) + P(A_n) - 1 && \text{using the } k = n-1 \text{ case} \\ &\quad = P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1). \end{aligned}$$

By the Principle of Mathematical Induction, the result follows.

22. (a) $\dfrac{365(364)(363)\cdots(365-(n-1))}{(365)^n}$

(b) When $n = 23$, the fraction in (a) is .492703 and this value is closer to .5 than with any other $n$.

23. ${}_nP_x$ is the probability of the event "a person aged $x$ survives to age $x + n$" while ${}_mP_{x+n}$ is the probability of the event "a person aged $x + n$ survives to age $x + n + m$. The intersection of these events is "a person aged $x$ survives to age $x + n + m$," because that is precisely what happens when both events occur. The formula ${}_{n+m}P_x = ({}_nP_x)({}_mP_{x+n})$ is equivalent to assuming that these events are independent.

24. [BB] (a) ${}_{50}P_{20} = ({}_{30}P_{20})({}_{20}P_{50}) = .8(.7) = 0.56$

(b) ${}_{30}P_{40} = ({}_{10}P_{40})({}_{20}P_{50}) = .9(.7) = 0.63$

(c) $({}_{10}P_{40})({}_{20}P_{20}) = ({}_{30}P_{20})$, so ${}_{20}P_{20} = \frac{.8}{.9} = 0.\dot{8}$

25. (a) ${}_{50}P_{40} = ({}_{10}P_{40})({}_{40}P_{50}) = .9(.5) = 0.45$

(b) $({}_{30}P_{20})({}_{40}P_{50}) = ({}_{70}P_{20})$, so ${}_{30}P_{20} = \frac{.3}{.5} = 0.6$

(c) ${}_{20}P_{20}({}_{10}P_{40}) = ({}_{30}P_{20})$ so, using (b), ${}_{20}P_{20} = \frac{.6}{.9} = 0.\dot{6}$.

26. [BB] Since the lives are assumed independent, the first condition says ${}_{10}P_{20}({}_{10}P_{30}) = .9$, that is, ${}_{20}P_{20} = .9$. We are also give that ${}_{15}P_{20} = .94$ and ${}_{60}P_{20} = .55$. We want ${}_5P_{35} - {}_{45}P_{35}$. Now ${}_5P_{35}({}_{15}P_{20}) = {}_{20}P_{20}$, so ${}_5P_{35} = \frac{.9}{.94}$. Also, ${}_{45}P_{35}({}_{15}P_{20}) = {}_{60}P_{20}$, so ${}_{45}P_{35} = \frac{.55}{.94}$. The answer is $\frac{.9-.55}{.94} = .372$.

27. We are told that ${}_{40}P_{20} = .95$. The second statement can be rewritten as $.8(1 - {}_{60}P_{20}) = {}_{40}P_{20} - {}_{60}P_{20}$. This can be rewritten as ${}_{60}P_{20} = \frac{{}_{40}P_{20} - .8}{.2} = .75$. The third statement says that ${}_{15}P_{60}{}_{15}P_{75} = .3$, which is equivalent to ${}_{30}P_{60} = .3$. We now have ${}_{70}P_{20} = {}_{40}P_{20}\,{}_{30}P_{60} = .95(.3) = .285$. But ${}_{70}P_{20} = {}_{60}P_{20}\,{}_{10}P_{80}$, so ${}_{10}P_{80} = \frac{.285}{.75} = .38$.

28. We use induction on $n$, the strong form. If $n = 2$, the right hand side of the equation is just $P(A_1)P(A_2 \mid A_1) = P(A_1)\dfrac{P(A_2 \cap A_1}{P(A_1)} = P(A_2 \cap A_1)$, which is the left hand side, so the statement is true. Now assume $n > 2$ and the result holds for all $k$, $1 \leq k < n$. We have

$$\begin{aligned}
&P(A_1 \cap A_2 \cap \cdots \cap A_n) \\
&\quad = P((A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cap A_n) \\
&\quad = P(A_1 \cap A_2 \cap \cdots \cap A_{n-1})P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \text{ by the } k = 2 \text{ case} \\
&\quad = [P(A_1)P(A_2 \mid A_1) \cdots P(A_{n-1} \mid A_1 \cap A_2 \cap \cdots \cap A_{n-2})]\, P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})
\end{aligned}$$

by the $k = n - 1$ case, and this is the right hand side. By the Principle of Mathematical Induction, the result follows.

29. [BB] Let $A$, $B$, $C$ be the events of eating at Leonce's, Harold's, and Vegan Delights, respectively, and let $D$ be the event a customer is satisfied. We are given that $P(A) = .4$, $P(B) = .35$, $P(C) = .25$, $P(D \mid A) = .85$, $P(D \mid B) = .9$, $P(D \mid C) = .95$.

    (a) $P(D \cap C) = P(C)P(D \mid C) = .25(.95) = .2375$.

    (b) $P(B \cap D^c) = P(B) - P(B \cap D) = P(B) - P(B)P(D \mid B) = .35(1 - .9) = .035$.

    (c) $P(D) = P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C)$

    $$= .4(.85) + .35(.9) + .25(.95) = .8925.$$

    (d) $P(A \mid D) = \dfrac{P(A)P(D \mid A)}{P(D)} = \dfrac{.4(.85)}{.8925} = .381$, using (c).

    (e) $P(D^c \mid C^c) = \dfrac{P(D^c \cap C^c)}{P(C^c)} = \dfrac{P((D \cup C)^c}{1 - P(C)} = \dfrac{1 - P(D \cup C)}{1 - P(C)}$

    $= \dfrac{1 - (P(D) + P(C) - P(D \cap C))}{1 - P(C)} = \dfrac{1 - (.8925 + .25 - .2375)}{.75} = .127$, using (a) and (c).

30. In part (d) of Exercise 29, we saw that $P(A \mid D) = .381$. Similarly, $P(B \mid D) = \dfrac{P(B)P(D \mid B)}{P(D)} = \dfrac{.35(.9)}{.8925} = .353$. Also, $P(C \mid D) = \dfrac{P(C)P(D \mid C)}{P(D)} = \dfrac{.25(.95)}{.8925} = .266$. It is most likely the satisfied customer ate at Leonce's.

31. Let $A$, $B$, $C$ be the events of attending Rarer Birds, Codroy Vampire Project or La Vie d'un Elan, respectively, and let $D$ be the event of purchasing popcorn. We are given that $P(A) = .5$, $P(B) = .3$, $P(C) = .2$, $P(D \mid A) = .7$, $P(D \mid B) = .9$, $P(D \mid C) = .4$.

    (a) $P(C \cap D) = P(C)P(D \mid C) = .2(.4) = .08$.

    (b) $P(A \cap D^c) = P(A) - P(A \cap D) = P(A) - P(A)P(D \mid A) = .5(1 - .7) = .15$.

    (c) $\begin{aligned} P(D) &= P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C) \\ &= .5(.7) + .3(.9) + .2(.4) = .7. \end{aligned}$

    (d) $P(D^c \mid A \cup B) = P(D^c \mid C^c) = \dfrac{P(D^c \cap C^c)}{P(C^c)} = \dfrac{P((D \cup C)^c}{P(C^c)} = \dfrac{1 - P(D \cup C)}{1 - P(C)}$

    $= \dfrac{1 - (P(D) + P(C) - P(D \cap C))}{1 - P(C)} = \dfrac{1 - (.7 + .2 - .08)}{.8} = .225$, using (a) and (c).

(e) $P(C \mid D) = \dfrac{P(C)P(D \mid C)}{P(D)} = \dfrac{.2(.4)}{.7} = .114$, using (c).

## Exercises 7.5

1. [BB] Think of the dolls as identically colored marbles and the shelves as boxes.
   The answer is $\binom{30+7-1}{30} = \binom{36}{30}$.

2. (a) $5^{15}$
   (b) There are $\binom{15}{8}$ ways to select the questions to be answered correctly. Now the other seven questions must be answered incorrectly; there are $4^7$ ways in which this can be done since there are four wrong answers to each question. The answer is $\binom{15}{8}4^7$.

3. [BB] Each outcome can be signaled by dropping five identical marbles into six boxes labeled 1–6. The answer is $\binom{6+5-1}{5} = \binom{10}{5} = 252$.

4. [BB] $\binom{10+4-1}{10} = \binom{13}{10} = 286$

5. (a) Each bunch is specified by dropping six marbles into five boxes, each labeled with the color of a rose. The answer is $\binom{5+6-1}{6} = \binom{10}{6} = 210$.
   (b) There is only one additional rose to be selected after the mandatory five different colored roses are selected. There are only five possible bunches.

6. (a) [BB] The varieties are boxes. A selection can be indicated by dropping marbles into boxes. There are $\binom{27}{8} = 2{,}220{,}075$ possibilities.
   (b) The number of selections which exclude the cherry centre variety is $\binom{26}{8} = 1{,}562{,}275$, so the number which include at least one cherry centre is $\binom{27}{8} - \binom{26}{8} = 657{,}800$.
   (c) We determine the number of selections which involve none of the caramel centre variety, the number which include exactly one of this type, the number which include exactly two of this type, and add. The answer is $\binom{26}{8} + \binom{25}{7} + \binom{24}{6} = 2{,}177{,}571$.

7. (a) We wish to arrange 12 different books in a line, so the answer is 12!.
   (b) There are 3! ways of arranging the subjects in order. Then there are 5! ways of arranging the mathematics books, 3! ways of arranging the physics books and 4! ways of arranging the chemistry books. The answer is $3!(5!3!4!) = 103680$.
   (c) [BB] Now we wish to arrange 12 books, three of which are the same. This is the number of ways to rearrange the 12 letters, three of which are the same and the others all different. The answer is $\frac{12!}{3!} = 79{,}833{,}600$.
   (d) Now we have 12 marbles, three colored red, three colored blue and the other six colored with six different colors. The answer is $\frac{12!}{3!3!} = 13{,}305{,}600$.

8. There are $\binom{18}{6}$ ways of selecting the six books and, for each such selection, four possible students to receive the six. Next, there are $\binom{12}{2}$ ways to select two books from the twelve that remain and three possible students to receive them. There are now $\binom{10}{5}$ ways to select five books for one of the two remaining students. The other student gets the remaining five books. The answer is $\binom{18}{6}(4)\binom{12}{2}(3)\binom{10}{5} = \frac{12(18!)}{6!2!5!5!}$.

9. (a) There are $\binom{20}{5}$ ways for Philadelphia to select its players and then $\binom{15}{5}$ ways for Boston to choose, then $\binom{10}{5}$ ways for Miami, after which Toronto's players are determined. The answer is $\binom{20}{5}\binom{15}{5}\binom{10}{5} = \frac{20!}{5!5!5!5!} = 11{,}732{,}745{,}024$.

   (b) Each answer to this question gives rise to 4! answers to part (a) since for each division into four teams, there are 4! ways to assign them to four different cities. So the answer here is $\frac{20!}{5!5!5!5!4!} = 488{,}864{,}376$.

10. There are $\binom{12}{6}$ ways in which San Diego can choose its players, then $\binom{6}{4}$ ways for Houston to choose, after which Toronto gets the remaining two players. By the multiplication rule, there are altogether $\binom{12}{6}\binom{6}{4} = 924(15) = 13860$ possible drafts.

11. (a) [BB] Select the boxes for the red balls in one of $\binom{60}{10}$ ways, then the boxes for the white balls in one of $\binom{50}{10}$ ways and finally the boxes for the blue balls in one of $\binom{40}{10}$ ways. The answer is $\binom{60}{10}\binom{50}{10}\binom{40}{10} = \frac{60!}{10!10!10!30!}$.

    (b) There are $\binom{60+10-1}{10}$ ways to select the boxes for the balls of each color. The answer is $\binom{69}{10}^3$.

12. $\binom{20}{2}\binom{18}{2}\binom{16}{8} = \frac{20!}{2!2!8!8!} = 374{,}130{,}900$

13. [BB] $\binom{13}{1}\binom{12}{2}\binom{10}{5} = \frac{13!}{1!2!5!5!} = 216{,}216$

14. (a) [BB] $\frac{10!}{2!3!} = 302{,}400$ (b) $\frac{9!}{2!2!2!} = 45{,}360$ (c) $\frac{9!}{4!2!2!} = 3780$ (d) $\frac{10!}{2!3!3!} = 50{,}400$

15. [BB] This is the number of permutations of 30 flags with ten of one color, five of another, seven of another and eight of another. The answer is $\frac{30!}{10!5!7!8!}$.

16. There are $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = P(30,5)$ ways to assign the single rooms. There are then $\binom{25}{10}$ ways to select the ten students for the double rooms and $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$ ways to assign the rooms. There are then $\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}$ ways to assign the remaining 15 students to triple rooms. The answer is

$$P(30,5)\binom{25}{10}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3} = \frac{30!}{2!2!2!2!2!3!3!3!3!3!}.$$

17. (a) Label the boxes $x$, $y$, $z$. Then any way of putting ten marbles into the boxes corresponds to a solution of $x + y + z = 10$ in nonnegative integers.

    (b) By part (a), the answer is $\binom{12}{10} = 66$.

    (c) The solutions to $x + y + z + u + v = 19$ in nonnegative integers are in 1-1 correspondence with the number of ways to put 19 identical marbles into five boxes labeled $x, y, z, u, v$. This number is $\binom{23}{19} = 8855$.

18. (a) [BB] Given any solution $(x_1, x_2, x_3, x_4)$ of (*) with $x_1 \geq 8$, then $(x_1 - 8, x_2, x_3, x_4)$ is a solution of (**) with $x_1 - 8, x_2, x_3, x_4$ nonnegative integers. Conversely, given a solution $(x_1, x_2, x_3, x_4)$ of (**) in nonnegative integers, then $(x_1 + 8, x_2, x_3, x_4)$ is a solution to (*) with $x_1 + 8 \geq 8$. The number of solutions to (**) in nonnegative integers corresponds to the number of ways to put 13 identical marbles into four boxes labeled $x_1, x_2, x_3, x_4$. This number is $\binom{16}{13} = 560$.

    (b) Given any solution $(x_1, x_2, x_3, x_4)$ of (*) with $x_1, x_2 \geq 8$, then $(x_1 - 8, x_2 - 8, x_3, x_4)$ is a solution of (***) with $x_1 - 8, x_2 - 8, x_3, x_4 \geq 0$. Conversely, given any solution $(x_1, x_2, x_3, x_4)$ of (***) with all variables nonnegative, $(x_1 + 8, x_2 + 8, x_3, x_4)$ is a solution of (*) with $x_1 + 8, x_2 + 8 \geq 8$. The number of solutions of (***) in nonnegative integers corresponds to the number of ways to put 5 identical marbles into four boxes labeled $x, y, z, u$. This number is $\binom{8}{5} = 56$.

(c) Let $A_1$ be the set of solutions to (*) with $x_1 \geq 8$. Define $A_2$, $A_3$ and $A_4$ similarly. The question asks for the number of elements in ${A_1}^c \cap {A_2}^c \cap {A_3}^c \cap {A_4}^c = (A_1 \cup A_2 \cup A_3 \cup A_4)^c$ (by one of the laws of De Morgan). By the Principle of Inclusion-Exclusion,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

By part (a), $|A_1| = 560$. Similarly, each of the four terms in $\sum_i |A_i|$ is 560. By part (b), $|A_1 \cap A_2| = 56$ and similarly, each of the $\binom{4}{2} = 6$ terms in $\sum_{i<j} |A_i \cap A_j|$ is 56. Now $A_1 \cap A_2 \cap A_3$ is the set of solutions to (*) with $x_1, x_2, x_3 \geq 8$ and $z \geq 0$. There are no such solutions! Thus, $|A_1 \cup A_2 \cup A_3 \cup A_4| = 4(560) - 6(56) = 1904$. Since the number of solutions to (*) in nonnegative integers is $\binom{24}{21} = 2024$, the number of solutions with each variable at most seven is $2024 - 1904 = 120$.

(d) We proceed as in part (c). Let $A_i$ be the set of all solutions to the given equation in which variable $x_i \geq 11$. The question asks for $|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c| = |(A_1 \cup A_2 \cup A_3 \cup A_4)^c|$. Now the solutions to

(*) $\qquad x_1 + x_2 + x_3 + x_4 = 35$

in which $x_1 \geq 11$ are in 1–1 correspondence with the solutions to

(**) $\qquad x_1 + x_2 + x_3 + x_4 = 24$

in nonnegative integers since any solution to (*) with $x_1 \geq 11$ becomes a solution to (**) by replacing $x_1$ by $x_1 - 11$. So $|A_1|$ is the number of solutions to (**), namely, $\binom{27}{24}$. Similarly $|A_i| = \binom{27}{24}$ for all $i$. The solutions to (*) in which both $x_1$ and $x_2$ are $\geq 11$ are in 1–1 correspondence with the solutions to

(***) $\qquad x_1 + x_2 + x_3 + x_4 = 13$

in nonnegative integers since any solution $(x_1, x_2, x_3, x_4)$ to (*) with $x_1, x_2 \geq 11$ gives rise to the solution $(x_1 - 11, x_2 - 11, x_3, x_4)$ to (***). Thus, $|A_1 \cap A_2| = \binom{16}{13}$. Similarly, $|A_i \cap A_j| = \binom{16}{13}$ for all distinct $i, j$. Finally, since the solutions to (*) in which $x_1, x_2, x_3 \geq 11$ are in 1–1 correspondence with solutions to

(****) $\qquad x_1 + x_2 + x_3 + x_4 = 2$

we obtain that $|A_i \cap A_j \cap A_k| = \binom{5}{2}$ for all distinct triples $i, j, k$. It is trivial that $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$. Thus $|A_1 \cup A_2 \cup A_3 \cup A_4|$

$$= \textstyle\sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$
$$= 4\tbinom{27}{24} - \tbinom{4}{2}\tbinom{16}{13} + \tbinom{4}{3}\tbinom{5}{2} = 4(2925) - 6(560) + 4(10) = 8380.$$

Since there are $\binom{38}{35} = 8436$ solutions to (*) in nonnegative integers, the number of solutions in which each $x_i \leq 10$ is $8436 - 8380 = 56$.

19. $\binom{20-(1+2+3+4)+4-1}{20-(1+2+3+4)} = \binom{13}{10} = 286$

## Exercises 7.6

1. [BB] $D_6 = 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$
$= 6! - 6! + 6 \cdot 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 6 \cdot 5 - 6 + 1$
$= 360 - 120 + 30 - 5 = 265$

$$\begin{aligned} D_7 &= 7!\left(1-\tfrac{1}{1!}+\tfrac{1}{2!}-\tfrac{1}{3!}+\tfrac{1}{4!}-\tfrac{1}{5!}+\tfrac{1}{6!}-\tfrac{1}{7!}\right) \\ &= 7!-7!+7\cdot6\cdot5\cdot4\cdot3-7\cdot6\cdot5\cdot4+7\cdot6\cdot5-7\cdot6+7-1 \\ &= 2520-840+210-42+6=1854 \end{aligned}$$

$$\begin{aligned} D_8 &= 8!\left(1-\tfrac{1}{1!}+\tfrac{1}{2!}-\tfrac{1}{3!}+\tfrac{1}{4!}-\tfrac{1}{5!}+\tfrac{1}{6!}-\tfrac{1}{7!}+\tfrac{1}{8!}\right) \\ &= 8!-8!+8\cdot7\cdot6\cdot5\cdot4\cdot3-8\cdot7\cdot6\cdot5\cdot4+8\cdot7\cdot6\cdot5-8\cdot7\cdot6 \\ &\qquad +8\cdot7-8+1=20{,}160-6720+1680-336+56-7=14{,}833 \end{aligned}$$

2. This is $D_{26}=26!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{1}{26!}\right)$.

3. [BB] This is $D_{11}=11!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots-\frac{1}{11!}\right)$.

4. $D_{50}$

5. (a) $D_7$ (b) [BB] $7!-D_7$ (c) 1

6. (a) $D_{20}$ (b) $20!-D_{20}$

   (c) [BB] There are 20 choices for the person who receives his or her own hat. For each choice, the 19 other people can get their hats in $D_{19}$ ways. Hence, the answer is $20D_{19}$.

   (d) This is the answer to (b) less the answer to (c); that is, $20!-D_{20}-20D_{19}$.

   (e) This means either no person receives his/her own hat or exactly one person receives his/her own hat or exactly two people receive their own hats. The answer is $D_{20}+20D_{19}+\binom{20}{2}D_{18}$.

7. (a) [BB] Let $A_1$ be the set of permutations of 1–9 such that 1 is in position 1; $A_3$ the set of permutations of 1–9 such that 3 is in position 3. Define $A_5$, $A_7$ and $A_9$ similarly. Then

$$\begin{aligned} \left|\bigcup_i A_i\right| &= \sum_i |A_i| - \sum_{i<j} |A_i\cap A_j| + \sum_{i<j<k} |A_i\cap A_j\cap A_k| \\ &\qquad - \sum_{i<j<k<\ell} |A_i\cap A_j\cap A_k\cap A_\ell| + |A_1\cap A_3\cap A_5\cap A_7\cap A_9| \\ &= 5(8!)-\binom{5}{2}7!+\binom{5}{3}6!-\binom{5}{4}5!+4!. \end{aligned}$$

   So the answer is $9!-[5(8!)-10(7!)+10(6!)-5(5!)+4!]=362{,}880-157{,}824=205{,}056$.

   (b) Arguing as in (a), we obtain $9!-4(8!)+\binom{4}{2}7!-\binom{4}{3}6!+5!=229{,}080$.

   (c) We can choose the four fixed integers in $\binom{9}{4}$ ways. Then there are $D_5$ ways of permuting the remaining five integers such that none is in its correct position. Hence, the answer is $\binom{9}{4}D_5=126(44)=5544$.

8. Any permutation of $n$ symbols either moves all the symbols (there are $D_n$ of these), or fixes precisely one symbol (there are $\binom{n}{1}D_{n-1}$ of these) or fixes precisely two symbols (there are $\binom{n}{2}D_{n-2}$ of these), ..., or fixes precisely $n-1$ symbols (there are $\binom{n}{n-1}D_1$ of these) or fixes all $n$ symbols (there is just one of these). By the addition rule, the number of permutations of $n$ symbols, $n!$, is the given sum.

9. Let $A_1$ be the set of permutations of $1,2,\ldots,n$ in which the number 1 is in position 1. Let $A_2$ be the set of permutations of $1,2,\ldots,n$ in which the number 2 is in position 2. In general, for $1\le i\le n$, let $A_i$ be the set of permutations of $1,2,\ldots,n$ in which number $i$ is in position $i$. The set of permutations in which at least one of the numbers $1,2,\ldots,n$ is left in its natural position is $\bigcup_{i=1}^n A_i$ and the complement of this set consists precisely of the derangements of $1,2,\ldots,n$. It is the cardinality of this complement that we want and, since there are $n!$ permutations altogether, this cardinality is $D_n=n!-\left|\bigcup_{i=1}^n A_i\right|$.

The Inclusion-Exclusion Principle (6.1.2) says

$$\Big|\bigcup_{i=1}^{n} A_i\Big| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Now $|A_1| = (n-1)!$ (1 goes into the first position, the remaining numbers go anywhere in the remaining $n-1$ positions) and, similarly, each $A_i$ has cardinality $(n-1)!$. The set $A_1 \cap A_2$ contains those permutations of $1, 2, \ldots, n$ in which 1 and 2 are in the right positions; there are $(n-2)!$ of these corresponding to the arbitrary placement of $3, 4, \ldots, n$ in the last $n-2$ positions of a permutation. Similarly, each of the $\binom{n}{2}$ terms $|A_i \cap A_j|$ is $(n-2)!$. Similar reasoning shows that each of the $\binom{n}{3}$ terms $|A_i \cap A_j \cap A_k|$ equals $(n-3)!$, and so on. We get

$$\begin{aligned}\Big|\bigcup_{i=1}^{n} A_i\Big| &= n(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n+1}\binom{n}{n}0! \\ &= n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n+1}\end{aligned}$$

and so

$$D_n = n! - \Big[n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n+1}\Big] = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\Big)$$

as required.

10. (a) [BB] Using Proposition 7.6.2,

$$\begin{aligned}D_n &= n! - n! + [n(n-1)(n-2)\cdots 3] - [n(n-1)(n-2)\cdots 4] + \cdots + (-1)^{n-1}n + (-1)^n \\ &= n[\,(n-1)(n-2)\cdots 3 - (n-1)(n-2)\cdots 4 + \cdots + (-1)^{n-1}(n-1)\,] + (-1)^n \\ &\equiv (-1)^n \pmod{n}.\end{aligned}$$

(b) [BB] This follows immediately from part (a).

11. (a) [BB] $(n-1)(D_{n-1} + D_{n-2})$

$$\begin{aligned}&= (n-1)\Big\{\,(n-1)!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\Big] \\ &\qquad +(n-2)!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-2}\frac{1}{(n-2)!}\Big]\,\Big\} \\ &= n(n-1)!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\Big] \\ &\qquad -(n-1)!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\Big] \\ &\qquad +(n-1)!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-2}\frac{1}{(n-2)!}\Big] \\ &= n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\Big) - n!(-1)^n\frac{1}{n!} - (n-1)!(-1)^{n-1}\frac{1}{(n-1)!} \\ &= D_n - (-1)^n - (-1)^{n-1} = D_n\end{aligned}$$

(b) $nD_{n-1}+(-1)^n = n\Big[(n-1)!(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^{n-1}\frac{1}{(n-1)!})\Big]+(-1)^n$
$= n!\Big(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^n\frac{1}{n!}\Big)-n!(-1)^n\frac{1}{n!}+(-1)^n$
$= D_n-(-1)^n+(-1)^n=D_n$

(c) $D_1=0$ while the values of $D_2$,$D_3$, $D_4$, $D_5$ appear in the text.
Using (b) $D_6=6D_5+1=6(44)+1=265$; $D_7=7D_6-1=7(265)-1=1854$;
$D_8=8D_7+1=8(1854)+1=14{,}833$; $D_9=9D_8-1=133{,}496$;
$D_{10}=10D_9+1=1{,}334{,}961$.
The information requested is tabulated below.

| $n$ | $D_n$ | $D_n/n!$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | .5 |
| 3 | 2 | .333333 |
| 4 | 9 | .375000 |
| 5 | 44 | .366667 |

| $n$ | $D_n$ | $D_n/n!$ |
|---|---|---|
| 6 | 265 | .368056 |
| 7 | 1854 | .367857 |
| 8 | 14,833 | .367882 |
| 9 | 133,496 | .367879 |
| 10 | 1,334,961 | .367879 |

Note that $\frac{1}{e}\approx .367879$.

12. By equation (4) of this section, $e^{-x}=1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\cdots$

and by (12) of Section 5.4, $\dfrac{1}{1-x}=1+x+x^2+x^3+\cdots$

Thus, $\dfrac{e^{-x}}{1-x} = \Big(1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\cdots\Big)\Big(1+x+x^2+x^3+\cdots\Big)$
$= 1+0x+(1-1+\dfrac{1}{2!})x^2+(1-1+\dfrac{1}{2!}-\dfrac{1}{3!})x^3+\cdots$
$+(1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!})x^n+\cdots$

and we note that for any $n\geq 0$, the coefficient of $x^n$ is $\frac{D_n}{n!}$ by Proposition 7.6.2.

## Exercises 7.7

1. (a) [BB] $(x+y)^6 = x^6+\binom{6}{1}x^5y+\binom{6}{2}x^4y^2+\binom{6}{3}x^3y^3+\binom{6}{4}x^2y^4+\binom{6}{5}xy^5+y^6$
$= x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

(b) [BB] $(2x+3y)^6 = (2x)^6+\binom{6}{1}(2x)^5(3y)+\binom{6}{2}(2x)^4(3y)^2+\binom{6}{3}(2x)^3(3y)^3$
$+\binom{6}{4}(2x)^2(3y)^4+\binom{6}{5}(2x)(3y)^5+(3y)^6$
$= 64x^6+6(32x^5)(3y)+15(16x^4)(9y^2)+20(8x^3)(27y^3)$
$+15(4x^2)(81y^4)+6(2x)(243y^5)+729y^6$
$= 64x^6+576x^5y+2160x^4y^2+4320x^3y^3+4860x^2y^4$
$+2916xy^5+729y^6$

2. $(a+4b)^5 = a^5+5a^4(4b)+10a^3(4b)^2+10a^2(4b)^3+5a(4b)^4+(4b)^5$
$= a^5+20a^4b+160a^3b^2+640a^2b^3+1280ab^4+1024b^5$

3. (a) $(2x^3-y^2)^8 = (2x^3)^8+\binom{8}{1}(2x^3)^7(-y^2)+\binom{8}{2}(2x^3)^6(-y^2)^2$
$+\binom{8}{3}(2x^3)^5(-y^2)^3+\binom{8}{4}(2x^3)^4(-y^2)^4+\binom{8}{5}(2x^3)^3(-y^2)^5$
$+\binom{8}{6}(2x^3)^2(-y^2)^6+\binom{8}{7}(2x^3)(-y^2)^7+(-y^2)^8$
$= 256x^{24}-8(128x^{21})y^2+28(64x^{18})y^4-56(32x^{15})y^6$
$+70(16x^{12})y^8-56(8x^9)y^{10}+28(4x^6)y^{12}-8(2x^3)y^{14}+y^{16}$
$= 256x^{24}-1024x^{21}y^2+1792x^{18}y^4-1792x^{15}y^6$
$+1120x^{12}y^8-448x^9y^{10}+112x^6y^{12}-16x^3y^{14}+y^{16}$

(b) [BB] $(2x^3-x^2)^8 = (2x^3)^8+\binom{8}{1}(2x^3)^7(-x^2)+\binom{8}{2}(2x^3)^6(-x^2)^2$
$+\binom{8}{3}(2x^3)^5(-x^2)^3+\binom{8}{4}(2x^3)^4(-x^2)^4+\binom{8}{5}(2x^3)^3(-x^2)^5$
$+\binom{8}{6}(2x^3)^2(-x^2)^6+\binom{8}{7}(2x^3)(-x^2)^7+(-x^2)^8$
$= 256x^{24}-8(128x^{23})+28(64x^{22})-56(32x^{21})$
$+70(16x^{20})-56(8x^{19})+28(4x^{18})-8(2x^{17})+x^{16}$
$= 256x^{24}-1024x^{23}+1792x^{22}-1792x^{21}+1120x^{20}$
$-448x^{19}+112x^{18}-16x^{17}+x^{16}$

(c) $\left(2x^3-\frac{1}{x^2}\right)^8 = (2x^3)^8+\binom{8}{1}(2x^3)^7(-\frac{1}{x^2})+\binom{8}{2}(2x^3)^6(-\frac{1}{x^2})^2+\binom{8}{3}(2x^3)^5(-\frac{1}{x^2})^3$
$+\binom{8}{4}(2x^3)^4(-\frac{1}{x^2})^4+\binom{8}{5}(2x^3)^3(-\frac{1}{x^2})^5+\binom{8}{6}(2x^3)^2(-\frac{1}{x^2})^6$
$+\binom{8}{7}(2x^3)(-\frac{1}{x^2})^7+(-\frac{1}{x^2})^8$
$= 256x^{24}-8(128x^{21})\frac{1}{x^2}+28(64x^{18})\frac{1}{x^4}-56(32x^{15})\frac{1}{x^6}$
$+70(16x^{12})\frac{1}{x^8}-56(8x^9)\frac{1}{x^{10}}+28(4x^6)\frac{1}{x^{12}}-8(2x^3)\frac{1}{x^{14}}+\frac{1}{x^{16}}$
$= 256x^{24}-1024x^{19}+1792x^{14}-1792x^9+1120x^4-\frac{448}{x}+\frac{112}{x^6}-\frac{16}{x^{11}}+\frac{1}{x^{16}}$

4. [BB] $\binom{12}{3}(x^3)^9(-2y^2)^3 = -1760x^{27}y^6$

5. (a) $x^{20}$, $20x^{19}$, $\binom{20}{2}x^{18}y^2 = 190x^{18}y^2$ (b) $\binom{20}{18}x^2y^{18} = 190x^2y^{18}$, $20xy^{19}$, $y^{20}$

(c) [BB] The binomial coefficients appear in the order $\binom{20}{0}$, $\binom{20}{1}$, ... . The coefficient appearing in the seventh term is $\binom{20}{6}$, so the seventh term is $\binom{20}{6}x^{14}y^6 = 38{,}760x^{14}y^6$. Similarly, the fifteenth term is $\binom{20}{14}x^6y^{14} = 38{,}760x^6y^{14}$.

(d) $\binom{20}{7} = 77{,}520$

6. (a) [BB] 17 terms

(b) [BB] There is a middle term (since 16 is even), namely, $\binom{16}{8}(2x)^8(-y)^8 = 12{,}870(256)x^8y^8 = 3{,}294{,}720x^8y^8$.

7. (a) 18 terms

   (b) There are two middle terms (since 17 is odd): $\binom{17}{8}x^9(3y)^8$ and $\binom{17}{9}x^8(3y)^9$.

8. [BB] The required term is $\binom{10}{7}(4x)^3(5y)^7 = 120(64x^3)(78125y^7)$. The coefficient is 600,000,000 .

9. [BB] The term in question is $\binom{17}{4}x^{13}(3y^2)^4 = 2380x^{13}(81y^8)$. The coefficient is $2380(81) = 192{,}780$.

10. The general term is $\binom{20}{k}x^{20-k}(-2x^{-2})^k = \binom{20}{k}(-2)^k(x^{20-3k})$.

    We want $20 - 3k = 5$, so $k = 5$ and the coefficient is $\binom{20}{5}(-2)^5 = -496{,}128$.

11. [BB] The general term is $\binom{18}{k}\left(\frac{3}{x}\right)^{18-k}(x^2)^k = \binom{18}{k}3^{18-k}x^{3k-18}$.

    We want $3k - 18 = 27$, so $k = 15$ and the coefficient is $\binom{18}{15}3^3 = (816)(27) = 22{,}032$.

12. The general term is $\binom{58}{k}(2x)^{58-k}\left(\frac{-3}{x^2}\right)^k$.

    We want $x^{58-k}(x^{-2})^k = x^{25}$; that is, $x^{58-3k} = x^{25}$. Thus, $58 - 3k = 25$, $3k = 33$, $k = 11$. The term we seek is

    $$\binom{58}{11}(2x)^{58-11}\left(\frac{-3}{x^2}\right)^{11} = \binom{58}{11}(2x)^{47}(-3)^{11}x^{-22}$$

    and the coefficient of $x^{25}$ is $-\binom{58}{11}2^{47}3^{11}$.

13. We use mathematical induction. If $n = 5$, $4^5 = 1024 > 625 = 5^4$ so the result is true. Now assume that $4^k > k^4$ for some integer $k \geq 5$. Then $4^{k+1} = 4 \cdot 4^k > 4 \cdot k^4$ by hypothesis. To prove the desired inequality $4^{k+1} > (k+1)^4$ it is, therefore, sufficient to prove that $4 \cdot k^4 \geq (k+1)^4$. We know that $(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$. Since $4 < k$, $4k^3 < k^4$ and since $6 < k^2$, $6k^2 < k^4$. Also, $4k + 1 < 4k + k = 5k \leq kk < k^4$. Thus, $(k+1)^4 < k^4 + k^4 + k^4 + k^4 = 4k^4$ as desired.

14. (a) [BB] If $n = 0$, $(1+\sqrt{2})^0 = 1 = 1 + 0\sqrt{2}$, so $x_0 = 1$, $y_0 = 0$.

    If $n = 1$, $(1+\sqrt{2})^1 = 1 + \sqrt{2}$, so $x_1 = 1$, $y_1 = 1$.

    If $n = 2$, $(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$ so $x_2 = 3$, $y_2 = 2$.

    If $n = 5$, $\begin{aligned}(1+\sqrt{2})^5 &= 1 + 5\sqrt{2} + 10(\sqrt{2})^2 + 10(\sqrt{2})^3 + 5(\sqrt{2})^4 + (\sqrt{2})^5 \\ &= 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2} = 41 + 29\sqrt{2}\end{aligned}$

    and so $x_5 = 41$, $y_5 = 29$.

    (b) We prove that, for $n$ even,

    $$\begin{aligned}x_n &= 1 + 2\tbinom{n}{2} + 4\tbinom{n}{4} + 8\tbinom{n}{6} + \cdots + 2^{n/2} \\ y_n &= n + 2\tbinom{n}{3} + 4\tbinom{n}{5} + 8\tbinom{n}{7} + \cdots + 2^{(n-2)/2}\tbinom{n}{n-1}\end{aligned}$$

    while for $n$ odd,

    $$\begin{aligned}x_n &= 1 + 2\tbinom{n}{2} + 4\tbinom{n}{4} + 8\tbinom{n}{6} + \cdots + 2^{(n-1)/2}\tbinom{n}{n-1} \\ y_n &= n + 2\tbinom{n}{3} + 4\tbinom{n}{5} + 8\tbinom{n}{7} + \cdots + 2^{(n-1)/2}\end{aligned}$$

Using the Binomial Theorem,

$$\begin{aligned}(1+\sqrt{2})^n &= 1+n\sqrt{2}+\tbinom{n}{2}(\sqrt{2})^2+\tbinom{n}{3}(\sqrt{2})^3+\tbinom{n}{4}(\sqrt{2})^4\\ &\quad +\tbinom{n}{5}(\sqrt{2})^5+\tbinom{n}{6}(\sqrt{2})^6+\cdots+(\sqrt{2})^n\\ &= 1+n\sqrt{2}+2\tbinom{n}{2}+2\sqrt{2}\tbinom{n}{3}+4\tbinom{n}{4}+4\sqrt{2}\tbinom{n}{5}\\ &\quad +8\tbinom{n}{6}+8\sqrt{2}\tbinom{n}{7}+\cdots+(\sqrt{2})^n\end{aligned}$$

from which the result follows.

(c) $x_{n+1}+y_{n+1}\sqrt{2} = (1+\sqrt{2})^{n+1} = (1+\sqrt{2})(1+\sqrt{2})^n = (1+\sqrt{2})(x_n+y_n\sqrt{2})$
$= x_n+y_n\sqrt{2}+\sqrt{2}x_n+2y_n = (x_n+2y_n)+(x_n+y_n)\sqrt{2}.$

Equating coefficients of 1 and $\sqrt{2}$, we have the desired formulas.

15. (a) Using the Binomial Theorem,

$$(\sqrt{2}-1)^n = (\sqrt{2})^n++\binom{n}{1}(\sqrt{2})^{n-1}(-1)+\binom{n}{2}(\sqrt{2})^{n-2}(-1)^2 + \cdots+\binom{n}{k}(\sqrt{2})^{n-k}(-1)^k+\cdots+(-1)^n.$$

When $n-k$ is even, $(\sqrt{2})^{n-k} = 2^{\frac{n-k}{2}}$, which is an integer. When $n-k$ is odd, $(\sqrt{2})^{n-k} = \sqrt{2}(\sqrt{2})^{n-k-1} = \sqrt{2}\sqrt{2}^{\frac{n-k-1}{2}}$ which is of the form $c\sqrt{2}$ for some integer $c$. It follows that every term in the expansion of $(\sqrt{2}-1)^n$ is either an integer or of the form $c\sqrt{2}$. Collecting similar terms gives the result.

(b) Since $1<2<4$, we know $1<\sqrt{2}<2$, hence $0<\sqrt{2}-1<1$. So $\lim_{n\to\infty}(\sqrt{2}-1)^n = 0$.

(c) Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some integers $a$ and $b$, with $b\neq 0$. Using part (a), $(\sqrt{2}-1)^n = x_n+y_n\frac{a}{b} = \frac{x_nb+y_na}{b}$. Using (b), we have $\lim_{n\to\infty}\frac{x_nb+y_na}{b} = 0$. This means that $\lim_{n\to\infty}(x_nb+y_na) = 0$. But $x_nb+y_na$ is an integer, for any $n$. Thus, there exists an integer $n_0$ such that $x_nb+y_na = 0$ for all $n\geq n_0$.
Now $y_n\neq 0$: otherwise, $0 = x_nb+y_na = x_nb$ would imply $x_n = 0$ too (since $b\neq 0$), contradicting $(\sqrt{2}-1)^n>0$. Thus $\frac{x_n}{y_n} = -\frac{a}{b}$ for all $n\geq n_0$. But this says that $x_n = -\sqrt{2}y_n$ and (a) then tells us that $(\sqrt{2}-1)^n = 0$ for all $n\geq n_0$. Thus $\sqrt{2} = 1$, a contradiction.

16. [BB] With the given interpretation, row $n$ of Pascal's triangle becomes the single number

$$10^n\binom{n}{0}+10^{n-1}\binom{n}{1}+\cdots+10\binom{n}{n-1}+1 = \sum_{r=0}^{n}10^{n-r}\binom{n}{r}.$$

So we are trying to prove that $\sum_{r=0}^{n}10^{n-r}\binom{n}{r} = 11^n$. This follows immediately from the Binomial Theorem since $11^n = (10+1)^n = \sum_{r=0}^{n}\binom{n}{r}(10)^{n-r}(1)^r$.

17. We use induction on $n$. For $n=0$, row zero is a single 1; this equals $\binom{0}{0}$, so the result holds. Now assume that the numbers in row $k$ are $\binom{k}{0},\binom{k}{1},\ldots,\binom{k}{k}$ and consider the numbers in row $k+1$. The first number is 1 which equals $\binom{k+1}{0}$; the last number is 1 which equals $\binom{k+1}{k+1}$. The second number is the sum of the numbers to the left and right immediately above it; that is, $\binom{k}{0}+\binom{k}{1}$ (by the induction

hypothesis). This sum is $\binom{k+1}{1}$, by (5). The third number is the sum of the numbers to the left and right immediately above it; that is, $\binom{k}{1}+\binom{k}{2}$. This sum is $\binom{k+1}{2}$ by (5). In general, for any $r$, $1 < r \leq k+1$, the $r$th number is the sum of the numbers to the left and right immediately above it. Using the induction hypothesis and the identity in 5, this sum is $\binom{k}{r-2}+\binom{k}{r-1}=\binom{k+1}{r-1}$. So the numbers in row $k+1$ are the binomial coefficients, as asserted.

18. If $n=1$, $x+y=\binom{1}{0}x^1y^0+\binom{1}{1}x^0y^1$, as required. Assume the result is true for $n=\ell$; that is, assume

$$(x+y)^\ell=\sum_{k=0}^{\ell}\binom{\ell}{k}x^{\ell-k}y^k.$$

Then $(x+y)^{\ell+1}=(x+y)(x+y)^\ell=(x+y)\displaystyle\sum_{k=0}^{\ell}\binom{\ell}{k}x^{\ell-k}y^k$

$$\begin{aligned}
&=\sum_{k=0}^{\ell}\binom{\ell}{k}x^{\ell+1-k}y^k+\sum_{k=0}^{\ell}\binom{\ell}{k}x^{\ell-k}y^{k+1}\\
&=x^{\ell+1}+\sum_{k=1}^{\ell}\binom{\ell}{k}x^{\ell+1-k}y^k+\sum_{k=0}^{\ell-1}\binom{\ell}{k}x^{\ell-k}y^{k+1}+y^{\ell+1}\\
&=x^{\ell+1}+\sum_{k=1}^{\ell}\binom{\ell}{k}x^{\ell+1-k}y^k+\sum_{k=1}^{\ell}\binom{\ell}{k-1}x^{\ell+1-k}y^k+y^{\ell+1}\\
&=x^{\ell+1}+\sum_{k=1}^{\ell}\left[\binom{\ell}{k}+\binom{\ell}{k-1}\right]x^{\ell+1-k}y^k+y^{\ell+1}\\
&=x^{\ell+1}+\sum_{k=1}^{\ell}\binom{\ell+1}{k}x^{\ell+1-k}y^k+y^{\ell+1}\quad\text{by identity (5)}\\
&=\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}x^{\ell+1-k}y^k.
\end{aligned}$$

By the Principle of Mathematical Induction, the proof is complete.

19. (a) $\displaystyle\sum_{k=2}^{6}\binom{k}{2}=\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}=1+3+6+10+15=35=\binom{7}{3}.$

The sum here is the sum of the entries of Pascal's triangle on the third southwest-northeast diagonal (/) beginning with the entry above and to the left of $35=\binom{7}{3}$ and moving up and right.

(b) The formula is true if $r = n$ (since then both sides equal 1), so we assume $r < n$. Then

$$\sum_{k=r}^{n}\binom{k}{r} = 1 + \sum_{k=r+1}^{n}\binom{k}{r} = 1 + \sum_{k=r+1}^{n}\left[\binom{k+1}{r+1} - \binom{k}{r+1}\right] \qquad \text{using identity (5)}$$

$$= 1 + \left[\binom{r+2}{r+1} - \binom{r+1}{r+1} + \binom{r+3}{r+1} - \binom{r+2}{r+1}\right.$$

$$+ \binom{r+4}{r+1} - \binom{r+3}{r+1} + \cdots$$

$$\left. + \binom{n}{r+1} - \binom{n-1}{r+1} + \binom{n+1}{r+1} - \binom{n}{r+1}\right]$$

$$= 1 + \left[-1 + \binom{n+1}{r+1}\right] = \binom{n+1}{r+1}$$

as desired. This formula says that any given entry (other than the first 1) in a row of Pascal's triangle is equal to the sum of the entries on the southwest-northeast diagonal (/) beginning with the entry above and to the left of the given one.

20. [BB] Consider $(x-y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-y)^k = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-1)^k y^k$. Setting $x = y = 1$, we obtain

$$0 = (1-1)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}(-1)^k 1^k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n.$$

21. (a) $1 + 2\binom{4}{1} + 4\binom{4}{2} + 8\binom{4}{3} + 16 = 1 + 2(4) + 4(6) + 8(4) + 16 = 81 = 3^4$.

(b) Setting $x = 1$, $y = 2$ in $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$, we obtain

$$3^n = (1+2)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}2^k = \sum_{k=0}^{n}2^k\binom{n}{k}.$$

22. (a) [BB] $\displaystyle\sum_{k=1}^{n}k\binom{n}{k} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \cdots + (n-1)\binom{n}{n-1} + n$

$$= n + n(n-1) + \frac{n(n-1)(n-2)}{2} + \frac{n(n-1)(n-2)(n-3)}{3!} + \cdots + n(n-1) + n$$

$$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)(n-3)}{3!} + \cdots + n - 1 + 1\right]$$

$$= n\left[1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + 1\right] = n2^{n-1}$$

using the result of Problem 34 in the last step.

(b) $\displaystyle\sum_{k=1}^{n}k^2\binom{n}{k} = \binom{n}{1} + 4\binom{n}{2} + 9\binom{n}{3} + \cdots + (n-2)^2\binom{n}{n-2} + (n-1)^2\binom{n}{n-1} + n^2$

$$= n + 2n(n-1) + 3\frac{n(n-1)(n-2)}{2} + \cdots + (n-1)^2\frac{n(n-1)}{2} + (n-1)^2 n + n^2$$
$$= n\left[1 + 2(n-1) + 3\frac{(n-1)(n-2)}{2} + \cdots + (n-2)^2\frac{(n-1)}{2} + (n-1)^2 + n\right]$$
$$= n\left[1 + 2\binom{n-1}{1} + 3\binom{n-1}{2} + \cdots + (n-2)\binom{n-1}{2}1 + (n-1)\binom{n-1}{1} + n\right]$$
$$= n\sum_{k=1}^{n-1}(k+1)\binom{n-1}{k} = n\left[\sum_{k=0}^{n-1}k\binom{n-1}{k} + \sum_{k=0}^{n-1}\binom{n-1}{k}\right]$$
$$= n[(n-1)2^{n-2} + 2^{n-1}] = n2^{n-2}(n-1+2) = n(n+1)2^{n-2}$$

using (a) and the result of Problem 34 in the second last line.

23. The remarks at the beginning of this section suggest that the terms of the Fibonacci sequence are special kinds of diagonal sums. Using Figures 7.8 and 7.6, for instance, we see that $2 = a_2 = \binom{2}{0} + \binom{1}{1}$, $3 = a_3 = \binom{3}{0} + \binom{2}{1}$, $5 = a_4 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2}$. We conjecture that

$$a_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots = \sum_k \binom{n-k}{k}$$

where we sum over all $k$ for which $n - k \geq k$. Using identity (5) and the strong form of the Principle of Mathematical Induction, we can verify this result. For $n = 1$, $a_1 = 1 = \binom{1}{0}$, while for $n = 2$, $a_2 = 2 = \binom{2}{0} + \binom{1}{1}$. Now assume that $k > 2$ and the result is true for all $n$ in the range $1 \leq n < k$. We wish to prove the result for $k$. Note that the identity we wish to prove has two different types of final terms depending on whether $n$ is even or odd. When $n$ is odd, the final term in the sum is $\binom{\frac{n+1}{2}}{\frac{n-1}{2}}$ while, if $n$ is even, the final term is $\binom{\frac{n}{2}}{\frac{n}{2}}$. We establish the odd and even cases separately.

**Case 1:** $n$ is odd.

In this case, we wish to prove that $a_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}}$.

But $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}}$

$$= \binom{n-1}{0} + \left[\binom{n-2}{0} + \binom{n-2}{1}\right] + \left[\binom{n-3}{1} + \binom{n-3}{2}\right] + \cdots + \left[\binom{\frac{n-1}{2}}{\frac{n-3}{2}} + \binom{\frac{n-1}{2}}{\frac{n-1}{2}}\right]$$
$$= \left[\binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{\frac{n-1}{2}}{\frac{n-1}{2}}\right] + \left[\binom{n-2}{0} + \binom{n-3}{1} + \cdots + \binom{\frac{n-1}{2}}{\frac{n-3}{2}}\right]$$

(Note how identity (5) was employed at the very first step.) Now applying the induction hypothesis to $n - 1$, the term in the first set of brackets here is $a_{n-1}$ and applying the induction hypothesis to $n - 2$, the other term is $a_{n-2}$.

**Case 2:** $n$ is even.

This time we wish to prove $a_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\frac{n}{2}}{\frac{n}{2}}$.

$$\text{But } \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\frac{n}{2}+1}{\frac{n}{2}-1} + \binom{\frac{n}{2}}{\frac{n}{2}}$$

$$= \binom{n-1}{0} + \left[\binom{n-2}{0} + \binom{n-2}{1}\right] + \cdots + \left[\binom{\frac{n}{2}}{\frac{n}{2}-2} + \binom{\frac{n}{2}}{\frac{n}{2}-1}\right] + \binom{\frac{n}{2}-1}{\frac{n}{2}-1}$$

$$= \left[\binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{\frac{n-1+1}{2}}{\frac{n-1-1}{2}}\right] + \left[\binom{n-2}{0} + \binom{n-3}{1} + \cdots + \binom{\frac{n-2}{2}}{\frac{n-2}{2}}\right]$$

$= a_{n-1} + a_{n-2}$, by the induction hypothesis

$= a_n$, as before.

In either case, we obtain $a_n = a_{n-1} + a_{n-2}$, the recurrence relation which we know defines the Fibonacci numbers. By the Principle of Mathematical Induction we conclude that this is true for all $n \geq 1$.

24. (a) There is just one way to select 0 books from $n$.

(b) We prove this by contradiction. So assume the result is not true. Then we can find $k \geq \frac{n}{2}+1$ such that $\begin{bmatrix} n \\ k \end{bmatrix} \neq 0$; that is, a selection of the required type can be made. Label the books $1, 2, \ldots, n$ from left to right and first assume $n$ is even. If we partition the books into the $\frac{n}{2}$ sets $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{n-1, n\}$, the Pigeon-Hole Principle immediately tells us that two of the $k$ books selected must be in the same set, contradicting the fact that no two adjacent books are to be chosen.

Now assume $n$ is odd. Since $k$ is an integer, the condition $k \geq \frac{n}{2}+1$ actually means $k \geq \frac{n+1}{2}+1$ in this case. Partitioning the books into the $\frac{n+1}{2}$ sets $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{n-2, n-1\}$, $\{n\}$, the Pigeon-Hole Principle again gives a contradiction.

(c) We will see that it is necessary to start the induction with both $n = 1$ and $n = 2$. If $k = 0$, the result follows from part (a). The only other possibility in these cases is $k = 1$, and here direct verification easily shows that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 = \binom{1-1+1}{1}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 = \binom{2-1+1}{1}$.

Now assume $n \geq 3$ and that the result holds for all $m < n$ and all $k < \frac{m}{2} + 1$. Assume that $k < \frac{n}{2} + 1$. Using (a), we may also assume that $k \geq 1$. The selection of $k$ books from $n$ with no two adjacent can be accomplished in two mutually exclusive ways, either the first book on the left is selected (in which case the second book cannot be chosen and we must choose $k-1$ books from $n-2$ with no two adjacent) or the first book is not selected (in which case we must select $k$ books from $n-1$ with no two adjacent). We obtain the formula

$$(*) \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Note that the conditions $k \geq 1$ and $n \geq 3$ guarantee that both terms on the right hand side make sense. First assume that $k < \frac{n-1}{2} + 1$. Since $0 \leq k-1 < \frac{n-2}{2} + 1$ is always true under this assumption, the induction hypothesis can be applied to both terms on the right. We obtain

$$\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n-2-(k-1)+1}{k-1} + \binom{n-1-k+1}{k} = \binom{n-k}{k-1} + \binom{n-k}{k} = \binom{n-k+1}{k}$$

by (5).

Finally assume that $k \geq \frac{n-1}{2} + 1$. Then $\begin{bmatrix} n-1 \\ k \end{bmatrix} = 0$ by (b), so $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}$. Since $k < \frac{n}{2} + 1$, it follows that $k = \frac{n-1}{2} + 1$. But the induction hypothesis then says that

$$\begin{bmatrix} n-2 \\ k-1 \end{bmatrix} = \binom{n-2-\frac{n-1}{2}+1}{\frac{n-1}{2}} = \binom{\frac{n-1}{2}}{\frac{n-1}{2}} = 1.$$

Since

$$\binom{n-k+1}{k} = \binom{n-(\frac{n-1}{2}+1)+1}{\frac{n-1}{2}+1} = \binom{\frac{n+1}{2}}{\frac{n+1}{2}} = 1$$

also, we again conclude that $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n-k+1}{k}$. By the Principle of Mathematical Induction, we have the result.

25. As suggested, we create a triangle like that in Fig. 7.8, replacing $\binom{n}{k}$ with $\begin{bmatrix} n \\ k \end{bmatrix}$ and obtain the following.

$$\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & & 1 & & & & & & & \\
 & & & & 1 & & 2 & & 0 & & & & & & \\
 & & & 1 & & 3 & & 1 & & 0 & & & & & \\
 & & 1 & & 4 & & 3 & & 0 & & 0 & & & & \\
 & 1 & & 5 & & 6 & & 1 & & 0 & & 0 & & & \\
1 & & 6 & & 10 & & 4 & & 0 & & 0 & & 0 & & \\
1 & & 7 & & 15 & & 10 & & 1 & & 0 & & 0 & & 0 \\
 & & & & & & \vdots & & & & & & & &
\end{array}$$

As proved in Exercise 24(a), the left-most entry in each row is 1. Also, the result in (b) gives a substantial number of 0s at the end of each row.

The result in (c) is more interesting. It tells us, for example, that $\begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \ldots$ are just the binomial coefficients $\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \ldots$. But $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5$, $\begin{bmatrix} 6 \\ 2 \end{bmatrix} = 10$, $\begin{bmatrix} 7 \\ 3 \end{bmatrix} = 10$ are also the elements on one of the downward left to right diagonals. In this way, we see that the present triangle is just a skewed version of Pascal's (with diagonals replacing rows), padded with 0s.

Recalling that any entry in Pascal's triangle is the sum of two entries directly above it, we conclude that the same holds here, but to diagonals instead of rows. The reader is invited to reflect on the identity $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}$ used in the solution to Exercise 24(c).

# Chapter 7 Review

1. For $n = 1$, $P(2,1) = 2$ is divisible by $2^1$, so the result is true. Assume that $k \geq 1$ and that $2^k \mid P(2k,k)$. We have

$$\begin{aligned}
P(2(k+1),k+1) &= \frac{[2(k+1)]!}{(k+1)!} = \frac{(2k+2)!}{(k+1)!} \\
&= \frac{(2k+2)(2k+1)(2k)!}{(k+1)!} \\
&= \frac{2(k+1)(2k+1)(2k)!}{(k+1)k!} = \frac{2(2k+1)(2k)!}{k!} = 2(2k+1)P(2k,k).
\end{aligned}$$

Since $P(2k,k)$ is divisible by $2^k$ (by the induction hypothesis), we see that $P(2(k+1),k+1)$ is divisible by $2^{k+1}$, the desired result with $n = k+1$. By the Principle of Mathematical Induction $2^n \mid P(2n,n)$ for all $n \geq 1$.

2. As a group, the girls can be arranged in 6! ways. Then, together with the boys, they form 8 "units" to be arranged in a row. Since the girls cannot appear first, there are seven choices for the first position in the row. Then there are six choices for the last position (again, not the girls). There are 6! ways to fill out the remaining positions, so the answer is $7(6)(6!)(6!) = 21{,}772{,}800$.

3. The answer is zero! In a circle, if the six girls are together, then the boys must also be together.

4. Let $A$ be the set of permutations containing $aeb$ and $B$ the set of permutations containing $bcf$. We want $|A \cup B|$. By the Inclusion-Exclusion Principle, this is $|A| + |B| - |A \cap B|$. Now $|A| = 4!$ ($aeb$ is a single unit in a string of four "units"), $|B| = 4!$ and $|A \cap B| = 2!$ (if both $aeb$ and $bcf$ appear, then $aebcf$ must appear). Thus $|A \cup B| = |A| + |B| - |A \cap B| = 24 + 24 - 2 = 46$.

5. (a) Let $W$, $P$ and $J$ denote, respectively, those permutations in which "wendy", "patrick" and "josh" appear. Then $|W| = 22!$ (arrange the 22 blocks consisting of "wendy" and the 21 letters of the alphabet excluding "w,e,n,d,y"). Similarly, $|P| = 20!$, $|J| = 23!$, $|W \cap P| = 16!$, $|W \cap J| = 19!$, $|P \cap J| = 17!$ and $|W \cap P \cap J| = 13!$. By the Principle of Inclusion-Exclusion, the answer is

$$\begin{aligned} |W \cup P \cup J| &= |W| + |P| + |J| - |W \cap P| - |W \cap J| - |P \cap J| + |W \cap P \cap J| \\ &= 22! + 20! + 23! - 16! - 19! - 17! + 13!. \end{aligned}$$

   (b) Let $W$, $P$ and $J$ denote, respectively, those permutations in which "wendy", "patrick" and "john" appear. Then $|W| = 22!$ (arrange the 22 blocks consisting of "wendy" and the 21 letters of the alphabet excluding "w,e,n,d,y"). Similarly, $|P| = 20!$, $|J| = 23!$, $|W \cap P| = 16!$, and $|P \cap J| = 17!$. Note, however, that $|W \cap J| = 0 = |W \cap P \cap J|$. By the Principle of Inclusion-Exclusion, the answer is

$$\begin{aligned} |W \cup P \cup J| &= |W| + |P| + |J| - |W \cap P| - |W \cap J| - |P \cap J| + |W \cap P \cap J| \\ &= 22! + 20! + 23! - 16! - 17!. \end{aligned}$$

6. We must select three kings from four and then another two cards from 48. The answer is $\binom{4}{3}\binom{48}{2} = (4)\frac{48(47)}{2} = 4512$.

7. (a) We want the number of committees which contain exactly six women, which is $\binom{9}{6}\binom{8}{1} = 8\binom{9}{6}$, or exactly seven women, which is $\binom{9}{7}$. Since the corresponding sets are pairwise disjoint, the number required is the sum $8\binom{9}{6} + \binom{9}{7} = 708$.

   (b) Let $A$ be the set of committees which contain Bob and $B$ the set of committees which contain Alice. The question asks for $|A \oplus B|$. We have $|A| = \binom{16}{6} = |B|$ and $|A \cap B| = \binom{15}{5}$. The answer is $|A| + |B| - 2|A \oplus B| = \binom{16}{6} + \binom{16}{6} - 2\binom{15}{5} = 10{,}010$.

   Alternatively, the number of committees which contain both Bob and Alice is $\binom{15}{5}$, while the number that contain neither Bob nor Alice is $\binom{15}{7}$. So the answer is $\binom{17}{7} - \binom{15}{7} - \binom{15}{5}$.

8. (a) $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 369{,}600$ (b) $\frac{369{,}600}{4!} = 15{,}400$

9. (a) $\binom{20}{7}$ (b) $\binom{20}{7}\binom{7}{4}$

10. The groups will have sizes 2, 2, 6 or 2, 3, 5 or 2, 4, 4 or 3, 3, 4. There are

- $\frac{1}{2}\binom{10}{6}\binom{4}{2} = 1260$ groups of 2, 2 and 6; the $\frac{1}{2}$ to avoid counting the groups {Peter, Scott}, {Amanda, Amy} twice;

- $\binom{10}{5}\binom{5}{3} = 2520$ groups of 2, 3 and 5;
- $\frac{1}{2}\binom{10}{4}\binom{6}{4} = 1575$ groups of 2, 4 and 4; and
- $\frac{1}{2}\binom{10}{4}\binom{6}{3} = 2100$ groups of 3, 3 and 4.

By the addition rule, $630 + 2520 + 1575 + 2100 = 6825$ groups can be formed.

11. She can purchase the books in $\binom{12}{8}$ ways. For each of these purchases, there are three possibilities. In Case 1, one boy friend receives 4 books and each of the other boys just one. There are $5 \times \binom{8}{4} \times 4! = 8400$ ways this can occur. In Case 2, one boy receives three books, one boy receives two and three boys receive one each. There are $5 \times 4 \times \binom{8}{3} \times \binom{5}{2} \times 3! = 67200$ ways in which this can occur. In Case 3, three boys each receive two books and two each receive one book. This can occur in $\binom{5}{3} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times 2! = 50400$ ways. The answer is $\binom{12}{8}(8400+67200+50400) = 495 \times 126{,}000 = 6{,}237{,}000$.

12. Seat the Israeli delegation at the table first. There are 4 choices for the delegation to the right of Israel and then 3 for the delegation to the left. This leaves 7 delegations which can be seated in 7! ways. The answer is $4 \times 3 \times 7! = 60{,}480$ ways.

13. (a) A journey from $A$ to $B$ requires seven decisions—down or right—three of which are down. The answer is $\binom{7}{3}$.
    (b) The answer is $\binom{m+n}{m} = \binom{m+n}{n}$.

14. This is just like Problem 27. The answer is $\binom{12+7-1}{12} = \binom{18}{12}$.

15. (a) With no restrictions, the answer is the number of orderings of 12 objects: 12!.
    (b) Arrange the Canadian ships in 3! ways and the Icelandic ships in 2! ways; then arrange all nine units in 9! ways. The answer is 3!2!9!.
    (c) Arrange the Canadian and Icelandic ships in 5! ways; then arrange eight units in 8! ways. The answer is 5!8!.
    (d) There are $2(11!)$ ways in which the ships enter the harbour one after the other. The alternative happens in $12! - 2(11!)$ ways.

16. (a) This is the number of ways to put $r = 10$ identical marbles into $n = 7$ boxes: the answer is $\binom{10+7-1}{10} = \binom{16}{10}$.
    (b) This is the number of ways to put $r = 7$ identical marbles into $n = 6$ boxes: the answer is $\binom{7+6-1}{7} = \binom{12}{7}$.
    (c) This is the number of ways to put $r = 3$ identical marbles into $n = 7$ boxes: the answer is $\binom{3+7-1}{3} = \binom{9}{3}$.
    (d) $\binom{16}{10} - \binom{9}{3}$

17. This is just like Problem 29. The answer is $\frac{7!}{2!2!} = 1260$.

18. (a) There are six cases to consider. The rings all go on one finger in $4 \times 5! = 480$ ways. Four go on one finger and one on another in $4(3)(5!) = 1440$ ways. Three go on one and two on another in $4(3)(5!) = 1440$ ways. Three go on one and the other two go on different fingers in $4\binom{3}{2}(5!) = 1440$ ways. Two go on two fingers and one goes on a third in $\binom{4}{2}\binom{2}{1}(5!) = 1440$ ways while two go on one finger and each of the remaining three go on one finger in $4(5!) = 480$ ways. The total number of possibilities is 6720.

(b) This is the problem of putting $r = 5$ identical marbles into $n = 4$ boxes. The answer is $\binom{4+5-1}{5} = 56$.

19. The general power of $x$ in the binomial expansion is $x^{20-k}(x^{-2})^k = x^{20-3k}$. We have $20 - 3k = 5$ for $k = 5$. The coefficient in question is $\binom{20}{5}(-2)^5$.

20. The general term of the binomial expansion in question is $\binom{12}{k}(16x^2)^{12-k}\left(-\frac{1}{2x}\right)^k$, the exponent of $x$ here being $(24 - 2k) - k = 24 - 3k$. We want $24 - 3k = -6$, so $3k = 30$ and $k = 10$. The term in question is

$$\binom{12}{10}(16x^2)^2\left(-\frac{1}{2x}\right)^{10} = \frac{12(11)}{2}(2^8x^4)\frac{1}{2^{10}x^{10}} = 66\frac{1}{2^2}x^{-6}.$$

The coefficient is $\frac{66}{4} = \frac{33}{2}$.

21. There are two statements here. We prove each by mathematical induction. Consider first the inequality,

$$(*) \qquad 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 < \frac{n^4}{4}.$$

For $n = 1$, the left side is $0^3$, the right side is $\frac{1}{4}$. Since $0 < \frac{1}{4}$, the statement is true. Now assume that $k \geq 1$ and the statement is true for $n = k$; that is, assume

$$1^3 + 2^3 + 3^3 + \cdots + (k-1)^3 < \frac{k^4}{4}.$$

We must prove that the statement is true for $n = k + 1$; that is,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 < \frac{(k+1)^4}{4}.$$

We have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 &= 1^3 + 2^3 + 3^3 + \cdots + (k-1)^3 + k^3 \\ &< \frac{k^4}{4} + k^3, \qquad \text{using the induction hypothesis,} \\ &= \frac{k^4 + 4k^3}{4} < \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} = \frac{(k+1)^4}{4}, \end{aligned}$$

which is the desired result. By the Principle of Mathematical Induction, inequality $(*)$ holds for all $n \geq 1$.

Next we prove that

$$(**) \qquad 1^3 + 2^2 + 3^3 + \cdots + n^3 > \frac{n^4}{4}$$

for all $n \geq 1$. Since $1 > \frac{1}{4}$, the inequality is true for $n = 1$. Now assume $k \geq 1$ and the inequality is true for $n = k$; that is,

$$1^3 + 2^2 + 3^3 + \cdots + k^3 > \frac{k^4}{4}.$$

With $n = k + 1$, we have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 &= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\ &> \frac{k^4}{4} + (k+1)^3, \qquad \text{using the induction hypothesis,} \\ &= \frac{k^4 + 4(k+1)^3}{4} \\ &= \frac{k^4 + 4k^3 + 12k^2 + 12k + 4}{4} \\ &> \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} = \frac{(k+1)^4}{4}, \end{aligned}$$

which is the desired result. By the Principle of Mathematical Induction, inequality $(**)$ holds for all $n \geq 1$.

22. (a) $\binom{n}{k}$ is the number of ways of choosing $k$ objects from a set of $n$ objects. We must show that $\binom{n-1}{k-1} + \binom{n-1}{k}$ counts the same set of possibilities. Let $X$ be one of the $n$ objects. The number of ways to choose $k$ objects, including $X$, is the number of ways to select $k-1$ objects from $n-1$, namely, $\binom{n-1}{k-1}$. The number of ways to choose $k$ objects, excluding $X$, is the number of ways to select $k$ objects from $n-1$, namely, $\binom{n-1}{k}$. Since the two cases—include $X$, exclude $X$—are mutually exclusive and exhaustive (one must occur), the total number of ways of selecting $k$ objects from $n$ is the sum $\binom{n-1}{k-1} + \binom{n-1}{k}$, as desired.

(b) Use the identity in (a) twice to obtain

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} = \left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right] + \left[\binom{n-2}{k-1} + \binom{n-2}{k}\right] \\ &= \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}. \end{aligned}$$

Thus $\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2} = 2\binom{n-2}{k-1}$ is even.

23. This is just $\displaystyle\sum_{k=0}^{n} \binom{n}{k}(3^{n-k})(-1)^k = (3-1)^n = 2^n$. (Note that a different expansion for $2^n$ was found in Problem 34.)

24. If $n = 1$, $\binom{2}{1} = 2$ is even. If $n = 2$, $\binom{4}{2} = 6$ is even, so assume $n \geq 3$. Then

$$\begin{aligned} \binom{2n}{n} &= \binom{2n-1}{n} + \binom{2n-1}{n-1} \\ &= \binom{2n-2}{n} + \binom{2n-2}{n-1} + \binom{2n-2}{n-1} + \binom{2n-1}{n-2} \\ &= \binom{2n-2}{n} + \binom{2n-2}{n-2} + 2\binom{2n-2}{n-1} \end{aligned}$$

so it suffices to prove that $\binom{2n-2}{n} + \binom{2n-2}{n-2}$ is even, which is the case because these numbers are equal: $\binom{2n-2}{n} = \dfrac{(2n-2)!}{n!(n-2)!} = \binom{2n-2}{n-2}$.

25. Both students are right and we can see this with several applications of formula (5), p. 350: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Since $\binom{30}{19} + \binom{30}{20} = \binom{31}{20}$, X's answer is $\binom{30}{19} + \binom{31}{20}$. Since $\binom{32}{20} = \binom{31}{19} + \binom{31}{20}$ and $\binom{31}{19} = \binom{30}{18} + \binom{30}{19}$, Y's answer is $\binom{31}{19} + \binom{31}{20} - \binom{30}{18} = \binom{31}{20} + \binom{30}{19}$, in agreement with X.

26. The required probability is the sum

$$(*) \qquad P(\text{one head}) + P(3 \text{ heads}) + \cdots + P(k \text{ heads}) = \frac{\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{k}}{2^n}$$

where $k = \begin{cases} n & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$

If $n$ is odd (so $k = n-1$), we can use the identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ to rewrite the expression on the right of (*) as

$$\frac{\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-2} + \binom{n-1}{n-1}}{2^n}.$$

Problem 34 of Section 7.7 tells us this is $\dfrac{2^{n-1}}{2^n} = \frac{1}{2}$. If $n$ is even (so $k = n$), the identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ allows us to rewrite the right side of (*) as

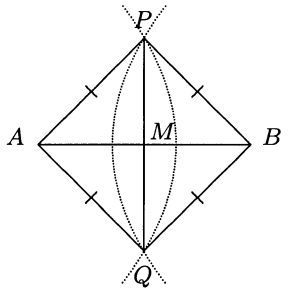
$$\frac{\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-3} + \binom{n-1}{n-2} + \binom{n}{n}}{2^n}.$$

Since $\binom{n}{n} = \binom{n-1}{n-1}$, this is again $\frac{2^{n-1}}{2^n} = \frac{1}{2}$, giving the result.

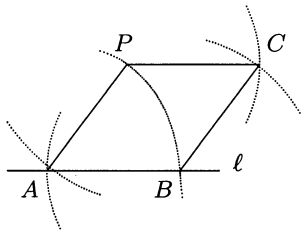
## Exercises 8.1

1. [BB] To find the mid-point of a line segment $AB$, choose a radius $r$ (for example, $r = |AB|$) such that the arcs with centers $A$ and $B$ and radius $r$ meet at two distinct points, $P$ and $Q$. The point of intersection, $M$, of $AB$ and $PQ$ is the required mid-point.

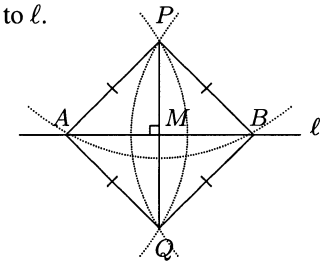
   To see why this works, consider the diagram to the right, in which the labels are given as before and line segments $AP$, $BP$, $AQ$, $BQ$ are joined. Note that $\angle PAB = \angle PBA$ since $\triangle APB$ is isosceles. Since $\triangle BPQ$ is congruent to $\triangle APQ$ (three pairs of sides of equal length), $\angle BPQ = \angle APQ$. Thus, triangles $PAM$ and $PBM$ are congruent (two equal pairs of sides and equal contained angles), so $|AM| = |BM|$ as required.




2. Let $A$ be any point of $\ell$. Draw an arc with center $A$ and radius $r = |AP|$. Let $B$ be the intersection of this arc with $\ell$. Draw arcs with radius $r$ and centers $P$ and $B$ meeting in $A$ and $C$. Then the line through $PC$ is the desired line. To see this, note that $PABC$ has four equal sides, so it's a parallelogram.

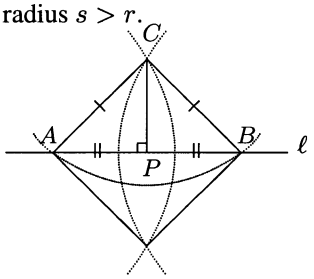


3. (a) [BB] Let $A$ be any point on $\ell$ and let $r = |AP|$. Draw the circle with center $P$ and radius $r$. If this meets $\ell$ in only the single point $A$, then $PA$ is the desired perpendicular to $\ell$. Otherwise, the circle meets $\ell$ in two points, $A$ and $B$. Draw arcs with centers $A$ and $B$ and radius $r$ meeting in $P$ and $Q$. Then $PQ$ is the desired perpendicular to $\ell$.




To see this, consider the diagram at the right, with $M$ the point of intersection of $PQ$ and $\ell$. Since triangles $PAQ$ and $PBQ$ are congruent, $\angle APM = \angle BPM$. Then it follows that $\triangle APM \equiv \triangle BPM$ and thus, $\angle AMP = \angle BMP$. Since these angles have sum 180°, each is a right angle.

(b) Draw an arc with center $P$ and any radius $r$ and let $A$ and $B$ be the points of intersection of this arc with $\ell$. Then draw arcs with centers $A$ and $B$ and any radius $s > r$.




Letting $C$ be one of the points of intersection of these arcs, $CP$ is the required perpendicular. To see this, note that triangles $CPA$ and $CPB$ have three pairs of equal sides and so are congruent. Thus, $\angle CPA = \angle CPB$ and, because these angles sum to 180°, they are right angles.

4. (a) [BB]

| | |
|---|---|
| ~~168~~ | ~~413~~ |
| ~~84~~ | ~~826~~ |
| ~~42~~ | ~~1652~~ |
| 21 | 3304 |
| ~~10~~ | ~~6608~~ |
| 5 | 13216 |
| ~~2~~ | ~~26432~~ |
| 1 | 52864 |
| | 69384 |

(b) We can reduce our work by putting the smaller number on the left.

| | |
|---|---|
| 461 | 973 |
| ~~230~~ | ~~1946~~ |
| 115 | 3892 |
| 57 | 7784 |
| ~~28~~ | ~~15568~~ |
| ~~14~~ | ~~31136~~ |
| 7 | 62272 |
| 3 | 124544 |
| 1 | 249088 |
| | 448553 |

(c)

| | |
|---|---|
| 141 | 141 |
| ~~70~~ | ~~282~~ |
| 35 | 564 |
| 17 | 1128 |
| ~~8~~ | ~~2256~~ |
| ~~4~~ | ~~4512~~ |
| ~~2~~ | ~~9024~~ |
| 1 | 18048 |
| | 19881 |

5. [BB] The basic idea behind this is the simple fact that $xy = (\frac{x}{2})(2y)$.

When $x$ is even, the product $xy$ can be equally well determined as the product of $x/2$ and $2y$. This explains why one crosses out a line where the first term is even: The product of the two numbers in a crossed out line equals the product of the numbers in the line below.

When $x$ is odd, however, the entry below $x$ is $\frac{x-1}{2}$ while the entry below $y$ is $2y$. Their product is $(\frac{x-1}{2})(2y) = xy - y$, so the product $xy$ is the product of the numbers in the line below, **plus $y$**. Note that $y$ is the number in the right column corresponding to the odd $x$ on the left. Thus, we do not cross out lines where $x$ is odd; the $y$ terms must be added at the end to give the correct product.

6. (a) If $k = 1$, then $n = 1$ or 2 and, in each case, the Russian peasant method is complete with at most one step. Assuming the result for $k$, let $n$ be an integer with $2^k \leq n \leq 2^{k+1}$. After one step, the integer on the left is $\lfloor \frac{n}{2} \rfloor$, so we can show that the entire process will require at most $k + 1$ steps

(completing the induction) if we can prove that at most $k$ steps are required in the calculations which begin with second line. This will follow by the induction hypothesis if $2^{k-1} \leq \lfloor \frac{n}{2} \rfloor < 2^k$. Now $n \leq 2^{k+1}$, so $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < 2^k$. Also, $\lfloor \frac{n}{2} \rfloor > \frac{n}{2} - 1 \geq 2^{k-1} - 1$ and this implies $\lfloor \frac{n}{2} \rfloor \geq 2^{k-1}$, as desired.

(b) Apply the Russian peasant method to compute the product of $n = n \times 1$. The numbers which appear on the right in the computation are $2, 4, 8, \ldots$, namely, powers of 2. Eventually, the product, $n = n \times 1$, is obtained as the sum of certain of these powers. By part (a), the number of powers involved is at most $k$, where $2^{k-1} \leq n < 2^k$. Since $k - 1 \leq \log_2 n$, we know $k \leq 1 + \log_2 n$.

7. [BB] $\sqrt{3} \approx 1.7321$ as shown.

| | 1. | 7 | 3 | 2 | 0 | 5 |
|---|---|---|---|---|---|---|
| | 3. 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| 1 | 1 | | | | | |
| 27 | 2 0 0 | | | | | |
| | 1 8 9 | | | | | |
| 343 | 1 1 | 0 0 | | | | |
| | 1 0 2 9 | | | | | |
| 3462 | | 7 1 | 0 0 | | | |
| | | 6 9 2 4 | | | | |
| 34640 | | 1 7 | 6 0 | 0 | | |
| | | | | 0 | | |
| 346405 | | 1 7 | 6 0 | 0 | 0 | 0 |
| | | 1 7 | 3 2 | 0 | 2 | 5 |
| | | | 2 7 | 9 | 8 | 5 |

$\sqrt{12} \approx 3.4641$ as shown.

| | 3. | 4 | 6 | 4 | 1 | 0 |
|---|---|---|---|---|---|---|
| | 12. 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| 3 | 9 | | | | | |
| 64 | 3 0 0 | | | | | |
| | 2 5 6 | | | | | |
| 686 | 4 4 | 0 0 | | | | |
| | 4 1 1 6 | | | | | |
| 6924 | 2 8 | 4 0 | 0 | | | |
| | 2 7 | 6 9 | 6 | | | |
| 69281 | | 7 0 | 4 0 | 0 | | |
| | | 6 9 | 2 8 | 1 | | |
| 692820 | | 1 | 1 1 | 9 | 0 | 0 |
| | | | | | | 0 |
| | | 1 | 1 1 | 9 | 0 | 0 |

$\sqrt{153} \approx 12.3693$ as shown.

| | 12. | 3 | | 6 | | 9 | | 3 | | 1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 153. | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 144 | | | | | | | | | | |
| 243 | 9 | 0 | 0 | | | | | | | | |
| | 7 | 2 | 9 | | | | | | | | |
| 2466 | 1 | 7 | 1 | 0 | 0 | | | | | | |
| | 1 | 4 | 7 | 9 | 6 | | | | | | |
| 24729 | | 2 | 3 | 0 | 4 | 0 | 0 | | | | |
| | | 2 | 2 | 2 | 5 | 6 | 1 | | | | |
| 247383 | | | | 7 | 8 | 3 | 9 | 0 | 0 | | |
| | | | | 7 | 4 | 2 | 1 | 4 | 9 | | |
| 2473861 | | | | | 4 | 1 | 7 | 5 | 1 | 0 | 0 |
| | | | | | 2 | 4 | 7 | 3 | 8 | 7 | 1 |
| | | | | | 1 | 7 | 0 | 1 | 2 | 3 | 9 |

8. [BB] We find the sum of the numbers and divide by $n$. Thus, to find the average of $n$ numbers $a_1, a_2, \ldots, a_n$,

   **Step 1:** set $S = 0$;

   **Step 2:** for $i = 1$ to $n$, replace $S$ by $S + a_i$;

   **Step 3:** output $\frac{S}{n}$.

9. [BB] To find the maximum of $n$ numbers $a_1, a_2, \ldots, a_n$,

   **Step 1:** let $M = a_1$;

   **Step 2:** for $i = 2$ to $n$, if $a_i > M$, replace $M$ by $a_i$;

   **Step 3:** output $M$.

   The value of $M$ output in Step 3 is the maximum of the $a_i$.

10. We establish a counter, $T$, which is initially set to 0 and is increased by 1 each time an occurrence of $x$ is found in the list. To determine how many times the element $x$ appears in the list $a_1, a_2, \ldots, a_n$,

    **Step 1:** set $T = 0$;

    **Step 2:** for $i = 1$ to $n$, if $a_i = x$, replace $T$ by $T + 1$;

    **Step 3:** output $T$.

    The output value of $T$ is the number of times $x$ appears in the list.

11. To determine how many elements from among $a_1, a_2, \ldots, a_n$ lie in the range $85 \leq x \leq 90$,

    **Step 1:** set $t = 0$;

    **Step 2:** for $i = 1$ to $n$, if $a_i \geq 85$ and $a_i \leq 90$, replace $t$ by $t + 1$;

    **Step 3:** output $t$.

    The value of $t$ output in Step 3 is the desired number.

12. [BB] If $x \leq a_1$, we wish to output $x, a_1, a_2, \ldots, a_n$. If $a_1 < x \leq a_2$, we wish to output $a_1, x, a_2, \ldots, a_n$, and so on. If $x \geq a_n$, we wish to output $a_1, a_2, \ldots, a_n, x$. Here is one suitable algorithm.

    To insert $x$ into its correct position in the ordered list $a_1 \leq a_2 \leq \cdots \leq a_n$,

**Step 1:** set $F = 0$;

**Step 2:** for $i = 1$ to $n$,
if $x \leq a_i$ and $F = 0$, output $x$ and set $F = 1$;
output $a_i$.

**Step 3:** if $F = 0$, output $x$.

The variable $F$ introduced here is a *flag*, whose purpose is to tell us whether or not $x$ has been output. At the end of the loop in Step 2, if $x$ has not been output (that is, $x$ is larger than all the $a_i$'s), then we will know this, because $F$ will not have changed from its initial value of 0; hence, we must output $x$ as the final element. Note that the "and" in Step 2 is the logical "and" introduced in Section 0.1. We output $x$ and set $F = 1$ only if both $x \leq a_i$ and $F = 0$ are true.

13. The idea is to test each integer $x$ between 0 and $n-1$ for the possibility $n \mid (ax - b)$. After we have checked $x = n-1$, however, we need a way to ascertain whether or not any solutions have been found. For this purpose, we introduce a flag $F$ which is set initially to 0 and, if a solution to $ax \equiv b \pmod{n}$ is found, reset to 1. If the value of $F$ at the end is still 0, we know there are no solutions. Here is an algorithm.

**Step 1:** Set $F = 0$.

**Step 2:** For $x = 0$ to $n-1$, if $n \mid (ax - b)$, output $x$ and set $F = 1$.

**Step 3:** If $F = 0$, output "no solutions."

14. [BB] Here's the idea. Divide $a$ by 2, thereby obtaining $a = 2q_1 + a_1$, $0 \leq a_1 < 2$. Then divide $q_1$ by 3, obtaining $q_1 = 3q_2 + a_2$, $0 \leq a_2 < 3$. Note that, at this point, we have $a = 2(3q_2 + a_2) + a_1 = 3!q_2 + 2!a_2 + a_1$. Next, divide $q_2$ by 4, obtaining $q_2 = 4q_3 + a_3$ and $a = 4!q_3 + 3!a_3 + 2!a_2 + a_1$. Eventually we have

$$a = n!q_{n-1} + a_{n-1}(n-1)! + \cdots + a_2 2! + a_1,$$

which, with $a_n = q_{n-1}$, is the required expression. (Note that if at some stage $q_{k-1} = 0$, we have $a_n = a_{n-1} = \cdots = a_k = 0$, but the procedure is still valid.)

Here is the algorithm for writing $a = a_n n! + a_{n-1}(n-1)! + \cdots + a_2 2! + a_1$.

**Step 1:** Let $q_0 = a$.

**Step 2:** For $i = 1$ to $n-1$, write $q_{i-1} = (i+1)q_i + a_i$ with $0 \leq a_i < i+1$.

**Step 3:** Let $a_n = q_{n-1}$.

15. To determine the digits of a number $n$ given in base 10,

**Step 1:** Let $a = n$.

**Step 2:** Write $a = 10q + r$ with $0 \leq r < 10$. Output $r$ and, if $q = 0$, stop; otherwise, let $a = q$ and repeat Step 2.

The digits of $n$, in order, beginning with the units digit, are the numbers $r$ output by this procedure.

16. [BB] A number between 0 and $2^n - 1$, when expressed in base 2, is a string of at most $n$ 0's and 1's. By "padding" with initial 0's, if necessary, we can assume that all such strings have length exactly $n$. Then such a string, $\epsilon_1\epsilon_2\cdots\epsilon_n$, each $\epsilon_i = 0$ or 1, determines the subset $A$ of $\{a_1, a_2, \ldots, a_n\}$ as follows: $a_i \in A \leftrightarrow \epsilon_i = 1$. Thus, the string $00\ldots0$ of $n$ 0's corresponds to the empty set, the string $11\ldots1$ of $n$ 1's to the entire set $\{a_1, a_2, \ldots, a_n\}$, the string $11010\ldots0$ to the subset $\{a_1, a_2, a_4\}$, and so on. Enumerating the subsets of $\{a_1, \ldots, a_n\}$ now simply amounts to listing the integers between 0 and $2^n - 1$ in base 2. Here is an algorithm.

**Step 1:** Let $M = 0$. Output the words "empty set."

**Step 2:** for $i = 1$ to $2^n - 1$

- replace $M$ by $M + 1$;
- write $M = \epsilon_1\epsilon_2 \cdots \epsilon_n$ as an $n$-digit number in base 2;
- for $k = 1$ to $n$, if $\epsilon_k = 1$, output $a_k$.

Each value of $i$ in Step 2 yields one subset of $\{a_1, \ldots, a_n\}$ and, when Step 2 is complete, all subsets have been output.

17. (a) [BB] i. $S = 1$; $S = 1 + (-3)(2) = -5$; $S = -5 + 2(2^2) = -5 + 8 = 3$.
    ii. $S = 2$; $S = -3 + 2(2) = 1$; $S = 1 + 1(2) = 3$.
    (b) i. $S = 1$; $S = 1 + 0(5) = 1$; $S = 1 + 3(5^2) = 1 + 75 = 76$.
    ii. $S = 3$; $S = 0 + 3(5)$=15; $S = 1 + 15(5) = 76$.
    (c) i. $S = -4$; $S = -4 + 5(-1) = -4 - 5 = -9$; $S = -9 + 6(-1)^2 = -9 + 6 = -3$;
    $S = -3 + (-4)(-1)^3 = -3 - 4(-1) = -3 + 4 = 1$.
    ii. $S = -4$; $S = 6 - 4(-1) = 10$; $S = 5 + 10(-1) = -5$; $S = -4 - 5(-1) = 1$.
    (d) i. $S = -7$; $S = -7 + 16(3) = -7 + 48 = 41$; $S = 41 + 0(3^2) = 41$;
    $S = 41 + (-40)(3^3) = 41 - 40(27) = 41 - 1080 = -1039$; $S = -1039 + 0(3^4) = -1039$;
    $S = -1039 + 17(3^5) = -1039 + 17(243) = -1039 + 4131 = 3092$.
    ii. $S = 17$; $S = 0 + 17(3) = 51$; $S = -40 + 51(3) = 113$; $S = 0 + 113(3) = 339$;
    $S = 16 + 339(3) = 1033$; $S = -7 + 1033(3) = 3092$.

18. [BB] The first value of $S$ is $S = a_n$.
    With $i = 1$ in Step 2, the value of $S$ is $a_{n-1} + Sx = a_{n-1} + a_n x$.
    With $i = 2$ in Step 2, the value of $S$ is $a_{n-2} + Sx = a_{n-2} + (a_{n-1} + a_n x)x = a_{n-2} + a_{n-1}x + a_n x^2$.
    With $i = 3$ in Step 2, the value of $S$ is $a_{n-3} + Sx = a_{n-3} + a_{n-2}x + a_{n-1}x^2 + a_n x^3$.
    With $i = n$, the value of $S$ is $a_{n-n} + a_{n-(n-1)}x + a_{n-(n-2)}x^2 + \cdots + a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ as desired.

19. We need a mechanical way to implement the "crossing out" required by the Sieve of Eratosthenes. To do this, we first assign a 1 to each integer between 1 and $n$ and change the 1 to a 0 if we are required to "cross out." Eventually, the primes between 1 and $n$ will be those integers whose labels have not been changed.

    **Step 1:** For $i = 2$ to $n$, set $a_i = 1$.

    **Step 2:** Let $k = \lfloor \sqrt{n} \rfloor$. (Because of Lemma 4.3.4, we need only check for divisibility by integers no larger than $\sqrt{n}$.)

    **Step 3:** For $i = 2$ to $k$, if $a_i = 1$, change $a_j$ to 0 for any $j$, $i < j \leq n$ for which $i \mid j$.

    **Step 4:** Output those $j$ for which $a_j = 1$.

20. Calculus is full of algorithms; for computing the derivative of $x^n$, for computing the derivative of a product or quotient, for integrating a polynomial. In a beginning course on linear algebra, students learn some variation of an algorithm called Gaussian elimination for solving linear systems of equations. Other algorithms associated with linear algebra are cross product, distance from point to plane, and Gram-Schmidt orthogonalization. In statistics, students quickly learn the formulas for mean and standard deviation, for the slope and $y$-intercept of the "least squares line" through a set of points in the plane.

## Exercises 8.2

1. [BB] Let addition be the basic operation. We need $n$ subtractions, $n$ squarings (each equivalent to two basic operations), $n-1$ additions, and one square root (the equivalent of 20 basic operations). Hence, $f(n) = n + 2n + n - 1 + 20 = 4n + 19$.

2. The multiplication of $a = (a_{n-1}a_{n-2}\ldots a_1a_0)_{10}$ by the single digit $b$ requires initially the multiplication of $b$ and $a_0$, one operation. Then we multiply $b$ and $a_1$ and perhaps add a carry from the first step. Since two single-digit additions may be needed in this carry (e.g. $89 \times 7$), at most three operations are required in the multiplication of $b$ and $a_1$ and, similarly, in the multiplication of $b$ and each $a_i$, $i \geq 1$. An upper bound for the total number of operations required is $1 + 3(n-1) = 3n - 2$.

3. [BB] Consider the division of a number $a$, which has at most two digits, by a single-digit number $b$, where $a < 10b$. Since the quotient $q$ has just one digit, the division counts as one operation, the product $qb$ counts as another, and since $a - qb < b$, the difference $a - qb$ has just a single digit, so the subtraction requires a third operation. In the division of an $n$-digit number by a single-digit number, this process is repeated at most $n-1$ or $n$ times, depending on whether or not $b$ is greater than the first digit of $a$. So at most $3n$ operations are required.

$$\begin{array}{r|l} & q \\ b & a \\ & qb \\ \hline & a - qb \end{array}$$

4. The obvious way to find the minimum of $n$ numbers $n$ numbers $a_1, a_2, \ldots, a_n$ uses the following algorithm.

   **Step 1:** Let $m = a_1$;

   **Step 2:** for $i = 2$ to $n$, if $a_i < m$, replace $m$ by $a_i$;

   **Step 3:** output $m$.

   The value of $m$ output in Step 3 is the minimum of the $a_i$'s. This algorithm involves $n-1$ comparisons, so $f(n) = n - 1$ is the complexity function.

5. [BB] The number of multiplications required by Algorithm $\mathcal{A}$ is the number of times Step 2 is executed, which is $2^n$. On the other hand, Algorithm $\mathcal{B}$ requires just $n$ multiplications. Algorithm $\mathcal{B}$ is more efficient because $n \prec 2^n$.

6. (a) [BB] With reference to Horner's algorithm as described in (8.1.1), each iteration of Step 2 requires two operations. Since Step 2 is repeated $n$ times, this method requires $2n$ operations. This is fewer operations than are needed by the obvious method of polynomial evaluation, as we now show.

   Assuming $i$ multiplications to compute $a_ix^i$, $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ multiplications are involved in the evaluation of $a_0 + a_1x + \cdots + a_nx^n$. As well, $n$ additions are required, for a total of $\dfrac{n(n+1)}{2} + n = \dfrac{n^2+3n}{2}$ operations. This estimate can be improved. Problem 18 shows that $x^i$ can be computed with approximately $\log i$ multiplications. So $1 + \log i$ multiplications are required for $a_ix^i$ and

   $$1 + (1 + \log 2) + (1 + \log 3) + \cdots + (1 + \log n) = n + \log n!$$

   for the polynomial. (Recall that $\log a + \log b = \log ab$.) Including additions, approximately $2n + \log n!$ operations are required.

   (b) If powers of $x$ can be stored, then $n-1$ multiplications are required to compute $x^2, \ldots, x^n$. A further $n$ multiplications are required for the products $a_ix^i$ and finally $n$ additions are needed for a total of $3n - 1$ operations. This still exceeds the $2n$ operations required by Horner's method.

7. (a) [BB] Since $5n \leq n^3$ for all $n \geq 3$, we can take $c = 1$, $n_0 = 3$, or $c = 5$, $n_0 = 1$.

   (b) For $n \geq 1$, we have $17n^4 + 8n^3 + 5n^2 + 6n + 1 \leq 17n^4 + 8n^n + 5n^4 + 6n^4 + n^4 = 37n^4$. So we can take $c = 37$ and $n_0 = 1$.

   (c) [BB] For $n \geq 1$, we have $8n^3 + 4n^2 + 5n + 1 \leq 9n^4 + 18n^2 + 24n + 6 = 3(3n^4 + 6n^2 + 8n + 2)$, so we can take $c = 3$, $n_0 = 1$.

   (d) For $n \geq 1$, $2^n \leq 3^n$, so take $c = 1$, $n_0 = 1$.

   (e) For $n \geq 1$, we have $f(n) = 2n^2 + 3n + 1 \leq 2n^2 + 3n^2 + 5 = 5n^2 + 5 = 5g(n)$, so take $n_0 = 1$ and $c = 5$.

   (f) We have $f(n) = 2n^2 - 3n + 5 \leq 2n^2 + 3n + 5 \leq 2n^2 + 3n^2 + 5n^2 = 10n^2$ for any $n \geq 1$. Now $n^2 \leq 2(n^2 - 7n) = 2n^2 - 14n = 2g(n)$ for $n \geq 14$ since $(2n^2 - 14n) - n^2 = n^2 - 14n = n(n - 14) \geq 0$ for $n \geq 14$. So, for $n \geq 14$, $f(n) \leq 10n^2 \leq 20g(n)$. So $f = \mathcal{O}(g)$ with $n_0 = 14$ and $c = 20$.

   (g) Since $\log_2 n^5 = 5 \log_2 n$ and since $\log_2 n \leq n$ for any $n \geq 1$ we have $\log_2 n^5 \leq 5n$ for $n \geq 1$. So take $c = 5$ and $n_0 = 1$.

8. (a) [BB] By Proposition 8.2.7, for instance, $5n \asymp n$, and by Proposition 8.2.8, $n \prec n^3$. Thus, $5n \prec n^3$.

   (b) By Proposition 8.2.7, $f(n) \asymp n^4$. Thus, $f \asymp g$.

   (c) [BB] By Proposition 8.2.7, $f(n) \asymp n^3$ and $g(n) \asymp n^4$. By Proposition 8.2.8, $n^3 \prec n^4$, so $f \prec g$.

   (d) Suppose $3^n = \mathcal{O}(2^n)$. Then there is a constant $c$ such that $3^n \leq c2^n$ for all sufficiently large $n$; that is, $(\frac{3}{2})^n < c$ for all large enough $n$. Since $\frac{3}{2} > 1$, $(\frac{3}{2})^n$ grows with $n$ and for large $n$ exceeds any constant. Thus, $3^n$ is not $\mathcal{O}(2^n)$. Hence, $2^n \prec 3^n$.

   (e) By Proposition 8.2.7, $f(n) \asymp n^2$ and $g(n) \asymp n^2$, so $f(n) \asymp g(n)$, since $\asymp$ is transitive.

   (f) By Proposition 8.2.7, $f(n) \asymp n^2$ and $g(n) \asymp n^2$, so $f(n) \asymp g(n)$, since $\asymp$ is transitive.

   (g) Suppose $n = \mathcal{O}(\log_2 n^5)$. Then there is a constant $c$ such that $n \leq c \log_2 n^5 = 5c \log_2 n$ for all sufficiently large $n$. Hence, $n = \mathcal{O}(\log_2 n)$, contradicting $\log_2 n \prec n$ (Proposition 8.2.9). Thus, $n$ is not $\mathcal{O}(\log_2 n^5)$, so $\log_2 n^5 \prec n$.

9. There exist an integer $n_1$ and a real number $c_1$ such that $|f(n)| \leq c_1|h(n)|$ for all $n \geq n_1$. There exist an integer $n_2$ and a real number $c_2$ such that $|g(n)| \leq c_2|h(n)|$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then, for $n \geq n_0$, we have $|f(n) + g(n)| \leq |f(n)| + |g(n)| \leq c_1|h(n)| + c_2|h(n)| = (c_1 + c_2)|h(n)|$. With $c = c_1 + c_2$ in Definition 8.2.1, we see that $f + g = \mathcal{O}(h)$.

10. (a) Let $f, g, h$: $\mathsf{N} \to \mathsf{R}$ and suppose that $f = \mathcal{O}(g)$ and $g = \mathcal{O}(h)$. We must prove that $f = \mathcal{O}(h)$. There exists a constant $c$ and an integer $n_0$ such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$. There exists also a constant $d$ and an integer $n_1$ such that $|g(n)| \leq d|h(n)|$ for all $n \geq n_1$. Then if $n \geq \max\{n_0, n_1\}$, we have $|f(n)| \leq c|g(n)| \leq cd|h(n)|$ so, using $cd$ and $N = \max\{n_0, n_1\}$, we see that $f = \mathcal{O}(h)$.

    (b) This follows immediately from part (a) since $f = \mathcal{O}(g)$ and $g = \mathcal{O}(h)$ and $\mathcal{O}$ is transitive.

    (c) [BB] Suppose $f \prec g$ and $g \prec h$. We must prove $f \prec h$. By part (a), $f = \mathcal{O}(h)$, so it remains only to prove that $h \neq \mathcal{O}(f)$. Assume to the contrary that $h = \mathcal{O}(f)$. Then, since $f = \mathcal{O}(g)$, using part (a) again, we would have that $h = \mathcal{O}(g)$, contradicting $g \prec h$.

11. [BB] If $f \asymp g$, then $f = \mathcal{O}(g)$. Since $g \prec h$, $g = \mathcal{O}(h)$ and so, by Exercise 10(a), $f = \mathcal{O}(h)$. To obtain $f \prec h$, we have now to prove that $h$ is not $\mathcal{O}(f)$. But if $h = \mathcal{O}(f)$, then, because $f = \mathcal{O}(g)$, we would have that $h = \mathcal{O}(g)$, contradicting $g \prec h$.

12. [BB] We have $a^n < b^n$ for $n \geq 1$ since $a < b$, so $a^n = \mathcal{O}(b^n)$. We must prove $b^n \neq \mathcal{O}(a^n)$. If $b^n = \mathcal{O}(a^n)$, then there would exist a constant $c$ such that $b^n \leq ca^n$ for all sufficiently large $n$. Dividing by $a$, we obtain $(\frac{b}{a})^n \leq c$, for all sufficiently large $n$, but this is not true because $\frac{b}{a} > 1$ implies that $(\frac{b}{a})^n$ grows without bound as $n$ increases.

13. (a) [BB] We established $2^n < n!$ for $n \geq 4$ in Problem 6 of Chapter 5. Thus, $2^n = \mathcal{O}(n!)$ ($c = 1$, $n_0 = 4$). On the other hand, $n! \neq \mathcal{O}(2^n)$, for consider

$$\frac{n!}{2^n} = \frac{1}{2}\frac{2}{2}\frac{3}{2}\frac{4}{2}\cdots\frac{n}{2}.$$

The product of the first three factors on the right is $\frac{3}{4}$ and each of the remaining $n-3$ terms is bigger than 2. So $\frac{n!}{2^n} > \frac{3}{4}(2^{n-3})$ for $n > 3$. If $n! = \mathcal{O}(2^n)$, then $n! < c2^n$ for some constant $c$ and all sufficiently large $n$ and so $\frac{n!}{2^n} < c$. We have shown this is not possible. Since $2^n = \mathcal{O}(n!)$ but $n! \neq \mathcal{O}(2^n)$, we have $2^n \prec n!$ as required.

(b) Since $b^n \leq \lceil b \rceil^n$, we can assume that $b$ is an integer. We have

$$\begin{aligned}\frac{b^n}{n!} &= \frac{b}{n}\frac{b}{n-1}\frac{b}{n-2}\cdots\frac{b}{b+1}\frac{b}{b}\frac{b}{b-1}\cdots\frac{b}{2}\frac{b}{1} \\ &= \frac{b}{n}\frac{b}{n-1}\frac{b}{n-2}\cdots\frac{b}{b+1}\,k\end{aligned}$$

with $k = \dfrac{b^b}{b!}$. Thus, if $n > b$, $\dfrac{b^n}{n!} < k$, so $b^n = \mathcal{O}(n!)$. On the other hand,

$$\frac{n!}{b^n} = \frac{n}{b}\frac{n-1}{b}\cdots\frac{b+1}{b}\frac{b!}{b^b} = \frac{n}{b}\frac{n-1}{b}\cdots\frac{b+1}{b}\frac{1}{k} > \frac{n}{bk}$$

for $n > b+1$. So, there is no constant $c$ with $n! < cb^n$ for arbitrarily large $n$. Thus $n! \neq \mathcal{O}(b^n)$ and we have the desired result.

14. (a) We proceed by induction on $n$. Since $10^3 = 1000$ and $2^{10} = 1024$ the statement holds for $n = 10$. Now suppose the statement is true for some $k > 10$. Then

$$\begin{aligned}(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ &\leq k^3 + 3k^2 + 3k^2 + k^2 \\ &\leq k^3 + 7k^2 \\ &< k^3 + k^3 = 2k^3 && \text{since } 7 < k \\ &\leq 2(2^k) = 2^{k+1} && \text{by the induction hypothesis}\end{aligned}$$

which is the statement for $n = k+1$. By the Principle of Mathematical Induction, the statement is true for all $n \geq 10$.

(b) Since $n^2 < n^3$ and $n^3 < 2^n$ for $n \geq 10$, certainly $n^2 < 2^n$ for $n \geq 10$. (In fact, as mentioned in this section, $n^2 < 2^n$ for $n \geq 5$.) Thus, $n^2 = \mathcal{O}(2^n)$. We have now to prove that $2^n \neq \mathcal{O}(n^2)$. Assume to the contrary that $2^n = \mathcal{O}(n^2)$. Then $2^n < cn^2$ for some constant $c$ and all sufficiently large enough $n$. Since $2^n > n^3$ for sufficiently large $n$ (by part (a)), we would have $n^3 < 2^n < cn^2$ and hence $n < c$ for large $n$. This is clearly not the case.

15. [BB] For any $n \geq 1$, $|kf(n)| = |k||f(n)|$. Taking $n_0 = 1$ and $c = |k|$ in Definition 8.2.1, we see that $kf$ is $\mathcal{O}(f)$. Also, $|f(n)| = \frac{1}{|k|}|k(f(n)|$ says that $f = \mathcal{O}(kf)$ and so $f \asymp kf$.

16. (a) No. By its very definition, $\prec$ is not reflexive.

    (b) Yes, $(A, \preceq)$ is a partially ordered set.

    **Reflexivity:** Since $f = f$, we have $f \preceq f$.

    **Antisymmetric:** If $f \preceq g$ but $f \neq g$, then $f \prec g$. This means $f = \mathcal{O}(g)$ but $g \neq \mathcal{O}(f)$, so $g \not\prec f$. It follows that if $f \preceq g$ and $g \preceq f$, then $f = g$.

    **Transitivity:** See the solution to Exercise 10.

17. [BB] **Reflexivity:** For all $n \geq 1$, $|f(n)| \leq |f(n)|$, so with $c = 1$, $n_0 = 1$ in Definition 8.2.1 we see that $f$ is $\mathcal{O}(f)$ and hence $f \asymp f$.

    **Symmetry:** If $f \asymp g$, then $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$. Hence, $g = \mathcal{O}(f)$ and $f = \mathcal{O}(g)$, so $g \asymp f$.

    **Transitivity:** If $f \asymp g$ and $g \asymp h$, then $f = \mathcal{O}(g)$ and $g = \mathcal{O}(h)$. Therefore, $f = \mathcal{O}(h)$ by Exercise 10(a). Since $h = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$, an argument identical to the one just given yields that $h = \mathcal{O}(f)$ as well. Thus, $f \asymp h$.

18. [BB] Since $\log_2 n = \mathcal{O}(n)$ by Proposition 8.2.9 and $n = \mathcal{O}(n)$, we have $n \log_2 n = \mathcal{O}(n^2)$ by Proposition 8.2.3. Thus, it remains only to show that $n^2 \neq \mathcal{O}(n \log_2 n)$. Assume to the contrary that for some positive constant $c$, we have $n^2 \leq cn \log_2 n$ for all sufficiently large $n$. Then, for large $n$, we would have $n \leq c \log_2 n$. This says $n = \mathcal{O}(\log_2 n)$, contradicting Proposition 8.2.9.

19. (a) By Proposition 8.2.7, $f$ has the same order as $n^4$.

    (b) [BB] Since $\log n = \mathcal{O}(n)$, $n \log n = \mathcal{O}(n^2)$, hence, $f = \mathcal{O}(n^2)$. Since $n^2 \leq f(n)$ for all $n \geq 1$, $f \asymp n^2$.

    (c) $5 \asymp 1$ is a direct consequence of Exercise 15 with $f(n) = 1$, $k = 5$.

    (d) Since $2^n = \mathcal{O}(n^n)$ and $n^n \asymp 4n^n$, we have that $2^n = \mathcal{O}(4n^n)$ by transitivity of $\mathcal{O}$ (Exercise 10(a)) and so $f(n) = \mathcal{O}(n^n)$ by Proposition 8.2.3. Since $n^n \leq f(n)$ for any $n \geq 1$, we have easily $n^n = \mathcal{O}(f(n))$ and so $f(n) \asymp n^n$.

    (e) [BB] Note first that $3n! - 17n^4 < 3n!$, so $3n! - 17n^4 = \mathcal{O}(n!)$ using $c = 3$, $n_0 = 1$. For the converse, recall (see just before Proposition 8.2.9) that $n^4 < 2^n$ for large enough $n$ (here, $n \geq 17$), so that $n! - 17n^4 > n! - 17(2^n)$ for $n \geq 17$. However, for such values of $n$, it is clear that $n! - 17(2^n) > 0$, so $3n! - 17n^4 > n!$. Thus, $n!$ is $\mathcal{O}(3n! - 17n^4)$, taking $c = 1$, $n_0 = 17$.

    (f) For $n \geq 1$, $n \leq n + \log n$, thus, $n = \mathcal{O}(n + \log n)$. On the other hand, since $\log n = \mathcal{O}(n)$ (Proposition 8.2.9), $n + \log n = \mathcal{O}(n)$ by Proposition 8.2.3. Thus, $n + \log n \asymp n$.

20. (a) [BB] $n^5$ (b) $2^n$ (c) $\pi^n$ (d) $n^2$

    (e) [BB] Since $2 + 4 + 6 + \cdots + 2n = n^2 + n$, the answer is $n^2$.

    (f) Since $2 + 4 + 8 + \cdots + 2^n = 2(2^n - 1) = 2(2^n) - 2$, the answer is $2^n$.

    (g) $n \log_2 n$

21. [BB] Since $\log_a n = (\log_a b)(\log_b n)$, we have $|\log_a n| = |\log_a b||\log_b n|$ for all $n \geq 1$. With $c = \log_a b$ and $n_0 = 1$ in Definition 8.2.1 we see that $\log_a n$ is $\mathcal{O}(\log_b n)$. By symmetry, $\log_b n = \mathcal{O}(\log_a n)$; hence, $\log_a n \asymp \log_b n$.

22. Since $A$ and $B$ are positive, $n \leq k(An + B)$ for some positive constant $k$. Since $\log_2$ is an increasing function, $\log n \leq \log(k(An + B)) = \log k + \log(An + B)$. If we choose $n_0 > \frac{k-B}{A}$, then for all $n \geq n_0$, $\log(An + B) > \log k$, so $\log n < 2 \log(An + B)$. Setting $c = 2$ and using $n_0$ as above, we see that $\log n = \mathcal{O}(\log(An + B))$. Conversely, we must show that $\log(An + B) \leq c \log n$ for

some constant $c$ and all sufficiently large $n$. Now $c \log n = \log n^c$ and log is an increasing function, so $\log(An + B) \leq c \log n \iff An + B \leq n^c$. But $An + B \leq (A + B)n \leq n^2$ for $n > A + B$ so $c = 2$, works.

23. Since $n = ab$ with $1 < a, b < n$ implies either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$, $n$ is prime if does not have a proper factor less than or equal to $\sqrt{n}$. So, for each integer $k$, $1 < k < \lceil \sqrt{n} \rceil$, use the Euclidean algorithm to determine if $\gcd(k, n) = 1$. As shown in Problem 19, this algorithm requires $\mathcal{O}(\ln k) = \mathcal{O}(\ln \sqrt{n}) = \mathcal{O}(\ln n)$ divisions. The integer $n$ is prime if and only if $\gcd(k, n) = 1$ for all such $k$. The entire procedure is $\mathcal{O}(\sqrt{n} \ln n)$.

24. [BB] Since $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 < n^n$, $\log n! \leq n \log n$. With $c = 1$, $n_0 = 1$ in Definition 8.2.1, we see that $\log n! = \mathcal{O}(n \log n)$ as required.

25. (a) If $a = (a_k a_{k-1} \ldots a_2 a_1 a_0)_2$, then the base 2 digits $a_0, a_1, \ldots, a_k$ are the remainders given by the following process of successive divisions by 2:

$$\begin{aligned} a &= 2q_0 + a_0 \\ q_0 &= 2q_1 + a_1 \\ q_1 &= 2q_2 + a_2 \\ &\vdots \\ q_{k-1} &= 2q_k + a_k \end{aligned}$$

The process terminates when $q_k = 0$. Assuming that each line here is a single operation, the number of operations required is $k + 1$.

Now

$$\begin{aligned} a &= 2q_0 + a_0 \\ &= 2(2q_1 + a_1) + a_0 = 4q_1 + 2a_1 + a_0 \\ &= 4(2q_2 + a_2) + 2a_1 + a_0 = 8q_2 + 4a_2 + 2a_1 + a_0 \\ &\vdots \\ &= 2^k q_{k-1} + 2^{k-1} a_{k-1} + 2^{k-2} a_{k-2} + \cdots + 2a_1 + a_0 \geq 2^k. \end{aligned}$$

Applying $\log_2$, $k \leq \log_2 a$. So the number of operations required is $k + 1 \leq (\log_2 a) + 1$ and the complexity of this algorithm is $\mathcal{O}(\log a)$.

(b) If a division is three operations, then the number of operations is at most $3(k + 1) \leq 3 \log_2 a + 3$. Since $3 \log a \asymp \log a$, the Big Oh estimate is still $\mathcal{O}(\log a)$.

26. (a) An $n$-digit number is less than or equal to $10^n$. If we choose $k$ such that $2^{k-1} < 10^n \leq 2^k$, then at most $k$ rows are required and, since $k - 1 < n \log_2 10$, we have $k < 1 + n \log_2 10$.

(b) As shown in Exercises 2 and 3, the number of basic operations required to multiply and divide an $n$-digit number by 2 are at most $3n - 2$ and $3n$, respectively.

In the Russian peasant method of multiplication, the first step involves multiplying an $n$-digit number by 2 and dividing an $n$-digit number by 2. The multiplication requires at most $3n - 2$ operations and the division at most $3n$ operations, giving at most $6n - 2$ operations in all. At the second step, the number on the left could still have $n$ digits while the number on the right could have as many as $n + 1$ digits. Thus, the second step could require as many as $3n + [3(n + 1) - 2] = 6n + 1$ operations. While the numbers on the left have a decreasing sequence of digits, it is difficult to decide precisely when these numbers drop. Thus we continue to assume that these have $n$ digits, hence that division by 2 requires $3n$ operations. Thus the third step, which

also involves the multiplication of a number with perhaps $n+2$ digits by 2, requires at most $3n + [3(n+2) - 2] = 6n + 4$ operations. The next step requires at most $6n + 7$ operations, and so on. Assuming $k$ steps, the number of multiplications and divisions by 2 is at most

$$\begin{aligned}(6n-2) + (6n+1) + (6n+4) + (6n+7) + \cdots + (6n+3k-5) \\ &= k(6n) + [-2 + 1 + 4 + \cdots + (3k-5)] \\ &= k(6n) + \tfrac{k}{2}[2(-2) + (k-1)3] \\ &= 6kn + \tfrac{k}{2}(3k-7) = \tfrac{3}{2}k^2 - \tfrac{7}{2}k + 6kn \le \tfrac{3}{2}k^2 + 6kn,\end{aligned}$$

using the fact that the sum of the first $k$ terms of the arithmetic sequence $a, a+d, a+2d, \ldots$ is $\frac{k}{2}[2a + (k-1)d]$—see (7) on p. 234. By part (a), $k < 1 + n\log_2 10$, so the number of multiplications and divisions by 2 is at most

$$\tfrac{3}{2}(1 + n\log_2 10)^2 + 6(1 + n\log_2 10)n\mathcal{O}(n^2).$$

(c) We must add at most $k$ integers, each with at most $n+k$ digits. Using Problem 16, we see that this requires at most

$$k[2(n+k) + k] \le (1 + n\log_2 10)((2n + 3 + 3n\log_2 10) = \mathcal{O}(n^2)$$

basic operations.

(d) We use the result of Exercise 9. Since the number of operations required by each of the two stages of the algorithm is $\mathcal{O}(n^2)$, the total number of operations is $\mathcal{O}(n^2)$, as desired.

27. (a) For real numbers $a$ and $b$, we have $(a+b)^2 = a^2 + 2ab + b^2 \le |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$. Taking the square root of each side and using $\sqrt{x^2} = |x|$ for any real number $x$ gives $|a+b| \le ||a| + |b|| = |a| + |b|$ because $|a| + |b|$ is nonnegative.

(b) We use mathematical induction. For $n = 1$, we have $|x_1| \le |x_1|$ and the result is true. Now assume $k \ge 1$ and that $|x_1 + x_2 + \cdots + x_k| \le |x_1| + |x_2| + \cdots + |x_k|$ for any numbers $x_1, x_2, \ldots, x_k$. Given $x_1, x_2, \ldots, x_{k+1}$, we have

$$\begin{aligned}|x_1 + x_2 + \cdots + x_{k+1}| \\ &= |(x_1 + x_2 + \cdots + x_k) + x_{k+1}| \\ &\le |x_1 + x_2 + \cdots + x_k| + |x_{k+1}| && \text{using part (a)} \\ &\le (|x_1| + |x_2| + \cdots + |x_k|) + |x_{k+1}| && \text{using the induction hypothesis} \\ &= |x_1| + |x_2| + \cdots + |x_{k+1}|\end{aligned}$$

which is the desired result for $n = k+1$. By the Principle of Mathematical Induction, we conclude that $|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|$ for all $n \ge 1$.

28. (a) There exists an integer $n_0$ such that for $n \ge n_0$, $|\frac{f(n)}{g(n)}| \le \epsilon = 1$ and hence $|f(n)| \le |g(n)|$. With $c = 1$ and $n_0$ in Definition 8.2.1, we have that $f = \mathcal{O}(g)$. We have now to show that $g$ is not $\mathcal{O}(f)$. But were this the case, there would exist a constant $c$ and an integer $n_0$ such that for $n \ge n_0$, $|g(n)| \le c|f(n)|$; that is, $|\frac{f(n)}{g(n)}| \ge \frac{1}{c}$. This contradicts $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| = 0$.

(b) There exists an integer $n_0$ such that for all $n \ge n_0$, $|\frac{f(n)}{g(n)}| \ge 1$; that is, $|g(n)| \le |f(n)|$. Thus, $g = \mathcal{O}(f)$. We must show that $f$ is not $\mathcal{O}(g)$. But were this the case, there would exist a constant $c$ and positive integer $n_0$ such that for all $n \ge n_0$, $|f(n)| \le c|g(n)|$; that is, $|\frac{f(n)}{g(n)}| \le c$. This contradicts $\lim_{n\to\infty} = +\infty$.

(c) There exists a positive integer $n_0$ such that for all $n \geq n_0$, $|\frac{f(n)}{g(n)} - L| < 1$ and hence $|\frac{f(n)}{g(n)}| - |L| < 1$ (using the fact that $|a| - |b| \leq |a - b|$ for any real numbers $a$ and $b$). Thus, for $n \geq n_0$, $|\frac{f(n)}{g(n)}| < |L| + 1$ so, with $c = |L| + 1$ we see that $f = \mathcal{O}(g)$. We now must show that $g$ is also $\mathcal{O}(f)$. But from the theory of limits, $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{1}{L}$ (since $L \neq 0$). So the argument which showed $f = \mathcal{O}(g)$ also shows that $g = \mathcal{O}(f)$.

## Exercises 8.3

1. (a) [BB] Since $n = 9 \neq 1$, we set $m = \lfloor \frac{9}{2} \rfloor = 4$. Since $2 = x \leq a_4 = 4$, we set $n = m = 4$ and change the list to $1, 2, 3, 4$.

   Since $4 = n \neq 1$, we set $m = \lfloor \frac{4}{2} \rfloor = 2$. Since $2 = x \leq a_2 = 2$ we set $n = m = 2$ and change the list to $1, 2$.

   Since $2 = n \neq 1$, we set $m = \lceil \frac{2}{2} \rceil = 1$. Since $2 = x > a_1 = 1$, we replace $n$ by $n - m = 1$ and change the list to 2.

   Since $n = 1$ and $x = a_1$, we output "true" and stop.

   This search required four comparisons of $x$ with an element in the list; a linear search would have used two comparisons.

   (b) Since $n = 9 \neq 1$, we set $m = \lfloor \frac{9}{2} \rfloor = 4$. Since $7 = x > a_4 = 4$, we replace $n$ by $n - m = 5$ and change the list to $5, 6, 7, 8, 9$.

   Since $5 = n \neq 1$, we set $m = \lfloor \frac{5}{2} \rfloor = 2$. Since $7 = x > a_2 = 6$, we replace $n$ by $n - m = 3$ and change the list to $7, 8, 9$.

   Since $3 = n \neq 1$, we set $m = \lfloor \frac{3}{2} \rfloor = 1$. Since $7 = x \leq a_1$, we let $n = m = 1$ and change the list to 7.

   Since $n = 1$ and $x = a_1$, we output "true" and stop.

   This search required four comparisons of $x$ with an element in the list; a linear search would have used seven comparisons.

   (c) Since $n = 9 \neq 1$, the algorithm sets $m = \lfloor \frac{9}{2} \rfloor = 4$. Since $x = 10 > a_4 = 4$, the algorithm replaces $n$ by $n - m = 5$ and changes the list to $5, 6, 7, 8, 9$.

   Since $n \neq 1$, the algorithm sets $m = \lfloor \frac{5}{2} \rfloor = 2$. Since $x = 10 > a_2 = 6$, the algorithm replaces $n$ with $n - m = 3$ and changes the list to $7, 8, 9$.

   Since $n \neq 1$, the algorithm sets $m = \lfloor \frac{3}{2} \rfloor = 1$. Since $x = 10 > a_1 = 7$, the algorithm replaces $n$ with $n - m = 2$ and changes the list to $8, 9$.

   Since $n \neq 1$, the algorithm sets $m = \lfloor \frac{2}{2} = 1$. Since $x = 10 > a_1 = 8$, the algorithm replaces $n$ with $n - m = 1$ and changes the list to 9.

   Since $n = 1$ and $x = 10 \neq a_1 = 9$, the algorithm outputs "false," sets $n = 0$ and stops.

   This search required five comparisons of $x$ with other elements; a linear search would have required nine.

2. (a) Since $n = 11 \neq 1$, we set $m = \lfloor \frac{11}{2} \rfloor = 5$. Since $7 = x \leq a_5 = 12$, we set $n = m = 5$ and change the list to $3, 5, 7, 10, 12$.

Since $5 = n \neq 1$, we set $m = \lfloor \frac{5}{2} \rfloor = 2$. Since $7 = x > a_2 = 5$ we replace $n$ by $n - m = 3$ and change the list to $7, 10, 12$.

Since $3 = n \neq 1$, we set $m = \lceil \frac{3}{2} \rceil = 1$. Since $7 = x \leq a_1 = 7$, we replace $n$ by $m = 1$ and change the list to 7.

Since $n = 1$ and $x = a_1$, we output "true" and stop.

This search required four comparisons of $x$ with an element in the list; a linear search would have required three.

(b) Since $n = 11 \neq 1$, we set $m = \lfloor \frac{11}{2} \rfloor = 5$. Since $30 = x > a_5 = 12$, we replace $n$ with $n - m = 6$ and change the list to $15, 17, 22, 24, 30, 31$.

Since $6 = n \neq 1$, we set $m = \lfloor \frac{6}{2} \rfloor = 3$. Since $30 = x > a_3 = 22$ we replace $n$ by $n - m = 3$ and change the list to $24, 30, 31$.

Since $3 = n \neq 1$, we set $m = \lceil \frac{3}{2} \rceil = 1$. Since $30 = x > a_1 = 24$, we replace $n$ by $n - m = 2$ and change the list to $30, 31$.

Since $2 = n \neq 1$, we set $m = \lceil \frac{2}{2} \rceil = 1$. Since $30 = x \leq a_1 = 30$, we replace $n$ by $m = 1$ and change the list to 30.

Since $n = 1$ and $x = a_1$, we output "true" and stop.

This search required five comparisons of $x$ with an element in the list; a linear search would have required ten.

(c) Since $n = 11 \neq 1$, we set $m = \lfloor \frac{11}{2} \rfloor = 5$. Since $16 = x > a_5 = 12$, we replace $n$ with $n - m = 6$ and change the list to $15, 17, 22, 24, 30, 31$.

Since $6 = n \neq 1$, we set $m = \lfloor \frac{6}{2} \rfloor = 3$. Since $16 = x \leq a_3 = 22$, we replace $n$ by $m = 3$ and change the list to $15, 17, 22$.

Since $3 = n \neq 1$, we set $m = \lceil \frac{3}{2} \rceil = 1$. Since $16 = x > a_1 = 15$, we replace $n$ by $n - m = 2$ and change the list to $17, 22$.

Since $2 = n \neq 1$, we set $m = \lceil \frac{2}{2} \rceil = 1$. Since $16 = x \leq a_1 = 17$, we replace $n$ by $m = 1$ and change the list to 17.

Since $n = 1$ and $x \neq a_1$, we output "false" and stop.

This search required five comparisons of $x$ with an element in the list; a linear search would have required eleven.

3. (a) To search an ordered list $a_1 \leq a_2 \leq \cdots \leq a_n$ for an element $x$, proceed as follows.

```
while n > 0
        if n = 1 then
            if x = a_1 output "true" and set n = 0
            else output "false" and set n = 0
        else if n = 2 then
            if x = a_1 or x = a_2 output "true" and set n = 0
            else output "false" and set n = 0
        else
            set ℓ = ⌊n/3⌋ and m = 2ℓ
            if x ≤ a_ℓ
                set n = ℓ and replace current list with a_1, ..., a_ℓ
            else if x > a_m
                replace n with n − m and the current list with a_{m+1}, ..., a_n
            else replace n with m − ℓ and the current list with a_{ℓ+1}, ..., a_m.
end while
```

Note that we check both possibilities $n = 1$ and $n = 2$ so that there is no possibility that $\ell = 0$ in what follows.

(b) To determine the number of comparisons required by the algorithm, first suppose that $n = 3^k$ is a power of 3. The algorithm proceeds by progressively dividing the list into thirds, so it terminates after the while statement—$n > 0$?—has been encountered $k$ times. As long as $n > 2$, three comparisons are needed to check $n > 0$? $n = 1$? $n = 2$? and a further two comparisons are needed to determine if $x \leq a_\ell$ and $x > a_m$, giving at most $5k$ comparisons in all. Finally, a list with one or two elements will require at most an additional four comparisons. Thus, for a list of length $3^k$, our algorithm requires $5k + 4$ comparisons. If $n$ is not a power of 3, find $k$ such that $3^{k-1} < n \leq 3^k$ and increase the length of the input list, if necessary, by adding terms all equal to $a_n$ until the extended list has length $3^k$. This list, and hence the original, is searched after $5k + 4$ comparisons. Since $3^{k-1} < n$, we know $k - 1 < \log_3 n$, so $5k + 4 < 5(1 + \log_3 n) + 4$. Since $11 + 5 \log_3 n \asymp \log_3 n$, the ternary search algorithm is $\mathcal{O}(\log_3 n)$.

4. (a) Since $n = 9 \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{9}{3} \rfloor = 3$ and $m = \lfloor \frac{18}{3} \rfloor = 6$. Since $2 = x \leq a_3 = 3$, we set $n = \ell = 3$ and change the list to $1, 2, 3$.

Since $n \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{3}{3} \rfloor = 1$ and $m = \lfloor \frac{6}{3} \rfloor = 2$. Since $a_\ell = 1 < x = 2 \leq a_m = 2$, we replace $n$ by $m - \ell = 1$ change the list to 2.

Since $n = 1$ and $x = a_1$, we output "true" and stop.

This search required three comparisons of $x$ with an element in the list; a linear search would have used two comparisons.

(b) Since $n = 9 \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{9}{3} \rfloor = 3$ and $m = \lfloor 183 \rfloor = 6$. Since $7 = x > a_6 = 6$, we replace $n$ by $n - m = 3$ and change the list to $7, 8, 9$.

Since $3 = n \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{3}{3} \rfloor = 1$ and $m = \lfloor \frac{6}{3} \rfloor = 2$. Since $7 = x \leq a_1$, we replace $n$ by $\ell = 1$ and change the list to 7.

Since $n = 1$ and $x = a_1$, we output "true" and stop.

This search required two comparisons of $x$ with an element in the list; a linear search would have used seven comparisons.

(c) Since $n = 9 \neq 1$ and $n \neq 2$, the algorithm sets $\ell = \lfloor \frac{9}{3} \rfloor = 3$ and $m = \lfloor \frac{18}{3} \rfloor = 6$. Since $x = 10 > a_6 = 6$, the algorithm replaces $n$ by $n - m = 3$ and changes the list to $7, 8, 9$.

Since $n \neq 1$ and $n \neq 2$, the algorithm sets $\ell = \lfloor \frac{3}{2} \rfloor = 1$ and $m = \lfloor \frac{6}{3} \rfloor = 2$. Since $x = 10 > a_2 = 8$, the algorithm replaces $n$ with $n - m = 1$ and changes the list to 9.

Since $n = 1$ and $x \neq a_1$, the algorithm outputs "false" and stops.

This search required three comparisons of $x$ with other elements; a linear search would have required nine.

5. (a) Since $n = 11 \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{11}{3} \rfloor = 3$ and $m = \lfloor \frac{22}{3} \rfloor = 7$. Since $7 = x \leq a_3 = 7$, we set $n = \ell = 3$ and change the list to $3, 5, 7$.

Since $n \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{3}{3} \rfloor = 1$ and $m = \lfloor \frac{6}{3} \rfloor = 2$. Since $7 = x > a_m = 5$, we replace $n$ by $n - m = 1$ change the list to 7.

Since $n = 1$ and $x = a_1$, we output "true" and stop.

This search required three comparisons of $x$ with an element in the list; a linear search would have used three comparisons.

(b) Since $n = 11 \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{11}{3} \rfloor = 3$ and $m = \lfloor 223 \rfloor = 7$. Since $30 = x > a_7 = 17$, we replace $n$ by $n - m = 4$ and change the list to $22, 24, 30, 31$.

Since $4 = n \neq 1$ and $n \neq 2$, we set $\ell = \lfloor \frac{4}{3} \rfloor = 1$ and $m = \lfloor \frac{8}{3} \rfloor = 2$. Since $30 = x > a_m$, we replace $n$ by $n - m = 2$ and change the list to $30, 31$.

Since $n = 2$, $x \neq a_1$ and $x = a_2$, we output "true" and stop.

This search required four comparisons of $x$ with an element in the list; a linear search would have used ten comparisons.

(c) Since $n = 11 \neq 1$ and $n \neq 2$, the algorithm sets $\ell = \lfloor \frac{11}{3} \rfloor = 3$ and $m = \lfloor \frac{22}{3} \rfloor = 7$. Since $7 = a_3 < x = 16 \leq a_7 = 17$, the algorithm replaces $n$ by $m - \ell = 4$ and changes the list to $10, 12, 15, 17$.

Since $4 = n \neq 1$ and $n \neq 2$, the algorithm sets $\ell = \lfloor \frac{4}{3} \rfloor = 1$ and $m = \lfloor \frac{8}{3} = 2$. Since $x = 16 > a_2 = 12$, the algorithm replaces $n$ with $n - m = 2$ and changes the list to $15, 17$.

Since $n = 2$ and $x \neq a_1$ and $x \neq a_2$, the algorithm outputs "false" and stops.

This search required five comparisons of $x$ with other elements; a linear search would have required eleven.

6. (a) [BB] To find the complement of the set $A = \{a_1, a_2, \ldots, a_k\}$ with respect to $U = \{1, 2, \ldots, 100\}$,

**Step 1:** For $i = 1$ to 100, search $A$ for $i$ and if $i$ is not found, output $i$.

The output numbers are the elements of $A^c$.

(b) To find the complement of the set $A = \{a_1, a_2, \ldots, a_k\}$ with respect to $U = \{1, 2, \ldots, n\}$,

**Step 1:** For $i = 1$ to $n$, search $A$ for $i$ and if $i$ is not found, output $i$.

The output numbers are the elements of $A^c$.

Using the efficient binary search, each search requires $\mathcal{O}(\log k)$ comparisons. There are $n$ steps, so the algorithm is $\mathcal{O}(n \log k)$.

7. [BB]

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $1, 2, 3, 4, 5$ | $2, 4, 6, 8, 10$ | |
| $2, 3, 4, 5$ | $2, 4, 6, 8, 10$ | $1$ |
| $3, 4, 5$ | $2, 4, 6, 8, 10$ | $1, 2$ |
| $3, 4, 5$ | $4, 6, 8, 10$ | $1, 2, 2$ |
| $4, 5$ | $4, 6, 8, 10$ | $1, 2, 2, 3$ |
| $5$ | $4, 6, 8, 10$ | $1, 2, 2, 3, 4$ |
| $5$ | $6, 8, 10$ | $1, 2, 2, 3, 4, 4$ |
| | $6, 8, 10$ | $1, 2, 2, 3, 4, 4, 5, 6, 8, 10$ |

The algorithm required seven comparisons.

8.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $2, 4, 4, 6, 8$ | $1, 5, 7, 9, 10$ | |
| $2, 4, 4, 6, 8$ | $5, 7, 9, 10$ | $1$ |
| $4, 4, 6, 8$ | $5, 7, 9, 10$ | $1, 2$ |
| $4, 6, 8$ | $5, 7, 9, 10$ | $1, 2, 4$ |
| $6, 8$ | $5, 7, 9, 10$ | $1, 2, 4, 4$ |
| $6, 8$ | $7, 9, 10$ | $1, 2, 4, 4, 5$ |
| $8$ | $7, 9, 10$ | $1, 2, 4, 4, 5, 6$ |
| $8$ | $9, 10$ | $1, 2, 4, 4, 5, 6, 7$ |
| | | $1, 2, 4, 4, 5, 6, 7, 8, 9, 10$ |

The algorithm required seven comparisons.

9.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $2,4,7,7,9,11$ | $1,4,9,12$ | |
| $2,4,7,7,9,11$ | $4,9,12$ | $1$ |
| $4,7,7,9,11$ | $4,9,12$ | $1,2$ |
| $7,7,9,11$ | $4,9,12$ | $1,2,4$ |
| $7,7,9,11$ | $9,12$ | $1,2,4,4$ |
| $7,9,11$ | $9,12$ | $1,2,4,4,7$ |
| $9,11$ | $9,12$ | $1,2,4,4,7,7$ |
| $11$ | $9,12$ | $1,2,4,4,7,7,9$ |
| $11$ | $12$ | $1,2,4,4,7,7,9,9$ |
| | | $1,2,4,4,7,7,9,9,11,12$ |

The algorithm required nine comparisons.

10. (a) [BB] The lists $1$ and $2,3,4$ can be merged with one comparison.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $1$ | $2,3,4$ | |
| | | $1,2,3,4$ |

(b) [BB] The lists $5$ and $2,3,4$ can be merged with three comparisons.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $5$ | $2,3,4$ | |
| $5$ | $3,4$ | $2$ |
| $5$ | $4$ | $2,3$ |
| | | $2,3,4,5$ |

(c) [BB] The lists $1,3,5$ and $2,4,6$ can be merged with $3+3-1=5$ comparisons.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $1,3,5$ | $2,4,6$ | |
| $3,5$ | $2,4,6$ | $1$ |
| $3,5$ | $4,6$ | $1,2$ |
| $5$ | $4,6$ | $1,2,3$ |
| $5$ | $6$ | $1,2,3,4$ |
| | | $1,2,3,4,5,6$ |

11. Two such lists are $1,2,3,4$ and $5$, and here's why.

| $\mathcal{L}_1$ | $\mathcal{L}_2$ | $\mathcal{L}_3$ |
|---|---|---|
| $1,2,3,4$ | $5$ | |
| $2,3,4$ | $5$ | $1$ |
| $3,4$ | $5$ | $1,2$ |
| $4$ | $5$ | $1,2,3$ |
| | | $1,2,3,4,5$ |

12. To merge two given sorted lists

$$\mathcal{L}_1 : a_1 \le a_2 \le \cdots \le a_s \quad \text{and} \quad \mathcal{L}_2 : b_1 \le b_2 \le \cdots \le b_t,$$

of lengths $s$ and $t$, into a single sorted list

$$\mathcal{L}_3 : c_1 \le c_2 \le \cdots \le c_{s+t}$$

of length $s+t$, proceed as follows.

**Step 1:** Set $F = 0$.

**Step 2:** If $\mathcal{L}_1$ is empty, set $\mathcal{L}_3 = \mathcal{L}_2$ and $F = 1$. If $\mathcal{L}_2$ is empty, set $\mathcal{L}_3 = \mathcal{L}_1$ and $F = 1$; otherwise let $\mathcal{L}_3$ be the empty list and proceed to Step 2.

**Step 3:** If $F = 0$, do the following:

Suppose $a_1 \leq b_1$. Then append $a_1$ to $\mathcal{L}_3$ and, if this empties $\mathcal{L}_1$, append the elements of $\mathcal{L}_2$ to $\mathcal{L}_3$ and stop. If $r$ elements remain in $\mathcal{L}_1$ and $r \neq 0$, then label them $a_1, a_2, \ldots, a_r$ in increasing order and repeat Step 2.

Suppose $a_1 > b_1$. Then append $b_1$ to $\mathcal{L}_3$ and, if this empties $\mathcal{L}_2$, append the elements of $\mathcal{L}_1$ to $\mathcal{L}_3$ and stop. If $r$ elements remain in $\mathcal{L}_2$ and $r \neq 0$, then label them $b_1, b_2, \ldots, b_r$ in increasing order and repeat Step 2.

Eventually, one of the lists is empty, so $F$ is set to 1 at Step 2 and the lines following the "If" statement in Step 3 are ignored.

13. (a) [BB] Here's the bubble sort:

$k = 5$: $\underline{3,1},7,2,5,4 \to 1,\underline{3,7},2,5,4 \to 1,3,\underline{7,2},5,4 \to 1,3,2,\underline{7,5},4$
$\to 1,3,2,5,\underline{7,4}$

$k = 4$: $\underline{1,3},2,5,4,7 \to 1,\underline{3,2},5,4,7 \to 1,2,\underline{3,5},4,7 \to 1,2,3,\underline{5,4},7$

$k = 3$: $\underline{1,2},3,4,5,7 \to 1,\underline{2,3},4,5,7 \to 1,2,\underline{3,4},5,7$

$k = 2$: $\underline{1,2},3,4,5,7 \to 1,\underline{2,3},4,5,7$

$k = 1$: $\underline{1,2},3,4,5,7 \to 1,2,3,4,5,7.$

This required a total of $5 + 4 + 3 + 2 + 1 = 15$ comparisons.

(b) [BB] Here's the merge sort.

**Step 2:** $3;\quad 1;\quad 7;\quad 2;\quad 5;\quad 4$

**Step 3:** $1,3;\quad 2,7;\quad 4,5$

**Step 3:** $1,2,3,7;\quad 4,5$

**Step 3:** $1,2,3,4,5,7$

Merging two lists of length 1 to one of length 2 requires $1 + 1 - 1 = 1$ comparison. Thus, the initial merging of six lists of length 1 to three lists of length 2 requires $1+1+1 = 3$ comparisons. Merging three lists of length 2 to one of length 2 and one of length 2 requires $2 + 2 - 1 = 3$ comparisons. The final merging of lists of lengths 4 and 2 requires $4 + 2 - 1 = 5$ comparisons, for a total of $3 + 3 + 5 = 11$ comparisons.

14. (a) Here's the bubble sort:

$k = 6$: $\underline{7,2},2,5,3,5,4 \to 2,\underline{7,2},5,3,5,4 \to 2,2,\underline{7,5},3,5,4 \to 2,2,5,\underline{7,3},5,4 \to 2,2,5,3,\underline{7,5},4 \to 2,2,5,3,\underline{57,4}$

$k = 5$: $\underline{2,2},5,3,5,4,7 \to 2,\underline{2,5},3,5,4,7 \to 2,2,\underline{5,3},5,4,7 \to 2,2,3,\underline{5,5},4,7 \to 2,2,3,5,\underline{5,4},7$

$k = 4$: $\underline{2,2},3,5,4,5,7 \to 2,\underline{2,3},5,4,5,7 \to 2,2,\underline{3,5},4,5,7 \to 2,2,3,\underline{5,4},5,7$

$k = 3$: $\underline{2,2},3,4,5,5,7 \to 2,\underline{2,3},4,5,5,7 \to 2,2,\underline{3,4},5,5,7$

$k = 2$: $\underline{2,2},3,4,5,5,7$ $\rightarrow$ $2,\underline{2,3},4,5,5,7$

$k = 1$: $\underline{2,2},3,4,5,5,7$ $\rightarrow$ $2,2,3,4,5,5,7$

This required a total of $6+5+4+3+2+1 = \frac{(6)(7)}{2} = 21$ comparisons.

(b) Here's the merge sort.

**Step 2:** 7; 2; 2; 5; 3; 5; 4;

**Step 3:** 2,7; 2,5; 3,5; 4

**Step 3:** 2,2,5,7; 3,4,5

**Step 3:** 2,2,3,4,5,5,7

Merging two lists of length 1 to one of length 2 requires $1+1-1 = 1$ comparison. Thus, the initial merging of seven lists of length 1 to three of length 2 and one of length 1 required $1+1+1 = 3$ comparisons. The merging of three lists of length 2 and one of length 1 to one list of length 4 and another of length 3 required $(2+2-1)+(2+1-1) = 5$ comparisons. The final merging required $4+3-1 = 6$ comparisons. The total is $3+5+6 = 14$ comparisons.

15. (a) Here's the bubble sort:

$k = 7$: $\underline{10,11},15,3,18,14,7,1$
$\rightarrow$ $10,\underline{11,15},3,18,14,7,1$
$\rightarrow$ $10,11,\underline{15,3},18,14,7,1$
$\rightarrow$ $10,11,\underline{3,15},18,14,7,1$
$\rightarrow$ $10,11,3,\underline{15,18},14,7,1$
$\rightarrow$ $10,11,3,15,\underline{18,14},7,1$
$\rightarrow$ $10,11,3,15,14,\underline{18,7},1$
$\rightarrow$ $10,11,3,15,14,\underline{7,18,1}$

$k = 6$: $\underline{10,11},3,15,14,7,1,18$
$\rightarrow$ $10,\underline{11,3},15,14,7,1,18$
$\rightarrow$ $10,3,\underline{11,15},14,7,1,18$
$\rightarrow$ $10,3,11,\underline{15,14},7,1,18$
$\rightarrow$ $10,3,11,14,\underline{15,7},1,18$
$\rightarrow$ $10,3,11,14,7,\underline{15,1},18$

$k = 5$: $\underline{10,3},11,14,7,1,15,18$
$\rightarrow$ $3,\underline{10,11},14,7,1,15,18$
$\rightarrow$ $3,10,\underline{11,14},7,1,15,18$
$\rightarrow$ $3,10,11,\underline{14,7},1,15,18$
$\rightarrow$ $3,10,11,7,\underline{14,1},15,18$

$k = 4$: $\underline{3,10},11,7,1,14,15,18$
$\rightarrow$ $3,\underline{10,11},7,1,14,15,18$
$\rightarrow$ $3,10,\underline{11,7},1,14,15,18$
$\rightarrow$ $3,10,7,\underline{11,1},14,15,18$

$k = 3$: $\underline{3,10},7,1,11,14,15,18$
$\rightarrow$ $3,\underline{10,7},1,11,14,15,18$
$\rightarrow$ $3,7,\underline{10,1},11,14,15,18$

$k = 2$: $\underline{3,7},1,10,11,14,15,18$
$\rightarrow$ $3,\underline{7,1},10,11,14,15,18$

$k = 1$: $\underline{3,1},7,10,11,14,15,18$
$\rightarrow$ $1,3,7,10,11,14,15,18.$

This required a total of $7+6+5+4+3+2+1 = \frac{(7)(8)}{2} = 28$ comparisons.

(b) Here's the merge sort.

**Step 2:** 10; 11; 15; 3; 18; 14; 7; 1

**Step 3:** 10,11; 3,15; 14,18; 1,7

**Step 3:** 3,10,11,15; 1,7,14,18

**Step 3:** 1,3,7,10,11,14,15,18

Merging two lists of length 1 to one of length 2 requires $1+1-1 = 1$ comparison. Thus, the initial merging of eight lists of length 1 to four of length 2 requires $1+1+1+1 = 4$ comparisons. The merging of four lists of length 2 to two of length 4 requires $3+3 = 6$ comparisons and the final merging requires $4+4-1 = 7$ comparisons. The total is $4+6+7 = 17$ comparisons.

16. [BB] Sort the list into increasing order $a_1 \le a_2 \le \cdots \le a_n$. If $n = 2m + 1$ is odd, the median is $a_m$; if $n = 2m$ is even, the median is $\frac{1}{2}(a_m + a_{m+1})$. Since determining the parity of $n$ adds only another operation, the complexity of this algorithm is the complexity of the sort, at best $\mathcal{O}(n \log n)$.

17. (a) [BB] Here's the bubble sort.

$k = 7$: $\underline{a, b}, c, d, u, v, w, x \to a, \underline{b, c}, d, u, v, w, x \to a, c, \underline{b, d}, u, v, w, x$
$\to a, c, d, \underline{b, u}, v, w, x \to a, c, d, u, \underline{b, v}, w, x$
$\to a, c, d, u, b, \underline{v, w}, x \to a, c, d, u, b, v, \underline{w, x}$

$k = 6$: $\underline{a, c}, d, u, b, v, x, w \to a, \underline{c, d}, u, b, v, x, w \to a, d, \underline{c, u}, b, v, x, w$
$\to a, d, u, \underline{c, b}, v, x, w \to a, d, u, c, \underline{b, v}, x, w \to a, d, u, c, b, \underline{v, x}, w$

$k = 5$: $\underline{a, d}, u, c, b, x, v, w \to d, \underline{a, u}, c, b, x, v, w \to d, a, \underline{u, c}, b, x, v, w$
$\to d, a, u, \underline{c, b}, x, v, w \to d, a, u, c, \underline{b, x}, v, w$

$k = 4$: $\underline{d, a}, u, c, x, b, v, w \to d, \underline{a, u}, c, x, b, v, w \to d, a, \underline{u, c}, x, b, v, w$
$\to d, a, u, \underline{c, x}, b, v, w$

$k = 3$: $\underline{d, a}, u, c, x, b, v, w \to d, \underline{a, u}, c, x, b, v, w \to d, a, \underline{u, c}, x, b, v, w$

$k = 2$: $\underline{d, a}, u, c, x, b, v, w \to d, \underline{a, u}, c, x, b, v, w$

$k = 1$: $\underline{d, a}, u, c, x, b, v, w \to d, a, u, c, x, b, v, w$

A total of $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ comparisons is needed.

(b) Here's the merge sort.

**Step 2:** $a$; $b$; $c$; $d$; $u$; $v$; $w$; $x$

**Step 3:** $a, b$; $d, c$; $u, v$; $x, w$

**Step 3:** $d, a, c, b$; $u, x, v, w$

**Step 3:** $d, a, u, c, x, b, v, w$

Merging two lists of length one to one of length two requires $1 + 1 - 1 = 1$ comparison. Thus, the initial merging of eight lists of length one to four of length two requires $1 + 1 + 1 + 1 = 4$ comparisons. The merging of four lists of length two to two of length four requires $3 + 3 = 6$ comparisons and the final merging requires $4+4-1 = 7$ comparisons. The total is $4+6+7 = 17$ comparisons.

18. (a) [BB] $\underline{c, a}, e, b, d \to a, \underline{c, e}, b, d \to a, c, \underline{e, b}, d \to a, c, b, \underline{e, d}$
$\underline{a, c}, b, d, e \to a, \underline{c, b}, d, e \to a, b, \underline{c, d}, e$
$\underline{a, b}, c, d, e \to a, \underline{b, c}, d, e$
$\underline{a, b}, c, d, e \to a, b, c, d, e$

(b) $\underline{1, 2}, 3, 4, 5 \to 2, \underline{1, 3}, 4, 5 \to 2, 1, \underline{3, 4}, 5 \to 2, 1, 4, \underline{3, 5}$
$\underline{2, 1}, 4, 5, 3 \to 2, \underline{1, 4}, 5, 3 \to 2, 4, \underline{1, 5}, 3$
$\underline{2, 4}, 1, 5, 3 \to 2, \underline{4, 1}, 5, 3$
$\underline{2, 4}, 1, 5, 3 \to 2, 4, 1, 5, 3$

(c) As suggested in parts (a) and (b), this task can be accomplished by mimicking the bubble sort but, where a swap of $a_i$ and $a_{i+1}$ is required, swapping $i$ and $i + 1$ instead. Here's the algorithm:

**Step 1:** For $k = n - 1$ to 1,

- For each $i = 1$ to $k$, if $a_i > a_{i+1}$, swap $i$ and $i + 1$.

19. As mentioned in the text and illustrated in Problem 26, the maximum number of comparisons occurs when just one element remains in one list at the time the other is emptied. Thus an example exists for any $s$ and $t$. Let $\mathcal{L}_2\colon b_1 \le b_2 \le \cdots \le b_t$ be any ordered list of length $t$. Then choose $\mathcal{L}_1\colon a_1 \le a_2 \le \cdots \le a_s$ with $a_{s-1} < b_t < a_s$. This ensures that $a_s$ is the only remaining element at the time that $\mathcal{L}_2$ is emptied.

20. The most efficient sorting algorithm is $\mathcal{O}(n \log n)$. Thus sorting and then applying a binary search is $\mathcal{O}(n \log n + \log n) = \mathcal{O}(n \log n)$. Since $n \preceq n \log n$, this is less efficient than a linear search.

21. [BB] The answer is $\min\{s, t\}$. It is impossible to have fewer than this number of comparisons since until $\min\{s, t\}$ of comparisons have been made, elements remain in each list. To see that $\min\{s, t\}$ can be achieved, consider ordered lists $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_t$ where $s < t$ and $a_s < b_1$. After $s$ comparisons, the first list is empty.

22. Let $k = 2$ and consider the list $4, 2, 3, 1$ of length $2^k$. Two comparisons are needed to merge into pairs $2, 4;\ 1, 3$. Three more comparisons are needed to complete the merge sort, giving a total of five. Note that $k2^k - 2^k + 1 = 2(4) - 4 + 1 = 5$ as well.

23. [BB] Using the efficient binary search, $\mathcal{O}(\log k)$ comparisons are needed to search the predecessors of $a_k$. In all, the number of comparisons is $\mathcal{O}(\log 1 + \log 2 + \cdots + \log n)$. Since $\log a + \log b = \log ab$, we have $\log 1 + \log 2 + \cdots + \log n = \log n!$. The result follows.

24. Remove the phrase "if $a_i > a_{i+1}$" from the bubble sort algorithm but everything else stays the same; that is, **always** swap $a_i$ and $a_{i+1}$. The algorithm requires $(n-1)+(n-2)+\cdots+3+2+1 = \frac{1}{2}n(n-1)$ swaps and hence is $\mathcal{O}(n^2)$.

25. [BB] This is less efficient. A merge sort is $\mathcal{O}(n \log n)$ and a binary search is $\mathcal{O}(\log n)$. Since $n + n \log n \asymp n \log n$, the suggested procedure is $\mathcal{O}(n \log n)$, while a linear search is $\mathcal{O}(n)$, which is better.

26. Testing each $a_i$ to see if it is 0 and increasing a counter by 1 whenever $a_i = 0$, we can determine the number of 0's in the list with $n$ comparisons. Assuming $k$ 0's, output $k$ 0's followed by $n - k$ 1's. This method is $\mathcal{O}(n)$, a considerable improvement on $\mathcal{O}(n^2)$ for a bubble sort and $\mathcal{O}(n \log n)$ for a merge sort.

## Exercises 8.4

1. (a) [BB]

| $t$ | Perm($t$) | $j$ | $m$ | $S$ |
|---|---|---|---|---|
| 1 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 2\ 3\ 4}$ | 3 | 4 | $\{1,2,4\}^c = \{3\}$ |
| 2 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 2\ 4\ 3}$ | 2 | 3 | $\{1,3\}^c = \{2,4\}$ |
| 3 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 3\ 2\ 4}$ | 3 | 4 | $\{1,3,4\}^c = \{2\}$ |
| 4 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 3\ 4\ 2}$ | 2 | 4 | $\{1,4\}^c = \{2,3\}$ |
| 5 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 4\ 2\ 3}$ | 3 | 3 | $\{1,4,3\}^c = \{2\}$ |
| 6 | $\underset{\pi_1\pi_2\pi_3\pi_4}{1\ 4\ 3\ 2}$ | 1 | 2 | $\{2\}^c = \{1,3,4\}$ |
| 7 | $\underset{\pi_1\pi_2\pi_3\pi_4}{2\ 1\ 3\ 4}$ | 3 | 4 | $\{2,1,4\}^c = \{3\}$ |
| 8 | 2 1 4 3 | | | |

(b)

| $t$ | Perm($t$) | $j$ | $m$ | $S$ |
|---|---|---|---|---|
| 9 | $\underset{\pi_1\pi_2\pi_3\pi_4}{2\ 3\ 1\ 4}$ | 3 | 4 | $\{2,3,4\}^c = \{1\}$ |
| 10 | $\underset{\pi_1\pi_2\pi_3\pi_4}{2\ 3\ 4\ 1}$ | 2 | 4 | $\{2,4\}^c = \{1,3\}$ |
| 11 | $\underset{\pi_1\pi_2\pi_3\pi_4}{2\ 4\ 1\ 3}$ | 3 | 3 | $\{2,4,3\}^c = \{1\}$ |
| 12 | $\underset{\pi_1\pi_2\pi_3\pi_4}{2\ 4\ 3\ 1}$ | 1 | 3 | $\{3\}^c = \{1,2,4\}$ |
| 13 | $\underset{\pi_1\pi_2\pi_3\pi_4}{3\ 1\ 2\ 4}$ | 3 | 4 | $\{3,1,4\}^c = \{2\}$ |
| 14 | $\underset{\pi_1\pi_2\pi_3\pi_4}{3\ 1\ 4\ 2}$ | 2 | 2 | $\{3,2\}^c = \{1,4\}$ |
| 15 | $\underset{\pi_1\pi_2\pi_3\pi_4}{3\ 2\ 1\ 4}$ | 3 | 4 | $\{3,2,4\}^c = \{1\}$ |
| 16 | 3 2 4 1 | | | |

(c)

| $t$ | Perm($t$) | $j$ | $m$ | $S$ |
|---|---|---|---|---|
| 17 | $\underset{\pi_1\pi_2\pi_3\pi_4}{3\ 4\ 1\ 2}$ | 3 | 2 | $\{3,4,2\}^c = \{1\}$ |
| 18 | $\underset{\pi_1\pi_2\pi_3\pi_4}{3\ 4\ 2\ 1}$ | 1 | 4 | $\{4\}^c = \{1,2,3\}$ |
| 19 | $\underset{\pi_1\pi_2\pi_3\pi_4}{4\ 1\ 2\ 3}$ | 3 | 3 | $\{4,1,3\}^c = \{2\}$ |
| 20 | $\underset{\pi_1\pi_2\pi_3\pi_4}{4\ 1\ 3\ 2}$ | 2 | 2 | $\{4,2\}^c = \{1,3\}$ |
| 21 | $\underset{\pi_1\pi_2\pi_3\pi_4}{4\ 2\ 1\ 3}$ | 3 | 3 | $\{4,2,3\}^c = \{1\}$ |
| 22 | $\underset{\pi_1\pi_2\pi_3\pi_4}{4\ 2\ 3\ 1}$ | 2 | 3 | $\{4,3\}^c = \{1,2\}$ |
| 23 | $\underset{\pi_1\pi_2\pi_3\pi_4}{4\ 3\ 1\ 2}$ | 3 | 2 | $\{4,3,2\}^c = \{1\}$ |
| 24 | 4 3 2 1 | | | |

2. (a)

| | | | | |
|---|---|---|---|---|
| 12345 | 12354 | 12435 | 12453 | 12534 |
| 12543 | 13245 | 13254 | 13425 | 13452 |
| 13524 | 13542 | 14235 | 14253 | 14325 |
| 14352 | 14523 | 14532 | 15234 | 15243 |

(b) [BB] 42531, 43125, 43152, 43215, 43251

(c) 41532, 42135, 42153, 42315, 42351

3. (a) [BB] We consider each part of Step 2. Finding the largest $j$ such that $\pi_j < \pi_{j+1}$ requires at most $n$ comparisons. At most another $n$ comparisons are needed to find the minimum of $\{\pi_i \mid i < j, \pi_i > \pi_j\}$. To find the complement of a subset $A$ of $\{1, 2, \ldots, n\}$ requires searching $A$ for each of the elements $1, 2, \ldots, n$ and noting those which are not in $A$. Using the efficient binary search, each search is $\mathcal{O}(\log_2 n)$, adding another $n \log_2 n$ comparisons. An efficient merge sort adds another $n \log_2 n$ comparisons, so each pass through Step 2 requires at most $n + n + n \log_2 n + n \log_2 n = 2(n + n \log_2 n)$ comparisons. Since this step is executed $n! - 1$ times and $n + n \log_2 n \sim n \log_2 n$, the algorithm is $\mathcal{O}(n! n \log_2 n)$.

   (b) Parts (iii) and (iv) of Step 2 can be accomplished much more efficiently than their wording suggests. In fact, no searching or sorting is required. The $j$ found in part (i) divides the permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ into three parts; an initial segment $\pi_1 \ldots \pi_{j-1}$, a middle term $\pi_j$, and a terminal segment $\pi_{j+1} \ldots \pi_n$. By definition of $j$, we have $\pi_{j+1} > \pi_{j+2} \cdots > \pi_n$. Thus, parts (iii) and (iv) are equivalent simply to reversing the order of the terminal segment and then interchanging $\pi_j$ and $m$.

   For example, consider how the successor of 1524763 is determined. Here $j = 4$, $m = 6$, $S = \{1, 5, 2, 6\}^c = \{3, 4, 7\}$ and we obtain 1526347. But this permutation can be determined without the use of $S$ or a sort by reversing the terminal segment 763 to get 1524367 and then interchanging $\pi_j = \pi_4 = 4$ and $m = 6$.

   Omitting searching and sorting allows Step 2 to be accomplished with $\mathcal{O}(n)$ comparisons. Since this step is executed $n! - 1$ times, the algorithm can be accomplished with $\mathcal{O}(n!n)$ comparisons.

4. (a) Since $x = \pi_j$ is replaced by an integer larger than itself and all integers to the left of $x$ remain unchanged, $\pi \prec \pi'$.

(b) Recall from (a) that all numbers to the left of $x = \pi_j$ are the same in $\pi$ and $\pi'$ and hence in $\sigma$. Also, since $x$ is replaced (in $\pi'$) by the smallest number $m$ to its right which is greater than $x$, and the sequence of numbers following $m$ is strictly increasing in $\pi'$, it follows that $x$ must also be in the same position in $\sigma$ as in $\pi$. But, since the sequence of integers following $x$ in $\pi$ is strictly decreasing and since $\pi \preceq \sigma$, it can only be that $\pi = \sigma$.

5. (a) [BB]

| | | | | | |
|---|---|---|---|---|---|
| 1234 | 1235 | 1236 | 1245 | 1246 |
| 1256 | 1345 | 1346 | 1356 | 1456 |
| 2345 | 2346 | 2356 | 2456 | 3456 |

(b)

| | | | | |
|---|---|---|---|---|
| 123 | 124 | 125 | 126 | 134 |
| 135 | 136 | 145 | 146 | 156 |
| 234 | 235 | 236 | 245 | 246 |
| 256 | 345 | 346 | 356 | 456 |

6. (a) [BB] 23459, 23467, 23468 precede; 23478, 23479, 23489 follow.

(b) 13478, 13479, 13489 precede; 13568, 13569, 13578 follow.

(c) 45678, 45679, 45689 precede; 46789 and 56789, the last combination in the list, follow.

7. (a)

| | | | | | | |
|---|---|---|---|---|---|---|
| 12 | 13 | 14 | 15 | 16 | 17 | 23 |
| 24 | 25 | 26 | 27 | 34 | 35 | 36 |
| 37 | 45 | 46 | 47 | 56 | 57 | 67 |

(b) [BB] First list the combinations of $1, 2, \ldots, n$ taken $r$ at a time by the method of Proposition 8.4.3. Then, for each of these combinations, enumerate all permutations of its elements using Algorithm 8.4.2.

(c)

| | | | | | | |
|---|---|---|---|---|---|---|
| 12 | 21 | 13 | 31 | 14 | 41 | 15 |
| 51 | 16 | 61 | 17 | 71 | 23 | 32 |
| 24 | 42 | 25 | 52 | 26 | 62 | 27 |
| 72 | 34 | 43 | 35 | 53 | 36 | 63 |
| 37 | 73 | 45 | 54 | 46 | 64 | 47 |
| 74 | 56 | 65 | 57 | 75 | 67 | 76 |

8. (a) [BB]

| | | | | | | |
|---|---|---|---|---|---|---|
| 123456 | 123457 | 123458 | 123467 | 123468 | 123478 | 123567 |
| 123568 | 123578 | 123678 | 124567 | 124568 | 124578 | 124678 |
| 125678 | 134567 | 134568 | 134578 | 134678 | 135678 | 145678 |
| 234567 | 234568 | 234578 | 234678 | 235678 | 245678 | 345678 |

(b) The first ten combinations are 123 124 125 126 127
128 134 135 136 137.

The last ten are 456 457 458 467 468
478 567 568 578 678.

(c) The first ten permutations are 123 132 213 231 312
321 124 142 214 241.

The last ten are 758 785 857 875 678
687 768 786 867 876.

9. [BB] If $a_1 a_2 \ldots a_r$ is one of the combinations of $1, 2, \ldots, n$ taken $r$ at a time, $a_1$ cannot be less than 1, $a_2$ cannot be less than 2 and, in general, $a_i$ cannot be less than $i$. Thus, $123 \ldots r$ is the smallest combination.

Now let $\pi$ be a combination and $\pi'$ the combination determined by $\pi$ as in the proposition. We show that $\pi'$ is the immediate successor of $\pi$ with respect to lexicographic order. First, since all numbers to the left of $k-1$ are the same in both combinations, while $k-1$ is increased to $k$, we conclude that

$\pi \prec \pi'$. Now suppose that $\pi \prec \sigma \preceq \pi'$ for some combination $\sigma$. All integers to the left of $k-1$ (in $\pi$) are the same in $\pi'$ as well, and hence also in $\sigma$. If $k-1$ were also the same in $\sigma$, then we would have $\sigma = \pi$ since no number to the right of $k-1$ can be increased in $\pi$. Thus, in $\sigma$, $k-1$ must be increased to $k$ (and no more, since $\sigma \prec \pi'$). It then follows that $\sigma = \pi'$ since $\pi'$ is the smallest sequence whose initial segment (up to $k$) consists of the integers of $\pi'$.

10. (a) [BB] $n$; $n-r+1$; $n-r+j$.

    (b) [BB] It must be $123\ldots r$; (c) [BB] It must be $(n-r+1)\ (n-r+2)\ \ldots\ n$.

11. [BB] Given $n \geq r > 0$, to enumerate the $\binom{n}{r}$ combinations of $1, 2, \ldots, n$ taken $r$ at a time, proceed as follows.

    **Step 1:** Set $t = 1$. Output $\text{Comb}(1) = 123\ldots r$. If $r = n$, stop.

    **Step 2:** For $t = 1$ to $\binom{n}{r} - 1$, given combination $\text{Comb}(t) = a_1 a_2 \ldots a_r$, determine the next combination $\text{Comb}(t+1)$ as follows.

    (i) Find the largest $j$ such that $a_j < n - r + j$.

    (ii) Output $\text{Comb}(t+1) = a_1\ a_2\ \ldots\ a_{j-1}\ a_j + 1\ a_j + 2\ \ldots\ a_j + r - j + 1$.

12. Let $S = \{x_1, x_2, \ldots, x_{kn}\}$. Generate the combinations of $1, 2, \ldots, kn$ taken $k$ at a time. As soon as combination $a_1 a_2 \ldots a_k$ is generated, output the subset $\{x_{a_1}, x_{a_2}, \ldots, x_{a_k}\}$.

# Chapter 8 Review

1. (a) $S = 2$, $S = 0 + 2(-3) = -6$, $S = -4 + (-6)(-3) = 14$, $S = 1 + 14(-3) = -41 = f(-3)$.

   (b) $S = 1$, $S = -2 + 1(2) = 0$, $S = 1 + 0(2) = 1$, $S = -5 + 1(2) = -3$, $S = 6 + (-3)(2) = 0 = f(2)$.

2. (a)

| 149 | × | 712 |
|---|---|---|
| ~~74~~ | | ~~1424~~ |
| 37 | | 2848 |
| ~~18~~ | | ~~5696~~ |
| 9 | | 11392 |
| ~~4~~ | | ~~22784~~ |
| ~~2~~ | | ~~45568~~ |
| 1 | | 91136 |
| | | 106088 |

(b)

| ~~1018~~ | × | ~~72~~ |
|---|---|---|
| 509 | | 144 |
| ~~254~~ | | ~~288~~ |
| 127 | | 576 |
| 63 | | 1152 |
| 31 | | 2304 |
| 15 | | 4608 |
| 7 | | 9216 |
| 3 | | 18432 |
| 1 | | 36864 |
| | | 73296 |

3. In each case, we set up a flag $F$ which is 0 if the relation has the desired property and is 1 otherwise.

   (a) **Step 1:** Set $F = 0$ and $i = 1$.

   **Step 2:** while $F = 0$ and $i < t$,
   if $a_i \neq b_i$ set $F = 1$;
   replace $i$ by $i + 1$.
   end while

**Step 3:** if $F = 0$, output "reflexive"; otherwise, output "not reflexive".

The algorithm requires at most $t$ comparisons.

(b) We set up a matrix $A = [a_{ij}]$ whose $(i, j)$ entry is a 1 if and only if $(i, j) \in \mathcal{R}$, and otherwise 0.

**Step 1:** for $i = 1$ to $t$
  for $j = 1$ to $t$
    set $a_{ij} = 0$.

**Step 2:** For $i = 1$ to $t$, set the $(a_i, b_i)$ entry of $A$ equal to 1.

**Step 3:** Set $F = 0$ and $i = 1$.

**Step 4:** while $F = 0$ and $i < t$,
  for $j = i$ to $t$
    if $a_{ij} \neq a_{ji}$ set $F = 1$;
  replace $i$ by $i + 1$.
 end while

**Step 5:** If $F = 0$, output "symmetric" and stop; otherwise output "not symmetric" and stop.

When $i = 1$, the loop inside Step 4 requires $t$ comparisons. When $i = 2$, the loop requires $t - 1$ comparisons, and so on. The number of comparisons is at most $t+(t-1)+\cdots+2+1 = \frac{1}{2}t(t+1)$, so the algorithm is $\mathcal{O}(t^2)$.

(c) We set up a matrix $A$ whose $(i, j)$ entry is a 1 if and only if $(i, j) \in \mathcal{R}$, and otherwise 0. The relation fails to be antisymmetric if and only if some $a_{ij} = 1 = a_{ji}$ with $i \neq j$, and this occurs if and only if $a_{ij} + a_{ji} = 2$.

**Step 1:** for $i = 1$ to $t$
  for $j = 1$ to $t$
    set $a_{ij} = 0$.

**Step 2:** For $i = 1$ to $t$, set the $(a_i, b_i)$ entry of $A$ equal to 1.

**Step 3:** Set $F = 0$ and $i = 1$.

**Step 4:** while $F = 0$ and $i < t$,
  for $j = i + 1$ to $t$
    if $a_{ij} + a_{ji} = 2$ set $F = 1$;
  replace $i$ by $i + 1$.
 end while

**Step 5:** If $F = 0$, output "antisymmetric" and stop; otherwise output "not antisymmetric" and stop.

When $i = 1$, the loop inside Step 4 requires $t - 1$ comparisons. When $i = 2$, the loop requires $t-2$ comparisons, and so on. The number of comparisons is at most $(t-1)+(t-2)\cdots+2+1 = \frac{1}{2}(t-1)t$, so the algorithm is $\mathcal{O}(t^2)$.

(d) We set up a matrix $A$ whose $(i, j)$ entry is a 1 if and only if $(i, j) \in \mathcal{R}$, and otherwise 0. The relation fails to be transitive if and only if some $a_{ij} = a_{jk} = 1$ for some $i, j, k$ while $a_{ij} = 0$.

**Step 1:** for $i = 1$ to $t$
  for $j = 1$ to $t$
    set $a_{ij} = 0$.

**Step 2:** For $i = 1$ to $t$, set the $(a_i, b_i)$ entry of $A$ equal to 1.

**Step 3:** Set $F = 0$ and $i = 1$.

**Step 4:** for $i = 1$ to $t$
  for $j = 1$ to $t$
    for $k = 1$ to $t$
      if $a_{ij} + a_{jk} = 2$ and $a_{ik} = 0$ set $F = 1$

**Step 5:** If $F = 0$, output "transitive" and stop; otherwise output "not transitive" and stop.

The loops in Step 4 are executed $t^3$ times, each iteration requiring one addition and two comparisons. So there are at most $3t^3$ additions and comparisons. The algorithm is $\mathcal{O}(t^3)$.

(e) **Step 1:** Set $F = 0$.

**Step 2:** while $F = 0$
for $i = 1$ to $t - 1$
for $j = i + 1$ to $t$
if $a_j = a_i$ and $b_j \neq b_i$ set $F = 1$.
end while

**Step 3:** If $F = 1$ output "Not a function";
else output "The relation is a function."

When $i = 1$, the $j$-loop in Step 2 requires $2(t - 1)$ comparisons. When $i = 2$, the loop requires $2(t-2)$ comparisons, and so on. The number of comparisons is $2[(t-1)+(t-2)+\cdots+2+1] = t(t - 1)$, so the algorithm is $\mathcal{O}(t^2)$.

4. (a) The best sorting algorithm (for example, a merge sort) is $\mathcal{O}(n \log n)$.

(b) There certainly are better ways to proceed. We could, for example, apply just the first two iterations of a bubble sort, or use the following $\mathcal{O}(n)$ algorithm (which requires at most $n - 3$ comparisons.)

Initially we set $M_1$ to be the larger of $a_1, a_2$ and $M_2$ the lesser. Then, in turn, we compare $M_2$ with each $a_i$. When $a_i > M_2$, we check whether or not $a_i$ is also larger than $M_1$. Eventually, $M_1$ is the largest and $M_2$ the second largest integer.

**Step 1:** if $a_1 > a_2$, set $M_1 = a_1$ and $M_2 = a_1$;
else set $M_1 = a_2$ and $M_2 = a_1$.

**Step 2:** if $n > 2$,
for $i = 3$ to $n$,
if $a_i > M_2$
if $a_i > M_1$, replace $M_1$ by $a_i$ and $M_2$ by $M_1$;
else replace $M_2$ by $a_i$.

**Step 3:** output $M_1$ and $M_2$.

5. **Step 1:** Let $M = a_1$ and $m = a_1$.

**Step 2:** for $i = 2$ to $n$,

- if $a_i > M$, replace $M$ by $a_i$;
- if $a_i < m$, replace $m$ by $a_i$;

**Step 3:** Output "The largest number is $M$ and the smallest is $m$."

6. **Step 1:** Set $n = 0$.

**Step 2:** If $x > 0$
while $x > n$ replace $n$ by $n + 1$;
output $n - 1$;
else while $-x > n$ replace $n$ by $n + 1$;
output $-n$

7. (a) $1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots n^k = n(n^k) = n^{k+1}$, so the result follows with $c = 1$ and $n_0 = 1$.

(b) Since $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$—see Problem 4 of Section 5.1—the result follows because a polynomial has the same order as its highest power, by Proposition 8.2.7.

8. (a) Let $f(n) = 3n^3-5n^2+2n+1$. By the triangle inequality, for $n \geq 1$, $|f(n)| \leq 3n^3+5n^2+2n+1 \leq 3n^3+5n^3+2n^3+n^3 = 11n^3$, so $f(n) = \mathcal{O}(n^3)$ with $n_0 = 1$ and $c = 11$ in Definition 8.2.1. Conversely (and here we borrow an idea from the proof of Proposition 8.2.7),

$$f(n) = n^3\left[3-\frac{5}{n}+\frac{2}{n^2}+\frac{1}{n^3}\right] \geq n^3\left[3-\frac{5}{n}-\frac{2}{n^2}-\frac{1}{n^3}\right].$$

Choose $n_0$ such that $\frac{5}{n}$, $\frac{2}{n^2}$ and $\frac{1}{n^3}$ are all less than $\frac{1}{3}$ for $n \geq n_0$, for example, $n_0 = 16$. Then

$$-\frac{5}{n} \geq -\frac{1}{3}, \quad -\frac{2}{n^2} \geq -\frac{1}{3} \quad \text{and} \quad -\frac{1}{n^3} \geq -\frac{1}{3}$$

so that $f(n) \geq n^3\left[3-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}\right] = 2n^3$; hence $n^3 \leq \frac{1}{2}f(n)$ for $n \geq n_0$. So $n^3 = \mathcal{O}(f(n))$ with $c = \frac{1}{2}$ in Definition 8.2.1.

(b) Let $f(n) = n^4-2n^3+3n^2-5n+7$. By the triangle inequality, for $n \geq 1$, $|f(n)| \leq n^4+2n^3+3n^2+5n+7 \leq n^4+2n^4+3n^4+5n^4+7n^4 = 18n^4$, so $f(n) = \mathcal{O}(n^4)$ with $n_0 = 1$ and $c = 18$ in Definition 8.2.1. Conversely (and here we borrow an idea from the proof of Proposition 8.2.7),

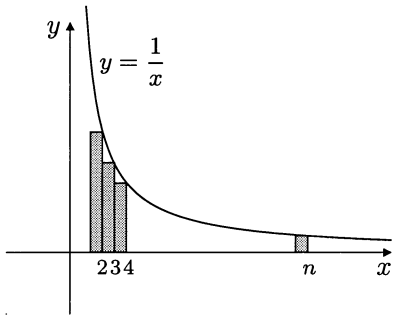
$$f(n) = n^4\left[1-\frac{2}{n}+\frac{3}{n^2}-\frac{5}{n^3}+\frac{7}{n^4}\right] \geq n^4\left[1-\frac{2}{n}-\frac{3}{n^2}-\frac{5}{n^3}-\frac{7}{n^4}\right].$$

Choose $n_0$ such that $\frac{2}{n}$, $\frac{3}{n^2}$, $\frac{5}{n^3}$ and $\frac{7}{n^4}$ are all less than $\frac{1}{8}$ for $n \geq n_0$, for example, $n_0 = 17$. Then

$$-\frac{2}{n} \geq -\frac{1}{8}, \quad -\frac{3}{n^2} \geq -\frac{1}{8}, \quad -\frac{5}{n^3} \geq -\frac{1}{8} \quad \text{and} \quad -\frac{7}{n^4} \geq -\frac{1}{8}$$

so that $f(n) \geq n^4\left[1-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}\right] = \frac{1}{2}n^4$; hence $n^4 \leq 2f(n)$ for $n \geq n_0$. So $n^4 = \mathcal{O}(f(n))$ with $c = 2$ in Definition 8.2.1.

9. (a) The rectangles in the graph to the right have base 1 and heights $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$. We see that $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is less than the area under the graph of $y = \frac{1}{x}$ from 1 to $n$. Thus $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \int_1^n \frac{1}{x}\,dx = \log n$ and $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\log n = \mathcal{O}(\log n)$.




(b) Let $M = \sqrt{n}$. The heart of the Sieve of Eratosthenes is a pair of loops:

```
for i from 2 to M do
    if $i$ is prime
       for j from 1 to n/i  cross out ji
```

When $i = 2$, there is one comparison followed by $\frac{n}{2}$ cross outs. When $i = 3$, there is one comparison and $\frac{n}{3}$ cross outs. For each $i$ between 2 and $M$ there is one comparison and at most $\frac{n}{i}$

cross outs. So the number of basic operations is at most

$$\left(1+\frac{n}{2}\right)+\left(1+\frac{n}{3}\right)+\cdots+\left(1+\frac{n}{M}\right)$$
$$=(M-1)+n\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{M}\right)$$
$$=M-1+n\mathcal{O}(\log M)=\sqrt{n}-1+n\mathcal{O}(\tfrac{1}{2}\log n)=\mathcal{O}(n\log n).$$

10. Since $n=11\neq 1$, we set $m=\lfloor\frac{11}{2}\rfloor=5$. Since $5=x=a_5=1$ and $x<a_5$ are both false, we replace $n$ by $11-5=6$ and change our list to $2,4,5,8,9,12$.

    Since $6=n\neq 1$, we set $m=\lfloor\frac{6}{2}\rfloor=3$. Since $5=x$ output "true" and stop.

    This search required three comparisons of 5 with an element in the list; a linear search would have used two comparisons.

11.

| | | |
|---|---|---|
| $-1,3,7,10,12,15$ | $-3,-1,2$ | $-4$ |
| $-1,3,7,10,12,15$ | $-1,2$ | $-4,-3$ |
| $3,7,10,12,15$ | $-1,2$ | $-4,-3,-1$ |
| $3,7,10,12,15$ | $2$ | $-4,-3,-1,-1$ |
| $3,7,10,12,15$ | | $-4,-3,-1,-1,2$ |
| | | $-4,-3,-1,-1,2,3,7,10,12,15$ |

    The algorithm required five comparisons.

12. (a) Here's the bubble sort:

    $k=6$: $\underline{9,-3},1,0,-4,5,3\to -3,\underline{9,1},0,-4,5,3\to -3,1,\underline{9,0},-4,5,3$
    $\to -3,1,0,\underline{9,-4},5,3\to -3,1,\underline{0,-4},\underline{9,5},3\to -3,1,0,\underline{-4,5},\underline{9,3}$

    $k=5$: $\underline{-3,1},0,-4,5,3,9\to -3,\underline{1,0},-4,5,3,9\to -3,0,\underline{1,-4},5,3,9$
    $\to -3,0,-4,\underline{1,5},3,9\to -3,0,\underline{-4,1},\underline{5,3},9$

    $k=4$ $\underline{-3,0},-4,1,3,5,9\to -3,\underline{0,-4},1,3,5,9\to -3,-4,\underline{0,1},3,5,9$
    $\to -3,-4,0,\underline{1,3},5,9$

    $k=3$: $\underline{-3,-4},0,1,3,5,9\to -4,\underline{-3,0},1,3,5,9\to -4,-3,\underline{0,1},3,5,9$

    $k=2$: $\underline{-4,-3},0,1,3,5,9\to -4,\underline{-3,0},1,3,5,9$

    $k=1$: $\underline{-4,-3},0,1,3,5,9\to -4,-3,0,1,3,5,9$

    This required a total of $6+5+4+3+2+1=21$ comparisons.

    (b) Here's the merge sort.

    **Step 1:** $9;\ -3;\ 1;\ 0;\ -4;\ 5;\ 3$

    **Step 2:** $-3,9;\ 0,1;\ -4,5;\ 3$

    **Step 2:** $-3,0,1,9;\ -4,3,5$

    **Step 2:** $-4,-3,0,1,3,5,9$

    Merging two lists of length 1 to one of length 2 requires $1+1-1=1$ comparison. Thus, the initial merging of seven lists of length 1 to three lists of length 2 and one of length 1 requires $1+1+1=3$ comparisons. Merging two lists of length 2 requires $2+2-1=3$ comparisons while merging one list of length 2 and another of length 1 requires $2+1-1=2$ comparisons. So the second merging requires $3+2=5$ comparisons. The final merging of lists of lengths 4 and 3 requires $4+3-1=6$ comparisons, for a total of $3+5+6=14$ comparisons.

13. 463152, 463512, 463521, 465123, 465132, 465213, 465231 465312, 465321, 512346, 512364, 512436, 512463, 512634, 512643, 513246, 513264, 513426, 513462, 513624, 513642.

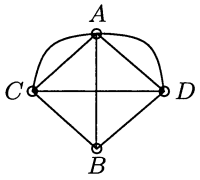
14. (a) 2456, 3456, 3457 precede; 3567 and 4567 follow.

    (b) 1234 and 1235 precede; 1237, 1247 and 1257 follow.

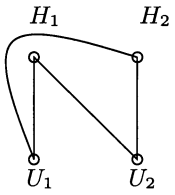
    (c) 1567, 2345 and 2346 precede; 2357, 2367 and 2456 follow.

## Exercises 9.1

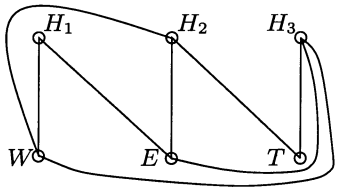
1. [BB]



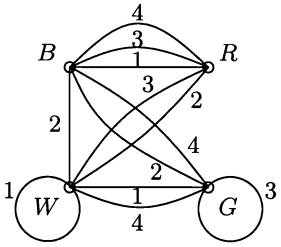

2. [BB]

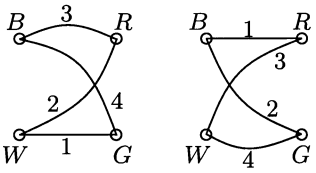



3. Yes it is possible, as shown in the following diagram (the owner of house 1 does not have telephone service):




4. (a) [BB] We present the graph which corresponds to the cubes, and two edge disjoint subgraphs.

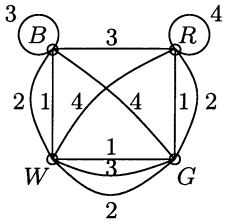


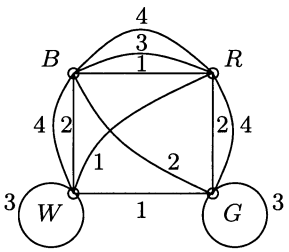

Here is the solution indicated.

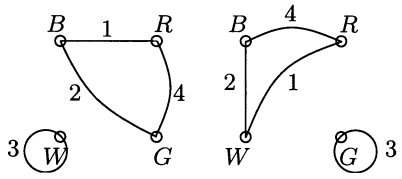
| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $B$ | $G$ | $W$ | $G$ |
| Cube 3 | $R$ | $B$ | $R$ | $W$ |
| Cube 2 | $W$ | $R$ | $G$ | $B$ |
| Cube 1 | $G$ | $W$ | $B$ | $R$ |

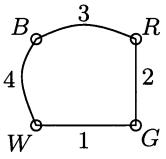
(b) No suitable pair of edge disjoint subgraphs can be found here; there is no solution.



(c) To the right, we show the graph which corresponds to the cubes. There are two pairs of edge disjoint subgraphs, as shown on the right below (the first and second, and first and third subgraphs).



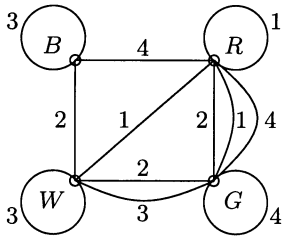

The solutions indicated are

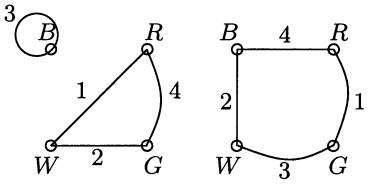
| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $G$ | $B$ | $R$ |
| Cube 3 | $W$ | $W$ | $G$ | $G$ |
| Cube 2 | $G$ | $B$ | $W$ | $B$ |
| Cube 1 | $B$ | $R$ | $R$ | $W$ |

and

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $G$ | $B$ | $W$ |
| Cube 3 | $W$ | $W$ | $R$ | $B$ |
| Cube 2 | $G$ | $B$ | $G$ | $R$ |
| Cube 1 | $B$ | $R$ | $W$ | $G$ |

(d) Here are the graph corresponding to the cubes and two edge disjoint subgraphs.

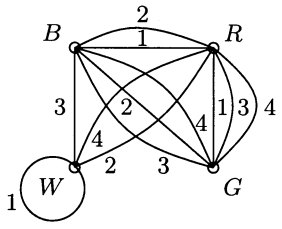


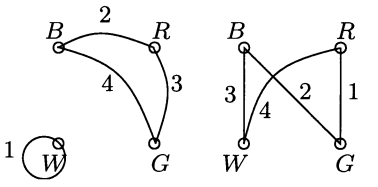




Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $G$ | $B$ | $R$ |
| Cube 3 | $B$ | $B$ | $G$ | $W$ |
| Cube 2 | $G$ | $W$ | $W$ | $B$ |
| Cube 1 | $W$ | $R$ | $R$ | $G$ |

(e) [BB] Here are the graph corresponding to the cubes and two edge disjoint subgraphs.

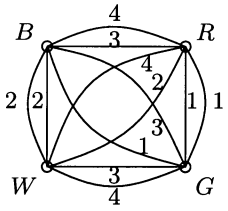


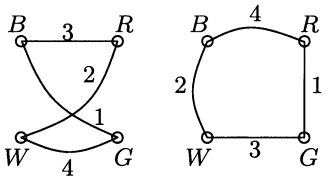



Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $G$ | $B$ | $W$ | $R$ |
| Cube 3 | $R$ | $G$ | $B$ | $W$ |
| Cube 2 | $B$ | $R$ | $G$ | $B$ |
| Cube 1 | $W$ | $W$ | $R$ | $G$ |

(f) Here are the graph corresponding to the cubes and two edge disjoint subgraphs.

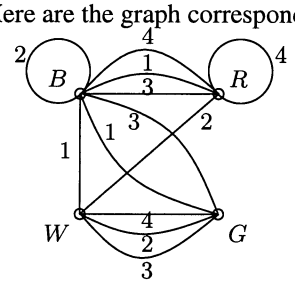


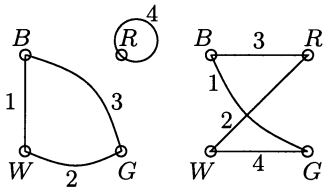




Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $B$ | $R$ | $G$ | $W$ |
| Cube 3 | $G$ | $W$ | $R$ | $B$ |
| Cube 2 | $W$ | $B$ | $W$ | $R$ |
| Cube 1 | $R$ | $G$ | $B$ | $G$ |

(g) Here are the graph corresponding to the cubes and two edge disjoint subgraphs.

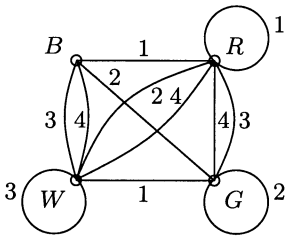


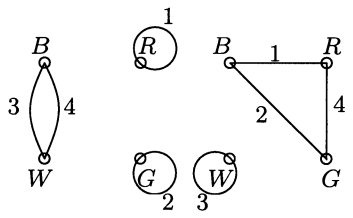




Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $R$ | $G$ | $W$ |
| Cube 3 | $G$ | $B$ | $R$ | $B$ |
| Cube 2 | $W$ | $G$ | $W$ | $R$ |
| Cube 1 | $B$ | $W$ | $B$ | $G$ |

(h) Here are the graph corresponding to the cubes and two edge disjoint subgraphs.



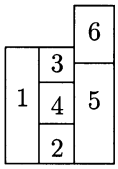




Here is the solution indicated.

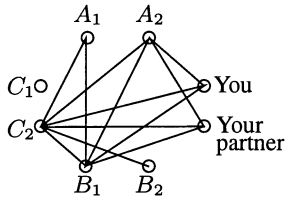
| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $G$ | $W$ | $B$ |
| Cube 3 | $W$ | $W$ | $B$ | $W$ |
| Cube 2 | $G$ | $B$ | $G$ | $G$ |
| Cube 1 | $B$ | $R$ | $R$ | $R$ |

5. (a) [BB] Beta index $= \frac{7}{6}$; (b) Beta index $= \frac{7}{7} = 1$.

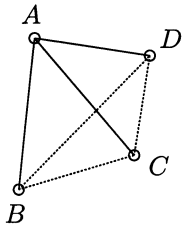
6. [BB]

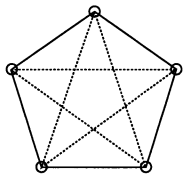



7. The other couples are labeled $A_1$ and $A_2$, $B_1$ and $B_2$, $C_1$ and $C_2$. You shook three hands; your partner shook three hands.




8. (a) Let $A$ be any vertex and consider the five edges which meet at $A$. At least three of these have the same color. Suppose these edges are $AB$, $AC$, $AD$ and that these are all red. If any of the edges $BC$, $BD$, $CD$ is red, we would have a red triangle; for example, if $BC$ were red, triangle $ABC$ would be red. On the other hand, if $BC$, $BD$ and $CD$ are all white, then triangle $BCD$ is white. In any case, we have a monochromatic triangle. (This exercise is a restatement of the mutual friends/mutual strangers problem, Problem 17, discussed in Section 6.3.)



(b) As shown to the right above, the result of (a) is not true for a graph with five vertices.

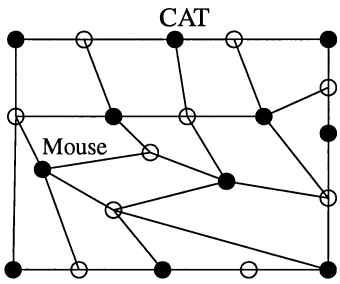
9. [BB] If there are four or more vertices of one color, say red, then there are at least $\binom{4}{3} = 4$ red triangles. Otherwise, there are three red and three white vertices, hence, one triangle of each color.

10. By Exercise 8, we know that the graph contains at least one monochromatic triangle. Select such a triangle and name it $ABC$. We may assume all its edges are colored white. If there were any any other white triangle, we would be done, so assume $\triangle ABC$ is the unique white triangle in the graph.

    Let $D$ be any vertex different from $A$, $B$, $C$. Three edges incident with $D$ have the same color—call them $DX$, $DY$, $DZ$. If one of the edges $XY$, $YZ$, $ZX$ were of the same color as $DX$, we would have a second monochromatic triangle. So we are finished unless $\{X, Y, Z\} = \{A, B, C\}$. Thus $XYZ$ is white and $DA$, $DB$, $DC$ are all red.

    The last paragraph applies to all three vertices $D$, $E$, $F$ which are different from $A$, $B$, $C$. Thus $DA$, $DB$, $DC$, $EA$, $EB$, $EC$, $FA$, $FB$, $FC$ are all red. Since $\triangle ABC$ is the only white triangle, one of $DE$, $EF$, $FD$ must be red giving us a red, and a second monochromatic, triangle.

11. (a) Such a coloring is shown at the right.

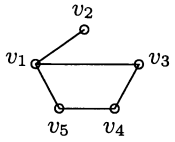



(b) Suppose the cat and the mouse are each at black positions, as they are initially, and it is the cat's move. The cat must move to a white position, so she doesn't have the mouse since the mouse is at a black position.

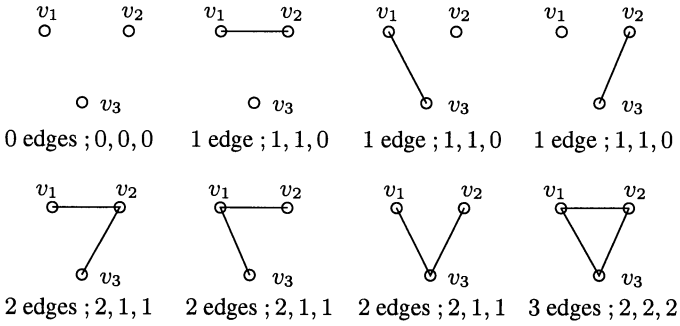
Since every vertex in the graph is joined to at least two others, the mouse can always move to a vertex where the cat isn't. Now both animals are at white vertices and it's the cat's turn to move. The cat moves to a black vertex where the mouse isn't (the mouse is at a white position). Again, the mouse can move to a (necessarily black) position different from the cat's. And so on.

## Exercises 9.2

1. [BB] Here is one possibility.




2.



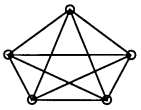

The question doesn't make sense for pseudographs because we could just keep adding edges indefinitely.

3. [BB]

4. (a) There are six vertices, 12 edges and the degree sequence is 6, 5, 4, 4, 3, 2.

(b) Proposition 9.2.5: $\sum \deg v_i = 6 + 5 + 4 + 4 + 3 + 2 = 2(12) = 2|\mathcal{E}|$.
Corollary 9.2.6: There are **two** odd vertices (of degrees 3 and 5), two being even.

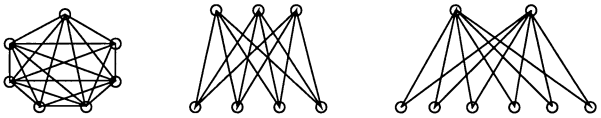
5. [BB] 10 edges. This is $\mathcal{K}_5$, the complete graph on five vertices.



6. (a) $n-1$

   (b) For $n = 1$, the answer is 0. For $n \geq 2$, the largest number of edges will occur when every pair of vertices is joined. The number of pairs of $n$ vertices is $\binom{n}{2}$.

   (c) For $n = 1$, $\circ$ is a graph with the maximum number of edges. For $n \geq 2$, $\mathcal{K}_n$ has the maximum number of edges.

7. 

   $\mathcal{K}_7$ $\mathcal{K}_{3,4}$ $\mathcal{K}_{2,6}$

8. (a) The graph is left to the student. To see that it is bipartite, associate with each vertex the color—red or black—of the square it represents and note that a knight always moves to a square of a different color. Thus, in the graph, each edge joins a red vertex and a black vertex. (The red vertices and the black vertices are the bipartition sets.)

   (b) The degrees of the vertices are shown to right inside the squares they represent. There are four vertices of degree 2, eight vertices of degree 3, twenty vertices of degree 4, sixteen vertices of degree 6 and sixteen vertices of degree 8.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 4 | 4 | 4 | 3 | 2 |
| 3 | 4 | 6 | 6 | 6 | 6 | 4 | 3 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 3 | 4 | 6 | 6 | 6 | 6 | 4 | 3 |
| 2 | 3 | 4 | 4 | 4 | 4 | 3 | 2 |

9. (a) i. [BB]

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 2 | 2 | 2 | 1 |
| $v_2$ | 1 | 0 | 1 | 2 | 3 | 2 |
| $v_3$ | 2 | 1 | 0 | 1 | 2 | 1 |
| $v_4$ | 2 | 2 | 1 | 0 | 2 | 1 |
| $v_5$ | 2 | 3 | 2 | 2 | 0 | 1 |
| $v_6$ | 1 | 2 | 1 | 1 | 1 | 0 |

   ii. [BB]

| Vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| column total | 8 | 9 | 7 | 8 | 10 | 6 |
| vertex degree | 2 | 2 | 3 | 2 | 1 | 4 |
| accessibility index | 4 | 4.5 | $2.\dot{3}$ | 4 | 10 | 1.5 |

   The accessibility index is a measure of average path length from a given vertex to the others. If a vertex were connected to every other vertex, its accessibility index would be 1; the corresponding city would be as accessible as possible. In the present case, $v_6$ has the lowest accessibility index; it is the most accessible. City $v_5$ is least accessible.

   iii. [BB] Joining $v_1$ and $v_3$ gives now eight edges, with still six vertices.
   New beta index $= \frac{8}{6} = 1.\dot{3}$.
   New accessibility indices are $2.\dot{3}$, 4.5, 1.5, 4, 10, 1.5.
   Cities $v_6$ and $v_3$ are now tied for most accessible; $v_5$ is least accessible.

iv. [BB] Joining $v_2$ and $v_6$ gives eight edges and still six vertices.
New beta index $= \frac{8}{6} = 1.\dot{3}$.
New accessibility indices are 4, $2.\dot{3}$, $2.\dot{3}$, 4, 9, 1.
City $v_6$ is most accessible; $v_5$ is least accessible.

(b) i.

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|---|---|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 2 | 2 | 2 | 3 | 3 |
| $v_2$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 |
| $v_3$ | 2 | 1 | 0 | 2 | 2 | 3 | 3 |
| $v_4$ | 2 | 1 | 2 | 0 | 2 | 1 | 2 |
| $v_5$ | 2 | 1 | 2 | 2 | 0 | 2 | 1 |
| $v_6$ | 3 | 2 | 3 | 1 | 2 | 0 | 1 |
| $v_7$ | 3 | 2 | 3 | 2 | 1 | 1 | 0 |

ii.

| vertex | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|---|---|---|---|---|---|---|---|
| column total | 13 | 8 | 13 | 10 | 10 | 12 | 12 |
| vertex degree | 1 | 4 | 1 | 2 | 2 | 2 | 2 |
| accessibility index | 13 | 2 | 13 | 5 | 5 | 6 | 6 |

Here, $v_2$ is most accessible; $v_1$ and $v_3$ are least accessible.

iii. New beta index $= \frac{8}{7}$;
New accessibility indices are 6, 2, 6, 5, 5, 6, 6.
City $v_2$ is still most accessible; now $v_6$ and $v_7$ are tied with $v_1$ and $v_3$ for least accessible.

iv. New beta index $= \frac{8}{7}$;
New accessibility indices are 12, $1.\dot{4}$, 12, 5, 5, 3, 6.
$v_2$ is still most accessible; $v_1$ and $v_3$ are least accessible.

10. [BB] $\mathcal{K}_n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Each of the $n$ vertices has degree $n-1$, so the sum of the degrees is $n(n-1)$. This is twice the number of edges, as asserted by Proposition 9.2.5. If $n$ is even there are $n$ odd vertices; if $n$ is odd, there are 0 odd vertices. In either case the number of odd vertices is even, in accordance with Corollary 9.2.6. The beta index is $\frac{n(n-1)/2}{n} = \frac{n-1}{2}$.

11. (a) $\mathcal{K}_{m,n}$ has $mn$ edges. It has $n$ vertices of degree $m$ and $m$ vertices of degree $n$, thus, $\sum \deg v_i = nm + mn = 2mn = 2|\mathcal{E}|$, in accordance with Proposition 9.2.5.

(b) Suppose first that $m$ and $n$ are both odd. Then every vertex is odd, so the number of odd vertices is $n + m$ is even. If $m$ and $n$ are both even, the number of odd vertices is 0, which is even. If one of $m$, $n$ is even and the other is odd (say $n$ is even, $m$ is odd), then the odd vertices are those joined to all the vertices of the bipartition set of size $m$; the number of odd vertices is $n$, which is even. In every case, we have verified Corollary 9.2.6.

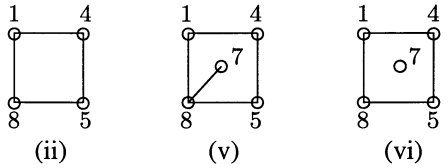
(c) The beta index of $\mathcal{K}_{m,n}$ is $|\mathcal{E}|/|\mathcal{V}| = \frac{mn}{m+n}$.

12. (a) [BB] Suppose $\mathcal{G}$ has $a$ vertices of degree 3 and $b$ vertices of degree 4. Then $\sum \deg v_i = 3a+4b = 56 = 2|\mathcal{E}|$ and $a + b = 12$. These equations imply $a < 0$, which cannot be. No such $\mathcal{G}$ exists.

(b) Suppose $\mathcal{G}$ has $a$ vertices of degree 3 and $b$ vertices of degree 6. Then $\sum \deg v_i = 3a+6b = 56 = 2|\mathcal{E}|$ and $a + b = 12$. These equations imply $3b = 20$, which cannot be (since $b$ is an integer). No such $\mathcal{G}$ exists.

13. [BB] Consider the graph in which the vertices correspond to the people at the party and an edge between vertices indicates that the corresponding people shook hands. The degree of a vertex in this graph is the number of hands that person shook. The result is an immediate consequence of Corollary 9.2.6.

14. Only graphs (ii), (iv) and (v) are subgraphs of the graph in Fig. 9.21, as shown.

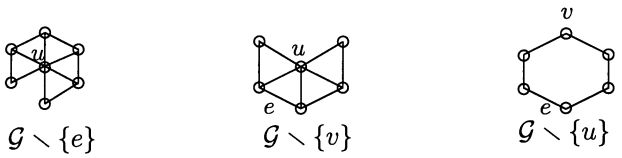



15. Yes for (a) [BB] and (c), as indicated.

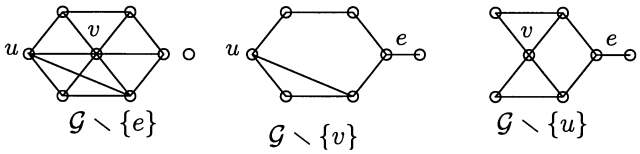
(b) No. The graph on the left contains two vertices of degree 3 while the one on the right has just one vertex of degree $\geq 3$.

(d) No. In the graph on the right, every two triangles share a vertex.

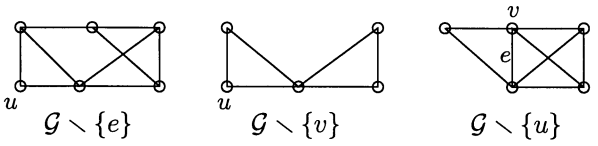
16. (a) [BB]



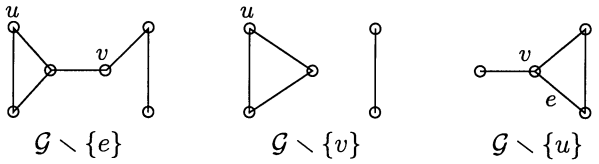

(b)




(c)




(d)




17. (a) [BB] Degree of $v_1$ is 1; degree of $v_2$ is 3; degree of $v_3$ is 4; degree of $v_4$ is 2.

(b) [BB] No. The maximum degree of a vertex in a graph with four vertices is 3. (Loops are not allowed in graphs.)

18. (a) [BB] No such graph exists. The sum of the degrees of the vertices is an odd number, 17, which is impossible.

    (b) No such graph exists:

$$2+2+2+3+4+\cdots+100 = 1+(1+2+3+\cdots+100) = 3+\frac{100(101)}{2} \quad \text{is odd.}$$

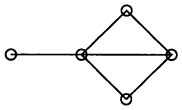
    (c) [BB] Impossible. A vertex of degree 5 in a graph with six vertices must be adjacent to all other vertices. Two vertices of degree 5 means all other vertices have degree at least 2, but the given degree sequence contains a 1.

    (d) o—o o—o o—o

    (e) Not possible. Since there is a vertex of degree 5 there must be at least six vertices in the graph.

    (f) Not possible. The vertex of degree 5 would be joined to all other vertices. But now, the degrees of the two vertices of degree 1 have been accounted for, and it is not possible to have a vertex of degree 4.

    (g) Not possible. The vertices of degree 6 use up all the degrees 1, 1, 2, 2, 2, leaving a vertex of degree 4 impossible.

    (h) Yes. 

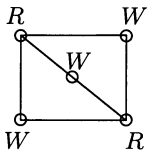
19. No. Since there are five vertices, we would have to have at least three edges in order that every vertex be incident with at least one of them, but then some vertex is incident with two edges, contradicting no adjacent edges.

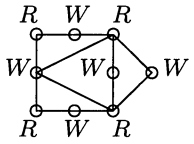
20. (a) [BB] $\sum \deg v_i = 5(4) + 2(2) = 24 = 2|\mathcal{E}|$, so there are 12 edges.

    (b) $\sum \deg v_i = 4(3) + 2(4) + 2(5) = 30 = 2|\mathcal{E}|$, so there are 15 edges.

21. (a) [BB] This is not bipartite because it contains a triangle.

    (b) This is bipartite with bipartition sets indicated $R$ and $W$ on the graph at the left below.

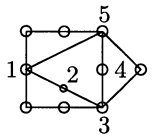


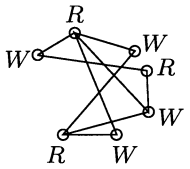




    (c) [BB] This is bipartite with bipartition sets indicated $R$ and $W$ on the graph at the right above.

    (d) This is not bipartite. Consider the vertices labeled 1, 2, 3, 4, 5 in the picture at the left below. Vertices 1 and 2 are would have to lie in different bipartition sets and, thus, 1 and 3 would lie in the same set. Since 3 and 4, and 4 and 5 lie in different sets, 3 and 5 lie in the same set. But this

puts 1 and 5 in the same set, a contradiction.




(e) This is bipartite, with bipartition sets indicated $R$ and $W$ on the graph at the right above.

(f) This is not bipartite because it contains a triangle.

22. (a) [BB] At least two of the three vertices must lie in one of the bipartition sets. Since these two are joined by an edge, the graph cannot be bipartite.

(b) A graph which contains an $n$-cycle with $n$ odd cannot be partite. To see why, let $v_1v_2v_3\cdots v_{2k+1}v_1$ be an odd cycle. Then $v_1$ and $v_2$ must lie in different bipartition sets, so $v_1$ and $v_3$ lie in the same bipartition set. Continuing this way, $v_1, v_3, v_5, \ldots, v_{2k+1}$ all lie in the same bipartition set. Since $v_1$ and $v_{2k+1}$ are adjacent, we have a contradiction.

23. (a) Yes. Assign to each vertex of the subgraph the bipartition set it had in the larger graph. Since each edge in the subgraph is an edge in the graph, the subgraph can only contain edges joining vertices in different bipartition sets.

(b) Again we say "yes". If bipartition sets were required to be nonempty, then a one vertex graph (no edges) would not be bipartite, but such a graph is a subgraph of every graph (with at least one vertex).

24. [BB] Let $x$ be the number of vertices in one of the bipartition sets. Then $n-x$ is the number of vertices in the other. The largest number of edges occurs when all $x$ vertices in one set are joined to all $n-x$ vertices in the other; so the number of edges is at most $x(n-x)$. The function $f(x) = x(n-x)$ (whose graph is a parabola) has a unique maximum at $(\frac{n}{2}, \frac{n^2}{4})$, so $x(n-x) \leq \frac{n^2}{4}$ for all $x$, and the result follows.

25. $\lfloor \frac{n}{2} \rfloor$, corresponding to the complete bipartite graphs $\mathcal{K}_{1,n-1}, \mathcal{K}_{2,n-2}, \ldots, \mathcal{K}_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

26. (a) [BB] There are $\binom{n}{2}$ possible edges which could be drawn. Every subset of this set of edges yields a graph. Hence, the number of possible graphs is $2^{\binom{n}{2}}$.

(b) In this case, three of the edges are determined to be in the graph, but we still have a free choice from among the remaining $\binom{n}{2} - 3$. So the number is $2^{\binom{n}{2}-3}$.

(c) [BB] There are $\binom{n}{3}$ possible triangles to be considered and, by (b), each triangle is in $2^{\binom{n}{2}-3}$ different graphs. Hence, the total number of triangles in all the graphs is $\binom{n}{3}2^{\binom{n}{2}-3}$.

(d) $$\text{average} = \frac{\text{total number of triangles in all graphs}}{\text{total number of graphs}} = \frac{\binom{n}{3}2^{\binom{n}{2}-3}}{2^{\binom{n}{2}}} = \frac{\binom{n}{3}}{8}$$

27. Say it is not true that at least five vertices have degree 6. This means that fewer than five vertices have degree 6. If exactly four vertices had degree 6, then we would have five vertices of degree 5, and this is impossible since the number of odd vertices must be even. If three or fewer vertices have degree 6, then at least six vertices have degree 5, as required.

28. [BB] Let $n$ be the number of vertices. Since all vertices have degree $\geq 3$, it follows that $\sum \deg v_i \geq 3n$. But $\sum \deg v_i = 2|\mathcal{E}| = 70$, so $70 \geq 3n$ and $23\frac{1}{3} \geq n$. Since $n$ must be an integer, the largest possible number is 23.

29. We know that $m|\mathcal{V}| \leq \sum \deg v_i \leq M|\mathcal{V}|$. But $\sum \deg v_i = 2|\mathcal{E}|$. So the result follows.

30. [BB] $\sum \deg v_i = k|\mathcal{V}|$. But also, $\sum \deg v_i = 2|\mathcal{E}|$. Therefore, $2|\mathcal{E}| = k|\mathcal{V}|$ and so $k$ divides $2|\mathcal{E}|$. But $k$ is odd, so $k \mid |\mathcal{E}|$.

31. Suppose $\mathcal{G}$ has vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Since $\sum_{v \in \mathcal{V}} \deg v = 2|\mathcal{E}|$, we have $dn = 34$, so $d = 1, 2, 17, 34$ and $n = 34, 17, 2, 1$, respectively. Clearly the cases $d = 34, n = 1$ and $d = 17, n = 2$ are impossible. The case $d = 1, n = 34$ defines a graph with 17 components each of which is a single edge. The case $d = 2$, $n = 17$ arises when each (connected)component of $\mathcal{G}$ is a cycle.

32. [BB] There are $n$ vertices and $n$ possible degrees for the vertices; namely, $0, 1, 2, \ldots, n-1$. If, however, we have a vertex of degree 0, then it is not possible to have another vertex of degree $n-1$. Hence, there are really only $n-1$ possible "holes" into which the $n$ vertices can fit. Hence, some vertex degree is repeated.

33. If $\{d_1, d_2, \ldots, d_n\}$ is the set of degrees of a pseudograph, then $\sum d_i$ is even by Proposition 9.2.5. Conversely, suppose $d_1, d_2, \ldots, d_n$ are given nonnegative integers whose sum is even. We must construct a pseudograph with these degrees. To start, we associate a vertex $v_i$ with each $d_i$. If each $d_i$ is even, we simply add $\frac{1}{2}d_i$ loops at $v_i$ and we have the desired pseudograph. Suppose, on the other hand, that some $d_i$ are odd. Since $\sum d_i$ is even, just as in the proof of Corollary 9.2.6, we deduce that the number of odd $d_j$ is even, say $2m$, with $m \geq 1$. Without loss of generality, assume that the odd $d_j$ are $d_1, d_2, \ldots, d_{2m}$. Join $v_1$ to $v_2$, $v_3$ to $v_4$, ..., $v_{2m-1}$ to $v_{2m}$. Then at each $v_j$, $1 \leq j \leq 2m$, add $\frac{1}{2}(d_j - 1)$ loops. Finally, add $\frac{1}{2}d_i$ loops at each of the even vertices $v_{2m+1}, \ldots, v_n$.

34. Suppose $v_1, v_2, \ldots, v_t$ are vertices with the degrees $d_1, d_2, \ldots, d_t$. The sum in question, $\sum_1^t d_i$, is the number of edges incident with $v_1$, plus the number incident with $v_2$ plus, finally, the number of edges incident with $v_t$. These edges are of two types: (i) edges joining two of $v_1, \ldots, v_t$ and (ii) edges joining one of $v_1, \ldots, v_t$ to a vertex $v \notin \{v_1, v_2, \ldots, v_t\}$. For each $i$, $1 \leq i \leq i$, write $d_i = d_i' + d_i''$, where $d_i'$ is the number of edges of type (i) incident with $v_i$ and $d_i''$ is the number of edges of type (ii) incident with $v_i$. Thinking of the subgraph with vertices $v_1, v_2, \ldots, v_t$ and all edges from the whole graph amongst them, $\sum_1^t d_i'$ is twice the number of edges in this subgraph (by Proposition 9.2.5) and this number is at most the number of edges in the complete graph $\mathcal{K}_t$, that is, $2\binom{t}{2} = t(t-1)$. Now $\sum_1^t d_i''$ is the sum of all edges of type (ii) which is the sum, over all $v \notin \{v_1, v_2, \ldots, v_t\}$, of the number of edges of the form $vv_i$ for some $1 \leq i \leq t$. For each $v$, this number cannot exceed $t$, nor can it exceed $\deg v$. Thus, $\sum_{i=1}^t d_i'' \leq \sum_{i=t+1}^n \min\{t, d_i\}$. Since $\sum_1^t d_i = \sum_1^t d_i' + \sum_1^t d_i''$, the result follows.

35. No, there cannot. The sums of the degrees for the ten vertices given is $3(1) + 7(4) = 31$. But $\sum \deg v_i = 2|\mathcal{E}| = 62$ for the whole graph. It follows that the three remaining vertices must have degrees which sum to 31. Since the graph has just 13 vertices, the degree of each of these is at most 12 and at least one of them has degree more than 10.

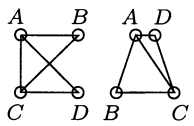
    **Case 1:** There is a vertex of degree 12.
    In this case, the vertex of degree 12 is joined to all other vertices, in particular to the three vertices of degree one. These three vertices can be joined to no other vertex. Hence, the degree of any other vertex is at most nine, and the sum of the three degrees is $\leq 12 + 9 + 9 = 30$, which is a contradiction. So this case is impossible.

**Case 2:** There is no vertex of degree 12.
In this case, there is a vertex $v$ of degree 11. Then $v$ must be joined to at least two of the vertices of degree one. These vertices cannot be connected to any other vertex, so the other two vertices must have degree at most 10. If one had degree 10, then it would be connected to the third vertex of degree one, and the final vertex would be of degree at most 9, which is impossible because $11+10+9 = 30 < 31$. Otherwise, the other two vertices have degrees at most 9 and then the three vertices have degree sum $\leq 11 + 9 + 9 = 29 < 31$, again a contradiction.

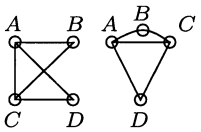
## Exercises 9.3

1. [BB] (i) and (ii) are not isomorphic because (i) has five edges and (ii) has four.

   (i) and (iii) are isomorphic, as shown by the labeling.



(i) and (iii)



(i) and (v)

   (i) and (iv) are not isomorphic because (iv) has a vertex of degree 1 and (i) does not. (Also, (iv) has only four edges.)

   (i) and (v) are isomorphic, as shown by the labeling.

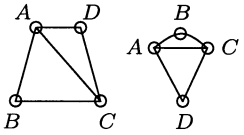
   (ii) and (iii) are not isomorphic because (ii) has four edges and (iii) has five edges.

   (ii) and (iv) are not isomorphic because (iv) has a vertex of degree 1 and (ii) does not.

   (ii) and (v) are not isomorphic because (v) has five edges and (ii) has four edges.

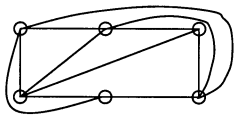
   (iii) and (iv) are not isomorphic because (iv) has a vertex of degree 1 and (iii) does not.

   (iii) and (v) are isomorphic, as shown by the labeling.

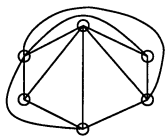



   (iv) and (v) are not isomorphic because (iv) has a vertex of degree 1 [or because (v) has five edges].

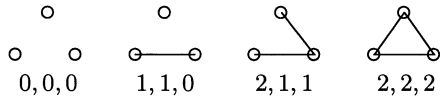
2. (a)



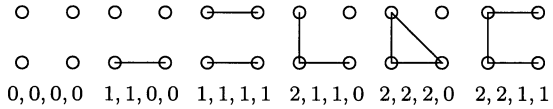
(b)

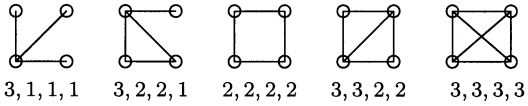


3. (a) [BB]




(b)



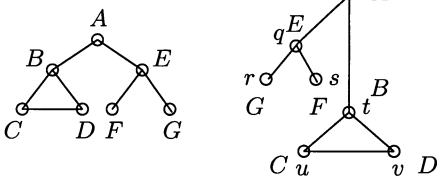




4. (a) These graphs are not isomorphic. Vertex $q$ has degree 4 but the other graph has no vertex of degree 4.

(b) [BB] These graphs are isomorphic. One possible isomorphism is given by

$$\varphi(A) = p, \quad \varphi(B) = t, \quad \varphi(C) = u, \quad \varphi(D) = v,$$
$$\varphi(E) = Q, \quad \varphi(F) = s, \quad \varphi(G) = r$$

as illustrated.

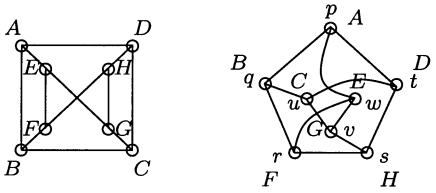



(c) These graphs are not isomorphic. Consider the unique vertices $B$ and $r$ of degree 3 in each graph. Vertex $B$ is adjacent to two vertices of degree 1, but vertex $r$ is not.

(d) These graphs are isomorphic. One possible isomorphism is given by

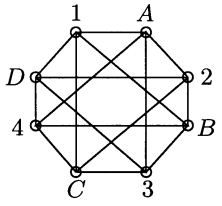
$$\varphi(A) = p, \quad \varphi(B) = q, \quad \varphi(C) = u, \quad \varphi(D) = t,$$
$$\varphi(E) = w, \quad \varphi(F) = r, \quad \varphi(G) = v, \quad \varphi(H) = s$$

as illustrated.

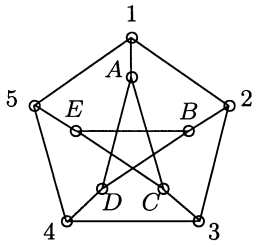


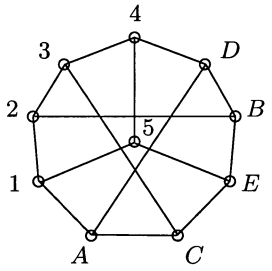

5. (a) [BB] No; it has a vertex of degree 5.

(b) Yes. In the graph at the right, the bipartition sets are labeled $\{1, 2, 3, 4\}$ and $\{A, B, C, D\}$.




6. The labelings show that the graphs are isomorphic.




7. [BB] Any graph $\mathcal{G}$ with $n$ vertices is a subgraph of $\mathcal{K}_n$, as is easily seen by joining any pair of vertices of $\mathcal{G}$ where there is not already an edge.

8. (a) [BB] Suppose that $\mathcal{G}$ and $\mathcal{H}$ are isomorphic graphs and $\varphi\colon \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{H})$ is the isomorphism of vertex sets given by Definition 9.3.1. Label the edges of $\mathcal{G}$ arbitrarily, $g_1, g_2, \ldots, g_m$, and then use $\varphi$ to label the edges of $\mathcal{H}$ as $h_1, h_2, \ldots, h_m$; that is, if edge $g_1$ has end vertices $v$ and $w$ in $\mathcal{G}$, let $h_1$ be the edge joining $\varphi(v)$ and $\varphi(w)$ in $\mathcal{H}$. Repeat to obtain $h_2, \ldots, h_m$. Now a triangle in $\mathcal{G}$ is a set of three edges $\{g_i, g_j, g_k\}$ each two of which are adjacent. It follows from the way we labeled the edges of $\mathcal{H}$ that $\{g_i, g_j, g_k\}$ is a triangle in $\mathcal{G}$ if and only if $\{h_i, h_j, h_k\}$ is a triangle in $\mathcal{H}$ . Thus, the number of triangles in each graph is the same.

   (b) Let $n$ be a given natural number $\geq 4$. To prove that isomorphic graphs $\mathcal{G}$ and $\mathcal{H}$ contain the same number of $n$-cycles, we label the edges of $\mathcal{G}$ and $\mathcal{H}$ as in part (a) and note that $\{g_{i1}, g_{i2}, \ldots, g_{in}\}$ is an $n$-cycle in $\mathcal{G}$ if and only if $\{h_{i1}, h_{i2}, \ldots, h_{in}\}$ is an $n$-cycle in $\mathcal{H}$.

   (c) The two graphs each have six vertices and nine edges; the degree of every vertex in each graph is 3, yet the graphs are **not** isomorphic. Vertices 1, 4 and 5 in $\mathcal{G}_2$ form a triangle, but there is no triangle in $\mathcal{G}_1$.

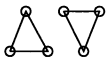
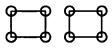
9. (a) [BB] Each graph has seven vertices and 14 edges. The degree sequence of $\mathcal{G}_1$ is 4, 4, 4, 4, 4, 4, 4; the degree sequences of $\mathcal{G}_2$ and $\mathcal{G}_3$ are 5, 4, 4, 4, 4, 4, 3. The numbers of triangles in $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ are 7, 8, 8, respectively; specifically,

$$\begin{aligned} \mathcal{G}_1 &: \ ABC, ABG, AFG, BCD, CDE, DEF, EFG \\ \mathcal{G}_2 &: \ abc, abd, acd, adg, bcd, bcf, deg, efg \\ \mathcal{G}_3 &: \ 125, 126, 156, 167, 236, 256, 347, 367 \end{aligned}$$

   (b) Because of the degree situation discussed in (a), $\mathcal{G}_1$ is isomorphic to neither of the other graphs. As for $\mathcal{G}_2$ and $\mathcal{G}_3$, they aren't isomorphic because in $\mathcal{G}_2$, the vertices of degrees 3 and 5 ($d$ and $e$) are adjacent while this is not the case in $\mathcal{G}_3$ (vertices 4 and 6 are not adjacent). Alternatively, $\mathcal{G}_2$ contains nine 4-cycles while $\mathcal{G}_3$ has only eight; specifically,

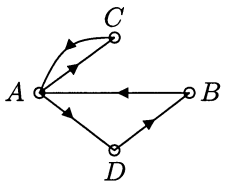
$$\begin{aligned} \mathcal{G}_2 &: \ abcd, abdg, abfg, acdg, acfg, bdef, cdef, defg, adeg \\ \mathcal{G}_3 &: \ 1236, 1236, 1256, 1256, 1567, 2345, 3456, 4567 \end{aligned}$$

10. The graph on the left has two quadrilaterals, $ABHG$ and $CDFE$; while the other has three quadrilaterals, 1287, 3465, 1357.

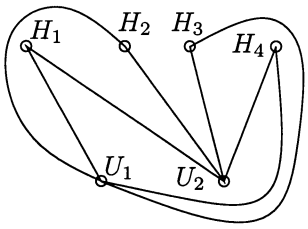
11. (a) [BB] No. $\mathcal{G}$  $\mathcal{H}$ (b) No. $\mathcal{G}$  $\mathcal{H}$

## Chapter 9 Review

1. The new graph is drawn to the right. Following the arrows gives a walk of the type required.

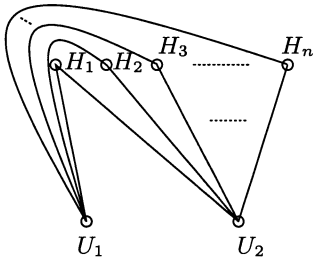



2. (a)



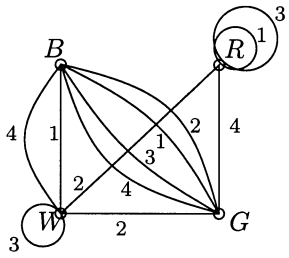

We show one of several possible configurations.

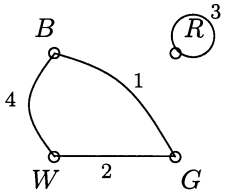
(b)

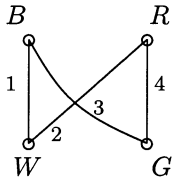



Any number of houses can be connected to two utilities without crossovers, as shown.

3. (a) We present below the graph which corresponds to the cubes and two edge disjoint subgraphs.



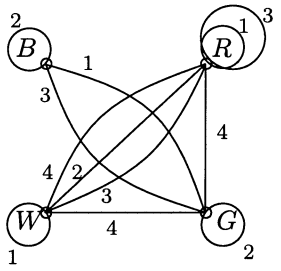



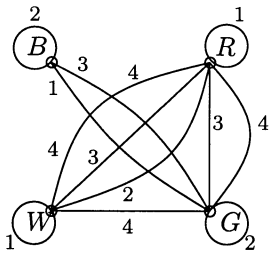

Here is the solution indicated.

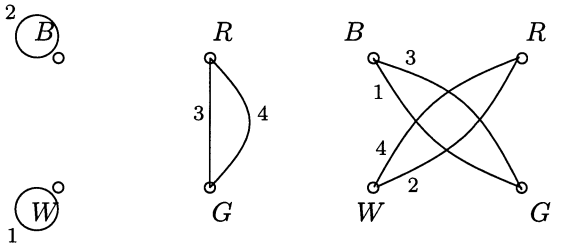
| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $B$ | $W$ | $R$ | $G$ |
| Cube 3 | $R$ | $R$ | $G$ | $B$ |
| Cube 2 | $W$ | $G$ | $W$ | $R$ |
| Cube 1 | $G$ | $B$ | $B$ | $W$ |

(b) The graph corresponding to the cubes is drawn to the right. It is not possible to find two edge disjoint subgraphs, so there is no solution.




(c) Here are the graph corresponding to the cubes and two edge disjoint subgraphs.

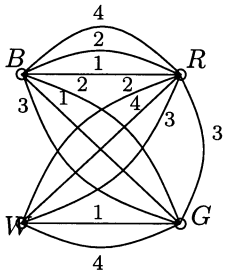


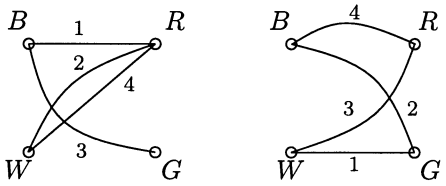




Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $R$ | $G$ | $W$ | $R$ |
| Cube 3 | $G$ | $R$ | $B$ | $G$ |
| Cube 2 | $B$ | $B$ | $R$ | $W$ |
| Cube 1 | $W$ | $W$ | $G$ | $B$ |

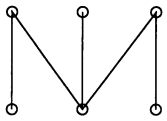
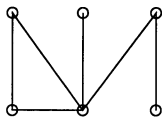
(d) We present below the graph which corresponds to the cubes and two edge disjoint subgraphs.




Here is the solution indicated.

| | $F$ | $B$ | $R$ | $L$ |
|---|---|---|---|---|
| Cube 4 | $B$ | $R$ | $W$ | $G$ |
| Cube 3 | $R$ | $W$ | $G$ | $B$ |
| Cube 2 | $G$ | $B$ | $R$ | $W$ |
| Cube 1 | $W$ | $G$ | $B$ | $R$ |

4. (a)  (b) 

(c) It is not possible. For the conditions in 4(a) and 4(b) to be hold simultaneously,the graph would have to have exactly three odd vertices. This is impossible by Corollary 9.2.6.

5. (a) $\mathcal{K}_4$, the complete graph on 4 vertices has this degree sequence.

   (b) No such graph exists because the sum of the degrees in a graph must be even, but the sum here is 15, which is not even.

   (c) Draw two triangles, with vertices $A_1, B_1, C_1$ and $A_2, B_2, C_2$, respectively. Then add edges $A_1A_2$, $B_1B_2$, $C_1C_2$. This gives a graph with the given degree sequence.

   (d) Drawing a graph by the approach of part (c), but starting with two squares, gives a graph with the given degree sequence.

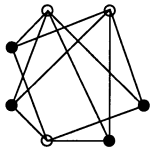
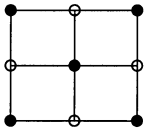
6. (a) The sum of the degrees is 41, an odd number, contradicting Proposition 9.2.5.

   (b) Suppose such a graph existed. It would have nine vertices two of which have degree 8. Hence, these two would be joined to all vertices except themselves, and thus every vertex would have degree at least 2. This contradicts the fact that there is a vertex of degree 1.

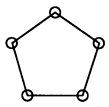
7. (a) This is the complete bipartite graph $\mathcal{K}_{3,4}$.

   (b) This is not bipartite. It contains an odd cycle (the vertices along the top and right side).

   (c) This is bipartite, but not complete. There are black and white vertices not joined by an edge.

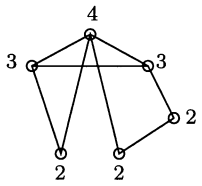
   (a)  (c) 

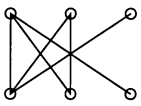
8. (a) Yes to both: $\mathcal{K}_{2,2}$

   (b) Here's a graph: 

   There is no such bipartite graph. The graph must have five edges and the bipartition sets have sizes 1 and 4, or 2 and 3. Neither is possible since a bipartition set of size 4 accounts for eight edges and one of size 3 accounts for six.

   (c) We show a graph with the given degree sequence. No **bipartite** graph with this degree sequence can exist. Since there is a vertex of degree 4, one bipartition set contains at least four vertices and hence must contain a vertex of degree 3. It follows that the vertex of degree 4 is in a bipartition set of size of least 3. We have accounted for seven vertices when there are only six. This contradiction shows that a bipartite graph with degree sequence $4, 3, 3, 2, 2, 2$ cannot exist.




   (d) Yes to both. 

9. There exists a bipartite graph with $n$ vertices and every vertex of degree $d$ if and only if $n$ is even and $d \leq \frac{n}{2}$.

**Proof.** ( $\leftarrow$ ) If $n = 2m$, the complete bipartite graph $\mathcal{K}_{d,d}$ has the required property.

( $\leftarrow$ ) Now assume that $\mathcal{G}$ is a bipartite graph with $n$ vertices each of degree $d$. Label the bipartition sets $\mathcal{V}_1$ and $\mathcal{V}_2$, and suppose they contain $n_1$ and $n_2$ vertices, respectively. Now count the edges in $\mathcal{G}$ and note that each edge contributes 1 to the sum of the degrees of the vertices in $\mathcal{V}_1$ and one to the sum of the degrees of the vertices in $\mathcal{V}_2$. Thus, we discover

$$dn_1 = \sum_{v \in \mathcal{V}_1} \deg v = \sum_{v \in \mathcal{V}_2} \deg v = dn_2,$$

so $n_1 = n_2$ and $n = n_1 + n_2$ is even. Now a vertex $v$ in $\mathcal{V}_1$ is adjacent only to vertices in $\mathcal{V}_2$, so $\deg v = d \leq n_2 = \frac{n}{2}$. This completes the argument.

10. (a) Assume this is possible. Then two of the vertices of $\mathcal{H}$ would have to belong to the same bipartition set in $\mathcal{G}$ (since $\mathcal{H}$ has at least three vertices). In a bipartite graph, however, vertices in the same bipartition set are not adjacent. This contradicts the fact that $\mathcal{H}$ is complete, so no such situation is possible.

    (b) This is indeed possible. For example, let $\mathcal{G}$ be $\mathcal{K}_4$ and let $\mathcal{H}$ be $\mathcal{K}_{2,2}$.

11. Suppose it is not true that there are at least 25 vertices of degree 4. Then the number of vertices of degree 4 is at most 24. Since the graph has 49 vertices, the number of vertices of degree 5 is at least 25. But the number of odd vertices is even, so the number of vertices of degree 5 is at least 26.

12. We are given that $\deg v \geq 2$ for all vertices $v$. Suppose $\deg v_0 \geq 3$ for some $v_0$. Then $2n = \sum_{v \in \mathcal{V}} \deg v \geq 3 + 2(n-1) = 2n + 1$, a contradiction.

13. Euler's formula says $2|\mathcal{E}| = \sum \deg v$, so $100 = 4(2)+6(5)+8(4)+6x$. Thus $6x = 100-8-30-32 = 30$, so there are $x = 5$ vertices of degree 6. Altogether, $\mathcal{G}$ has 23 vertices.

14. Make a graph with 24 vertices, one for each player. Let an edge between vertices signify that the corresponding players have played on the same team. The graph has 78 edges.

    (a) If this is wrong, then the degree of every vertex is at most 6. Then $2|\mathcal{E}| = 156 = \sum \deg v \leq 24(6) = 144$ gives a contradiction.

    (b) If this is wrong, then then the degree of every vertex is at least 7. Then $2|\mathcal{E}| = 156 = \sum \deg v \geq 24(7) = 168$ gives a contradiction.

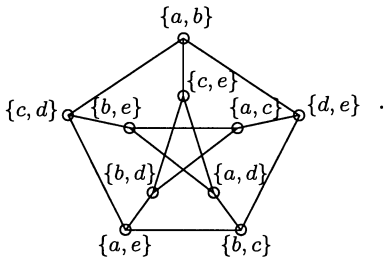
15. (a) These graphs are isomorphic. One possible isomorphism $\varphi$ is defined by

$$\varphi(A) = p, \quad \varphi(B) = q, \quad \varphi(D) = r, \quad \varphi(E) = s, \quad \varphi(C) = t.$$

    (b) These graphs are not isomorphic. In the graph drawn on the left, vertex $B$ has degree 4 while the graph on the right has no such vertex.

    (c) These graphs are not isomorphic. In the graph on the left, the two vertices of degree 3 ($C$ and $F$) are not adjacent while in the graph on the right, the two vertices of degree 3 ($q$ and $t$) are adjacent.

16. The labels show that the graph is isomorphic to the Petersen graph.



17. (a) Not isomorphic: $\mathcal{K}_4$ has four vertices while $\mathcal{K}_{3,8}$ has 11.
    (b) Not isomorphic: $\mathcal{K}_{11}$ has $\binom{11}{2} = 55$ edges while $\mathcal{K}_{3,8}$ has 24.
    (c) Not isomorphic: $\mathcal{K}_{4,6}$ has ten vertices while $\mathcal{K}_{2,12}$ has 14.
    (d) Not isomorphic: $\mathcal{K}_{4,6}$ has 24 edges while $\mathcal{K}_{5,5}$ has 25.
    (e) Isomorphic: in fact, these are the same graph simply viewed in different ways. In each case, the graph consists of a set of four vertices each of which is joined in all possible ways to the vertices in a second set of six.

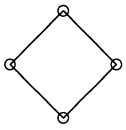
18. No. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be any pair of graphs which are not isomorphic and set $\mathcal{G}_3 = \mathcal{G}_1$. The conditions hold but $\mathcal{G}_1 \cong \mathcal{G}_3$.

19. In this case, George's deduction is correct. To see why most easily, recall that the notion of "isomorphism" is an equivalence relation. Thus, if $\mathcal{G}_1$ is not isomorphic to $\mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are in different equivalence classes. If $\mathcal{G}_2 \cong \mathcal{G}_3$, then $\mathcal{G}_2$ and $\mathcal{G}_3$ are in the same equivalence class. Thus $\mathcal{G}_1$ and $\mathcal{G}_3$ are in different equivalence classes and hence not isomorphic.

20. ($\longrightarrow$) Assume $\mathcal{K}_{a,b} \cong \mathcal{K}_{c,d}$. Note that every vertex in $\mathcal{K}_{a,b}$ has degree $a$ or degree $b$ while every vertex in $\mathcal{K}_{c,d}$ has degree $c$ or degree $d$. The result follows.

    ($\longleftarrow$) Assume $\{a, b\} = \{c, d\}$ There are two possibilities. If $a = c$ and $b = d$, then $\mathcal{K}_{a,b} = \mathcal{K}_{c,d}$. Suppose $a = d$ and $b = c$. Let $\{x_1, x_2, \ldots, x_a\}$ and $\{y_1, y_2, \ldots, y_b\}$ be the bipartition sets for $\mathcal{K}_{a,b}$ and let $\{u_1, u_2, \ldots, u_c\}$ and $\{v_1, v_2, \ldots, v_d\}$ be the bipartition sets for $\mathcal{K}_{c,d}$. Define $\varphi\colon \mathcal{K}_{a,b} \to \mathcal{K}_{c,d}$ by $\varphi(x_i) = v_i$, $i = 1, 2, \ldots, a$, and $\varphi(y_j) = u_j$, $j = 1, 2, \ldots b$. Since $a = d$ and $b = c$ and, in a complete bipartite graph, all vertices in one bipartition set are adjacent to all vertices in the other, $\varphi$ is an isomorphism.

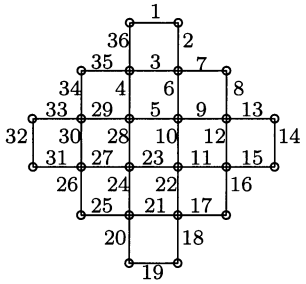
## Exercises 10.1

1. (a) [BB]  (b) 

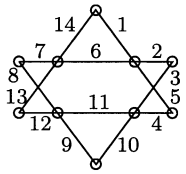
2. [BB] Since the graph describing the Königsberg Bridge Problem (Fig. 9.2) has several odd vertices, it is not Eulerian. It is not possible to walk over the bridges of Königsberg exactly once and return to the starting position.

3. (a) [BB] The graph is Eulerian because it is connected and each vertex has even degree 4. Piecing the circuits $ABCDEFGA$ and $ACFBDGEA$ together at $A$ gives the Eulerian circuit $ABCDEFGACFBDGEA$.

   (b) The graph is Eulerian because it is connected and each vertex has even degree.

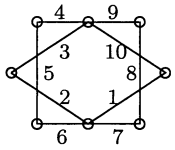


4. (a) [BB]

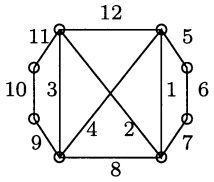



(b) No Eulerian circuit. The graph is not connected.

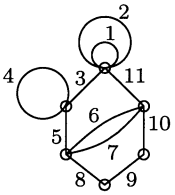
(c)




(d)

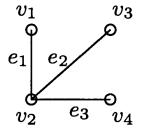



(f)

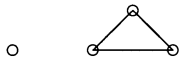



(e) No Eulerian circuit. There are (four) vertices of odd degree.

5. [BB] This definition does not work. In the graph to the right, the sequence $e_1e_2e_3$ of edges fits Gerard's criteria, but does not define a walk.

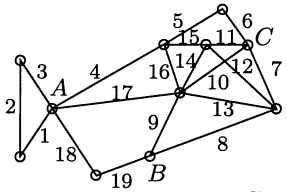



6. With the proposed definition, an Eulerian graph would no longer necessarily be connected, as the example shows.

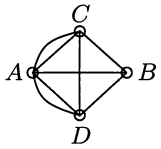


7. (a) [BB] Yes, there is because $A$ and $B$ are the only vertices of odd degree.

(b) No. $C$ has even degree.

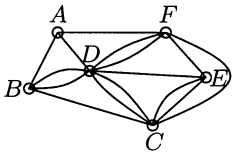



8. (a) [BB] No. In the graph representing the modified Königsberg Bridge Problem there are two vertices of odd degree.




(b) [BB] This question asks about the possibility of an Eulerian trail and there is one, between the two vertices $A$ and $B$ of odd degree. One possibility is $ACBDADCAB$.

9. The graph depicting the situation is shown at the right.




(a) This question asks whether the graph is Eulerian. It is not: there are vertices of odd degree, $A$, for instance.

(b) This question asks whether the graph has an Eulerian trail. It does, between $A$ and $F$, since these are the unique vertices of odd degree in a connected graph. One possible trail is $ABDBCDCECFEDFADF$.

10. [BB] Yes. In this case, both $\mathcal{G}$ and $\mathcal{H}$ must be cycles.

11. The resulting graph has an Eulerian trail (between $v_1$ and $v_2$) but no Eulerian circuit. This is because, in the new graph, $v_1$ and $v_2$ are the only vertices of odd degree.

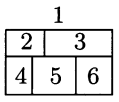
12. (a) [BB] $\mathcal{K}_n$ is Eulerian $\leftrightarrow$ $n$ is odd.

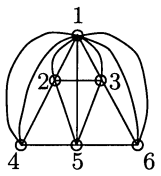
    (b) [BB] $\mathcal{K}_n$ has an Eulerian trail if and only if $n = 2$. For $n = 2$, certainly $\circ$—$\circ$ has an Eulerian trail. For $n > 2$, if two vertices have odd degree, then there are other vertices of odd degree so no Eulerian trail can exist.

13. (a) Since $\mathcal{K}_{m,n}$ is connected, we seek only necessary and sufficient conditions for every vertex to be even. In $\mathcal{K}_{m,n}$, every vertex in the bipartition set with $m$ vertices has degree $n$ and every vertex in the other bipartition set has degree $m$. Hence, $\mathcal{K}_{m,n}$ is Eulerian if and only if $m$ and $n$ are both even.

    (b) $\mathcal{K}_{m,n}$, $m \leq n$ has an Eulerian trail if and only if $m = 1$, $n = 2$ or $m = 2$ and $n$ is odd. By Theorem 10.1.5, complete bipartite graphs with the specified conditions on $m$ and $n$ have Eulerian trails. On the other hand, since $\mathcal{K}m, n$ has $n$ vertices of degree $m$ and $m$ of degree $n$, if $m > 2$, then each bipartition set contains more than two vertices (since $n \geq m$) so it is impossible for there to exist precisely two vertices of odd degree.

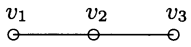
14. Labeling the six regions determined by this figure as shown, we draw a pseudograph whose vertices correspond to these regions and where each edge signifies the crossing of an edge in the given figure in a route which passes between the corresponding regions. The specified route exists if and only if the pseudograph has an Eulerian circuit or trail. Since the pseudograph contains odd vertices, there is no Eulerian circuit and since it contains more than two odd vertices, there is no Eulerian trail either. No route of the desired sort exists.




15. [BB] Suppose the vertices of the circuit are $v_0, v_1, \ldots, v_n, v_0$. Consider all subcircuits of the form $v_i, v_{i+1}, v_{i+2}, \ldots, v_i$. (There is at least one such, taking $i = 0$.) That subcircuit $v_i, v_{i+1}, v_{i+2}, \ldots, v_i$ which uses the fewest number of vertices is a cycle. If the original circuit was not a cycle, then those vertices not on the first chosen subcircuit $v_i, v_{i+1}, v_{i+2}, \ldots, v_i$, together with $v_i$, form another subcircuit, and the same argument as before shows that this contains a second cycle.

16. [BB] False. In the graph shown, $v_1v_2v_3v_2v_1$ is a closed walk that does not contain a cycle.




17. One direction is obvious. What requires proof is that the existence of a walk from $u$ to $v$ implies the existence of a path from $u$ to $v$. Suppose then that

$$u = u_0, u_1, \ldots, u_k = v$$

is a walk from $u$ to $v$. If there are no repeated vertices then this walk is already a path. Suppose then that some vertex is repeated; thus, there exist $i$ and $j$, $i < j$, with $u_j = u_i$. Deleting the vertices $u_{i+1}, u_{i+2}, \ldots, u_j$, we obtain another walk

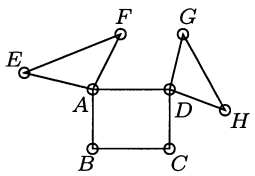
$$u = u_0, u_1, \ldots, u_i, u_{j+1}, \ldots, u_k = v$$

from $u$ to $v$ with fewer vertices than the first walk. If it contains no repeated vertices, it's the desired path; otherwise, repeating the above procedure, we obtain a shorter walk. This process continues until we get a walk without repeated vertices, in other words, a path.

18. [BB] By definition, $u \sim u$, so $\sim$ is reflexive. If $u \sim v$, then there is a walk $u = u_0, u_1, \ldots, u_k = v$ from $u$ to $v$. But then $v = u_k, u_{k-1}, \ldots, u_1, u_0 = u$ is a walk from $v$ to $u$, so $v \sim u$ and $\sim$ is

symmetric. Finally, if $u \sim v$ and $v \sim w$, then there is a walk $u = u_0, u_1, \ldots, u_k = v$ from $u$ to $v$ and a walk $v = v_0, v_1, \ldots, v_\ell = w$ from $v$ to $w$. But then $u = u_0, u_1, \ldots, u_k = v = v_0, v_1, \ldots, v_\ell = w$ is a walk from $u$ to $w$, proving $u \sim w$ and establishing transitivity.

19. No. Let $\mathcal{G}$ be as shown and let circuit $\mathcal{C}$ be $ABCDA$. After removal of the edges of $\mathcal{C}$, the remaining graph $\mathcal{G}_1$ consists of the 3-cycles $AFEA$ and $DHGD$. This is not connected.




20. Let $A$ be a vertex in $\mathcal{G}_1$ and $B$ be a vertex in the circuit $\mathcal{C}_1$. Since $\mathcal{G}$ is connected, there is a walk

$$A, u_1, u_2, \ldots, u_k, B$$

in $\mathcal{G}$ from $A$ to $B$. If the edge $Au_1$ is in $\mathcal{C}_1$, then $A$ is a vertex in both $\mathcal{G}_1$ and $\mathcal{C}_1$ and we are done. Otherwise, the edge $Au_1$ is not in $\mathcal{C}_1$, so $Au_1$ is in $\mathcal{G}_1$ and hence $u_1$ is in $\mathcal{G}_1$. Repeat this argument. Eventually, either we find a vertex $u_i$ in both $\mathcal{G}_1$ and $\mathcal{C}_1$ or else none of the edges along the above path is in $\mathcal{C}_1$. In this case, $B$ is in $\mathcal{G}_1$ as well as $\mathcal{C}_1$ and again, we are done.

21. We are given that there is an isomorphism $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$. Suppose that $\mathcal{G}_1$ is connected. Then $\mathcal{G}_2$ must be connected for if $v$ and $w$ are any two vertices of $\mathcal{G}_2$, there exist vertices $x$ and $y$ of $\mathcal{G}_1$ such that $v = \varphi(x)$ and $w = \varphi(y)$. Since $\mathcal{G}_1$ is connected, there is a walk in $\mathcal{G}_1$ of the form $x, v_1, v_2, \ldots, v_k, y$. Thus, $v, \varphi(v_1), \varphi(v_2), \ldots, \varphi(v_k), w$ is a walk from $v$ to $w$ in $\mathcal{G}_2$ because each of $v\varphi(v_1)$, $\varphi(v_1)\varphi(v_2), \ldots \varphi(v_{k-1})\varphi(v_k)$, $\varphi(v_k)y$ is an edge. (Remember the definition of isomorphism in §9.3.1.) So we have shown that if $\mathcal{G}_1$ connected, then so is $\mathcal{G}_2$. An identical argument shows the converse. Thus, if one of the graphs is connected, so is the other and also, if one is not connected, neither can the other be. This gives the result.

22. [BB] We must show that there is a walk between any two vertices $x$ and $y$ of $\mathcal{G}$. We show that there is, in fact, a walk of length at most 2 between $x$ and $y$. If $xy$ is an edge, then obviously there is a walk from $x$ to $y$, so suppose that $xy$ is not an edge. Let $\mathcal{S}$ be the set of vertices adjacent to $x$ and $\mathcal{T}$ be the set of vertices adjacent to $y$. Thus $x \notin \mathcal{S} \cup \mathcal{T}$ and $y \notin \mathcal{S} \cup \mathcal{T}$ so the number of vertices in $\mathcal{G}$ is $20 \geq |\mathcal{S} \cup \mathcal{T}| + 2$. Using the Principle of Inclusion-Exclusion, $20 \geq |\mathcal{S}| + |\mathcal{T}| - |\mathcal{S} \cap \mathcal{T}| + 2 = \deg x + \deg y - |\mathcal{S} \cap \mathcal{T}| + 2 \geq 21 - |\mathcal{S} \cap \mathcal{T}|$. It follows that $\mathcal{S} \cap \mathcal{T} \neq \emptyset$, so there is a vertex $u$ in both $\mathcal{S}$ and $\mathcal{T}$ and, thus, a walk $xuy$ from $x$ to $y$.

23. (a) [BB] Since $\mathcal{G}$ is connected and $n > 1$, no vertices have degree zero. Therefore, if there are no vertices of degree one, every vertex of $\mathcal{G}$ has degree at least 2. Using $\mathcal{E}$ to denote the set of edges of $\mathcal{G}$, it follows that $2|\mathcal{E}| = \sum \deg v_i \geq 2n$ (by Proposition 9.2.5) and the number of edges $|\mathcal{E}| \geq n$ as required.

    (b) We use induction on $n$. If $n = 2$, $\mathcal{G}$ must have an edge (by connectedness), so the number of edges $|\mathcal{E}| = 1 = n - 1$ and the result is true. Now assume the result for connected graphs with $k$ vertices and suppose $\mathcal{G}$ is connected with $k + 1$ vertices. We must show the number of edges in $\mathcal{G}$ is $\geq k$. If $\mathcal{G}$ has no vertex of degree one, then $\mathcal{G}$ has $\geq (k + 1)$ edges, by (a), and we are done. On the other hand, if $\mathcal{G}$ has a vertex of degree one, then removing this vertex and the edge $e$ with which it is incident produces a connected graph (because a vertex of degree one cannot be an intermediate vertex of a path) with $k$ vertices. By the induction hypothesis, the smaller graph has $\geq k - 1$ edges. So $\mathcal{G}$ has $\geq (k - 1) + 1 = k$ edges, as required. (The "+1" counts the edge $e$.)

24. (a) If $\mathcal{G}$ has a vertex of degree 0, $\mathcal{G}$ would have the most edges in the case that each pair of the remaining $n-1$ vertices were joined with an edge. The graph in all would have at most $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2} < m$ edges.

    (b) Assume to the contrary that $\mathcal{G}$ is not connected. Then we can find vertices $u$ and $v$ not joined to each other by a walk. In particular, $u$ and $v$ are not adjacent to any common vertex and they are not adjacent to each other. We conclude that $\deg u + \deg v \leq n-2$. [If $\deg u = k$, then $u$ is adjacent to $k$ vertices, $v$ is not adjacent to these; neither is it adjacent to $u$ or to itself, so $\deg v \leq n-k-2$.] The remaining $n-2$ vertices each have degree at most $n-2$ (none can be adjacent to both $u$ and $v$) so the sum of the degrees of all vertices is at most

$$n-2+(n-2)(n-2) = n^2-3n+2 = (n-1)(n-2).$$

    Proposition 9.2.5 then says $|\mathcal{E}| \leq \frac{(n-1)(n-2)}{2}$, a contradiction.

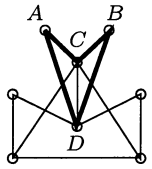
25. Assume that $\mathcal{G}$ is bipartite. Then any vertex of $\mathcal{G}$ is joined only to vertices in the other bipartition set. Hence, in any path in $\mathcal{G}$, the vertices must alternate between the two different bipartition sets. Hence, any cycle must be even. Conversely, assume $\mathcal{G}$ is a graph with no odd cycles. We build the two bipartition sets $\mathcal{V}_1$, $\mathcal{V}_2$ as follows. First let $v$ be any vertex in $\mathcal{G}$ and put $v$ in $\mathcal{V}_1$. Next, if $w$ is any vertex adjacent to $v$, put $w$ in $\mathcal{V}_2$. Then, if $u$ is any vertex adjacent to $w$ put $u$ into $\mathcal{V}_1$. In general, if $x$ is a vertex not yet assigned to either $\mathcal{V}_1$ or $\mathcal{V}_2$ and $x$ is adjacent to a vertex $y$ already assigned to $\mathcal{V}_1$ or $\mathcal{V}_2$, put $x$ into the set to which $y$ does *not* belong. Since there are no odd cycles, $x$ cannot be adjacent to vertices in both $\mathcal{V}_1$ and $\mathcal{V}_2$, so there is never any confusion as to where to place $x$. If the graph is connected, the process stops only when all vertices have been assigned to $\mathcal{V}_1$ or $\mathcal{V}_2$, which are bipartition sets of an evidently bipartite $\mathcal{G}$. If $\mathcal{G}$ is not connected, the process stops when all the vertices in one connected component of $\mathcal{G}$ have been assigned; it must then be repeated in each of the other components of $\mathcal{G}$. The final bipartition sets are then the unions of those produced in the components.

26. Construct any circuit $\mathcal{C}_1$ and let $\mathcal{G}_1$ be $\mathcal{G}$ less the edges of $\mathcal{C}$ and any vertices which are left isolated by this procedure, as in Theorem 10.1.4. If $\mathcal{C}_1$ is not Eulerian, then $\mathcal{G}_1$ contains edges. Attempt to find such an edge which is incident with a vertex $B$ in $\mathcal{C}_1$. If such an edge and vertex $B$ exist, construct a circuit $\mathcal{C}$ in $\mathcal{G}_1$ which contains $B$, then combine with $\mathcal{C}_1$ to produce a larger circuit $\mathcal{C}_2$. Continue until this is no longer possible. At this point, an Eulerian circuit $\mathcal{C}'$ has been constructed in one *component* of $\mathcal{G}$. If there remain edges in $\mathcal{G}$ which are not part of $\mathcal{C}'$, take one and enlarge to a maximal circuit containing it as before. This circuit will identify a second component. Continue until every edge is part of some maximal circuit. The number of maximal circuits is the number of components.

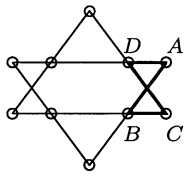
27. [BB] Suppose that $\mathcal{G}_1$ has $n_1$ components and that $\mathcal{G}_2$ has $n_2 < n_1$ components. Let $v_1, \ldots, v_{n_1}$ be vertices of $\mathcal{G}_1$ each in a different component. Then there is no walk between any pair of these vertices. On the other hand, if $w_1, \ldots, w_{n_1}$ are any $n_1$ vertices of $\mathcal{G}_2$, at least two of these must lie in the same component (by the Pigeon-Hole Principle) and hence there is a walk between these two. Thus, the vertices $v_i$ in $\mathcal{G}_1$ do not correspond to any $n_1$ vertices of $\mathcal{G}_2$, so these graphs cannot be isomorphic.

## Exercises 10.2

1. [BB] This graph is not Hamiltonian. To see this, suppose $\mathcal{H}$ were a Hamiltonian cycle. Since vertices $A$ and $B$ have degree 2, the two edges incident with each of these vertices would be in $\mathcal{H}$. Thus, $\mathcal{H}$ would contain the cycle $ACBDA$, which cannot be the case since this does not contain all vertices of the graph. The graph is not Eulerian because it contains vertices of odd degree.

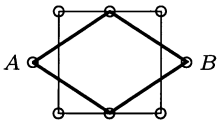



2. (a) This is not Hamiltonian. Since vertices $A$ and $C$ each have degree 2, the proper cycle $ABCDA$ would have to be included in any Hamiltonian cycle, and this is not possible.

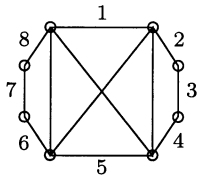



   (b) [BB] This is not Hamiltonian, since it isn't connected.

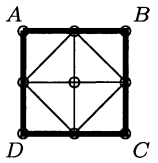
   (c) This is not Hamiltonian. Since the vertices labeled $A$ and $B$ are of degree two, both edges incident with them would be on a Hamiltonian cycle, but this forms a smaller cycle not containing all vertices, that is, a proper cycle.




   (d) [BB]

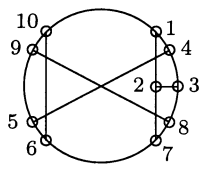



   (e) This is not Hamiltonian. Since the vertices labeled $A$, $B$, $C$ and $D$ are of degree two, both edges incident with each of them would be on a Hamiltonian cycle, but these eight edges would then form a proper cycle, as shown.

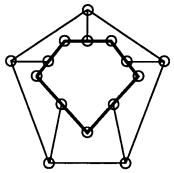



   (f) This is a pseudograph, but not a graph, so the question does not apply. The definitions of Section 10.2 were given only for graphs.

3. (a)



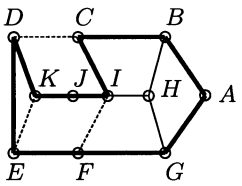

   (b)



   (a) The graph in (a) is Hamiltonian; a Hamiltonian cycle is indicated. Any Hamiltonian graph has a Hamiltonian path.

(b) The graph in (b) is not Hamiltonian. Both edges incident with a vertex of degree two must be on any Hamiltonian cycle. Thus, the inner cycle would have to be part of any Hamiltonian cycle. Since this inner cycle excludes some vertices, it would be a proper cycle, a contradiction. The graph does have a Hamiltonian path, however. Start at any outer vertex, go around the outside stopping one short of the vertex where you started, then enter and go round the inner cycle.

(c) The graph in (c) is not Hamiltonian. If $\mathcal{H}$ were a Hamiltonian cycle, then it would contain vertices $A$ and $J$, these having degree 2, and hence edges $AB$, $AG$, $JI$ and $JK$. Precisely two of the edges incident with $K$ would be in $\mathcal{H}$, hence, precisely one of $DK$, $EK$. Suppose $DK$ were in and $EK$ out. Then edges $DE$ and $EF$ would be part of $\mathcal{H}$. At this point, two edges incident at $D$ are in $\mathcal{H}$, so $DC$ could not be in $\mathcal{H}$. As the only other edges incident with $C$, both $CB$ and $CI$ would have to be $\mathcal{H}$.

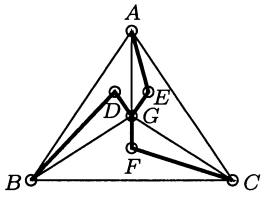



Since we now have two edges incident with $I$ part of $\mathcal{H}$, $IF$ could not be in $\mathcal{H}$, so $FG$ would have to be in. Now, however, $\mathcal{H}$ would contain the proper cycle $AGFEDKJICBA$, a contradiction. A similar argument, assuming $EK$ is in $\mathcal{H}$ and $DK$ is not, also leads to a proper cycle. The graph does have a Hamiltonian path, however: $DCBAGFEKJIH$.

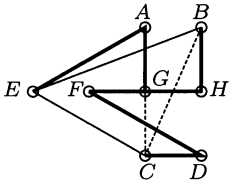
(d) This graph has no Hamiltonian path, hence also no Hamiltonian cycle. Any Hamiltonian path would have to begin at one of the two vertices of degree 1 and finish at the other. Having entered the right 4-cycle, however, there is no way to return without repeating a vertex.

(e) This graph is not Hamiltonian. To see why, suppose $\mathcal{H}$ were a Hamiltonian cycle. Since $D$, $E$, and $F$ have degree 2, the two edges incident with each of these vertices would be in $\mathcal{H}$, as shown. Now exactly one of $AB$, $AG$, $AC$ must also be in $\mathcal{H}$, but the inclusion of any of these produces a proper cycle within $\mathcal{H}$, which is not possible.

(e)



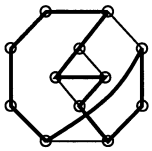

(f)

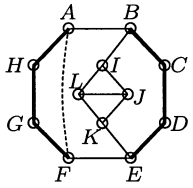



(f) There is a Hamiltonian path, $AECDFGHB$, but no Hamiltonian cycle. To see why, suppose $\mathcal{H}$ were a Hamiltonian cycle. Since vertices $A$, $D$, $F$, and $H$ have degree 2, the edges incident with these vertices would have to be part of $\mathcal{H}$. In order to avoid proper cycles, $\mathcal{H}$ could not contain $BC$ or $CG$. Since two edges incident with $C$ must be in $\mathcal{H}$, $EC$ would have to be in $\mathcal{H}$, giving a proper cycle (avoiding $B$).

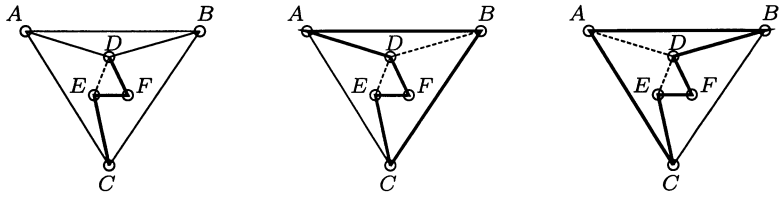
(g) This is Hamiltonian. A Hamiltonian cycle is shown. So the graph also has a Hamiltonian path.

(h) There is a Hamiltonian path, $AHGFEDCBILKJ$, but no Hamiltonian cycle. To see why not, suppose that $\mathcal{H}$ is a Hamiltonian cycle. Since $G$, $H$, $C$, and $D$ have degree 2, the edges incident with these vertices would have to be in $\mathcal{H}$, in particular, $AH$, $HG$, and $GF$. Since $\mathcal{H}$ cannot contain a proper cycle, edge $AF$ is not in $\mathcal{H}$, but then, since $\mathcal{H}$ must contain two edges incident with $A$ and with $F$, we see that $AB$ and $FE$ would be in $\mathcal{H}$. But $\mathcal{H}$ contains the proper cycle around the outer edge.

(g) 

(h) 


4. Let $\mathcal{H}$ be a Hamiltonian cycle. Vertex $F$ has degree 2, so $\mathcal{H}$ contains both edges incident with $F$. Since $\mathcal{H}$ can contain no proper cycles, edge $DE$ is not in $\mathcal{H}$ and therefore $CE$ is in. The present situation is depicted in the graph on the left below.




Exactly one of the edges $AD$, $BD$ is in $\mathcal{H}$, so we consider two cases.

**Case 1:** $AD$ is in; $BD$ is out. In this case, since two edges incident with $B$ must be in $\mathcal{H}$, both $AB$ and $BC$ are in. So $\mathcal{H}$ is $ABCEFDA$ as shown in the middle above.

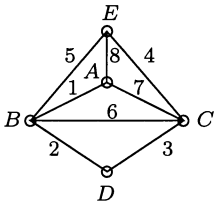
**Case 2:** $BD$ is in; $AD$ is out. In this case, since two edges incident with $A$ must be in $\mathcal{H}$, both $AB$ and $AC$ are in. So $\mathcal{H}$ is $ABDFECA$ as shown on the right above.

5. (a) [BB] Yes, it is Hamiltonian: $ABDCEA$ is a Hamiltonian cycle.

   (b) Any Hamiltonian graph has a Hamiltonian path; in the graph here, $ABDCE$ is a Hamiltonian path, for instance.

   (c) [BB] The graph is not Eulerian. Vertices $A$ and $E$ have odd degree.

   (d) The trail indicated is Eulerian.

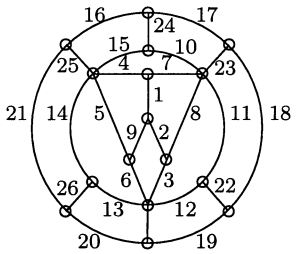



6. (a) Yes, it is possible. Here is a Hamiltonian cycle for the knight.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 64 | 25 | 28 | 41 | 44 | 53 | 46 |
| 24 | 27 | 4 | 63 | 52 | 47 | 40 | 43 |
| (1) | 6 | 17 | 26 | 29 | 42 | 45 | 54 |
| 18 | 23 | 62 | 3 | 48 | 51 | 34 | 39 |
| 7 | 2 | 19 | 16 | 35 | 30 | 55 | 50 |
| 22 | 13 | 10 | 61 | 58 | 49 | 38 | 33 |
| 11 | 8 | 15 | 20 | 31 | 36 | 59 | 56 |
| 14 | 21 | 12 | 9 | 60 | 57 | 32 | 37 |

(b) Such a tour is not possible because a $7 \times 7$ board has an odd number of squares. A knight's tour corresponds to an alternating set of distinct red and black squares. Since we require that the knight return to the square from which he left, the number of squares of the tour must be even.

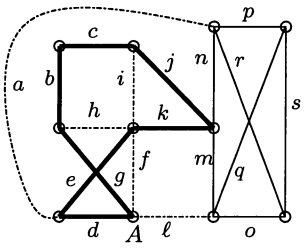
7. No, it isn't. Label edges as shown. Two of the three middle edges 1, 2, 9 must be in any Hamiltonian cycle. By symmetry, we may assume that these are 1 and 2. (A solution for any other choice could be found with an argument similar to what follows.) Edge 9 is not in the cycle, hence, 5 and 6 both must be in. Now, one of 4 and 7 is in the cycle, and one of 3 and 8. Not both 7 and 8 can be in, however, or the cycle would contain the smaller cycle 1, 2, 7, 8. Also, as we saw in PAUSE 6, edges 3 and 4 cannot both be in the cycle. Hence, both 3 and 7 are in, or 4 and 8 are in. Assume that 3 and 7 are in. (The other case follows similarly.)



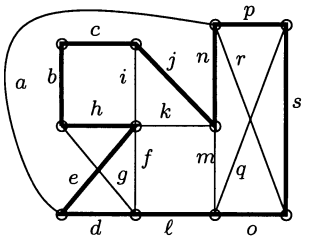

We now have 7, 1, 2, 3, 6 and 5 in the cycle, while 9, 4, 8 are not in the cycle. Since 3 and 6 are in, edges 12, 13 and 21 are not, so edges 11, 22, 14, 26, 19 and 20 are. But now we have found a smaller cycle, 7, 1, 2, 3, 6, 5, 14, 26, 20, 19, 22, 11 which does not contain all vertices. Hence, there is no Hamiltonian cycle.

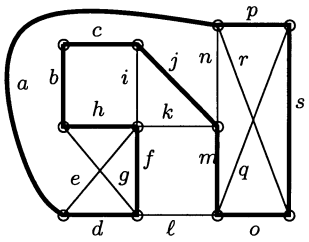
8. (a) Suppose a Hamiltonian cycle $\mathcal{H}$ excludes edges $a$ and $\ell$. Since a Hamiltonian cycle contains precisely two edges adjacent with any vertex, $\mathcal{H}$ contains edges $b$, $c$, $d$, and $e$. Edge $f$ cannot be in $\mathcal{H}$; otherwise, $\mathcal{H}$ would contain the proper cycle $def$.

Now $\mathcal{H}$ contains two edges incident with $A$. We have $d$ already. Since $\ell$ is not a possibility, $g$ must be in $\mathcal{H}$. This means that $h$ is excluded (to avoid the cycle $hedg$). Edge $i$ must also be excluded to avoid the proper cycle $bciedg$. Thus $\mathcal{H}$ contains $j$ and $k$, giving again a proper cycle. No such $\mathcal{H}$ exists.

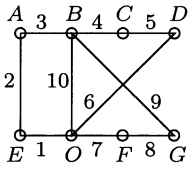



(b) $bcjnpsoldeh$ and $apsomjcbhfd$ are both Hamiltonian cycles.




9. In the graph to the right, vertices correspond to rooms and edges to doorways.




(a) [BB] No, because the graph is not Hamiltonian. Since vertices $A$, $C$, $D$, and $E$ have degree 2, the edges labeled 3, 4, 5, 6, 1, and 2 would have to be part of any Hamiltonian cycle. These edges, however, define a proper cycle, which is not allowed.

(b) Yes, because the graph is Eulerian. An Eulerian circuit is $ABCDOFGBOEA$.

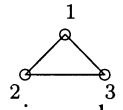
10. [BB] The result is obvious if $n = 1$ since in this case there are just two people who are friends. So we assume that $n > 1$. Consider the graph whose vertices correspond to people and where an edge between vertices $v$ and $u$ signifies that $v$ and $u$ are friends. The question asks us to prove that this graph is Hamiltonian. This is an immediate consequence of Dirac's Theorem since the graph in question has $2n \geq 3$ vertices each of degree $d \geq n = \frac{2n}{2}$.

11. (a) [BB] $n$ edges.

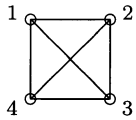
(b) If $n = 2$, there are no cycles, so assume $n \geq 3$. Numbering the vertices $1, 2, \ldots, n$ and beginning at 1, there are $n - 1$ choices for the second vertex in a cycle (since all edges $1i$ exist), then $n - 2$ choices for the third vertex and so on. Altogether, there are $(n - 1)!$ different Hamiltonian cycles in $\mathcal{K}_n$.

(c) [BB] $\mathcal{K}_n$ has $\frac{n(n-1)}{2}$ edges, so the maximum is $\frac{n-1}{2}$ edge disjoint cycles.

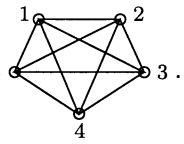
(d) If $n = 2$, there are no Hamiltonian cycles (and therefore no edge disjoint ones).

If $n = 3$, the graph is 

. 123 and 132 are the only Hamiltonian cycles; so there are no edge disjoint Hamiltonian cycles.

If $n = 4$, the graph is 

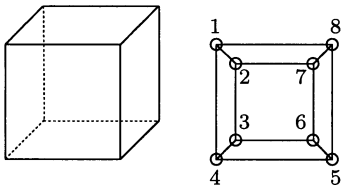
. The Hamiltonian cycles are 1234, 1243, 1324, 1342, 1423 and 1432. No pair are edge disjoint.

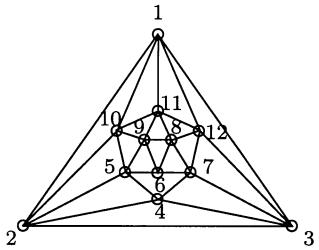
If $n = 5$, the graph is 

. The Hamiltonian cycles are 12345, 12354, 12435, 12453, 12534, 12543, 13245, 13254, 13425, 13452, 13524, 13542, 14235, 14253, 14325, 14352, 14523, 14532, 15234, 15243, 15324, 15342, 15423, 15432.

Since $\frac{n-1}{2} = 2$, we could possibly have two edge disjoint cycles and we do; for instance, 12345 and 13524.

12. [BB] The cube is indeed Hamiltonian; the labels $1, \ldots, 8$ on the vertices exhibit a Hamiltonian cycle.

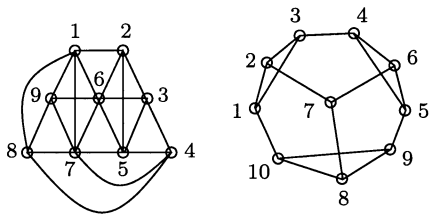



13. 

14. The reader will appreciate that it is difficult to insert three-dimensional models within the covers of a book!

15. (a) [BB] As suggested, add an extra vertex $v$ to $\mathcal{G}$ and join it to all other vertices. Then $\deg v = n \geq \frac{n+1}{2}$, and $\deg w \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$ for all other vertices. By Theorem 10.2.4, this new graph with $n+1$ vertices has a Hamiltonian cycle. Deleting $v$ and all the new edges incident with $v$ leads to a Hamiltonian path in our original graph.

    (b) The first graph satisfies the condition in (a), so it has a Hamiltonian path. One is shown to the left below.




    (c) The second graph has a Hamiltonian path as shown on the right above.

    (d) No. The graph on the right above is a counterexample since it has a Hamiltonian path while $\frac{n-1}{2} = \frac{9}{2} = 4\frac{1}{2}$ and not all vertices have degree at least 5. Another example is the graph ○—○—○—○—○—○ . This certainly has a Hamiltonian path, but $\frac{n-1}{2} = \frac{5-1}{2} = 2$ and the end vertices are of degree one.

    Also, the converse to Dirac's Theorem is false, as we can see by the graph on the right above. This graph has the Hamiltonian cycle 1, 3, 4, 6, 5, 9, 10, 8, 7, 2, 1 yet not all vertices have degree at least $5 = \frac{n}{2}$.

    (e) [BB] ○—○ is an example!

16. As in the proof of Dirac's Theorem all vertices adjacent to $v_1$ are in $\mathcal{P}$.

    (a) [BB] If $v_1$ and $v_t$ are adjacent, $\mathcal{C}_1\colon v_1 v_2 \cdots v_t v_1$ is a cycle. If $t < n$, then there exists a vertex $w$ not in $\mathcal{C}_1$. As noted, $w$ is not adjacent to $v_1$, hence the hypothesis tells us that $\deg v_1 + \deg w \geq n$. It follows that there exists some vertex $u$ which is adjacent to both $v_1$ and $w$; otherwise, the graph has at least $n+2$ vertices, counting $v_1$ and $w$ and the disjoint sets of vertices adjacent to $v_1$ and to $w$. Since $u$ is adjacent to $v_1$, $u = v_i$ for some $i$. Now $w v_i v_{i+1} \cdots v_t v_1 \cdots v_{i-1}$ is a path longer than $\mathcal{P}$, a contradiction. Thus $t = n$, so $\mathcal{C}_1$ is a Hamiltonian cycle and we're done.

    (b) Suppose $v_1$ and $v_t$ are not adjacent. By hypothesis, $\deg v_1 + \deg v_t \geq n$. As in the proof of Dirac's Theorem, we claim that there is a pair of vertices $v_k, v_{k+1}$ in $\mathcal{P}$, $1 \leq k < t$, such that $v_1$ is adjacent to $v_{k+1}$ and $v_t$ is adjacent to $v_k$. If this were not the case, then each vertex of $\mathcal{P}$ adjacent to $v_1$ would determine a vertex not adjacent to $v_t$, namely, its predecessor in $\mathcal{P}$. Counting $v_t$ as well, this guarantees at least $1 + \deg v_1$ vertices not adjacent to $v_t$, hence at most $n - 1 - \deg v_1$ vertices adjacent to $v_t$. Thus $\deg v_1 + \deg v_t \leq n - 1$, a contradiction. It follows that $\mathcal{C}_2\colon v_1 v_{k+1} v_{k+2} \cdots v_t v_k v_{k-1} \cdots v_1$ is a cycle, so there is path of length $t$ beginning at $v_k$ and ending at $v_t$. The argument given in (a) for the case where $v_1$ and $v_t$ are adjacent now shows that $\mathcal{C}_2$ is Hamiltonian.

17. By Ore's Theorem, it is sufficient to show that the sum of the degrees of any two nonadjacent vertices is at least $n$. So let $u$ and $v$ be nonadjacent vertices. If we delete these vertices (and all edges incident

with $u$ or $v$), we are left with a graph $\mathcal{G}'$ with $|\mathcal{E}| - \deg u - \deg v$ edges. (By $\mathcal{E}$, we mean the set of edges of $\mathcal{G}$.) Since $\mathcal{G}'$ has $n-2$ vertices, $\mathcal{G}'$ has at most $\binom{n-2}{2}$ edges, so

$$\binom{n-1}{2} + 2 - \deg u - \deg v \leq |\mathcal{E}| - \deg u - \deg v \leq \binom{n-2}{2}$$

and

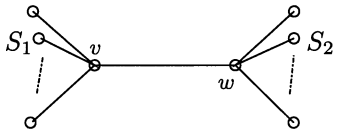
$$\begin{aligned}\deg u + \deg v &\geq \binom{n-1}{2} - \binom{n-2}{2} + 2 \\ &= \frac{(n-1)(n-2)}{2} - \frac{(n-2)(n-3)}{2} + 2 \\ &= \frac{(n-2)[(n-1)-(n-3)]}{2} + 2 = n - 2 + 2 = n\end{aligned}$$

and the result follows.

18. [BB] Add a new vertex adjacent to all existing vertices and apply the result of Exercise 16.

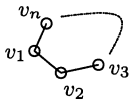
19. (thanks to Michael Watson) Let $v$ and $w$ be two adjacent vertices. Each of these vertices is adjacent to at least $\frac{n}{2} - 1 = \frac{n-2}{2}$ other vertices. If $n$ is odd, this means that both $v$ and $w$ are adjacent to at least $\frac{n-1}{2}$ other vertices. Since there are only $n-2$ other vertices in all, it follows that they must both be adjacent to a common vertex, and we have a triangle.

So assume $n$ is even and that $\mathcal{G}$ does not contain a triangle. Then $v$ must be adjacent to exactly half of the remaining $n-2$ vertices (let $S_1$ be the set of such vertices) and $w$ must be adjacent to the other half (denote this set $S_2$).




Since $\mathcal{G}$ does not contain a triangle, no two vertices of $S_1$ are adjacent and no two vertices of $S_2$ are adjacent, so $\mathcal{G}$ is isomorphic to $\mathcal{K}_{\frac{n}{2},\frac{n}{2}}$ (with bipartition sets $S_1 \cup \{w\}$ and $S_2 \cup \{v\}$).

20. [BB] By Dirac's Theorem, we know that $\mathcal{G}$ contains a Hamiltonian cycle $v_1, v_2, \ldots, v_n$.




First assume that $v_1$ and $v_2$ are not the vertices of a triangle, equivalently, that $v_1$ and $v_2$ are not adjacent to a common vertex. Then $v_1$ must be adjacent to exactly half of $v_3, \ldots, v_n$ while $v_2$ is adjacent to the other half. Since $v_2$ and $v_3$ are adjacent, there exists a minimal $i > 3$ such that $v_2$ is not adjacent to $v_i$. But then $v_1$ is adjacent to $v_i$ and $v_2$ is adjacent to $v_{i-1}$, so $v_1 v_2 v_{i-1} v_i$ is a 4-cycle.

Next assume that $v_1$ and $v_2$ are adjacent to two common vertices $v_i$ and $v_j$. Then $v_1 v_i v_2 v_j$ is a 4-cycle.

Finally, we reach the case that $v_1$ and $v_2$ are adjacent to precisely one vertex $v_j$. We may also assume that $n \geq 6$ since if $n = 4$, the Hamiltonian cycle is a 4-cycle and if $n = 5$, any edge added to the Hamiltonian cycle produces a 4-cycle. As before, choose the least $i > 3$ such that $v_2$ is not adjacent to $v_i$. If $v_1$ and $v_i$ are adjacent, the previous argument still holds, so assume this is not the case. But then, for any $k \neq i, j$, $k \geq 3$, exactly one of $v_1, v_2$ must be adjacent to $v_k$. We finish the proof by considering a number of cases.

i. If $i > 4$ and $j = 3$, then $v_1 v_2 v_4 v_3$ is a 4-cycle.

ii. If $i > j > 3$, then $v_1v_2v_{j-1}v_j$ is a 4-cycle.

iii. If $i = 4$ and $j = 3$, then $v_1$ is adjacent to exactly half of $v_5, \ldots, v_n$ while $v_2$ is adjacent to the other half. If $v_2$ is adjacent to $v_5$, choosing $k > 5$ minimal with $v_2$ not adjacent to $v_k$, we have the 4-cycle $v_1v_2v_{k-1}v_k$. If $v_1$ is adjacent to $v_5$, choose $k > 5$ minimal with $v_1$ and $v_k$ not adjacent. Then $v_1v_2v_kv_{k-1}$ is a 4-cycle.

iv. If $j = n$ and $v_1$ is adjacent to $v_{n-1}$, then $v_1v_2v_nv_{n-1}$ is a 4-cycle.

v. If $j = n$ and $v_2$ is adjacent to $v_{n-1}$, then $v_1v_2v_{n-1}v_n$ is a 4-cycle.
We note that the case $j = n$, $i = n - 1$ is not possible.

vi. If $n > j > i$ and $v_1$ is adjacent to $v_{j+1}$, then $v_1v_2v_jv_{j+1}$ is a 4-cycle.

vii. If $n > j > i$ and $v_2$ is adjacent to $v_{j+1}$, then $v_1v_2v_{j+1}v_j$ is a 4-cycle.

21. (a) False. A Hamiltonian cycle in a graph contains no proper cycles. The graph itself, on the other hand, can be expected to contain many proper cycles, for example, any complete graph $\mathcal{K}_n$ ($n > 3$).

(b) The complete graph $\mathcal{K}_n$, which is Hamiltonian, shows that vertices in a Hamiltonian graph can have arbitrary degree. (They only have degree 2 in the subgraph which is the Hamiltonian cycle.)

(c) [BB] False. is Eulerian but not Hamiltonian.

(d) False. is Hamiltonian, but not Eulerian.

22. (a) [BB] Yes! Since there is a path between any two vertices, the graph is connected. Thus, there exists an edge $e$ in the graph, joining, say, vertices $v$ and $w$. Now let $\mathcal{P}$ be a Hamiltonian path from $v$ to $w$. Then starting at $v$, following $\mathcal{P}$ to $w$ and then $e$ to $v$ produces a Hamiltonian cycle from $v$ to $v$.

(b) No! The Petersen graph is a counterexample.

(c) No! This question is silly. If there is an Eulerian trail between any two vertices, then every vertex of $\mathcal{G}$ has odd degree, so there can exist no Eulerian circuit.

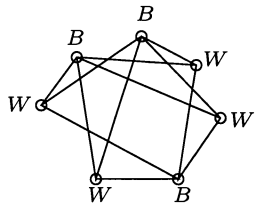
23. Note that $\sum_{i=1}^{11} \deg v_i = 106$ and that the maximum degree of a vertex in $\mathcal{G}$ is 10. If $\mathcal{G}$ had a vertex of degree $d \leq 5$, then $\sum \deg(v_i) \leq 10(10) + 5 = 105$. Thus, each vertex has degree at least $6 > \frac{11}{2} = \frac{n}{2}$, so $\mathcal{G}$ is Hamiltonian by Dirac's Theorem. It remains to show that $\mathcal{G}$ is not Eulerian. Since $\mathcal{G}$ is connected, if it is Eulerian, it has only even vertices and, as we have just seen, 6, 8 and 10 are the only possibilities for vertex degrees. In order that $\sum \deg v_i = 106$, $\mathcal{G}$ has at least nine vertices of degree 10 and it is easily seen that either $\mathcal{G}$ has nine vertices of degree 10 and two of degree 8, or ten vertices of degree 10 and one of degree 6. Neither case is possible. In the first case, each vertex of degree 10 is adjacent to every other vertex, in particular, to those of degree 6. The second case leads to a similar impossibility. Since not all vertices can be even, $\mathcal{G}$ is not Eulerian.

24. (a) $\mathcal{K}_{m,n}$ is Hamiltonian if and only if $n > 1$ and $m = n$.

**Proof.** Assume first that $n > 1$ and $m = n$. Let $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_m$ be the vertices in the two bipartition sets. Then $v_1, w_1, v_2, w_2, \ldots, v_m, w_m, v_1$ is a Hamiltonian cycle. Conversely, assume $\mathcal{K}_{m,n}$ is Hamiltonian. If $n = 1$, $\mathcal{K}_{m,n}$ has a vertex of degree 1 and hence cannot be Hamiltonian. Thus, $n > 1$. Assume $v_1, v_2, \ldots, v_\ell, v_1$ is a Hamiltonian cycle in $\mathcal{K}_{m,n}$. Let the bipartition sets (of sizes $m$ and $n$) be called EVEN and ODD and let ODD be the set which

contains $v_1$. Since there is an edge between $v_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n-1$, it follows that $v_1, v_3, v_5, \ldots$ are in ODD and that $v_2, v_4, v_6, \ldots$ are in EVEN. Now $v_\ell$ is in EVEN; otherwise, it is in ODD along with $v_1$, an impossibility because there is an edge between $v_\ell$ and $v_1$. So EVEN and ODD have the same number of elements; $m = n = \frac{\ell}{2}$. ■

(b) The graph is bipartite, as shown, and a subgraph of $\mathcal{K}_{3,4}$. Since $\mathcal{K}_{3,4}$ is not Hamiltonian, neither is any subgraph.




## Exercises 10.3

1. In Fig. 10.2, [BB] $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. In Fig. 10.4, $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

2. The adjacency matrix of $\mathcal{K}_n$ is $\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$.

   If we order the vertices of $\mathcal{K}_{m,n}$ so that the bipartition sets are $v_1, \ldots, v_m$ and $v_{m+1}, \ldots, v_{m+n}$, the adjacency matrix is $\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$.

3. (a) [BB] The $(3,5)$ entry of $A^3$ is the number of walks of length 3 from $C$ to $E$ and, hence, equals 5. The $(2,2)$ entry of $A^3$ is the number of walks of length 3 from $B$ to itself, hence, equals 2.

   (b) The $(3,5)$ entry of $A^3$ equals 2. The $(2,2)$ entry of $A^3$ equals 4.

4. [BB] Each 1 represents an edge. Each edge $v_iv_j$ contributes two 1's to the matrix, in positions $(i,j)$ and $(j,i)$. The number of 1's is twice the number of edges.

5. The $i$th entry on the diagonal of $A^3$ is the number of walks of length 3 from $v_i$ to itself. However, a closed walk of length 3 in a graph must give a triangle. Every triangle can be walked in exactly two different directions. Hence, the number of walks of length 3 is twice the number of triangles.

6. (a) [BB] The $(i, j)$ entry in $A^2$ is the number of walks of length 2 from $i$ to $j$. Hence, the sum of all such entries is the total number of walks of length 2.

   (b) Given a vertex $v_i$ of degree $d_i$, there are $d_i$ ways of choosing an edge incident with $v_i$ and $d_i$ ways of choosing a second (not necessarily distinct) edge incident with $v_i$. These two edges, in the order chosen, define a walk of length 2 "through" $v_i$. Conversely, any walk of length 2 will be chosen in this manner for some $v_i$. Hence, the total number of walks of length 2 equals $\sum d_i^2$, the total number of ways of making such a choice.

7. $A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

   Interchanging vertices 1 and 2 in graph $\mathcal{G}_1$ gives graphs $\mathcal{G}_2$, so $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

8. (a) [BB] $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$.

   (b) [BB] The function $\varphi$ is an isomorphism because, if the vertices of $\mathcal{G}_1$ are relabeled, $v_i$ being replaced by $\varphi(v_i) = u_i$, then the adjacency matrix of $\mathcal{G}_1$ relative to the $u_i$'s is $A_2$. (See Theorem 10.3.3.)

   (c) [BB] $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

9. (a) $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

   (b) The function $\varphi$ is an isomorphism because, if the vertices of $\mathcal{G}_1$ are relabeled, $v_i$ being replaced by $\varphi(v_i) = u_i$, then the adjacency matrix of $\mathcal{G}_1$ relative to the $u_i$'s is $A_2$. (See Theorem 10.3.3.)

   (c) $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

10. (a) $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(b) The function $\varphi$ is an isomorphism because if the vertices of $\mathcal{G}_1$ are relabeled, $v_i$ being replaced by $\varphi(v_i) = u_i$, then the adjacency matrix of $\mathcal{G}_1$ relative to the $u_i$'s is $A_2$. (See Theorem 10.3.3.)

(c) $P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

11. (a) [BB] $PA = \begin{bmatrix} p & q & r \\ x & y & z \\ a & b & c \end{bmatrix}$ is $A$, but with rows written in the order $2, 3, 1$, the order in which the rows of the $3 \times 3$ identity matrix were rearranged to give $P$.

(b) $AP^T = \begin{bmatrix} b & c & a \\ q & r & p \\ y & z & x \end{bmatrix}$ is $A$, but with columns written in the order $2, 3, 1$, the order in which the rows of the $3 \times 3$ identity matrix were rearranged to give $P$.

(c) $PAP^T = \begin{bmatrix} q & r & p \\ y & z & x \\ b & c & a \end{bmatrix}$ is $A$, but with rows and columns written in the order $2, 3, 1$.

12. $PA = \begin{bmatrix} x & y & z & w \\ a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ p & q & r & s \end{bmatrix}$ is $A$, but with rows written in the order $3, 1, 4, 2$, the order in which the rows of the $4 \times 4$ identity matrix were written to make $P$.

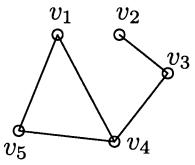
$AP^T = \begin{bmatrix} c & a & d & b \\ r & p & s & q \\ z & x & w & y \\ \gamma & \alpha & \delta & \beta \end{bmatrix}$ is $A$, but with columns written in the order $3, 1, 4, 2$, the order in which the rows of the identity matrix were written to make $P$.

$PAP^T = \begin{bmatrix} z & x & w & y \\ c & a & d & b \\ \gamma & \alpha & \delta & \beta \\ r & p & s & q \end{bmatrix}$ is $A$, but with rows and columns written i the order $3, 1, 4, 2$.

13. (a) [BB] The matrices are the adjacency matrices of graphs $\mathcal{G}_1$, $\mathcal{G}_2$, respectively. Since $\mathcal{G}_2$ has three

vertices of degree 1 while $\mathcal{G}_1$ has only one, the graphs are not isomorphic, so no such $P$ exists, by Theorem 10.3.4.

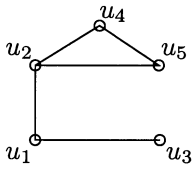
(b) The matrices are the adjacency matrices of the graphs $\mathcal{G}_1$, $\mathcal{G}_2$, respectively, shown to the right. The mapping $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ defined by




$\mathcal{G}_1$

$$v_1 \mapsto u_4, \quad v_2 \mapsto u_3, \quad v_3 \mapsto u_1, \quad v_4 \mapsto u_2, \quad v_5 \mapsto u_5$$

is an isomorphism. Let $P$ be the corresponding permutation matrix, that is, $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.



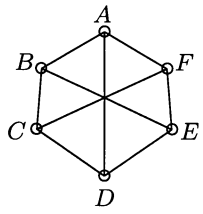

$\mathcal{G}_2$

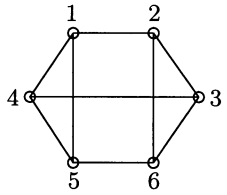
Then $PA_1P^T = A_2$.
This solution is not unique. Another isomorphism is

$$v_1 \mapsto u_5, \quad v_2 \mapsto u_3, \quad v_3 \mapsto u_1, v_4 \mapsto u_2, \quad v_5 \mapsto u_4,$$

and this gives $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \end{bmatrix}$.

(c) The matrices are, respectively, the adjacency matrices of the graphs $\mathcal{G}_1$, $\mathcal{G}_2$ shown in Exercise 8 of Section 9.3, which we reproduce here. The graphs are not isomorphic ($\mathcal{G}_2$ on the right has triangles), so no such $P$ exists.




14. [BB] The $i$th entry on the diagonal of $A^{37}$ is the number of walks of length 37 from $v_i$ to itself. But in a bipartite graph, you can only get from $v_i$ back to itself in an even number of steps. Hence, the entry is 0.

15. (a) [BB] $A^2$ is an adjacency matrix $\leftrightarrow$ $A$ is the 0 matrix.

**Proof.** For $A^2$ to be an adjacency matrix, it must have all diagonal entries equal to 0. But the $i$th diagonal entry of $A^2$ is the number of walks of length 2 from $v_i$ to itself. Now, if $v_iv_j$ is an edge of $\mathcal{G}$, then $v_iv_jv_i$ is a walk of length 2 from $v_i$ to itself, and the $i$th diagonal entry would not be 0. We conclude that $\mathcal{G}$ cannot have any edges; that is, $A$ is the zero matrix. On the other hand, if $A$ is the zero matrix, certainly $A^2 = A$ is an adjacency matrix. ■

(b) $A^3$ is the adjacency matrix of a graph $\mathcal{G} \leftrightarrow$ the connected components of $\mathcal{G}$ are of the form $\circ$ or $\circ$—$\circ$.

**Proof.** Suppose that $A^3$ is the adjacency matrix of a graph $\mathcal{G}$. Then all entries of $A^3$ must be 0 or 1. If $\mathcal{G}$ contains any vertex of degree $d > 1$, say $v$, then we have $w$—$v$—$x$. But now, there are two walks of length 3 from $v$ to $w$; namely, $vxvw$ and $vwvw$, so the $(v, w)$ entry of $A^3$ would be bigger than 1. We conclude that every vertex of $\mathcal{G}$ is of degree 0 or 1; that is, the connected components of $\mathcal{G}$ are of the form $\circ$ or $\circ$—$\circ$. Conversely, if $\mathcal{G}$ is a graph whose connected components are all of this form, then all entries of $A^3$ are 0 or 1 ($a_{ij} = 1 \leftrightarrow v_i \circ$—$\circ v_j$). Note that the diagonal entries are all 0 since there are no triangles. Hence, $A^3$ is the adjacency matrix of some graph. ■

16. ($\longrightarrow$) Suppose the graph is connected. Then there is a walk between every pair of vertices. Let $n$ be the length of the longest such walk. Then for any $i$ and $j$, there is a walk of length $k \leq n$ from $i$ to $j$, and so the $(i, j)$ entry of $A^k$ is 1. Since the entries of any power of $A$ are nonnegative, the $(i, j)$ entry of $B$ is $\geq 1$.

($\longleftarrow$) Conversely, if for some $n$, the matrix $B$ has no zero entries, then given $i$ and $j$, some $A^k$, $1 \leq k \leq n$, has its $(i, j)$ entry nonzero, so there is a walk of length $k$ from $i$ to $j$.

17. The characteristic polynomial of an $n \times n$ matrix $A$ is $\det(A - \lambda I)$ where $I$ is the $n \times n$ identity matrix and $\lambda$ is an indeterminate. This problem requires knowing that the characteristic polynomials of similar matrices are the same.

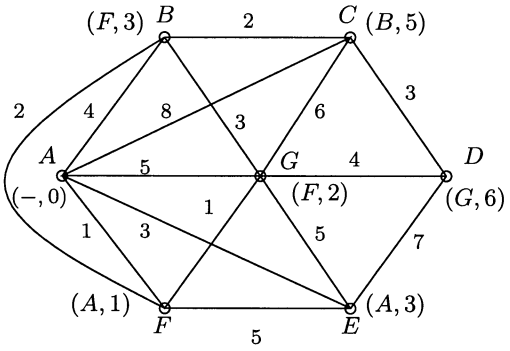
Now the $(i, j)$ entry of $PP^T$ is the dot product of row $i$ of $P$ and column $j$ of $P^T$, that is, the dot product of rows $i$ and $j$ of $P$. Since each row of $P$ is a row of the identity matrix $I$, row $i \cdot$ row $j = 1$ if and only if $i = j$; otherwise, it is 0. Therefore, $P^T P = I$, so $P^T = P^{-1}$. But now, by Theorem 10.3.4, if $\mathcal{G}_1 \cong \mathcal{G}_2$, then $A_2 = PA_1P^{-1}$. Thus, $A_2$ and $A_1$ are similar matrices from which the result follows.

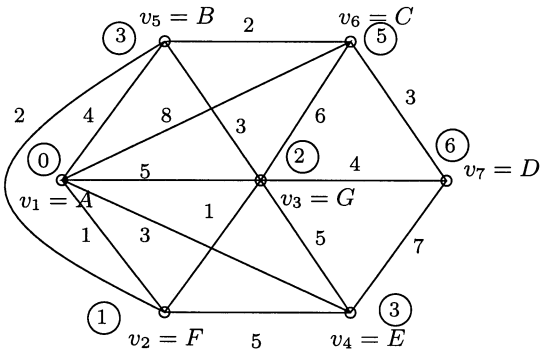
18. [BB] We could store the *incidence matrix*, whose columns correspond to edges and rows to vertices. The $(i, j)$ entry is 1 if vertex $i$ is incident with edge $j$, and otherwise 0. We could also simply store the edges as a linear list, perhaps of numbers: Assuming less than 100 edges, we could store the edge $ij$ as $100i + j$ and recover $i$ as $\lfloor \frac{100i+j}{100} \rfloor$ and $j$ as $(100i + j) - 100i$.

## Exercises 10.4

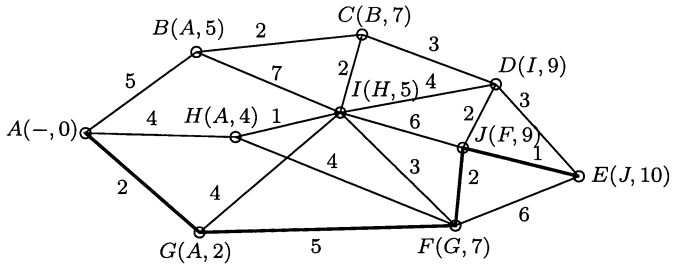
1. It may not be the case that the map of towns which a salesman must visit will be Hamiltonian. Even if it is, it is not clear that the route requiring the least time or taking the shortest distance is necessarily Hamiltonian. It may well be quicker on occasion to travel between small towns via some large metropolitan city rather than to travel on a small airline directly between the towns.

2. [BB]

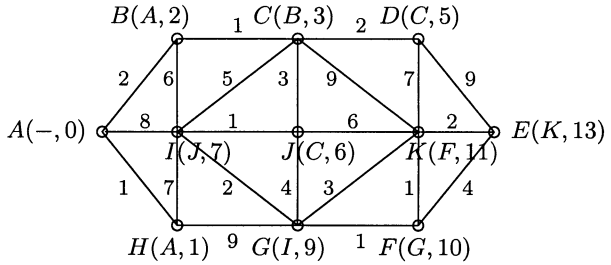


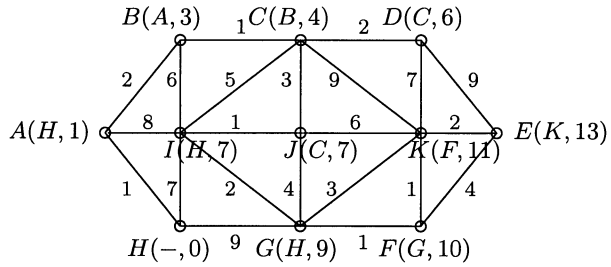

3.




4. (a) [BB] The final labeling starting at $A$ is shown to the right. The shortest path from $A$ to $E$ is $ABCJIGFKE$ and has length 13.

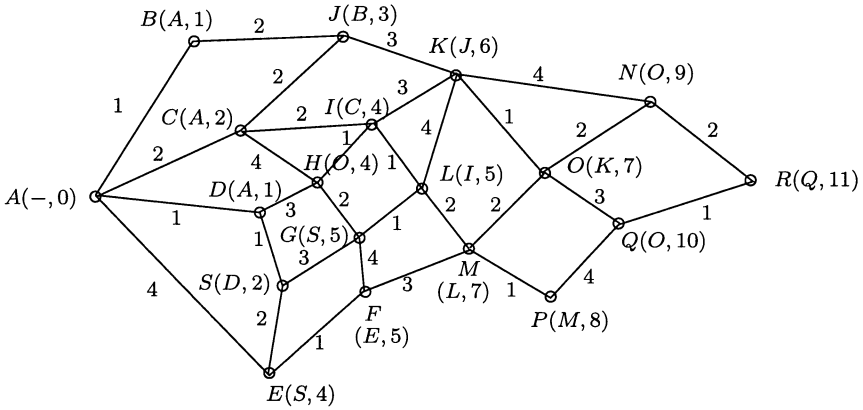



(b) The final labeling, starting at $H$ is shown to the right. The shortest path from $H$ to $D$ is $HABCD$ and has length 6.



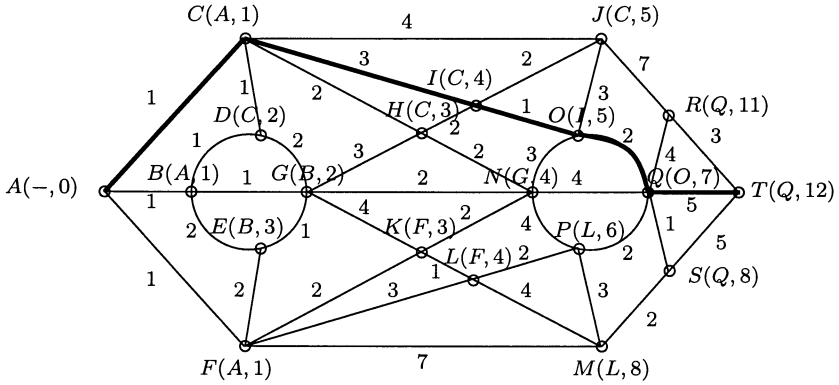
5. [BB; $A$ only] If we start at $A$, permanent labels will be assigned in the order $A$, $H$, $B$, $C$, $D$, $J$, $I$, $G$, $F$, $K$, $E$. If we start at $H$, permanent labels will be assigned in the order $H$, $A$, $B$, $C$, $D$, $I$ or $J$ (at same time), $G$, $F$, $K$, $E$.

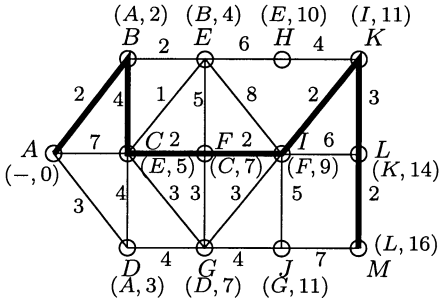
6. A shortest path is $ABJKOQR$, of length 11, as shown.



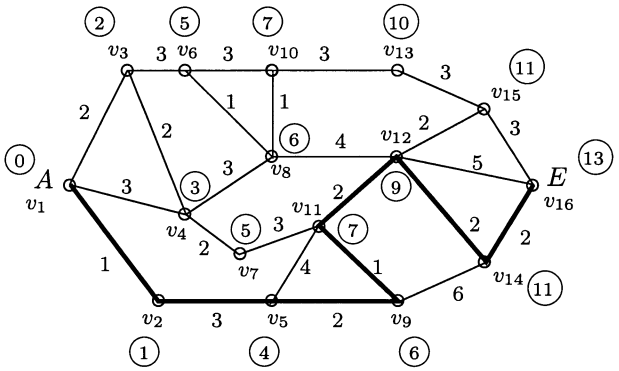

7. [BB] A shortest path has length 11. One way in which permanent labels might be assigned is in the order $A$, $B$, $D$, $C$, $S$, $J$, $I$, $H$, $E$, $L$, $G$, $F$, $K$, $M$, $O$, $P$, $N$, $Q$, $R$.

8. The shortest path is $ACIOQT$, of length 12.



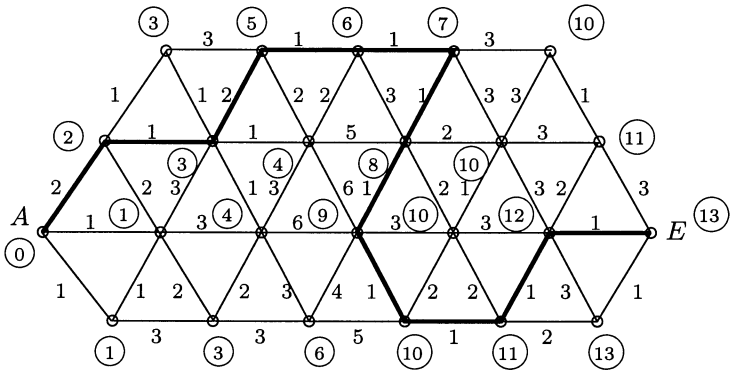

9. 


A shortest path $A$ to $M$ has length 16. One such route is $ABCFIKLM$.



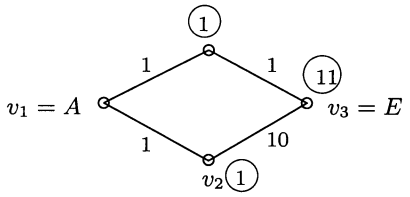

10. [BB] The shortest path has length 13.

11. The shortest path has length 13 and is shown on the graph.




12. [BB] In each case the answer is yes, if $A$ and $E$ were in different components of a graph which was not connected.

13. The idea was interesting, but I don't think it works. Here is a graph where the "new improved" algorithm could give 11 for the distance from $A$ to $E$ by making a bad choice of $v_2$.

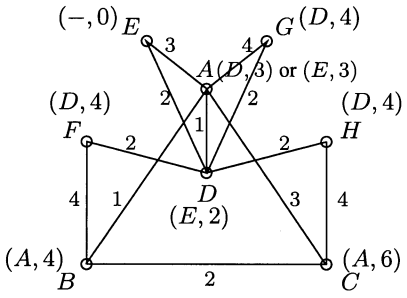



14. (a) The graph is not Hamiltonian.

    (b) Both edges incident with vertices $E$, $F$, $G$ and $H$ must be in any circuit, that is, $AE$, $DE$, $BF$, $DF$, $AG$, $DG$, $CH$ and $DH$. Vertices $B$ and $C$ appear only once among these edges, so if they are to be part of a circuit, additional edges incident with $B$ and $C$ must appear. The most economical way to do this is simply to add edge $BC$, of weight 2, obtaining the circuit $AEDFBCHDGA$ of total weight 25.

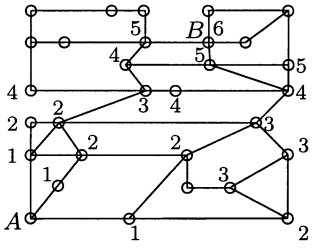
    (c) No! For instance, $ABCHDFDEDGA$ provides a route of length 23.

(d)




15. (a) [BB] Assign each edge a weight of 1.

(b) The shortest path requires 6 edges. Most labels are shown.




16. [BB] As explained in the text, the complexity function for determining the shortest distance from a given vertex to each of the others is $\mathcal{O}(n^2)$; that is, for sufficiently large $n$ and some constant $c$, the algorithm requires at most $cn^2$ comparisons. Applying the algorithm to each of the $n$ vertices (after which all shortest distances are known) requires at most $n(cn^2) = cn^3$ comparisons. This process is $\mathcal{O}(n^3)$.

17. (a) [BB] The final values are the shortest distances from $A$ to itself and to the other vertices. From Fig. 10.23, we deduce these to be 0, 7, 15, 20, 21, 15, 8, 5, 10 and 17.

(b) [BB] After $k = 4$, the value of $d(1, 5)$ is the length of a shortest path from $A$ to $E$ via the vertices $A, B, C, D$. The only such path is $ABCDE$ of length 27: $d(1, 5) = 27$.

The value of $d(1, 6)$ is the length of a shortest path from $A$ to $F$ via $A, B, C, D$. There is no such path, so $d(1, 6)$ is still $\infty$.

The value of $d(3, 4)$ is 8.

The value of $d(8, 5)$ is the length of a shortest path from $H$ to $E$ via $A, B, C, D$. The shortest such path is $HBCDE$: $d(8, 5) = 24$.

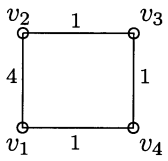
(c) [BB] The initial value of $d(2, 5)$ is $\infty$ since $BE$ is not an edge. After $k = 1, 2, \ldots, 10$, the values of $d(2, 5)$ are $\infty$, $\infty$, $\infty$, 20, 20, 20, 20, 20, 18, 17.

18. (a) These are the shortest distances from $v_7$ to each of $v_1, \ldots, v_8$; namely, 4, 3, 1, 2, 3, 3, 0, 4

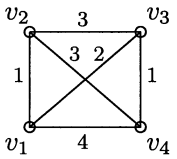
(b) $d(1, 2) = 1$; $d(3, 4) = 1$; $d(2, 5) = 4$; $d(8, 6) = 10$.

(c) $\infty$, $\infty$, $\infty$, 12, 10, 5, 5, 5, 5

19. It certainly does! In the graph at the right, with $i = 1$ and $j = 2$, the algorithm finds that the smallest value of $d(1, k) + d(k, 2)$ is 4 (because at this stage, $d(1, 3) = d(4, 2) = \infty$) and this value never changes.



20. [BB] It is indeed necessary to continue. An identical set of $d(i,j)$ for $k = r$ and $k = r+1$ simply indicates that the shortest path from each $v_i$ to each $v_j$ passing through $v_1, \ldots, v_r$ has the same length as the shortest path through $v_1, \ldots, v_{r+1}$. In the graph shown to the right, for instance, the values of $d(i,j)$ do not change until $k = 3$ since the shortest path between pairs of vertices cannot be reduced until vertex $v_3$ is used.




21. It is sufficient to prove that for each $r = 1, 2, \ldots, n$, after the outer loop has completed the $k = r$ iteration, the number $d(i,j)$ is the length of a shortest path from $v_i$ to $v_j$ through the intermediate vertices $v_1, v_2, \ldots, v_r$. We use mathematical induction on $r$. Suppose $r = 1$. If there is a path from $v_i$ to $v_j$ requiring at most the intermediate vertex $v_1$, the shortest path is $v_iv_j$ or $v_iv_1v_j$. With $k = 1$, the algorithm compares the lengths of these paths and makes $d(i,j)$ the smaller of these numbers. Thus the result is true. Assume the result is true for $k = r$. Thus, when the algorithm has completed the $k = r$ step, the value of each $d(i,j)$ is the length of a shortest path from $v_i$ to $v_j$ through intermediate vertices $v_1, v_2, \ldots, v_r$. If some shortest path from $v_i$ to $v_j$ through $v_1, \ldots, v_{r+1}$ does not use $v_{r+1}$, then $d(i,j) \leq d(i,r+1)+d(r+1,j)$, $d(i,j)$ does not change and remains the length of a shortest path $v_i$ to $v_j$. On the other hand, suppose there is a path $\mathcal{P}$ from $v_i$ to $v_j$ through $v_1, \ldots, v_{r+1}$ which does use $v_{r+1}$ and is shorter than any path through just $v_1, \ldots, v_r$. This path can be viewed as a shortest path from $v_i$ to $v_{r+1}$ through $v_1, \ldots, v_r$ followed by a shortest path from $v_{r+1}$ to $v_j$ through $v_1, \ldots, v_r$. Since the length of $\mathcal{P}$ is the sum of the lengths of these two paths, its length is $d(i,r+1)+d(r+1,j)$, which replaces the former $d(i,j)$. This gives the result for $k = r+1$ and completes the induction.

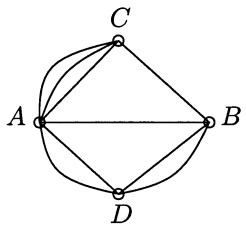
22. For each value of $k$, $i$ and $j$, the algorithm determines one sum, $d(i,k) + d(k,j)$, and makes one comparison to find the minimum of this sum and the current value of $d(i,j)$. Since there are $n^3$ possibilities for $k, i, j$, the number of additions and comparisons is $2n^3$. This function is $\mathcal{O}(n^3)$, as required.

23. Edsger Wybe Dijkstra is currently Schlumberger Professor of Computer Science and Mathematics in the Department of Mathematics at the University of Texas in Austin, where he has worked since 1984. E. W. Dijkstra was born in 1930 in the Netherlands. He obtained his PhD from the University of Amsterdam in 1959 and is renowned for his advocacy of programming as a scientific discipline. As a point of interest, his application for a marriage licence was rejected when he listed "programmer" as his profession, but later accepted when he wrote "theoretical physicist" (his area of PhD research) instead. The author of a famous "GOTO considered harmful" letter, he was one of the first proponents of structured programming. He was the first to solve the "single-source shortest path problem" by providing an algorithm for finding simultaneously shortest paths from one vertex to all others in a weighted graph. He has won a number of prestigious awards including the Turing Award from the Association of Computing Machinery (ACM) in 1972 and the Harry Goode Memorial Award from AFIPS in 1974. In his honor, the Edsger W. Dijstra Scholarship has been established at the Technical University of Eindhoven.

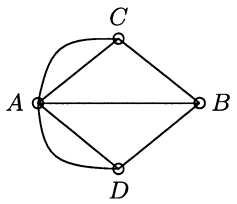
24. Robert W. Floyd received a B.A. and B.S. from the University of Chicago (1955, 1958), but never a Ph.D. although his many accomplishments would easily have earned him one. In 1978, he won the Turing award, the highest award given by the ACM (Association for Computing Machinery). He was one of the inventors of the deterministic linear time selection algorithm. He also made early improvements to quicksort and quickselect. His research involved the analysis of algorithms and computational complexity theory. He also studied parsing, game theory, program size complexity, and mathematical theories of computation. Professor of Computer Science at Stanford University for many years, Robert Floyd passed away on September 25, 2001.

# Chapter 10 Review

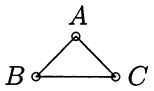
1. The graphical representation of the new problem is shown at the right. Since the degree of every vertex is now even, Theorem 10.1.4 tells us that the connected pseudograph is Eulerian. So the answer is yes.



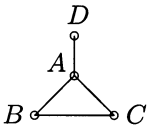

2. Recall the graphical representation of the Königsberg Bridge Problem shown at the right. There are four vertices of odd degree. Building a single new bridge must still leave two vertices of odd degree and the resulting graph will still not be Eulerian. Our mayoralty candidate is **not** telling the truth!




3. Since $\mathcal{G}_1$ has an Eulerian trail, it has exactly two vertices of odd degree, by Theorem 10.1.5. Let $v$ be one of these. Similarly, let $w$ be one of the two vertices of odd degree in $\mathcal{G}_2$. When $v$ and $w$ are joined, the new graph will be connected, and $v$ and $w$ will now be vertices of even degree. Hence the new graph will have exactly two vertices of odd degree and so will possess an Eulerian trail.

4. (a) False. In this graph, 

, $ABCA$ is a trail which is not a path. (Vertex $A$ appears twice).

   (b) False. In this graph, 

, $ABCAD$ is a trail which is not a path.

   (c) True. Suppose $vu_1u_2\cdots u_nw$ is an open trail from $v$ to $w$. If all vertices are distinct, this is a path and we're done. Otherwise, we will show how to construct a shorter open trail from $v$ to $w$. Since this shortening procedure can't continue indefinitely, we will eventually have an open trail all of whose vertices are distinct.

   If vertex $v$ repeats in the trail, say $v = u_i$, the $u_iu_{i+1}\cdots u_nw$ will be a shorter open trail from $v$ to $w$. If vertex $w$ repeats, say $w = u_i$, then $vu_1u_2\cdots u_i$ will be a shorter open trail from $v$ to $w$. Finally, if $u_i = u_j$ with $i < j$, then $vu_1\cdots u_iu_{j+1}\cdots u_nw$ will be the desired shorter open trail.

   (d) True. No repeated vertices imply no repeated edges.

   (e) True. This follows immediately from 4(d).

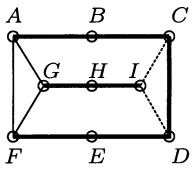
5. We will show that the conditions of Theorem 10.1.4 are met. Let $v$ be any vertex of $\mathcal{G}$. If $e_1$ is any edge incident with $v$, then we are given that $e_1$ belongs to some cycle. This cycle must contain a second edge $e_2$ incident with $v$. The edges $e_1, e_2$ do not belong to any other cycle. In other words, every edge incident with $v$ determines a second such edge, and the edges determined in this way do not repeat. Hence $\deg v$ is even, and the result follows.

6. The graph is not Hamiltonian. Since $E$ is adjacent to three vertices of degree two, the three edges $ED$, $EF$, $EH$ would have to be part of a Hamiltonian cycle, contradicting one of the basic properties of a cycle. The graph is Eulerian. It is connected and all vertices have even degree.

7. (a) This graph is Hamiltonian: $AFGBCDHEA$ is a Hamiltonian cycle.

   (b) This graph is not Hamiltonian. The corner vertices, $A$, $G$, $E$, and $C$ each have degree 2, so any Hamiltonian cycle would have to include the eight edges adjacent with these vertices. Thus any Hamiltonian cycle would have to include the proper cycle $ABCDEFGHA$, which cannot be.

8. The graph is not Hamiltonian as we explain. Since $\mathcal{H}$ were a Hamiltonian circuit. Since vertices $B$, $E$, and $H$ have degree 2, $\mathcal{H}$ would have to contain edges $AB$, $BC$, $GH$, $HI$, $FE$, and $ED$. Since $\mathcal{H}$ cannot contain a proper cycle, not both edges $AG$ and $IC$ could be in $\mathcal{H}$. By symmetry, we may assume that $IC$ is not in $H$.

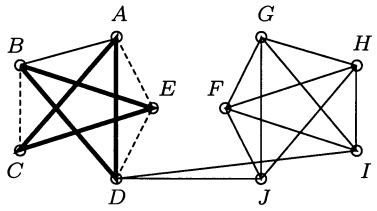


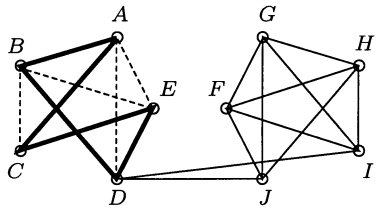

   Then, since $\mathcal{H}$ contains precisely two edges incident with $C$, $CD$ is in $\mathcal{H}$. Since $DC$ and $DE$ are now in $\mathcal{H}$, $DI$ cannot be in $H$. But now only one edge incident with $I$ is in $\mathcal{H}$, a contradiction.

9. $EACBDJGIHFE$ is a Hamiltonian cycle.

10. (a) Any Hamiltonian cycle must contain precisely two edges incident with $C$.

    (b) **Case 1:** $\mathcal{H}$ does not contain $BC$; hence $AC$ and $CE$ must be in $\mathcal{H}$. In this case, $\mathcal{H}$ does not contain $AE$ (else, $\mathcal{H}$ would contain the proper cycle $ACEA$).




    Since $\mathcal{H}$ contains precisely two edges incident with $E$, $\mathcal{H}$ contains precisely one of $BE$, $DE$.

    **Case 1a:** $BE$ is in $\mathcal{H}$ while $DE$ is not. (See graph on the left above.) Here $AB$ cannot be in $\mathcal{H}$ (else $\mathcal{H}$ would contain the proper cycle $ABECA$) and, since $\mathcal{H}$ contains precisely two edges incident with $D$, both $AD$ and $BD$ are in $\mathcal{H}$. This puts the proper cycle $ADBECA$ in $\mathcal{H}$, which cannot be.

    **Case 1b:** $DE$ is in $\mathcal{H}$ while $BE$ is not. (See graph on the right above.) Here, since $\mathcal{H}$ must contain two edges incident with $B$, both $AB$ and $BD$ must be in $\mathcal{H}$. Since $\mathcal{H}$ contains precisely two edges incident with $D$ and we already have $DE$ and $BD$ included, neither $DI$ nor $DJ$ is in $\mathcal{H}$. Thus $\mathcal{H}$ cannot contain a path from $D$ to any of $F, G, H, I, J$, a contradiction.

    We briefly sketch the arguments in the remaining cases, leaving graphs to the reader.

    **Case 2:** $AC$ is not in $\mathcal{H}$ while $BC$ and $CE$ are in. Here $BE$ is out, so either $AE$ or $DE$ are in.

    **Case 2a:** $AE$ is in; $DE$ is out. Now $\mathcal{H}$ cannot contain $AB$ (else it would contain a proper cycle). So both $BD$ and $AD$ are in $\mathcal{H}$. Since $\mathcal{H}$ contains precisely two edges incident with $D$, it cannot contain a path from $D$ to any of the vertices on the right, which cannot be the case.

    **Case 2b:** $DE$ is in; $AE$ is out. Since $\mathcal{H}$ contains exactly two edges incident with $A$, it contains both $AB$ and $AD$. As before, $\mathcal{H}$ cannot contain a path from $D$ to any of the vertices $F, G, H, I, J$.

    **Case 3.** $CE$ is not in $\mathcal{H}$ while both $AC$ and $BC$ are in. Thus $AB$ is not in $\mathcal{H}$ and there are two subcases since precisely one of $AD$, $AE$ is in $\mathcal{H}$.

**Case 3a.** $AD$ is in $\mathcal{H}$ while $AE$ is not. Since $\mathcal{H}$ contains precisely two edges incident with $E$, both $BE$ and $DE$ are in $\mathcal{H}$ and we have a proper cycle in $\mathcal{H}$.

**Case 3b.** $AE$ is in $\mathcal{H}$ while $AD$ is not. This time $BE$ is not in $\mathcal{H}$ (else, $\mathcal{H}$ would contain a proper cycle). We need two edges incident with $E$ and with $B$, so both $DE$ and $BD$ are in $\mathcal{H}$, so $\mathcal{H}$ does not contain a path from $D$ to any of the vertices $F, G, H, I, J$.

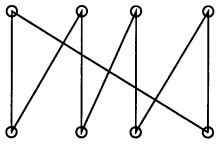
This concludes the proof.

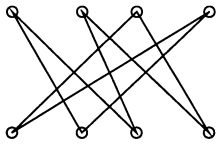
11. (a) $2n$ edges.

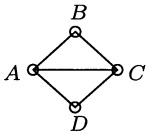
    (b) If a graph is Hamiltonian, any vertex is the starting point of a Hamiltonian cycle. So let $v$ be a fixed vertex. There are $n$ choices for the second vertex in the cycle, then $n-1$ for the third, $n-1$ for the fourth (which is in the bipartition set not containing $v$), and so on. Altogether, there are $n[(n-1)!]^2$ different Hamiltonian cycles in $\mathcal{K}_{n,n}$.

    (c) $\mathcal{K}_{n,n}$ has $n^2$ edges and each cycle requires $2n$ edges, so the maximum is $\dfrac{n^2}{2n} = \dfrac{n}{2}$ edge disjoint cycles.

    (d) Part 11(c) predicts $\frac{4}{2} = 2$ possible edge disjoint Hamiltonian cycles. Here are two.





12. (a) False. The graph 

is Hamiltonian, but $AC$ is not part of any Hamiltonian cycle.

    (b) True. Say $uv$ is an edge. If $uv$ is part of a Hamiltonian cycle, we're done. Otherwise, there must be some path from $u$ to $v$ which is part of of a Hamiltonian cycle (since the graph is Hamiltonian) This path, together with $uv$, gives the required cycle.

    (c) True. Every circuit contains a cycle which is itself a Hamiltonian subgraph.

    (d) True. A Hamiltonian graph contains a Hamiltonian cycle and this cycle (when viewed as a subgraph) is Eulerian.

13. The complete graph $\mathcal{K}_t$ has $(t-1)!$ Hamiltonian cycles. (If the vertices are numbered $1, 2, \ldots, n$ and we begin a cycle with 1, there are $t-1$ choices for the second vertex, $t-2$ for the third, and so on.) The number of $t$-cycles in $\mathcal{K}_n$ is therefore $\binom{n}{t}(t-1)! = \frac{n!}{t(n-t)!}$.

14. Let $v_1, \ldots, v_{14}$ be the vertices of $\mathcal{G}$ and note that $\sum_{i=1}^{14} \deg v_i = 176$. If $\mathcal{G}$ had a vertex of degree $d \leq 6$, then $\sum \deg v_i \leq 13(13) + 6 = 175$, a contradiction. So, each vertex has degree $d \geq 7 = \frac{14}{2}$. Dirac's Theorem now says that $\mathcal{G}$ is Hamiltonian.

    If $\mathcal{G}$ were Eulerian, every vertex would have to have even degree and hence degree $d \leq 12$. Then $\sum \deg v_i \leq 12(14) = 168$, a contradiction.

15. (a) A connected graph is Eulerian if and only if every vertex has even degree. The graph in question has 10 vertices, so each vertex has degree at most 9, hence at most 8 if it is Eulerian. But then $\sum_{v \in \mathcal{V}} \deg v \leq 10(8) = 80$, contradicting $\sum_{v \in \mathcal{V}} \deg v = 2|\mathcal{E}| = 82$.

(b) Dirac's Theorem says that a graph with $n$ vertices is Hamiltonian provided the degree of every vertex is at least $\frac{n}{2}$. This must be the case with the graph in question. To see this, assume $\deg v < \frac{10}{2} = 5$ for some $v \in \mathcal{V}$. This implies that $\deg w \leq 8$ for five other vertices $w$ in $\mathcal{V}$ (those not adjacent to $v$). So $\sum_{v\in\mathcal{V}} \deg v < 5 + 5(8) + 4(9) = 81$, again contradicting $\sum_{v\in\mathcal{V}} \deg v = 2|\mathcal{E}| = 82$.

16. (a) $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

(b) The $(1,5)$ entry of $A^2$ is 1, because there is precisely one walk of length two from $v_1$ to $v_5$. Similarly, the $(1,5)$ entry of $A^3$ is one ($v_1v_6v_3v_5$ is the unique walk of length three), the $(1,5)$ entry of $A^4$ is 5 ($v_1v_2v_5v_3v_5$, $v_1v_2v_5v_2v_5$, $v_1v_2v_1v_2v_5$, $v_1v_4v_1v_2v_5$, $v_1v_6v_1v_2v_5$), and the $(1,5)$ entry of $A^5$ is 6 ($v_1v_2v_1v_6v_3v_5$, $v_1v_4v_1v_6v_3v_5$, $v_1v_6v_1v_6v_3v_5$, $v_1v_6v_3v_5v_2v_5$, $v_1v_6v_3v_5v_3v_5$, $v_1v_6v_3v_6v_3v_5$).

(c) Here is one possibility: $\varphi(v_1) = w_5$, $\varphi(v_2) = w_3$, $\varphi(v_3) = w_1$, $\varphi(v_4) = w_6$, $\varphi(v_5) = w_2$, $\varphi(v_6) = w_4$.

(d) $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

The adjacency matrix for $\mathcal{H}$ is $B = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

We note that

$$PAP^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = B.$$

17. (a) The answer is 0 since there is no edge from $v_1$ to $v_5$.

(b) The answer is 1 since there is an edge from $v_8$ to $w_1$.

(c) The answer is 8 since there are 8 walks of length two from $w_2$ to $w_4$ (via any of $v_1, \ldots, v_8$).

(d) The answer is 12 for the same reason as (c).

(e) The answer is 0 because there is no walk of length two from $w_{12}$ to $v_6$.

(f) The answer is $(12)(8)(12) = 1152$.

(g) The answer is 0 because there is no walk of odd length from $v_5$ to $v_7$.

18. Yes it is. By Theorem 10.1.4, a connected graph is Eulerian if and only if the degree of every vertex is even and this can easily be determined from the adjacency matrix. Remember that the degree of vertex $i$ is the number of 1's in row (or column) $i$.

19. There are at least four 1's in each row. So the degree of each vertex is $d \geq 4 \geq \frac{7}{2}$. The graph has 7 vertices, so the result follows from Dirac's Theorem.

20. Let the vertices of $\mathcal{G}$ be $v_1, \ldots, v_n$. The number $b_{ii}$ is the number of walks of length 2 from $v_i$ to $v_i$, that is, the number of edges incident with $v_i$, equivalently, $\deg v_i$. The given sum is $\frac{1}{2}\sum \deg v_i$. This is the number of edges in $\mathcal{G}$, by Proposition 9.2.5.

21. (a) i. No. The diagonal of an adjacency matrix consists entirely of 0s.

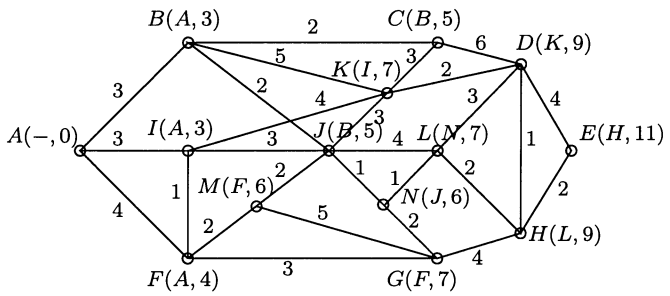
ii. Yes.

iii. No. An adjacency matrix must be symmetric.

(b) The $(1,3)$-entry is the number of walks of length 3 from $v_1$ to $v_3$. This is 0. The $(2,4)$-entry is the number of walks of length 3 from $v_2$ to $v_4$. This is 3 ($v_2v_1v_2v_4$, $v_2v_4v_2v_4$, $v_2v_3v_2v_4$).

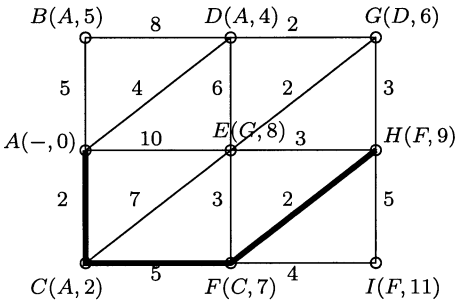
(c) Let the vertices of $\mathcal{G}$ be $v_1, v_2, \ldots, v_n$ and let $m$ be the largest number of edges in shortest paths between all pairs of vertices. Thus, for given $i$ and $j$, there is a path from $v_i$ to $v_j$ of length $m_{ij} \leq m$. Thus, the $(i,j)$ term of $C = B + B^2 + B^3 + \cdots + B^n$ is the $(i,j)$ term of $B^{m_{ij}}+$ other nonnegative terms. This is at least 1, so $C$ has no zero entries.

22. A shortest path from $A$ to $E$ has length 11 and can be realized with $ABJNLHE$.

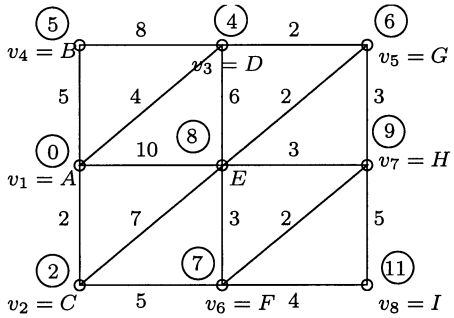



23. One way in which permanent labels might be assigned is $v_1(A)$, $v_2(B)$, $v_3(I)$, $v_4(F)$, $v_5(J)$, $v_6(C)$, $v_7(M)$, $v_8(N)$, $v_9(K)$, $v_{10}(L)$, $v_{11}(G)$, $v_{12}(H)$, $v_{13}(D)$, $v_{14}(E)$.

24. $A$ is given the label $(-, 0)$. Next $C$ is labelled $(A, 2)$, then $D$ is labelled $(A, 4)$ and $B$ is $(A, 5)$. Then $G$ is labelled $(D, 6)$ and $F$ is $(C, 7)$ Then $E$ is labelled $(G, 8)$ and $H$ is $(F, 9)$ or $(G, 9)$. Finally, $I$ is labelled $(F, 11)$. These labels give the required lengths. One shortest path from $A$ to $H$ is $ACFH$; $ADGH$ is another, since $H$ could also have been labelled $(G, 9)$.




25. First, $A$ is given the label 0 and all other vertices are given temporary labels of $\infty$. Next, $B$, $C$, $D$ and $E$ are given temporary labels of 5, 2, 4 and 10, respectively, and 2 becomes the permanent label on $C$. Now the temporary label on $E$ is decreased to 9 while $F$ is given a temporary label of 7. So $D$ gets the next permanent label, 4. Now $G$ acquires a temporary label of 6 ($E$'s temporary label of 9 does not change) and $B$ gets the next permanent label, 5. No new temporary labels are assigned, so the next permanent label is 6 on $G$. Now $E$'s temporary label is decreased to 8, while $H$ gets a temporary label of 9.




The lowest temporary label is now 7 on $F$, so this becomes permanent and $I$ obtains a temporary label of 11 ($H$'s label does not decrease). Next, $E$ gets a permanent label of 8 and then $H$ obtains a permanent label of 9. Since this does not decrease $I$'s temporary label, $I$ then obtains a permanent label of 11. These permanent labels now give the required lengths. The shortest path from $A$ to $H$ given by this procedure is $ADGH$, since $H$ obtained its permanent label in the first instance from $G$. The path $ACFH$ is another shortest one.

26. The solution appears in Fig. 1.

27. In Fig. 2, we show the initial values of $d(i, j)$ and the values for $k = 1, 3, 5, 9$. The required distances appear in the first row of the final array. To find the shortest path from $A$ to $H$, we have to examine how entry 9, in position $(1, 8)$, was obtained. It first appeared when $k = 6$ as $9 = 7 + 2$, the sum of the $(1, 6)$ and $(6, 8)$ entries when $k = 5$. The 2 was present initially while 7 arose as $7 = 2 + 5$, the sum of the $(1, 3)$ and $(3, 6)$ entries when $k = 3$. The 2 and 5 were present initially. The path indicated is thus $v_1v_3v_6v_8$, that is, $ACFH$, as before.

28. A complete graph on 10 vertices has $\binom{10}{2} = 45$ edges. Viewing the towns as vertices and the direct connections as edges, we see that six edges are missing from the complete graph. If a particular vertex $v$ were an endpoint for all six of these edges, then three vertices would not be endpoints for any of them, and this contradicts the fact that only one vertex is joined to all the others. If $v$ were an endpoint for five of these edges, there would still be two vertices of degree 9, again a contradiction. We conclude that each vertex is an endpoint for at most four of the removed edges. Hence each vertex $v$ has degree $d \geq 5$. Since $5 = \frac{10}{2}$, Dirac's Theorem tells us that the graph is Hamiltonian, so the Travelling Salesman's Problem has a solution.

Initial values of $d(i,j)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | $\infty$ | $\infty$ | $\infty$ | 1 | $\infty$ | $\infty$ |
| $v_2$ | 1 | 0 | 3 | $\infty$ | $\infty$ | $\infty$ | 1 | $\infty$ |
| $v_3$ | $\infty$ | 3 | 0 | 2 | $\infty$ | $\infty$ | $\infty$ | 2 |
| $v_4$ | $\infty$ | $\infty$ | 2 | 0 | 1 | $\infty$ | $\infty$ | $\infty$ |
| $v_5$ | $\infty$ | $\infty$ | $\infty$ | 1 | 0 | 3 | $\infty$ | 1 |
| $v_6$ | 1 | $\infty$ | $\infty$ | $\infty$ | 3 | 0 | 4 | $\infty$ |
| $v_7$ | $\infty$ | 1 | $\infty$ | $\infty$ | $\infty$ | 4 | 0 | 1 |
| $v_8$ | $\infty$ | $\infty$ | 2 | $\infty$ | 1 | $\infty$ | 1 | 0 |

$\longrightarrow$

After $k = 1$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | $\infty$ | $\infty$ | $\infty$ | 1 | $\infty$ | $\infty$ |
| 1 | 0 | 3 | $\infty$ | $\infty$ | 2 | 1 | $\infty$ |
| $\infty$ | 3 | 0 | 2 | $\infty$ | $\infty$ | $\infty$ | 2 |
| $\infty$ | $\infty$ | 2 | 0 | 1 | $\infty$ | $\infty$ | $\infty$ |
| $\infty$ | $\infty$ | $\infty$ | 1 | 0 | 3 | $\infty$ | 1 |
| 1 | 2 | $\infty$ | $\infty$ | 3 | 0 | 4 | $\infty$ |
| $\infty$ | 1 | $\infty$ | $\infty$ | $\infty$ | 4 | 0 | 1 |
| $\infty$ | $\infty$ | 2 | $\infty$ | 1 | $\infty$ | 1 | 0 |

$\longrightarrow$

After $k = 2$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | $\infty$ | $\infty$ | 1 | 2 | $\infty$ |
| 1 | 0 | 3 | $\infty$ | $\infty$ | 2 | 1 | $\infty$ |
| 4 | 3 | 0 | 2 | $\infty$ | 5 | 4 | 2 |
| $\infty$ | $\infty$ | 2 | 0 | 1 | $\infty$ | $\infty$ | $\infty$ |
| $\infty$ | $\infty$ | $\infty$ | 1 | 0 | 3 | $\infty$ | 1 |
| 1 | 2 | 5 | $\infty$ | 3 | 0 | 3 | $\infty$ |
| 2 | 1 | 4 | $\infty$ | $\infty$ | 3 | 0 | 1 |
| $\infty$ | $\infty$ | 2 | $\infty$ | 1 | $\infty$ | 1 | 0 |

$\longrightarrow$

After $k = 3$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 6 | $\infty$ | 1 | 2 | 6 |
| 1 | 0 | 3 | 5 | $\infty$ | 2 | 1 | 5 |
| 4 | 3 | 0 | 2 | $\infty$ | 5 | 4 | 2 |
| 6 | 5 | 2 | 0 | 1 | 7 | 6 | 4 |
| $\infty$ | $\infty$ | $\infty$ | 1 | 0 | 3 | $\infty$ | 1 |
| 1 | 2 | 5 | 7 | 3 | 0 | 3 | 7 |
| 2 | 1 | 4 | 6 | $\infty$ | 3 | 0 | 1 |
| 6 | 5 | 2 | 4 | 1 | 7 | 1 | 0 |

$\longrightarrow$

After $k = 4$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 6 | 7 | 1 | 2 | 6 |
| 1 | 0 | 3 | 5 | 6 | 2 | 1 | 5 |
| 4 | 3 | 0 | 2 | 3 | 5 | 4 | 2 |
| 6 | 5 | 2 | 0 | 1 | 7 | 6 | 4 |
| 7 | 6 | 3 | 1 | 0 | 3 | 7 | 1 |
| 1 | 2 | 5 | 7 | 3 | 0 | 3 | 7 |
| 2 | 1 | 4 | 6 | 7 | 3 | 0 | 1 |
| 6 | 5 | 2 | 4 | 1 | 7 | 1 | 0 |

$\longrightarrow$

After $k = 5$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 6 | 7 | 1 | 2 | 6 |
| 1 | 0 | 3 | 5 | 6 | 2 | 1 | 5 |
| 4 | 3 | 0 | 2 | 3 | 5 | 4 | 2 |
| 6 | 5 | 2 | 0 | 1 | 4 | 6 | 2 |
| 7 | 6 | 3 | 1 | 0 | 3 | 7 | 1 |
| 1 | 2 | 5 | 4 | 3 | 0 | 3 | 4 |
| 2 | 1 | 4 | 6 | 7 | 3 | 0 | 1 |
| 6 | 5 | 2 | 2 | 1 | 4 | 1 | 0 |

$\longrightarrow$

After $k = 6$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 5 | 4 | 1 | 2 | 5 |
| 1 | 0 | 3 | 5 | 5 | 2 | 1 | 5 |
| 4 | 3 | 0 | 2 | 3 | 5 | 4 | 2 |
| 5 | 5 | 2 | 0 | 1 | 4 | 6 | 2 |
| 4 | 5 | 3 | 1 | 0 | 3 | 6 | 1 |
| 1 | 2 | 5 | 4 | 3 | 0 | 3 | 4 |
| 2 | 1 | 4 | 6 | 6 | 3 | 0 | 1 |
| 5 | 5 | 2 | 2 | 1 | 4 | 1 | 0 |

$\longrightarrow$

After $k = 7$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 5 | 4 | 1 | 2 | 3 |
| 1 | 0 | 3 | 5 | 5 | 2 | 1 | 2 |
| 4 | 3 | 0 | 2 | 3 | 5 | 4 | 2 |
| 5 | 5 | 2 | 0 | 1 | 4 | 6 | 2 |
| 4 | 5 | 3 | 1 | 0 | 3 | 6 | 1 |
| 1 | 2 | 5 | 4 | 3 | 0 | 3 | 4 |
| 2 | 1 | 4 | 6 | 6 | 3 | 0 | 1 |
| 3 | 2 | 2 | 2 | 1 | 4 | 1 | 0 |

$\longrightarrow$

After $k = 8$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 5 | 4 | 1 | 2 | 3 |
| 1 | 0 | 3 | 4 | 3 | 2 | 1 | 2 |
| 4 | 3 | 0 | 2 | 3 | 5 | 3 | 2 |
| 5 | 4 | 2 | 0 | 1 | 4 | 3 | 2 |
| 4 | 3 | 3 | 1 | 0 | 3 | 2 | 1 |
| 1 | 2 | 5 | 4 | 3 | 0 | 3 | 4 |
| 2 | 1 | 3 | 3 | 2 | 3 | 0 | 1 |
| 3 | 2 | 2 | 2 | 1 | 4 | 1 | 0 |

**FIGURE 1**: Solution to Review Exercise 26, Chapter 10

Initial values of $d(i,j)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $v_1$ | 0 | 5 | 2 | 4 | 10 | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| $v_2$ | 5 | 0 | $\infty$ | 8 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| $v_3$ | 2 | $\infty$ | 0 | $\infty$ | 7 | 5 | $\infty$ | $\infty$ | $\infty$ |
| $v_4$ | 4 | 8 | $\infty$ | 0 | 6 | $\infty$ | 2 | $\infty$ | $\infty$ |
| $v_5$ | 10 | $\infty$ | 7 | 6 | 0 | 3 | 2 | 3 | $\infty$ |
| $v_6$ | $\infty$ | $\infty$ | 5 | $\infty$ | 3 | 0 | $\infty$ | 2 | 4 |
| $v_7$ | $\infty$ | $\infty$ | $\infty$ | 2 | 2 | $\infty$ | 0 | 3 | $\infty$ |
| $v_8$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 3 | 2 | 3 | 0 | 5 |
| $v_9$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 4 | $\infty$ | 5 | 0 |

$\longrightarrow$

After $k = 1$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 2 | 4 | 10 | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 5 | 0 | 7 | 8 | 15 | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 2 | 7 | 0 | 6 | 7 | 5 | $\infty$ | $\infty$ | $\infty$ |
| 4 | 8 | 6 | 0 | 6 | $\infty$ | 2 | $\infty$ | $\infty$ |
| 10 | 15 | 7 | 6 | 0 | 3 | 2 | 3 | $\infty$ |
| $\infty$ | $\infty$ | 5 | $\infty$ | 3 | 0 | $\infty$ | 2 | 4 |
| $\infty$ | $\infty$ | $\infty$ | 2 | 2 | $\infty$ | 0 | 3 | $\infty$ |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | 3 | 2 | 3 | 0 | 5 |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 4 | $\infty$ | 5 | 0 |

$\longrightarrow$

After $k = 3$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 2 | 4 | 9 | 7 | $\infty$ | $\infty$ | $\infty$ |
| 5 | 0 | 7 | 8 | 14 | 12 | $\infty$ | $\infty$ | $\infty$ |
| 2 | 7 | 0 | 6 | 7 | 5 | $\infty$ | $\infty$ | $\infty$ |
| 4 | 8 | 6 | 0 | 6 | 11 | 2 | $\infty$ | $\infty$ |
| 9 | 14 | 7 | 6 | 0 | 3 | 2 | 3 | $\infty$ |
| 7 | 12 | 5 | 11 | 3 | 0 | $\infty$ | 2 | 4 |
| $\infty$ | $\infty$ | $\infty$ | 2 | 2 | $\infty$ | 0 | 3 | $\infty$ |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | 3 | 2 | 3 | 0 | 5 |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 4 | $\infty$ | 5 | 0 |

$\longrightarrow$

After $k = 5$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 2 | 4 | 9 | 7 | 6 | 12 | $\infty$ |
| 5 | 0 | 7 | 8 | 14 | 12 | 10 | 17 | $\infty$ |
| 2 | 7 | 0 | 6 | 7 | 5 | 8 | 10 | $\infty$ |
| 4 | 8 | 6 | 0 | 6 | 9 | 2 | 9 | $\infty$ |
| 9 | 14 | 7 | 6 | 0 | 3 | 2 | 3 | $\infty$ |
| 7 | 12 | 5 | 9 | 3 | 0 | 5 | 2 | 4 |
| 6 | 10 | 8 | 2 | 2 | 5 | 0 | 3 | $\infty$ |
| 12 | 17 | 10 | 9 | 3 | 2 | 3 | 0 | 5 |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 4 | $\infty$ | 5 | 0 |

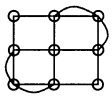
$\longrightarrow$

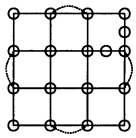
After $k = 9$

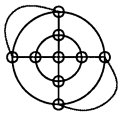
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 2 | 4 | 8 | 7 | 6 | 9 | 11 |
| 5 | 0 | 7 | 8 | 12 | 12 | 10 | 13 | 16 |
| 2 | 7 | 0 | 6 | 7 | 5 | 8 | 7 | 9 |
| 4 | 8 | 6 | 0 | 4 | 7 | 2 | 5 | 10 |
| 8 | 12 | 7 | 4 | 0 | 3 | 2 | 3 | 7 |
| 7 | 12 | 5 | 7 | 3 | 0 | 5 | 2 | 4 |
| 6 | 10 | 8 | 2 | 2 | 5 | 0 | 3 | 8 |
| 9 | 13 | 7 | 5 | 3 | 2 | 3 | 0 | 5 |
| 11 | 16 | 9 | 10 | 7 | 4 | 8 | 5 | 0 |

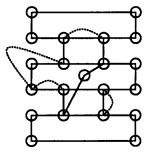
**FIGURE 2**: Solution to Review Exercise 27, Chapter 10

## Exercises 11.1

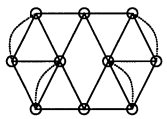
1. (a) [BB] 

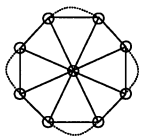
(b) 

(c) 

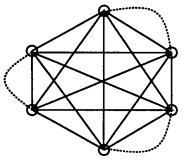
(d) 

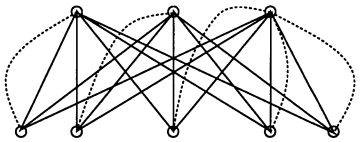
(e) and (f) In each of these graphs, there are four pairs of odd vertices, so the best we can possibly do is to duplicate four edges. In each case, we show such a pseudograph.

(e) 

(f) 

2. [BB; $\mathcal{K}_5$] $\mathcal{K}_5$ is Eulerian, so no additional edges are needed. The solution for $\mathcal{K}_6$ is on the left and for $\mathcal{K}_{3,5}$ on the right.

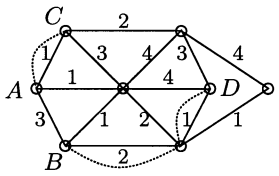




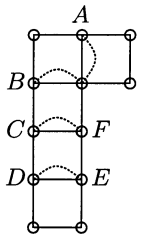
3. [BB] There are four odd vertices and the table shows the requisite calculations.

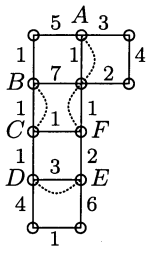
| Partition into pairs | Sum of lengths of shortest paths |
|---|---|
| $\{A,B\},\{C,D\}$ | $2+5=7$ |
| $\{A,C\},\{B,D\}$ | $1+3=4$ |
| $\{A,D\},\{B,C\}$ | $4+3=7$ |

The unique solution is shown.




4. [BB] There are six odd vertices, labeled $A,\ldots,F$ in the figures to the right. In the unweighted case, we know that at least four additional edges are required (since $A$ is not adjacent to any of $B,C,D,E,F$) and we also observe that there is a solution using four extra edges. So we have an immediate answer. The table shows the calculations required for the weighted graph and, just for interest's sake, for the unweighted graph as well.



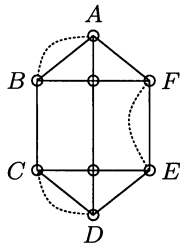




One of three solutions to the unweighted problem, each requiring the duplication of four edges, is shown above. One of several solutions to the weighted problem, each requiring the duplication of edges of total weight 6, is also shown.

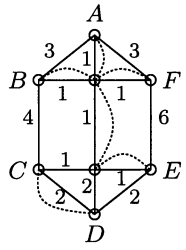
| Partition into pairs | Sum of lengths of shortest paths | |
|---|---|---|
| | Unweighted graph | Weighted graph |
| $\{A,B\},\{C,D\},\{E,F\}$ | $2+1+1=4$ | $4+1+2=7$ |
| $\{A,B\},\{C,E\},\{D,F\}$ | $2+2+2=6$ | $4+3+2=9$ |
| $\{A,B\},\{C,F\},\{D,E\}$ | $2+1+1=4$ | $4+1+3=8$ |
| $\{A,C\},\{B,D\},\{E,F\}$ | $3+2+1=6$ | $3+2+2=7$ |
| $\{A,C\},\{B,E\},\{D,F\}$ | $3+3+2=8$ | $3+4+2=9$ |
| $\{A,C\},\{B,F\},\{D,E\}$ | $3+2+1=6$ | $3+2+3=8$ |
| $\{A,D\},\{B,C\},\{E,F\}$ | $4+1+1=6$ | $4+1+2=7$ |
| $\{A,D\},\{B,E\},\{C,F\}$ | $4+3+1=8$ | $4+4+1=9$ |
| $\{A,D\},\{B,F\},\{C,E\}$ | $4+2+2=8$ | $4+2+3=9$ |
| $\{A,E\},\{B,C\},\{D,F\}$ | $3+1+2=6$ | $4+1+2=7$ |
| $\{A,E\},\{B,D\},\{C,F\}$ | $3+2+1=6$ | $4+2+1=7$ |
| $\{A,E\},\{B,F\},\{C,D\}$ | $3+2+1=6$ | $4+2+1=7$ |
| $\{A,F\},\{B,C\},\{D,E\}$ | $2+1+1=4$ | $2+1+3=6$ |
| $\{A,F\},\{B,D\},\{C,E\}$ | $2+2+2=6$ | $2+2+3=7$ |
| $\{A,F\},\{B,E\},\{C,D\}$ | $2+3+1=6$ | $2+4+1=7$ |

5. In graphs (a), (b), and (c), there are six odd vertices, labeled $A,\ldots,F$. In case (a), we must add at least three edges. A solution with just three edges added is shown. In case (b), edges of total weight 7 must be added. In case (c), edges of total weight 11 must be added. In case (d), there are only four odd vertices, labeled $A$, $B$, $C$, $D$. The solution is to duplicate four edges, with a total weight of 8, as shown.

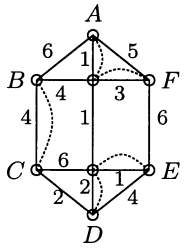
(a)



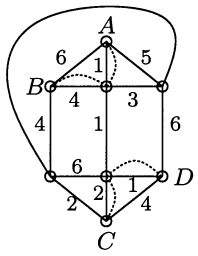

(b)




(c)




(d)



| Partition into pairs | Sum of lengths of shortest paths | | |
|---|---|---|---|
| | Graph (a) | Graph (b) | Graph (c) |
| $\{A,B\},\{C,D\},\{E,F\}$ | $1+1+1=3$ | $2+2+3=7$ | $5+2+5=12$ |
| $\{A,B\},\{C,E\},\{D,F\}$ | $1+2+2=5$ | $2+2+4=8$ | $5+5+6=16$ |
| $\{A,B\},\{C,F\},\{D,E\}$ | $1+3+1=5$ | $2+3+2=7$ | $5+8+3=16$ |
| $\{A,C\},\{B,D\},\{E,F\}$ | $2+2+1=5$ | $3+4+3=10$ | $6+6+5=17$ |
| $\{A,C\},\{B,E\},\{D,F\}$ | $2+3+2=7$ | $3+3+4=10$ | $6+6+6=18$ |
| $\{A,C\},\{B,F\},\{D,E\}$ | $2+2+1=5$ | $3+2+2=7$ | $6+7+3=16$ |
| $\{A,D\},\{B,C\},\{E,F\}$ | $3+1+1=5$ | $4+3+3=10$ | $4+4+5=13$ |
| $\{A,D\},\{B,E\},\{C,F\}$ | $3+3+3=9$ | $4+3+3=10$ | $4+6+8=18$ |
| $\{A,D\},\{B,F\},\{C,E\}$ | $3+2+2=7$ | $4+2+2=8$ | $4+7+5=16$ |
| $\{A,E\},\{B,C\},\{D,F\}$ | $2+1+2=5$ | $3+3+4=10$ | $3+4+6=13$ |
| $\{A,E\},\{B,D\},\{C,F\}$ | $2+2+3=7$ | $3+4+3=10$ | $3+6+8=17$ |
| $\{A,E\},\{B,F\},\{C,D\}$ | $2+2+1=5$ | $3+2+2=7$ | $3+7+2=12$ |
| $\{A,F\},\{B,C\},\{D,E\}$ | $1+1+1=3$ | $2+3+2=7$ | $4+4+3=11$ |
| $\{A,F\},\{B,D\},\{C,E\}$ | $1+2+2=5$ | $2+4+2=8$ | $4+6+5=15$ |
| $\{A,F\},\{B,E\},\{C,D\}$ | $1+3+1=5$ | $2+3+2=7$ | $4+6+2=12$ |

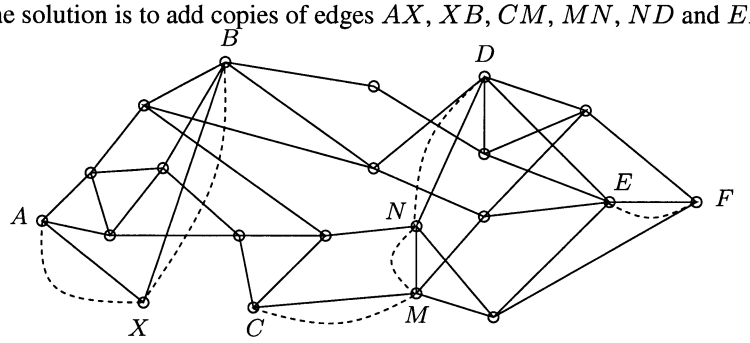
| Graph (d) | |
|---|---|
| Partition into pairs | Sum of lengths of shortest paths |
| $\{A,B\},\{C,D\}$ | $5+3=8$ |
| $\{A,C\},\{B,D\}$ | $4+6=10$ |
| $\{A,D\},\{B,C\}$ | $3+6=9$ |

6. [BB] The shortest route from $X$ to $X$ is a circuit (perhaps in a pseudograph) passing through $Y$. Hence, it can also be viewed as a circuit from $Y$ to $Y$. Any other route from $Y$ to $Y$ could also be viewed as a route from $X$ to $X$. So there cannot be any shorter route from $Y$ to $Y$.

7. The six vertices of odd degree are $\{A,B,C,D,E,F\}$ and the shortest path totals for all partitions are given.

| | | | |
|---|---|---|---|
| $\{A,B\},\{C,D\},\{E,F\}$ | $2+3+1=6$ | $\{AB,CE,DF\}$ | $2+3+2=7$ |
| $\{A,B\},\{C,F\},\{D,E\}$ | $2+3+1=6$ | $\{AC,BD,EF\}$ | $3+2+1=6$ |
| $\{A,C\},\{B,E\},\{D,F\}$ | $3+3+2=8$ | $\{AC,BF,DE\}$ | $3+4+1=8$ |
| $\{A,D\},\{B,C\},\{E,F\}$ | $4+3+1=8$ | $\{AD,BE,CF\}$ | $4+3+3=10$ |
| $\{A,D\},\{B,F\},\{C,E\}$ | $4+4+3=11$ | $\{AE,BC,DF\}$ | $5+3+2=10$ |
| $\{A,E\},\{B,D\},\{C,F\}$ | $5+2+3=10$ | $\{AE,BF,CD\}$ | $5+4+3=12$ |
| $\{A,F\},\{B,C\},\{D,E\}$ | $6+3+1=10$ | $\{AF,BD,CE\}$ | $6+2+3=11$ |
| $\{A,F\},\{B,E\},\{C,D\}$ | $6+3+3=12$ | | |

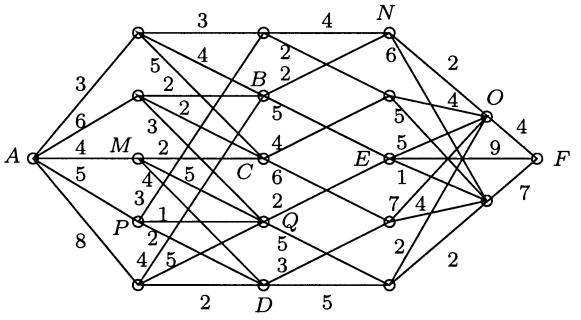
So we see that one solution is to add copies of edges $AX$, $XB$, $CM$, $MN$, $ND$ and $EF$, as shown.



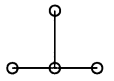
8. The six vertices of odd degree are $\{A, B, C, D, E, F\}$ and the shortest path labels for all partitions are shown.

| | | | |
|---|---|---|---|
| $\{A,B\},\{C,D\},\{E,F\}$ | $7+6+8=21$ | $\{AB, CE, DF\}$ | $7+9+11=27$ |
| $\{A,B\},\{C,F\},\{D,E\}$ | $7+12+5=24$ | $\{AC, BD, EF\}$ | $6+6+8=20$ |
| $\{A,C\},\{B,E\},\{D,F\}$ | $6+5+11=22$ | $\{AC, BF, DE\}$ | $6+8+5=19$ |
| $\{A,D\},\{B,C\},\{E,F\}$ | $7+4+8=19$ | $\{AD, BE, CF\}$ | $7+5+12=24$ |
| $\{A,D\},\{B,F\},\{C,E\}$ | $7+8+9=24$ | $\{AE, BC, DF\}$ | $8+4+11=23$ |
| $\{A,E\},\{B,D\},\{C,F\}$ | $8+6+12=26$ | $\{AE, BF, CD\}$ | $8+8+6=22$ |
| $\{A,F\},\{B,C\},\{D,E\}$ | $15+4+5=24$ | $\{AF, BD, CE\}$ | $15+6+9=30$ |
| $\{A,F\},\{B,E\},\{C,D\}$ | $15+5+6=26$ | | |

So we see that one of two solutions is to add copies of edges $AM$, $MC$, $BN$, $NO$, $OF$, $DP$, $PQ$ and $QE$.




9. [BB] It certainly is possible and this is illustrated by the graph shown (all of whose vertices are odd).



10. [BB] Each even vertex in $\mathcal{G}$ either has unchanged degree in $\mathcal{G}'$ or, as an intermediate vertex on one or more paths between odd vertices, has its degree increased by a multiple of 2, so its degree remains even in $\mathcal{G}'$. Each odd vertex in $\mathcal{G}$ is the end vertex of precisely one new path constructed by the algorithm. In addition, an odd vertex could be an intermediate vertex on a path between odd vertices, so it is even in $\mathcal{G}'$.

11. **Solution 1.** The first odd vertex can be paired with any of the remaining $n-1$. For each pairing, one of the remaining vertices can be paired with any of $n-3$ vertices. Then one of the remaining vertices can be paired with any of $n-5$ and so on. The number of pairs is

$$(n-1)(n-3)(n-5)\cdots 5\cdot 3\cdot 1$$

Note that $n$ is even so the product here really does finish as indicated.

**Solution 2.** Let $m = n/2$. There are $\binom{n}{2}$ ways to select a pair $\{v_1, w_1\}$ of odd vertices; for each of these ways, there are $\binom{n-2}{2}$ ways to select another pair $\{v_2, w_2\}$, and so on. Altogether, there are

$$\binom{n}{2}\times\binom{n-2}{2}\times\cdots\times\binom{4}{2}\times\binom{2}{2}=\frac{n!}{2^m}$$

ways to select $\{v_1, w_1\}, \ldots, \{v_m, w_m\}$. Disregarding the order in which these pairs are selected, there are just $\dfrac{n!}{2^m m!}$ ways to divide the odd vertices into pairs.

12. (a) Let $v_1$ be any odd vertex. Since $v_1$ acquires even degree in $\mathcal{G}''$, it is incident with a new edge $v_1w$. If $w$ is odd, then $v_1w$ is the desired path. Otherwise, $w$ is even in $\mathcal{G}$ and, still even in $\mathcal{G}''$, and hence incident with another new edge $wu$. If $u$ is odd, then $v_1wu$ is the desired path. Otherwise, continue until the first odd vertex $w_1$ is reached.

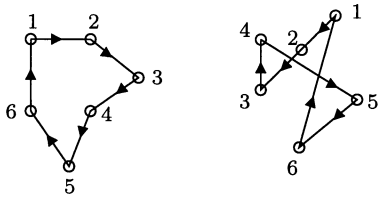
    (b) Continuing as in (a), let $\mathcal{P}_1$ be a path of new edges from $v_1$ to $w_1$ and suppose there is another odd vertex $v_2$ ($\notin \{v_1, w_1\}$). Apply the procedure outlined in (a) to this vertex, but with two changes. If the path of new edges starting at $v_2$ reaches a vertex $w$ on $\mathcal{P}_1$ which was even in $\mathcal{G}$, then there must be another new edge incident with $w$ (since $w$ still has even degree in $\mathcal{G}''$). Follow from $w$ that edge which was not in $\mathcal{P}_1$. Also, if the first odd vertex encountered on the path of new edges from $v_2$ should happen to be $v_1$ (or $w_1$ or $v_2$), then observe that there must be a third new edge incident with $v_1$ ($w_1$, $v_2$ respectively) because $v_1$ has even degree in $\mathcal{G}''$. Take this third new edge and continue on a path of new edges not in $\mathcal{P}_1$ until the first odd vertex not encountered before is reached. In this way, we obtain a path $\mathcal{P}_2$, edge disjoint with $\mathcal{P}_1$, consisting of new edges, starting at $v_2$ and ending at an odd vertex $w_2$ ($\notin \{v_1, w_1, w_2\}$). Continue.

    (c) Since the paths constructed in (b) are edge disjoint and consist entirely of new edges, the number of new edges duplicated for $\mathcal{G}''$ is the sum of the lengths of these paths. And the number of edges duplicated for $\mathcal{G}'$ was the minimum of the sum of the lengths of shortest paths between all possible pairs of vertices in a partition of the odd vertices into pairs.

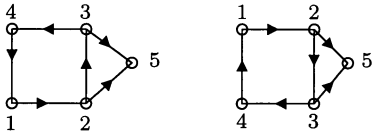
## Exercises 11.2

1. (a) [BB] These digraphs are not isomorphic because one has six vertices while the other has five.

    (b) The following labels show that these digraphs are isomorphic.




    (c) The following labels show that these digraphs are isomorphic.

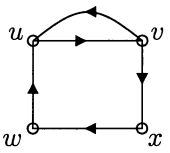



    (d) These digraphs are not isomorphic. The left one has a vertex of indegree 2 and outdegree 0 while the right one has no such vertex.

2. No. The digraphs in Exercise 1(c) for example, have indegree sequence $2, 1, 1, 1, 1$ and outdegree sequence $2, 2, 1, 1, 0$.

3. [BB] Every arc comes out of one vertex and goes into another, hence, adds one to the sum of all indegrees and one to the sum of all outdegrees.

4. The digraph is strongly connected—you can get from any vertex to any other along the circuit $uvxwu$—but it is not Eulerian because, for example, $\text{indeg}\, u = 2 \neq 1 = \text{outdeg}\, u$.




5. [BB] The answer is yes. Since $\mathcal{G}$ is an Eulerian **graph**, there exists an Eulerian circuit. Now just orient the edges of this circuit in the direction of a walk along it.

6. Since there are no cycles, any walk is a path, so it contains at most $n$ vertices and $n-1$ arcs.

7. The sum of the entries in row $i$ of an adjacency matrix is the outdegree of vertex $i$; the sum of the entries in column $i$ is the indegree of vertex $i$.

8. [BB] (a) $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

   (b) The $(3,3)$ entry of $A^2$ is 2 because there are two directed walks of length 2 from vertex 3 to vertex 3; namely, 323 and 343. The $(1,4)$ entry of $A^2$ is 1 because the only directed walk of length 2 from vertex 1 to vertex 4 is 134.

   (c) The $(4,2)$ entry of $A^3$ is 4 because there are four directed walks of length 3 from vertex 4 to vertex 2; namely, 4212, 4232, 4132, and 4342. The $(1,3)$ entry of $A^4$ is 6. There are six walks of length 4 from vertex 1 to vertex 3; namely, 13423, 13413, 13213, 12123, 12323, and 12343.

   (d) The digraph is strongly connected; 12341 is a directed circuit which permits travel in the right direction between any two vertices.

   (e) The digraph is not Eulerian. Vertices 2, 3, and 4 have different indegree and outdegree; vertex 3, for instance, has indegree 3 and outdegree 2.

9. (a) $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

   (b) The $(3,3)$ entry of $A^2$ is 1 because there is just one directed walk of length 2 from vertex 3 to vertex 3; namely, 343.

   The $(1,4)$ entry of $A^2$ is 0 because there are no directed walks of length 2 from vertex 1 to vertex 4.

   (c) The $(4,2)$ entry of $A^3$ is 1 because there is just one walk of length 3 from vertex 4 to vertex 2; namely, 4352.

   The $(1,3)$ entry of $A^4$ is 3 because there are three directed walks of length 4 from vertex 1 to vertex 3; namely, 14343, 12143, and 14143.

   (d) The digraph is strongly connected; 143521 is a directed cycle which allows travel respecting arrows between any two vertices.

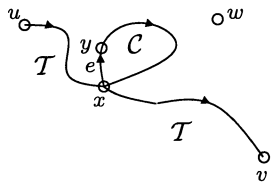
   (e) The digraph is not Eulerian. Only vertex 1 has indegree and outdegree equal.

10. Copy the proof of Theorem 10.1.4. The only change is that we have to be sure it is possible to follow an arc in the proper direction.

11. [BB] A digraph has an Eulerian trail between vertices $u$ and $v$ if and only if

    - one of these vertices, say $u$, has outdegree one more than its indegree, the other vertex, $v$, has indegree one more than its outdegree,
    - the indegrees and outdegrees of every vertex except $u$ and $v$ are equal, and
    - for every pair of vertices $x, y$ there is a (directed) path from $x$ to $y$ or from $y$ to $x$.

    **Proof.** If there is an Eulerian trail from $u$ to $v$, it is clear that the three given conditions are necessary. On the other hand, suppose the three conditions hold in a digraph $\mathcal{G}$. We show that $\mathcal{G}$ has an Eulerian trail. The result is obvious if $u$ and $v$ are the only vertices of $\mathcal{G}$, so we may assume $\mathcal{G}$ has at least three vertices.

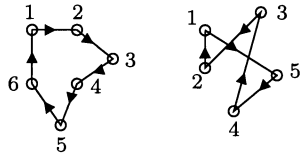
    
    

    By hypothesis, there is an arc $ux$. If $x \neq v$, there is an arc $xy$, and so on. By hypothesis, this process can be continued without repeating an arc. It must eventually terminate, however, and only at $v$, at which point we have a directed trail $\mathcal{T}$ from $u$ to $v$. Since $\mathcal{G}$ has at least three vertices, we may assume that $\mathcal{T}$ contains at least three vertices. If $\mathcal{T}$ contains every vertex and every arc, it is an Eulerian trail. Otherwise, there is a vertex $x$ not in $\mathcal{T}$ and a vertex $y$ in $\mathcal{T}$, $y \neq u$, $y \neq x$. By hypothesis, there is a directed path from $x$ to $y$ or from $y$ to $x$. In the former case, the existence of a terminal arc $zy$ implies another arc $yw$. In either event, there is an arc $yw$ not in $\mathcal{T}$. Continue along a trail beginning $yw$ and note that every time we reach a vertex different from $y$ it is possible to leave on a new arc. Thus this trail can only terminate at $y$, with a circuit $C$. Now $uyCyv$ is a trail longer than $\mathcal{T}$. If it contains every vertex and every edge, it is Eulerian. Otherwise, continue and eventually an Eulerian trail is achieved. ■

12. $A^T$ is the adjacency matrix of the digraph obtained by reversing the arrows on the edges of $\mathcal{G}$.

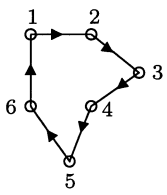
    Yes it is possible for $\mathcal{G}$ and $\mathcal{G}^T$ to be isomorphic. Such is the case with graph $\mathcal{G}_1$ in Fig. 11.7, for example.

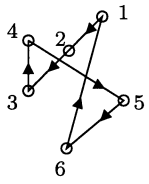
13. (a) [BB]

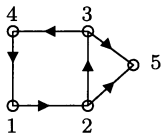
    With the graphs labeled as shown, the adjacency matrices are

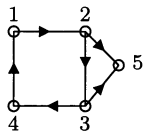
$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) 



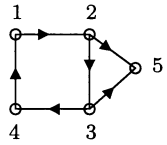

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

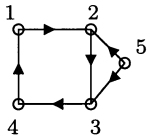
(c) 




$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) With the graphs labeled as shown, the adjacency matrices are






$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The row of 0's in $A_1$ and the lack of such a row in $A_2$ is a further indication of the fact that the corresponding graphs are not isomorphic. (One has a vertex of outdegree 0 and the other does not.)

14. (a) [BB] $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(b) [BB] With the vertices of $\mathcal{G}_1$ relabeled according to $\varphi$, its adjacency matrix becomes that of $\mathcal{G}_2$.

(c) [BB] $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(d) [BB] The digraphs are strongly connected: In $\mathcal{G}_1$, for instance, $v_1v_2v_3v_4v_1$ is a circuit which respects arrows and $\mathcal{G}_2$ is isomorphic to $\mathcal{G}_1$ hence, also strongly connected.

(e) [BB] The digraphs are not Eulerian. In $\mathcal{G}_1$, for instance, vertex $v_2$ has indegree 2 but outdegree 1.

15. (a) $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$.

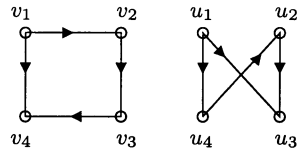
(b) With the vertices of $\mathcal{G}_1$ relabeled according to $\varphi$, the adjacency matrix of $\mathcal{G}_1$ becomes that of $\mathcal{G}_2$.

(c) $P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(d) The digraphs are not strongly connected; in $\mathcal{G}_1$, for instance, there is no path from $v_5$ to $v_4$ which respects directions and $\mathcal{G}_2$, being isomorphic to $\mathcal{G}_1$ cannot be strongly connected either.

(e) The digraphs are not Eulerian because they are not strongly connected. (In addition, there are vertices whose indegree differs from their outdegree; $v_2$ in $\mathcal{G}_1$, for example.)

16. (a) The given matrices are the adjacency matrices of the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ shown.




$\mathcal{G}_1$ $\mathcal{G}_2$

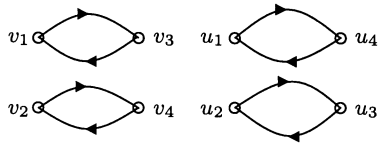
The map $\varphi$ defined by

$$\varphi(v_1) = u_1,\ \varphi(v_2) = u_4,\ \varphi(v_3) = u_2,\ \varphi(v_4) = u_3$$

is an isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$. Thus, $A_2 = PA_1P^T$ where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) The given matrices are the adjacency matrices of the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ shown.




$\mathcal{G}_1$ $\mathcal{G}_2$

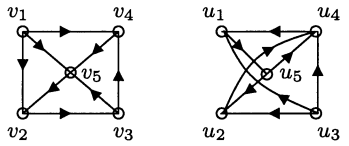
The map $\varphi$ defined by

$$\varphi(v_1) = u_1,\ \varphi(v_2) = u_2,\ \varphi(v_3) = u_4,\ \varphi(v_4) = u_3$$

is an isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$. Thus, $A_2 = PA_1P^T$ where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(c) The given matrices are the adjacency matrices of the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ shown.




$\mathcal{G}_1$ $\mathcal{G}_2$

The map $\varphi$ defined by

$$\varphi(v_1) = u_3,\ \varphi(v_2) = u_1,\ \varphi(v_3) = u_5,\ \varphi(v_4) = u_2,\ \varphi(v_5) = u_4$$

is an isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$. Thus, $A_2 = PA_1P^T$ where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

17. (a) [BB] Each of these graphs is strongly connected; each is a cycle.

(b) Each of these graphs is strongly connected; each is a cycle.

(c) Neither graph is strongly connected. Each has a vertex of outdegree 0 from which travel is clearly not possible.

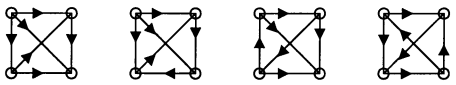
(d) The graph on the left is not strongly connected because it has a vertex of outdegree 0 from which travel is not possible. The one on the right is strongly connected, as can be seen by checking all pairs of vertices (in each direction).

18. (a) [BB] There are just two possibilities for the outdegree sequence; $1, 1, 1$ and $2, 1, 0$. The corresponding graphs are shown.

(b) There are just four possibilities for the outdegree sequence:

$$3, 2, 0, 1; \quad 3, 1, 1, 1; \quad 2, 2, 2, 0 \text{ and } 2, 2, 1, 1.$$

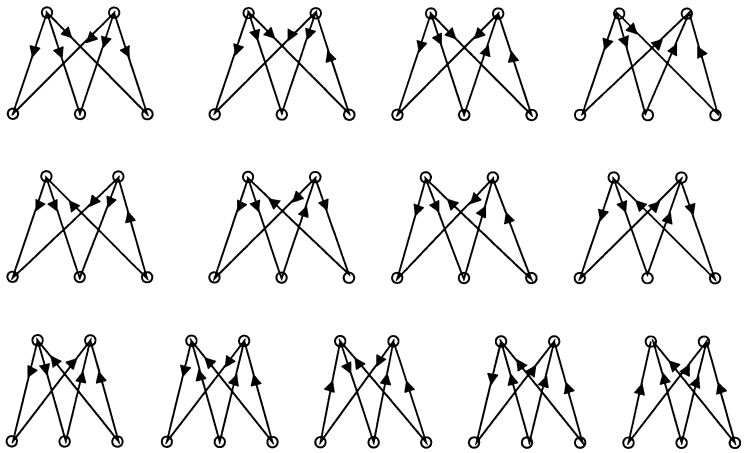
The correspondence graphs are these.

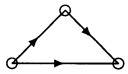


(c) The possible outdegree sequences listed for the vertices of the bipartition sets of sizes 2 and 3 respectively are

| | | | | |
|---|---|---|---|---|
| $3,3,0,0,0$ | $3,2,0,0,1$ | $3,1,0,1,1$ | $3,0,1,1,1$ | $2,2,2,0,0$ |
| $2,2,1,1,0$ | $2,1,2,1,0$ | $2,1,1,1,1$ | $2,0,1,1,2$ | $1,1,2,2,0$ |
| $1,1,2,1,1$ | $1,0,2,2,1$ | $0,0,2,2,2$ | | |

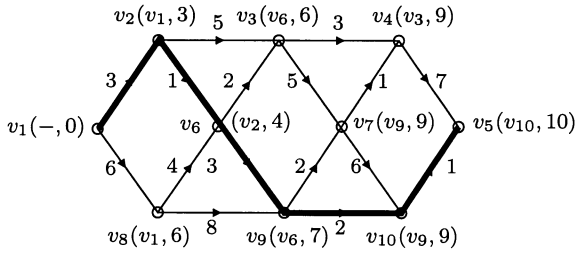
The corresponding graphs are shown.



19. [BB] No. $\mathcal{K}_3$ is connected, but  is not strongly connected.

20. [BB] True. Let $v$ be a vertex. Since $\mathcal{G}$ has at least two vertices and $\mathcal{G}$ is strongly connected there is some arc of the form $vw$. Since $\mathcal{G}$ is strongly connected, there is a path from $w$ to $v$. This path does not use arc $vw$ since all vertices of a path are distinct. For the same reason, all arcs on this (or any) path are distinct. Thus, arc $vw$ followed by the path from $w$ to $v$ gives the desired circuit.

21. (a) A shortest path has length 10 along $v_1, v_2, v_6, v_9, v_{10}, v_5$, as shown.



(b)

| Vertices | Max. no. of arcs: 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $v_2$ | $3, v_1$ | $3, v_1$ | $3, v_1$ | $3, v_1$ | $3, v_1$ |
| $v_3$ | $\infty$ | $8, v_2$ | $6, v_6$ | $6, v_6$ | $6, v_6$ |
| $v_4$ | $\infty$ | $\infty$ | $11, v_3$ | $9, v_3$ | $9, v_3$ |
| $v_5$ | $\infty$ | $\infty$ | $\infty$ | $17, v_{10}$ | $10, v_4$ |
| $v_6$ | $\infty$ | $4, v_2$ | $4, v_2$ | $4, v_2$ | $4, v_2$ |
| $v_7$ | $\infty$ | $\infty$ | $13, v_3$ | $9, v_9$ | $9, v_9$ |
| $v_8$ | $6, v_1$ | $6, v_1$ | $6, v_1$ | $6, v_1$ | $6, v_1$ |
| $v_9$ | $\infty$ | $14, v_8$ | $7, v_6$ | $7, v_6$ | $7, v_6$ |
| $v_{10}$ | $\infty$ | $\infty$ | $16, v_9$ | $9, v_9$ | $9, v_9$ |

22. (a) [BB]

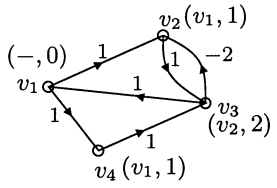
| | Max. no. of arcs: 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $v_2$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ |
| $v_3$ | $8, v_1$ | $2, v_2$ | $2, v_2$ | $2, v_2$ | $2, v_2$ | $2, v_2$ |
| $v_4$ | $\infty$ | $11, v_3$ | $5, v_3$ | $4, v_5$ | $4, v_5$ | $4, v_5$ |
| $v_5$ | $\infty$ | $4, v_2$ | $3, v_3$ | $3, v_3$ | $3, v_3$ | $3, v_3$ |
| $v_6$ | $7, v_1$ | $7, v_1$ | $7, v_1$ | $6, v_5$ | $6, v_5$ | $6, v_5$ |

(b)

| | Max. no. of arcs: 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $v_2$ | $4, v_1$ | $2, v_4$ | $2, v_4$ | $2, v_4$ | $2, v_4$ | $2, v_4$ |
| $v_3$ | $\infty$ | $5, v_2$ | $3, v_2$ | $3, v_2$ | $3, v_2$ | $3, v_2$ |
| $v_4$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ | $1, v_1$ |
| $v_5$ | $7, v_1$ | $7, v_1$ | $6, v_3$ | $4, v_3$ | $4, v_3$ | $4, v_3$ |
| $v_6$ | $8, v_1$ | $7, v_4$ | $7, v_4$ | $6, v_3$ | $5, v_5$ | $5, v_5$ |

23. [BB] Bellman-Ford works fine on undirected graphs without negative edges. It wouldn't make sense to apply Bellman-Ford to an undirected graph with a negative edge weight, since any walk could be shortened by passing up and down that edge as often as desired.

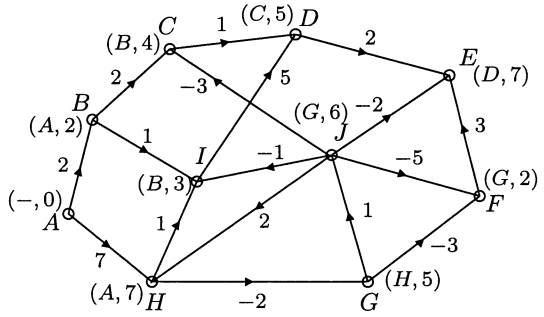
24. (a) [BB] Dijkstra incorrectly determines that the length of a shortest path to $v_2$ is 1. Dijkstra does not always work when applied to digraphs which have arcs of negative weight.




(b) [BB] No shortest path algorithm will work. There is a negative weight cycle, hence no shortest distance to $v_2$, for example.

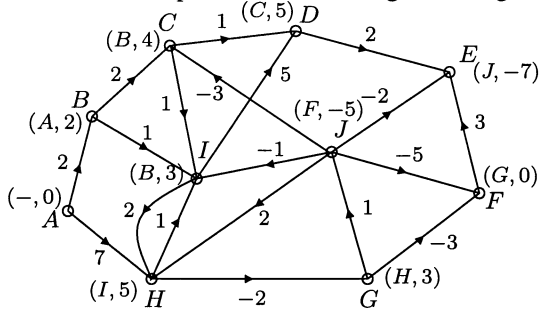
25. (a) Because of the negative weight arcs, Dijkstra's algorithm does not find the correct shortest distances to $C$, $D$, $E$, and $F$, these being 3, 4, 4 and 1, respectively.

(b) Bellman-Ford would work just fine, however, because there are no negative weight cycles.




26. (a) Applying the first version of Dijkstra's algorithm to the digraph produces the labels shown. Only the labels at $A$ and $B$ are correct. There is a negative weight cycle $HGJCIH$ from which every vertex except $A$ and $B$ is accessible. So, repeating the cycle various numbers of times before heading to a vertex (except $A$ or $B$) changes the labels of that vertex, so the labels on every vertex (except $A$ and $B$) are meaningless.

    (b) This time Bellman-Ford does not help, because of the negative weight cycle.

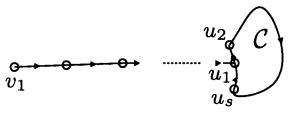



27. [BB] As described in the text, the second last vertex on a shortest path is $p(j)$. Since a shortest path to $v_j$ makes use of a shortest path to $p(j)$ following by edge $p(j)v_j$, the third last vertex on a shortest path to $v_j$ is $p(p(j))$, this being the second last vertex on a shortest path to $p(j)$. So the vertices of a shortest path to $v_j$, in reverse order, are $v_j, p(j), p(p(j)), p(p(p(j))), \ldots, v_1$.

28. Yes it can. When $d_n(j) < d_{n-1}(j)$ for some vertex $v_j$, output $j_1 = p(j)$, the predecessor vertex of $v_j$, then $j_2 = p(j_1)$, the predecessor of $v_{j_1}$, and so on, until some $j_k = j$.

29. (a) [BB] As Step 2 of the algorithm shows, for each $j$, the values $d_i(j)$ depend only on the values $d_{i-1}(k)$ and the arc weights. It follows that if the values $d_{i-1}(j) = d_i(j)$ are identical, then $d_{i-1}(j) = d_i(j) = d_{i+1}(j) = \cdots = d_{n-1}(j)$.

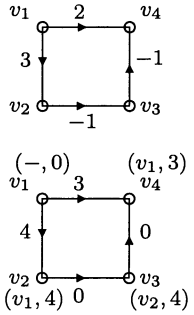
    (b) If a weighted digraph has a negative weight cycle, choose such a cycle $\mathcal{C}$ and let $u_1$ be a vertex in $\mathcal{C}$ which can be reached from $v_1$ by a (directed) path containing no other vertices of $\mathcal{C}$. Suppose this path contains $k$ vertices (including $v_1$ and $u_1$) and that $\mathcal{C}$ is $u_1u_2\cdots u_su_1$. Note that $k-1+s \le n$, the total number of vertices of $\mathcal{G}$. The Bellman-Ford algorithm assigns the finite distance $d = d_{k-1}(u_1)$ to $u_1$ at stage $k-1$. By stage $k$, $d_k(u_2)$ will be finite, by stage $k+1$, $d_{k+1}(u_3)$ will be finite, by stage $k+s-2 < n$, $d_{k+s-2}(u_s)$ will be finite and—most important—by stage $k+s-1 \le n$, $d_{k+s-1}(u_1) < d$ will definitely have decreased, since $\mathcal{C}$ has negative weight. For a similar reason, every $s$ stages thereafter, $d(u_1)$ must decrease. It follows that columns $n-1$ and $n$ are not the same, by part (a).



30. The answer is no, unless there are negative weight cycles. Suppose there is a walk $u \cdots \underbrace{v \cdots v}_{C} \cdots w$ with a repeated vertex $v$. We may assume that the part of the walk from $v$ back to $v$ (which we have denoted $C$) is a cycle. If this walk has length less than $u \cdots v \cdots u$ (removing $C$), then $C$ must have had negative weight.

31. At Step 2, on each of $n^2$ occasions, the algorithm finds the minimum of $n-1$ numbers. This requires $n-2$ comparisons. In addition, one more comparison is required to determine the value of $d_i(j)$. Finally, Step 3 requires a further $n$ comparisons. In all, our algorithm requires $n^2[(n-2)+1]+n = \mathcal{O}(n^3)$ comparisons.

32. The proposed procedure fails since paths which use more edges will be rejected in favor of higher weight paths using fewer edges. In the digraph shown at the left, the proposal is first to add 1 to the weight of each arc. This gives the digraph on the right. Also shown on the right are the final labels obtained by the original version of Dijkstra's algorithm. The shortest distance (on the right) from $v_1$ to $v_4$ is 3. Since a shortest path uses one edge, we would subtract 1 and obtain 2 for the length of a shortest path $v_1$ to $v_4$ in the original graph. This is incorrect.

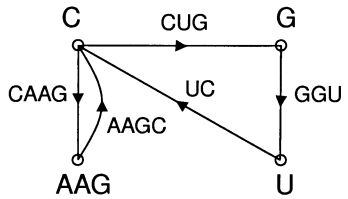



33. Richard Ernest Bellman was born in 1920 and is known for pioneering the development of systems theory in the 1950s and 60s. Credited with creating the discipline of dynamic programming, he won the John von Neumann Prize in 1976 for "leadership in the mathematical theory of multi-stage decision processes". He enjoyed a distinguished career at the University of Southern California and died young in 1984, the result of complications from brain surgery. The Bellman-Ford shortest path algorithm is based on a concept called "dynamic programming" created by Bellman. It is significant because it can be applied to weighted digraphs with negative weight arcs, provided there are no negative weight cycles which, if any, are identified.

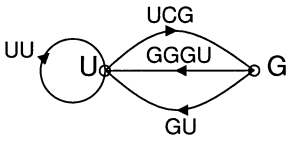
    Lester Ford Jr., after whom the Bellman-Ford and Ford-Fulkerson algorithms are named, was born in 1927 and worked for the RAND Corporation. His father, Lester Randolph Ford Sr., was editor of the *American Mathematical Monthly* from 1942–46 and served as President of the Mathematical Association of America from 1947–48. The "Lester R. Ford Awards" were created in 1964 to recognize expository excellence in the *American Mathematical Monthly*.

## Exercises 11.3

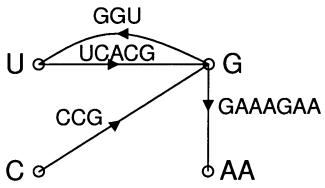
1. (a) [BB] Since the chain ends UC, the given G-fragments arise from any of $3! = 6$ chains. The given U,C-fragments arise from any of $5!/2! = 60$ chains. The better estimate is 6. The abnormal fragment is UC, and this ends the chain. There is one Eulerian circuit ending in UC, so the only answer is CAAGCUGGUC.



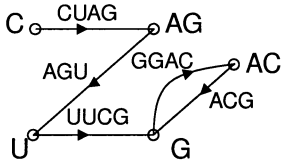
(b) Since the chain ends UU, the given G-fragments arise from any of $4!/3! = 4$ chains. The U,C-fragments arise from any of $4! = 24$ chains. The better estimate is 4. The abnormal fragment is UU; this ends the chain. We can find two Eulerian trails ending in UU, beginning at G. There are two answers: GUCGGGUU and GGGUCGUU.



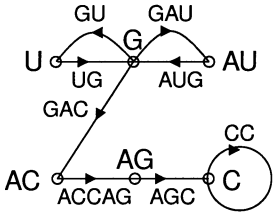

(c) The chain ends with GAAAGAA, so the given G-fragments arise from any of $4! = 24$ chains while the U,C-fragments arise from any of $5!/3! = 20$ chains. The better estimate is 20. There is one Eulerian trail ending at AA, and it begins at C. The only possible chain is CCGGUCACGAAAGAA.



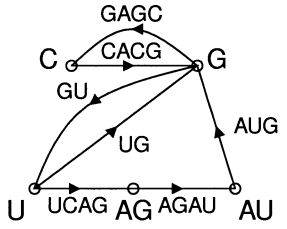

(d) The G-fragments arise from any of $4! = 24$ possible chains. Observing that the chain ends G, the U,C-fragments could have come from one of $6!/(2!2!) = 180$ chains. The better estimate is 24 chains. There is one Eulerian trail ending at G, and it begins at C. The only possible chain is CUAGUUCGGACG.



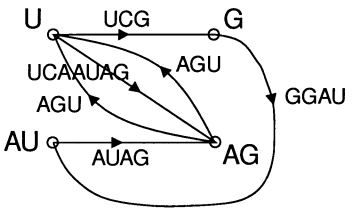

(e) [BB] Since the chain ends CC, the G-fragments arise from any of $4! = 24$ chains. The U,C-fragments arise from any of $6!/2! = 360$ chains. The better estimate is 24. We can find two Eulerian trails ending with CC, beginning at G. There are two answers: GUGAUGACCAGCC and GAUGUGACCAGCC.




(f) The G-fragments arise from any of $5! = 120$ possible chains. Since the chain ends G, the U,C-fragments arise from any of $6! = 720$ chains. The better estimate is 120. There are four Eulerian trails ending at G, and they begin at U. There are four answers: UCAGAUGAGCACGUG, UCAGAUGUGAGCACG, UGAGCACGUCAGAUG and UGUCAGAUGAGCACG.



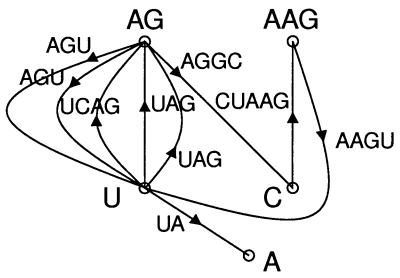

(g) The G-fragments arise from any of $5! = 120$ possible chains. Since the chain ends AG, the U,C-fragments arise from any of $6!/(2!2!) = 180$ chains. The better estimate is 120 chains. Note that there are four Eulerian circuits ending at AG, but only two of these represent different solutions (because arc AGU is repeated). The solutions are AGUCAAUAGUCGGAUAG and AGUCGGAUAGUCAAUAG.




(h) The list of G-fragments could arise from any of $5!/2! = 60$ possible RNA chains since UA is abnormal. Noting that A must be the final fragment, the list of U,C-fragments could arise from

any of $7!/(2!2!) = 1260$ RNA chains. The better estimate is 60 chains. In this case, we can find 36 Eulerian trails ending at A, each one beginning at U. Repeated labels on arcs mean that there are nine possible answers:

UCAGGCUAAGUAGUAGUA,
UCAGUAGGCUAAGUAGUA,
UCAGUAGUAGGCUAAGUA,
UAGGCUAAGUCAGUAGUA,
UAGUCAGGCUAAGUAGUA,
UAGUCAGUAGGCUAAGUA,
UAGUAGGCUAAGUCAGUA,
UAGGCUAAGUAGUCAGUA,
UAGUAGUCAGGCUAAGUA




2. (a) [BB] The last letter in the chain is U, C, G, or A. If it is U or C, the last G-fragment will be abnormal. If it is G, the last U,C-fragment will be abnormal. If it is A, the last G and the last U,C-fragments will each be abnormal.

   (b) [BB] There are two abnormal fragments if and only if the chain ends with A.

3. [BB] The chains GGUGU and GUGGU have the same G- and U,C-fragments. We leave it to you to check that no shorter chains exist.

4. (a) [BB] An abnormal fragment must have arisen from the use of either a G-enzyme or a U,C-enzyme. But a G-enzyme cuts after each G, so an **abnormal** fragment containing a G must have come from a U,C-enzyme. But this enzyme cuts after each U or C, so the **abnormal** fragment cannot contain either of these.

   A similar argument holds if the abnormal fragment contains a U or a C.

   (b) The shorter abnormal fragment is either a G-fragment or a U,C-fragment. Assuming the former (the argument is the same either way), we conclude that in the original chain, a G appears just before this fragment and no G appears in the fragment. Hence G is part of the longer abnormal fragment and it follows from part (a) that this longer fragment cannot contain a U or a C; hence, the same is true of the shorter abnormal fragment. So the shorter fragment can only contain As.

5. The first extended base in the chain is the first extended base of a fragment (of the other type) which splits. The last extended base of this fragment is the first extended base of a fragment (of the first type). And so we move through the chain, considering all the splittable fragments in order, the last extended base of one being the first extended base of the next, until there are no splittable fragments remaining, and we reach the last extended base of the last splittable fragment. Thus, in the pseudograph, each time we arrive at a vertex, we are able to leave along a new arc and every arc corresponding to a splittable fragment is covered, until we reach an extended base of a splittable fragment which is not the first extended base of another splittable fragment. But then we are done!

## Exercises 11.4

1. (a) [BB] See Exercise 18(a) of Section 11.2. The score sequences (left to right) are $1, 1, 1$ and $2, 1, 0$. The tournament on the left is not transitive; the one on the right is.

(b) See Exercise 18(b) of Section 11.2. The score sequences, left to right, are $3,2,1,0$, $3,1,1,1$, $2,2,2,0$ and $2,2,1,1$. Only the leftmost tournament is transitive.

2. [BB] Since for each pair of (distinct) vertices $v_i$, $v_j$ precisely one of $v_iv_j$, $v_jv_i$ is an arc, in the adjacency matrix $A$, for each $i \neq j$, precisely one of $a_{ij}$, $a_{ji}$ is 1. Thus, $A + A^T$ has 0's on the diagonal and 1's in every off-diagonal position.

3. (a) If $A$ is the adjacency matrix of a tournament, then $A + A^T$ has 0's on the diagonal and 1's everywhere else, by Exercise 2. This matrix is $J - I$. On the other hand, if $A + A^T = J - I$ for some adjacency matrix $A$, then $2a_{ii} = 0$, so each diagonal entry of $A$ is 0. Also, if $i \neq j$, $a_{ij} + aji = 1$ with each $a_{ij}$ either 0 or 1 says that exactly one of these entries is 1, while the other is 0. Thus $A$ is the adjacency matrix of a tournament.

   (b) Let $A$ be the adjacency matrix of a tournament and let $B = A^T$. Then $B + B^T = A^T + A$, so $B$ is also the adjacency matrix of a tournament, by Exercise 3(a)

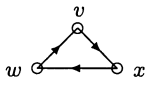
   (c) Using Exercise 3(a), it suffices to show that $PAP^T = J - I$. We know $A = J - I$, so $PAP^T = PJP^T - PIP^T = PJP^T - PP^T$. Since $P$ is a permutation matrix, $P^T = P^{-1}$, so $PP^T = I$. Also, $PJ$ is just $J$ with its rows rearranged in the same way that the rows of the identity were rearranged to form $P$. Since all rows of $J$ are the same, $PJ = J$, so $PJP^T = JP^T$. Now $JP^T$ is $J$ with its columns rearranged, so $JP^T = J$ too. The result follows.

4. (a) There are five: $ABECD$, $ACBED$, $BEACD$, $CBEAD$, $EACBD$.

   (b) Now there are nine: $ABECD$, $ADBEC$, $ACBED$, $ACDBE$, $BEACD$, $CBEAD$, $CDBEA$, $DBEAC$, and $EACDB$. (Did you get them all?)

5. (a) [BB] The sum of the scores is the number of arcs, by Proposition 11.2.2; thus, $\sum_{i=1}^n s_i = \binom{n}{2} = \frac{1}{2}n(n-1)$. Since $\sum_{i=1}^n (n-1-s_i) = \sum_{i=1}^n (n-1) - \sum_{i=1}^n s_i = n(n-1) - \frac{1}{2}n(n-1) = \frac{1}{2}n(n-1)$, we have the desired result. In a tournament with $n$ players, each player plays $n-1$ games, so if a player wins $s_i$ games, he loses $n-1-s_i$ games. The result says that the sum of the numbers of wins equals the sum of the numbers of losses.

   (b) $\sum_{i=1}^n (n-1-s_i)^2 = \sum_{i=1}^n [(n-1)^2 - 2(n-1)s_i + s_i^2] = \sum_{i=1}^n (n-1)^2 - 2(n-1)\sum_{i=1}^n s_i + \sum_{i=1}^n s_i^2$. Using the fact that $\sum_{i=1}^n (n-1)^2 = n(n-1)^2$ and $\sum_{i=1}^n s_i = \binom{n}{2} = \frac{1}{2}n(n-1)$ (the sum of the scores is just the number of arcs, by Proposition 11.2.2) the result follows. In a tournament with $n$ players, each player plays $n-1$ games, so if a player wins $s_i$ games, he loses $n-1-s_i$ games. The result says that the sum of the squares of all the wins in a tournament equals the sum of the squares of all the losses.

6. [BB] Let $w$ be any other vertex. If $v$ beats $w$, there is a path of length 1 from $v$ to $w$ and we're fine. Hence, assume that $w$ beats $v$, that is, $w \longrightarrow v$. Among those vertices that $v$ beats there must be one, say $x$, which beats $w$, since otherwise $s(w) \geq s(v) + 1$ (recall that $w$ beats $v$), contradicting the maximum score of $v$. Hence, for some $x$, we have 

, but then there is a path of length 2 from $v$ to $w$.

7. (a) [BB] No. The sum of the scores, $\sum s(v)$, is the number of arcs. Here, $1+1+2+3 = 7$, but the number of arcs is $\binom{4}{2} = 6$.

   (b) No. The vertices with scores of 1 determine a digraph which is $\mathcal{K}_4$ with a direction assigned to each arc. The number of arcs in this digraph is $\binom{4}{2} = 6$, but the sum of the outdegrees is only 4, a contradiction to Proposition 11.2.2.

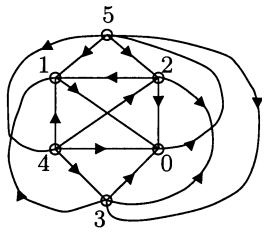
8. (a) The sum of *all* scores is the number of arcs (by Proposition 11.2.2) and in a tournament, this number is $\binom{n}{2} = \frac{1}{2}n(n-1)$ because every two vertices are connected by a single arc.

   (b) Any $t$ vertices $v_1, \ldots, v_t$ determine a digraph which is $\mathcal{K}_t$ with an arrow on each arc. The number of arcs in this digraph is $\binom{t}{2} = \frac{1}{2}t(t-1)$. So, by Proposition 11.2.2, the sums of the scores of the vertices $v_1, \ldots, v_t$ just counting games between these players is $\frac{1}{2}t(t-1)$. But the score of each $v_i$ (in the entire tournament) is at least its score in the smaller digraph, so $\sum s_i \geq \frac{1}{2}t(t-1)$.

9. (a) Suppose $v$ and $w$ are vertices in a transitive tournament $\mathcal{T}$. Either $vw$ or $wv$ must be an arc so, without loss of generality, we assume $vw$ is an arc. If $x$ is any vertex such that $wx$ is an arc, so also is $vx$ an arc, by transitivity. Remembering that $vw$ is an arc, it follows that $s(v) \geq s(w) + 1$. In particular, $s(v) \neq s(w)$. Thus, the scores are all different.

   (b) If every player in a tournament $\mathcal{T}$ has a different score, then the score sequence is $\{0, 1, 2, \ldots, n-1\}$ by PAUSE 9. Let $v_i$ be the player with score $i$. By PAUSE 11, $v_{n-1}v_{n-2}\cdots v_1v_0$ is a Hamiltonian path. It is unique because in any other listing of the vertices, there would be an adjacent pair $v_iv_j$ with $j > i$, implying an arc from $v_i$ to $v_j$, implying in turn that $v_i$ beat $v_j$. This is untrue since the proof of PAUSE 11 shows that $v_j$ beats $v_i$ if $j > i$.

10. The tournament has a unique Hamiltonian path 543210 because it has a distinct score sequence!




11. [BB; $\rightarrow$] If a tournament contains no cycles, then it cannot contain the cycle shown in Fig. 11.21, so it is transitive. Conversely, suppose $\mathcal{T}$ is a transitive tournament that contains the cycle $v_1v_2\ldots v_nv_1$. Since $\mathcal{T}$ contains no 3-cycle, upon considering the vertices $v_1$, $v_2$ and $v_3$, we see that $v_1v_3$ must be an arc. Then, considering $v_1$, $v_3$, $v_4$, we see that $v_1v_4$ is an arc. Continuing, we eventually have an arc $v_1v_n$, contradicting the fact that $v_nv_1$ is an arc. (In a tournament, for any pair $u, v$ of vertices, precisely one of $uv$, $vu$ is an arc.)

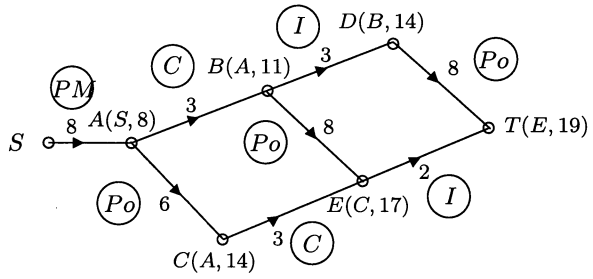
12. (a) Since in a digraph, it is possible for $uv$ and $vu$ to be arcs, the relation need not be antisymmetric. Moreover, there is no reason why the relation should be transitive.

    (b) The relation is reflexive by definition. It is antisymmetric since, in a tournament, it is not possible for both $uv$ and $vu$ to be arcs for $u \neq v$. Thus, the relation is a partial order if and only if $\mathcal{G}$ is transitive. Since, given any two vertices $u$, $v$ in a tournament, either $uv$ or $vu$ is an arc, the partial order is a total order.

## Exercises 11.5

1. [BB] With reference to Fig. 11.24, the arc $BD$ now has weight 4. This does not change any of the labels shown, but it does eliminate the possibility of a second shortest path. Now the only shortest path is $SADFT$, requiring 11 units of time as before.

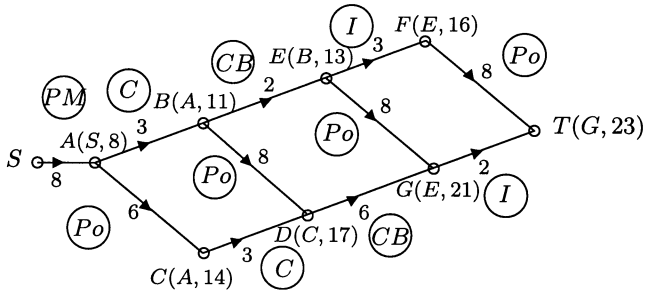
2. (a) [BB] Since the times required for the tasks depend on which other tasks have already been completed and since tasks cannot take place at the same time, this is a Type I scheduling problem.



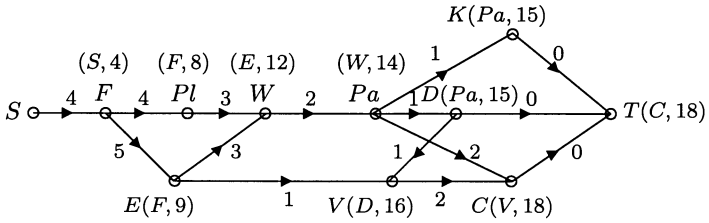

   (b) [BB] The critical path is $SACET$; that is, pour the mold, polish, calibrate, inspect. The shortest time for the project is the length of this path: 19 units of time.

3. The weight on arc $AC$ becomes 9, changing the label on $C$ to $(A, 17)$. Thus, the label on $E$ changes to $(B, 19)$ and the label at the finish changes to $(E, 21)$. The critical path is now $SABET$; that is, first pour the mold, then calibrate, then polish and finally inspect. Now 21 time units are required.

4. The critical path is $SABEGT$: Pour the mold, calibrate, have a coffee break, polish and inspect. The shortest time is 23 units.

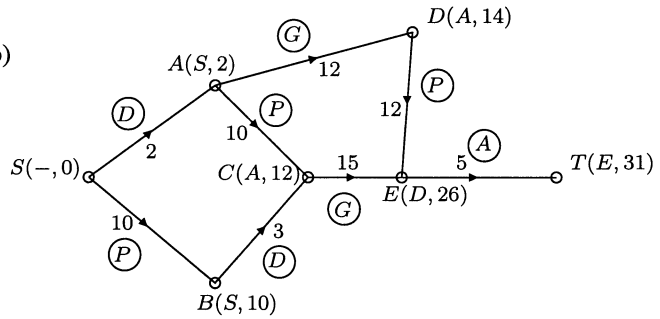



5. [BB] The new digraph is shown. The project now requires 18 units of time. The critical path is $SFEWPaDVCT$. The slack in $Pl$ is now 1; the slack in the installation of kitchen cabinets is 3. All other tasks have slack 0 since they lie on the critical path.




6. (a) Since the time for each task depends on the tasks which have already been completed, this is a type I scheduling problem.

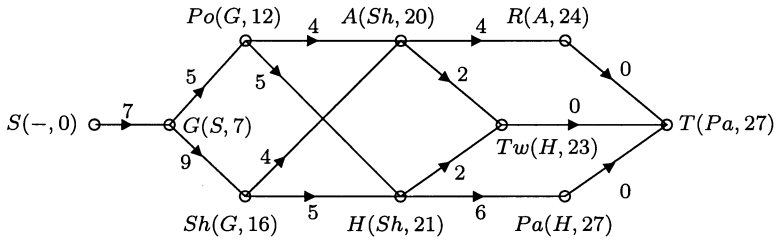
(b)




The ship can be assembled in 31 units of time along $SADET$, that is, by first detaching and sorting the pieces, then gluing, painting and applying the decals.

7. [BB] Since the time required for a task does not depend on which other tasks have been completed and since some tasks can occur simultaneously, this a Type II scheduling problem.

(a)

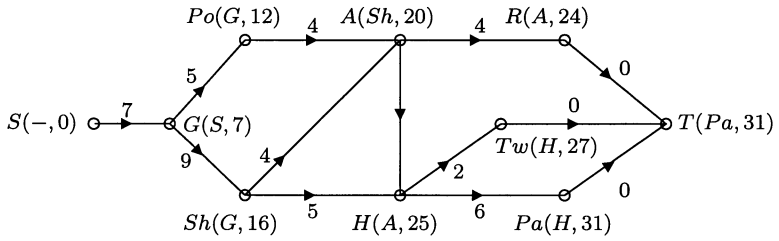



(b) The only critical path is $SGShHPaT$, taking a total of 27 months.

(c) The slack of $Po$ is 4. This job could take as much as 9 units of time without affecting the time (21) by which the task $H$, subsequent to $Po$ and on the critical path, would be accomplished. There is no slack in $Sh$. It's on the critical path. The slack in $A$ is 3. If this task took 7 units of time, its label would be $(Sh, 23)$ and the label on $R$ would become $(A, 27)$. There would be another critical path, $SGShART$, but the time for the job would be unaffected. The slack in $Tw$ is 4. Twisting could take as long as 6 units of time and the project would still be completed in 27 units. The slack in $R$ is 3.

(d) No, the project will not be delayed. Delaying $Po$ by three months changes $Po$'s label to $(G, 15)$, but this does not affect any other labels on the digraph. Delaying $A$ by three months changes $A$'s label to $(Sh, 23)$, but does not affect the critical path, or its length.

8. (a) Now the critical path is $SGShAHPaT$, taking 31 months. Note that the arcs $PoH$ and $ATw$ are no longer necessary since the corresponding precedences in tasks can be inferred from other arcs.



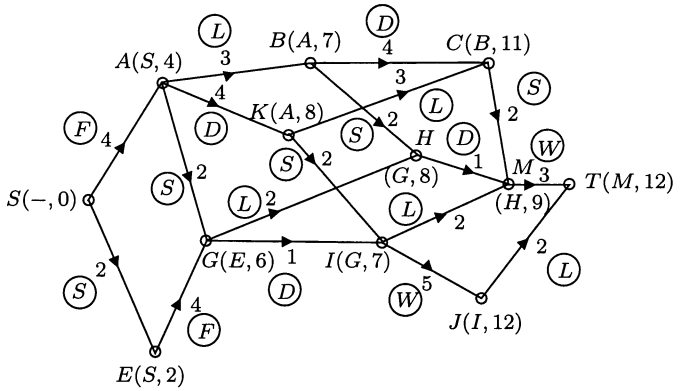

(b) The slack in $Po$ is still 4. If $Po$ takes nine months, the label on $Po$ would be $(G, 16)$ and the label on $A$ could be either $(Sh, 20)$ or $(Po, 20)$. There would then be a second critical path. If $Po$ took more than nine months, the label on $A$ would increase and the total time of the project would increase. There is no slack in $Sh$ and $A$ since these are on the critical path. The slack in $Tw$ is 4. The slack in $R$ is 7.

(c) If $Po$ and $A$ are each delayed three months, the project will be delayed three months as well. The label on $Po$ would become $(G, 15)$ and the label on $A$ would become $(Sh, 23)$. Then $H$ would be $(A, 28)$ and $Pa$, $(H, 34)$. The critical path would be $SGShAHPaT$ as before, but the project would now require 34 months.

9. Looking at the answer to Exercise 7, we see that the label on $A$ would become $(Sh, 22)$ while $R$ would acquire the new label $(A, 29)$. Vertex $Tw$ would be labeled $(A, 24)$. The critical path would now be $SGShART$ requiring 29 months.

10. Have you ever seen a Möbius strip? Take a strip of paper, give it a single twist and glue the ends together. What you get is a strip with just one side! Notice that the original strip was 2-dimensional, but the Möbius strip only exists in three dimensions. A Klein bottle is a similar sort of thing, but unfortunately it exists only in four dimensions so, while there are many pictures of it (there is a Klein bottle web site, www.kleinbottle.com, containing some marvelous pictures), it is in fact an imaginary object. It's a bottle without a boundary. You can't tell inside from out!

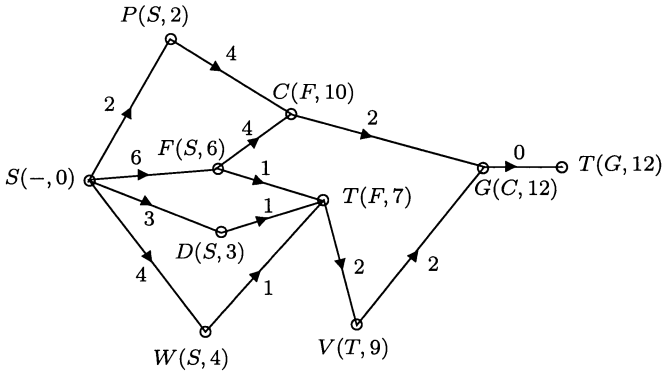
11. (a) [BB] This type I scheduling problem is described by the following digraph.




(b) [BB] The shortest time is 12 units.

(c) [BB] One of several ways in which this time can be achieved is to first conduct the library search, then do the field work, then the laboratory analysis, then create the database and finally do the write-up.

12. (a) Since the times for the various tasks are independent of which tasks have already been completed, and since various tasks can occur simultaneously, this is a problem of type II .



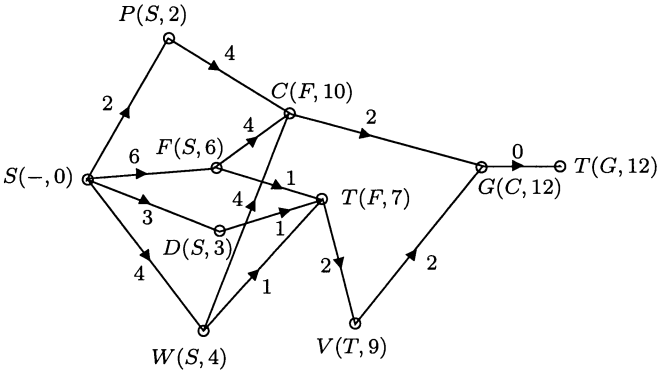

(b) As the digraph shows, the shortest time in which dinner can be prepared is 12 units. The critical path is $FCG$: Catch the fish, cook the food and greet the guests.

(c) The slack in $W$ is 3 since buying the wine could take as long as seven units with dinner still being prepared in 12 units.

The slack in $C$ is 0 because $C$ is on the critical path.

The slack in $D$ is four. Dusting could take as long as seven units without increasing the minimal time (12) for the whole job. (If it took more than seven, then $DTVG$ would become the critical path.)

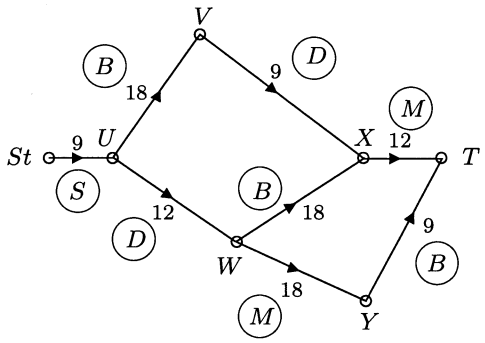
13. (a) The new digraph is shown.



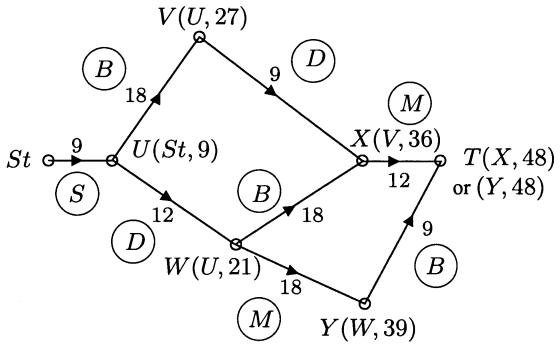

(b) As shown, the critical path is still $FCG$ and the time is still 12 units.

(c) The slack in $W$ is now two. The slack in $C$ is still 0 and the slack in $D$ is still four.

14. (a) [BB] Since this is clearly a one-man project (!), it is of type I. The appropriate directed network is shown.

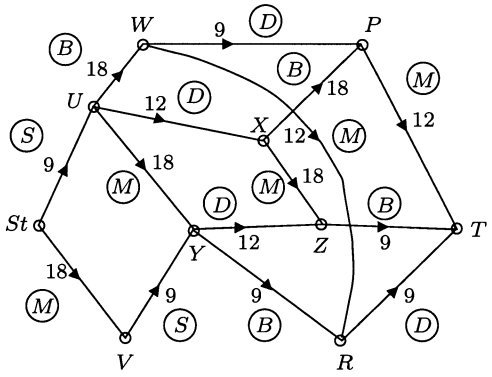


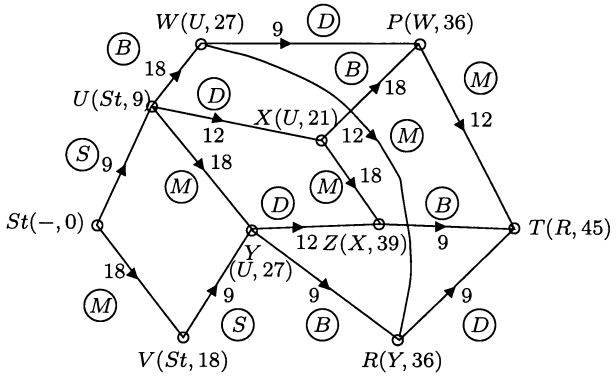

(b) Carrying out Dijkstra's algorithm (first version) gives the graph shown. There are two critical paths: $SBDM$ and $SDMB$, each of length 48 minutes. Thus, John can either shower, eat breakfast, dress and do his math assignment, or shower, dress, finish his assignment and eat breakfast.




15. (a) The appropriate directed network and the digraph obtained by using Dijkstra's algorithm are shown.

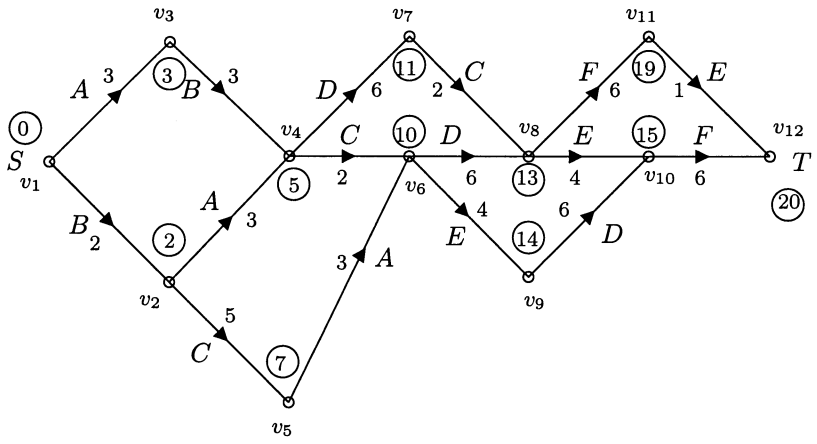
(b) The critical path is $SMBD$ of length 45 minutes corresponds to showering, finishing math assignment, eating breakfast and dressing. Note that there were choices available in the labeling of vertices $Y$ and $Z$. Since $Y$ is the only one of these on the critical path, we conclude that the only other critical path is $MSBD$.



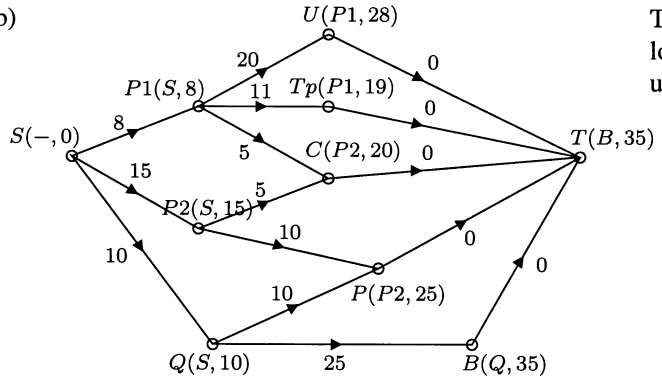

16. This is a type I scheduling problem. We draw a digraph whose vertices represent stages in the process

and where arcs correspond to tasks. The weight of an arc is the cost of the task. One shortest path costs \$2000 and passes through vertices $v_1$, $v_2$, $v_4$, $v_7$, $v_8$, $v_{11}$, $v_{12}$. On this path, tasks should be completed in the order $B$, $A$, $D$, $C$, $F$, $E$. The other shortest path goes through vertices $v_1$, $v_2$, $v_4$, $v_6$, $v_8$, $v_{11}$, $v_{12}$. On this path, the tasks are completed in the order $B$, $A$, $C$, $D$, $F$, $E$.




17. (a) Since the time for a given task is independent on which other tasks have already been completed, and since some tasks may occur simultaneously, this is a type II scheduling problem.

(b)



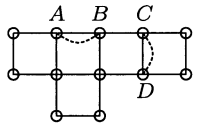

The shortest time is the length of the longest path from $S$ to $T$. This is 35 units of time.

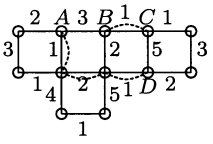
(c) The critical path is $Q, B$. There is no slack in any task on the critical path. The table shows the slack in the other activities.

| Task | Slack | Task | Slack |
|---|---|---|---|
| $P1$ | 7 | $Tp$ | 16 |
| $P2$ | 10 | $C$ | 15 |
| $U$ | 7 | $P$ | 10 |

# Chapter 11 Review

1. There are four odd vertices, labeled $A$, $B$, $C$, $D$ in the figures to the right. In the unweighted case, at least two additional edges are required; in fact, two suffice, as shown. The table shows the calculations required for the weighted graph. It is necessary to duplicate four edges, as shown.



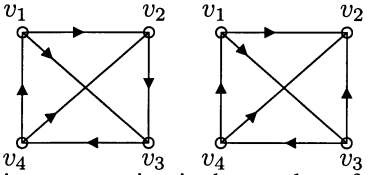

| Weighted Case | |
|---|---|
| Partition into pairs | Sum of lengths of shortest paths |
| $\{A,B\},\{C,D\}$ | $3+4=7$ |
| $\{A,C\},\{B,D\}$ | $4+3=7$ |
| $\{A,D\},\{B,C\}$ | $4+1=5$ |

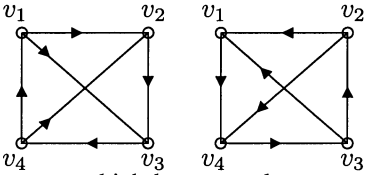
2. This graph has exactly two vertices of odd degree, $C$ and $G$. The shortest path from $C$ to $G$ is $CADG$, of length 8. (This follows from Exercise 27 of the Review Problems for Chapter 10; it also follows quickly by inspection.) Hence, we solve the problem by duplicating edges $CA$, $AD$ and $DG$.

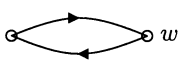
3. Let us label the bipartition sets $\{u_1,u_2,u_3\}$ and $\{v_1,v_2,v_3,v_4,v_5,v_6\}$. Our graph has six vertices of odd degree, namely, $v_1,\ldots,v_6$. The shortest path between any two of these will always be of length two, passing through one of $u_1,u_2,u_3$. So one solution here would be to duplicate $v_1u_1$, $u_1v_2$, $v_3u_1$, $u_1v_4$, $v_5u_1$ and $u_1v_6$.

4. If a graph $\mathcal{G}$ has an Eulerian trail between vertices $v$ and $w$, then $v$ and $w$ are the unique vertices of odd degree in $\mathcal{G}$. So, to solve the Chinese Postman Problem, it suffices to find a shortest path from $v$ to $w$ and to duplicate the edges on this path.

5. Proposition 11.2.2 says that the sum of the indegrees of all vertices of $\mathcal{G}$ equals the sum of the outdegrees. So it is not possible for every vertex to have smaller indegree than outdegree, and we can choose a vertex $v_1$ with $\text{indeg}\, v_1 > \text{outdeg}\, v_1$. Assume $\mathcal{G}$ has vertices $v_1,\ldots,v_n$. If $v_1$ were the only vertex with indegree greater than outdegree, then $\sum_{i=2}^{n}\text{outdeg}\, v_i \geq (n-1)+\sum_{i=2}^{n}\text{indeg}\, v_i$. Proposition 11.2.2 then says $\text{indeg}\, v_1 \geq (n-1)+\text{outdeg}\, v_1$. Since $\mathcal{G}$ is strongly connected, $\text{outdeg}\, v_1 \geq 1$. Thus $\text{indeg}\, v_1 \geq n$, which is impossible. Thus $v_1$ is not the only such vertex, so there are at least two whose indegrees are greater than their outdegrees.

6. (a) 


(b) 

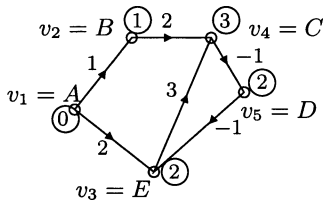

7. The given expression is the number of pairs of vertices $v,w$ which have arcs between them in both directions; that is, 

. To see why, note that $b_{ii}$ is the number of directed walks

of length two from $v_i$ to $v_i$, and such a walk is only possible when the above configuration occurs. Thus $\sum b_{ii}$ counts the number of such configurations twice.

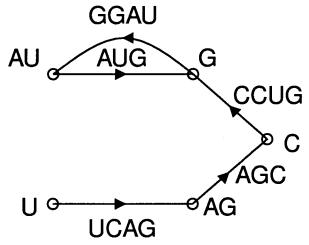
8. (a) The final labels assigned by Dijkstra's algorithm, the improved version, are shown. The algorithm incorrectly determines that the shortest path from $A$ to $E$ has length 2, when the path $ABCDE$ has length 1. It fails because Dijkstra's algorithm does not apply to digraphs where some arcs have negative weights.

   (b) For $k = 1$ to $n$, where $n$ is the number of vertices, the Bellman-Ford Algorithm proceeds by determining shortest paths along $k$ arcs from $v_1$ to each other vertex. We show the successive stages of the algorithm in the table.

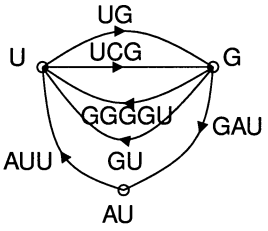



| Vertices | Max. no. of arcs: 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $B$ | $1, A$ | $1, A$ | $1, A$ | $1, A$ | $1, A$ |
| $C$ | $\infty$ | $3, B$ | $3, B$ | $3, B$ | $3, B$ |
| $D$ | $\infty$ | $\infty, A$ | $2, C$ | $2, C$ | $2, C$ |
| $E$ | $2, A$ | $2, A$ | $2, A$ | $1, D$ | $1, D$ |

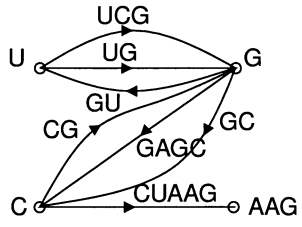
9. (a) The only abnormal fragment is G, so the chain ends with G. Since the abnormal fragment doesn't split, we look for Eulerian trails or circuits whose last vertex is G. The unique chain is UCAGCCUGGAUG.



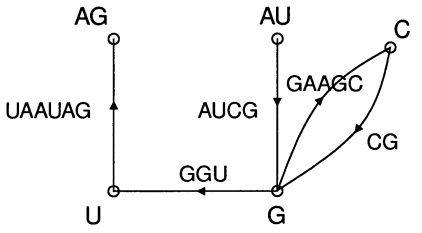

   (b) The only abnormal fragment is AUU, so the chain ends with this fragment. Since the abnormal fragment splits, we look for Eulerian trails or circuits whose final arc is labeled AUU. There are four chains: GUCGGGGUGAUU, GUGGGGUCGAUU, GGGGUGUCGAUU, GGGGUCGUGAUU.




   (c) The only abnormal fragment is AAG, so the chain ends with this. Since the abnormal fragment does not split, we look for Eulerian trails or circuits whose last vertex is AAG. There are eight chains: conditions are UGUCGCGAGCUAAG, UGUCGAGCGCUAAG, UGCGUCGAGCUAAG, UGAGCGUCGCUAAG, UCGUGCGAGCUAAG, UCGUGAGCGCUAAG, UCGCGUGAGCUAAG, UCGAGCGUGCUAAG.



(d) The only abnormal fragment is AG and this does not split, so look for Eulerian trails or circuits whose last vertex is AG. There is a unique Eulerian trail and the chain is AUCGAAGCGGUAAUAG.




10. The score of vertex $w_i$ is $n - i$. Since all scores are different, the result follows from Theorem 11.4.4.

11. (a) No. The sum of all the scores is $23 \neq 21 = \binom{7}{2}$.

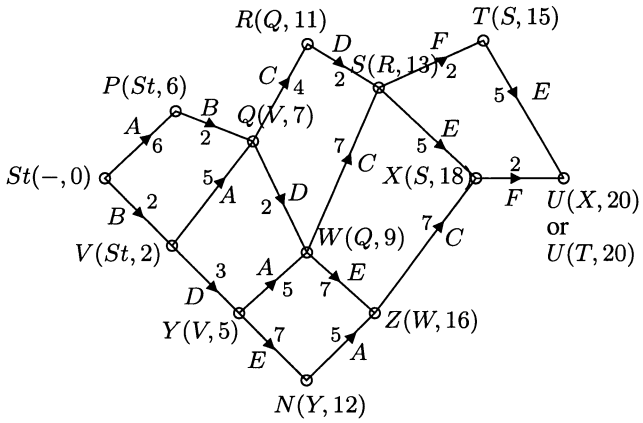
    (b) No. Consider the five players with the lowest scores. Among themselves, they play a total of $\binom{5}{2} = 10$ games and yet their total scores add only to 9.

12. Reversing the orientation on all the arcs of $\mathcal{T}$ gives a tournament in which $v$ has maximum score. Exercise 6 of Section 11.4 says that the length of the shortest path from $v$ to any other vertex in the new tournament is 1 or 2, so our result follows.

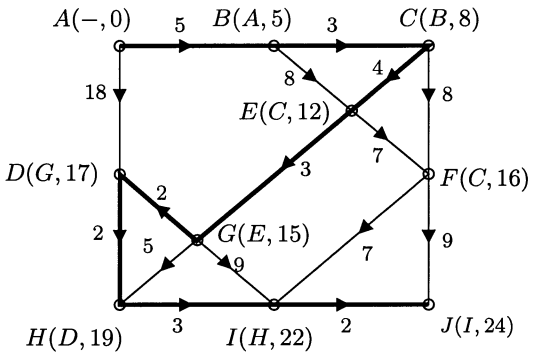
13. Let $n$ be the number of players competing and assume first that the $n - 2$ players tied for second all had winning records. The sum of the outdegrees of these vertices would then exceed the sum of the indegrees by at least $n - 2$. If we add Alice's score to the sums, the sum of the outdegrees would exceed the sum of the indegrees by at least $n$ (since Alice has a better record than the players tied for second). In any digraph, however, the outdegree and indegree sums are the same, so it follows that George's indegree must be at least $n$ greater than his outdegree. This is impossible.

    A similar argument shows that the players tied for second cannot have losing records, so these players must have an equal number of wins and losses. Since each player has $n - 1$ matches, it follows that $n - 1$ is even, so $n$ is odd.

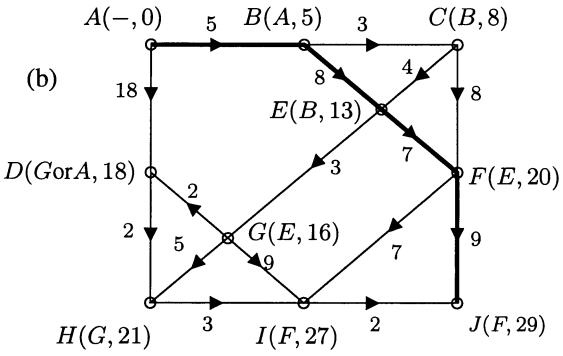
14. This is a type I scheduling problem. The shortest time is 20 hours. There are two critical paths: $St$, $V$, $Q$, $R$, $S$, $T$, $U$ and $St$, $V$, $Q$, $R$, $S$, $X$, $U$.



15. (a) A shortest path of length 24 passes through $A, B, C, E, G, D, H, I, J$, as shown.

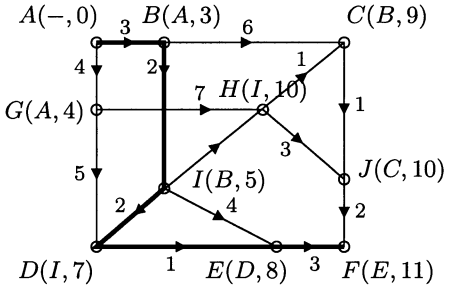



(b)

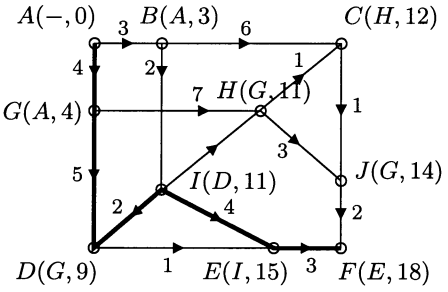



There is a critical path of 29 days through $A$, $B$, $E$, $F$, $J$ as shown.

16. (a) A shortest path has length 11, through $A, B, I, D, E, F$, as indicated.

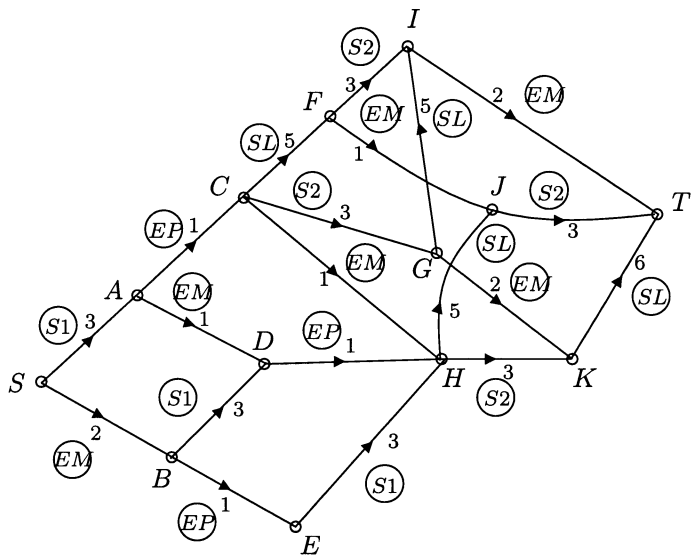



(b)




A longest path has length 18, through $A, G, D, I, E, F$, as indicated.

17. In the directed network for this type I scheduling problem, we will let $S1$, $S2$, $SL$, $EP$ and $EM$ denote, respectively, study session 1, study session 2, sleep, eating pizza and checking e-mail.



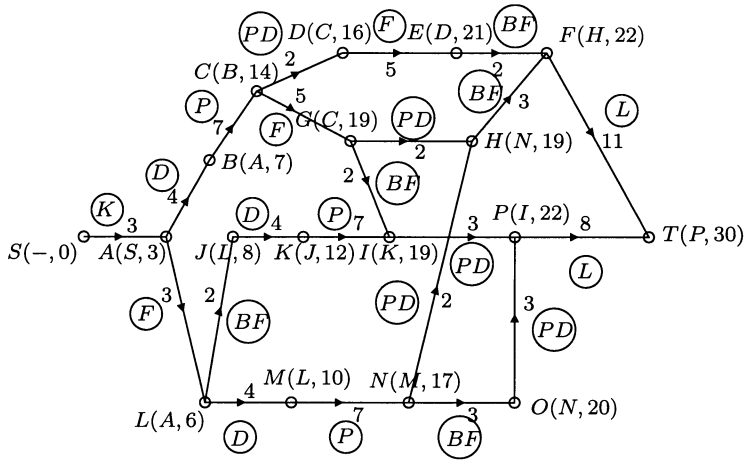

(a) Applying Dijkstra's algorithm (first version), we obtain the labels $B(S,2)$, $A(S,3)$, $E(B,3)$, $C(A,4)$, $D(A,4)$, $H(C \text{ or } D,5)$,$G(C,7)$, $K(H,8)$, $F(C,9)$, $J(F \text{ or } H,10)$, $I(F \text{ or } G,12)$, $T(J,13)$. So George can complete his activities in thirteen hours using any of the following paths: $SACFJT$, $SACHJT$, $SADHJT$.

(b) No. One of the pahts, SADHJT, has him doing this.

(c) Yes. In this case, George must do another activity after checking his e-mail and before ordering the pizza, and none of the paths in (a) satisfy this. From the network, we can see that this means he must begin in the order $SBDH$. It is clear then by inspection that the shortest path will be $SBDHJT$, requiring fourteen hours.

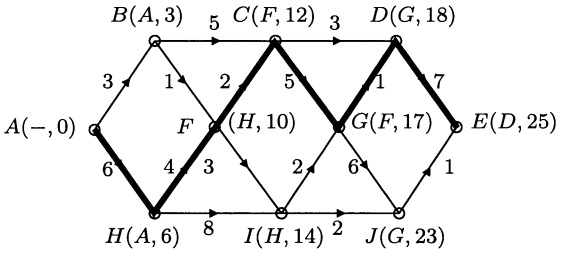
18. (a) Since the time for various tasks depends on which tasks have already been completed, this is a Type I scheduling problem.

(b)

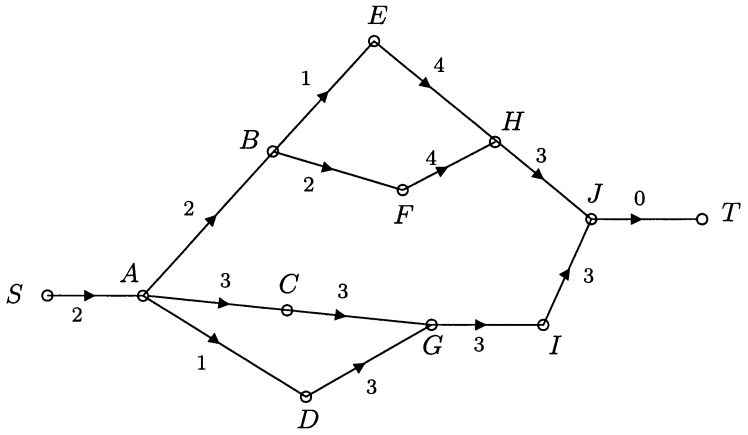



(c) The groceries can be purchased, bagged and on the truck in 30 units time, along the route $SALJKIPT$.

19. A critical path takes 25 days, along $A, H, F, C, G, D, E$, as shown. The slacks in $B$, $I$, and $J$ are 4, 1, and 1, respectively.




20. The directed network for this type II scheduling problem is shown.

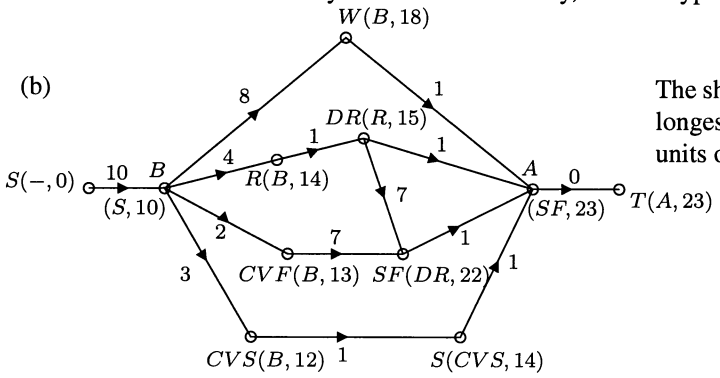



(a) Modifying Dijskstra's algorithm as in the text, we obtain the labels $A(S, 2)$, $D(A, 3)$, $B(A, 4)$, $C(A, 5)$, $E(B, 5)$, $F(B, 6)$, $G(C, 8)$, $H(F, 10)$, $I(G, 11)$, $J(I, 14)$, $T(J, 14)$. It takes fourteen days to build a glynskz. The only critical path is $SACGIJT$.

(b) Since $C$ is on the critical path, its slack is 0. The slack in $D$ is 2, the same as $E$.

21. (a) Since the time for a given task is independent on which other tasks have already been completed, and since some tasks may occur simultaneously, this is a type II scheduling problem.

(b)

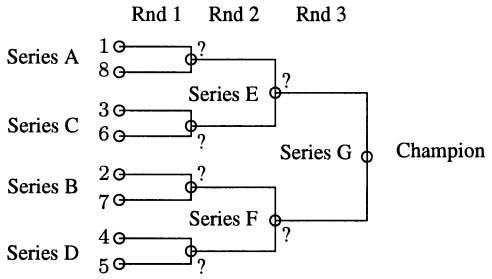



The shortest time is the length of the longest path from $S$ to $T$. This is 23 units of time.

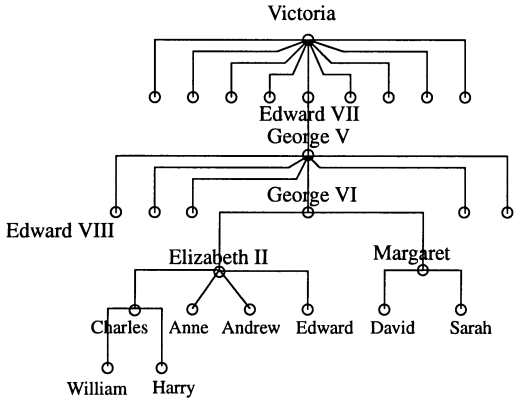
(c) The critical path is $B, R, DR, SF, A$. The slack in cooking the wings is 4. The slack in chopping the vegetables for the salad and making the salad is 8 in each case. The slack in chopping the vegetables for the stirfry is 3.

## Exercises 12.1

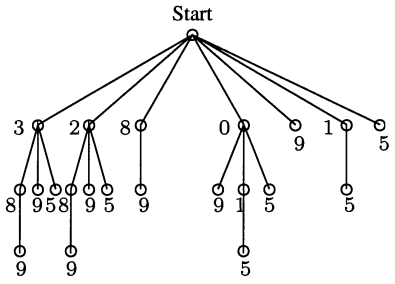
1. [BB]



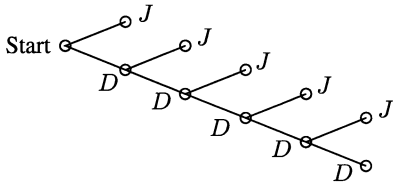

2.



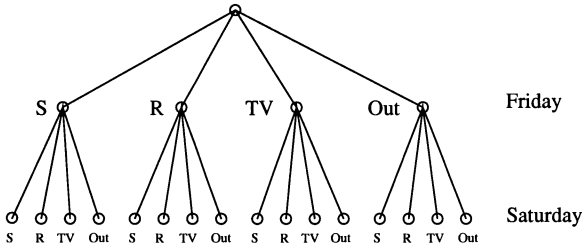

3. There are thirteen increasing subsequences, as shown. Seven have level one, eleven have level two and three have level three.



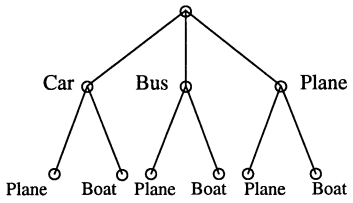

4. There are six possible outcomes corresponding to the five paths from start to vertices denoted $J$ together with the path Start—$D$—$D$—$D$—$D$—$D$. In five of these, John wins a game.



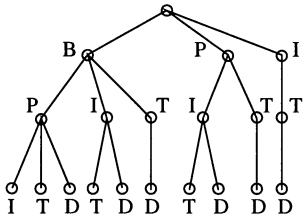
5. [BB]




6. Mary has six different ways to travel, avoiding the bus in four of these.

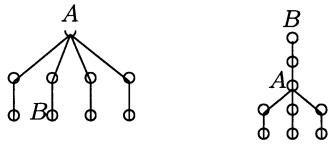



7. [BB]



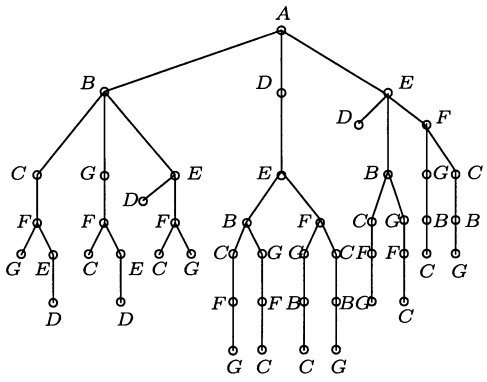

There are $\binom{5}{3} = 10$ committees.

8.




9. (a) The paths terminating at $C$ are $ABC$, $ABGFC$, $ABEFC$, $ADEBC$, $ADEBGFC$, $ADEFGBC$, $ADEFC$, $AEBC$, $AEBGFC$, $AEFC$ and $AEFGBC$.

The Hamiltonian paths are $ADEBCFG$, $ADEBGFC$, $ADEFGBC$ and $ADEFCBG$.

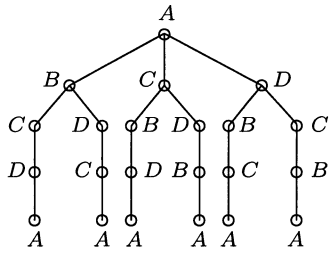



(b) No Hamiltonian paths end at a vertex adjacent to $A$ so the graph is not Hamiltonian. (Alternatively, vertices $C$ and $G$ of degree 2 impose a proper circuit.)

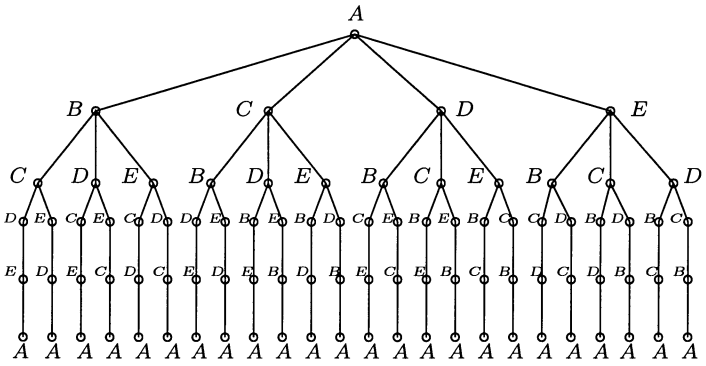
(c) Since there are Hamiltonian paths from $A$, we seek a paving of roads which allows us to visit *every* town with paved roads from $A$. To do this at minimal cost, we should pave the roads along the Hamiltonian path $ADEFCBG$ at a cost of 15. This path is the best of the Hamiltonian paths,

but it also must be cheaper than any other selection of edges connecting the six other towns to $A$. This is in fact the case, because $ADEFCBG$ consists of the six edges of smallest weight in the entire graph!

10. (a) The minimum cycle is $ACBDA$ (or $ADBCA$, in reverse) with a total of 90.



(b) The minimum path is $ADCBEA$ which, in reverse, is the same as $AEBCDA$, with a total of 220.



11. (a) Give the tree $n$ vertices. Then there are $n-1$ edges, so the sum of the degrees is $2(n-1)$. This gives

$$100 + 20(6) + \frac{1}{2}(n-120)4 + \frac{1}{2}(n-120)2 = 2(n-1) = 2n-2$$

so $n = 138$. The number of vertices of degree 2 is $\frac{1}{2}(n-120) = 9$.

(b) Let there be $k$ vertices of degree 1 and $n$ in all. Since the sum of the degrees is $2|\mathcal{E}| = 2(n-1)$, we obtain

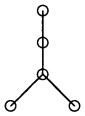
$$k + 20(6) + \frac{1}{2}(n-k-20)4 + \frac{1}{2}(n-k-20)2 = 2n-2,$$

so $n = 2k - 62$. We know that there are at least $k+20$ vertices, so $2k - 62 \geq k + 20$, giving $k \geq 82$. This shows, in particular, that if $k < 80$, no tree as specified can exist.

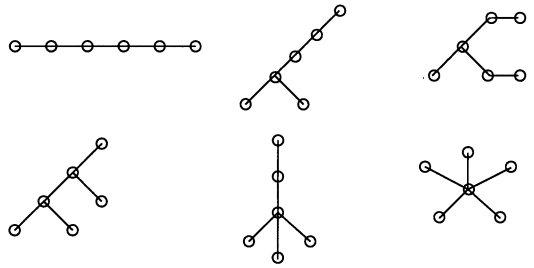
12. [BB] We think of a path as a sequence of vertices. There are at least two vertices of $\mathcal{P}_2$ which are also in $\mathcal{P}_1$ ($v$ and $w$, for example). Let $u \neq w$ be the first vertex of $\mathcal{P}_1$ which is encountered on the reversed path $\mathcal{P}_2$ from $w$ back to $v$. (It is possible that $u$ might equal $v$.) It follows that the path from from $u$ to $w$ along $\mathcal{P}_1$ followed by the path back to $u$ along $\mathcal{P}_2$ in reverse from $w$ is a cycle.

13. (a) [BB]





(b) [BB] Since each tree with five vertices has all vertices of degree at most four, there is one isomer for each such tree, the $C$ atoms corresponding to the vertices. There are three isomers of $C_5H_{12}$.

14. (a) 

(b) There is one isomer for each tree (with six vertices) in which all vertices have degree at most four, the $C$ atoms corresponding to the vertices. There are five isomers of $C_6H_{14}$.

15. [BB] A beta index less than 1 says that there are fewer edges than vertices. One possibility is that the graph is not connected; in other words, there exist two cities such that it is impossible to fly from one city to the other. If the graph is connected, then it must be a tree by Theorem 12.1.6. This means there is a unique way of flying from any city to any other city.

16. By Proposition 9.2.5, the sum of the degrees of the vertices is twice the number of edges. According to Theorem 12.1.6, a connected graph with $n$ vertices is a tree if and only if it has $n-1$ edges, but this occurs if and only if the sum of the vertex degrees is $2(n-1)$.

17. [BB] ($\longrightarrow$) A tree with $n$ vertices has $n-1$ edges by Theorem 12.1.6 and no cycles by Proposition 12.1.2.

($\longleftarrow$) Suppose $\mathcal{G}$ is an acyclic graph with $n$ vertices and $n-1$ edges. Since $\mathcal{G}$ has no cycles, Proposition 12.1.2 shows that we have only to prove that $\mathcal{G}$ is connected. Let then $C_1, C_2, \ldots, C_k$ be the connected components of $\mathcal{G}$ and suppose that $C_i$ has $n_i$ vertices. (Thus, $\sum n_i = n$.) Since $\mathcal{G}$ has no cycles, there are no cycles within each $C_i$. It follows that each $C_i$ is a tree with $n_i - 1$ edges. The number of edges in $\mathcal{G}$ is, therefore, $\sum_1^k (n_i - 1) = (\sum_1^k n_i) - k = n - k$. So $n - k = n - 1$, $k = 1$, $\mathcal{G}$ has only one component; that is, $\mathcal{G}$ is connected.

18. The new graph is connected, so it is a tree if its edges number one less than the number of vertices. Suppose $\mathcal{T}_1$ has $n_1$ vertices and $\mathcal{T}_2$ has $n_2$ vertices. Then $\mathcal{T}_1$ has $n_1 - 1$ edges and $\mathcal{T}_2$ has $n_2 - 1$ edges so the new graph has $(n_1 - 1) + (n_2 - 1) + 1 = (n_1 + n_2) - 1$ edges, and $n_1 + n_2$ vertices.

19. (a) [BB] $\mathcal{T}$ has $n$ vertices and $n-1$ edges. Hence $\mathcal{T} \setminus \{e\}$ has $n$ vertices and $n-2$ edges; $\mathcal{T} \setminus \{e\}$ has no circuits. Hence $\mathcal{T}$ cannot be connected or it would be a tree (with the wrong number of edges).

(b) It is true for $n = 1$. Assume it is true for all $n \leq k$ and say that $\mathcal{T}$ has $k+1$ vertices. Let $\{e\}$ be any edge of $\mathcal{T}$. Then $\mathcal{T} \setminus \{e\}$ is not connected by (a), and has two connected components, $\mathcal{T}_1$ and $\mathcal{T}_2$, both of which are trees. Say $\mathcal{T}_1$ has $a$ vertices and $\mathcal{T}_2$ has $b$ vertices. Note that $a + b = k + 1$. Since $a \leq k$ and $b \leq k$, the strong induction hypothesis tells us that $\mathcal{T}_1$ has $a - 1$ edges and $\mathcal{T}_2$ has $b - 1$ edges. Then $\mathcal{T}$ has $(a-1) + (b-1) + 1(\text{for } e) = a + b - 1 = k + 1 - 1 = k$ edges as required.

20. Suppose there are $k_i$ vertices of degree $i$. Since $\sum \deg v_i$ is twice the number of edges in $\mathcal{T}$, we have $k_1 + 2k_2 + 3k_3 + \cdots = 2(n-1)$. Since $k_1 + k_2 + \cdots = n$,

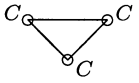
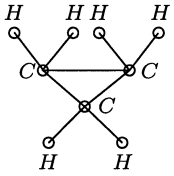
$$k_1 + 2k_2 + 3k_3 + \cdots = 2(k_1 + k_2 + \cdots - 1) = 2k_1 + 2k_2 + 2k_3 + \cdots - 2.$$

Solving for $k_1$, we get $k_1 = (3-2)k_3 + (4-2)k_4 + (5-2)k_5 + \cdots + 2$. This is exactly what we were asked to establish since $(i-2)k_i$ is just $\sum[\deg v_j - 2]$, the sum over all vertices of degree $i$.

21. [BB] There are no cycles in the subgraph since there are no cycles in $C_k H_{2k+2}$. Also, given any two $C$ vertices, there is a path between them in $C_k H_{2k+2}$ (because $C_k H_{2k+2}$ is connected). Any $H$ vertex on this path would have degree at least 2. Thus, there is none; the path consists entirely of $C$ vertices and hence lies within the subgraph. The subgraph is connected, hence a tree.

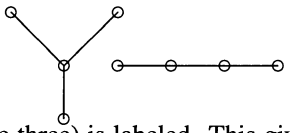
22. (a) Let $x$ be the number of $H$ vertices adjoined. Since $\mathcal{T}$ had $k-1$ edges, and one new edge is added for each $H$, $\mathcal{G}$ has $(k-1)+x$ edges. Therefore, $\sum \deg v_i = 2(k-1+x)$. But $\sum \deg v_i = 4k+x$ since each $C$ has degree 4 and each $H$ has degree 1. Therefore, $4k + x = 2k - 2 + 2x$ and $x = 2k + 2$.

    (b) The above proof depends on $\mathcal{T}$ being a tree. The result is false otherwise.

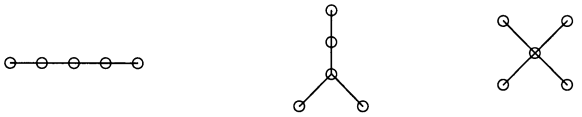
Consider  which becomes 

Here $2k + 2 = 8$, but only six $H$'s are needed.

23. (a) With reference to the two unlabeled trees with four vertices shown to the right, the tree on the right determines $4!/2 = 12$ distinct labeled trees. Any labeled tree derived from the one on the left is completely determined once the central vertex (of degree three) is labeled. This gives another four labeled trees, so there are 16 in all. Notice that $16 = 4^{4-2}$, in agreement with the formula $n^{n-2}$ which appears in the text.



    (b) The three unlabeled trees with five vertices are shown below.



The one on the left determines $5!/2 = 60$ distinct labeled trees and the one on the right five. To label the tree in the middle, first assign a label to the vertex of degree three (in one of five ways), then choose labels for the two vertices of degree one below it (in $\binom{4}{2} = 6$) and then assign the remaining two labels to the top two vertices in two ways. Thus, the tree in the middle determines $5 \cdot 6 \cdot 2 = 60$ labeled trees giving altogether $60 + 5 + 60 = 125$ distinct labeled trees. Notice that $125 = 5^{5-2}$, in agreement with the formula $n^{n-2}$ which appears in the text.

24. We use induction on $n$. The result is clear for $n = 2$. Assume it is true for $n = k$ and suppose $\mathcal{T}$ is a tree with $k+1$ vertices. Let $v$ be a vertex of degree one in $\mathcal{T}$. As in the proof of Theorem 12.1.6, the pruned tree obtained by removing $v$ and the edge $vu$ with which it is incident leaves a subgraph which is also a tree. By the induction hypothesis this pruned tree is bipartite. Putting $v$ and $u$ into different bipartition sets shows that $\mathcal{T}$ is also bipartite.

25. [BB] A tree is a complete bipartite graph if and only if it is $\mathcal{K}_{1,n}$ for some $n$.

**Proof.** Certainly $\mathcal{K}_{1,n}$ is a tree. Conversely, if a tree is complete bipartite, then it is $\mathcal{K}_{m,n}$ for some $m$ and $n$. But such a graph has no vertices of degree one unless $m$ or $n$ is 1. The result follows. ■

26. (a) [BB] Remember that a *component* in a graph is a maximal connected subgraph. Also, any graph is the disjoint union of its components. If a graph has no cycles, neither does any component, so each component is a tree.

(b) By Corollary 12.1.7, each of the $c$ components has at least two vertices of degree 1. So there are at least $2c$ vertices of degree 1 all together.

(c) The graph o o is a forest consisting of two components and two (not four) vertices.

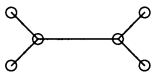
(d) Let the number of vertices in the $c$ components be $n_1, n_2, \ldots, n_c$ respectively. We are given that $n_1+n_2+\cdots+n_c = n$. Since each component is a tree, the number of edges in the $i$th component is $n_i - 1$. Thus, the total number of edges is $(n_1 - 1) + (n_2 - 1) + \cdots + (n_c - 1) = n - c$.

27. (a) [BB] Using Corollary 12.1.7, we have $\sum \deg(v_i) \geq 8$, so the tree has at least four edges and hence at least five vertices. If the result is not true, then there are two vertices of degree 3, at most three vertices of degree 1, and the rest have degree at least 2. Then

$$\sum \deg v_i \geq 2(3) + 3(1) + (n-5)2 = 2n - 1,$$

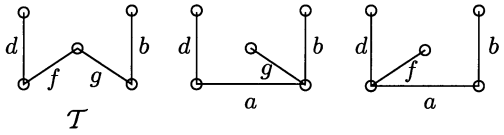
contradicting the fact that $\sum \deg v_i = 2(n-1)$.

(b) Here is a tree with two vertices of degree 3 and exactly four vertices of degree 1.

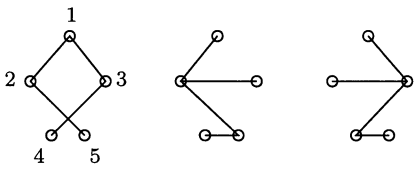


## Exercises 12.2

1. [BB] We show $\mathcal{T}$ and two other spanning trees found by adding $a$ and then successively deleting $f$ and $g$ are shown to the right.




2. (a) [BB] Here is one possible answer. The second and third trees are isomorphic, but neither is isomorphic to the first.



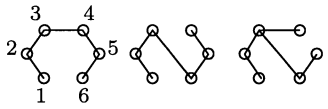
Kirchhoff's matrix is $\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$. The value of the $(5,1)$ cofactor is

$$\begin{vmatrix} -1 & -1 & 0 & 0 \\ 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -[(-1)(1) + (-1)(5)] + [3(1) - 1(-2)] = 11.$$

We conclude that there are 11 spanning trees.

(b) Here is one possible answer. The first and second trees are isomorphic, but neither is isomorphic to the third.




Kirchhoff's matrix is $\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$.

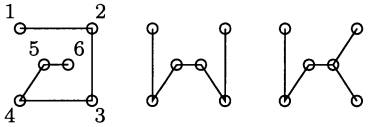
The $(1,1)$ cofactor is $\begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{vmatrix}$

$$= 2\left\{3\begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 3 & -1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{vmatrix}\right\}$$
$$+ (-1)\begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2\{3[3(3) + 1(-2)] + (-1)3 + 1(-1) + (-1)(1) + (-1)(5)\} + (-1)[3(3) + 1(-2)]$$
$$= 2(11) + (-1)(7) = 15.$$

We conclude that there are 15 spanning trees.

(c) Here is one possible answer. The first and second trees are isomorphic, but neither is isomorphic to the third.



Kirchhoff's matrix is $\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$.

The $(1,1)$ cofactor is $\begin{vmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{vmatrix}$

$$= 3\begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 0 & 0 & -1 \\ 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{vmatrix}$$

$$= 3\left\{3\begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix}\right\}$$

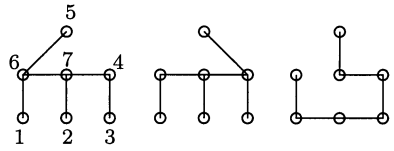
$$+ (-1)\begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 3 & -1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{vmatrix}$$

$$- \left\{(-1)\begin{vmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix}\right\}$$

$$= 3\{3[3(5) + (-3)] + [(-1)5 + (-1)] + [(-1)1 + -5]\}$$
$$- [3(5) + (-3)] + [(-1)1 - 5] - \{(-1)[-1(1) - 5] + 3(5) + (-2)\}$$
$$= 3(24) - 12 - 6 - 19 = 35.$$

We conclude that there are 35 spanning trees.

(d) Here is one possible answer. The first and second trees are isomorphic, but neither is isomorphic to the third.




Kirchhoff's matrix is $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$.

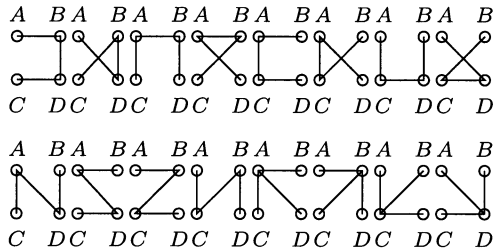
The value of the $(1,1)$ cofactor is $\begin{vmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & -1 & 4 \end{vmatrix} = 65.$

We conclude that there are 65 spanning trees.

3. $\mathcal{K}_n$ has $n^{n-2}$ spanning trees each with $n-1$ edges. Hence, the total number of all edges used in all spanning trees is $(n-1)n^{n-2}$. Now each of the $\binom{n}{2}$ edges in $\mathcal{K}_n$ is equally likely to be included in a spanning tree. Hence, the number of spanning trees containing $e$ is $\dfrac{(n-1)n^{n-2}}{\binom{n}{2}} = 2n^{n-3}$.

4. [BB] By Theorem 12.2.3, the numbers are $1^{-1} = 1$, $2^0 = 1$, $3^1 = 3$, $4^2 = 16$, $5^3 = 125$, and $6^4 = 1296$.

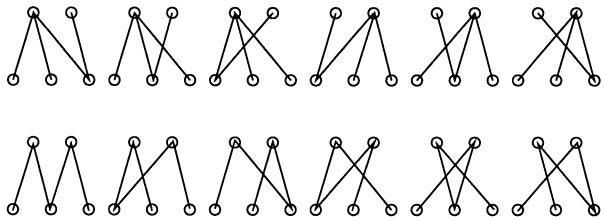
5. [BB]




6. $\mathcal{K}_7$ (labeled) has $7^5$ spanning trees. (See Theorem 12.2.3.)

7. (a) [BB] $\mathcal{K}_{2,2}$ has four spanning trees (obtained by deleting each edge in succession). They are all isomorphic to .

   (b) $\mathcal{K}_{2,3}$ has 12 spanning trees as shown.



The graphs in the top row form one isomorphism class and those in the bottom row a second.

8. Let the bipartition sets be $\mathcal{V}$ and $\mathcal{W}$ and let $\mathcal{V} = \{v_1, v_2\}$ be the one of order two. In any spanning tree of $\mathcal{K}_{2,n}$, if two vertices of $\mathcal{W}$ are adjacent to both $v_1$ and $v_2$, there would be a cycle, which is not allowed. On the other hand, if no vertex of $\mathcal{W}$ was adjacent (in a spanning tree) to both $v_1$ and $v_2$, then the tree would not be connected, another contradiction. Thus, exactly one of the $n$ vertices in $\mathcal{W}$ is adjacent to both $v_1$ and $v_2$. It follows that a spanning tree with $k$ vertices joined to $v_1$ is obtained by selecting one of the $n$ vertices of $\mathcal{W}$ to be adjacent to both $v_1$ and $v_2$, then (if $k > 1$) $k-1$ of the remaining $n-1$ vertices of $\mathcal{W}$ to join to $v_1$ and finally joining the remaining $n-k$ vertices of $\mathcal{W}$ to $v_2$ (no choice involved here). The number of spanning trees with $k > 1$ vertices joined to $v_1$ is $n\binom{n-1}{k-1}$.

When $k = 1$, there are $n$ spanning trees. Thus, the total number of spanning trees is

$$n + n\sum_{k=2}^{n}\binom{n-1}{k-1} = n + n\sum_{k=1}^{n-1}\binom{n-1}{k} = n\Big(1 + \sum_{k=0}^{n-2}\binom{n-1}{k}\Big) = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}.$$

9. [BB] The edge in question is a *bridge* (see Definition 12.6.2); that is, its removal disconnects the graph. To see why, call the edge $e$. If $\mathcal{G} \setminus \{e\}$ were connected, it would have a spanning tree. However, since $\mathcal{G} \setminus \{e\}$ contains all the vertices of $\mathcal{G}$, any spanning tree for it is also a spanning tree for $\mathcal{G}$. We have a contradiction.

10. The only spanning tree of $\mathcal{G}$ is $\mathcal{G}$ itself. Any spanning tree different from $\mathcal{G}$ would have to contain some edge not in $\mathcal{G}$ and there are no such edges.

11. The addition of any edge to a tree produces a circuit. We are told that if $\mathcal{T}$ is a spanning tree, the addition of $e$ does not produce a circuit, so $e$ must already belong to $\mathcal{T}$.

12. (a) [BB] Say the edge is $e$ and $\mathcal{T}$ is any spanning tree. If $e$ is not in $\mathcal{T}$, then $\mathcal{T} \cup \{e\}$ must contain a circuit. Deleting any edge of this circuit other than $e$ gives another spanning tree which includes $e$.

    (b) Let $e_1$ and $e_2$ be the given edges and let $\mathcal{T}$ be any spanning tree. By part (a), we may assume that $\mathcal{T}$ contains $e_1$. If it also contains $e_2$, we are done; otherwise, $\mathcal{T} \cup \{e_2\}$ must contain a circuit $\mathcal{C}$ to which $e_2$ belongs. Since circuits contain at least three edges, $\mathcal{C}$ contains some edge $g$ different from $e_1$ and different from $e_2$. Now $(\mathcal{T} \cup \{e_2\}) \setminus \{g\}$ is a spanning tree containing both $e_1$ and $e_2$.

    (c) [BB] No. If the three edges form a circuit, no spanning tree can contain them.

13. Let $\mathcal{T}$ be a spanning tree for $\mathcal{G}$ and let $v$ be a vertex of degree 1 **in** $\mathcal{T}$. Since $\mathcal{T} \setminus \{v\}$ is connected, so is $\mathcal{G} \setminus \{v\}$. The final statement follows from the fact that any tree has at least two vertices of degree 1.

14. (a) [BB] The subgraph $\mathcal{T}_0 \cup \{e\}$ is connected because $\mathcal{T}_0$ is. Since it has the same number of vertices and edges it cannot be a tree; therefore, it contains a circuit. Let $f$ be an edge on this circuit such that $f \notin \mathcal{T}$. (Since $\mathcal{T}$ does not contain a circuit, this is possible.) Then $(\mathcal{T}_0 \cup \{e\}) \setminus \{f\}$ is still connected (you can go either way around a circuit), and it has $n$ vertices and $n - 1$ edges, so it's a spanning tree.

    (b) The procedure of (a) produces from $\mathcal{T}_0$ a tree $\mathcal{T}_1$ which has one more edge in common with $\mathcal{T}$ than did $\mathcal{T}_0$. If $\mathcal{T}_1$ is not $\mathcal{T}_0$, repeat (a) with $\mathcal{T}_1$ in place of $\mathcal{T}_0$, thereby obtaining a tree $\mathcal{T}_2$ which has two edges more in common with $\mathcal{T}$ than did $\mathcal{T}_0$. Continuing, we obtain a sequence of trees, each with one more edge in common with $\mathcal{T}$ than its predecessor. The process must eventually stop.

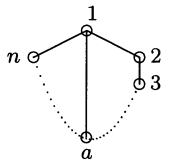
15. (a) There is a spanning tree obtained by deleting exactly one edge of the circuit, and all spanning trees are obtained in this way. Therefore, the number of spanning trees is $n$.

    Alternatively, the matrix is $\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ 0 & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$, which is $n \times n$. So the number

of spanning trees is the determinant of the $(n-1)\times(n-1)$ matrix $\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$.

It is a straightforward exercise in mathematical induction to prove that this determinant is $n$.

(b) There are $n$ spanning trees not containing edge $1a$ (by part (a)). Now suppose $1a$ is in some spanning tree. We must now delete exactly one of the edges $12, 23, \ldots, (a-1)a$ and exactly one of the edges $a(a+1), \ldots, n1$. There are $a-1$ choices in the first list and $n-(a-1)$ choices in the second. So the number of trees containing $1a$ is $(a-1)(n-(a-1))$ and the total number of trees is $n+(a-1)(n-(a-1)) = an-(a-1)^2$. (Alternatively, use Kirchhoff's Theorem!)




16. [BB] There are $\binom{n}{2}$ possible edges from which we choose $n-1$. The number of graphs is, therefore, $\binom{\binom{n}{2}}{n-1}$. The number of trees on $n$ labeled vertices is $n^{n-2}$. For $n \le 6$, the table shows the numbers of trees as compared with the numbers of graphs.

| $n$ | No. of trees | No. of graphs |
|---|---|---|
| 2 | 1 | 1 |
| 3 | 3 | 3 |
| 4 | 16 | 20 |
| 5 | 125 | 210 |
| 6 | 1296 | 3003 |

17. Begin by assigning a weight of 1 to all edges. Then a shortest path is a path requiring the least number of edges.

**Solution 1.** (This solution works with any shortest path algorithm.)

Call the graph $\mathcal{G}$. Choose any two vertices and find the shortest path $\mathcal{P}$ between them. Since paths don't contain circuits, if $\mathcal{P}$ contains all vertices of $\mathcal{G}$, it is a spanning tree. On the other hand, if there is a vertex $v$ not in $\mathcal{P}$, find the shortest path $\mathcal{P}_v$ from $v$ to $\mathcal{P}$; that is, find the shortest paths from $v$ to each vertex of $\mathcal{P}$ and let $\mathcal{P}_v$ be the shortest of these. The critical point is that there can be no cycles among the vertices and edges of $\mathcal{P}$ and $\mathcal{P}_v$. Here is why.

Suppose $\mathcal{P}_v$ meets $\mathcal{P}$ at vertex $x$. Then $\mathcal{P}_v$ shortest guarantees that $x$ is the only vertex in both paths and $\mathcal{P}_v$ a path guarantees that $\mathcal{P}_v$ meets $\mathcal{P}$ via a unique edge $yx$. Any cycle in $\mathcal{P}\cup\mathcal{P}_v$ would have to include $x$ and other vertices of $\mathcal{P}_v$. Starting the cycle at $x$ and moving into $\mathcal{P}_v$, we see that the cycle would have to begin $xy$ and end $yx$. But this implies a cycle in $\mathcal{P}_v$ from $y$ to $y$, a contradiction.

Thus, the vertices and edges of $\mathcal{P}$ and $\mathcal{P}_v$ form a tree $\mathcal{T}$. If $\mathcal{T}$ contains all vertices of $\mathcal{G}$, we are done. Otherwise, let $w$ be a vertex not in $\mathcal{T}$ and let $\mathcal{P}_w$ be the shortest path from $w$ to $\mathcal{T}$. Continue.

**Solution 2.** (Dijkstra's algorithm—first version)

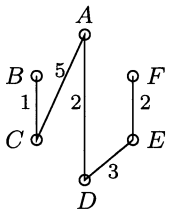
Select any vertex $A$ and apply the first version of Dijsktra's Algorithm to label all vertices in the graph. For each vertex, the algorithm determines a unique shortest path back to $A$. The edges of all such shortest paths must determine a spanning tree.

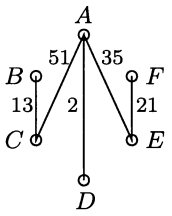
Why? The graph that these edges determine contains all vertices and is connected since there is a path between any two vertices via $A$. Also it contains no circuits since any vertex of a circuit would be

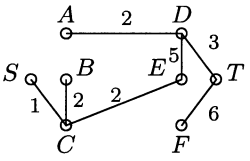
connected to $A$ via two paths of edges and, as we have noted, Dijkstra's algorithm provides just one path. Also note that if a vertex $X$ is encountered on the shortest path from $A$ to $B$, as given by the algorithm, then the shortest path from $A$ to $X$ given by the algorithm agrees with the first part of the shortest path from $A$ to $B$.

## Exercises 12.3

1. (a) [BB] We want five edges (since there are six vertices). Choose $BC$, then $AD$, $FE$, and $DE$. We would like next to choose $AE$, but this would complete a circuit with $AD$ and $DE$, so we choose $AC$ and obtain the spanning tree shown, of weight 13.

(a) 

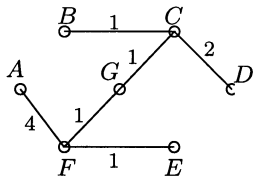

(b) 


(c) 

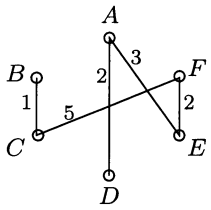

   (b) As in part (a), we need five edges. Choose $AD$, then $BC$, then $EF$, then $AE$. Now $DE$ is no good, so $AC$ will finish the job. The weight of a minimum spanning tree is 122 as shown.

   (c) We want seven edges (since there are eight vertices). First choose $SC$, then $CE$, then $AD$, then $BC$ and $DT$. Now $BE$ cannot be chosen since it would complete a circuit with previously chosen edges $BC$ and $CE$. So we choose next $DE$ and then either $CF$ or $FT$ to complete our spanning tree. We obtain the graph shown, of minimum weight 21.

   (d) We want six edges (since there are seven vertices). First choose $BC$, then $CG$, $GF$ and $FE$, all of weight 1. We cannot choose the last edge, $GE$, of weight 1 since it would complete a circuit with previously chosen edges $GF$ and $FE$. There is only one edge of weight 2 which does not complete a circuit with edges already chosen, $CD$. There is no eligible edge of weight 3. Of the edges of weight 4, only $AF$ does not complete a circuit with previous edges. So we take it and obtain a spanning tree of weight 10.

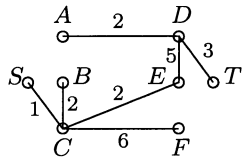



2. (a) [BB] The edge of least weight incident with $E$ is $EF$. The least weight of those edges adjacent to $EF$ is 3; we choose one of them, say $AE$ (in an effort to obtain a different tree from before). There is just one edge of least weight among whose which, together with $EF$ and $AE$, form a tree; namely, $AD$. Now those edges which together with $EF$, $AE$, and $AD$ form a tree have weights 5 and 6. We choose one of least weight, say $CF$. Finally, we choose $BC$, obtaining the tree of weight 13 shown above right.

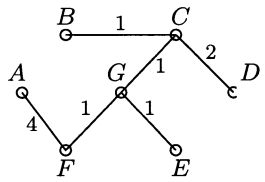



   (b) The edge of least weight incident with $E$ is $FE$. The edge of least weight among those adjacent to $FE$ is $AE$. The edge of least weight among those which, together with $FE$ and $AE$, form a tree is $AD$. Continuing to form a sequence of ever-growing trees, we are forced to select $CA$ and then $BC$. We obtain the same tree as before, as could have been predicted. (See Exercise 11.)

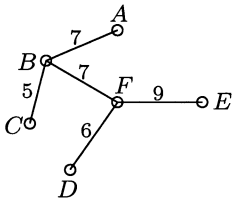
(c) The edge of least weight incident with $E$ is $CE$. The edge of least weight among those adjacent to $CE$ is $SC$. The edge of least weight among those which, together with $CE$ and $SC$, form a tree is $BC$. Then we must add, in order, $DE$, $AD$, and $DT$. Finally, so as not to duplicate the tree we obtained in Exercise 1, we select $CF$.




(d) Starting at vertex $E$, we select first edge $GE$ of weight 1 and then $GF$ (to change the tree we had before). Of the edges incident with $G$, $E$ and $F$, those of least weight are $FE$ and $GC$. We cannot select $FE$ because it is incident with **two** of the vertices already chosen, so we choose $GC$, then $BC$, $CD$ and $AF$. We obtain another spanning tree of weight 10.

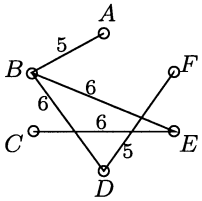


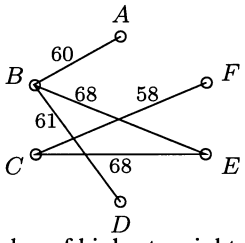

3. [BB] We need five edges because there are six vertices in the graph. We use Kruskal's algorithm. First select $BC$ of weight 5, then $DF$ of weight 6. Next we select $BF$ and $AB$, both of weight 7. We would like to select $AF$ next, but cannot since it completes the circuit $ABFA$, so we select $EF$ of weight 9 next. The five edges we have selected comprise a minimum spanning tree. The smallest length of pavement required is $5 + 6 + 7 + 7 + 9 = 34$.




4. (a) Choose the edge of highest weight first. Then choose the edge of highest weight from those remaining. Continue, at each step choosing an edge of highest weight from among those edges not yet chosen which do not complete a circuit with previously selected edges.

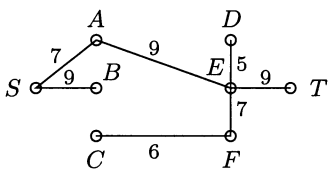
(b) i. [BB] Choose $BD$, $BE$, and $CE$ each of weight 6. The remaining edge, $CD$, of maximum weight (6) cannot be chosen because it would complete the circuit $BDCEB$ with the previously chosen edges. So we choose $DF$ and then $AB$ obtaining the spanning tree shown on the left, of maximum weight 28.

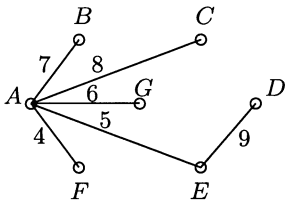



ii. Choose $CE$, $BE$, $BD$ and $AB$. Now the remaining edge of highest weight, $CD$, cannot be used since it would complete circuit $CDBEC$ whose edges have already been chosen. So we choose $CF$ obtaining the spanning tree shown on the right, of maximum weight 315.

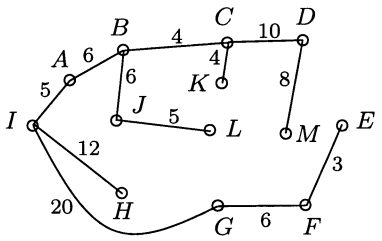
iii. Choose $AE$, $ET$, $SB$, $AS$, $EF$, and $CF$. At this point, the remaining edges of highest weight 6, $FT$ and $AB$, cannot be used since the addition of either completes a circuit with edges already chosen. So we choose $DE$, obtaining the spanning tree shown on the left, of maximum weight 52.



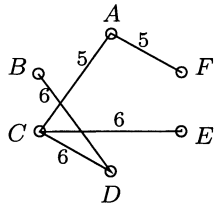
iv. Choose $DE$, $AC$, $AB$, $AG$ and $AE$ in order. There are two possibilities for an edge of weight 4. We choose $AF$, obtaining the spanning tree on the right, of maximal weight 39.

v. Choose $IG, IH, CD, DM, AB, BJ, GF, IA$ and $JL$. At this point, the next edge of highest weight, $HG$, cannot be selected since it would complete a circuit with previously chosen edges. So choose $BC$ and $CK$. We are not able to choose $KM$ or $GL$, so we select $FE$, giving a spanning tree of maximum weight 89.




5. (a) Start with any vertex and the edge of maximum weight incident with that vertex. Then pick an edge of maximum weight from among those edges adjacent to the first. In general, given a subgraph of the given graph which is a tree but not a spanning tree, add to this an edge of maximum weight from those edges one of whose end vertices is not in the tree and the other of which is (so that the augmented tree is also a tree). Continue until there are no vertices outside the tree, that is, until the tree is a spanning tree.

(b) i. Select first $AC$, one of the three edges of maximum weight incident with $A$. Then select the two edges, $CD$ and $CE$, of maximum weight incident with $C$. There are two edges of maximum weight (6) which form a tree with $AC$, $CD$ and $CE$; we select one of them, say $BD$. Finally we select any edge of weight 5 incident with $F$, say $AF$, obtaining the tree shown at the right of maximum weight 28.




ii. Starting at vertex $A$, the first edge we must select is $AB$. Then, in order, we select $BE$, $CE$, $BD$ and $CF$. This is the tree we had before.

iii. Starting at vertex $A$, the first edge we must include is $AE$. Then we might select $ET$, $FE$, $SA$ and then $SB$, $CF$ and $DE$. We obtain the same tree as before.

iv. Starting at vertex $A$, we select in order $AC$, $AB$, $AG$, $AE$ and $ED$ and then, either $DF$ or $AF$, obtaining a tree of maximum weight 39, as before.

v. Starting at vertex $A$, the first edge we must select is $AB$. Then, in order, we select $BJ$, $JL$ and $AI, IG, IH, GF, BC, CD, DM, KC$ and $EF$, obtaining the tree we had before.

6. [BB] Assign all edges of the graph a weight of 1. Then carry out the algorithm.

7. Let $\mathcal{G}$ be the graph and $e$ the edge. Give $e$ a weight of 2 and all other edges a weight of 1. Since there exists a spanning tree excluding $e$, the weight of a minimum spanning tree is $n - 1$ (assuming $\mathcal{G}$ has $n$ vertices) whereas the weight of any spanning tree containing $e$ is $2 + (n - 2)1 = n$. It follows the tree found by any minimum spanning tree algorithm will have weight $n - 1$ and hence exclude $e$. (Alternatively, apply the spanning tree algorithm to $\mathcal{G} \setminus \{e\}$.)

8. (a) [BB] If the graph is unweighted, put a weight of 1 on edge $e$ and 2 on every other edge. If the graph is weighted, ensure (by temporarily changing weights if necessary) that the weights of the edges different from $e$ are all larger than the weight of $e$. In either case, Kruskal's algorithm will select $e$ first.

(b) If the graph is unweighted, put weights of 1 on the edges of the subgraph and 2 on all other edges. If the graph is weighted, ensure (by temporarily changing weights if necessary) that the weights

on edges of the subgraph are all less than the weights on all other edges. In either case, Kruskal's algorithm selects first the edges of the subgraph since these have weights smaller than other edges and there are no circuits among them.

9. (a) If the graph is unweighted, put a weight of 1 on edge $e = uv$ and 2 on every other edge. If the graph is weighted, ensure (by temporarily changing weights if necessary) that the weights of the edges different from $e$ are all larger than the weight of $e$. In either case, if Prim's algorithm starts at $u$ or $v$, it will select $e$ first and, if it starts at any other vertex, then as soon as the first of $u$, $v$ is included in one of the sequence of growing trees (to be specific, suppose this is $u$ and the tree is $\mathcal{T}$), edge $e$ will be selected since it is eligible for selection ($\mathcal{T} \cup \{e\}$ is also a tree) and $e$ is the edge of least weight in the graph.

   (b) If the graph is unweighted, put weights of 1 on the edges of the subgraph and 2 on all other edges. If the graph is weighted, ensure (by temporarily changing weights if necessary) that the weights on edges of the subgraph are all less than the weights on all other edges. To see why Prim's algorithm is bound to include the edges of the subgraph (in either the weighted or unweighted case), note that the connected components of the subgraph are trees. Thus, as soon as the first vertex in any of these components is included in one of the sequence of trees produced by Prim's algorithm, the rest of the component will be included next as edges become adjacent to already selected edges because they have lowest weights in the graph and will not produce circuits with previously chosen edges.

10. [BB] Let $d$ be the lowest weight among the edges incident with vertex $v$. The first time that an edge incident with $v$ is considered for selection (because it has least weight among remaining edges), no edge incident with $v$ will complete a circuit with edges previously selected. Always seeking edges of lowest weight, the algorithm must select an edge of weight $d$.

11. (a) Let $\mathcal{T}_m$ be any minimum spanning tree. We prove that this tree could have been found with Kruskal's algorithm. Suppose that $\mathcal{T}_{K_1}$ is any tree produced by Kruskal's algorithm with edges $e_1, e_2, \ldots, e_n$ selected in this order. If $\mathcal{T}_{K_1} = \mathcal{T}_m$, there is nothing to prove. Otherwise, let $e_i$ be the first edge of $\mathcal{T}_{K_1}$ which is not in $\mathcal{T}_m$. Arguing as in the text, there exists an edge $e \in \mathcal{T}_m$ of weight $w(e) = w(e_i)$ such that $e_1, e_2, \ldots, e_{i-1}, e$ do not include a circuit. Thus, Kruskal's algorithm could equally well have selected $e$ at the time it selected $e_i$. We conclude that $e_1, e_2, \ldots, e_{i-1}, e$ are part of a tree $\mathcal{T}_{K_2}$ which can be selected by Kruskal's algorithm and which has its first $i$ edges in $\mathcal{T}_m$. If $\mathcal{T}_{K_2} = \mathcal{T}_m$ we are done; otherwise, we proceed as before to find with Kruskal's algorithm, a tree $\mathcal{T}_{K_3}$ whose first $i+1$ edges are in $\mathcal{T}_m$. Eventually this process must terminate with some $\mathcal{T}_{K_s} = \mathcal{T}_m$.

   (b) [BB] If the weights of a graph with $n$ vertices are distinct, then Kruskal's algorithm selects at each stage a unique edge of lowest weight. Since Kruskal's algorithm yields only one tree, the graph has only one minimum spanning tree, by (a).

12. (a)

| Component | Size | Vertices | $\xrightarrow{\text{add } LK}$ | Comp. | Size | Vertices |
|---|---|---|---|---|---|---|
| 1 | 2 | $A, J$ | | 1 | 2 | $A, J$ |
| 2 | 1 | $B$ | | 2 | 1 | $B$ |
| 3 | 1 | $C$ | | 3 | 1 | $C$ |
| 7 | 5 | $G, M, F, D, E$ | | 7 | 5 | $G, M, F, D, E$ |
| 8 | 1 | $H$ | | 8 | 1 | $H$ |
| 9 | 1 | $I$ | | 9 | 1 | $I$ |
| 11 | 1 | $K$ | | 11 | 2 | $K, L$ |
| 12 | 1 | $L$ | | | | |

| | Comp. | Size | Vertices |
|---|---|---|---|
| add $HJ$ ⟶ | 1 | 3 | $A, J, H$ |
| | 2 | 1 | $B$ |
| | 3 | 1 | $C$ |
| | 7 | 5 | $G, M, F, D, E$ |
| | 9 | 1 | $I$ |
| | 11 | 2 | $K, L$ |

| | Comp. | Size | Vertices |
|---|---|---|---|
| add $KM$ ⟶ | 1 | 3 | $A, J, H$ |
| | 2 | 1 | $B$ |
| | 3 | 1 | $C$ |
| | 7 | 7 | $G, M, F, D, E, K, L$ |
| | 9 | 1 | $I$ |

| | Comp. | Size | Vertices |
|---|---|---|---|
| add $CK$ ⟶ | 1 | 3 | $A, J, H$ |
| | 2 | 1 | $B$ |
| | 7 | 8 | $G, M, F, D, E, K, L, C$ |
| | 9 | 1 | $I$ |

| | Comp. | Size | Vertices |
|---|---|---|---|
| add $BC$ ⟶ | 1 | 3 | $A, J, H$ |
| | 7 | 9 | $G, M, F, D, E, K, L, C, B$ |
| | 9 | 1 | $I$ |

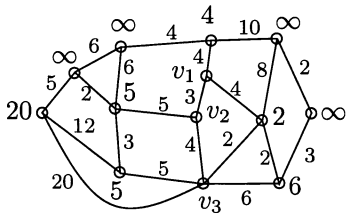
| | Comp. | Size | Vertices |
|---|---|---|---|
| add $JL$ ⟶ | 7 | 12 | $G, M, F, D, E, K, L, C, B, A, J, H$ |
| | 9 | 1 | $I$ |

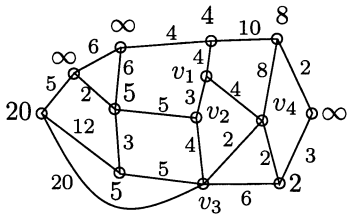
| | Comp. | Size | Vertices |
|---|---|---|---|
| add $IA$ ⟶ | 7 | 13 | $G, M, F, D, E, K, L, C, B, A, J, H, I$ |

(b) [BB] Each time a vertex is relabeled, the component to which it belongs contains at least twice as many vertices as before. Thus, if the label on a vertex changes $t$ times, $2^t \leq n$; so $t \leq \log_2 n$ as required.

(c) [BB] Since the initial sorting of edges requires $\mathcal{O}(N \log N)$ comparisons, we have only to show that the relabeling process described in (a) and (b) can also be accomplished within this bound. By (b), the total number of vertex relabelings is $\mathcal{O}(n \log n)$. If $n \leq N$, we are done. If $n = N+1$, then $n \log n \leq (2N) \log N^2 = 4N \log N$ and again we are done, except when $n = 1$ or 2 when no steps are needed. Finally, if $N \leq n-2$, no spanning tree is possible and a check for this could be included at the beginning of the algorithm.

13. (a) The labels after $v_3 = G$ was selected are shown on the left.




(b) For each vertex $v$ which is adjacent to $v_3$, change the label of $v$ to the minimum of its old label and the weight of edge $vv_3$. Select $v_4$ with minimum label and add the corresponding edge—$v_4v_1$, $v_4v_2$ or $v_4v_3$. With reference to Fig. 12.16, with $v_1 = K$, $v_2 = L$ and $v_3 = G$, we must have $v_4 = M$ and labels as shown on the right.

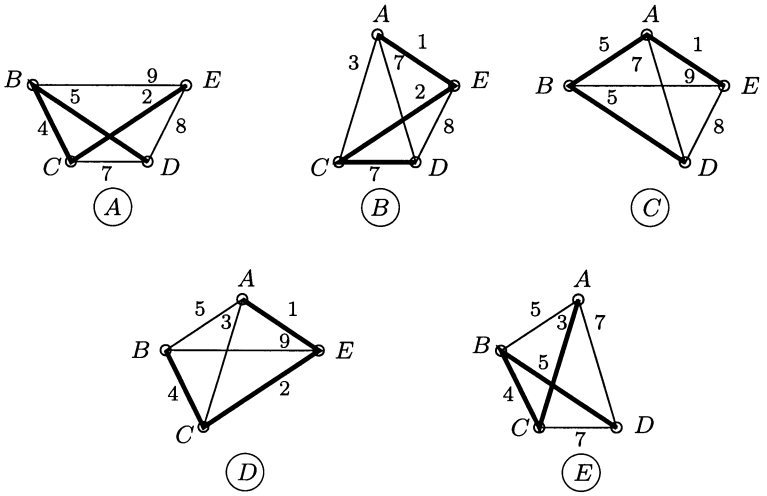
(c) The selection of $v_2$ requires finding the minimum of, at worst (every vertex is adjacent to $v_1$), $n-1$ numbers and requires $n-2$ comparisons. If every vertex is adjacent to $v_2$, we would have to compare the weight of $vv_2$ with the old label of $v$ for $n-2$ vertices $v$. This would require another $n-2$ comparisons. Then we would have to find the minimum of $n-2$ numbers, accounting for another $n-3$ comparisons. In general, after $v_1, v_2, \ldots, v_k$ have been identified, we might, at

worst, have to compare the old label on each other vertex $v$ with the weight of edge $vv_k$ ($n-k$ comparisons) and then find the minimum of $n-k$ numbers (another $n-k-1$ comparisons). In all, this implementation could require

$$\begin{aligned}(n-2)+[(n-2)+(n-3)]+[(n-3)+(n-4)]+\cdots+[2+1]+[1+0]\\ &=(n-2)+(2n-5)+(2n-7)+\cdots+3+1\\ &=(n-2)+\textstyle\sum_{k=0}^{n-3}(2k+1)=(n-2)+(n-2)^2\end{aligned}$$

comparisons. This function is $\mathcal{O}(n^2)$ as required.

14. The five subgraphs shown are those obtained by successively removing the vertices $A, B, C, D, E$. In each case, a minimum spanning tree is shown.

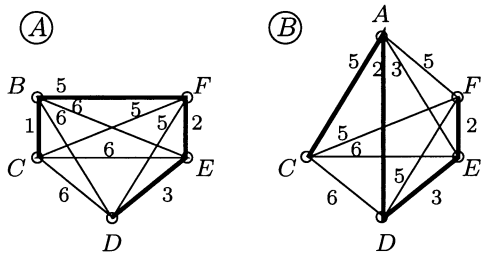



| Vertex removed | Weight of minimum tree | Minimum sum of weights of two incident edges | Sum |
|---|---|---|---|
| $A$ | 11 | 4 | 15 |
| $B$ | 10 | 9 | 19 |
| $C$ | 11 | 5 | 16 |
| $D$ | 7 | 12 | 19 |
| $E$ | 12 | 3 | 15 |

As the table shows, any Hamiltonian cycle must have weight at least 19. The Hamiltonian cycle $AECBDA$ has weight $1+2+4+5+7=19$ and hence is a minimum.

15. (a) [BB] The graph is complete, so an obvious approach is to try to choose the lowest weight available edge at each vertex. One such cycle is $ADEFBCA$, which has weight $2+3+2+5+1+5=18$.

(b) [BB] As shown on the left, the minimum weight of a spanning tree after $A$ is removed is 11. The two edges of least weight at $A$ have weights 2 and 3 so we obtain an estimate of $11+2+3=16$ as a lower bound for the weight of any Hamiltonian cycle.




(c) [BB] As shown on the right, the minimum weight of a spanning tree after $B$ is removed is 12. The two least edges at $B$ have weights 1 and 5, so we obtain $12 + 1 + 5 = 18$ as a lower bound.

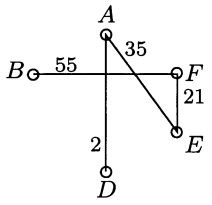
(d) It is not possible to improve the lower bound in (c), as we can see from the cycle found in (a).

16. (a) The graph is complete, so an obvious approach is to try to choose the lowest weight available edge at each vertex. Such a cycle is $ADEFBCA$, which has weight $2+36+21+55+13+51 = 178$.

(b) As shown on the right, the minimum weight of a spanning tree after $D$ is removed is 120. The two least edges at $D$ have weights 2 and 36, so we obtain $120 + 2 + 36 = 158$ as a lower bound.

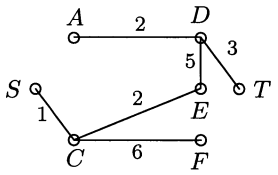
(c) As shown on the left, the minimum weight of a spanning tree after $F$ is removed is 101. The two edges of least weight at $F$ have weights 21 and 52, so we obtain an estimate of $101+21+52 = 174$ as a lower bound for the weight of any Hamiltonian cycle.

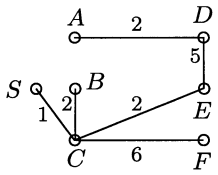
(d) As shown at the right, if vertex $C$ is removed, the minimum weight of a spanning tree is 113. The two least edges at $C$ have weights 13 and 51 so we get an estimate of $113 + 13 + 51 = 177$ as a lower bound. To see that this is best possible, the student can check that if vertex $A$ is removed, a minimum weight spanning tree has weight 125 and we get an estimate of $125 + (2 + 35) = 162$ for the weight of a Hamiltonian cycle; if vertex $B$ is removed, a minimum weight spanning tree has weight 109 and we get an estimate of $109 + (13 + 55) = 177$ for the weight of a Hamiltonian cycle; if vertex $E$ is removed, a minimum weight spanning tree has weight 118 and we get an estimate of $118 + (21 + 35) = 174$ for the weight of a Hamiltonian cycle.




17. (a) One (relatively) low weight Hamiltonian cycle is $SCEFTDABS$, of weight $1+2+7+6+3+2+6+9 = 36$.

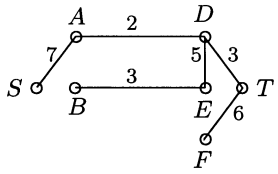
(b) As shown on the left, the minimum weight of a spanning tree after $B$ is removed is 19. The two least edges at $B$ have weights 2 and 3, so we obtain an estimate of $19 + 2 + 3 = 24$ as a lower bound for the weight of any Hamiltonian cycle.



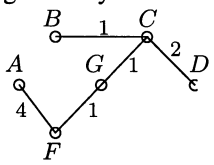

(c) As shown on the right, the minimum weight of a spanning tree after $T$ is removed is 18. The two least edges at $T$ have weights 3 and 6, so we obtain $18 + 3 + 6 = 27$ as a lower bound.

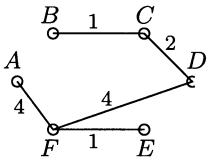
(d) As shown to the right, if vertex $C$ is removed, the minimum weight of a spanning tree is 26. The two least edges at $C$ have weights 1 and 2 so we get an estimate of $26 + 1 + 2 = 29$ as a lower bound. By way of comparison, the student should check that if vertex $A$ is removed, the weight of a minimum spanning tree is 19, implying minimum Hamiltonian cycle of weight $19 + 2 + 6 = 27$; if vertex $D$ is removed, the weight of a minimum spanning tree is 23, implying minimum Hamiltonian cycle of weight $23 + 2 + 3 = 28$; if vertex $E$ is removed, the weight of a minimum spanning tree is 20, implying minimum Hamiltonian cycle of weight $20 + 2 + 3 = 25$; if vertex $F$ is removed, the weight of a minimum spanning tree is 15, implying minimum Hamiltonian cycle of weight $15 + 6 + 6 = 27$; and if vertex $S$ is removed, the weight of a minimum spanning tree is 20, implying minimum Hamiltonian cycle of weight $20 + 1 + 7 = 28$. Thus the best estimate is indeed 29.

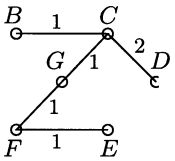



18. (a) One (relatively) low weight Hamiltonian cycle is $AFGEDCBA$, of weight $4 + 1 + 1 + 9 + 2 + 1 + 7 = 25$.

(b) As shown on the left, the minimum weight of a spanning tree after $E$ is removed is 9. The two least edges at $E$ each have weight 1, so we obtain an estimate of $9 + 1 + 1 = 11$ as a lower bound for the weight of any Hamiltonian cycle.




(c) As shown in the middle, the minimum weight of a spanning tree after $G$ is removed is 12. The two least weight edges at $G$ each have weight 1, so we obtain $12 + 1 + 1 = 14$ as a lower bound.

(d) As shown to the right, if vertex $A$ is removed, the minimum weight of a spanning tree is 6. The two least weight edges at $A$ have weights 4 and 5 so we get an estimate of $6 + 4 + 5 = 15$ as a lower bound. To see that this is best, the student should have checked that if vertex $B$ is removed, the weight of a minimum spanning tree is 9, implying minimum Hamiltonian cycle of weight $9 + 1 + 2 = 12$; if vertex $C$ is removed, the weight of a minimum spanning tree is 11, implying minimum Hamiltonian cycle of weight $11 + 1 + 1 = 13$; if vertex $D$ is removed, the weight of a minimum spanning tree is 8, implying minimum Hamiltonian cycle of weight $8 + 2 + 3 = 13$; and if vertex $F$ is removed, the weight of a minimum spanning tree is 10, implying minimum Hamiltonian cycle of weight $10 + 1 + 1 = 12$. Thus the best estimate is indeed 15, obtained by removing $A$.

## Exercises 12.4

1. (a) [BB] This is acyclic: $F, B, D, H, A, E, C, G$ is a canonical labeling.
   (b) This has cycle $DHFAED$.
   (c) This is acyclic: $C, H, I, F, B, G, D, E, A$ is a canonical labeling.
   (d) This digraph has cycle $CEFDC$.

2. $\text{indeg}\, v_0 = 0$ because there are no arcs $v_i v_0$ with $i > 0$. There are no restrictions on the outdegree of $v_0$, which can be any integer between 0 and $n-1$ (inclusive).

3. [BB] Let $\mathcal{G}$ be a digraph with $n$ vertices and let $A$ be the adjacency matrix of $\mathcal{G}$. The indegree of vertex $i$ is the sum of the entries in column $i$. This requires $n-1$ two-number additions. Repeating for $n$ vertices involves $n(n-1) = \mathcal{O}(n^2)$ additions.

4. (a) We use the algorithm described in Theorem 12.4.3.
   Given an acyclic digraph with vertex set $\mathcal{V} = \{0, 1, 2, \ldots, n-1\}$ to find a canonical labeling,

   **Step 1:** Let $V = \{0, 1, \ldots, n-1\}$ be the vertex set and let $t = 0$.

   **Step 2:** While $t \leq n-1$
   - Let $v_t$ be a vertex in $V$ of indegree 0.
   - Replace $V$ by $V \setminus \{v_t\}$ and compute the indegrees of the vertices of (this new) $V$.
   - Replace $t$ by $t+1$.

   end while

   **Step 3:** Output $v_0, v_1, \ldots, v_{n-1}$.

   (b) First we must compute the indegrees of the $n$ vertices. This operation requires $\mathcal{O}(n^2)$ additions. (See Exercise 3.) Finding a vertex of indegree 0 can be accomplished with at most $n$ comparisons. The deletion of vertex $v_i$ and computation of the indegrees of the $n-1$ vertices of $\mathcal{V} \setminus \{v_i\}$ can be done with $2n$ comparisons and additions by means of a loop like this.

   for $j = 0$ to $n-1$, if $a_{ji} = 1$, then $a_{j0} = 0$ and $\text{indeg}_i = \text{indeg}_i - 1$.

   Thus the deletion of $n-1$ vertices requires at most $2n^2$ comparisons and additions. Our algorithm is $\mathcal{O}(n^2)$.

   (c) Here is Maple code (output suppressed) which does the job for an acyclic graph with $n$ vertices. Since Maple will not permit 0 to be the index of a row or column of an array, we assume the vertices of the graph are $\{1, 2, \ldots, n\}$ and produce a canonical labeling $v_1, v_2, \ldots, v_n$.

```
> restart:
> with(linalg):   #Permit reference to matrix
> V := {seq( i, i=1..n )};  #vertex set
> v := vector(n);   #v[1], v[2], ..., v[n] will be the
                        #canonical labeling
> for i to n do     #initialize v
    v[i] := -1:
  od:
> for i to n do       #indeg[i] will be indegree of
                           #vertex i
     indeg[i] := 0:   #here we initialize
  od:
> for i to n do       #find the original indegrees
```

```
      for j to n do
         indeg[i] := indeg[i] + A[j,i]:
      od:
   od:
> t := 1:
> found := 0:     #found = 1 when we have found
                     #the next vertex
> while(t <= n) do
      i := 0:
      found := 0:
      while((found =0) and (i <n)) do
         i := i+1:
         if (member(i,V)=true) then
             if (indeg[i]=0) then
                v[t] := i;
                t := t+1:
                found := 1:
             fi:
         fi:
      od:
      V := V minus {i};      #remove vertex i
         for j in V do       #find indegrees for new V
            if (A[i,j]=1) then
              indeg[j] := indeg[j] - 1:
            fi:
         od:
     od:
     S := seq(v[i],i=1..n):
     lprint('A canonical labeling is ', S);
```

5. (a) [BB] If $\mathcal{G}$ has a cycle, then there is no canonical labeling so (with notation as in Theorem 12.4.3) some $\mathcal{S} = \mathcal{V} \setminus \{v_0, v_1, v_2, \ldots, v_k\}$ has the property that $\mathcal{S}_v \neq \emptyset$ for all $v \in \mathcal{S}$; that is, for every $v \in \mathcal{S}$, there exists an arc of the form $xv$ with $x \in \mathcal{S}$. Let $u_0 \in \mathcal{S}$ be arbitrary and choose $u_1 \in \mathcal{S}$ such that $u_1u_0$ is an arc. Choose $u_2 \in \mathcal{S}$ such that $u_2u_1$ is an arc. If $u_2 = u_0$, then $u_2u_1u_0u_2$ is a cycle. Otherwise, choose $u_3 \in \mathcal{S}$ so that $u_3u_2$ is an arc. In general, having chosen distinct vertices $u_0, u_1, \ldots, u_k \in \mathcal{S}$ such that $u_{i+1}u_i$ is an arc for $i = 0, 1, \ldots k-1$, choose $u_{k+1} \in \mathcal{S}$ such that $u_{k+1}u_k$ is an arc. Since $\mathcal{S}$ is finite, there exists $k$ such that $u_{k+1} = u_\ell$ with $\ell < k$. Then $u_{k+1}u_ku_{k-1}\cdots u_\ell$ is a cycle.

   (b) Given a digraph with vertex set $\mathcal{V} = \{0, 1, 2, \ldots, n-1\}$ in which we know there is a cycle, to find this cycle,

   **Step 1:** Let $V = \{0, 1, \ldots, n-1\}$ be the vertex set and let $t = 0$.

   **Step 2:** While $t \leq n-1$

   If there exists a vertex in $V$ of indegree 0

   - call this vertex $v_t$;
   - replace $V$ by $V \setminus \{v_t\}$ and compute the indegrees of the vertices of (this new) $V$.
   - Replace $t$ by $t+1$.

   end while

**Step 3:** Otherwise (all the vertices of the current $V$ have indegree at least 1, so $V$ contains a cycle)
Let $u_0$ be any vertex of $V$, let $U = \{u_0\}$ and let $k = 0$;
While $k \leq n - 1$,

- Find $x \in V$ for which $xu_k$ is an arc and let $u_{k+1} = x$;
- Replace $V$ by $V \setminus \{x\}$ and $U$ by $U \cup \{x\}$.
- If $u_{k+1} \in U$, find the largest $j$ with $u_j = u_{k+1}$, output the cycle $u_j, u_{j+1}, \ldots, u_{k+1}$ and stop; otherwise replace $k$ by $k + 1$.

end while

(c) Here is Maple code (output suppressed) which does the job for a digraph, known to possess a cycle, with vertices $1, 2, \ldots, n$ and adjacency matrix $A$. The output is a cycle $u_i, u_{i+1}, \ldots, u_j$.

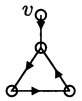
```
> restart:
> with(linalg): #to permit definition of matrix
> V := {seq( i, i=1..n )};    #vertex set
> v := vector(n);     #v[1], v[2], ..., are the vertices
                         #at the start of a canonical
                         #labeling
> for i to n do       #initialize v
    v[i] := -1:
  od:
> for i to n do
    indeg[i] := 0:
  od:
> for i to n do           #find original indegrees
     for j to n do
        indeg[i] := indeg[i] + A[j,i]:
     od:
  od:
> t := 1:
> found := 1:   #found = 1 as long as we continue to find
                    #vertices of a canonical labeling
> while((found=1) and (t <= n)) do
       found := 0:
       i := 0:
       while(i < n) do
             i := i+1:
             if ((member(i,V)=true) and (indeg[i]=0)) then
                   v[t] := i;
                   t := t+1:
                   V := V minus {v[t]}:
                   for j in V do    #find indegrees for new V
                      if (A[i,j]=1) then
                         indeg[j] := indeg[j] - 1:
                      fi:
                   od:
                   found := 1:
             fi:
        od:
```

```
   od:   #Since we assume the given digraph has a cycle,
              #eventually found=1
   u := vector(n+1):  #the vertices of V contain a cycle
                           #because each has indegree 1.
                           #We just have to find it
   u[1] := V[1]:
   U := {u[1]}:
   k := 1:
   j := 1:
   while (j<=n) do
      if ((member(j,V)=true) and (A[j,u[k]]=1)) then
         u[k+1] := j:      #if there is an arc from j to u[k]
                           #with j in V, then j is the next
                           #vertex on the desired cycle
         if(member(u[k+1],U)=true) then
            j := n+1:
         else
            U := U union {j}:
            V := V minus {j}:
            k := k+1:
            j := 0:
         fi:
      fi:
      j := j+1:
  od:
>  j := k:      #since u[k+1] is first repeated vertex
               #of U, there is a cycle from u[j]=u[k+1]
               #to u[k+1], we just have to find j
>  while(u[j]<>u[k+1]) do
      j := j - 1:
   od:
>  S := seq(u[i], i=j..k+1):
>  lprint('Here's a cycle:', S);
```

6. There are two versions of an algorithm by Dijkstra for computing the shortest distance from a specified vertex in a graph to every other vertex in weighted graph. These were described in Section 10.4. The Floyd-Warshall algorithm (discussed in the same section) computes the shortest distances between every pair of vertices in a (weighted) graph. In Section 11.2, we discussed the Bellman-Ford algorithm which finds the shortest path from a specified vertex to every other vertex in a weighted digraph devoid of negative weight cycles. In this section, we learned about Bellman's algorithm, which determines the shortest distance from a specified vertex in an acyclic digraph to every other vertex.

7. [BB] Since the undirected graph is a tree with $n$ vertices, $\mathcal{T}$ has $n-1$ arcs, by Theorem 12.1.6.

8. The answer is no. The digraph shown has a unique vertex $v$ of indegree 0 but it is not a rooted tree because the unoriented graph has a cycle.



9. (a) [BB] Here are the distances $d_t$ and the corresponding values of $p_t$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $d_t$ | 0 | 5 | 7 | 4 | 5 | 7 | 6 | 6 |
| $p_t$ | $-1$ | 0 | 0 | 0 | 0 | 4 | 4 | 3 |

(b) Here are the distances $d_t$ and the corresponding values of $p_t$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | 0 | 1 | 2 | 3 | 4 | 4 | 5 | 5 | 5 |
| $p_t$ | $-1$ | 0 | 0 | 0 | 1 | 3 | 0 | 2 | 1 |

10. We use induction on $t$. If $t = 1$, there is a path to $v_1$ if and only if $v_0v_1$ is an arc. At the first iteration of the loop in Step 2, with $t = 1$, the algorithm checks the value of $d_0 + w(v_0, v_1) = w(v_0, v_1)$ and, if this is finite, assigns this number to $d_1$ and sets $p_1 = 0$, as we claimed. Now assume $t > 1$ and the algorithm has worked as claimed for each $j < t$. Consider a path from $v_0$ to to $v_t$. The last arc on this is of the form $v_jv_t$ with $j < t$ (since the vertices are labeled canonically). Moreover, since $v_j$ lies on a path from $v_0$, the induction hypothesis implies that the algorithm has identified the length $d_j$ of a shortest path to $v_j$. A shortest path to $v_t$ has length the least value of $d_j + w(v_j, v_t)$ for all $j = 0, 1, \ldots, t-1$. This is exactly what the algorithm computes in Step 2 at iteration $t$ of the loop. By the Principle of Mathematical Induction, the algorithm works as advertised for all $t$.

11. [BB] The arc $v_{p_t}v_t$ is the last on a shortest path which the algorithm has found from $v_0$ to $v_t$. Let $\mathcal{G}$ be the digraph which consists of precisely these arcs (and their end vertices). Vertex $v_0$ has indegree 0 since there is no arc $v_iv_0$ with $i > 0$. Every other vertex in $\mathcal{G}$ has indegree 1 since it is the last on a shortest directed path. Suppose $\mathcal{G}$ has $r$ vertices. Then it has $r - 1$ edges, one for each of its vertices other than $v_0$. By Theorem 12.1.6, it suffices to show that $\mathcal{G}$ is connected. This will certainly be the case if we show that every vertex of $\mathcal{G}$ is connected to $v_0$ by some path in $\mathcal{G}$. Suppose this is not the case and let $t$ be minimal with the property that there is no directed path from $v_0$ to $v_t$. Clearly $t > 0$. Then $\mathcal{G}$ contains an arc $v_{p_t}v_t$, the last on a directed path to $v_t$. Since $p_t < t$, there is a directed path from $v_0$ to $v_{p_t}$, and the arc $v_{p_t}v_t$ is directed $v_{p_t} \to v_t$. Thus there is a path from $v_0$ to $v_t$.

12. When $t = 1$, there is 1 addition, $d_0 + w(v_0, v_1)$. When $t > 1$, there are $t$ additions of the form $d_j + w(v_j, v_t)$, $t = 0, 1, \ldots, t-1$. Finding the minimum of these numbers requires $t - 1$ comparisons, giving $2t - 1$ additions/comparisons at the general step. The number of basic operations required is $1 + \sum_{t=1}^{n-1}(2t - 1) = 1 + 3 + 5 + \cdots + [2(n-1) - 1]^2 = (n-1)^2$. Bellman's algorithm is $\mathcal{O}(n^2)$.

13. [BB] It is sufficient simply to modify Step 1 by setting $d_k = 0$ and $d_i = \infty$ for $i < k$. (We could also modify Step 2 by starting the for loop at $t = k + 1$, but there is little change in efficiency unless $k$ is roughly the size of $n$.)

14. (a) i. [BB] Here are the distances $d_t$ from $v_1$ and the corresponding values of $p_t$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | 0 | $\infty$ | $\infty$ | $\infty$ | 6 | 4 | 2 |
| $p_t$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 |

ii. [BB] Here are the distances $d_t$ from $v_2$ and the corresponding values of $p_t$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | $\infty$ | 0 | $\infty$ | $\infty$ | 3 | 2 | 7 |
| $p_t$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | 2 | 2 | 5 |

(b) i. Here are the distances $d_t$ from $v_1$ and the corresponding values of $p_t$.

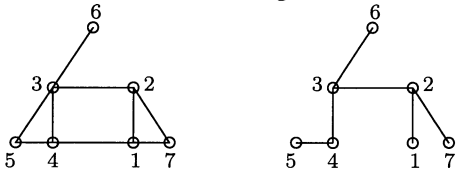
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | 0 | 1 | 2 | 3 | 3 | 4 | 4 | 4 |
| $p_t$ | $-1$ | $-1$ | 1 | 2 | 1 | 3 | 1 | 2 | 1 |

ii. Here are the distances $d_t$ from $v_2$ and the corresponding values of $p_t$.

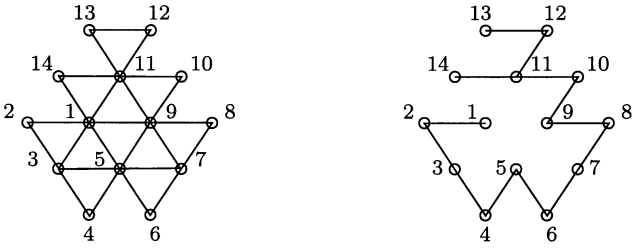
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | $\infty$ | 0 | 1 | 4 | 2 | 3 | 3 | 4 |
| $p_t$ | $-1$ | $-1$ | $-1$ | 2 | 3 | 3 | 2 | 2 | 7 |

## Exercises 12.5

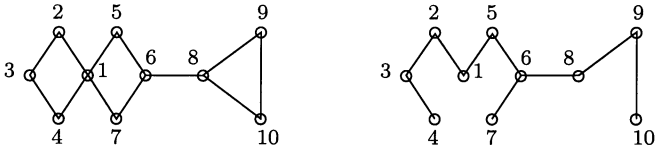
1. (a) [BB] The final backtracking is 7, 2, 1.



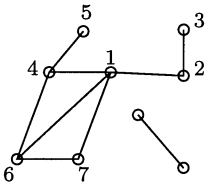

(b) The final backtracking is 14, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1.



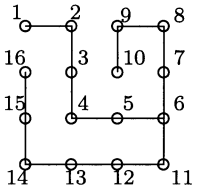

(c) The final backtracking is 10, 9, 8, 6, 5, 1



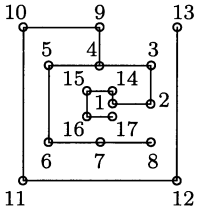

(d) The final backtracking is 7, 6, 4, 1. There is no spanning tree. The graph is not connected.



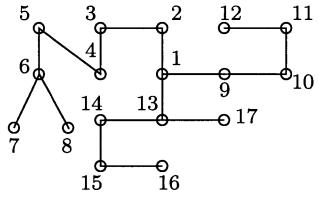

2. (a) [BB] The final backtracking is 16, 15, 14, 13, 12, 11, 6, 5, 4, 3, 2, 1.



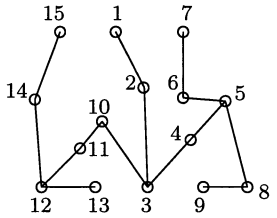

(b) The final backtracking is 17, 16, 15, 14, 1.



(c) The final backtracking is 17, 13, 1.




(d) The final backtracking is 15, 14, 12, 11, 10, 3, 2, 1.




3. We use the strong form of mathematical induction on $k$, a vertex label, the result being clearly true for $k = 1$. Suppose $k > 1$ and the result is true for all $\ell$ in the interval $1 \leq \ell < k$; thus, there is a path from 1 to $\ell$ which uses only edges required by the algorithm. We must prove the result for $k$: We must prove that there is a path from 1 to $k$ which uses only edges required by the algorithm. Vertex $k$ acquires its label because the algorithm finds this vertex unlabeled but adjacent to a previously labeled vertex $\ell$. Since at the time a vertex is labeled, the algorithm always picks the smallest unused label, we must have $\ell < k$. So, by the induction hypothesis, there is a path from 1 to $\ell$ using edges required by the algorithm; also, the algorithm uses the edge $\ell k$ to label $k$. Thus, there is a path from 1 to $k$ along edges which the algorithm requires.

4. (a) [BB] Consider the status of the algorithm at the time that $v$ is labeled. If there are no unlabeled vertices adjacent to $v$ at this stage, we have nothing to prove. So we assume that there are some. Let $\mathcal{G}'$ denote the subgraph of $\mathcal{G}$ consisting of those vertices as yet unlabeled which are connected to $v$ via a path of unlabeled vertices, and all the edges among these vertices. After $v$ is labeled, the algorithm moves to an unlabeled vertex adjacent to $v$ and hence into the subgraph $\mathcal{G}'$. It then either returns to $v$ or moves to another vertex in $\mathcal{G}'$ and gives it a label.

   Now the depth-first search algorithm covers any edge at most twice, once when it labels a vertex and once when it backtracks. Since there are only finitely many edges in $\mathcal{G}'$, the algorithm must eventually backtrack to $v$. If there remain any adjacent unlabeled vertices, the algorithm chooses one to label and moves again into $\mathcal{G}'$. As before, it must eventually backtrack to $v$. The process continues until all vertices adjacent to $v$ are labeled.

   (b) Let $v$ be any vertex in the connected graph $\mathcal{G}$. Let $1v_1v_2 \ldots v_kv$ be a path from 1 to $v$. By part (a), $v_1$ is labeled by the algorithm, hence, so is the adjacent vertex $v_2$. Since $v_i$ is adjacent to $v_{i-1}$, each $v_i$ is labeled [again by (a)] and, hence, $v$ is as well, because it is adjacent to $v_k$.

5. From Exercise 3 we know that there is a path from vertex 1 to each labeled vertex $k$ which uses only edges required by the algorithm. Exercise 4 says that if $\mathcal{G}$ is connected, every vertex gets labeled. It follows that if $\mathcal{G}$ is connected then the subgraph whose vertices are the $n$ vertices of $\mathcal{G}$ and whose edges are just those employed by the algorithm is connected. Moreover the subgraph has just $n - 1$ edges (so it's a tree) since the only edges used by the algorithm are those required at the time a vertex other than 1 is labeled.

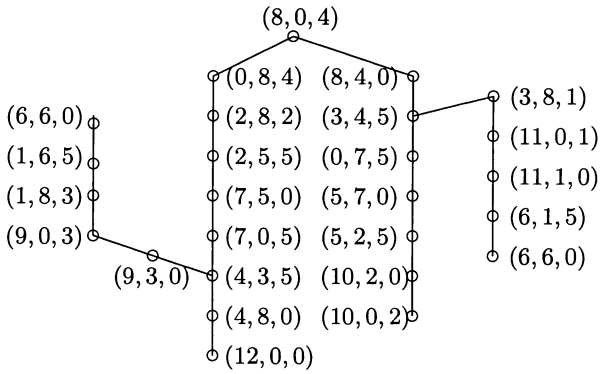
6. [BB] If $\mathcal{G}$ is , this spanning tree, , is not obtainable by depth-first search.

7. (a) [BB] Each time the procedure terminates, start it again with an unlabeled vertex. The number of connected components is equal to the number of times the procedure terminates.

   (b) Depth-first search assigns the labels $1, 2, 3, 4$ to the vertices $v_1$, $v_2$, $v_{11}$, $v_6$ respectively and then terminates. The next search assigns the labels $1, 2, 3$ to vertices $v_3, v_8, v_{13}$ respectively and terminates. The next search assigns the labels $1, 2, 3, 4, 5, 6$ to the vertices $v_4$, $v_5$, $v_{10}$, $v_7$, $v_9$, $v_{12}$ respectively and terminates. At this point, all the vertices have been labeled. There are three components.

8. The graph shows a spanning tree which results from one possible depth-first search applied to a graph for this problem (formed in the same way that the graph in the solution to Problem 5 was formed). Vertex $(a, b, c)$ indicates the state where there are $a$ ounces in the 12-ounce flagon, $b$ ounces in the eight-ounce flagon and $c$ ounces in the five-ounce flagon. Various graphs are possible, but all should show that the shortest method involves **seven** steps and is achieved with the following sequence of operations:

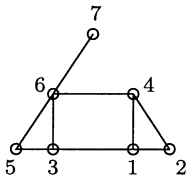
$$(12,0,0) \to (4,8,0) \to (4,3,5) \to (9,3,0) \to (9,0,3) \to (1,8,3) \to (1,6,5) \to (6,6,0)$$



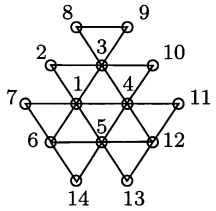

9. [BB] No. For instance, you can get from $(7, 1, 0)$ to $(3, 5, 0)$ in one step, but not back.

10. [BB] Breadth-first search is another way of systematically moving through all the vertices of a graph. Suppose a graph has $n$ vertices. The procedure assigns to these vertices labels from the set $\{1, 2, \ldots, n\}$. Assign label 1 to any vertex. Then label **all** vertices adjacent to vertex 1 with consecutive labels starting at 2. Then pass to vertex 2 and label **all** vertices adjacent to 2 which have not yet been labeled with consecutive labels from the unused members of $\{1, 2, \ldots, n\}$, always using the smallest available integer first. Then pass to vertex 3 and so on, at each stage moving to the next highest labeled vertex and labeling all adjacent vertices not yet labeled with the unused members of $\{1, 2, \ldots, n\}$, smallest unused integer first. Continue until some vertex acquires label $n$ or all vertices adjacent to the vertex with the highest label are already labeled.

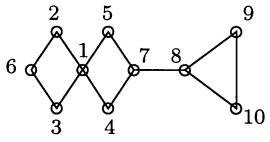
11. (a) [BB]



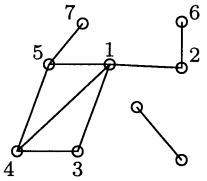

(b)




(c)




(d)




12. (a) The argument that we used to answer Exercise 3 works here as well. Use the strong form of mathematical induction on $k$, a vertex label, the result being clearly true for $k = 1$. Suppose $k > 1$ and the result is true for all $\ell$ in the interval $1 \leq \ell < k$; thus, there is a path from vertex 1 to the vertex labeled $\ell$ which uses only edges required by the algorithm. We must prove the result for $k$: We must prove that there is a path from 1 to the vertex with label $k$ which uses only edges required by the algorithm. Vertex $k$ was so labeled because the algorithm found this vertex unlabeled but adjacent to a previously labeled vertex $\ell$. At the time a vertex is labeled, the algorithm chooses the smallest unused label, so $\ell < k$. By the induction hypothesis, there is a path from 1 to $\ell$ using edges required by the algorithm; also, the algorithm uses the edge $\ell k$ to label vertex $k$. Thus, there is a path from 1 to $k$ along edges which the algorithm requires.

    (b) Let $v$ be any vertex of $\mathcal{G}$. Let $1v_1v_2 \ldots v_kv$ be a path from 1 to $v$. Since $v_1$ is adjacent to 1, $v_1$ was labeled at the first stage of the algorithm. Note also that whenever a vertex is labeled, all adjacent vertices are also labeled. Thus $v_2$, and hence $v_3$, and finally $v_k$ and $v$ are all labeled.

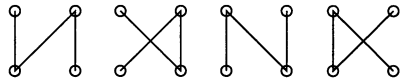
    (c) By parts (a) and (b), the subgraph $\mathcal{T}$ of $\mathcal{G}$ which consists of the $n$ vertices of $\mathcal{G}$ and the edges used by the algorithm is connected. The algorithm uses precisely $n - 1$ edges, one for each vertex except vertex 1. Since $\mathcal{T}$ is connected with $n$ vertices and $n - 1$ edges, it is a tree, and a spanning tree because it includes all vertices of $\mathcal{G}$.

13. We argue that the tree $\mathcal{T}$ produced by the breadth-first algorithm starting at vertex $u$ has the required property. We find it convenient to refer to the length of a shortest path between two vertices as the *distance* between them and to assign labels to vertices which reflect distances from $u$. To be precise, at the first step, we label all vertices adjacent to $u$ with a 1; then we label 2 all vertices adjacent to 1's which are not yet labelled; then we label 3 all vertices adjacent to 2 which are not yet labelled; we continue the process until all vertices are labelled. Now we use mathematical induction to show that the label $n$ on a vertex is its shortest distance from $u$ and that the breadth-first search tree contains a path of $n$ edges to $u$. First, if $v$ is a vertex labelled 1, then $v$ has distance 1 from $u$ and the tree contains the edge $uv$. So our claim is true for $n = 1$. Suppose $k \geq 1$, that any vertex labelled $k$ is distance $k$ from $u$, and that the breadth-first search tree contains a path of $k$ edges between this vertex and $u$. Suppose $v$ is labelled $k + 1$. By the way vertices were labeled, $v$ is adjacent to a vertex $w$ with label $k$. By the induction hypothesis, $w$ is distance $k$ from $u$ and there is a path from $w$ to $u$ which uses $k$ edges of $\mathcal{T}$. Since edge $wv$ is in $\mathcal{T}$, there is a path from $v$ to $u$ which uses $k+1$ edges and the distance from $v$ to $u$ is at most $k + 1$. We complete the induction argument and the proof by showing that the distance is precisely $k + 1$. Let $uv_1v_2 \cdots v_rv$ be any path from $u$ to $v$. By the way vertices were labeled, $v_1$ has label 1 and, if $v_i$ has label at most $i$, then $v_{i+1}$ has label at most $i + 1$. It follows that $v_r$ has label at most $r$ and $v$, being adjacent to $v_r$, at most $r + 1$. Since the label of $v$ was, in fact, $k + 1$, we have

$k+1 \leq r+1$. Thus the number of edges on the path $uv_1v_2 \cdots v_rv$ is $r \geq k$, which is precisely the fact we wanted to show.

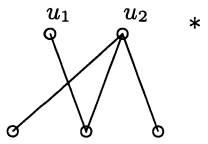
14. Let the bipartition sets of $\mathcal{K}_{2,n}$ be $\{u_1, u_2\}$ and $\{v_1, v_2, \ldots, v_n\}$. To show that all spanning trees are obtained as depth-first search spanning trees with $n = 2, 3, 4$ we could simply list the possible trees.

For example, the four spanning trees of $\mathcal{K}_{2,2}$ are shown to the right, and these are all depth-first search trees.

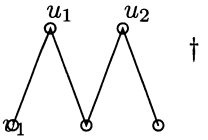


This approach is laborious, however, especially when $n = 4$, so we will give a general argument instead.

Any spanning tree of $\mathcal{K}_{2,n}$ must have some $v_i$ adjacent to both $u_1$ and $u_2$, and the remaining $v_j$ are partitioned into those adjacent to $u_1$ and those adjacent to $u_2$. If $n = 3$ and the two other vertices (different from $v_i$) are both adjacent to the same $u_i$, say $u_2$, we get a tree of the type shown. This tree comes from a depth-first search starting at $u_1$.



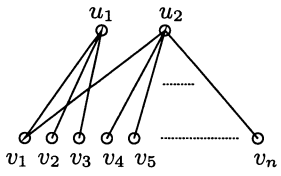

If $n = 3$ and the two other vertices are adjacent to different $u_i$, we obtain a tree of the type shown. Again, this comes from a depth-first search, starting at $v_1$.




If $n = 4$ and all remaining $v_i$ are adjacent to the same $u_j$, we get a case similar to that labeled $*$ previously, while if two are adjacent to $u_2$, say, while the third is adjacent to $u_1$, we get a case similar to †.

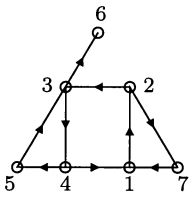
Finally, if $n \geq 5$, we will show that the tree shown to the right cannot be obtained from a depth-first search. If a depth-first search started at $u_1$, say, then it would first label a $v_i$, then $u_2$ and, from that point, it would never need to backtrack to $u_1$. Thus $u_1$ would have degree 1 in the tree.

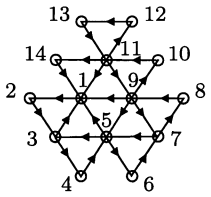
On the other hand, if a depth-first search started at a $v_i$, then it would first label $u_1$ or $u_2$, say $u_1$, and then continue to some $v_j$ and then $u_2$. Again, $u_1$ would never be encountered again, so this vertex would have degree 2 in the tree. In the example shown, both $u_1$ and $u_2$ have degree greater than 2. Hence this tree could not come from a depth-first search.




## Exercises 12.6

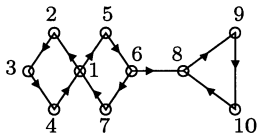
1. (a) [BB] Not strongly connected; edge 36 is a bridge as shown at the left below.



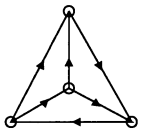
(b) Strongly connected. A strongly connected orientation is shown at the right above.

(c) Not strongly connected; edge 68 is a bridge.

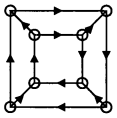



(d) Since the graph is not connected, a complete orientation cannot be obtained based on a depth-first search. Also, no orientation will be strongly connected.

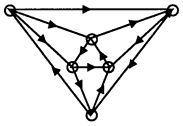
2.



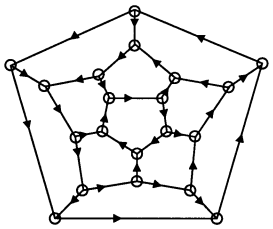
Tetrahedron [BB]



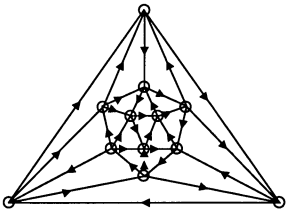
Cube



Octahedron

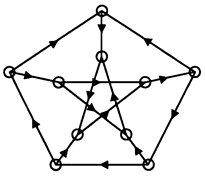


Dodecahedron



Icosahedron

3. (a) [BB] A strongly connected orientation for the Petersen graph is shown.



(b) A Hamiltonian graph always has a strongly connected orientation. Orient the edges of a Hamiltonian cycle in the direction of the cycle and other edges arbitrarily.

(c) The converse of (b) is false. The Petersen graph is not Hamiltonian.

4. Since the removal of any edge from a tree leaves the tree disconnected and since a tree with $n$ vertices has $n - 1$ edges, such a tree has $n - 1$ bridges.

5. [BB] No, it cannot. Every edge of an Eulerian graph is part of a circuit; the removal of an edge of a circuit certainly does not disconnect a graph.

6. [BB; ($\longrightarrow$)] ($\longrightarrow$) If $e$ is a bridge, we can let $u$ and $v$ be its ends. If there were a path from $u$ to $v$ which did not require $e$, together with $e$, we would have a circuit containing $e$. Since the deletion of an edge which is part of a circuit cannot disconnect a graph, we have a contradiction.

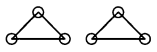
($\longleftarrow$) Conversely, if $e$ is not a bridge, then $e$ is part of a circuit, so there is a path between its ends which avoids $e$. This means that any path between vertices which requires $e$ can be replaced with a path which omits $e$, so vertices $u$ and $v$ as specified in the question could not exist.

7. (a) [BB] Choose a vertex $v$. For each vertex $u$ adjacent to $v$, orient the edge $uv$ in the direction $u \to v$.

   (b) The answer is yes. Since $\mathcal{G}$ is an Eulerian **graph**, there exists an Eulerian circuit. Now just orient the edges of this circuit in the direction of a walk along it.

8. [BB] Yes. To get from $a$ to $b$ in the new orientation, just find the path from $b$ to $a$ in the old orientation and follow it in reverse.

9. (a) [BB] False. The graph shown at the right cannot be given a strongly connected orientation, yet every edge is part of a circuit.



   (b) True. If $u$ has degree 1 and $u$ is incident with the edge $e$, then orienting $e$ away from $u$ makes it impossible to get to $u$ from another vertex and orienting $e$ towards $u$ makes it impossible to leave $u$. Either of these situations contradicts the fact that the graph is strongly connected.

   (c) True. By Theorem 10.1.4, a connected graph with no odd vertices is Eulerian. Orienting the edges along an Eulerian circuit in the direction of the circuit produces a strongly connected orientation.

10. (a) [BB] If there were a path $\mathcal{P}$ between $u$ and $v$ in $\mathcal{G} \setminus \{e\}$, then $\mathcal{P} \cup \{e\}$ would be a circuit, contradicting the fact that $e$ is a bridge.

    (b) If $w$ is in $\mathcal{G}_1$, then there would be a path connecting $w$ and $u$ in $\mathcal{G} \setminus \{e\}$. If $w$ were also in $\mathcal{G}_2$, then there would also be a path between $w$ and $v$ in $\mathcal{G} \setminus \{e\}$. Putting these paths together would give a walk between $u$ and $v$, contradicting (a).

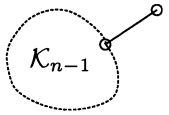
    (c) Suppose $\mathcal{H}$ is a connected subgraph of $\mathcal{G} \setminus \{e\}$ which contains $\mathcal{G}_1$. We prove all vertices of $\mathcal{H}$ are in $\mathcal{G}_1$ and hence $\mathcal{H} = \mathcal{G}_1$. Let $w$ be any vertex of $\mathcal{H}$. We know $u$ is in $\mathcal{G}_1$ and hence in $\mathcal{H}$. Since $\mathcal{H}$ is connected, there is a path in $\mathcal{H}$ between $w$ and $u$. Thus, $w$ is in $\mathcal{G}_1$.

11. [BB] Yes it can. Cutting all the bridges divides the city into connected components without bridges. Each component, therefore, has a strongly connected orientation (by Theorem 12.6.3), so it is possible to assign directions to the streets of each component which allows the possibility of (legal) travel between any two points. Now note that making each bridge a two-way street allows arbitrary travel between components.

12. To determine whether or not an edge $e$ is a bridge in a connected graph $\mathcal{G}$, we can use depth-first search to test $\mathcal{G} \setminus \{e\}$ for connectedness. Now depth-first search is $\mathcal{O}(n^2)$, so if our procedure for identifying bridges is to be $\mathcal{O}(n^3)$, we cannot afford to apply depth-first search to all possible graphs $\mathcal{G} \setminus \{e\}$. Notice, however, that if we apply depth-first search to the full graph $\mathcal{G}$, then any edge not used by this algorithm completes a circuit with those that are (since these comprise the edges of a spanning tree). Thus, an edge which is not identified by depth-first search is not a bridge. Otherwise put, the bridges of $\mathcal{G}$ are among the $n-1$ edges of the depth-first search spanning tree. Using depth-first searches on $\mathcal{G} \setminus \{e\}$ **just for these** $n-1$ **edges** $e$ gives a procedure which is $\mathcal{O}(n^3)$.

13. (a) We have only to prove that $\mathcal{G}$ has no bridges. Suppose then that $\mathcal{G}$ does have a bridge, say $e$. Then $\mathcal{G} \setminus \{e\}$ has two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$. Say $\mathcal{G}_1$ has $a$ vertices, so that $\mathcal{G}_2$ has $n-a$ vertices. Then $\mathcal{G}_1$ has at most $\binom{a}{2}$ edges and $\mathcal{G}_2$ has at most $\binom{n-a}{2}$ edges, so $\mathcal{G}$ has at most $\binom{a}{2} + \binom{n-a}{2} + 1$ edges. We see that this expression has a maximum value in the domain $1 \le a \le n-1$ of $\frac{1}{2}(n^2 - 3n + 4) = \frac{1}{2}(n-1)(n-2) + 1$, but this is in conflict with the given number of edges. Hence, $\mathcal{G}$ does not have a bridge.

(b) No. The graph at the right has $\frac{1}{2}(n-1)(n-2)+1$ edges but also a bridge.




14. (a) This is clear if the algorithm starts backtracking immediately after labeling $k$ since edge $\ell k$ was used to label $k$ and immediately afterwards in the backtracking. Suppose the algorithm continues to label vertices after $k$ (because there are unlabeled vertices adjacent to $k$ at the time label $k$ is used. The algorithm assigns $k+1$ to a vertex adjacent to $k$, and continues to assign labels $k+i$, $i>0$, until, at some point, after assigning $k+s$, $s>0$, the algorithm backtracks through vertices all of whose adjacent vertices have already been labeled to a vertex with label $t \leq k$. Let $t$ be the maximal such integer. The result follows if we can show that $t=k$ (since after the backtracking reaches $k$, it backtracks directly to $\ell$). But $t=k$ follows by a straightforward induction argument.

The result is clear if $s=1$. Assume $s>1$, $1 \leq r < s$ and the result is true for all such $r$. The vertex with label $k+s$ was labeled from a vertex $k+r$, $0 \leq r < s$. If $r=0$, the backtracking goes $k+s \to k \to \ell$. If $r>0$, the backtracking goes $k+s \to k+r \to k$ (by the induction hypothesis) $\to \ell$. In either case, $\ell k$ is used twice.

(b) **Solution 1.** Vertex $k$ gets labeled from $\ell$. By part (a), edge $\ell k$ is used twice by the algorithm, once when $k$ is labeled and once on a backtracking.

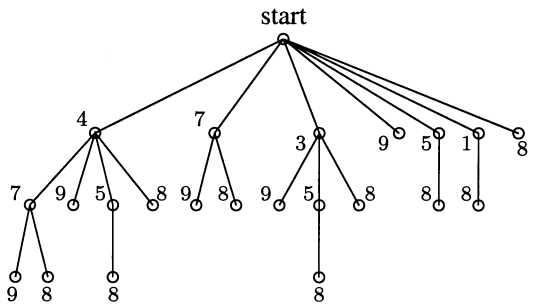
**Solution 2.** The result is clear for $k=1$, so assume $k>1$ and the result is true for all $\ell$, $1 \leq \ell < k$. Now $k$ received its label from $\ell$, with $1 \leq \ell < k$ and the algorithm backtracks to $\ell$ (by the induction hypothesis). Since $k\ell$ must be the last edge on such a backtracking $k$ also appears in this backtracking.

15. (a) [BB] Orient the depth first search spanning tree as in 12.6.4.

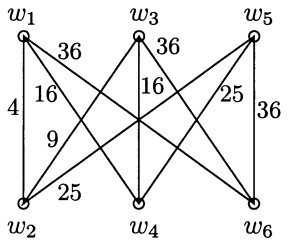
(b) [BB] This follows from the definitions.

# Chapter 12 Review

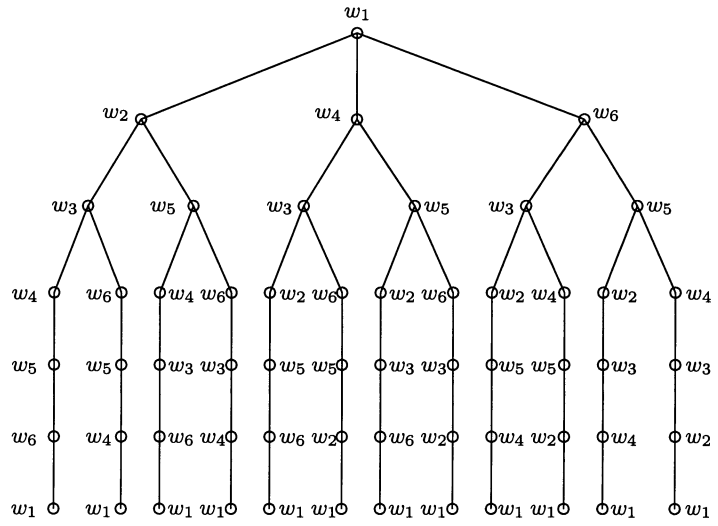
1. There are 22 such subsequences in all.



2. (a) This is $\mathcal{K}_{3,3}$.




(b) There are twelve Hamiltonian cycles. By checking all cases, we see that there are two pairs of solutions: $w_1w_2w_3w_4w_5w_6w_1$ (or its reverse) and $w_1w_2w_3w_6w_5w_4w_1$ (or its reverse).




3. (a) No. A tree with nine vertices has eight edges, and hence the sum of the degrees of its vertices should be 16. Here it is 18.

   (b) No. As in (a), the sum of the degrees of the vertices must be 16. Two vertices of degree 5 would leave a total of only 6 for the sum of the degrees of the remaining seven vertices. This forces some vertex to have degree 0, contradicting the fact that a tree is connected.

4. (i. $\rightarrow$ ii.) Assume $\mathcal{T}$ has an Eulerian trail. By Theorem 10.1.5, $\mathcal{T}$ must have exactly two vertices of odd degree. But Corollary 12.1.7 says $\mathcal{T}$ has at least two vertices of degree 1. It follows that $\mathcal{T}$ has exactly two vertices of degree 1.

   (ii. $\rightarrow$ iii.) Assume $\mathcal{T}$ has exactly two vertices of degree 1. Since $\mathcal{T}$ has $n-1$ edges, the sum of the degrees of its vertices is $2(n-1) = 2n-2$. This means that the remaining $n-2$ vertices must have degrees totalling $2n-4 = 2(n-2)$. But each of these has degree at least 2, and it follows that each must have degree exactly 2.

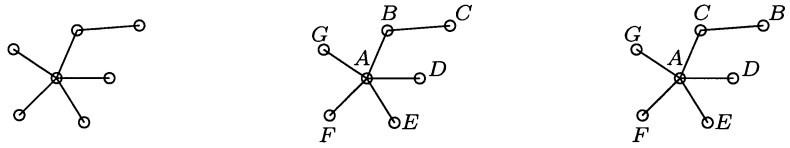
   (iii. $\rightarrow$ i.) Assume $\mathcal{T}$ has $n-2$ vertices of degree 2. Corollary 12.1.7 tells us that the other two vertices have degree 1. Since $\mathcal{T}$ is connected and has exactly two vertices of odd degree, Theorem 10.1.5 guarantees an Eulerian trail.

5. (a) $\mathcal{G}$ is connected because there is a path between any two vertices. If it contained a cycle, there would be two paths between any vertices of this cycle, contrary to fact. So $\mathcal{G}$ is acyclic and connected, hence a tree.

   (b) A tree with $n$ vertices has $n-1$ edges and $\sum \deg v = 2|\mathcal{E}|$. Here then, $33(1)+6(3)+(n-39)7 = 2(n-1)$. This gives $5n = 220$, so $n = 44$ and the number of vertices of degree 7 is $44-39 = 5$.

(c) Let there be $x$ vertices of degree 3 and $y$ of degree 5. Altogether, then, there are $n = 2 + x + y$ vertices, so $\sum \deg v = 2(n-1)$ says $2 + 3x + 5y = 2(1 + x + y)$. This simplifies to $x + 3y = 0$, so $x = y = 0$, giving the tree precisely two vertices, a contradiction.

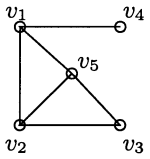
6. (a) Since the sum of the degrees of the vertices is 12, the remaining five vertices must all have degree 1. The vertex of degree 5 must be adjacent to the vertex of degree 2; otherwise, it would be adjacent to all five vertices of degree 1, leaving no vertex which could be adjacent to the other vertex of degree 2. Then the vertex of degree 5 will be adjacent to four of the vertices of degree 1, and the vertex of degree 2 will be adjacent to the other. The only possibility is shown on the left.




(b) Two labeled nonisomorphic trees are shown above.

7. $\mathcal{K}_8$ has $8^6$ spanning trees, by Theorem 12.2.3.

8. With vertices labeled as shown, the matrix defined in Kirchhoff's Theorem is




$$\begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{bmatrix}.$$

The value of the $(1,1)$ cofactor (expanding by cofactors along its third row) is

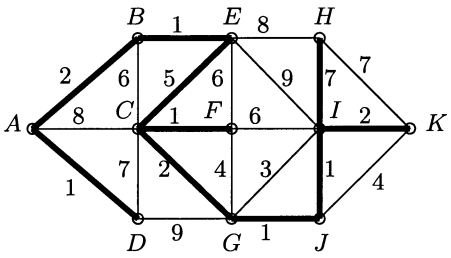
$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3(5) - (-1)(-4) - 1(3) = 8.$$

Thus, there are 8 spanning trees in all, as shown.

9. No. If our graph is a tree, it has exactly one spanning tree. Otherwise, let $\mathcal{T}$ be a spanning tree of a graph $\mathcal{G}$ with $\mathcal{T} \neq \mathcal{G}$. Thus $\mathcal{G}$ must contain an edge $e$ which is not in $\mathcal{T}$. Then $\mathcal{T} \cup \{e\}$ is not a tree and hence contains a circuit. Deleting any edge of this circuit will give a spanning tree for $\mathcal{G}$. Since any circuit contains at least three edges, we can produce at least three different spanning trees for $\mathcal{G}$.

10. Each spanning tree of $\mathcal{K}_{2,n}$ has $n+1$ edges. Hence the total number of edges used in all spanning trees is $(n+1)n2^{n-1}$. Now each of the $2n$ edges of $\mathcal{K}_{2,n}$ is equally likely to be included in a spanning tree. So the number containing a particular edge $e$ is $\dfrac{(n+1)n2^{n-1}}{2n} = (n+1)2^{n-2}$.

11. First select edges $AD$, $BE$, $CF$, $GJ$ and $IJ$ of weight 1. Then select edges $AB$, $CG$, $IK$ of weight 2. Since $GI$ would complete a circuit, we cannot select this edge. Since $JK$ and $FG$ would also complete circuits, we cannot select these either. We select next $CE$ of weight 5. Since $BC$ and $EF$ would complete circuits, we cannot select them, so we next select $HI$ or $HK$ of weight 7. We now have 10 edges, the necessary number for a spanning tree, so we are done. The weight of our tree is $5 + 6 + 5 + 7 = 23$.




12. Let us start at vertex $A$. We first select $AD$, then $AB$. Next $BE$ is selected and following that comes $EC$. Then come $CF$ and $CG$. Next we select $GJ$ and then $JI$. Then comes $IK$ and finally $IH$. The tree obtained by Prim is the same as that obtained by Kruskal in Exercise 11.

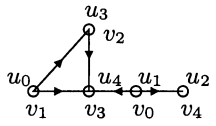
13. We use Kruskal's algorithm. The edge of least weight is $v_1v_2$, of weight 3 and the next edge chosen is $v_1v_3$, of weight 4. There are two edges of weight 5, $v_1v_4$ and $v_2v_3$, but $v_2v_3$ will complete a circuit. We choose $v_1v_4$. The process continues: if $v_1v_2, \ldots, v_1v_{i-1}$ have been chosen, the only available edge of least weight which will not complete a circuit is $v_1v_i$, of weight $i+1$. We obtain $v_1v_2, v_1v_3, \ldots, v_1v_n$ as the edges of a minimal spanning tree. The weight is

$$\begin{aligned}(1+2)+(1+3)+\cdots+(1+n) &= (1+2+3+\cdots+n)+(n-2)\\ &= \frac{n(n+1)}{2}+(n-2) = \frac{(n+4)(n-1)}{2}.\end{aligned}$$

14. (a) The graph is not acyclic: $J, F, G, I, D, C, J$ is a cycle.
    (b) The graph is acyclic: a canonical labeling is $I, J, F, B, N, H, C, A, D, E, L, K, G, M$.

15. (a) The digraph is now acyclic: a canonical labeling of vertices is $E, B, G, I, D, H, A, F, C, J$.
    (b) The digraph has cycle $G, F, J, N, H, D, E, L, G$.

16. The answer is yes. At the right, we show two different canonical orderings for a digraph.




17. A canonical ordering is $J, K, D, E, C, F, G, B, A, H, I$.

    (a) The table shows the lengths of shortest paths and predecessor vertices when root is $v_0$.

| | $J$ | $K$ | $D$ | $E$ | $C$ | $F$ | $G$ | $B$ | $A$ | $H$ | $I$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | 0 | 2 | 3 | 3 | 3 | 4 | 5 | 5 | 1 | 2 | 6 |
| $p_t$ | $-1$ | $J$ | $K$ | $K$ | $K$ | $K$ | $J$ | $C$ | $J$ | $A$ | $H$ |

    (b) Here are the lengths and predecessor vertices when root is $v_2$.

| | $J$ | $K$ | $D$ | $E$ | $C$ | $F$ | $G$ | $B$ | $A$ | $H$ | $I$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | $\infty$ | 0 | 1 | 4 | 3 | 4 | 6 | 8 | 4 | 8 |
| $p_t$ | $-1$ | $-1$ | $-1$ | $D$ | $D$ | $E$ | $F$ | $C$ | $B$ | $F$ | $H$ |

18. A canonical ordering is $L, C, K, J, I, H, G, M, D, E, F, A, B$.

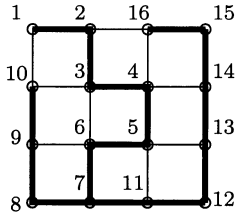
    (a) The table shows the lengths of shortest paths and predecessor vertices when root is $v_0$.

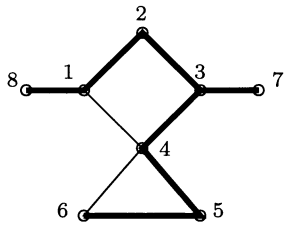
| | $L$ | $C$ | $K$ | $J$ | $I$ | $H$ | $G$ | $M$ | $D$ | $E$ | $F$ | $A$ | $B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | 0 | 1 | 2 | 3 | 1 | 1 | 2 | 3 | 2 | 3 | 5 | 4 | 3 |
| $p_t$ | $-1$ | $L$ | $C$ | $K$ | $L$ | $L$ | $H$ | $L$ | $L$ | $D$ | $M$ | $K$ | $C$ |

    (b) Here are the lengths and predecessor vertices when root is $v_1$.

| | $L$ | $C$ | $K$ | $J$ | $I$ | $H$ | $G$ | $M$ | $D$ | $E$ | $F$ | $A$ | $B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | $\infty$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 3 | 4 | 6 | 3 | 2 |
| $p_t$ | $-1$ | $-1$ | $C$ | $K$ | $K$ | $I$ | $H$ | $H$ | $C$ | $D$ | $E$ | $K$ | $C$ |

19. (a) The final backtracking is $16, 15, 14, 13, 12, 11, 7, 6, 5, 4, 3, 2, 1$. The edges on the resulting spanning tree are darkened.

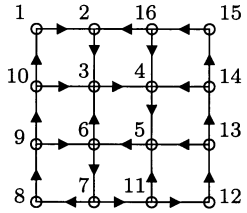


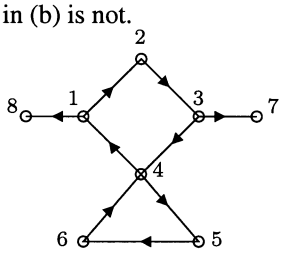

    (b) The final backtracking is $8, 1$. The edges on the resulting spanning tree are darkened.

20. The graph in (a) is strongly connected; the one in (b) is not.

(a)




(b)




21. If $v$ is incident with $e$, then we can start a depth-first search at $v$ and select edge $e$ first. So $v$ and $e$ would be in the eventual spanning tree. So assume $e = uw$, where neither of the end vertices $u$ and $w$ is equal to $v$. Using Dijkstra's algorithm, find a shortest path $v = v_1, v_2, \ldots, v_n = u$ from $v$ to $u$. Since the vertices on this path are all distinct, they could have been obtained as the first steps of a depth-first search, in the order given. If $v_{n-1} = w$, the edge $v_{n-1}v_n$ is $e$ and we are done. On the other hand, if $v_{n-1} \neq w$, then none of the $v_i$ can equal $w$: if $v_i = w$, then replacing $v_{i+1}$ by $u$ would give a shorter path from $v$ to $u$. Hence the next vertex chosen in a depth-first search (after $v_n$) could be $w$, and then $uw = v_n v_{n+1} = e$ would be an edge of the corresponding spanning tree.

22. These are exactly those graphs which have a Hamiltonian path from $v$ to some other vertex. Clearly, if a graph has a Hamiltonian path, then we can follow the path in the depth-first-search labeling. Conversely, if you have labeled all vertices in the way described, then the path you have followed repeats no vertices and so is Hamiltonian.

23. Let $e = uv$. If $\mathcal{T}$ is a spanning tree for $\mathcal{G}$, then $u$ and $v$ must be connected by some path in $\mathcal{T}$. But the only path between $u$ and $v$ is the edge $e$, since $e$ is a bridge. Hence $e$ is in $\mathcal{T}$.

24. ($\rightarrow$) Let $\mathcal{G}$ be a tree. Then $\mathcal{G}$ is connected, by definition. Suppose $e$ is an edge of $\mathcal{G}$. We want to show that $e$ is a bridge, that is, that $\mathcal{G} \setminus \{e\}$ is not connected. Suppose to the contrary that $\mathcal{G} \setminus \{e\}$ is connected. Then there is a path $\mathcal{P}$ in $\mathcal{G} \setminus \{e\}$ from one end vertex of $e$ to the other. This path, followed by $e$, is a circuit in $\mathcal{G}$, contradicting the fact that $\mathcal{G}$ is a tree.

    ($\leftarrow$) Suppose $\mathcal{G}$ is a connected graph and every edge of $\mathcal{G}$ is a bridge. To show $\mathcal{G}$ is a tree, we have only to show that $\mathcal{G}$ contains no cycles. Let $\mathcal{C}$ be a cycle and let $e$ be an edge of $\mathcal{C}$. The graph $\mathcal{G} \setminus \{e\}$ is connected because one can pass from one end vertex of $e$ to the other by following the edges of $\mathcal{C}$ other than $e$. So $e$ is not a bridge, a contradiction.

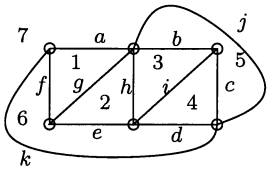
25. No, it is not possible. Let $uv$ be an edge of the graph. Since vertices $u$ and $v$ are part of a Hamiltonian cycle, there are two paths between $u$ and $v$ (either way around the cycle), so deleting $uv$ cannot disconnect the graph.

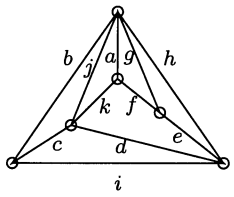
26. (a) No. For example, is not strongly connected.

    (b) Yes. The underlying graph for a tournament is a complete graph. Since $\mathcal{K}_n$ is connected and has no bridges (there are always at least two paths between vertices $u$ and $v$), it has a strongly connected orientation by Theorem 12.6.3.

## Exercises 13.1

1. (a) [BB] We draw the graph quickly as a planar graph and then, after some thinking, as a plane with straight edges.

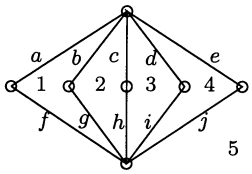



    (b) [BB] There are seven regions, numbered $1, 2, \ldots, 7$, with boundaries $afg$, $ghe$, $hbi$, $icd$, $bjc$, $fedk$, and $ajk$, respectively.

    (c) [BB] $E = 11$, $V = 6$, $R = 7$, $N = 22$; so $V - E + R = 6 - 11 + 7 = 2$; $N = 22 \leq 22 = 2E$ and $E = 11 \leq 12 = 3V - 6$.

2. (a)



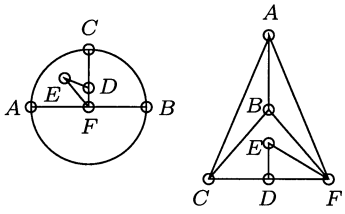

    (b) There are five regions, numbered $1, 2, 3, 4, 5$, with boundaries $abgf$, $bchg$, $cdih$, $deji$ and $aejf$.

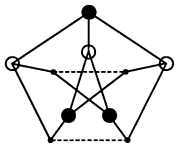
    (c) $E = 10$, $V = 7$, $R = 5$, $N = 20$; so, $V - E + R = 7 - 10 + 5 = 2$, $N = 20 \leq 20 = 2E$ and $E = 10 \leq 15 = 3V - 6$.

3. [BB]

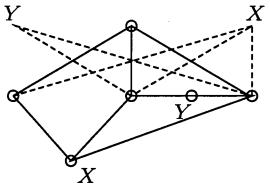
| Solid | $V$ | $E$ | $F$ | $V-E+F$ |
|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | 2 |
| cube | 8 | 12 | 6 | 2 |
| octahedron | 6 | 12 | 8 | 2 |
| dodecahedron | 20 | 30 | 12 | 2 |
| icosahedron | 12 | 30 | 20 | 2 |

4. The graph on the left is planar; it is redrawn, and as a plane graph with straight edges, on the left. The Petersen graph is not planar; it contains a subgraph homeomorphic to $\mathcal{K}_{3,3}$, as shown at the right.



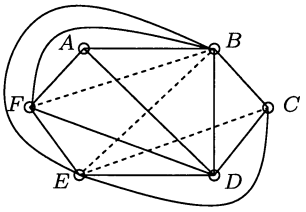



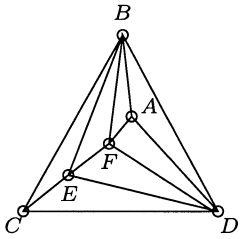
5. (a) [BB] This is planar. Here it is drawn as a plane graph (with straight edges).



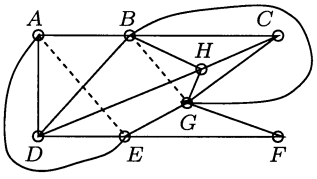

(b) This is not planar. There is a subgraph homeomorphic to $\mathcal{K}_5$ obtained by deleting the vertex of degree 3.

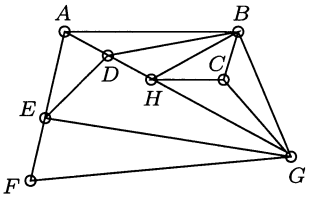
(c) This is planar.



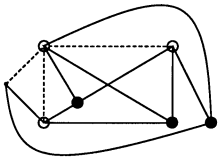




(d) This is planar.

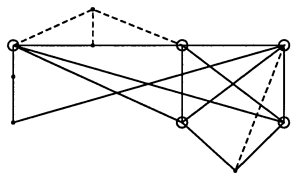



(e) [BB] This is not planar. There is a subgraph homeomorphic to $\mathcal{K}_{3,3}$ as shown.



(f) This is not planar. It has a subgraph homeomorphic to $\mathcal{K}_5$ as shown.



6. Let $N$ be as described in the proof of Corollary 13.1.3. Then $5R \leq N \leq 2E$, so $5R \leq 2E$. Thus, $5(E - V + 2) \leq 2E$, giving $3E \leq 5V - 10$.

7. [BB] We know from Theorem 13.1.4 that $E \leq 3V - 6$. Substituting in $E = V + R - 2$, we obtain $V + R - 2 \leq 3V - 6$, or $R \leq 2V - 4$, as required.

8. (a) [BB] $E = 3 = 3(3) - 6 = 3V - 6$.

   (b) Look at the proof of Theorem 13.1.4. We can only have $E = 3V - 6$ if $3R = N = 2E$, and $3R = N$ means that every region is a triangle.

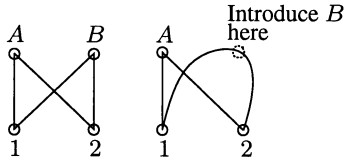
9. (a) If $\mathcal{G}$ has three vertices, it must have at least two edges since it's connected. It can't have three edges since it is not a triangle. Thus, $E = 2 \leq 2 = 2V - 4$ as required. If $\mathcal{G}$ has four vertices, it has at least three edges. Since it contains no triangles, it is either a tree or a 4-cycle. In either case, $E \leq 4 = 2V - 4$ as required.

   Now assume $V \geq 5$, so $E \geq 4$. As in the proof of Theorem 13.1.4, count the edges on the boundary of each region and sum over all regions. Denoting the sum $N$, we have $N \leq 2E$. Since the graph contains no triangles, there are at least four edges on the boundary of each region. Thus, $N \geq 4R$ and $4R \leq 2E$; thus, $2R \leq E$. Hence, $4 = 2V - 2E + 2R \leq 2V - E$, so $E \leq 2V - 4$ as required.

   (b) Recall that a bipartite graph contains no triangles so this question follows immediately from (a).

10. (a) [BB] $\mathcal{K}_n$ is planar if and only if $n \leq 4$.

    (b) $\mathcal{K}_{m,n}$ is planar if and only if $m \leq 2$ or $n \leq 2$.

11. [BB] 


12. Consider the case of $\mathcal{K}_5$. (The other case is identical.) The common graph specified by Definition 13.1.7 has to be $\mathcal{K}_5$ itself for added vertices have degree 2 and $\mathcal{K}_5$ has none.

13. Let $\mathcal{G}_1$ be obtained from $\mathcal{G}$ (with $V$ vertices and $E$ edges) by adding $k$ vertices to edges. Then $V_1 = V + k$ and $E_1 = E + k$ (since there is one additional edge for each vertex added to an edge). Similarly, if $\mathcal{G}_2$ is obtained from $\mathcal{G}$ by adding $\ell$ vertices, then $V_2 = V + \ell$ and $E_2 = E + \ell$. The result now follows easily.

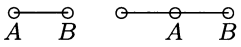
14. (a) [BB] Assume the result is not true. Then there is some counterexample $\mathcal{G}$ and subgraph $\mathcal{H}$; that is, for these graphs, $E_1 - V_1 < E_2 - V_2$. (In particular, $V_1 \neq V_2$.) Choose $\mathcal{H}$ such that $V_1 - V_2$ is as small as possible. Since $\mathcal{G}$ is connected, we can find a vertex $v$ which is in $\mathcal{G}$, but not $\mathcal{H}$, which is joined to some vertex in $\mathcal{H}$. Let $\mathcal{K}$ be that subgraph of $\mathcal{G}$ consisting of $\mathcal{H}$, $v$ and all edges joining $v$ to vertices in $\mathcal{H}$. Letting $V_3$ and $E_3$ denote the numbers of vertices and edges, respectively, in $\mathcal{K}$,

we have $V_3 = V_2 + 1$, while $E_3 \geq E_2 + 1$. Hence, $E_2 - V_2 \leq E_3 - V_3$ and so $E_1 - V_1 < E_3 - V_3$. Thus, $\mathcal{G}$ and its subgraph $\mathcal{K}$ provide another counterexample, but this contradicts the minimality of $V_1 - V_2$ since $V_1 - V_3 < V_1 - V_2$.

(b) Say $\mathcal{G}$ is not planar. Then $\mathcal{G}$ has a subgraph $\mathcal{H}$ with $V_1$ vertices and $E_1$ edges which is homeomorphic to $\mathcal{K}_{3,3}$ or $\mathcal{K}_5$. For $\mathcal{K}_{3,3}$, $E - V = 9 - 6 = 3$. For $\mathcal{K}_5$, $E - V = 10 - 5 = 5$. By Exercises 12 and 13, we conclude that $E_1 - V_1 = 3$ or 5, but we are given that $E_1 - V_1 \leq 2$, contradicting part (a).

(c) The graph need not be planar if $E = V + 3$ since $\mathcal{K}_{3,3}$ has six vertices and nine edges.

15. [BB] Yes. An example is shown to the right. The graphs are homeomorphic since the one on the right is obtainable from the other by adding a vertex of degree 2.

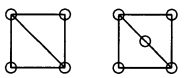



16. Doubtlessly Poland's most famous mathematician, Kazimierz Kuratowski was born in 1896 in Warsaw where he died on the 18th of June, 1980. Renowned for his lectures, Kuratowski's research was in topology and set theory. His classification of nonplanar graphs remains of fundamental importance in graph theory. Stefan Banach, Stanislaw Ulam and John von Neumann were amongst his collaborators.

    Intending a career as an engineer, he moved to Scotland in 1913 and started studies at the University of Glasgow, because of the reputation of its engineering school. He returned to Poland the next summer and with the outbreak of World War I found it impossible to return to Scotland. This was good for mathematics to which Kuratowski then turned his energy and talent. He received his PhD from the University of Warsaw in 1921 and was appointed as a professor at the Technical University of Lvov in 1927. In 1934, he became professor of mathematics at the University of Warsaw. He risked his life teaching at an illegal underground university during the Second World War and afterwards, as President of the Polish Mathematical Society, was a leader in rebuilding an educational system which had been devastated during the war years. For 19 years, he was Director of the Mathematical Institute of the Polish Academy of Sciences. He served on the editorial board of *Fundamenta Mathematicae* and was editor-in-chief from 1952 until his death. He made many visits to western Europe, Britain, the USA, and Canada.

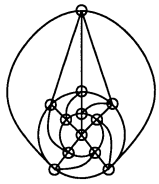
17. (a) True. Since $\mathcal{G}$ and $\mathcal{H}$ are homeomorphic, they can both be obtained from the same graph $\mathcal{K}$ by inserting vertices of degree 2. Let $w_1, w_2, \ldots, w_m$ be an Eulerian circuit in $\mathcal{G}$. By deleting those vertices which were inserted to get $\mathcal{G}$ from $\mathcal{K}$, we get an Eulerian circuit in $\mathcal{K}$ of the form $w_{i_1}, w_{i_2}, \ldots, w_{i_t}$. Now adding those vertices needed to get $\mathcal{H}$ from $\mathcal{K}$, we obtain an Eulerian circuit in $\mathcal{H}$.

    (b) [BB] False, as shown by the homeomorphic graphs at the right. The graph on the left is Hamiltonian, but the one on the right is not.



18. (a) If this result is true for connected planar graphs, it is true for planar graphs in general since any connected component would contain at least 12 vertices. Thus, it is sufficient to prove the result for connected graphs. Suppose $\mathcal{G}$ is a connected planar graph with every vertex of degree $d \geq 5$. Since $\sum \deg v_i = 2E$, then $2E \geq 5V$. Since $E \leq 3V - 6$, we have $6V - 12 \geq 2E \geq 5V$ and $V \geq 12$.

(b) The planar graph shown at the right, which is isomorphic to the graph of the icosahedron (see Exercise 13, Section 10.2), has twelve vertices, each of degree 5.



19. (a) [BB] Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ be the connected components of $\mathcal{G}$. Since $\mathcal{G}_i$ has at least three vertices, we have $E_i \leq 3V_i - 6$. Hence, $\sum E_i \leq 3\sum V_i - 6n$, so $E \leq 3V - 6n$ as required.

(b) Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ be the connected components of $\mathcal{G}$. If $\mathcal{G}_i$ has at least three vertices, then $E_i \leq 3V_i - 6$. If $\mathcal{G}_i$ has two vertices, then $E_i = 1$, so $E_i = 3V_i - 5$. If $\mathcal{G}_i$ has one vertex, then $E_i = 0$ and $E_i = 3V_i - 3$. In all cases, $E_i \leq 3V_i - 3$, so $E = \sum E_i \leq 3\sum V_i - 3n = 3V - 3n$. Finally, note that if $n \geq 2$, then $3V - 3n \leq 3n - 6$ so Theorem 13.1.4 holds while, if $n = 1$, the graph is connected; we established $E \leq 3V - 6$ for such a graph in the text.

20. We may assume in all parts of this question that $\mathcal{G}$ is connected.

(a) [BB] Say there is only one vertex of degree at most 5. Then $\sum \deg v_i \geq 6(V-1) = 6V - 6$, contradicting $\sum \deg v_i = 2E \leq 6V - 12$.

(b) Say there are only two vertices of degree at most 5. Then $\sum \deg v_i \geq 6(V-2) + 2$ (the "+2" because $\mathcal{G}$ is connected, so there are no vertices of degree 0). This says that $\sum \deg v_i \geq 6V - 10$, again contradicting $\sum \deg v_i \leq 6V - 12$.

(c) Assume there are just three vertices of degree at most 5. Then $\sum \deg v_i \geq 6(V-3) + 3$ (the "+3" because $\mathcal{G}$ is connected and so there can be no vertices of degree 0). Since $\sum \deg v_i = 2E$ must be even and $\sum \deg v_i = 2E \leq 6V - 12$, there are two possibilities to consider; namely, $\sum \deg v_i = 6V - 14$ and $\sum \deg v_i = 6V - 12$.

**Case 1:** $\sum \deg v_i = 2E = 6V - 14$.

In this case, note that $\mathcal{G}$ must have a vertex of degree 1; otherwise, $\sum \deg v_i \geq 6(V-3)+3(2) = 6V - 12$. The graph $\mathcal{G}^*$ obtained by deleting this vertex (and the single edge with which it is incident) is still connected and planar, so applying Theorem 13.1.4 to $\mathcal{G}^*$, we obtain $\sum \deg v_i \leq 6(V-1) - 12 = 6V - 18$. But $\mathcal{G}^*$ is obtained from $\mathcal{G}$ by removing one edge and one vertex, so for $\mathcal{G}^*$, we have $\sum \deg v_i = (6V - 14) - 2 = 6V - 16$, giving a contradiction.

**Case 2:** $\sum \deg v_i = 2E = 6V - 12$.

If there were a vertex of degree 1 in $\mathcal{G}$, the same argument as in Case 1 would give a contradiction. (Alternatively, Exercise 8(b) says no such vertex exists.) Thus, there is a vertex of degree 2. Let $\mathcal{G}^*$ be the graph obtained from $\mathcal{G}$ by deleting this vertex (and the two edges incident with it). Then $\mathcal{G}^*$ is planar, so by Theorem 13.1.4 if $\mathcal{G}^*$ is connected and otherwise Exercise 19, we have $\sum \deg v_i \leq 6(V-1) - 12 = 6V - 18$ in $\mathcal{G}^*$. But $\mathcal{G}^*$ is obtained from $\mathcal{G}$ by removing two edges and one vertex, so $\sum \deg v_i = (6V - 12) - 4 = 16$, giving again a contradiction.

21. (a) [BB] By Theorem 13.1.4, $E \leq 3V - 6$, so $E \leq 3(20) - 6 = 54$.

(b) Let $\mathcal{H}$ be the graph obtained from $\mathcal{G}$ by removing all vertices of degree 1. Then $\mathcal{H}$ is still planar and connected (removing vertices of degree 1 destroys only paths with these as end vertices). Hence $\mathcal{H}$ has at most $3(13) - 6 = 33$ edges. But $\mathcal{H}$ has exactly seven edges fewer than $\mathcal{G}$, so $\mathcal{G}$ has at most 40 edges.

22. (a) [BB] Say at most one region has at most five edges on its boundary. Then, with $N$ as in the proof of Corollary 13.1.3, $N \geq 6(R-1)$. But $N \leq 2E$, so $2E \geq 6R-6$, $3R \leq E+3$. Since $V-E+R=2$, $6=3V-3E+3R \leq 3V-2E+3$; that is, $2E \leq 3V-3$. But $2E=\sum \deg v_i \geq 3V$ by assumption, and this is a contradiction.

    (b) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are any two connected components of a planar graph, then each is planar, so each has at least two regions with at most five edges on the boundary, by part (a). Therefore, each has at least one **interior** region with at most five edges on the boundary. The result now follows because any interior region of a component is an interior region of the entire graph.

23. The given polygon has $n$ vertices and $n$ edges, so including possible new vertices introduced by triangulation, the graph produced by triangulation has $V \geq n$ vertices. This graph is a plane graph, so $V-E+R=2$. In it, every interior region is bounded by three edges while the exterior region is bounded by $n$ edges. Since every edge lies on the boundary of exactly two regions, counting the edges of each region and summing, we obtain $2E=3(R-1)+n$; that is, $3R+n=2E+3$. Since $V-E+R=2$, we have $3V-3E+(2E+3-n)=6$ and hence $E=3V-n-3 \geq 3n-n-3=2n-3$. Since there are $n$ exterior edges (those of the original polygon), triangulation must have required at least $n-3$ edges, as required.

24. [BB] Letting $x$ denote the number of connected components of $\mathcal{G}$, we have $V-E+R=1+x$.

    **Proof.** For each component $\mathcal{C}$, $V_\mathcal{C}-E_\mathcal{C}+R_\mathcal{C}=2$. Adding, we get $\sum V_\mathcal{C}-\sum E_\mathcal{C}+\sum R_\mathcal{C}=2x$. We have $\sum V_\mathcal{C}=V$ and $\sum E_\mathcal{C}=E$, but $\sum R_\mathcal{C}=R+(x-1)$ since the exterior region is common to all components. Thus, $V-E+R+x-1=2x$, $V-E+R=x+1$. ■

25. Let $d$ be the degree of each vertex and $a$ the number of edges on the boundary of each face. Note that $d \geq 3$ and $a \geq 3$. Summing the degrees of the vertices, we get $dV=2E$, and summing the edges on the boundary of each face, we get $aF=2E$ because each edge is the boundary of two faces. Substituting $V=\frac{2}{d}E$, $F=\frac{2}{a}E$ in $V-E+F=2$, we obtain $\frac{2}{d}E-E+\frac{2}{a}E=2$, then $E(\frac{2}{d}-1+\frac{2}{a})=2$, so $\frac{2}{d}-1+\frac{2}{a}>0$; that is, $\frac{1}{a}+\frac{1}{d}>\frac{1}{2}$. This equation has very few solutions for $a,d \geq 3$. In fact, its only solutions are $a=3, d=3,4,5$, $a=4, d=3$, $a=5, d=3$ and these correspond to the five platonic solids as shown:

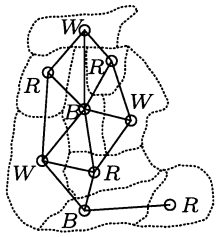
| | |
|---|---|
| $a=3, d=3$ | tetrahedron |
| $a=3, d=4$ | octahedron |
| $a=3, d=5$ | icosahedron |
| $a=4, d=3$ | cube |
| $a=5, d=3$ | dodecahedron |

## Exercises 13.2

1. The solution of the "Four-Color Problem" described in this section was certainly a major mathematical accomplishment. Every plane graph can be colored with at most four colors in such a way that bordering "countries" have different colors.

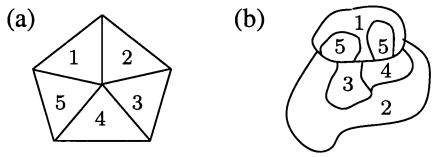
   Perhaps the most important mathematical accomplishment of the twentieth century was Andrew Wiles's solution to Fermat's Last Theorem, discussed in Chapter 4. For any positive integer $n>2$, there are no solutions to the equation $x^n+y^n=z^n$ in the nonzero integers.

2. (a) [BB] We show the graph superimposed over the given map. Since this graph contains triangles, at least three colors are necessary. A 3-coloring is shown, so the chromatic number is 3.

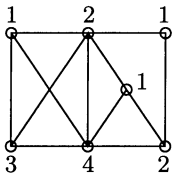



(b) [BB] False! The Four-Color Theorem says that the chromatic number of a planar graph is **at most** 4. The planar graph in (a) has chromatic number 3.

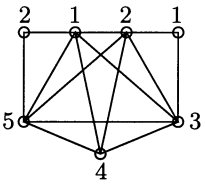
3. [BB] In neither case, will four colors necessarily suffice as the pictures to the right demonstrate.



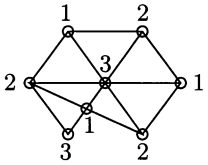

4. (a) A 4-coloring is shown; hence, $\chi(\mathcal{G}) \leq 4$. Since $\mathcal{K}_4$ is a subgraph, $\chi(\mathcal{G}) = 4$.



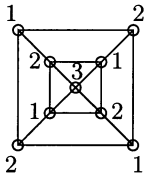

(b) [BB] A 5-coloring is shown; hence, $\chi(\mathcal{G}) \leq 5$. Since $\mathcal{K}_5$ is a subgraph, $\chi(\mathcal{G}) = 5$.



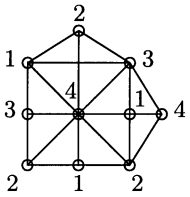

(c) A 3-coloring is shown; hence, $\chi(\mathcal{G}) \leq 3$. Since $\mathcal{K}_3$ is a subgraph, $\chi(\mathcal{G}) = 3$.



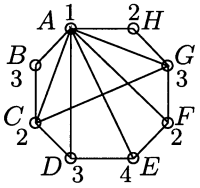

(d) A 3-coloring is shown; hence, $\chi(\mathcal{G}) \leq 3$. Since $\mathcal{G}$ contains $\mathcal{K}_3$ as a subgraph, $\chi(\mathcal{G}) = 3$.



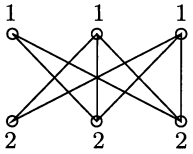

(e) [BB] A 4-coloring is shown; hence, $\chi(\mathcal{G}) \leq 4$. Since $\mathcal{G}$ contains $\mathcal{K}_4$ as a subgraph, $\chi(\mathcal{G}) = 4$.



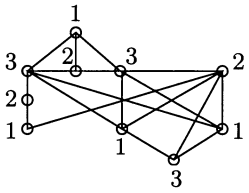
(f) A 4-coloring is shown and, in fact, $\chi(\mathcal{G}) = 14$. Since $ACG$ is a triangle, these vertices must have different colors. Since $ACD$ and $AFG$ are triangles, $D$ and $F$ must have different colors. But now a fourth color is needed for $E$, which is adjacent to $A$, $D$ and $F$.



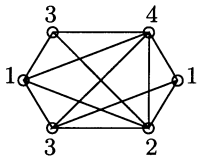

5. (a) This graph is bipartite, so $\chi(\mathcal{G}) \leq 2$. Since the graph has an edge, $\chi(\mathcal{G}) = 2$.



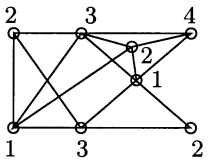

(b) A 3-coloring is shown. Since $\mathcal{G}$ contains triangles, $\chi(\mathcal{G}) = 3$.



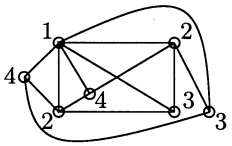

(c) A 4-coloring is shown. Since $\mathcal{G}$ contains $\mathcal{K}_4$ as a subgraph, $\chi(\mathcal{G}) = 4$.



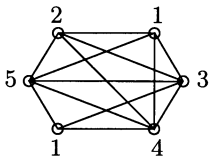

(d) A 4-coloring is shown. Since $\mathcal{G}$ contains $\mathcal{K}_4$ as a subgraph, $\chi(\mathcal{G}) = 4$.



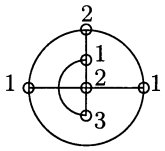

(e) A 4-coloring is shown. To see that there is no 3-coloring, start with three colors on a triangle and you will see that eventually a fourth color is required.

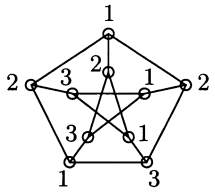



(f) A 5-coloring is shown. Since $\mathcal{G}$ contains $\mathcal{K}_5$ as a subgraph, $\chi(\mathcal{G}) = 5$.




6. (a) [BB] The graph $\mathcal{G}_1$ on the left has a 3-coloring, as shown. Since it contains a triangle, $\chi(\mathcal{G}_1) = 3$. The graph $\mathcal{G}_2$ on the right has a 3-coloring, as shown. By trying to label alternately the vertices of the outer pentagon, we see that two colors will not suffice, so $\chi(\mathcal{G}_2) = 3$ too.




(b) [BB] The converse of the Four-Color Theorem states that if the chromatic number of a graph is at most four, then the graph is planar. This result is false. The Petersen graph is not planar (Exercise 4 of Section 13.1), but, as we saw in part (a), $\chi = 3$.

7. (a) [BB] Yes, a tree is planar and we can prove this by induction on $n$, the number of vertices. Certainly a tree with one vertex is planar. Then, given a tree with $n$ vertices, removing a vertex of degree 1 (and the edge with which it is incident) leaves a tree with $n-1$ vertices which is planar by the induction hypothesis. The deleted vertex and edge can now be reinserted without destroying planarity.

   (b) Since a tree is connected and planar, we must have $V-E+R=2$. But $R=1$, hence, $V-E=1$ or $E=V-1$, as desired.

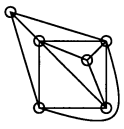
   (c) The proof that $\chi(\mathcal{T})=2$ is perhaps easiest by induction on $n$, the number of vertices. If $n=2$, the tree must be $\circ$—$\circ$ and this can be colored with 2 colors, but not with fewer. Assuming the result for trees with $k$ vertices, if we have a tree $\mathcal{T}$ with $k+1$ vertices, removing a vertex $v$ of degree 1 (with the lone edge $vw$ with which $v$ is incident), we are left with a tree with $k$ vertices. By the induction hypothesis, we can 2-color this tree. Then we assign to $v$ the color not used for $w$ and obtain a 2-coloring of $\mathcal{T}$.

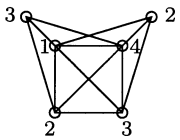
8. (a) [BB] We assume $\mathcal{G}$ has more than three vertices for otherwise there is nothing to prove. Let $\mathcal{T}$ be a spanning tree for $\mathcal{G}$. By Exercise 7(c) we know that $\chi(\mathcal{T})=2$. Since $\mathcal{G}$ has $n$ vertices, $\mathcal{G}=\mathcal{T}\cup\{e\}$ for some edge $e$. If the ends of $e$ have different colors in $\mathcal{T}$, then $\chi(\mathcal{G})=2$; otherwise, one of these ends must be given a third color and $\chi(\mathcal{G})=3$. In either case, $\chi(\mathcal{G})\leq 3$ as required.

   (b) Again we assume that $\mathcal{G}$ has more than 3 vertices and let $\mathcal{T}$ be a spanning tree. This time $\mathcal{G}=\mathcal{T}\cup\{e,f\}$ for two edges $e,f$ of $\mathcal{G}$. If $e$ and $f$ have a common vertex $v$, then giving $v$ a third color in $\mathcal{T}$ yields a coloring $\mathcal{G}$, so $\chi(\mathcal{G})\leq 3$. On the other hand, if $e=v_1v_2$ and $f=v_3v_4$ have no vertex in common, then since $\mathcal{T}$ contains no circuits, one of the edges $v_1v_3$, $v_3v_2$, $v_2v_4$, $v_4v_1$ is not in $\mathcal{T}$. If this edge is $v_iv_j$ then giving $v_i$ and $v_j$ the same third color yields a 3-coloring of $\mathcal{G}$.

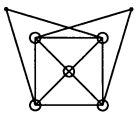
   (c) No. The complete graph on four vertices has $6=4+2$ edges, but $\chi(\mathcal{K}_4)=4$.

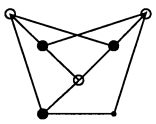
9. (a) The graph is planar. An isomorphic plane graph is shown.





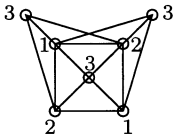

   (b) A 4-coloring is shown. Since the graph contains $\mathcal{K}_4$ as a subgraph, fewer than four colors will not suffice. The chromatic number is four. This result is consistent with the Four Color Theorem: The graph is planar, so its chromatic number cannot exceed 4, and it doesn't.

10. (a) [BB] 

(b) [BB] 

(c) [BB] The graph isn't planar, by Kuratowski's Theorem, as the results of (a) and (b) each illustrate.

(d) [BB] A 3-coloring is shown. Since the graph contains triangles, fewer than three colors will not suffice. The chromatic number is three.



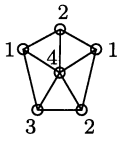

(e) [BB] The converse of the Four-Color Theorem says that a graph with $\chi \leq 4$ is planar. This is not true, as this graph illustrates: The chromatic number is 3, but the graph is not planar.

11. (a) [BB] By PAUSE 7 of this section, for any $n$, $\chi(\mathcal{K}_n) = n$ and for any $m$, $n$, $\chi(\mathcal{K}_{m,n}) = 2$. Thus, $\chi(\mathcal{K}_{14}) = 14$ and $\chi(\mathcal{K}_{5,14}) = 2$.

(b) A cycle with 38 edges is a cycle with an odd number (39) of vertices. We must start with two different colors $R$ and $W$. If we try to alternate these, the odd vertices $R$ and the even $W$, say, then we run into trouble because vertex 39 is adjacent to an even and an odd number vertex (38 and 1). Hence, $\chi(\mathcal{G}_1) = 3$. On the other hand, $\mathcal{G}_2$ is a cycle with an even number of vertices, so $\chi(\mathcal{G}_2) = 2$.

12. (a) [BB] False. If $\mathcal{G}$ is a 5-cycle, then $\chi(\mathcal{G}) = 3$ since 5 is odd but $\mathcal{G}$ contains no triangle.

(b) False, as shown by the graph to the right.




13. By coloring $\mathcal{K}_{n-3}$ with $n-3$ colors and $\mathcal{C}$ with another three, we obtain an $n$-coloring of $\mathcal{G}$. Thus, $\chi(\mathcal{G}) \leq n$. On the other hand, $\mathcal{K}_{n-3}$ requires $n-3$ colors none of which can be used for $\mathcal{C}$ since each vertex of $\mathcal{C}$ is adjacent to each vertex of $\mathcal{K}_{n-3}$. Thus, $\chi(\mathcal{G}) \geq n$ from which equality follows.

To show that $\mathcal{G}$ does not contain $\mathcal{K}_n$ as a subgraph suppose to the contrary. Then at least three of the vertices in this subgraph must come from $\mathcal{C}$ (since $\mathcal{K}_{n-3}$ has only $n-3$ vertices). But then $\mathcal{C}$ would contain a triangle, contrary to fact.

14. False. $\mathcal{G}$: is homeomorphic to $\mathcal{H}$: but $\chi(\mathcal{G}) = 3$, $\chi(\mathcal{H}) = 2$.

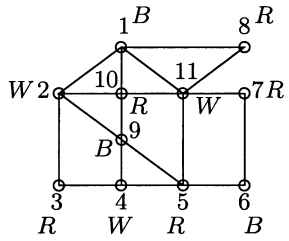
15. (a) [BB] False. $\mathcal{K}_{3,3}$ is Hamiltonian and has chromatic number two (by PAUSE 7) but is not planar.

(b) False. Any tree is planar and has chromatic number two (by Exercise 7(c)) but is not Hamiltonian.

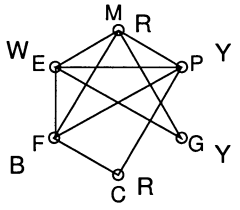
(c) False. $\mathcal{K}_{4,4}$ is both Hamiltonian and Eulerian and it has chromatic number two, but it is not planar because it contains $\mathcal{K}_{3,3}$ as a subgraph.

(d) [BB] False. $\mathcal{K}_7$ is Eulerian (since the degree of every vertex is six) and Hamiltonian (since every complete graph is Hamiltonian), yet $\chi(\mathcal{K}_7) = 7$.

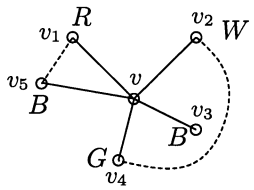
16. [BB] Three exam periods are required since this is the chromatic number of the graph.




17. We draw a graph whose vertices correspond to courses and an edge means that the two corresponding exams should not be scheduled at the same time (because there are students taking each course). The chromatic number of the graph is 4 (a 4-coloring is shown). This is best possible because the graph contains $\mathcal{K}_4$ ($E$, $M$, $F$, $P$). The minimum number of exam periods to avoid conflicts is **four**.

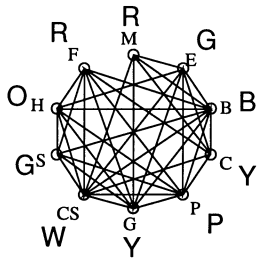



18. Our proof does **not** work for four colors. We show why by analyzing our proof of The Five Color Theorem. Let v be a vertex of degree five, adjacent to vertices $v_1, v_2, v_3, v_4, v_5$, and let $\mathcal{G}_0 = \mathcal{G} \setminus v$. The induction hypothesis would say that $\chi(\mathcal{G}_0) \leq 4$. Hence two of the $v_i$ would be assigned the same color. Suppose the colors $R$, $W$, $B$, $G$ were assigned as in the figure and also that $v_1v_5$ and $v_2v_4$ were edges (as shown). Here there is no path which alternates colors between $v_1$ and $v_3$, but there is such a path $v_2v_4$ between $v_2$ and $v_4$. However, if we exchange colors on $\mathcal{H}$—the vertices and edges of all paths through vertices $B$ and $R$ which start at $v_1$—then $v_1$ will become $B$ and $v_5$ will become $R$, so no color has been freed for use by v. Similarly, changing $v_3$ to $R$ doesn't help. The proof doesn't work in this case.

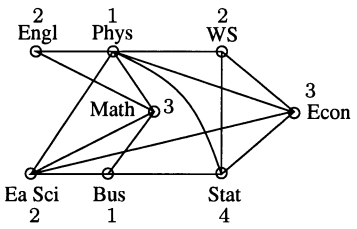



19. Certainly a new schedule is required since, for example, Spanish and French can no longer be written at the same time. Drawing the graph as before, notice that there are now **seven** vertices—$B, P, G, CS, S, H, F$—each two of which are joined by an edge. Thus, at least seven exam periods are necessary. In fact, seven is enough; the graph has chromatic number seven as the graph shows. One possible exam schedule which uses seven periods and avoids conflicts is also shown.

| | |
|---|---|
| Period 1 (Red) | Mathematics, French |
| Period 2 (Green) | English, Spanish |
| Period 3 (Blue) | Biology |
| Period 4 (Yellow) | Chemistry, Geography |
| Period 5 (Purple) | Psychology |
| Period 6 (White) | Computer Science |
| Period 7 (Orange) | History |

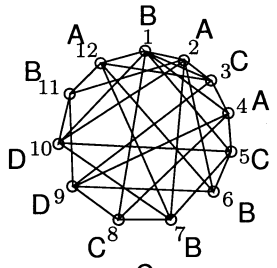



20. [BB] We draw a graph $\mathcal{G}$. The vertices are courses, and two vertices are joined if somebody is taking both courses. It turns out that $\chi(\mathcal{G}) = 4$, so four time periods are required.

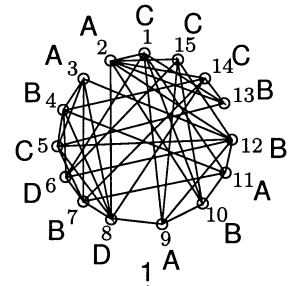


21. [BB] This means we can delete the edge joining "Econ" to "Stat," so "Stat" can be colored 3, $\chi(\mathcal{G}) = 3$ and three time periods now suffice.

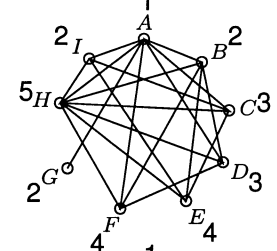
22. Put two varieties into different boxes if one attacks the other. Make a graph whose vertices are varieties and where an edge indicates that the variety represented by one end vertex attacks the other. Since $\chi(\mathcal{G}) = 4$, as shown, four is the minimum number of boxes required.



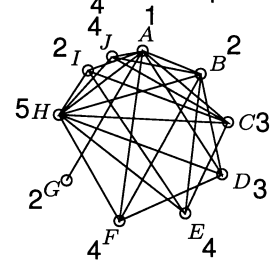

23. Two children must be put in different rooms if one fights with the other. We make a graph $\mathcal{G}$ whose vertices represent the 15 children. Two vertices are joined by an edge if the corresponding children fight. The graph contains $\mathcal{K}_4$ (consider vertices 3, 4, 5 and 8) so at least four colors are required. We show a 4-coloring, so we conclude that $\chi(\mathcal{G}) = 4$. The center can get away with four rooms.




24. Make a graph where vertices correspond to cities and an edge signifies that the corresponding cities are within 150 km and, therefore, must be assigned different channels. The number of channels required is the chromatic number of the graph which, in this problem, is five, as shown. (Note that the graph contains $\mathcal{K}_5$; vertices $A, B, D, F$ and $H$.)




25. No new channel is required since the chromatic number of the new graph is still five. City $J$ can be assigned the same channel as $E$ and $F$.



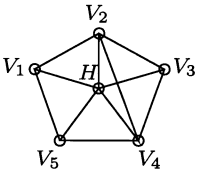

26. (a) [BB] The dual of the cube is the octahedron whose graph appears in Fig. 13.5.
    (b) The dual of the dodecahedron is the icosahedron whose graph appears in Fig. 10.19.

27. [BB] Yes, it is. Both the tetrahedron and its dual are $\mathcal{K}_4$.

## Exercises 13.3

1. (a) [BB] 36 nodes, 14 grid segments, 6 nets.

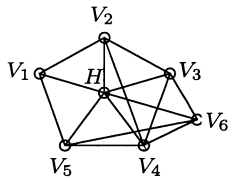
(c) [BB]



(b) [BB] $V_1$ and $V_3$, for example.

(d) [BB] We know $\chi(\mathcal{G}) \geq 4$ since $\mathcal{G}$ contains $\mathcal{K}_4$ (vertices $H$, $V_2$, $V_3$, $V_4$). On the other hand, a 4-coloring is given by $\{V_1, V_3\}$, $\{V_2, V_5\}$, $\{V_4\}$, $\{H\}$. Hence, $\chi(\mathcal{G}) = 4$. The above 4-coloring is also a partition of the nets.

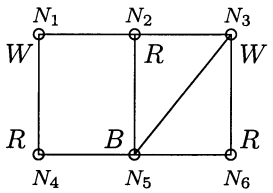
2. (a) 36 nodes, 19 grid segments, 7 nets.

(c) 


(b) $V_1$ and $V_3$, for example.

(d) We know $\chi(\mathcal{G}) \geq 4$ since $\mathcal{G}$ contains $\mathcal{K}_4$ (with vertices $H$, $V_2$, $V_3$, $V_4$ as in 1(d)). On the other hand, a 4-coloring, and partition of the nets, is $\{V_6, V_2\}$, $\{V_1, V_4\}$, $\{V_5, V_3\}$, $\{H\}$; so $\chi(\mathcal{G}) = 4$.

3. [BB] False. An example is the net pattern for which the line-of-sight graph is ○ ○.

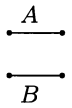
4. (a) 


(b) The graph contains a triangle, so $\chi(G) \geq 3$. A 3-coloring is shown, so $\chi(G) = 3$. The corresponding partition of nets is $\{N_1, N_3\}$, $\{N_2, N_4, N_6\}$, $\{N_5\}$.

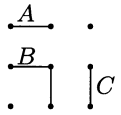
(c) The number of tests required would be $\binom{3}{2} = 3$.

5. (a) [BB] $\sim$ need not be reflexive. Consider 

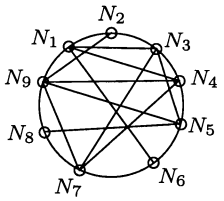
(b) $\sim$ is symmetric. If $A \sim B$, then there could be a short between $A$ and $B$, hence a short between $B$ and $A$, so $B \sim A$.

(c) $\sim$ is not antisymmetric. Consider 

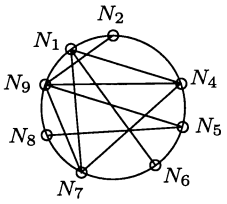
, where $A \sim B$ and $B \sim A$.

(d) $\sim$ is not transitive. Consider 

Here, $A \sim B$, $B \sim C$, but $A \not\sim C$.

6. (a) [BB] 


(b) Since $\mathcal{G}$ contains triangles, $\chi(\mathcal{G}) \geq 3$. On the other hand, the sets $\{N_1, N_5, N_7\}$, $\{N_2, N_4, N_8\}$ and $\{N_3, N_6, N_9\}$ define a 3-coloring and hence a partition of the nets. So $\chi(\mathcal{G}) = 3$.

7. (a) 


(b) Now we have $\chi(\mathcal{G}) \geq 4$, since $N_1$, $N_2$, $N_4$, $N_9$ determine $\mathcal{K}_4$. A 4-coloring, and hence a partition of the nets, is $\{N_1, N_5\}$, $\{N_2, N_7\}$, $\{N_4, N_6, N_8\}$, $\{N_9\}$, so $\chi(\mathcal{G}) = 4$.

8. [BB] False. The graph in Exercise 6 isn't planar since the subgraph with vertices $\{N_1, N_3, N_4, N_5, N_6, N_7\}$ and with edge $N_3N_4$ removed is isomorphic to $\mathcal{K}_{3,3}$.

9. (a) [BB] Imagine the possible shorts between nets as the edges in a graph: They won't cross since they're all parallel. Now compress each net to a point, and we have a plane graph. By the Four Color Theorem, $\chi(\mathcal{G}) \leq 4$.

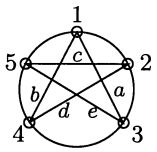
   (b) [BB] Draw separate graphs $\mathcal{G}(\mathcal{V}, \mathcal{E}_1)$, $\mathcal{G}(\mathcal{V}, \mathcal{E}_2)$ for horizontal lines of sight [as in (a)] and vertical lines of sight. These graphs have the same vertex set, so combine them. If an edge is repeated (that is, both horizontal and vertical lines exist between nets), then omit one of the original occurrences of it.

   (c) Color each of $\mathcal{G}(\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}(\mathcal{V}, \mathcal{E}_2)$ in (b) with four colors, from $\mathcal{S} = \{B, G, R, W\}$. The 16 different ordered pairs in $\mathcal{S} \times \mathcal{S}$ give a coloring for $\mathcal{G}$.

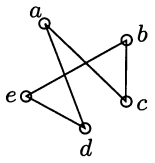
   (d) If $\mathcal{K}_{11}$ were a line-of-sight graph, it could be decomposed as in (b). Since $\mathcal{K}_{11}$ has 55 edges, either $\mathcal{E}_1 \geq 28$ or $\mathcal{E}_2 \geq 28$. But Theorem 13.1.4 says that $\mathcal{E}_i \leq 3(11) - 6$ by planarity, which is a contradiction.

10. [BB] It is possible for a short to exist between any pair of nets here. Hence, $\mathcal{G} \cong \mathcal{K}_8$, and $\chi(\mathcal{G}) = 8$, as required.

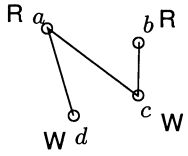
11. $\mathcal{K}_5$: [BB] $\mathcal{G}_1$

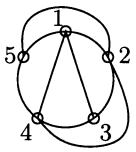


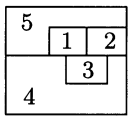

$\mathcal{H}$




$\mathcal{H}$ cannot be 2-colored since it is a cycle of odd length. If we delete any vertex, however, the resulting graph can be 2-colored. If we delete $e$, for example, we obtain the graph at the right. Shown are $\mathcal{G}'$ and a corresponding floor plan.

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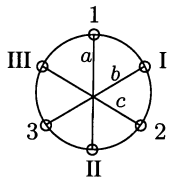
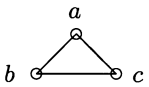
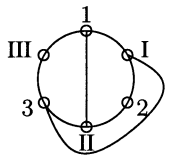
$\mathcal{K}_{3,3}$: $\mathcal{G}_1$

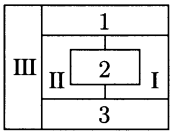



$\mathcal{H}$




Since $\mathcal{H}$ is $\mathcal{K}_3$, $\chi(\mathcal{H}) = 3$. But if we delete any vertex from $\mathcal{H}$, the resulting graph can be 2-colored. Say we delete $c$, obtaining the graph: $a$ R——W $b$. Here are $\mathcal{G}'$ and one possible floor plan.




Note that other relationships are satisfied here, although they weren't required. There is, however, no way to get 2 and III adjacent.

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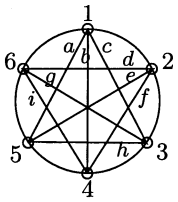
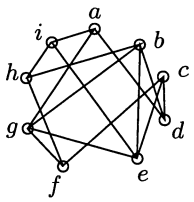
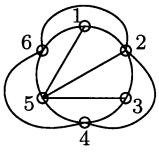
$\mathcal{K}_6$: $\mathcal{G}_1$

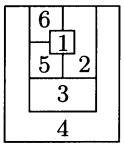



$\mathcal{H}$

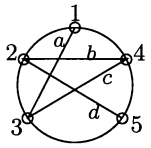



$\mathcal{H}$ is not 2-colorable because there are circuits of odd length. Removing one vertex will not be good enough. (Removing one edge from $\mathcal{K}_6$ does not make it planar.) Since removing two still does not make it planar (there will always be a subgraph isomorphic to $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$), we discover that three vertices must be removed from $\mathcal{H}$. You have to be careful which three. We remove $b$, $c$ and $g$ as shown. Here are $\mathcal{G}'$ and a floor plan.

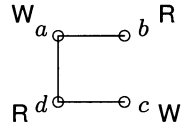



12. (a) $\mathcal{G}_1$

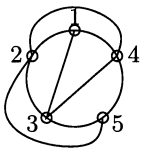


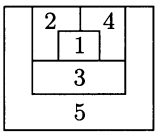

$\mathcal{H}$



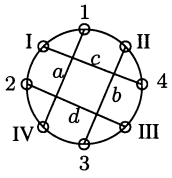

$\mathcal{H}$ is 2-colorable.

Here are $\mathcal{G}'$ and a possible floor plan.

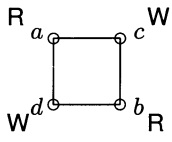



(b) $\mathcal{G}_1$

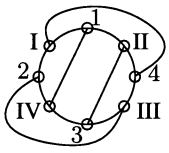


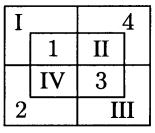

$\mathcal{H}$

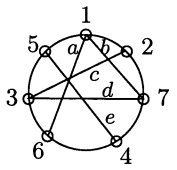


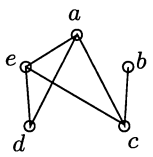

Note that $\chi(\mathcal{H}) = 2$, as shown, so no vertex needs to be removed.

Here are $\mathcal{G}'$ and a possible floor plan.

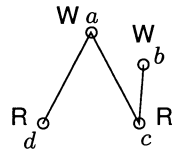


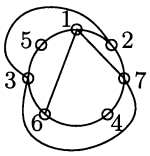



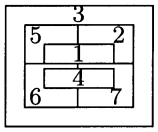
(c) $\mathcal{G}_1$ 


$\mathcal{H}$ 

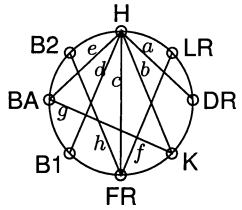

$\mathcal{H}$ is not 2-colorable since it contains a triangle. If $e$ is removed, we get the graph at the right. Shown are the graph $\mathcal{G}'$ and a possible floor plan.

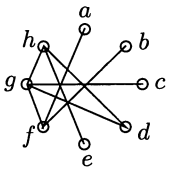


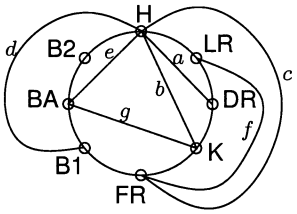



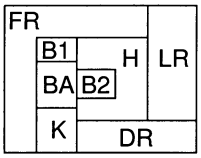

13. A graph depicting this situation is shown.

$\mathcal{G}_1$ 

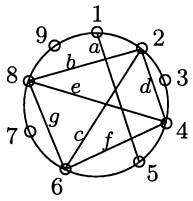

$\mathcal{H}$ 

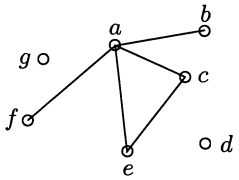

Since $\mathcal{H}$ has a triangle, $\chi(\mathcal{H}) = 3$ and it is impossible to construct the desired house. We could make $\chi(\mathcal{H}) = 2$ by removing any of $d, g, h$. The most practical of these to remove might be $h$ in which case $\mathcal{G}_1$ becomes as shown on the left. One possible corresponding house plan is at the right.




14. (a) [BB] We can immediately draw $\mathcal{G}_1$ for this problem (using the obvious Hamiltonian cycle that a golfer must follow). Both $\mathcal{G}_1$ and $\mathcal{H}$ are shown.

$\mathcal{G}_1$ 


$\mathcal{H}$ 


Since $\mathcal{H}$ contains a triangle, $\chi(\mathcal{H}) = 3$, so such a course is impossible.

(b) $\chi(\mathcal{H})$ would equal 2 if $a$, $c$ or $e$ were removed from $\mathcal{H}$, that is, if we removed one of the restrictions that holes 1 and 5 share a common border, or that holes 2 and 6, or holes 4 and 8 share a common border.

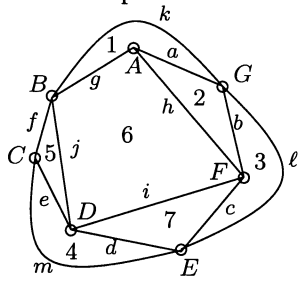
15. [BB] Deleting vertex 2 (and the three edges incident with 2) leaves $\mathcal{K}_5$.

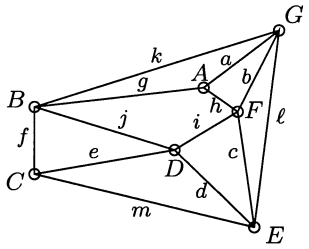
    Alternatively, $\{1, 4, 6\}$, $\{2, 3, 5\}$ are bipartition sets for $\mathcal{K}_{3,3}$ after deleting edges 14, 16, 46 and 35.

16. [BB] Since the graph is not $\mathcal{K}_n$ for any $n$, nor an odd cycle, Brooks's Theorem says $\chi(\mathcal{G}) \leq \Delta(\mathcal{G})$. Here $\Delta(\mathcal{G}) = 3$, so we have $\chi(\mathcal{G}) \leq 3$. But this graph contains cycles of odd length, so $\chi(\mathcal{G}) \geq 3$. We conclude that $\chi(\mathcal{G}) = 3$.

## Chapter 13 Review

1. (a) The graph is drawn first planar and then also with straight edges.



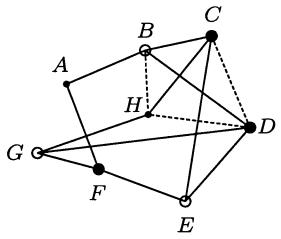




   (b) There are eight regions, numbered $1, 2, \ldots, 8$ with boundaries $agk$, $abh$, $bc\ell$, $dem$, $efj$, $ghij$, $cdi$ and $fk\ell m$ respectively.

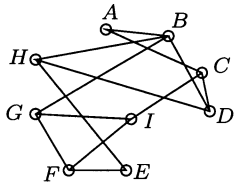
   (c) $E = 13$, $V = 7$, $R = 8$, $N = 26$, $V - E + R = 7 - 13 + 8 = 2$, $N = 26 \leq 26 = 2E$ and $E = 13 \leq 15 = 3V - 6$.

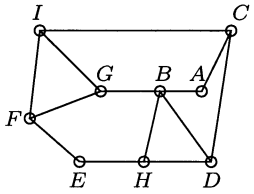
2. (a) This is not planar. The graph is homeomorphic to $\mathcal{K}_5$ ("removing" the two vertices of degree 2).

   (b) This is not planar. Labeling the graph as shown, the subgraph obtained by removing edges $BH$, $CD$ and $DH$ is homeomorphic to $\mathcal{K}_{3,3}$: the bipartition sets are $\{B, E, G\}$ and $\{C, D, F\}$, with vertices $A$ and $H$ "removed".

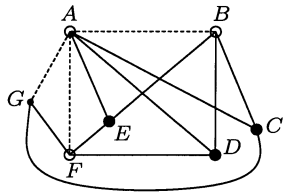



   (c) The graph is planar, as shown.

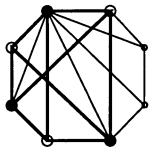


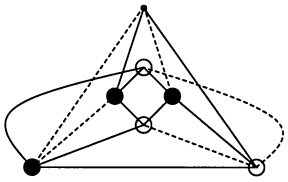

   (d) The graph contains a subgraph homeomorphic to $\mathcal{K}_{3,3}$ as shown. Deleting the edges shown with dotted lines gives the subgraph. What remains would be $\mathcal{K}_{3,3}$ except for the additional vertex at $G$.

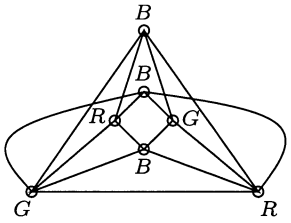


(e) The graph is not planar, by Kuratowski's Theorem, since it contains $\mathcal{K}_{3,3}$ as a subgraph, as shown.



3. (a) The graph is not planar. It has a subgraph homeomorphic to $\mathcal{K}_{3,3}$ as shown on the left.






(b) Since the graph contains triangles, its chromatic number is at 3. A 3-coloring is shown on the right, so the chromatic number is 3.

4. (a) It cannot contain a subgraph homeomorphic to $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$ because each of these graphs contain more than one circuit.

(b) A near-tree is planar, so $V - E + R = 2$. There is just one circuit, so $R = 2$ and $V = E$.

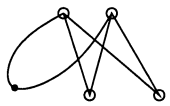
(c) We know that $\sum \deg v = 2|E| = 2|V|$. Suppose such a graph has a vertex of degree 3. If all others have degree greater than 2, then $\sum \deg v > 2|V|$, a contradiction.

5. Exercise 6 of Section 13.1 tells us that $3E \leq 5V - 10$. Substituting $E = V + R - 2$ gives the desired result.

6. (a) $E \leq 3V - 6 = 3(30) - 6 = 84$.

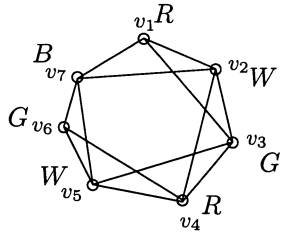
(b) If the vertices of degree 1 are removed together with the edges with which they are incident, we are left with a subgraph $\mathcal{H}$ which has 15 vertices, and which is still connected and planar. The number of edges in $\mathcal{H}$ is at most $3(15) - 6 = 39$. Since 15 edges were removed from $\mathcal{G}$, the number of edges in $\mathcal{G}$ is at most $39 + 15 = 54$.

7. Here is $\mathcal{K}_{2,3}$. This is homeomorphic to a graph with four vertices ("remove" a vertex of degree 2) and hence to a subgraph of $\mathcal{K}_4$.

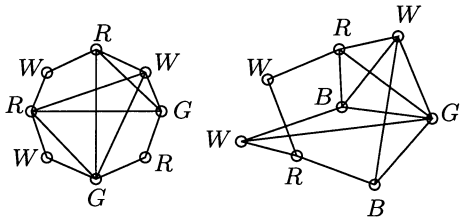


8. True. Since $\mathcal{G}$ and $\mathcal{H}$ are homeomorphic, each can be obtained from some graph $\mathcal{K}$ by adding vertices of degree 2. If $\mathcal{K}$ were not connected, then $\mathcal{G}$ would also not be connected. Also, if $\mathcal{K}$ contained a circuit, so would $\mathcal{G}$. Since $\mathcal{G}$ is a tree, it follows the $\mathcal{K}$ must also be a tree. But introducing vertices of degree 2 to a tree still leaves a tree. So $\mathcal{H}$ is a tree.

9. Here $\chi(\mathcal{G}) = 4$. To see this, start with vertex $v_1$ and proceed clockwise. Since $v_1, v_2, v_3$ form a triangle, they must be given different colors, say red, white and green. To avoid the use of a fourth colour, we would have to color $v_4$ red (since it has already been joined to green and white vertices). But then $v_5$ must be white and $v_6$ green. Now $v_7$ is adjacent to red, white and green vertices, so we need a fourth colour, blue, for this.




10. For the graph on the left, $\chi(\mathcal{G}) = 3$. Since the graph contains triangles, $\chi(\mathcal{G}) \geq 3$, and a 3-coloring is shown. For the graph on the right, $\chi(\mathcal{G}) = 4$. Since the graph contains a copy of $\mathcal{K}_4$, $\chi(\mathcal{G}) \geq 4$, and a 4-coloring is shown. Each graph reminds us that the converse to the Four-Color Theorem is false since in Exercise 2 we noted that neither is planar, yet $\chi(\mathcal{G}) \leq 4$ in each case.




11. (a) False. A counterexample is shown.

(b) True. Since $\mathcal{G}$ contains $\mathcal{K}_4$ as a subgraph, we know that $\chi(\mathcal{G}) \geq 4$. But since $\mathcal{G}$ is planar, $\chi(\mathcal{G}) \leq 4$, so $\chi(\mathcal{G}) = 4$.

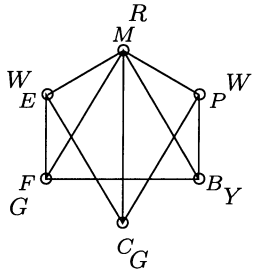
12. ($\longrightarrow$) Suppose $\chi(\mathcal{G}) = 2$ and that the vertices of $\mathcal{G}$ are colored red and white. Let $\mathcal{V}_1$ be the set of those vertices colored red and $\mathcal{V}_2$ the set of those colored white. Then $\{\mathcal{V}_1, \mathcal{V}_2\}$ is a partition of the vertices of $\mathcal{G}$ and any edge of the graph must join an edge of $\mathcal{V}_1$ to an edge of $\mathcal{V}_2$. Finally, $\chi(\mathcal{G}) \neq 1$ implies there is at least one edge.

($\longleftarrow$) Conversely, if $\mathcal{G}$ is bipartite with bipartition sets $\mathcal{V}_1$ and $\mathcal{V}_2$, then we can color the vertices of $\mathcal{G}$ with two colors using one color for $\mathcal{V}_1$ and the other for $\mathcal{V}_2$. This shows that $\chi(\mathcal{G}) = 2$. (Again, $\chi(\mathcal{G}) \neq 1$ because there is an edge.)

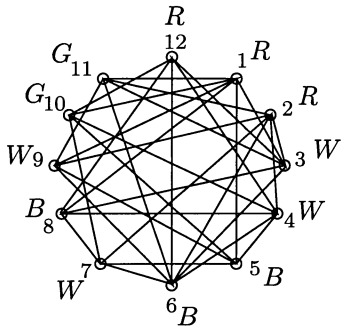
13. (a) With $\mathcal{G} = \mathcal{K}_{2,2}$, we have $\chi(\mathcal{G}) = 2$, although $\Sigma(\mathcal{G}) = 2$. In fact, we saw in the text that $\chi(\mathcal{K}_{m,n}) = 2$ for any $m, n$, so there is no possibility of changing the right hand side here and getting something which will always be true.

(b) Yes it is. For a tree $\mathcal{T}$, we know that $\Sigma(\mathcal{T}) = 1$ and $\chi(\mathcal{T}) = 2$.

14. We draw a graph whose vertices correspond to courses and an edge means that the two corresponding exams should not be scheduled at the same time (because there are students taking each course). The chromatic number of the graph is 4 (a 4-coloring is shown). The minimum number of exam periods to avoid conflicts is **four**.

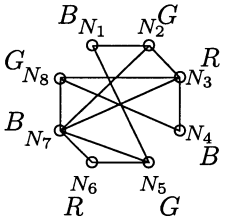


15. We must find the chromatic number of the graph shown. Note that 2, 3, 6, 11 determine a subgraph isomorphic to $\mathcal{K}_4$, so we need at least four colors. Let us select $R$, $W$, $B$, $G$ for these vertices, respectively. We try to color the remaining vertices with four colors. Note that 9 is joined to 2, 6 and 11, so it must be $W$. Since 4 and 7 are not joined to 3 or 9, we may color this $W$ as well. Since 1 and 12 are not joined to 2 or to each other, we may color them $R$. Since 5 and 8 are not joined to 6 nor to each other, we color them $B$. Finally, 10 can be colored $G$.



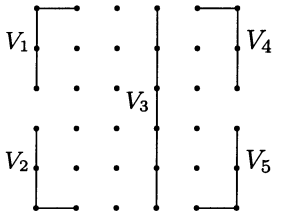

Since $\chi(\mathcal{G}) = 4$, the prisoners can be divided as follows: $\{1, 2, 12\}$, $\{3, 4, 7, 9\}$, $\{5, 6, 8\}$, $\{10, 11\}$.

16. The line-of-sight graph $\mathcal{G}$ will have chromatic number 3. To see why, note that $N_3$, $N_7$ and $N_8$ must be assigned different colors, say $R$, $B$, $G$, respectively. Then $N_2$ and $N_5$ can be colored $G$, and $N_1$ and $N_4$ can be colored $B$. Finally, $N_6$ can be colored $R$. The partition of nets is $\{N_1, N_4, N_7\}$, $\{N_2, N_5, N_8\}$, $\{N_3, N_6\}$.



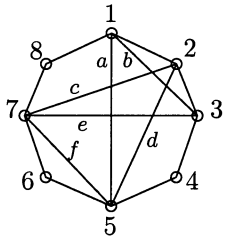

17. The line-of-sight graph here is as before except for an additional edge between $N_4$ and $N_7$. This means that $N_3$, $N_4$, $N_7$, $N_8$ require four different colors, say $R$, $W$, $B$, $G$, respectively. The other colors can remain as before, so $\chi(\mathcal{G}) = 4$. The partition of nets is $\{N_1, N_7\}$, $\{N_4\}$, $\{N_2, N_5, N_8\}$, $\{N_3, N_6\}$.

18. A nice example of a graph which is Eulerian but not Hamiltonian is . (See Exercise 21(c) of Section 10.2.) This is the line-of-sight graph for the net pattern shown to the right.

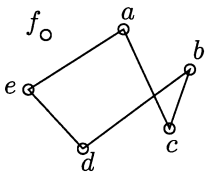



19. The bipartite graph $\mathcal{K}_{7,7}$ has 14 vertices and 49 edges. According to Exercise 9(b) of Section 13.3, if this graph is a line-of-sight graph, we can partition the edges into subsets $\mathcal{E}_1$, $\mathcal{E}_2$ such that $\mathcal{K}_{7,7}$ is planar with respect to each set of edges (taken separately). Then Exercise 9(a) of Section 13.1 tells us that each subset contains at most $2(14) - 4 = 24$ edges, contradicting the fact the $\mathcal{K}_{7,7}$ has 49 edges.

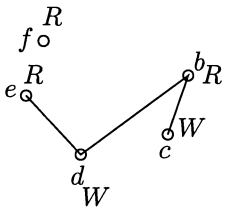
20. (a) The vertices are already arranged in a Hamiltonian cycle around the circumference of a circle, so we assign labels as shown. The graph $\mathcal{H}$ contains a 5-cycle and $\chi(\mathcal{H}) = 3$. Removing any vertex except $f$ will give us a best possible set of feasible relationships. Suppose we remove $a$. A 2-coloring of $\mathcal{H} \setminus \{a\}$ is shown.




$\mathcal{G}_1$

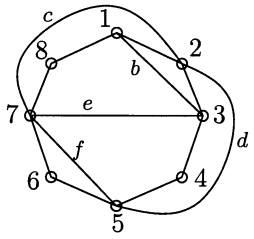


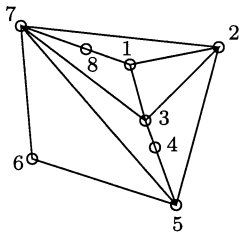

$\mathcal{H}$

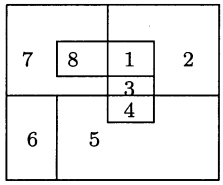



$\mathcal{H} \setminus \{a\}$

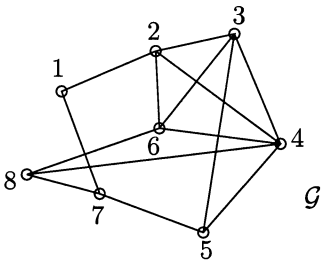
Below we draw $\mathcal{G}_1 \setminus \{a\}$ as a plane graph and then as a plane graph with straight edges. Finally, we produce the corresponding floor plan (which even satisfies an requirement not requested).

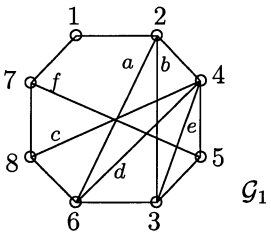


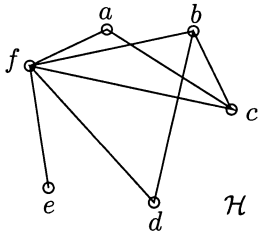



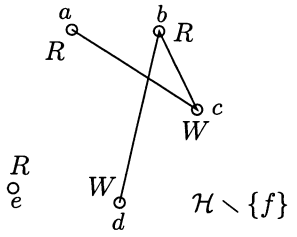

(b) We label the vertices of $\mathcal{G}$ as shown on the left and redraw it as $\mathcal{G}_1$.



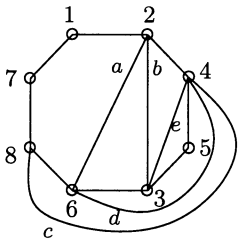



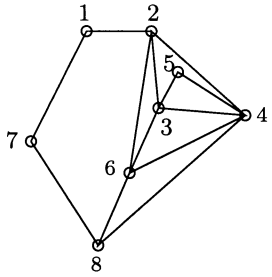

The chromatic number of the new graph $\mathcal{H}$ is 3 because $\mathcal{H}$ contains triangles. By removing vertex $f$, we obtain a graph whose chromatic number is 2.

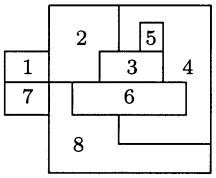



We draw $\mathcal{G}_1 \setminus \{f\}$ as a plane graph and then as a plane graph with straight edges. Finally, we produce a floor plan.

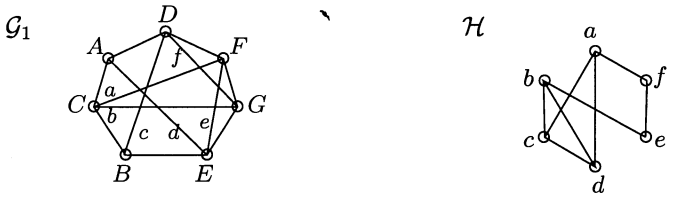



21. (a) By Kuratowski's Theorem, the graph here is not planar: It contains $\mathcal{K}_{3,3}$ as a subgraph, with bipartition sets $\{A, B, F\}$ and $\{C, D, E\}$.

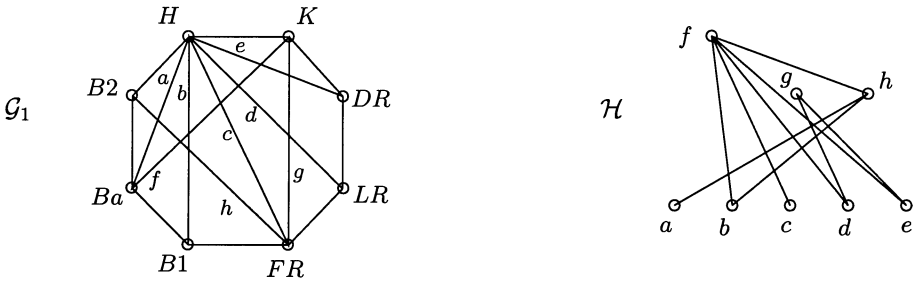
(b) Call the graph $\mathcal{G}$. $ACBEGFDA$ is a Hamiltonian cycle. On the left, we show these vertices in order on a circle and draw $\mathcal{G}$ again. Call the new graph $\mathcal{G}_1$. Next, we label the interior edges, $a$, $b$, $c$, $d$, $e$, and $f$, and draw a graph $\mathcal{H}$ (on the right) whose vertices correspond to $a, \ldots, f$.




We draw an edge between two vertices if the corresponding edges of $\mathcal{H}$ cross. The original plans can't be met because $\mathcal{H}$ does not have a 2-coloring. In fact, we must delete **two** vertices in order to obtain a subgraph with a 2-coloring. On the left below, we show $\mathcal{H} \setminus \{a, b\}$ and a 2-coloring. The adjacencies $CF$ (edge $a$) and $CG$ (edge $b$) are not met, but all others are.

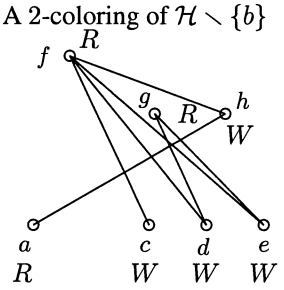
(c) On the right above, we draw $\mathcal{G}_1$ again, but as a plane graph achieved by pulling the white edges, $d$ and $e$, outside the circle. On the left below, we draw this graph as a plane graph with straight edges. Finally, on the right, we show a corresponding floor plan.

22. We draw the relationship graph $\mathcal{G}_1$ with the vertices of a Hamiltonian cycle—$H$, $K$, $DR$, $LR$, $FR$, $B1$, $Ba$, $B2$ around a circle and labeling interior edges $a, b, c, d, e, f, g, h$ as shown on the left. We then draw a new graph, $\mathcal{H}$, whose vertices are labeled $a, b, c, d, e, f, g, h$ and an edge indicates that the corresponding edges of $\mathcal{G}_1$ cross.

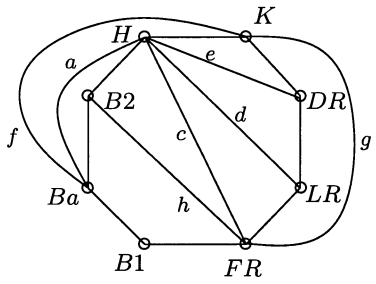



In order to 2-color $\mathcal{H}$, it is necessary to remove only one of $b$, $f$ or $h$. On the left below, we show the graph $\mathcal{H} \setminus \{b\}$ and a 2-coloring. On the right, we redraw $\mathcal{G}_1 \setminus \{b\}$ as a planar graph by pulling red edges $a$, $f$, $g$ outside the circle.

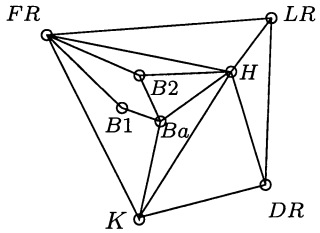
A 2-coloring of $\mathcal{H} \setminus \{b\}$

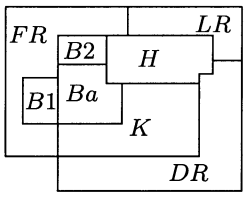



$\mathcal{G}_1 \setminus \{b\}$ as a plane graph

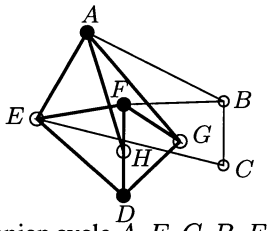



Finally, we draw $\mathcal{G}_1 \setminus \{b\}$ as a plane graph with straight edges (on the left) and then a corresponding floor plan.



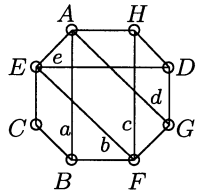




23. A graph is planar if and only if it has no subgraph homeomorphic to $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$. The given graph has $\mathcal{K}_{3,3}$ as a subgraph, with bipartition sets $\{A, D, F\}$ and $\{E, G, H\}$, as shown. The paths through $A, E, C, B, F, G, D, H, A$ and $E, D, G, F, H, A, B, C, E$ are Hamiltonian.

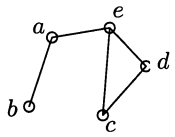



**Case 1:** We draw a graph $\mathcal{G}_1$ whose vertices are those of the Hamiltonian cycle $A, E, C, B, F, G, D, H, A$ arranged in a circle. There are five interior edges, which we label $a, b, c, d, e$ as shown on the left below. Now we draw a graph $\mathcal{H}$ whose vertices are labeled $a, b, c, d, e$ and two vertices are joined by an edge if and only if the corresponding edges cross in $\mathcal{G}_1$.

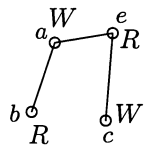
$\mathcal{G}_1$




$\mathcal{H}$

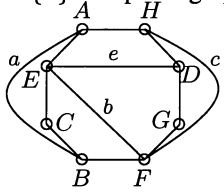



$\mathcal{H} \setminus \{d\}$



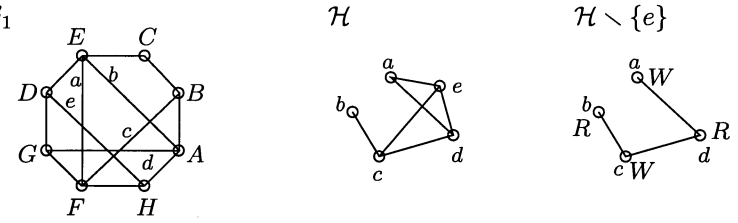

Since $\mathcal{H}$ contains a triangle, no 2-coloring exists, but there is a 2-coloring of $\mathcal{H} \setminus \{d\}$ as shown on the left. By pulling the white edges outside the circle, $\mathcal{G}_1 \setminus \{d\}$ can be redrawn as a planar graph, so removing the single adjacency constraint $AG$ gives the hospital board a floor plan which comes as close as possible to the one desired.

$\mathcal{G}_1 \setminus \{d\}$ as a plane graph



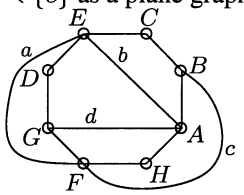
**Case 2:** We draw a graph $\mathcal{G}_1$ whose vertices are those of the Hamiltonian cycle $E, D, G, F, H, A, B, C, E$ arranged in a circle. There are five interior edges, which we label $a, b, c, d, e$ as shown on the left below. Now we draw a graph $\mathcal{H}$ whose vertices are labeled $a, b, c, d, e$ and two vertices are joined by an edge if and only if the corresponding edges cross in $\mathcal{G}_1$.

$\mathcal{G}_1$ $\mathcal{H}$ $\mathcal{H} \setminus \{e\}$

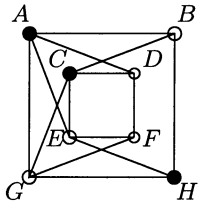



Since $\mathcal{H}$ contains a triangle, no 2-coloring exists, but there is a 2-coloring of $\mathcal{H} \setminus \{e\}$ as shown on the left. By pulling the white edges outside the circle, $\mathcal{G}_1 \setminus \{e\}$ can be redrawn as a planar graph, so removing the single adjacency constraint $DH$ gives the hospital board a floor plan which comes as close as possible to the one desired.

$\mathcal{G}_1 \setminus \{e\}$ as a plane graph

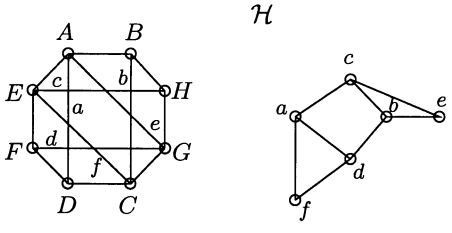



24. (a) Kuratowski's Theorem says that a graph is planar if and only if it has no subgraph homeomorphic to $\mathcal{K}_5$ or to $\mathcal{K}_{3,3}$. The given graph is not planar because it has a subgraph isomorphic to $\mathcal{K}_{3,3}$ with bipartition sets $\{A, C, H\}$ and $\{B, G, E\}$ as shown.




(b) One Hamiltonian cycle is $A$, $B$, $H$, $G$, $C$, $D$, $F$, $E$. We redraw $\mathcal{G}$ as a graph $\mathcal{G}_1$ by putting these vertices around a circle and labeling interior edges $a$, $b$, $c$, $d$, $e$, $f$. We then draw a new graph, $\mathcal{H}$, whose vertices are labeled $a, b, c, d, e, f$ and an edge indicates that the corresponding edges of $\mathcal{G}_1$ cross. In order to 2-color $\mathcal{H}$, we must remove **two** edges. On the right below, we show a 2-coloring of $\mathcal{H} \setminus \{a, c\}$.

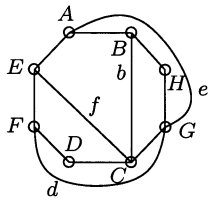
$\mathcal{G}_1$ $\mathcal{H}$ A 2-coloring of $\mathcal{H} \setminus \{a, c\}$

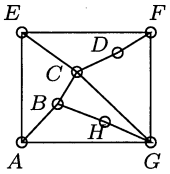


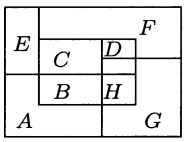

Then we redraw $\mathcal{G}_1 \setminus \{a, c\}$ as a planar graph by pulling white edges $d$ and $e$ outside the circle.

(c) Finally, we draw $\mathcal{G}_1 \setminus \{a, c\}$ as a plane graph with straight edges (on the left) and then a corresponding floor plan.

$\mathcal{G}_1 \setminus \{a, c\}$ as a plane graph then with straight edges and a floor plan




## Exercises 14.1

1. (a) [BB] At $a$, $\sum_v f_{va} = f_{sa} = 2$ and $\sum_v f_{av} = f_{ac} + f_{ae} = 2 + 0 = 2$. At $e$, $\sum_v f_{ve} = f_{ae} + f_{be} = 0 + 1 = 1$ and $\sum_v f_{ev} = f_{ec} + f_{ed} = 0 + 1 = 1$. At $d$, $\sum_v f_{vd} = f_{bd} + f_{ed} = 3 + 1 = 4$ and $\sum_v f_{dv} = f_{dt} = 4$.

   (b) [BB] The value of the flow is 6.

   (c) [BB] The capacity of the cut is $c_{ac} + c_{ae} + c_{be} + c_{bd} = 3 + 1 + 4 + 3 = 11$.

   (d) [BB] No. Arc $dt$ is saturated.

   (e) [BB] The flow is not maximum. For instance, it can be increased by adding 1 to the flow in the arcs along $sact$.

   (f) [BB] $(f_{ac} - f_{ca}) + (f_{ae} - f_{ea}) + (f_{be} - f_{eb}) + (f_{bd} - f_{db}) = (2-0) + (0-0) + (1-0) + (3-0) = 6$.

2. (a) At $a$, $\sum_v f_{va} = f_{sa} = 6$ and $\sum_v f_{av} = f_{ac} + f_{ae} = 3 + 3 = 6$. At $e$, $\sum_v f_{ve} = f_{ae} + f_{se} + f_{be} = 3+3+2 = 8$ and $\sum_v f_{ev} = f_{ec} + f_{et} + f_{ed} = 3+2+3 = 8$. At $d$, $\sum_v f_{vd} = f_{bd} + f_{ed} = 3+3 = 6$ and $\sum_v f_{dv} = f_{dt} = 6$.

   (b) [BB] The value of the flow is 14.

   (c) The capacity of the cut is $c_{ct} + c_{et} + c_{dt} = 6 + 2 + 6 = 14$.

   (d) The saturated arcs are $se$, $sb$, $ae$, $et$, $ed$, $ct$ and $dt$.

   (e) The flow is maximum, by Corollary 14.1.6.

   (f) $(f_{ct} - f_{tc}) + (f_{et} - f_{te}) + (f_{dt} - f_{td}) = (6 - 0) + (2 - 0) + (6 - 0) = 14$.

3. (a) At $a$, $\sum_v f_{va} = f_{sa} = 11$ and $\sum_v f_{av} = f_{ac} + f_{ad} = 3 + 8 = 11$. At $c$, $\sum_v f_{vc} = f_{ac} = 3$ and $\sum_v f_{cv} = f_{cd} + f_{cf} = 0 + 3 = 3$. At $d$, $\sum_v f_{vd} = f_{ad} + f_{bd} + f_{cd} = 8 + 1 + 0 = 9$ and $\sum_v f_{dv} = f_{dt} = 9$.

   (b) The value of the flow is 21.

   (c) [BB] $c_{ac} + c_{be} + c_{dt} = 3 + 9 + 9 = 21$.

   (d) The flow is maximum, by Corollary 14.1.6.

   (e) $(f_{ac} - f_{ca}) + (f_{be} - f_{eb}) + (f_{dt} - f_{td}) + (f_{dc} - f_{cd}) = (3-0) + (9-0) + (9-0) + (0-0) + (3-0) = 21$.

4. (a) At $a$, $\sum_v f_{va} = f_{sa} = 3$ and $\sum_v f_{av} = f_{ab} + f_{ac} + f_{ad} = 1 + 1 + 1 = 3$. At $b$, $\sum_v f_{vb} = f_{ab} = 1$ and $\sum_v f_{bv} = f_{bc} + f_{be} + f_{bt} = 1 + 0 + 0 = 1$. At $e$, $\sum_v f_{ve} = f_{be} + f_{de} = 0 + 3 = 3$ and $\sum_v f_{ev} = f_{et} = 3$.

(b) The value of the flow is 5.

(c) The capacity of the cut is $c_{ab} + c_{de} + c_{dt} = 2 + 5 + 4 = 11$.

(d) The flow can be increased by one unit along $sadt$ and by two units along $scdt$.

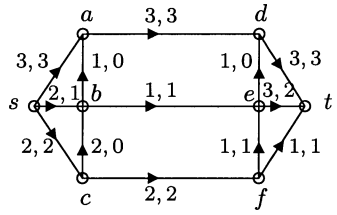
(e) The flow is not maximum, as noted in (d).

(f) $(f_{ab} - f_{ba}) + (f_{cb} - f_{bc}) + (f_{de} - f_{ed}) + (f_{dt} - f_{td}) = (1-0) + (0-1) + (3-0) + (2-0) = 5$.

5. (a) i. [BB] Send one unit through the path $sbet$.

ii. [BB] The flow in (a) has a saturated arc, $be$.

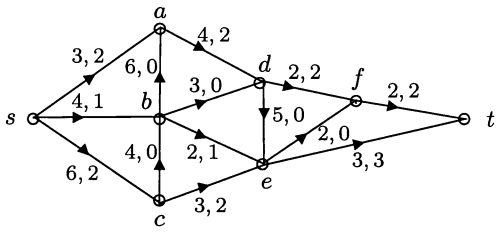
iii. [BB] Here is a maximum flow, of value 6. (Did you get it?) To see that the flow is maximum, consider the cut $\mathcal{S} = \{s, a, b\}$, $\mathcal{T} = \{c, d, e, f, t\}$.




(b) i. Send one unit through $scet$.

ii. Send three units through $scet$, saturating $ce$ and $et$.

iii. Here is a maximum flow, of value 5. (Did you find it?)

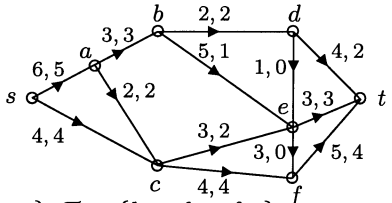



To see that this is maximum, consider the cut $\mathcal{S} = \{s, a, b, c, d, e, f\}$, $\mathcal{T} = \{t\}$.

(c) i. Send one unit through $scft$.

ii. Send four units through $scft$, saturating $sc$ and $cf$.

iii. Here is a maximum flow, of value 9. (Did you get it?)

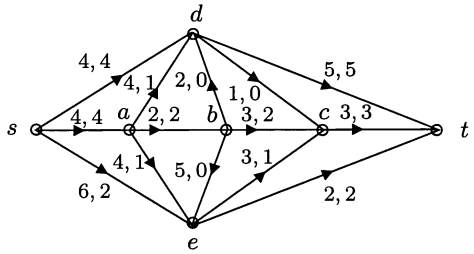



To see that this is maximum, consider the cut $\mathcal{S} = \{s, a\}$, $\mathcal{T} = \{b, c, d, e, f, t\}$.

(d) i. Send one unit through the path $set$.

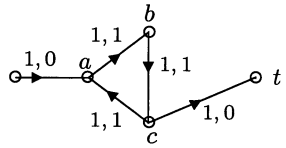
ii. Send two units through $set$, saturating $et$.

iii. Here is a maximum flow, of value 10. (Did you get it?)




To see that the flow is maximum, consider the cut $\mathcal{S} = \{s, a, b, c, d, e\}$, $\mathcal{T} = \{t\}$.

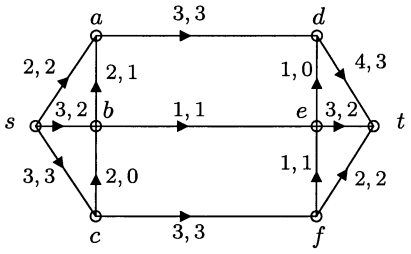
6. [BB]




7. [BB] Let $\{\mathcal{S}_1, \mathcal{T}_1\}$ be any cut. By Theorem 14.1.4, $\text{val}(\mathcal{F}) \leq \text{cap}(\mathcal{S}_1, \mathcal{T}_1)$. But $\text{val}(\mathcal{F}) = \text{cap}(\mathcal{S}, \mathcal{T})$. So $\text{cap}(\mathcal{S}, \mathcal{T}) \leq \text{cap}(\mathcal{S}_1, \mathcal{T}_1)$. Since $\{\mathcal{S}_1, \mathcal{T}_1\}$ was an arbitrary cut, the result is proven.

## Exercises 14.2

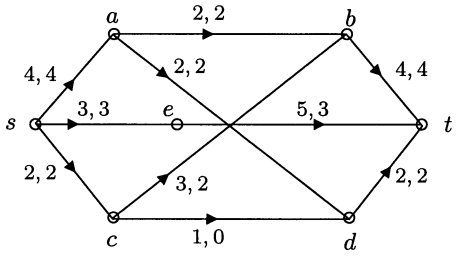
1. (a) i. [BB] One flow-augmenting chain is *sbadt* in which the slack is 1.




ii. [BB] Here is a maximum flow, of value 7. We can see this is maximum by examining the cut $\mathcal{S} = \{s, a, b\}$, $\mathcal{T} = \{c, d, e, f, t\}$, of capacity $c_{sc} + c_{ad} + c_{be} = 3 + 3 + 1 = 7$, the value of the flow.

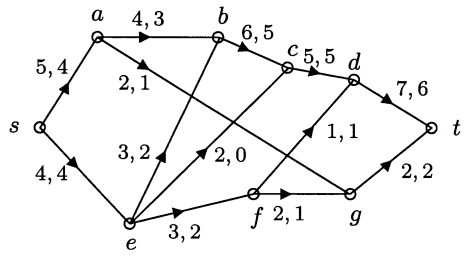
(b) i. One flow-augmenting chain is *sadt*, with slack 1. Another is *sadcbt*, also with slack 1.

ii. We show a maximum flow, of value 9. We can see that this is maximum by examining the cut $\mathcal{S} = \{s\}$, $\mathcal{T} = \{a, b, c, d, e, t\}$ which has capacity $c_{sa} + c_{se} + c_{sc} = 4 + 3 + 2 = 9$, the value of the flow.



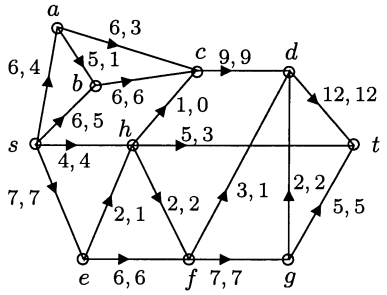

(c) i. One flow-augmenting chain is *sagt*, which has slack 1. Another is *sabcefgt*, which has slack 1 as well.

ii. We show a maximum flow, of value 8. We can see this is maximum by examining the cut $\mathcal{S} = \{s, a, b, c, e, f, g\}$, $\mathcal{T} = \{d, t\}$ which has capacity $c_{cd} + c_{fd} + c_{gt} = 5 + 1 + 2$, the value of the flow.



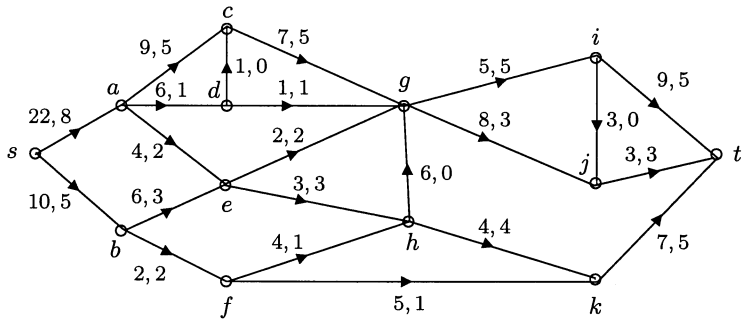

(d) i. [BB] One flow-augmenting chain is *sacdfgt*, which has slack 1.

ii. [BB] We show a maximum flow, of value 20. We can see this is maximum by examining the cut $\mathcal{S} = \{s, a, b, c\}$, $\mathcal{T} = \{d, e, f, g, h, t\}$. This has capacity $c_{se} + c_{sh} + c_{cd} = 7 + 4 + 9 = 20$, the value of the flow.




(e) i. One flow-augmenting chain is *sacghkt* in which the slack is 4.

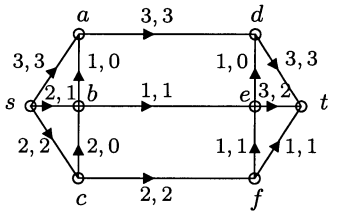
ii. We show a maximum flow, of value 13.



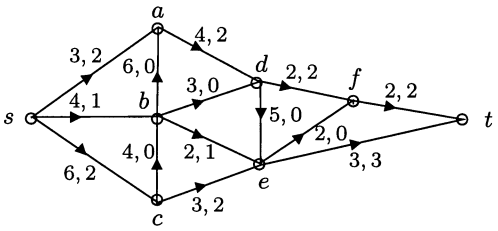

We can see that this is maximum by examining the cut $\mathcal{S} = \{s, a, b, c, d, e, g, j\}$, $\mathcal{T} = \{f, h, i, k, t\}$ which has capacity $c_{eh} + c_{bf} + c_{gi} + c_{jt} = 3 + 2 + 5 + 3 = 13$, the value of the flow.

2. The answers are given in the answers to Exercise 5 of Section 14.1 but, for convenience, we repeat them here.

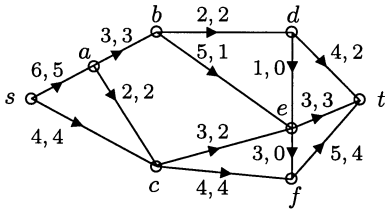
(a) [BB] We show a flow of value 6. To see that this is maximum, consider the cut $\mathcal{S} = \{s, a, b\}$, $\mathcal{T} = \{c, d, e, f, t\}$.



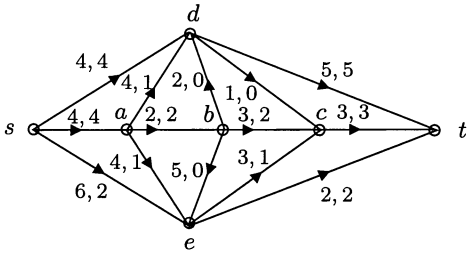

(b) We show a flow of value 5. To see that this is maximum, consider the cut $\mathcal{S} = \{s, a, b, c, d, e, f\}$, $\mathcal{T} = \{t\}$.



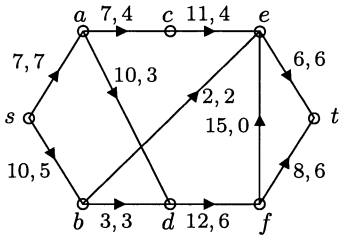

(c) We show a maximum of value 9. To see that this is maximum, consider the cut $\mathcal{S} = \{s, a\}$, $\mathcal{T} = \{b, c, d, e, f, t\}$.



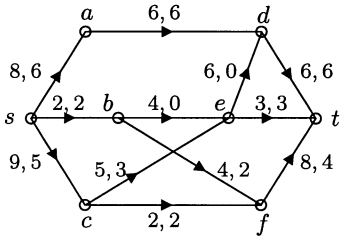

(d) We show a maximum of value 10. To see that the flow is maximum, consider the cut $\mathcal{S} = \{s, a, b, c, d, e\}$, $\mathcal{T} = \{t\}$.



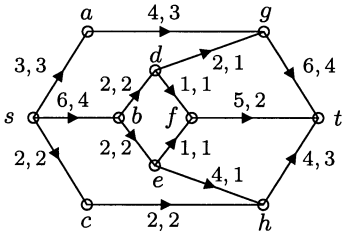
3. (a) [BB] To see that the pictured flow is maximum, consider the cut $\mathcal{S} = \{s, b\}$, $\mathcal{T} = \{a, c, d, e, f, t\}$.



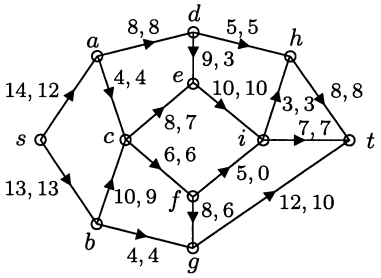

(b) To see that the pictured flow is maximum, consider the cut $\mathcal{S} = \{s, a, c, d, e\}$, $\mathcal{T} = \{b, f, t\}$.



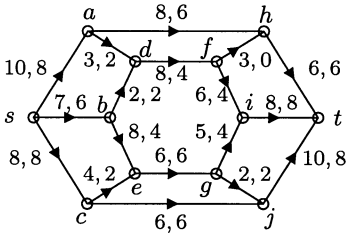

(c) [BB] To see that the pictured flow is maximum, consider the cut $\mathcal{S} = \{s, b\}$, $\mathcal{T} = \{a, c, d, e, f, g, h, t\}$.



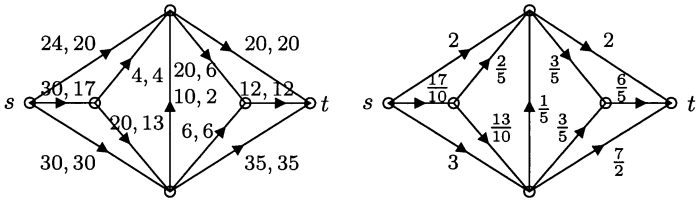

(d) To see that the pictured flow is maximum, consider the cut $\mathcal{S} = \{s, a\}$, $\mathcal{T} = \{b, c, d, e, f, g, h, i, t\}$.



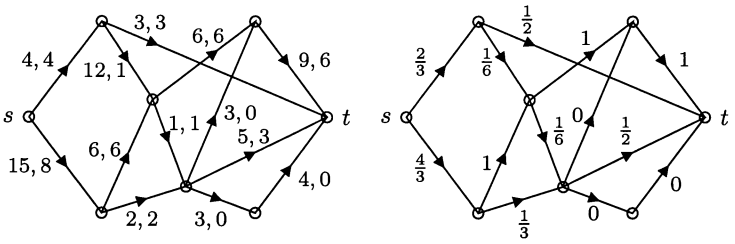

(e) To see that the pictured flow is maximum, consider the cut $\mathcal{S} = \{s, a, b, c, d, e, f, g, h, i\}$, $\mathcal{T} = \{j, t\}$.




4. (a) [BB] We multiply the capacities by 10, the greatest common divisor of the denominators, and find a maximum flow in the new network, as shown on the left. Then, dividing by 10, we find a maximum flow in the given network, of value $\frac{67}{10}$, as shown on the right. (Only the flow in each arc is shown.)

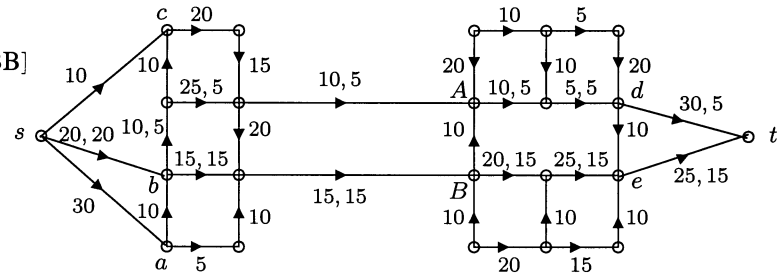



(b) We multiply the capacities by 6, the greatest common divisor of the denominators, and find a maximum flow in the new network, as shown on the left. Dividing by 6, we find a maximum flow in the given network, of value 2, as shown on the right. (Only the flow in each arc is shown.)



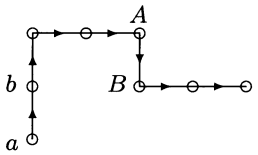

## Exercises 14.3

1. [BB]

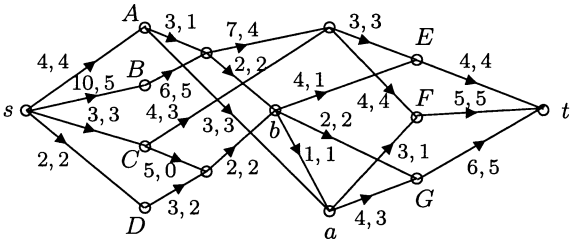



The demand cannot be met; the best that can be done is as shown. Warehouse $b$ sends out 20 Klein bottles and the others send out none. Five go to retail outlet $d$ and 15 to retail outlet $e$.

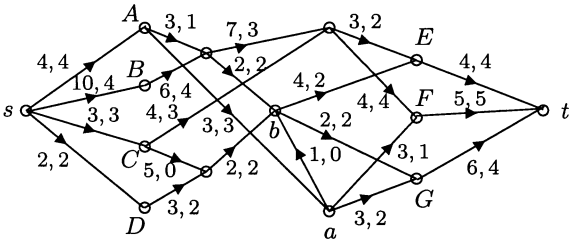
2. If the arrow on $AB$ is reversed, five more Klein bottles can be sent. They could go from warehouse $a$ to retail outlet $e$ as shown, or they could go from $c$ to $e$. This is clearly best possible now. A cut right down the middle of the network has capacity 25.



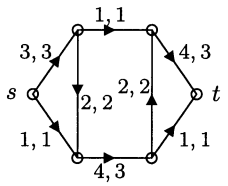
3. [BB] The needs of the retail outlets cannot be met. They can receive at most 14 units a month, as shown. To see that the flow there is a maximum, consider the cut in which $\mathcal{T} = \{a, F, G, t\}$ and $\mathcal{S}$ is the set of all other vertices.

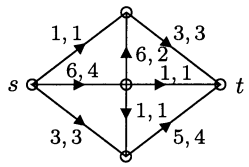



4. Now the retail outlets can receive at most 13 units. To see that the flow shown is maximum, consider the cut in which $\mathcal{T} = \{a, F, G, t\}$ and $\mathcal{S}$ is the set of all other vertices.



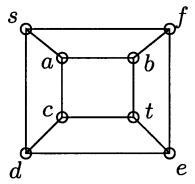

5. (a) [BB] A maximum flow is shown on the left, with directions added to the undirected edges.




(b) A maximum flow is shown on the right, with directions added.

6. The maximum number of edge disjoint paths from $s$ to $t$ is three: Observe that $sdct$, $sabt$, and $sfet$ are three edge disjoint paths from $s$ to $t$ and no more than three such paths can be found. The minimum number of edges whose removal severs all paths from $s$ to $t$ is three: If fewer than three edges incident with $s$ are not removed, then a path remains.




7. [BB] Four messengers can be sent.

8. Four messengers can be sent.

9. [BB] Let $u$ and $v$ be two vertices of $\mathcal{K}_n$. If $x$ is any other vertex, $u, x, v$ is a path from $u$ to $v$, while $u, v$ is also a path. These $n-1$ paths are edge disjoint. Also, there cannot be more than $n-1$ paths from $u$ to $v$ since $\deg u = n-1$ and edge disjoint paths must start with different edges. Thus, the maximum number of edge disjoint paths between $u$ and $v$ is $n-1$. But this is also the minimum number of edges which must be removed in order to sever all paths from $u$ to $v$ (in accordance with Menger's Theorem). To see this, note that deleting all edges incident with $u$ certainly severs all paths, so at most $n-1$ edges need be deleted. On the other hand, if we remove fewer than $n-1$ edges, then some path from $u$ to $v$ remains because we have already exhibited $n-1$ edge disjoint paths.

10. (a) Let $\mathcal{V}_1$ denote the bipartition set of order $n$ and $\mathcal{V}_2$ that of order $m$ in $\mathcal{K}_{n,m}$. Suppose $s$ and $t$ are in $\mathcal{V}_1$. For each vertex $v$ in $\mathcal{V}_2$, $s, v, t$ is a path from $s$ to $t$ and the $m$ paths produced this way

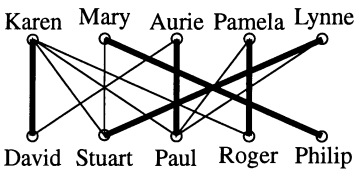
are edge disjoint. Thus, there are at least $m$ edge disjoint paths from $s$ to $t$. But there cannot be more since $\deg s = m$ and edge disjoint paths must start with different edges. We claim that $m$ is also the minimum number of edges which must be removed to sever all paths from $s$ to $t$ (in accordance with Menger's Theorem). To sever all such paths, we must remove at least $m$ edges since for every $v \in \mathcal{V}_2$, either $sv$ or $vt$ must be removed. On the other hand, removing all $m$ edges incident with $s$ clearly severs all paths from $s$ to $t$ so the minimum number of edges that have to be removed is $m$.

(b) Employing the notation of (a), suppose $s$ is in $\mathcal{V}_1$ and $t$ is in $\mathcal{V}_2$. The maximum number of edge disjoint paths between $s$ and $t$ and the minimum number of edges that must be removed in order to sever all paths from $s$ to $t$ is $\min\{m, n\}$ (in accordance with Menger's Theorem).

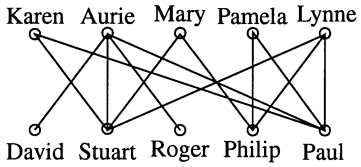
Without loss of generality, we assume that $m \leq n$, so that the minimum of $m$ and $n$ is $m$. Let $v_1, v_2, \ldots, v_{m-1}$ be the vertices of $\mathcal{V}_2$ other than $t$. Since $m \leq n$, there exist $m-1$ distinct vertices $u_1, u_2, \ldots, u_{m-1}$ in $\mathcal{V}_1$ which are different from $s$. Then we obtain $m$ edge disjoint paths $s, v_i, u_i, t$ from $s$ to $t$. There cannot be more since $\deg s = m$ and edge disjoint paths must start with different edges. This is also the least number of edges that must be removed in order to sever all paths from $s$ to $t$. Certainly this number is at most $m$ because removing all edges incident with $s$ does the job. And the number can't be less than $m$ since we have exhibited $m$ edge disjoint paths.

## Exercises 14.4

1. [BB] One possible matching is shown.




2. If $X = \{\text{Karen, Mary, Pamela, Lynne}\}$, for example, then $A(X) = \{\text{Stuart, Paul, Philip}\}$. Since $|X| > |A(X)|$, no matching exists, by Hall's Theorem.




3. [BB] $\{12, 34, 56, 78\}$ is a perfect matching.

4. (a) [BB] Let $X = \{\text{Angela, Brenda, Christine, Margaret, Phyllis, Renée}\}$. Then $A(X) = \{A, B, C, E, F\}$, so $|X| > |A(X)|$ and the condition fails.

   (b) The most we can hope for is six women to fill positions, and this can be done as follows: Angela fills $A$, Brenda fills $B$, Christine fills $C$, Helen fills $D$, Margaret fills $E$ and Phyllis fills $F$.

5. (a) Setting up a bipartite graph in the obvious way ($\mathcal{V}_1$ representing the males, $\mathcal{V}_2$ the females and the edges selections), it is clear that we are looking for a perfect matching. Note that we needed the choices to be mutual in order to have a graph (instead of a digraph). Since every vertex has degree 5, Problem 6 gives the result.

(b) After the first night, we remove the edges corresponding to the dates which occurred. We are left with a bipartite graph in which every vertex has degree 4. Again, Problem 6 guarantees a perfect matching. Continue for the five nights, reducing the degree by 1 each time.

6. (a) [BB] Bruce $\leftrightarrow$ Maurice, Edgar $\leftrightarrow$ Michael, Eric $\leftrightarrow$ Roland, Herb $\leftrightarrow$ Richard.

(b) If Roland and Bruce share a canoe, then Maurice and Michael would both have to be with Edgar.

(c) No. If Bruce and Roland were together, then both Maurice and Michael would have to be in the same canoe with Edgar. On the other hand, if Bruce and Roland were not in the same canoe, then Bruce and Roland would have to be in the same canoe with Maurice and Michael (in some order), leaving no one for Edgar since he refuses to be with Herb.

7. (a) [BB] Construct a bipartite graph with vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1$ has $n$ vertices corresponding to $A_1, \ldots, A_n$, $\mathcal{V}_2$ has one vertex for each element of $S$ and there is an edge joining $A_i$ to $s$ if and only if $s \in A_i$. Given a subset $X$ of $\mathcal{V}_1$, the set $A(X)$ is precisely the set of elements in $\bigcup_{x \in X} A_x$. Thus this question is just a restatement of Hall's Marriage Theorem.

(b) This is a specific instance of part (a). Let $S$ be the set of 13 denominations and $A_1, \ldots, A_{13}$ be the sets of denominations represented in columns $1, \ldots, 13$, respectively. For example, if column 12 consists of four queens, $A_{12} = \{\text{queen}\}$. Using the result of part (a), it is sufficient to show that for any subset $X$ of $\{1, 2, \ldots, 13\}$, $|X| \leq |\bigcup_{x \in X} A_x|$. This condition says that in any single column, at least one denomination must be represented, that in any two columns, at least two denominations must be represented and, in general, that in any $k$ columns, at least $k$ denominations must be represented. This is clear since there are $4k$ cards in $k$ columns and if there are only $\ell < k$ denominations in these columns, at most $4\ell$ cards are available.

8. This problem is much like Problem 6. We prove that $|X| \leq |A(X)|$ for every $X \subseteq \mathcal{V}_1$. Suppose $|X| = k$ and $|A(X)| = \ell$. Then

$$km \leq \sum_{x \in X} \deg x = \sum_{y \in A(X)} \deg_X y \leq \ell M$$

where by $\deg_X y$ we mean the sum of the number of edges joining $y$ to vertices in $X$. Since $\ell M \leq \ell m$, we get $k \leq \ell$ as desired.

9. (a) [BB] Exchange the roles of $\mathcal{V}_1$ and $\mathcal{V}_2$ in Theorem 14.4.2. It may be possible to find a matching which saturates $\mathcal{V}_2$. Here's a necessary and sufficient condition for this to happen: If $X$ is any subset of $\mathcal{V}_2$ and $A(X)$ is the set of all vertices of $\mathcal{V}_1$ which are adjacent to some vertex of $X$, then $|A(X)| \geq |X|$.

(b) The case $|\mathcal{V}_1| > |\mathcal{V}_2|$ is the case where there are fewer jobs available than applicants. To say that $\mathcal{V}_2$ is saturated is to say that every job has been filled. This is possible if and only if for every set of $k$ jobs, there are at least $k$ people qualified to fill them.

10. (a) [BB] $\mathcal{K}_{m,n}$ has a perfect matching if and only if $m = n$. To see this, first assume that $m = n$ and let the vertex sets be $\mathcal{V}_1 = \{u_1, u_2, \ldots, u_m\}$ and $\mathcal{V}_2 = \{v_1, v_2, \ldots, v_m\}$.
Then $\{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ is a perfect matching. Conversely, say we have a perfect matching and $m \leq n$. Since each edge in a matching must join a vertex of $\mathcal{V}_1$ to a vertex of $\mathcal{V}_2$, there can be at most $m$ edges. If $m < n$, some vertex in $\mathcal{V}_2$ would not be part of any edge in the matching, a contradiction. Thus, $m = n$.

(b) Suppose $m < n$, $\mathcal{V}_1 = \{u_1, u_2, \ldots, u_m\}$ and $\mathcal{V}_2 = \{v_1, v_2, \ldots, v_n\}$.
Then $\{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ saturates as many vertices as possible since it saturates $\mathcal{V}_1$ and all edges have a vertex in $\mathcal{V}_1$.

11. (a) The new bipartition sets $\mathcal{V}_1^*$ and $\mathcal{V}_2^*$ are defined as follows: $\mathcal{V}_1^* = \mathcal{V}_1 \cup \{y\}$; $\mathcal{V}_2^* = \mathcal{V}_2 \cup \{x\}$.

    (b) Choose any edge $ab$ in the perfect matching of $\mathcal{G}$, where $a \in \mathcal{V}_1$, $b \in \mathcal{V}_2$. For a matching of $\mathcal{G} \cup \{x, y\}$, take edges $ax$, $by$ and all edges in the perfect matching of $\mathcal{G}$ except for $ab$. We easily see that this is a perfect matching for $\mathcal{G} \cup \{x, y\}$.

12. (a) [BB] By Proposition 14.4.3, if $\mathcal{K}_n$ has a perfect matching, then $n$ must be even. Conversely, assume that $n = 2m$ is even and label the vertices $u_1, u_2, \ldots, u_{2m}$. Since every pair of vertices is joined by an edge, a perfect matching will be $\{u_1u_2, u_3u_4, \ldots, u_{2m-1}u_{2m}\}$.

    (b) The result is still true, and the proof of (a) works as long as we label the vertices carefully. For example, choose $u_1$ and $u_3$ to be the endpoints of $e$, so that $e$ is not required in the matching. If $n = 2$, the result is false—in fact, $\mathcal{K}_2 \setminus \{e\}$ has no edges!

13. (a) Each edge in a perfect matching uses two vertices, and the vertices can only be used once. Hence, $|\mathcal{V}|$ equals twice the number of edges in the matching and is even.

    (b) By (a), one direction is clear. Conversely, if $|\mathcal{V}| = 2m$ is even, let $u_1, u_2, \ldots, u_{2m}$ be the vertices, in order, along a Hamiltonian path. Then $\{u_1u_2, u_3u_4, \ldots, u_{2m-1}u_{2m}\}$ is a perfect matching.

14. [BB] No. The graph has $|\mathcal{V}|$ even, but it has no perfect matching.

15. [BB] No. The graph has a perfect matching, but each vertex has degree 2, which is less than $\frac{1}{2}|\mathcal{V}| = 3$.

16. We will prove this by induction on $n$, the number of vertices in the tree $\mathcal{T}$. Note that if $n$ is odd, the result is true vacuously, while if $n = 2$, ○—○ is the unique matching.

    Let $k \geq 4$ and assume the result is true for all trees with $n < k$ vertices. We prove the result for $n = k$. Let $\mathcal{T}$ then be a tree with $k$ vertices and assume that $\mathcal{T}$ has a perfect matching. By Corollary 12.1.5, $\mathcal{T}$ contains a vertex $v$ of degree 1. Let $w$ be the unique vertex adjacent to $v$. Note that $vw$ must be in the matching (since $v$ has degree 1) and, therefore, no other edge incident with $w$ is in the matching. Consider $\mathcal{T}' = \mathcal{T} \setminus \{v, w\}$, the subgraph obtained by deleting $v$, $w$ and all edges incident with $w$. Because of the remarks above, $\mathcal{T}'$ has a perfect matching, namely that matching of $\mathcal{T}$ with $vw$ deleted. Now $\mathcal{T}'$ contains no circuits and, hence, each of its connected components is a tree with fewer than $k$ vertices. Moreover, each of these components has a perfect matching since $\mathcal{T}'$ does. By the induction hypothesis, these matchings are unique, and it follows that the matching on $\mathcal{T}'$ is unique. Thus, the original matching on $\mathcal{T}$ is unique, and we have the result.

## Chapter 14 Review

1. (a) At $a$, we have $\sum_{v \in \mathcal{V}} \mathsf{f}_{av} = \mathsf{f}_{ad} + \mathsf{f}_{ae} = 2 + 3 = 5$, while $\sum_{v \in \mathcal{V}} \mathsf{f}_{va} = \mathsf{f}_{sa} = 5$.
    At $e$, we have $\sum_{v \in \mathcal{V}} \mathsf{f}_{ev} = \mathsf{f}_{et} = 4$, while $\sum_{v \in \mathcal{V}} \mathsf{f}_{ve} = \mathsf{f}_{ae} + \mathsf{f}_{be} + \mathsf{f}_{ce} = 3 + 1 + 0 = 4$.

    (b) $\text{val}(\mathcal{F}) = \sum_{v \in \mathcal{V}} \mathsf{f}_{sv} - \sum_{v \in \mathcal{V}} \mathsf{f}_{vs} = \mathsf{f}_{sa} + \mathsf{f}_{sb} + \mathsf{f}_{sc} = 5 + 5 + 1 = 11$.

    (c) $\text{cap}(\mathcal{S}, \mathcal{T}) = \sum_{u \in \mathcal{S}, v \in \mathcal{T}} \mathsf{c}_{uv} = \mathsf{c}_{sc} + \mathsf{c}_{ad} + \mathsf{c}_{ae} + \mathsf{c}_{bd} + \mathsf{c}_{be} + \mathsf{c}_{bf} = 6 + 4 + 5 + 1 + 4 + 4 = 24$.
    Illustrating Corollary 14.1.5, we note that $\sum_{u \in \mathcal{S}, v \in \mathcal{T}} (\mathsf{f}_{uv} - \mathsf{f}_{vu}) = \mathsf{f}_{sc} + \mathsf{f}_{ad} + \mathsf{f}_{ae} + \mathsf{f}_{bd} + \mathsf{f}_{be} + \mathsf{f}_{bf} = 1 + 2 + 3 + 0 + 1 + 4 = 11$, in agreement with part (b).

(d) No. This path contains the saturated arc $bf$.

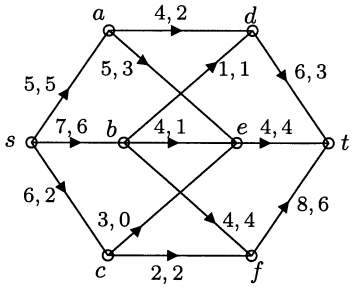
(e) No. The flow can be increased along the path $sbdt$ (or along $scft$).

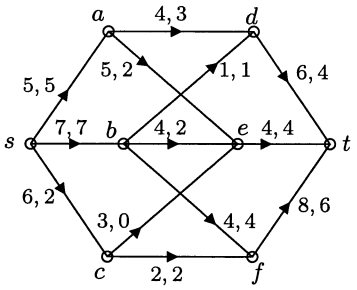
2. The network is now as shown on the left.

(a) $sadt$ and $saet$ both contain the saturated arc $sa$; $sbdt$ contains the saturated arc $bd$; $sbft$ contains the saturated arc $bf$; $sbet$ and $scet$ both contain the saturated arc $et$, and $scft$ contains the saturated arc $cf$.

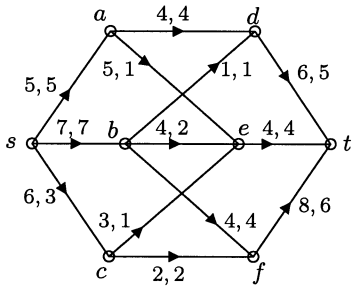
(b) The slack of $sbeadt$ is 1, since $c_{sb} - f_{sb} = 7 - 6$.

(c) Using the flow-augmenting chain in (b), we increase the flow as shown on the right.

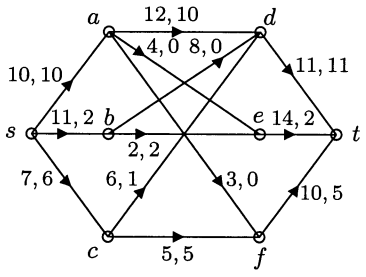


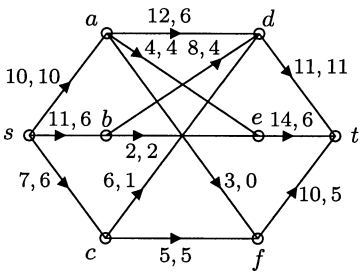




Note that we still have another flow-augmenting chain, namely $sceadt$. The slack in this chain is now 1: $c_{ad} - f_{ad} = 4 - 3$. We increase the flow as shown and claim that this flow is now maximal, of value 15. To prove this, we construct an $(s,t)$–cut using the method of Theorem 14.2.2. Thus $\mathcal{S} = \{s, c, e, a, b\}$ and $\mathcal{T} = \{d, f, t\}$. We have $\text{cap}(\mathcal{S}, \mathcal{T}) = c_{ad} + c_{bd} + c_{bf} + c_{cf} + c_{et} = 4+1+4+2+4 = 15$, which is the same as the value of our flow.



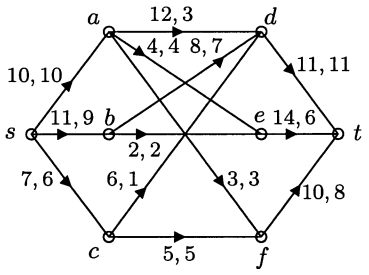

3. Pushing unit flows through until we have a saturated arc on each path from $s$ to $t$ gives the network on the left.



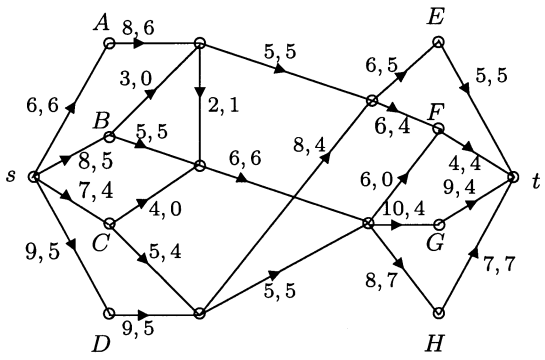

This is not a maximal flow; for example, $sbdaet$ is a flow-augmenting chain with slack 4. We increase the flow to that shown on the right above.

There is still another flow-augmenting chain, namely $sbdaft$, of slack 3. The flow increases to that shown on the right. The flow has value 25. We claim that this is maximal. Constructing an $(s,t)$–cut using the method of Theorem 14.2.2 gives $\mathcal{S} = \{s, b, d, a, c\}$, $\mathcal{T} = \{e, f, t\}$. Now $\text{cap}(\mathcal{S}, \mathcal{T}) = c_{ae} + c_{af} + c_{be} + c_{cf} + c_{dt} = 4 + 3 + 2 + 5 + 11 = 25$, agreeing with our value for the flow.




4. We are given that the value of the flow is $\text{cap}(\mathcal{S}, \mathcal{T}) = \sum_{u \in \mathcal{S}, v \in \mathcal{T}} c_{uv} = \sum_{v \in \mathcal{T}} c_{sv} = \sum_{v \in \mathcal{V}} c_{sv}$. However, $\text{val}(\mathcal{F}) = \sum_{v \in \mathcal{V}} f_{sv}$. Since $f_{sv} \leq c_{sv}$ always, those can only be equal if $f_{sv} = c_{sv}$ for all $v$, that is, each arc leaving $s$ is saturated.

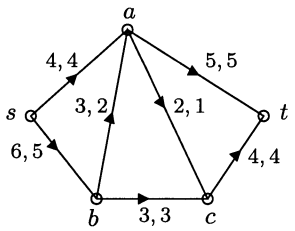
5. Add vertices $s$ and $t$ as shown. A maximal flow in this network is given. Note that all requirements have not been met.



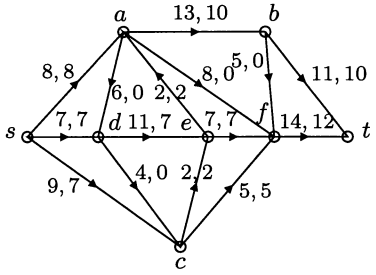

6. (a) With arcs $ba$ and $bc$ oriented as indicated, the maximum flow has value 9. We can see this is maximum by considering the cut $(\mathcal{S}, \mathcal{T})$, with $\mathcal{S} = \{s, a, b, c\}$ and $\mathcal{T} = \{t\}$ whose capacity is $c_{at} + c_{ct} = 5 + 4 = 9$.

   (b) With arcs $ab$, $ce$, $de$, and $ea$ oriented as indicated, the maximum flow has value 22. We can see this is maximum by considering the cut $(\mathcal{S}, \mathcal{T})$, with $\mathcal{S} = \{s, c\}$ and $\mathcal{T} = \{a, b, d, e, f, t\}$ whose capacity is $c_{sa} + c_{sd} + c_{ce} + c_{cf} = 8 + 7 + 2 + 5 = 22$.

(a)




(b)




7. (a) Trees! A graph is a tree if and only if there is precisely one path between any two vertices. (See Proposition 12.1.2.)

   (b) Cycles! To see this, note that a Hamiltonian graph $\mathcal{G}$ has a cycle containing all the vertices of $\mathcal{G}$, so there are at least two edge disjoint paths between any two vertices (either way round). If, in addition to the Hamiltonian cycle, there was another edge $uv$, then there would be three edge disjoint paths between $u$ and $v$ (edge $uv$ and either way round the cycle). So the graphs with the given property are precisely those which are cycles.

8. It is not possible for these people to select different teams. If $X = \{\text{Bruce, Craig, Camelia, Yuri}\}$, then $A(X) = \{\text{Russia, Slovakia, Romania}\}$. We have $|X| = 4$ while $|A(X)| = 3$: Hall's Marriage Theorem says the desirable cannot be done.

9. Now the desired betting scheme is possible. For example, we could have

| | | |
|---|---|---|
| Albert | betting on | USA |
| Bruce | betting on | Canada |
| Craig | betting on | Slovakia |
| Camelia | betting on | Romania |
| Oana | betting on | Finland |
| Yuri | betting on | Russia. |

10. ($\longrightarrow$) If $\mathcal{G}$ has a perfect matching, then Proposition 14.4.3 says $\mathcal{G}$ has an even number of vertices.

($\longleftarrow$) Conversely, suppose $\mathcal{G}$ has $2n$ vertices. It is not hard to see that a connected graph in which every vertex has degree 2 must be a cycle. But if the vertices of the cycle, written in order, are $v_1, v_2, \ldots, v_{2n}$, then edges $v_1v_2$, $v_3v_4$, $\ldots$, $v_{2n-1}v_n$ give a perfect matching.

11. (a) The sum of the degrees of the vertices of $\mathcal{G}$ is $3|\mathcal{V}|$. This is also $2|\mathcal{E}|$, so $2 \mid |\mathcal{V}|$.

(b) No, and we present a counterexample at the right. If this graph had a perfect matching, the middle vertex $v$ would have to be joined to an edge which is part of the matching, say $vw$ as shown. Then the five vertices in the leftmost group (or in the bottom group) could not all be saturated, since they could only be joined to each other.